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Probability theory II

Exercise Sheet 1

Submission is due on 10/16/2019 2 p.m.
Box 133

Exercise 1 (5 points)

(a) Let $\Omega = \{1, ..., 6\}$, $\mathbb{P}$ =Unif, and $A = \{4\}$, $B =$ "even number". If $X = 1_A$ and $\mathcal{F} = \sigma(B)$, then compute $\mathbb{E}[X|\mathcal{F}]$.

(b) Let X, Y be two real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f : \mathbb{R}^2 \to \mathbb{R}_+$ be the density for the joint distribution of (X, Y) (i.e. $\mathbb{P}(X \in A, Y \in B) = \int_{\mathbb{R}^2} 1_{\{x \in A\}} 1_{\{y \in B\}} f(x, y) \mathrm{d}x\mathrm{d}y$ for all $A, B \subset \mathbb{R}$).
Assume that $\int_{\mathbb{R}} f(x, y)\mathrm{d}x > 0$ for all $y \in \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $g \circ X \in L^1(\mathbb{P})$. Then show that $\mathbb{E}[g \circ X|\sigma(Y)] = h \circ Y$ where $h : \mathbb{R} \to \mathbb{R}$ is given by

$$h(y) = \frac{\int_{\mathbb{R}} g(x) f(x, y)\mathrm{d}x}{\int_{\mathbb{R}} f(x, y)\mathrm{d}x}.$$

Exercise 2 (2 points)

Let $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and $\mathbb{P}$ =Lebesgue. If $X(\omega) = \cos(\pi\omega)$, compute $\mathbb{E}[X|\mathcal{F}]$.

Exercise 3 (5 points)

Let $Z_1, ..., Z_n$ be iid $\mathcal{N}(0, 1)$ Gaussians on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define a new measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{Q}(\mathrm{d}\omega) = \frac{1}{Z_{n,\beta}} \exp\Big[\sum_{i=1}^{n} \beta_i Z_i(\omega)\Big] \mathbb{P}(\mathrm{d}\omega)$$

where $\beta = (\beta_1, ..., \beta_n) \in \mathbb{R}^n$ and $Z_{n,\beta}$ is a constant.

(a) How should we choose $Z_{n,\beta}$ so that $\mathbb{Q}$ is a probability measure? That is, which value of $Z_{n,\beta}$ makes $\mathbb{Q}(\Omega) = 1$?

(b) With the above choice of $Z_{n,\beta}$, what is the distribution of $(Z_1, ..., Z_n)$ under the new probability measure $\mathbb{Q}$?

Exercise 4 (3 points)

The following model describes the evolution of a population:

Let $(Y_{n,k})_{n\in\mathbb{N}_0,k\in\mathbb{N}}$ be iid random variables in $\mathbb{N}_0$, where $Y_{n,k}$ is the number of children of the k-th individual in the n-th generation. We assume $\mathbb{E}[Y_{n,k}] < \infty$ for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. After one step every individual of the last generation dies such that we can define the number of living individuals by

$$S_0 = 1 \quad S_n = \sum_{k=1}^{S_{n-1}} Y_{n-1,k}, \quad n \geq 1.$$

Prove that

$$Z_n := \frac{S_n}{\mu^n}, \quad n \geq 1$$

is a martingale with respect to $\mathcal{F}_n = \sigma(S_0, ..., S_n)$.

For the next exercise you can assume the following theorem.

Theroem: Let H be a Hilbert space (i.e., H is a vector space equipped with an inner-product $\langle \cdot, \cdot \rangle_H$ that defines a norm $\|x\|_H^2 := \langle x, x \rangle_H$ making H a complete metric space). Let $K \subset H$ be a closed subspace of H. Then for any $x \in H$, there exists $y \in K$ such that one of the equivalent properties hold:

(a) For any $z \in K$, $\langle x - y, z \rangle_H = 0$.

(b) For any $z \in K$, $\|y - x\|_H \leq \|z - x\|_H$.

Such y is unique, it is written as $y = \pi_K(x)$ and is called the orthogonal projection of x onto K.

Note that for any sigma algebra $\mathcal{A}$, $L^2(\mathcal{A})$ denotes all square integrable functions which are measurable w.r.t. the sigma algebra $\mathcal{A}$.

Exercise 5 (5 points)

Let $H = L^2(\mathcal{F})$ equipped with an inner product $\langle X, Y \rangle_H = \mathbb{E}[XY]$. Fix $X \in L^2(\mathcal{F})$ and let $K = L^2(\mathcal{G})$ where $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-algebra. Prove:

(a) $K \subset H$ is a closed subspace of H.

(b) The orthogonal projection of X onto K (which exists and is unique by the theorem above) is uniquely identified as

$$\pi_K(X) = \mathbb{E}[X|\mathcal{G}].$$