New perspectives in hermitian K-theory

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Algebraic K-theory

For a ring R, the algebraic K-group $K_0(R)$ is generated by isomorphism classes $[P]$ of finitely generated projective R-modules, under the relation $[P \oplus Q] = [P] + [Q]$.

- An algebraic analogue of the complex K-theory group $KU_0(X)$ of a topological space X.
- For R commutative, it captures rich geometric information about $\mathrm{spec}(R)$, related to its Picard group, Chow groups and motivic cohomology.
- For $R = \mathbb{Z}[G]$ a group ring the quotient $K_0(R)/K_0(\mathbb{Z})$ detects Wall's finiteness obstructions for a homotopy compact space with fundamental group G to be represented by a finite CW-complex.
- Can also be defined for sufficiently nice algebraic varieties and schemes by considering isomorphism classes of vector bundles and enforcing the relation $[F] = [E] + [G]$ for every short exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

of vector bundles.

The Grothendieck-Witt group

For a commutative ring R, the *Grothendieck-Witt group* $\mathrm{GW}_0^{\mathrm{q}}(R)$ is defined by isomorphism classes $[P,q]$ of finitely generated projective modules equipped with a unimodular quadratic form, under the relation $[P \oplus P', q \oplus q'] = [P,q] + [P',q']$.

- An algebraic analogue of real K-theory group $\mathrm{KO}_0(X)$ of a topological space X.
- Is related to algebraic K-theory via a pair of maps

$$\begin{array}{ccccc} \mathrm{GW}_0^{\mathrm{q}}(R) & \xrightarrow{\mathrm{fgt}} & \mathrm{K}_0(R) & \xrightarrow{\mathrm{hyp}} & \mathrm{GW}_0^{\mathrm{q}}(R) \\ [P,q] & \longmapsto & [P] & \longmapsto & [P \oplus P^*, h] \end{array}$$

- (Knebusch) For a sufficiently nice variety X we can define $\mathrm{GW}_0^{\mathrm{q}}(X)$ as the group generated by classes of vector bundles equipped with a unimodular quadratic forms under the direct sum relations and the relations $[E,q] = [L \oplus L^*, h]$ whenever E is a vector bundle with a unimodular form q and $L \subseteq E$ is a Lagrangian sub-bundle.

The Witt group

The cokernel of the map $\mathrm{hyp}\colon \mathsf{K}_0(R) \to \mathsf{GW}_0^{\mathrm{q}}(R)$ is known as the *Witt* group $\mathsf{W}^{\mathrm{q}}(R)$. One obtains an exact sequence

$$\mathsf{K}_0(R)_{\mathrm{C}_2} \to \mathsf{GW}_0^{\mathrm{q}}(R) \to \mathsf{W}^{\mathrm{q}}(R) \to 0$$

where the first term denotes the C_2-orbits of $\mathsf{K}_0(R)$ with respect to the action $[P] \mapsto [P^*]$, under which the hyperbolic map is invariant.

- This sequence is often used to obtain information on $\mathsf{GW}_0^{\mathrm{q}}(R)$ via the two outer groups, which are often more accessible.
- For example, for $R = \mathbb{Z}$ this sequence is split exact with an isomorphism $\mathsf{W}^{\mathrm{q}}(\mathbb{Z}) \cong \mathbb{Z}$ via the signature divided by 8 and an isomorphism $\mathsf{K}_0(\mathbb{Z}) \cong \mathbb{Z}$ via the rank.
- For R a field of characteristic not 2 the group $\mathsf{W}^{\mathrm{q}}(R)$ is highly accessible via a filtration by Galois $\mathbb{Z}/2$ cohomology groups. This was the subject of the famous Milnor conjecture, proven by Voevodsky.

Can we go higher?

$$\mathrm{K}_0(R)_{\mathrm{C}_2} \to \mathrm{GW}_0^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{q}}(R) \to 0$$

This sequence looks like it should be continued from the left to a long exact sequence. But with what groups?

Definition (Quillen)

The *algebraic* K*-theory space*

$$\mathcal{K}(R) := \mathrm{Proj}^{\simeq}(R)^{\mathrm{grp}}$$

is defined as the group completion of the symmetric monoidal groupoid (considered as an E_∞-monoid in spaces) of f. g. projective R-modules.

