On property (T) for Aut(F_n)

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Property (T)

G - group generated by a finite set S

Definition (Kazhdan 1966)

G has property (T) if there is $\kappa = \kappa(G, S) > 0$ such that

$$\sup_{s \in S} \|v - \pi_s v\| \geq \kappa \|v\|$$

for every unitary representation without invariant vectors.

Kazhdan constant = optimal $\kappa(G, S)$

Property (T)

Finite groups have (T)

Classical examples of infinite groups with (T):

- higher rank simple Lie groups and their lattices $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$, $n \geq 3$ (Kazhdan, 1966)
- automorphism groups of certain buildings, e.g. $\widetilde{A}_2$ (Cartwright-Młotkowski-Steger 1996, Pansu, Żuk, Ballmann-Świątkowski, 1997-98)
- certain random hyperbolic groups in the triangular and Gromov model (Żuk 2003)

Why is property (T) interesting?

Some applications:

- Margulis (1977): G with (T), $N_i \subseteq G$ family of finite index subgroups with trivial intersection $\Longrightarrow$ Cayley graphs of G/N_i form expander graphs

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- counterexamples to Baum-Connes type conjectures (Higson-Lafforgue-Skandalis 2003): K-theory classes represented by Kazhdan-type projections do not lie in the image of appropriate Baum-Connes assembly map

Question

Does $\mathrm{Aut}(F_n)$ *or* $\mathrm{Out}(F_n)$ *have property* (T)*?*

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For small values of n the answer is negative:

- $\mathrm{Aut}(F_2)$ maps onto $\mathrm{Out}(F_2) \simeq GL_2(\mathbb{Z})$
- $\mathrm{Aut}(F_3)$ maps onto $\mathbb{Z}$ (McCool 1989)
  and virtually onto F_2 (Grunewald-Lubotzky 2006)

$\implies$ no (T)

Ozawa's characterization

Laplacian in the real group ring $\mathbb{R}G$:

$$\Delta = |S| - \sum_{s \in S} s \qquad \in \mathbb{R}G$$

Theorem (Ozawa, 2014)

(G, S) has property (T) iff for some $\lambda > 0$ and a finite collection of $\xi_i \in \mathbb{R}G$

$$\Delta^2 - \lambda\Delta = \sum_{i=1}^{n} \xi_i^* \xi_i$$

This is a finite-dim condition - perhaps possible to solve with computer assistance.

Property (T) quantified

If the equation is satisfied:

$$\Delta^2 - \lambda\Delta = \sum_{i=1}^{n} \xi_i^* \xi_i$$

then relation to Kazhdan constants:

$$\sqrt{\frac{2\lambda}{|S|}} \leq \kappa(G, S)$$

we can also define the following notion

Definition

Kazhdan radius of (G, S) = smallest $r > 0$ such that $\operatorname{supp} \xi_i \subseteq B(e, r)$ for all ξ_i above.

Implementation for $\mathrm{SL}_n(\mathbb{Z})$

Implementation of this strategy pioneered by Netzer-Thom (2015):

a new, computer-assisted proof of (T) for $\mathrm{SL}_3(\mathbb{Z})$
(generators: elementary matrices)

Improvement of Kazhdan constant:

$$\simeq 1/1800 \quad \rightarrow \quad \simeq 1/6$$

Later also improved Kazhdan constants for $\mathrm{SL}_n(\mathbb{Z})$ for

$n = 3, 4$ (Fujiwara-Kabaya)

$n = 3, 4, 5$ (Kaluba-N.)

Setup: $\mathrm{SAut}(F_n)$

Subgroup of $\mathrm{Aut}(F_n)$, generated by Nielsen transformations:

$$R_{ij}^{\pm}(s_k) = \begin{cases} s_k s_j^{\pm 1} & \text{if } k = i \\ s_k & \text{oth.} \end{cases}, \quad L_{ij}^{\pm}(s_k) = \begin{cases} s_j^{\pm 1} s_k & \text{if } k = i \\ s_k & \text{oth.} \end{cases}$$

Equivalently,

$$\mathrm{SAut}(F_n) = ab^{-1}(\mathrm{SL}_n(\mathbb{Z}))$$

under the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ induced by the abelianization $F_n \to \mathbb{Z}^n$.

