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Probability Theory on Trees and Networks

Exercise Sheet 9

Submission is due on 01/18/2021 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de and to chiranjib.mukherjee@uni-muenster.de as a pdf-file.

Definition: An event is called a *tail event* if its occurrence does not depend on the first finitely many values of the random walk. More precisely, it is defined as follows. Let $\Omega = \Gamma^{\mathbb{N}}$ and define an equivalence relation by declaring $\omega_1 \sim \omega_2$ if ω_1 and ω_2 differ (as sequences) by finitely many values. Then, we say that A is a *tail event* if $\omega \in A$ implies $\omega' \in A$ for all $\omega' \sim \omega$. The set of all tail events forms a σ-algebra, known as the *tail-σ-algebra* $\mathcal{T}$.

Exercise 1 (5 points)

Show that $\mathcal{T} = \bigcap_{n\geq 0} \sigma(X_n, X_{n+1}, ...)$ where $\sigma(X_n, X_{n+1}, ...)$ is the smallest σ-algebra such that $X_n, X_{n+1}, ...$ are measurable.

Exercise 2 (6 points)

(i) Let X_n be the SRW on $\mathbb{Z}$ and define the event $A = \{X_n \text{ visits } o \text{ infinitely often}\}$. Prove that $A \in \mathcal{T}$. Also compute the probability of the event A, i.e. $\mathbb{P}_0(A)$.

(ii) Let X_n be the SRW on $\mathbb{F}_2$. Prove that the event that X_n ends in the top branch of the (4-regular) tree, which we denote by B, is a tail event. Also show that $\mathbb{P}_0(B) = 1/4$.

Exercise 3 (4 points)

Let X, Y, Z be three random variables and denote by $H(\cdot\,|\,\cdot)$ the conditional entropy as always. Prove that $H(X|Y, Z) \leq H(X|Y)$.

Exercise 4 (5 points)

Suppose u is a bounded harmonic function with respect to an irreducible Markov chain and c is a constant such that $\lim_{n\to\infty} u(X_n) = c$ almost surely with respect to $\mathbb{P}_0$. Show that $u \equiv c$.