Jun.-Prof. Dr. C. Mukherjee
Yannic Bröker

Winter Semester 2019/20
10/23/2019

Probability theory II

Exercise Sheet 3

Submission is due on 10/30/2019 2 p.m.
Box 133

Exercise 1 (4 points)

Let $(\xi_n)_{n\in\mathbb{N}}$ be iid with $\mathbb{P}(\xi_n = \pm 1) = \frac{1}{2}$, $S_0 = 0$ and $S_n = \sum_{j=1}^n \xi_j$. Let T be a stopping time with $\mathbb{E}[T] < \infty$. Show that

(a) $S^\star(\omega) = \sup_{1\le n\le T} |S_n(\omega)|$ is square integrable.
Hint: $S_n^2 - n$ is a martingale.

(b) $\mathbb{E}[S_T] = 0$.

Exercise 2 (4 points)

Let $(X_n)_{n\in\mathbb{N}_0}$ be a sub-martingale.

(a) Show that X_n can be written as $X_n = Y_n + A_n$ with the following properties:

  (i) $(Y_n)_{n\in\mathbb{N}_0}$ is a martingale

  (ii) $A_{n+1} \ge A_n$ for almost all ω and every $n \ge 0$

  (iii) $A_0 \equiv 0$

  (iv) for all $n \ge 1$, A_n is $\mathcal{F}_{n-1}$-measurable

(b) Show that the above decomposition of X_n with the prescribed properties is unique (i.e. if $X_n = Y_n' + A_n'$ with Y_n', A_n' satisfying (i)-(iv), then $Y_n = Y_n'$ and $A_n = A_n'$ almost everywhere).

Exercise 3 (6 points)

(a) Show that if X is any real valued random variable such that $\mathbb{E}[X] = 0$ and $a \le X \le b$ almost everywhere, then

$$\forall \lambda \in \mathbb{R}: \qquad \mathbb{E}[e^{\lambda X}] \le \exp\Big(\frac{\lambda^2 (b-a)^2}{8}\Big).$$

(b) Let $(X_n)_n$ be a martingale such that $|X_n - X_{n-1}| \le C_n$ for all n. Then, for all $\varepsilon > 0$, show that

$$\mathbb{P}(|X_N - X_0| \ge \varepsilon) \le 2\exp\Big(-\frac{\varepsilon^2}{2\sum_{k=1}^N C_k^2}.\Big).$$

(c) Show that if $(X_n)_n$ is a sub-martingale such that $|X_n - X_{n-1}| \le C_n$ for all n, then for all $\varepsilon > 0$,

$$\mathbb{P}(X_N - X_0 \le -\varepsilon) \le \exp\Big(-\frac{\varepsilon^2}{2\sum_{k=1}^N C_k^2}\Big).$$

Hint: For part (c), you may use exercise 2.

Exercise 4 (2 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale such that $X_0 = 0$ and $|X_n - X_{n-1}| \leq 1$ for all n. Show that $\frac{X_n}{n} \to 0$ almost surely.

Exercise 5 (4 points)

Consider the random walk $S_n = \xi_1 + ... + \xi_n$ and $S_0 = 0$ where the ξ_n are iid with $\mathbb{P}(\xi_n = \pm 1) = \frac{1}{2}$. Fix two real numbers $A, B > 0$. Let $N = \min\{n : S_n \geq A \text{ or } S_n \leq -B\}$. Compute $\mathbb{P}(S_N \geq A)$ and $\mathbb{E}[N]$.