$\mathrm{SAut}(F_n)$ has index 2 in $\mathrm{Aut}(F_n)$

Main results

Theorem (Kaluba - N. - Ozawa)

$\mathrm{SAut}(F_5)$ *has property* (T) *with Kazhdan constant*

$$0.18 \leq \kappa(\mathrm{SAut}(F_5))$$

Theorem (Kaluba - Kielak - N.)

$\mathrm{SAut}(F_n)$ *has property* (T) *for* $n \geq 6$ *with Kazhdan radius 2 and Kazhdan constant*

$$\sqrt{\frac{0.138(n-2)}{6(n^2-n)}} \leq \kappa(\mathrm{SAut}(F_n)).$$

Certifying positivity via semidefinite programming

$\eta \in \mathbb{R}G$ positive if η is a sum of squares:

$$\eta = \sum_{i=1}^{k} \xi_i^* \xi_i$$

where $\operatorname{supp} \xi_i \subseteq E$ - finite subset of G. (here always $E = B(e, 2)$)

$\iff$

there is a positive definite $E \times E$ matrix P such that for $\mathbf{b} = [g_1, \ldots, g_n]_{g_i \in E}$

$$\eta = \mathbf{b} P \mathbf{b}^T = \mathbf{b} Q Q^T \mathbf{b}^T = (\mathbf{b} Q)(\mathbf{b} Q)^T$$

i-th column of Q = coefficients of ξ_i

Using the solver

To check if η is positve we can use a semidefinite solver to perform convex optimization over positive definite matrices:

$$\text{find } P \in \mathbb{M}_{E\times E} \quad \text{such that:}$$
$$\eta = \mathbf{b}P\mathbf{b}^T$$
$$\text{positive semi-definite}$$

Assume now that a computer has found a solution P - we obtain

$$\eta \simeq \sum \xi_i^* \xi_i$$

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this is not a precise solution

However this can be improved if the error is small in a certain sense and η belongs to the augmentation ideal *IG*

Lemma (Ozawa, Netzer-Thom)

Δ is an order unit in IG: if $\eta = \eta^* \in IG$ then

$$\eta + R\Delta = \sum \xi_i^* \xi_i$$

for all sufficiently large $R \geq 0$.

Moreover, $R = 2^{2r-2}\|\eta\|_1$ is sufficient, where $\operatorname{supp}\eta \subseteq B(e, 2^r)$.

With this in mind we can try to improve our search of P to show that η is "strictly positive":

$$\text{maximize } \lambda \geq 0 \qquad \text{under the conditions}$$

$$\eta - \lambda\Delta = \mathbf{b}P\mathbf{b}^T$$

$$P \in \mathbb{M}_{E\times E} \text{ positive semi-definite}$$

Assume now that numerically on $B(e, 2^r)$

$$\eta - \lambda\Delta \simeq \mathbf{b}P\mathbf{b}^T$$

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Correct Q to $\overline{Q}$, where columns of Q sum up to 0.

$$\left\|\eta - \lambda\Delta - \mathbf{b}\overline{Q}\,\overline{Q}^T\mathbf{b}^T\right\|_1 \leq \varepsilon$$

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Then for $R = \varepsilon 2^{2r-2}$ or larger:

$$\eta - \underbrace{\lambda\Delta + R\Delta}_{(\lambda - R)\Delta} = \underbrace{\mathbf{b}\overline{Q}\,\overline{Q}^T\mathbf{b}^T}_{\sum \eta_i^* \eta_i} + \underbrace{(\eta - \lambda\Delta - \mathbf{b}\overline{Q}\,\overline{Q}^T\mathbf{b}^T + R\Delta)}_{\geq 0 \text{ by lemma}}$$

If $\lambda - R > 0$ then $\eta - (\lambda - R)\Delta \geq 0$ and in particular, $\xi \geq 0$.

Important:
The ℓ_1-norm is computed in interval artihmetic.

This gives a mathematically rigorous proof of positivity of η.

To prove (T): apply to $\eta = \Delta^2 - \lambda\Delta$ and maximize for $\lambda > 0$

We want to use this strategy in $\mathrm{SAut}(F_5)$.