Definition (Karoubi-Villamayor)

The *Grothendieck-Witt space*

$$\mathcal{GW}_{\mathrm{cl}}^{\mathrm{q}}(R) := \mathrm{Unimod}^{\mathrm{q},\simeq}(R)^{\mathrm{grp}}$$

is defined as the group completion of the symmetric monoidal groupoid of f. g. projective R-modules equipped with a unimodular quadratic form.

Can we go higher?

$$\mathrm{K}_0(R)_{\mathrm{C}_2} \to \mathrm{GW}_0^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{q}}(R) \to 0$$

When 2 is invertible in R, this sequence extends to a long exact sequence involving:

- The homotopy groups of the homotopy C_2-orbits $\mathcal{K}(R)_{\mathrm{hC}_2}$.
- The homotopy groups of $\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R)$, i.e., the higher Grothendieck-Witt groups.
- The quadratic L-groups of R defined by Wall and Ranicki.

L-groups

For a commutative ring R, the quadratic L-groups $\mathrm{L}^{\mathrm{q}}_n(R)$ are 4-periodic, with $\mathrm{L}^{\mathrm{q}}_0(R) = \mathrm{W}^{\mathrm{q}}(R)$ the Witt group of quadratic forms over R.

L-groups

Quadratic L-groups were defined by Wall and Ranicki in the context of surgery theory. The relevant ring R is then the group ring $\mathbb{Z}\pi_1(X)$ of the fundamental group of a given space.

What are quadratic forms over non-commutative rings?

Several proposals in varying levels of generality have been proposed in the literature (Wall's anti-structures, Karoubi's hermitian rings). They can all be described via the following formalism:

Modules with involution

Let R be an associative ring. A *module* with involution over R is an $(R \otimes R)$-module M, together with an involution $\sigma: M \to M$ satisfying $\sigma((r \otimes s)m) = (s \otimes r)\sigma(m)$.

Non-commutative quadratic forms

Modules with involution

Let R be an associative ring. A *module* with involution over R is an $(R \otimes R)$-module M, together with an involution $\sigma \colon M \to M$ satisfying $\sigma((r \otimes s)m) = (s \otimes r)\sigma(m)$.

For $P \in \mathrm{Proj}(R)$:

- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M) \Leftrightarrow$ bilinear M-valued forms on P.
- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M)^{\mathrm{C}_2} \Leftrightarrow$ symmetric M-valued forms on P.
- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M)_{\mathrm{C}_2} \Leftrightarrow$ quadratic M-valued forms on P.

The *polarization* of an M-valued form quadratic form on P is its image under the norm map

$$\begin{array}{ccc} \mathrm{Hom}_{R \otimes R}(P \otimes P, M)_{\mathrm{C}_2} & \longrightarrow & \mathrm{Hom}_{R \otimes R}(P \otimes P, M)^{\mathrm{C}_2} \\ [\beta] & \longmapsto & \beta(x,y) + \sigma\beta(y,x) \end{array}$$

Symmetric forms in the image of this map are called *even forms*.

Invertible modules with involution

Definition

A module with involution M over R is *invertible* if R is finitely generated and projective as an R-module, and the the map $R \to \mathrm{Hom}_R(M)$, induced by the two commuting R-actions, is an isomorphism.

For M an invertible module with involution over R one obtains an induced duality

$$\begin{aligned} \mathrm{D}_M\colon \mathrm{Proj}(R)^{\mathrm{op}} &\xrightarrow{\simeq} \mathrm{Proj}(R) \\ P &\longmapsto \mathrm{Hom}_R(P, M) \end{aligned}$$

Any bilinear or symmetric M-valued form $\beta\colon P \otimes P \to M$ induces a homomorphism $\beta_\sharp\colon P \to \mathrm{D}_M(P)$. The form β is called *unimodular* if $\beta_\sharp$ is an isomorphism. A quadratic form is *unimodular* if its polarization is.

Invertible modules with involution

Summary

An invertible module with involution M determines a duality $\mathrm{D}_M(P) = \mathrm{Hom}_R(P, M)$ on $\mathrm{Proj}(R)$. A form β is called unimodular if the induced map $\beta_\sharp \colon P \to \mathrm{D}_M(P)$ is an isomorphism.

