Cutoff on SL_2

Amitay Kamber

University of Cambridge

based on a joint work with Konstantin Golubev

7/6/2021

Introduction

Expansion tells us a lot about the geometry of combinatorial structures.
For example, expander graphs have logarithmic diameter.
However, expansion falls short of proving some more delicate results, such as more exact distribution of distances between points.
I will discuss the special case of Cayley and Schreier graphs of SL_2, which is a special case of a general theory.

The results of Bourgain-Gamburd

Theorem (Bourgain-Gamburd 2005, $\mathrm{SL}_2(\mathbb{Z})$ generators)

$S = \{s_1, ..., s_{q+1}\} \subset \mathrm{SL}_2(\mathbb{Z})$ a symmetric set which generates a free group, e.g., the Lubotzky generators

$$S = \left\{ \begin{pmatrix} 1 & \pm m \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ \pm m & 1 \end{pmatrix} \right\}, \quad m \geq 2.$$

For p prime, let $X_p = \mathrm{Cayley}(\mathrm{SL}_2(\mathbb{F}_p), S \mod p)$.
Then for p large enough, X_p is an expander.

Theorem (Bourgain-Gamburd 2005, random generators)

Let $S' = \{g_1, ..., g_{(q+1)/2}\}$ be random elements of $\mathrm{SL}_2(\mathbb{F}_p)$, $S = S' \cup S'^{-1}$, and $X_p = \mathrm{Cayley}(\mathrm{SL}_2(\mathbb{F}_p), S)$.
Then as $p \to \infty$ over primes, with probability $1 - o_p(1)$, X_p is an expander.

Schreier graphs

- The theorems also imply that any Schreier graph coming from the action of $\mathrm{SL}_2(\mathbb{F}_p)$ on $\mathrm{SL}_2(\mathbb{F}_p)/H$, $H \leq \mathrm{SL}_2(\mathbb{F}_p)$ is an expander.
- In particular, this is true for the Schreier graph on the *affine space* $A^2(\mathbb{F}_p) \cong \mathbb{F}_p^2$ and the *projective line* $P^1(\mathbb{F}_p) \cong A^2(\mathbb{F}_p)/\mathbb{F}_p^\times$.

Super Strong Approximation and Lifting

- Expansion implies that for p large enough the graphs have logarithmic diameter - the distance between every two points is bounded by $C \log |X|$.
- Let $\Gamma = \langle S \rangle \subset \mathrm{SL}_2(\mathbb{Z})$, and l:$\Gamma \to \mathbb{N}$ the length in terms of the generators.

Corollary (Lifting)

For p large enough, for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$, there is $\gamma \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that:

1. $\gamma \mod p = x$.
2. $l(\gamma) \leq C \ln p$.

Part 1 is called Strong Approximation. The theorem and its extensions are called Super Strong Approximation.

Optimal Almost-Diameter

We expect more to be true. Let X be a $(q+1)$-regular graph.

- The sphere of radius k has at most $(q+1)q^{r-1}$ points.
- Therefore, for every $x \in X$, for almost every $y \in X$, $d(x,y) \geq (1-o(1))\log_q(|X|)$.

Optimal almost-diameter

The graph has *optimal almost-diameter* if for all but $o(|X|^2)$ of the pairs $x, y \in X$, $d(x,y) \leq (1+o(1))\log_q(|X|)$.

Conjecture

All the Cayley and Schreier graphs we considered have optimal almost-diameter.

- The conjecture was verified experimentally by Rivin and Sardari (2017).
- Closely related to the cutoff phenomena, which concerns the speed on convergence of the random walk in L^1-norm.

Ramanujan graphs

Definition

Let $A\colon L^2(X) \to L^2(X)$ be the adjacency operator.
A $(q+1)$-regular graph is *Ramanujan* if the second largest eigenvalue in absolute value of A is bounded by $2\sqrt{q}$.

Almost-Ramanujan graph- second eigenvalue is bounded by $2\sqrt{q} + o(1)$.

Theorem (Lubetzky-Peres, Sardari 2015)

Almost-Ramanujan graphs have optimal almost-diameter.

Conjecture (Rivin-Sardari 2017)

In both cases, the graphs are almost-Ramanujan.

The almost-Ramanujan conjecture implies the previous conjecture, but seems to be out of reach.