Problem #1:

$$|B(e, 2)| = 4\,641$$

P has 10 771 761 variables - too large for a solver to handle

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We reduce the number of variables via symmetrization

$$
\begin{bmatrix}
\star & \bullet & & & & \\
\star & \bullet & \circ & & & \\
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& & \bullet & \bullet & \circ & \\
& & & \circ & \star & \bullet \\
& & & & \star & \star
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
\begin{bmatrix} * \end{bmatrix} & & & \\
& \begin{bmatrix} * & * \\ * & * \end{bmatrix} & & \\
& & \ddots & \\
& & & \begin{bmatrix} * \end{bmatrix}
\end{bmatrix}
$$

The reduction for $\mathrm{SAut}(F_5)$:

from 10 771 761 variables $\to$ **13 232** variables in 36 blocks

Theorem (Kaluba-N.-Ozawa)

$\mathrm{SAut}(F_5)$ *has* (T) *with Kazhdan constant* ≥ 0.18027*.*

Proof.

Find sum of squares decomposition for $\Delta^2 - \lambda\Delta$

on the ball of radius 2

Data from solver: P and $\lambda = 1.3$

$$8.30 \cdot 10^{-6} \quad \leq \quad \left\| \Delta^2 - \lambda\Delta - \sum \xi_i^* \xi_i \right\|_1 \quad \leq \quad 8.41 \cdot 10^{-6}$$

$\Longrightarrow R = 8.41 \cdot 10^{-6} \cdot 2^{4-2}$ suffices

$$\lambda - R = 1.2999 > 0$$

□

Property (T) **for** $\mathrm{Aut}(F_n)$**,** $n \geq 6$

G_n with generating set S_n will denote either one of the families

- $\mathsf{SAut}(F_n)$ generatred by Nielsen transformations $R^{\pm}_{i,j}$, $L^{\pm}_{i,j}$
- $\mathsf{SL}_n(\mathbb{Z})$ generated by elementary matrices $E^{\pm}_{i,j}$

G_n form a tower via the standard inclusions

$$F_n \subseteq F_{n+1} \qquad \text{and} \qquad \mathbb{Z}_n \subseteq \mathbb{Z}_{n+1}$$

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$\mathscr{C}_n$ = $(n-1)$-simplex on $\{1, \ldots, n\}$

E_n set of edges of $\mathscr{C}_n$ = (unoriented) pairs $e = \{i, j\}$

Alt_n acts on edges: $\sigma(e) = \sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$

Map

$$I_n : S_n \to E_n,$$

$$R_{ij}^{\pm}, L_{ij}^{\pm} \mapsto \{i, j\}, \qquad E_{ij}^{\pm} \mapsto \{i, j\}$$

$\Delta_n \in \mathbb{R}G_n$ - Laplacian of G_n

A copy of G_2 is attached at each edge.

For an edge $e = \{i, j\}$ let $S_e = \{s \in S_n : l_n(s) = e\}$ and

$$\Delta_e = |S_e| - \sum_{t \in S_e} t$$

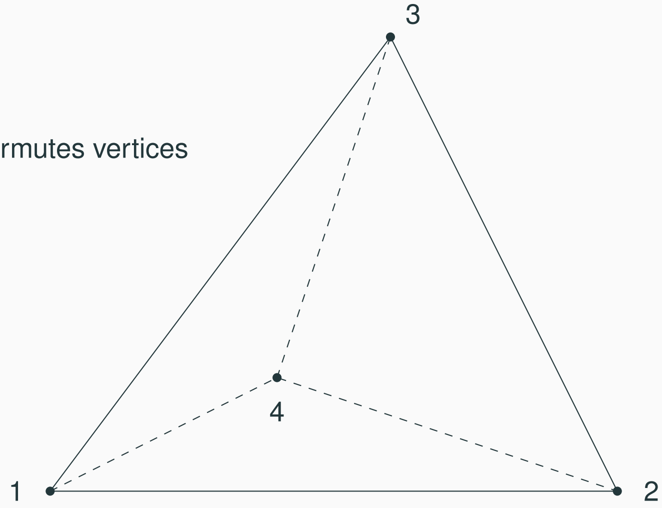
is the Laplacian of the copy of G_2 attached to e