Examples

- R commutative $M = R$ with trivial involution $\Rightarrow$ usual notion of unimodular symmetric and quadratic forms.
- R commutative $M = R$ with sign involution $\Rightarrow$ unimodular skew-symmetric and skew-quadratic forms.
- R with anti-involution $\sigma \colon R \xrightarrow{\cong} R^{\mathrm{op}}$ (e.g., group rings), $M = R$ with involution σ, or twisted by a central unit ε s.t. $\sigma(\varepsilon) = \varepsilon^{-1}$.
- When R is commutative one can take M to be any line bundle with involution over $\mathrm{spec}(R)$. This example naturally extends to the context of schemes.

L-groups with coefficients

The definition of quadratic L-groups extends to the setting of an invertible module with involution M. The associated quadratic L-groups $\mathrm{L}^{\mathrm{q}}_n(R, M)$ satisfy $\mathrm{L}^{\mathrm{q}}_{n+2}(R, M) = \mathrm{L}^{\mathrm{q}}_n(R, -M)$ with $\mathrm{L}^{\mathrm{q}}_0(R, M)$ the Witt group of M-valued forms. Here $-M$ is obtained from M by twisting the involution by a sign.

Grothendieck-Witt groups with coefficients

For R and M as above the associated *Grothendieck-Witt space*

$$\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M) := \mathrm{Unimod}^{\mathrm{q},\simeq}(R, M)^{\mathrm{grp}}$$

is defined by the group completion of the symmetric monoidal groupoid of unimodular M-valued forms. It's group of components $\mathrm{GW}^{\mathrm{q}}_0(R, M)$ is then the Grothendieck group of such forms.

An analogous definition can be made for symmetric and even forms. Polarization determines maps

$$\mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M) \to \mathcal{GW}^{\mathrm{ev}}_{\mathrm{cl}}(R, M) \to \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M)$$

which are equivalences when $\frac{1}{2} \in R$.

What's going on?

We obtain an exact sequence

$$\mathsf{K}_0(R)_{\mathrm{C}_2} \to \mathsf{GW}_0^{\mathrm{q}}(R, M) \to \mathsf{L}_0^{\mathrm{q}}(R, M) \to 0$$

When 2 is invertible this sequence continues on the left to a long exact sequence involving higher Grothendieck-Witt groups, the quadratic L-groups, and the homotopy groups of $\mathcal{K}(R, M)_{\mathrm{hC}_2}$. This can be used to reduce the study of Grothendieck-Witt groups to that of algebraic K-theory and the four groups $\mathsf{L}_0^{\mathrm{q}}(R, \pm M), \mathsf{L}_1^{\mathrm{q}}(R, \pm M)$, the latter being fairly accessible to computations.

This completely fails when 2 is not invertible.

In fact, when 2 is not invertible the relative homotopy groups of the map $\mathcal{K}(R)_{\mathrm{C}_2} \to \mathcal{GW}_{\mathrm{cl}}^{\mathrm{q}}(R, M)$ are generally not 4-periodic.

So what are these groups?

Answering this question is one of the main applications of the framework we are about to present.

Back to K-theory

To explain our approach, consider again algebraic K-theory.

Classical observation

The algebraic K-groups of R depends only on a certain category associated to R - the category of finitely generated projective R-modules - and the fact that this category admit direct sums.

Idea: define the algebraic K-group of an *additive* category. In particular, $\mathrm{K}_n(R) = \mathrm{K}_n(\mathrm{Proj}(R))$.

Variants

Consider additive categories with additional structure:
exact categories (Quillen), cofibration categories (Waldhausen).
Allows to take into account the case of vector bundles.

Modern perspective

Consider *stable* ∞*-categories*.

What is an ∞-category?

A notion of a category adapted to homotopy theory and homological algebra. One may speak of objects and morphisms, but also about homotopies between morphisms, homotopies between homotopies, etc.

- Every ordinary category can be considered an ∞-category.
- Every space can be considered as an ∞-category whose objects are points and morphisms are paths. This association identifies the notion of a space with that of an ∞-*groupoid*, that is, an ∞-category all of whose morphisms are invertible, an idea known as Grothendieck's *homotopy hypothesis*.

