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Probability theory II

Exercise Sheet 4

Submission is due on 11/06/2019 2 p.m.
Box 133

Exercise 1 (4 points)

(a) Let E be any complete and separable metric space equipped with transition probabilities $\{p(x,\cdot)\}_{x\in E}$. Then $(X_n)_n$ is a Markov chain with values in E with transition probabilities $\{p(x,\cdot)\}_x$ if and only if for all harmonic functions $h: E \to \mathbb{R}$, $(h(X_n))_n$ is a martingale (with respect to the canonical filtration).

(b) Let E be any complete and separable metric space equipped with transition probabilities $\{p(x,\cdot)\}_{x\in E}$. Then $(X_n)_n$ is a Markov chain with values in E with transition probabilities $\{p(x,\cdot)\}_x$ if and only if for all sub-harmonic (super-harmonic) functions $h: E \to \mathbb{R}$, $(h(X_n))_n$ is a sub-martingale (super-martingale) (with respect to the canonical filtration).

Exercise 2 (2 points)

Let $(X_n)_n$ be a Markov chain taking values in E, which is finite or countable, with transition probabilities $P = (p_{i,j})$. Define

$$f_{i,j} = \mathbb{P}(\tau_j < \infty)$$

where $\tau_j = \inf\{n \geq 1 : Z_n = j\}$. Show that

$$f_{i,j} = 1\, \forall i,j \in E \Leftrightarrow \text{every non-negative } P-\text{superharmonic function on } E \text{ is constant.}$$

Exercise 3 (4 points)

Let $(S_n)_{n\geq 0}$ be a random walk defined as $S_n = \xi_1 + ... + \xi_n$ with $S_0 = 0$ and $\mathbb{P}(\xi_1 = \pm 1) = 1/2$.

(a) Show that there exists $\sigma > 0$ such that $\mathbb{E}_0[e^{\sigma\tau_R}] < \infty$, where $\tau_R = \inf\{n \geq 1 : S_n \notin (-R,R)\}$ and $R > 0$.

(b) Is the estimate true for all $\sigma > 0$?

Exercise 4 (6 points)

Let Ω be an open set in $\mathbb{R}^n$.

(a) If $h : \Omega \to \mathbb{R}$ is twice continuous differentiable and $\Delta h = 0$, then show that for all balls $B_r(x) \subset \Omega$ of radius r around x,

$$u(x) = \frac{1}{n w_n r^{n-1}} \int_{\partial B_r(x)} u(y)\sigma(\mathrm{d}y) \tag{1}$$

and

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\mathrm{d}y \tag{2}$$

where σ is the $(n-1)$-dimensional surface measure on $\partial B_r(x)$ and $w_n = |B_1(0)|$.

(b) Let u be any locally integrable function (i.e. $\int_K |u(x)| \mathrm{d}x < \infty$) for all $K \subset \Omega$ compact) such that (1) holds for any $B_r(x) \subset \Omega$. Then show that u is infinitly many times differentiable and $\Delta u = 0$.

(c) Show that the function

$$h(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & \text{if } n = 2 \\ \frac{1}{n(n-2)w_n} \frac{1}{|x|^{n-2}}, & \text{if } n \geq 3 \end{cases}$$

defined on $\Omega = \mathbb{R}^n \setminus \{0\}$ is a harmonic function.

Exercise 5 (4 points)

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $\bar{\Omega}$ denote the closure of Ω. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic within Ω. Then show that

(a) $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

(b) Assume that Ω is also connected. If there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then show that u is constant inside Ω.