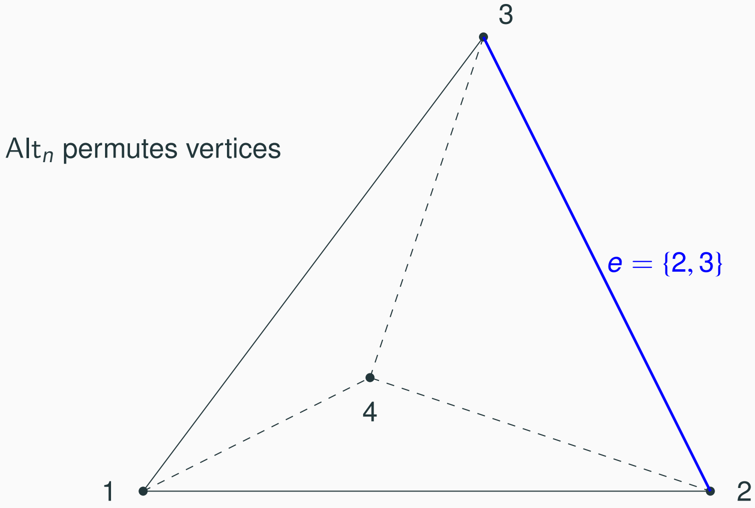
We have $\sigma(\Delta_e) = \Delta_{\sigma(e)}$ for any $\sigma \in \mathrm{Alt}_n$

Given 2 edges they can

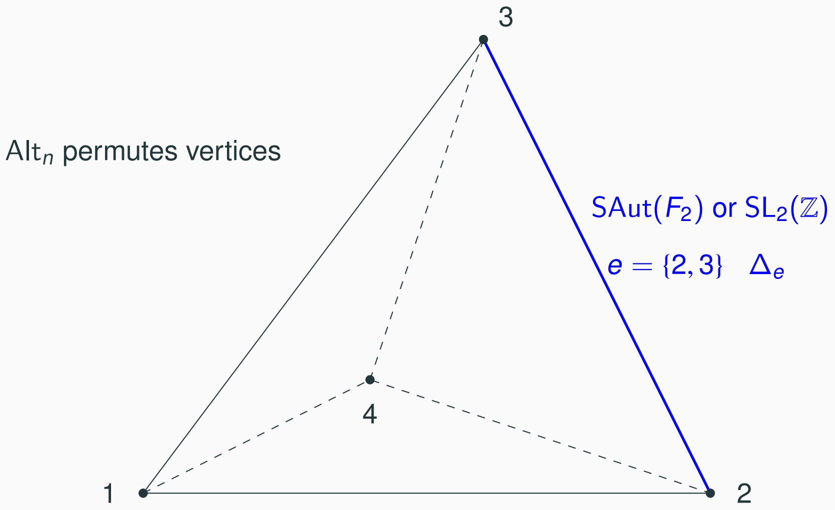
1. coincide
2. be adjacent (share a vertex)
3. be opposite (share no vertices) - corresponding copies of G_2 commute

