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Winter Semester 2019/20
11/13/2019

Probability theory II

Exercise Sheet 6

Submission is due on 11/20/2019 2 p.m.
Box 133

Please note that only the 4 best exercises will count for your marks. But we recommend to do all exercises, since they are all relevant for the exam.

Exercise 1 (5 points)

Consider the SRW on $\mathbb{Z}$ with a slight drift towards 0:

- $\pi(x, x+1) - \pi(x, x-1) \geq \frac{a}{|x|}$ if $x \leq -l$
- $\pi(x, x-1) - \pi(x, x+1) \geq \frac{a}{|x|}$ if $x \geq l$.

Show that the RW with these transition probabilities is positive recurrent.

Hint: Take X to be a countable set. Suppose you can find a function $V \geq 0$, a finite set $F = (-l, l)$ and a constant $C \geq 0$ such that for the transition operator $\mathbf{P}$,

$$\mathbf{P}V(x) - V(x) \leq \begin{cases} -1, & \text{if } x \notin F \\ C, & \text{if } x \in F. \end{cases}$$

Then show that for any $n \in \mathbb{N}$

$$-V(x) \leq -n + (1 + C) \sum_{j=1}^{n} \sum_{y \in F} \pi^{(j-1)}(x, y). \tag{1}$$

If a RW with transition operator $\mathbf{P}$ was null-recurrent, then show that (1) implies a contradiction to the assumptions imposed on V.

Definition: A sequence $(\mu_n)_{n\geq 1}$ of finite measures on (S, d) converges weakly to a finite measure on (S, d), shorthand $\mu_n \stackrel{w}{\to} \mu$, if

$$\lim_{n\to\infty} \int f \mathrm{d}\mu_n = \int f \mathrm{d}\mu$$

for all bounded, continuous functions $f : S \to \mathbb{R}$.

Exercise 2 (5 points)

Let $(\mu_n)_n$ be a sequence of probability measures on $\mathbb{R}$. Prove that the following statements are equivalent:

(i) $\mu_n \stackrel{w}{\to} \mu$

(ii) $\liminf_n \mu_n(G) \geq \mu(G)$ for all $G \subset \mathbb{R}$ open

(iii) $\limsup_n \mu_n(C) \leq \mu(C)$ for all $C \subset \mathbb{R}$ closed

(iv) $\lim_n \mu_n(A) = \mu(A)$ for all A with $\mu(\partial A) = 0$

(v) $\lim_{n\to\infty}\int f\mathrm{d}\mu_n = \int f\mathrm{d}\mu$ for all bounded and Lipschitz f.

Definition: A sequence $(\mu_n)_{n\geq 1}$ of finite measures on (S,d) converges vaguely to a finite measure on (S,d), shorthand $\mu_n \xrightarrow{v} \mu$, if

$$\lim_{n\to\infty}\int f\mathrm{d}\mu_n = \int f\mathrm{d}\mu$$

for all continuous functions $f : S \to \mathbb{R}$ with compact support.

Exercise 3 (5 points)

Prove whether the following sequences of probability measures converge weakly and/or vaguely:

(a) $\mu_n = \delta_n$

(b) $\mu_n = \frac{1}{n}\delta_1 + (1-\frac{1}{n})\delta_0$

(c) $\mu_n = (1-\frac{1}{n})\delta_{-1} + \frac{1}{n}\delta_{n^2}$

(d) $\mu_n = (1-n\sin(\frac{1}{n}))\mathcal{N}(0,n) + n\sin(\frac{1}{n})\mathcal{N}(\frac{1}{n},2)$

(e) $\mu_n = \frac{1}{2}\delta_{\sin(n)} + \frac{1}{2}\delta_{\cos(n)}$

Prokhorov's theorem: Let (S,d) be a separable metric space and $\mathcal{M}_1(S)$ denote the collection of all probability measures defined on S. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_1(S)$. Then $(\mu_n)_n$ has a convergent subsequence in the weak topology in $\mathcal{M}_1(S)$ if and only if $(\mu_n)_n$ is tight, i.e.

$$\forall \epsilon > 0, \exists K \subset S, \text{ such that } \sup_n \mu_n(K^{\mathrm{C}}) < \epsilon.$$

Exercise 4 (5 points)

Let $(f_n)_n$ be a sequence of continuous functions on $\mathbb{R}$ such that (f_n) is uniformly bounded and $f_n \to f$ uniformly on compact subsets of $\mathbb{R}$. Let $(\mu_n)_n$ be a sequence of probability measures such that $\mu_n \xrightarrow{w} \mu$. Prove,

$$\int f_n \mathrm{d}\mu_n \longrightarrow \int f\mathrm{d}\mu.$$

Hint: You may use Prokhorov's theorem without proving it.

Exercise 5 (5 points)

In exercise 3 of Exercise Sheet 2 you had to prove the identity

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}} = 1.$$

Now assume that the identity

$$\liminf_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}} = -1$$

holds true. Both identities together are called the "Law of the iterated logarithm".

Now let $(X_n)_{n\geq 1}$ be a sequence of iid Normal$(0,1)$-distributed random variables and $S_n = X_1 + ... + X_n$.

(a) Prove that $\frac{S_n}{\sqrt{2n\log\log n}} - \frac{S_{n+1}}{\sqrt{2(n+1)\log\log(n+1)}}$ converges to 0 a.s.

(b) Let $L = \{\text{all limit points of } \frac{S_n}{\sqrt{2n \log \log n}} \text{ as } n \to \infty\}$. Prove (e.g. using the Law of the iterated logarithm and part (a)) that $L = [-1, 1]$.