Results - Random Generators

Theorem (Golubev-K. 2020)

With high probability, a random Schreier graph on the projective line $P^1(\mathbb{F}_p)$ has optimal almost-diameter.

- The question is open for the Cayley graph and the affine plane.

Results - $\mathrm{SL}_2(\mathbb{Z})$ generators

Theorem

Let $S \leq \mathrm{SL}_2(\mathbb{Z})$ generate a free subgroup. Then:

1. For almost all p in the range $X \leq p \leq 2X$, the Schreier graph on $P^1(\mathbb{F}_p)$ has optimal almost-diameter.
2. Assuming the Lyapunov exponent determined by the generators is small enough, for every p large enough the Schreier graph on $P^1(\mathbb{F}_p)$ has optimal almost-diameter.

Results - $SL_2(\mathbb{Z})$ generators - remarks

- The generators S determine a random walk on $SL_2(\mathbb{R})$, with entropy $h = \frac{q-1}{q+1}\log(q)$ and some Lyapunov exponent χ, determining the growth of the norm of most elements.
- It also determines a Furstenberg stationary measure ν on $P^1(\mathbb{R})$, whose dimension is $\dim \nu = h/2\chi$ (Ledrappier 1983, Hochman-Solomyak 2017).
- The condition for cutoff on $P^1(\mathbb{F}_p)$ is $\dim \nu \geq 1/2$.
- It holds for $\left\{ \begin{pmatrix} 1 & \pm 3 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ \pm 3 & 1 \end{pmatrix} \right\}$, but not for $\left\{ \begin{pmatrix} 1 & \pm 4 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ \pm 4 & 1 \end{pmatrix} \right\}$.
- $\dim \nu \geq 4/5$ implies optimal almost-diameter for most of the primes for $A^2(\mathbb{F}_p)$.
- $\dim \nu = 1$ implies optimal almost-diameter for the Cayley graph...
- Can be generalized to the non-free case.

Proof Idea

- Based on the work of Sarnak and Xue (1991) on automorphic forms.
- Has a spectral definition ("density condition") and a geometric one.
- Geometric "weak injective radius property": Show that "on average", the number of fixed points of the action is $O\left(|X|^{\epsilon}\right)$. More specifically:

Proposition (Weak injective radius implies optimal almost-diameter)

Let $\Gamma = F_{S'}$ be the free group on the set S', and $S = S' \cup S'^{-1}$, $|S| = q + 1$.
Let X be a Γ-space, with a Schreier graph structure determined by S.
Assume that X is an expander and that for $k = 2\lfloor \log_q |X| \rfloor$,

$$\#\{(\gamma, x) \in \Gamma \times X : l(\gamma) = k, \gamma x = x\} \leq |X|^{o(1)} q^k = |X|^{2+o(1)}.$$

Then X has optimal almost-diameter.

The Weak Injective Radius Property

- There are a few cases where the weak injective property can be verified, such as random graphs.
- For Cayley graph on $\mathrm{SL}_2(\mathbb{F}_p)$, it can be verified elementarily for the LPS Ramanujan graphs (Davidoff-Sarnak-Valette 2003). Unknown for most of the other cases.
- The projective line and affine plane are easier.

Application for the projective line $P^1(\mathbb{F}_p)$

- Assume that $X = P^1(\mathbb{F}_q)$ and the Γ-action comes from a morphism $\phi : \Gamma \to \mathrm{SL}_2(\mathbb{F}_p)$.
- If $\phi(\gamma) \neq \pm I \in \mathrm{SL}_2(\mathbb{F}_p)$, then γ has at most two fixed points (corresponding to the eigenvectors).

Corollary

Let $X = P^1(\mathbb{F}_p)$, with a Schreier graph structure coming from $\phi : \Gamma \to \mathrm{SL}_2(\mathbb{F}_p)$.
Assume that X is an expander and that for $k = 2\lfloor \log_q p \rfloor$,

$$\#\{\gamma \in \Gamma : l(\gamma) = k, \gamma \equiv I \mod p\} \leq p^{o(1)} q^k / p = p^{1+o(1)}.$$

Then X has optimal almost diameter.