Alt_n permutes vertices





3
Alt_n permutes vertices
$e = \{2, 3\}$
4
1
2



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Alt$_n$ permutes vertices
SAut(F_2) or SL$_2$($\mathbb{Z}$)
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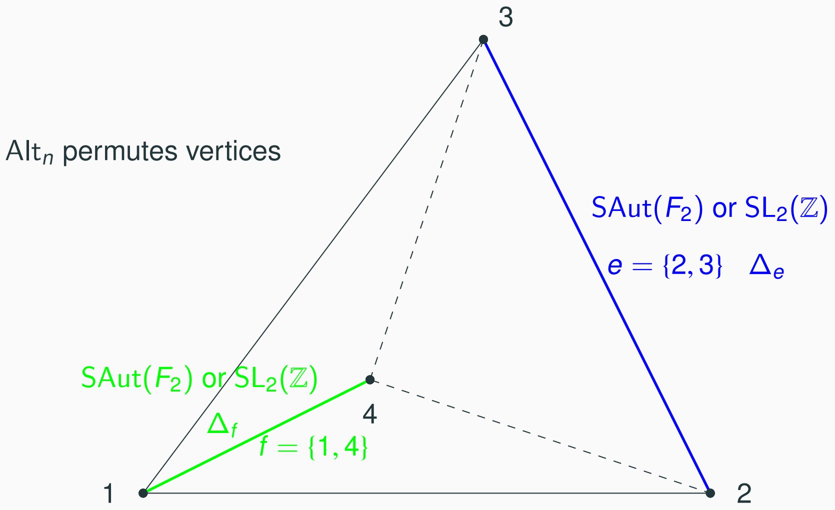
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Δ_f
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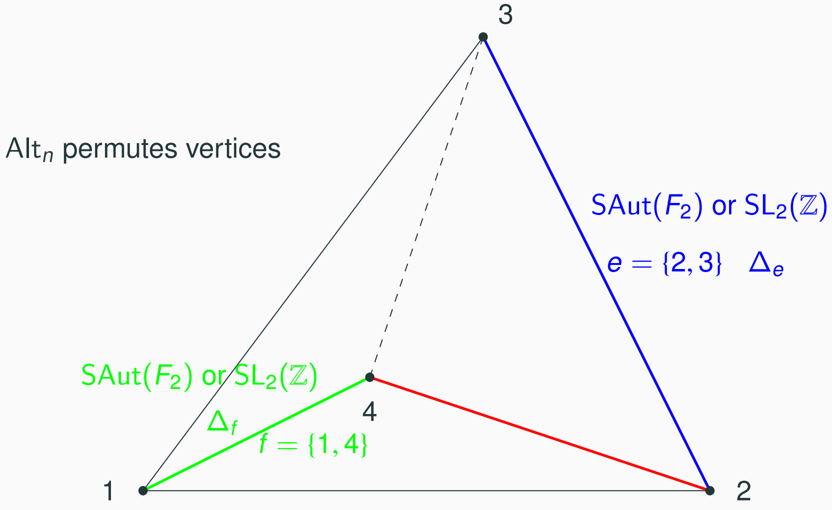
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Proposition

For $m \geq n \geq 3$ we have

$$\sum_{\sigma \in \mathrm{Alt}_m} \sigma(\Delta_n) = \binom{n}{2} (m-2)! \Delta_m.$$

Main step in proving property (T) for G_n:
a "stable decomposition" of $\Delta_n^2 - \lambda \Delta_n$

$$\Delta^2 = \left(\sum_e \Delta_e\right)\left(\sum_f \Delta_f\right) = \sum_{e,f} \Delta_e \Delta_f$$

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Define three elements of $\mathbb{R}G_n$:

1. $\mathsf{Sq}_n = \sum_{e \in E_n} \Delta_e^2$
2. $\mathsf{Adj}_n = \sum_{e \in E_n} \sum_{f \in \mathsf{Adj}_n(e)} \Delta_e \Delta_f$
3. $\mathsf{Op}_n = \sum_{e \in E_n} \sum_{f \in \mathsf{Op}_n(e)} \Delta_e \Delta_f$

Then

$$\mathsf{Sq}_n + \mathsf{Adj}_n + \mathsf{Op}_n = \Delta_n^2$$

Lemma

The elements Sq_n *and* Op_n *are sums of squares.*

Proof.

Obvious for Sq_n.

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For Op_n:

$$\Delta_e = \frac{1}{2} \sum_{t \in S_e} (1 - t)^* (1 - t)$$

e, f opposite edges then the generators associated to them commute and $\Delta_e \Delta_f$ can be rewritten as a sum of squares using

$$(1 - t)^*(1 - t)(1 - s)^*(1 - s) = \big((1 - t)(1 - s)\big)^* \big((1 - t)(1 - s)\big)$$

□

Stability for Adj and Op:

Proposition

For $m \geq n \geq 3$ we have

$$\sum_{\sigma \in \mathrm{Alt}_m} \sigma\left(\mathrm{Adj}_n\right) = \left(\frac{1}{2} n(n-1)(n-2)(m-3)!\right) \mathrm{Adj}_m$$

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Proposition

For $m \geq n \geq 4$ we have

$$\sum_{\sigma \in \mathrm{Alt}_m} \sigma\left(\mathrm{Op}_n\right) = \left(2\binom{n}{2}\binom{n-2}{n}(m-4)!\right) \mathrm{Op}_m$$

The following allows to prove (T) for $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ and for $\mathrm{SAut}(F_n)$, $n \geq 7$.

Theorem

Let $n \geq 3$ *and*

$$\mathrm{Adj}_n + k\,\mathrm{Op}_n - \lambda\Delta_n = \sum \xi_i^* \xi_i$$

where $\operatorname{supp} \xi_i \subset B(e, R)$, *for some* $k \geq 0$, $\lambda \geq 0$.