Some ∞-categories of interest

Example	Classical counterpart
$\mathcal{S}$ - the ∞-category of spaces	Sets
$\mathcal{S}p$ - the ∞-category of spectra	Abelian groups
$\mathcal{S}p^{\mathrm{f}} \subseteq \mathcal{S}p$ - subcategory of finite spectra	Finitely generated abelian groups
$\mathcal{D}(R)$ - the derived ∞-category of a ring R	R-modules
$\mathcal{D}^{\mathrm{p}}(R) \subseteq \mathcal{D}(R)$ - the subcategory of perfect complexes	f. g. projective R-modules
$\mathcal{D}^{\mathrm{p}}(X)$ - the ∞-category of perfect quasi-coherent sheaves on a scheme X	Vector bundles on X

Stable ∞-categories

Familiar notions from ordinary category theory usually have ∞-categorical counterparts with similar behaviors. For example, one may speak of limits and colimits, functors, adjunctions, etc.

Definition

An ∞-category $\mathcal{C}$ is said to be *stable* if it admits a zero object 0, pushouts and pullbacks, and the collection of pushout and pullback squares coincides. One then refers to such squares as *exact squares*. Exact squares with one corner zero are known as *exact sequences*.

Every stable ∞-category is additive.

Examples

- The ∞-categories $\mathcal{S}p$ and $\mathcal{S}p^{\mathrm{f}}$ are stable.
- For a ring R the derived ∞-category $\mathcal{D}(R)$ and its full subcategory $\mathcal{D}^{\mathrm{p}}(R)$ are both stable ∞-categories.
- The ∞-category of perfect of quasi-coherent sheaves on a scheme is stable.

Algebraic K-theory of stable ∞-categories

For $\mathcal{C}$ a stable ∞-category there is an ∞-category $\mathrm{Span}(\mathcal{C})$, whose objects are the objects of $\mathcal{C}$, and whose morphisms from X to Y are diagrams

$$X \xleftarrow{\quad} W \xrightarrow{\quad} Y$$

in $\mathcal{C}$, also known as *spans*. Spans are composed by forming the fiber product over the middle object.

Definition (Barwick-Rognes)

The K-theory space of a stable ∞-category $\mathcal{C}$ is given by

$$\mathcal{K}(\mathcal{C}) = \Omega|\mathrm{Span}(\mathcal{C})|,$$

where $|\bullet|$ denotes the realization, or classifying space of an ∞-category, and Ω denotes taking loop spaces.

The formation of direct sums makes $\mathcal{K}(\mathcal{C})$ into an E_∞-group.

Algebraic K-theory of stable ∞-categories

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- This definition is based on Quillen's definition of the algebraic K-theory space of an exact category using the Q-construction.
- K-theory of higher categories already appears in Waldhausen's work using the formalism of categories with cofibrations and weak equivalences.
- A direct adaptation of Waldhausen's $\mathcal{S}$-construction to the setting of stable ∞-categories was given by Blumberg-Gepner-Tabuada, and to Waldhausen ∞-categories by Barwick.
- The two approaches to higher K-theory of stable ∞-categories are equivalent (Barwick-Rognes).

Examples

Definition (Barwick-Rognes)

The K-theory space of a stable ∞-category $\mathcal{C}$ is given by

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where $|\bullet|$ denotes the realization, or classifying space of an ∞-category, and Ω denotes taking loop spaces.

- (Gillet-Waldhausen) There is a canonical equivalence $\mathcal{K}(\mathcal{D}^{\mathrm{p}}(R)) \simeq \mathcal{K}(R)$. More generally, for a sufficiently nice scheme there is a canonical equivalence $\mathcal{K}(\mathcal{D}^{\mathrm{p}}(X)) \simeq \mathcal{K}(\mathrm{Vect}(X))$.
- The space $\mathcal{K}(\mathcal{S}p^{\mathrm{f}})$ is also known as Waldhausen A-theory of the point, and plays an important role in geometric topology.

From stable to Poincaré ∞-categories

To study Grothendieck-Witt theory, we consider the framework of Poincaré ∞-categories, suggested by Lurie in his work on L-theory.

Definition

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. Then f is said to be:

- *reduced* if it preserves zero objects;
- *exact* or *linear* if it preserves zero objects and exact squares;
- *quadratic* if it preserves zero objects and sends strongly exact 3-cubes to exact 3-cubes.

Here, a 3-cube diagram in $\mathcal{C}$ is called exact if it is a limit/colimit cube, and strongly exact if its restriction to each 2-dimensional face is an exact square.

These notions are part of the general framework of *Goodwille calculus*.