Random Generators - Sketch

- $\Gamma = F_{(q+1)/2}$. Any $(q+1)/2$ elements from $\mathrm{SL}_2(\mathbb{F}_p)$ to determine a map $\phi : \Gamma \to \mathrm{SL}_2(\mathbb{F}_p)$.
- Let $\gamma \in \Gamma$. What is the probability that $\phi(\gamma) = I$?
- γ defines a *word map* $W_\gamma : \mathrm{SL}_2(\mathbb{F}_p)^{(q+1)/2} \to \mathrm{SL}_2(\mathbb{F}_p)$. We want to understand

$$V_\gamma = \{x \in \mathrm{SL}_2(\mathbb{F}_p)^{(q+1)/2} : W_\gamma(x) = I\}.$$

- This is a subvariety, and Gamburd, Hoory, Shahshahani, Shalev, and Virag observed that it is a proper subvariety and as a matter of fact

$$|V_\gamma| / \mathrm{SL}_2(\mathbb{F}_p)^{(q+1)/2} \leq \frac{l}{p} + O\left(p^{-2}\right).$$

- Therefore

$$E\left[\#\{\gamma \in \Gamma : l(\gamma) = 2\lfloor \log_q p \rfloor, \phi(\gamma) = I\}\right] \leq p^{1+o(1)}.$$

- Applying Markov's inequality we get the desired result.

Random Generators - Cont.

- We expect that for "most" $\gamma \in \Gamma$ with $l(\gamma) = O(\log(p))$,

$$|V_\gamma^*| / \mathrm{SL}_2(\mathbb{F}_p)^{(q+1)/2} = O(p^{-3}).$$

- Proving this sufficiently explicitly will imply optimal almost-diameter for the random Cayley graph.
- There are partial results in a work in progress by Becker-Breuillard-Varju.

Generators in $\mathrm{SL}_2(\mathbb{Z})$

- Let $S \leq \mathrm{SL}_2(\mathbb{Z})$ generate a free subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. We want to prove that

$$\{\gamma \in \Gamma : l(\gamma) = 2\lfloor \log_q p \rfloor, \gamma \equiv I \mod p\} \leq p^{1+o(1)}.$$

- If we average over p this is simple, since for $\gamma \neq I$, $\gamma \equiv I \mod$ p for $O(l(\gamma))$ different p-s.
- Can be bounded by bounding

$$\{\gamma \in \Gamma : l(\gamma) = 2\lfloor \log_q p \rfloor, \|\gamma\|_\infty \leq T, \gamma \equiv I \mod p\},$$

and using

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \|\gamma\|_\infty \leq T, \gamma \equiv I \mod p\} \leq T^{o(1)}\left(\frac{T^2}{p^3} + \frac{T}{p} + 1\right).$$

- By a variation on the method, it is enough to consider the Lyapunov exponent determined by S.
- Is there a better way to bound similar sets?

The Density Condition

- Consider the spectrum of $A : L^2(X) \to L^2(X)$. An eigenvalue λ_i with $|\lambda_i| \geq 2\sqrt{q}$ is called exceptional.
- Associate with each exceptional λ_i a $p_i \in [2, \infty)$ according to

$$\lambda_i = \pm \left(q^{1/p_i} + q^{1-1/p_i} \right).$$

- The Sarnak-Xue density condition is that for every $\epsilon > 0$, $p > 2$ it holds that

$$\# \{ i : p_i \geq p \} \ll_\epsilon n^{2/p+\epsilon}.$$

- It is equivalent to the weak injective radius condition.

Lifting for the matrix norm

1. Can ask for lifting for other measures of sizes of elements.
2. Sarnak (2015) proved that almost every $x \in \mathrm{SL}_2(\mathbb{F}_p)$ can be lifted to an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with coordinates bounded by $p^{3/2+o(1)}$. This is optimal.
3. Conjecturally, the same is true for SL_n as well - for example, we believe that one can lift almost every $x \in \mathrm{SL}_3(\mathbb{F}_p)$ to an element $\gamma \in \mathrm{SL}_3(\mathbb{Z})$ with coordinates bounded by $p^{4/3+o(1)}$.
4. Can be solved by (a variation on) a bound on the size of the set

$$\{\gamma \in \mathrm{SL}_3(\mathbb{Z}) : \gamma \equiv I \mod p, \|\gamma\| \leq T\}.$$

5. Closely related to "Sarnak's density conjecture" in automorphic forms, which is an approximations to the Generalized Ramanujan Conjecture.

Thank You!