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Winter Semester 2020/21
01/18/2021

Probability Theory on Trees and Networks

Exercise Sheet 10

Submission is due on 01/25/2021 9 p.m.

Please send the solutions to yannic.broeker@uni-muenster.de as a pdf-file.

Exercise 1 (4 points)

Let $\mathbb{T}_3$ be the 3-regular tree. Construct 3 non-constant bounded harmonic functions on $\mathbb{T}_3$ with respect to the simple random walk.

Exercise 2 (4 points)

Consider a network walk on $\mathbb{N}$ by assigning conductance $2^{\binom{k}{2}}$ to the edge between $k-1$ and k for all $k \geq 2$. Show that the σ-algebra $\mathcal{I}$ is $\mathbb{P}_0$-trivial, and show that the σ-algebra $\mathcal{T}$ is not $\mathbb{P}_0$-trivial.

Exercise 3 (4 points)

Prove the missing part of the theorem stated in today's lecture. That is, if P is the transition probability matrix of a transitive Markov chain $(X_n)_n$ on V and $\widetilde{P} = \frac{I+P}{2}$ is that of the corresponding lazy chain $(\widetilde{X}_n)_n$ and if V is equipped with an invariant graph metric d, then $\widetilde{\ell} = \ell/2$.

Exercise 4 (8 points)

Given a transitive Markov chain $(X_n)_n$ on a state space V. Let R_n be the number of distinct sites visited by time n. That is, $R_n = \text{range of } X_n \stackrel{\text{def}}{=} \#\{\text{distinct sites visited by } X_1, X_2, ..., X_n\}$.

Fix some $o \in V$ and show that

$$\lim_{n\to\infty} \mathbb{E}_0\left[\frac{R_n}{n}\right] = \mathbb{P}_0(\text{no return}) = \mathbb{P}_0(X_k \neq 0 \text{ for all } k \geq 1).$$

Deduce that R_n/n converges $\mathbb{P}_0$-almost surely to the same limit.