Hermitian ∞-categories

Definition

- A *hermitian ∞-category* is a pair $(\mathcal{C}, \Omega)$ where $\mathcal{C}$ is a stable ∞-category $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is a quadratic functor.
- A *hermitian functor* $(f, \eta)\colon (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ consists of an exact functor $f\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\eta\colon \Omega \Rightarrow f^*\Omega'$.

If $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is quadratic then:

- Its *polarization* $\mathrm{B}_\Omega(X, Y) := \mathrm{fib}[\Omega(X \oplus Y) \to \Omega(X) \oplus \Omega(Y)]$ is exact in each variable. We call B_Ω the *bilinear part* of Ω. It is *symmetric* in X and Y, i.e., admits a C_2-fixed structure with respect to the flip C_2-action on $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{S}p)$.
- The cofiber $\mathrm{L}_\Omega(X) := \mathrm{cof}[\mathrm{B}_\Omega(X, X)_{\mathrm{hC}_2} \to \Omega(X)]$ of the map induced by the C_2-equivariant diagonal $X \to X \oplus X$, is exact in X. We refer to L_Ω as the *linear part* of Ω.

We refer to quadratic functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ as *hermitian structures* on $\mathcal{C}$.

Hermitian forms

Given a hermitian ∞-category $(\mathcal{C}, \Omega)$ and an object $X \in \mathcal{C}$, we consider $\Omega(X)$ as encoding the notion of hermitian forms on X. In particular, we refer to maps $q: \mathbb{S} \to \Omega(X)$ as *hermitian forms* on X, and which case we call the pair (X, q) a *hermitian object*.

Example - homotopy symmetric/quadratic forms

Let R be a ring and M an invertible module with involution as discussed earlier. Then the functors

$$\Omega^{\mathrm{q}}_M : \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p \qquad X \mapsto \hom_{R\otimes R}(X \otimes X, M)_{\mathrm{hC}_2}$$

$$\Omega^{\mathrm{s}}_M : \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p \qquad X \mapsto \hom_{R\otimes R}(X \otimes X, M)^{\mathrm{hC}_2}$$

are hermitian structures on $\mathcal{D}^{\mathrm{p}}(R)$, encoding *homotopy coherent* variants of the notions of quadratic and symmetric M-valued forms, respectively. Here on the right we use the mapping *spectra* canonically attached to any stable ∞-category.

These hermitian structures have the same bilinear part

$$\mathrm{B}_M(X, Y) = \hom_{R\otimes R}(X \otimes Y, M).$$

Derived hermitian structures

Example - derived hermitian structures

There exists essentially unique hermitian structures

$$\Omega^{\mathrm{gq}}_M, \Omega^{\mathrm{ge}}_M, \Omega^{\mathrm{gs}}_M \colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$$

whose restriction to $\mathrm{Proj}(R) \subseteq \mathcal{D}^{\mathrm{p}}(R)$ are given by

$$\begin{aligned}
\Omega^{\mathrm{gq}}_M(P) &= \mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{\mathrm{C}_2} \\
\Omega^{\mathrm{gs}}_M(P) &= \mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{\mathrm{C}_2} \\
\Omega^{\mathrm{ge}}_M(P) &= \mathrm{im}[\Omega^{\mathrm{gq}}_M(P) \to \Omega^{\mathrm{gs}}_M(P)].
\end{aligned}$$

We refer to these as the *genuine quadratic*, *genuine symmetric* and *genuine even* structures, respectively.

We have a sequence of natural transformations

$$\Omega^{\mathrm{q}}_M \Rightarrow \Omega^{\mathrm{gq}}_M \Rightarrow \Omega^{\mathrm{ge}}_M \Rightarrow \Omega^{\mathrm{gs}}_M \Rightarrow \Omega^{\mathrm{s}}_M$$

which induce an equivalence on bilinear parts. When $\frac{1}{2} \in R$ these are all equivalences.

Let $\mathfrak{Q}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ be a hermitian structure on $\mathcal{C}$.

