Jun.-Prof. Dr. C. Mukherjee
Yannic Bröker

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Probability Theory on Trees and Networks

Exercise Sheet 3

Submission is due on 11/23/2020 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Recall that a group G is freely generated by a set $S \subset G$ if for any group Γ and any map $\phi : S \to \Gamma$ there is a unique homomorphism $\widetilde{\phi} : G \to \Gamma$ extending ϕ (i.e. $\widetilde{\phi}(x) = \phi(x)$ for all $x \in S$).

Exercise 1 (6 points)

(a) Suppose that G is freely generated by S and G' is freely generated by S' and $|S| = |S'|$. Then prove that G is isomorphic to G'.

(b) Prove that a free group is torsion-free (i.e. if $x^n = \mathrm{id}$ then $x = \mathrm{id}$).

Exercise 2 (4 points)

Recall that the normal closure of a subset A of a group G is the smallest normal subgroup of G containing A (written as $\langle\langle A \rangle\rangle$). Show that

$$\langle\langle A \rangle\rangle = \langle \{gag^{-1} : g \in G, a \in A\} \rangle.$$

Exercise 3 (4 points)

Let S and T be finite and symmetric generating sets in a group G and let d_S and d_T be the word metrics corresponding to S and T. Then show that there exists a real number $K > 1$ such that for all $g, h \in G$

$$\frac{1}{K}\ d_T(g,h) \leq d_S(g,h) \leq K\ d_T(g,h).$$

Exercise 4 (6 points)

(a) Let $\langle S|R \rangle$ is a presentation of a group G and $\langle S'|R' \rangle$ is a presentation of a group G'. Define $g * g' = \langle S \cup S' | R \cup R' \rangle$. Prove that the Cayley graph of $\mathbb{Z} * \mathbb{Z}_2$ is isomorphic to that of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ (i.e. $\mathrm{Cay}(\mathbb{Z} * \mathbb{Z}_2) \simeq \mathrm{Cay}(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$).

(b) Compute the velocity $\lim_{n\to\infty} \frac{d(e,X_n)}{n}$ of a simple random walk on this Cayley graph.