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Winter Semester 2020/21
11/23/2020

Probability Theory on Trees and Networks

Exercise Sheet 4

Submission is due on 11/30/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Recall Lecture 6 for definition of the Lamplighter group.

Exercise 1 (5 points)

Prove that the Lamplighter group is not finitely presentable.

Exercise 2 (5 points)

Prove that the Lamplighter group has exponential growth.

Exercise 3 (6 points)

Recall that the edge-isoperimetric constant of (G, c, D) is defined as

$$\Phi_E(G) = \Phi_E(G, c, D) := \inf \left\{ \frac{|\partial_E K|_c}{|K|_D} : \emptyset \neq K \subset V \text{ finite} \right\}.$$

(i) Show that $\mathbb{Z}^2$ is edge-amenable. That is, $\Phi_E(\mathbb{Z}^2) = 0$.

(ii) Show that $\Phi_E(T_{b+1}, \mathbf{1}, \mathbf{1}) = b - 1$ for all $b \geq 1$, where T_{b+1} is a $(b+1)$-regular tree.

Exercise 4 (4 points)

Recall that

$$\ell^2_-(E) = \left\{ \theta : E \to \mathbb{R} : \theta(-e) = -\theta(e) \text{ and } \sum_{e \in E} \theta^2(e) < \infty \right\}$$

is the space of all antisymmetric functions $\theta \in \ell^2(E)$. We further have

$$d : \ell^2(V) \to \ell^2_-(E) \quad \text{with} \quad (df)(e) = f(e^-) - f(e^+)$$

and

$$d^* : \ell^2_-(E) \to \ell^2(V) \quad \text{with} \quad (d^*\theta)(x) = \sum_{e : e^- = x} \theta(e).$$

Show that for all $f \in \ell^2(V)$ and for all $\theta \in \ell^2_-(E)$,

$$\langle \theta, df \rangle = \langle d^*\theta, f \rangle.$$