- We will say that $\mathfrak{Q}$ is *non-degenerate* if there exists a functor $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ together with a natural equivalence

$$\mathrm{B}_{\mathfrak{Q}}(X, Y) \simeq \mathrm{hom}_{\mathcal{C}}(X, \mathrm{D}Y).$$

  In this case D is determined by $\mathrm{B}_{\mathfrak{Q}}$ in an essentially unique manner, and we write $\mathrm{D}_{\mathfrak{Q}}$ to express its dependence on $\mathfrak{Q}$.
- We will say that $\mathfrak{Q}$ is *Poincaré* if it is non-degenerate and $\mathrm{D}_{\mathfrak{Q}}$ is an equivalence of ∞-categories. We will then say that $\mathrm{D}_{\mathfrak{Q}}$ is the *duality* associated to $\mathfrak{Q}$.

Definition

A *Poincaré ∞-category* is a hermitian ∞-category $(\mathcal{C}, \mathfrak{Q})$ such that $\mathfrak{Q}$ is Poincaré.

Example

The hermitian structures $\Omega^{q}_{M}, \Omega^{gq}_{M}, \Omega^{ge}_{M}, \Omega^{gs}_{M}, \Omega^{s}_{M}$ are all Poincaré and have the same duality

$$D_M(X) = \mathrm{Hom}^{\mathrm{cx}}(X, M)$$

given by the formation of mapping complexes into M.

Example

The formation of mapping spectra in $\mathcal{C}$ yields a quadratic functor

$$\Omega_{\mathrm{hyp}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}p \qquad (X, Y) \mapsto \mathrm{hom}_{\mathcal{C}}(X, Y).$$

The resulting hermitian ∞-category $\mathrm{Hyp}(\mathcal{C}) := (\mathcal{C} \times \mathcal{C}^{\mathrm{op}}, \Omega_{\mathrm{hyp}})$ is Poincaré with duality $(X, Y) \mapsto (Y, X)$.

Example

Ω^{u} - a hermitian structure on $\mathcal{S}p^{\mathrm{f}}$ which is initial among hermitian structures equipped with a hermitian form $\mathbb{S} \to \Omega(\mathbb{S})$. It sits in a fiber sequence

$$\mathrm{hom}(X \otimes X, \mathbb{S})_{\mathrm{hC}_2} \to \Omega^{\mathrm{u}}(X) \to \mathrm{hom}(X, \mathbb{S})$$

and has bilinear part $\mathrm{B}^{\mathrm{u}}(X, Y) = \mathrm{hom}(X \otimes Y, \mathbb{S})$. In particular, it is Poincaré with duality the Spanier-Whitehead duality $X \mapsto \mathrm{hom}(X, \mathbb{S})$.

Poincaré objects

Suppose (X, q) a hermitian object in a Poincaré ∞-category $(\mathcal{C}, Ϙ)$ with duality $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. Then the image of q in $\mathrm{B}_Ϙ(X, X) = \mathrm{hom}_{\mathcal{C}}(X, \mathrm{D}X)$ determines a map

$$q_\sharp\colon X \to \mathrm{D}X.$$

We say that (X, q) is a *Poincaré object* if $q_\sharp$ is an equivalence.

The collection of hermitian objects can be organized into an ∞-category $\mathrm{He}(\mathcal{C}, Ϙ)$, whose maximal ∞-groupoid we denote by $\mathrm{Fm}(\mathcal{C}, Ϙ)$. We let $\mathrm{Pn}(\mathcal{C}, Ϙ) \subseteq \mathrm{Fm}(\mathcal{C}, Ϙ)$ be the subspace spanned by the Poincaré objects.

Example (hyperbolic Poincaré objects)

For $V \in \mathcal{C}$ an object there is a hermitian form $h\colon \mathbb{S} \to Ϙ(V \oplus \mathrm{D}V)$ coming from the summand $\mathrm{hom}_{\mathcal{C}}(V, V) = \mathrm{B}_Ϙ(V, \mathrm{D}V)$. The resulting hermitian object $\mathrm{hyp}(V) := (V \oplus \mathrm{D}V, h)$ is always Poincaré.

Example

For the Poincaré ∞-category $\mathrm{Hyp}(\mathcal{C})$ one has $\mathrm{Pn}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathcal{C}^{\simeq}$.

Cobordisms

Let $(X, q), (X', q')$ be two Poincaré objects in $(\mathcal{C}, \mathfrak{P})$.

A *cobordism* from (X, q) to (X', q') is a span

$$X \xleftarrow{\alpha} W \xrightarrow{\beta} X'$$

together with a homotopy $\eta\colon \alpha^* q \sim \beta^* q'$ such that the induced map $W \to \mathrm{D}X \times_{\mathrm{D}W} \mathrm{D}X'$ is an equivalence.