Then G_m *has property* (T) *for every* $m \geq n$ *such that*

$$k(n-3) \leq m-3.$$

Moreover, the Kazhdan radius is bounded above by R.

Proof: Rewrite $\Delta_m^2 - \lambda\Delta_m$ using stability of Op, Adj and Δ.

$$\Delta_m^2 - \frac{\lambda(m-2)}{n-2}\Delta_m = \mathsf{Sq}_m + \mathsf{Adj}_m + \mathsf{Op}_m - \frac{\lambda(m-2)}{n-2}\Delta_m$$

$$= \mathsf{Sq}_m + \left(1 - \frac{k(n-3)}{m-3}\right)\mathsf{Op}_m + \frac{2}{n(n-1)(n-2)(m-3)!}\sum_{\sigma\in\mathsf{Alt}_m} \sigma\left(\mathsf{Adj}_n + k\,\mathsf{Op}_n - \lambda\Delta_n\right)$$

When $1 - \dfrac{k(n-3)}{m-3} \geq 0$ we obtain the claim. $\square$

Results

Theorem

SAut(F_n) *has* (T) *for* $n \geq 6$*, with Kazhdan radius* 2 *and Kazhdan constant estimate*

$$\sqrt{\frac{0.138(n-2)}{6(n^2-n)}} \leq \kappa(\mathrm{SAut}_n).$$

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Proof.

The case $n \geq 7$:

$$\mathrm{Adj}_5 + 2\,\mathrm{Op}_5 - 0.138\Delta_5$$

certified positive on the ball of radius 2. □

The case $n = 6$ needs a different computation.

The case $n = 5$ so far can only be proven directly.

Currently also $\mathrm{Adj}_4 + 100\,\mathrm{Op}_4 - 0.1\Delta \geq 0$ in $\mathrm{SAut}(F_4)$ proving (T) for $n \geq 103$.

Theorem

$\mathrm{SL}_n(\mathbb{Z})$ *has* (T) *for* $n \geq 3$*, with Kazhdan radius* 2 *and Kazhdan constant estimate*

$$\sqrt{\frac{0.157999(n-2)}{n^2-n}} \leq \kappa(\mathrm{SL}_n(\mathbb{Z})).$$

Proof.

$$\mathrm{Adj}_3 - 0.157999\Delta_3$$

certified positive on the ball of radius 2. □

This gives a new estimate on the Kazhdan constants of $\mathrm{SL}_n(\mathbb{Z})$.

We obtain an even better estimate by certifying positivity of

$$\mathrm{Adj}_5 + 1.5\,\mathrm{Op}_5 - 1.5\Delta_5$$

in $\mathbb{R}\,\mathrm{SL}_5(\mathbb{Z})$:

$$\sqrt{\frac{0.5(n-2)}{n^2-n}} \leq \kappa(\mathrm{SL}_n(\mathbb{Z})) \qquad \text{for } n \geq 6.$$

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Previously known bounds:

$$\text{(Kassabov 2005)} \quad \frac{1}{42\sqrt{n}+860} \leq \kappa(\mathrm{SL}_n(\mathbb{Z})) \leq \sqrt{\frac{2}{n}} \quad \text{(Żuk 1999)}$$

Asymptotically, the new lower bound is $1/2$ of the upper bound:

$$\frac{\text{Żuk's upper bound}}{\text{our lower bound}} = 2\sqrt{\frac{n-1}{n-2}} \quad \longrightarrow \quad 2 \qquad (n \geq 6)$$

Some applications

Product Replacement Algorithm generates random elements in finite groups

Lubotzky and Pak in 2001 showed that property (T) for $\mathtt{Aut}(F_n)$ would explain the fast performance of the Product Replacement Algorithm

Now proven.

Some applications

Question (Lubotzky 1994)

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Gilman 1977: $\mathrm{Aut}(F_n)$ for $n \geq 3$ residually alternating

$\implies$ sequences of alternating groups with $\mathrm{Aut}(F_n)$ generators are expanders

This gives an alternative answer to Lubotzky's questions with explicit generating sets

Final remarks

- property (T) for $\mathrm{Aut}(F_4)$ confirmed by Martin Nietsche
- jointly with Uri Bader we proved a generalization of Ozawa's characterization over $\mathbb{R}G$ of property (T) to higher cohomology with unitary coefficients