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Probability theory II

Exercise Sheet 11

Submission is due on 01/08/2020 2 p.m.
Box 133

In what follows, S is a complete separable metric space, and $\Omega = S^{\mathbb{Z}}$ equipped with the translation map $T : \Omega \to \Omega$ such that if $\omega = (x_n)_{n\in\mathbb{Z}} \in \Omega$, $T\omega = (x_{n+1})_{n\in\mathbb{Z}}$. Also, we will write, for all $\omega \in \Omega$, $X_n(\omega) := \omega(n)$ for the co-ordinate mapping process.

Exercise 1 (4 points)

(a) Let $\{p(x, \cdot)\}_{x\in S}$ be a transition probability function and $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is a stationary Markov process with respect to $p(\cdot, \cdot)$ (i.e. $\mathbb{P}$ is T-invariant and $\mathbb{P}(X_{n+1} \in A|\mathcal{F}_n) = p(X_n, A)$ almost surely with respect to $\mathbb{P}$). Then show that the 1-dimensional marginal distribution $\mu \in \mathcal{M}_1(S)$ which is given by $\mu(A) = \mathbb{P}(X_n \in A)$ for all $A \subset S$ (and is independent of n because of stationarity of $\mathbb{P}$) is p-invariant in the sense $\mu(A) = \int_S p(x, A)\mu(\mathrm{d}x)$.

(b) Conversely, given any transition probability $\{p(x, \cdot)\}_{x\in S}$ on S and any $\mu \in \mathcal{M}_1(S)$ such that μ is p-invariant, show that there exists a unique stationary Markov process $\mathbb{P}$ with transition probability $\{p(x, \cdot)\}_{x\in S}$ and 1-dimensional marginal μ.

Exercise 2 (4 points)

Let $F : \Omega \to \Omega$ such that with $\omega = (x_n)_{n\in\mathbb{Z}} \in \Omega$, $F\omega = (x_{-n})_{n\in\mathbb{Z}}$. Let $\mathbb{P}$ be a stationary Markov process with transition probability $p(\cdot, \cdot)$ and 1-dimensional marginal μ. If $\mathbb{Q} = \mathbb{P}F^{-1}$ is the push-forward of $\mathbb{P}$ under the map F, show that $\mathbb{Q}$ is a stationary Markov process and determine its transition probability $q(\cdot, \cdot)$ in terms of $p(\cdot, \cdot)$ and μ.

Exercise 3 (6 points)

(a) For any $p \in [1, \infty]$, prove that $(Pf)(x) = \int_S f(y)p(x, \mathrm{d}y)$ defines a contraction map $L^p(\mu)$ if μ is p-invariant.

(b) In the notation of Exercise 2, let $\mathbb{Q} = \mathbb{P}$. Then show that the operator $P : L^2(\mu) \to L^2(\mu)$ is symmetric (or self-adjoint in the sense $\langle Pf, g\rangle_{L^2(\mu)} = \langle f, Pg\rangle_{L^2(\mu)}$ for all $f, g \in L^2(\mu)$ (Here μ is again a p-invariant probability measure).

Notation: Let $\{p(x, \cdot)\}_{x\in S}$ be a transition probability function. Define $\mathcal{M}_s^{(p)} = \{\mu \in \mathcal{M}_1(S) : \mu \text{ is } p-\text{invariant i.e. } \mu(A) = \int_S p(x, A)\mu(\mathrm{d}x) \text{ for all } A \subset S\}$. Then obviously $\mathcal{M}_s^{(p)}$ is a convex set. Let $\mathcal{M}_{s,\mathrm{extr}}^{(p)}$ denote the (possibly empty) set of extreme points of the convex set $\mathcal{M}_s^{(p)}$.

If $\mathbb{P}$ is a stationary Markov chain with 1-dimensional marginal distribution $\mu = \delta_x$ (for any $x \in S$) and transition probability $p(x, \mathrm{d}y)$, it is customary to denote such a stationary Markov chain by

$\mathbb{P}_x$.

Exercise 4 (6 points)

Let $\mathbb{P}_x$ be a stationary Markov chain with transition probability $p(x, \mathrm{d}y)$. Let f be any bounded measurable function $f : S \to \mathbb{R}$. Show that for almost all x with respect to any $\nu \in \mathcal{M}^{(p)}_{s,\mathrm{extr}}$

$$\lim_{n\to\infty} \frac{1}{n}(f(X_1) + ... + f(X_n)) = \int f(y)\nu(\mathrm{d}y)$$

for almost all ω with respect to $\mathbb{P}_x$. That is, fix any $\nu \in \mathcal{M}^{(p)}_{s,\mathrm{extr}}$ and show that

$$\nu\Big(x \in S : \mathbb{P}_x\Big[\omega : \lim_{n\to\infty} \frac{1}{n}\big(f(X_1(\omega)) + ... + f(X_n(\omega))\big) = \mathbb{E}^{\nu}[f(\omega)]\Big] = 1\Big) = 1.$$