Cobordisms can be composed by first composing the spans and then composing the homotopies.

- We say that two Poincaré objects are *cobordant* if there is a cobordism between them. This is an equivalence relation.
- We say that a Poincaré object (X, q) is *metabolic* if it is cobordant to $(0, 0)$. Explicitly, this means that there is a map $L \to X$ and a null homotopy of $q|_L$ such that the resulting sequence $L \to X \simeq \mathrm{D}X \to \mathrm{D}L$ is exact. We then say that L is a *Lagrangian* in X.

The Q-construction

Recall that the *twisted arrow category* $\mathrm{TwAr}[n]$ of $[n]$ is the category of pairs $i \leq j \in [n]$ where there is a unique morphism from $i \leq j$ to $i' \leq j'$ if $i \leq i' \leq j' \leq j$, and no morphisms otherwise.

Definition

Let $(\mathcal{C}, \Omega)$ be a hermitian ∞-category. We define $\mathrm{Q}_n(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ to be the full subcategory spanned by those functors $\varphi \colon \mathrm{TwAr}[n] \to \mathcal{C}$ such that the square

$$\begin{array}{ccc} \varphi(i \leq l) & \longrightarrow & \varphi(j \leq l) \\ \downarrow & & \downarrow \\ \varphi(i \leq k) & \longrightarrow & \varphi(j \leq k) \end{array}$$

is exact for every $i \leq j \leq k \leq l \in [n]$. We refine this to a hermitian ∞-category $\mathrm{Q}_n(\mathcal{C}, \Omega) = (\mathrm{Q}_n(\mathcal{C}), \Omega_n)$ with $\Omega_n(\varphi) = \lim_{\mathrm{TwAr}[n]^{\mathrm{op}}} \Omega\varphi$.

Claim

If $(\mathcal{C}, \Omega)$ is Poincaré then $\mathrm{Q}_n(\mathcal{C}, \Omega)$ is Poincaré for all n.

The Q-construction and cobordisms

Example

In the Poincaré ∞-category $\mathrm{Q}_1(\mathcal{C}, \Omega)$ objects are spans $X \xleftarrow{\alpha} W \xrightarrow{\beta} X'$.

- A hermitian form on such a span is by definition a choice of hermitian forms q, q' on X and X' respectively, and a homotopy $\eta\colon \alpha^* q \sim \beta^* q'$.
- Such a hermitian form is Poincaré if and only if (X, q) and (X', q') are Poincaré and the induced map $W \to \mathrm{D}X \times_{\mathrm{D}W} \mathrm{D}X'$ is an equivalence.

In particular, Poincaré objects in $\mathrm{Q}_1(\mathcal{C}, \Omega)$ correspond to a pair of Poincaré objects in $(\mathcal{C}, \Omega)$ and a cobordism between them. More generally, Poincaré objects in $\mathrm{Q}_n(\mathcal{C}, \Omega)$ can be identified with the data of a sequence of n composable cobordisms.

Claim

If $(\mathcal{C}, \Omega)$ is a Poincaré ∞-category then the simplicial space given by $[n] \mapsto \mathrm{Pn}\,\mathrm{Q}_n(\mathcal{C}, \Omega)$ is a *complete Segal space.*

Cobordism ∞-categories

Definition

Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category. We define $\mathrm{Cob}(\mathcal{C}, \Omega)$ to be the ∞-category corresponding to the complete Segal space $\mathrm{Pn}\,\mathsf{Q}_\bullet(\mathcal{C}, \Omega^{[1]})$, and call it the *cobordism category* of $(\mathcal{C}, \Omega)$.

Here $\Omega^{[n]} = \Sigma^n \Omega$ is the shift of Ω. It is introduced to accommodate the dimension convention in geometric cobordism categories.

Example

For the Poincaré ∞-category $\mathsf{Hyp}(\mathcal{C})$ one has $\mathrm{Cob}(\mathsf{Hyp}(\mathcal{C})) \simeq \mathsf{Span}(\mathcal{C})$.

