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Probability theory II

Exercise Sheet 2

Submission is due on 10/23/2019 2 p.m.
Box 133

Exercise 1 (3 points)

Let $(X_k)_{k\in\mathbb{N}}$ be independent random variables such that $\mathbb{E}[X_k] = 0$ and $X_k \in L^2(\mathbb{P})$ for all $k \in \mathbb{N}$. If $\sigma_k^2 = \mathrm{Var}(X_k)$ and $S_n = \sum_{1\le k\le n} X_k$, then $\mathrm{Var}(S_n) = \sum_{1\le k\le n} \sigma_k^2$. Show that

$$\mathbb{P}(\max_{1\le k\le n} S_k > l) \le \frac{\sum_{1\le k\le n}\sigma_k^2}{l^2}.$$

Exercise 2 (4 points)

(a) Let $X \sim \mathrm{Normal}(0,1)$ and $a > 0$. Show

$$\frac{1}{\sqrt{2\pi}}\frac{a}{1+a^2}e^{-\frac{a^2}{2}} \le \mathbb{P}(X > a) \le \frac{1}{\sqrt{2\pi}}\frac{1}{a}e^{-\frac{a^2}{2}}.$$

(b) Now let $X \sim \mathrm{Normal}(\mathbf{0}_n, I_n)$ and again $a > 0$. Show

$$\frac{n|B_1(0)|}{(2\pi)^{n/2}}\frac{a^n}{1+a^2}e^{-\frac{a^2}{2}} \le \mathbb{P}(|X| > a) \le \begin{cases} \frac{n|B_1(0)|}{(2\pi)^{n/2}}\frac{a^n}{a^2-(n-1)}e^{-\frac{a^2}{2}}, & \text{if } a > \sqrt{n-1} \\ 1, & \text{if } a \in (0, \sqrt{n-1}]. \end{cases}$$

Exercise 3 (7 points)

Let $X_1, ..., X_n$ be iid with $X_1 \sim \mathrm{Normal}(0,1)$. The moment generating function of $S_n = X_1 + ... + X_n$ is $\mathbb{E}[e^{\theta S_n}] = e^{\frac{\theta^2 n}{2}}$.

(a) Show $\mathbb{P}(\sup_{1\le k\le n} S_k \ge c) \le e^{\frac{\theta^2 n}{2} - \theta c}$ for any $\theta > 0$.

(b) Maximize over θ to show $\mathbb{P}(\sup_{1\le k\le n} S_k \ge c) \le e^{-\frac{c^2}{2n}}$.

(c) For $h(n) = \sqrt{2n\log\log n}$, apply (b) for a sequence, $C_n = rh(r^{n-1})$ for some $r > 1$ and use Borel-Cantelli-Lemma 1 to conclude

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} \le 1 \quad \text{a.s.}$$

(d) Apply the estimate

$$\mathbb{P}(X > a) \ge \frac{1}{\sqrt{2\pi}}\frac{a}{1+a^2}e^{-\frac{a^2}{2}}$$

and use independence and Borel-Cantelli-Lemma 2, to conclude

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} \ge 1 \quad \text{a.s.}$$

Conclude that $\overline{\lim}_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1$ a.s.

Exercise 4 (2 points)

Construct a non-negative martingale $(X_n)_n$, such that $\mathbb{E}[X_n] = 1$ for all $n \in \mathbb{N}$ but $X^\star = \sup_{n\in\mathbb{N}} X_n \notin L^1$.

Hint: Use the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, $\mathbb{P} =$Unif$(0,1)$ with filtration $\mathcal{F}_n = \sigma$-alg. generated by intervals ending with $j/2^n$ for some positive integer j and the random variables

$$X_n = \begin{cases} 2^n, & \text{if } 0 \leq x \leq 2^{-n} \\ 0, & \text{if } 2^{-n} \leq x \leq 1 \end{cases}$$

Exercise 5 (4 points)

Show the following statements:

(a) Let $X \in L^1$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $F \in \mathcal{F}$,

$$\mathbb{P}(F) < \delta \Rightarrow \mathbb{E}[1_F |X|] < \varepsilon.$$

(b) Suppose $X \in L^1$ and $\varepsilon > 0$. There exists $K \in (0, \infty)$ such that

$$\mathbb{E}[1_{\{|X|>K\}} |X|] < \varepsilon.$$