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Winter Semester 2019/20
11/20/2019

Probability theory II

Exercise Sheet 7

Submission is due on 11/27/2019 2 p.m.
Box 133

Please note that only the 4 best exercises will count for your marks. But we recommend to do all exercises, since they are all relevant for the exam.

Exercise 1 (5 points)

Let $(\xi_n)_n$ be iid random variables such that $\mathbb{E}[\xi_n^+] = \infty$ and $\mathbb{E}[\xi_n^-] < \infty$. Does $S_n = \xi_1 + ... + \xi_n$ converge almost surely? If yes, prove the statement, if no, give a counterexample.

Exercise 2 (5 points)

Let T be a stopping time such that for some $N \in \mathbb{N}$, some $\varepsilon > 0$ and for all $n \in \mathbb{N}$,

$$\mathbb{P}(T \leq n + N | \mathcal{F}_n) > \varepsilon \quad \text{a.s.}$$

Prove that $\mathbb{E}[T] < \infty$.

Exercise 3 (5 points)

Let $(\xi_i)_i$ be iid such that $\xi_i \sim \text{Normal}(0,1)$. Let $M_n = e^{a\sum_{i=1}^n \xi_i - bn}$ for $a, b \in \mathbb{R}$. Prove:

(a)

$$M_n \longrightarrow 0 \quad \text{a.s.} \qquad \Longleftrightarrow \qquad b > 0$$

(b) For $p \geq 1$:

$$M_n \longrightarrow 0 \quad \text{in } L^p \qquad \Longleftrightarrow \qquad p < \tfrac{2b}{a^2}.$$

Exercise 4 (5 points)

Let $(S_n)_n$ be a sequence of $\mathbb{R}$-valued random variables such that for any closed set $K \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \in K) \leq - \inf_{x \in K} I(x)$$

where $I : \mathbb{R} \to [0, \infty]$ is some lower semicontinuous function such that $\{x : I(x) = 0\} = \{0\}$. Prove that $S_n \to 0$ almost surely.

Exercise 5 (5 points)

Let $(X_n)_n$ be a Markov chain taking values in a compact state space E. Let $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{X_j} \in \mathcal{M}_1(E)$. Prove that for all $K \subset \mathcal{M}_1(E)$ closed,

$$\sup_{x\in E} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x(\mu_n \in K) \leq - \inf_{\mu \in K} I(\mu)$$

where $I(\mu) = \sup_{u>0} \int_E \log\left(\frac{u(y)}{(\mathbf{P}u)(y)}\right)\mu(\mathrm{d}y)$.

Hint: You may use (exponential) Chebyshev's inequality and exercise 5 from the Exercise sheet 5.