Definition

The n-th L-group of $(\mathcal{C}, \Omega)$ is the group

$$\mathrm{L}_n(\mathcal{C}, \Omega) := \pi_0|\mathrm{Cob}(\mathcal{C}, \Omega^{[-n-1]})|$$

of cobordism classes of Poincaré objects in $(\mathcal{C}, \Omega^{[-n]})$. Addition is given by direct sum $[[X, q]] + [[X', q']] = [[X \oplus X', q + q']]$, and the inverse of $[[X, q]]$ is $[[X, -q]]$.

Example

In the case of $(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{q}}_M)$ these L-groups recover the classical Wall-Ranicki quadratic L-groups: $\mathrm{L}_n(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{q}}_M) \cong \mathrm{L}^{\mathrm{q}}_n(R, M)$.

Example

For the Poincaré ∞-category $\mathrm{Hyp}(\mathcal{C})$ all L-groups vanish.

The Grothendieck-Witt space

Definition

Let $(\mathcal{C}, \Omega)$ be a Poincaré ∞-category. We define its *Grothendieck-Witt space* by

$$\mathcal{GW}(\mathcal{C}, \Omega) := \Omega|\mathrm{Cob}(\mathcal{C}, \Omega)| = \Omega|\mathrm{Pn}\,\mathsf{Q}_\bullet(\mathcal{C}, \Omega^{[1]})|,$$

where the middle term $|\bullet|$ is the geometric realization of an ∞-category, corresponding in this case to the geometric realization of the simplicial space on the right.

Example

For the Poincaré ∞-category $\mathsf{Hyp}(\mathcal{C})$ one has

$$\mathcal{GW}(\mathsf{Hyp}(\mathcal{C})) \simeq \Omega|\,\mathsf{Span}(\mathcal{C})| = \mathcal{K}(\mathcal{C}).$$

The comparison theorem

For a ring R, an invertible module with involution M, and $r \in \{\mathrm{q}, \mathrm{gq}, \mathrm{ge}, \mathrm{gs}, \mathrm{s}\}$ we denote

$$\mathcal{GW}^r(R, M) := \mathcal{GW}(\mathcal{D}^{\mathrm{p}}(R), \Omega^r_M).$$

Theorem (Hebestreit-Steimle)

There are natural equivalences

$$\mathcal{GW}^{\mathrm{gq}}(R, M) \simeq \mathcal{GW}^{\mathrm{q}}_{\mathrm{cl}}(R, M)$$
$$\mathcal{GW}^{\mathrm{ge}}(R, M) \simeq \mathcal{GW}^{\mathrm{ev}}_{\mathrm{cl}}(R, M)$$
$$\mathcal{GW}^{\mathrm{gs}}(R, M) \simeq \mathcal{GW}^{\mathrm{s}}_{\mathrm{cl}}(R, M).$$

The Grothendieck-Witt space of the genuine Poincaré structures recovers classical Grothendieck-Witt spaces defined using unimodular forms and group completion.

$\mathcal{GW}^{\mathrm{q}}(R, M)$ and $\mathcal{GW}^{\mathrm{s}}(R, M)$ are new invariants of rings.

The Grothendieck-Witt group

The group of components $\mathrm{GW}_0(\mathcal{C}, \mathfrak{Q}) = \pi_0 \mathcal{GW}(\mathcal{C}, \mathfrak{Q})$ admits the following explicit presentation:

Generators and relations for GW_0

The group $\mathrm{GW}_0(\mathcal{C}, \mathfrak{Q})$ is generated by equivalence classes $[X, q]$ of Poincaré objects modulue the relation

$$[X, q] = [\mathrm{hyp}(L)]$$

whenever (X, q) is metabolic with Lagrangian $L \to X$.

After taking cobordism classes one has $[[X, q]] = [[\mathrm{hyp}(L)]] = 0$, and so the association $[X, q] \mapsto [[X, q]]$ determines a group homomorphism $\mathrm{GW}_0(\mathcal{C}, \mathfrak{Q}) \to \mathrm{L}_0(\mathcal{C}, \mathfrak{Q})$. This homomorphism fits in an exact sequence

$$\mathrm{K}_0(\mathcal{C})_{\mathrm{C}_2} \to \mathrm{GW}_0(\mathcal{C}, \mathfrak{Q}) \to \mathrm{L}_0(\mathcal{C}, \mathfrak{Q}) \to 0,$$

where the first map is induced by $Z \mapsto \mathrm{hyp}(Z)$.

Can we extend this exact sequence to the left?