Nonlinear spectral gaps and coarse non-universality

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(joint work with M. Mendel and A. Naor)

Interactions between expanders, groups and operator algebras
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Classical expanders

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It follows from Cheeger's inequality that a sequence of 3-regular graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ as $n \to \infty$ is an expander graph sequence if and only if there exists a universal constant $\gamma \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every function $f : V_n \to \ell_2$ satisfies the Poincaré inequality

$$\frac{1}{|V_n|^2} \sum_{x,y \in V_n} \big\|f(x) - f(y)\big\|_{\ell_2}^2 \leq \frac{\gamma}{|V_n|} \sum_{\{x,y\} \in E_n} \big\|f(x) - f(y)\big\|_{\ell_2}^2.$$

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Simple linear algebra shows that the optimal constant in this inequality satisfies $\gamma \asymp \sup_{n\geq 1} \frac{1}{1-\lambda_2(G_n)}$, where $\lambda_2(G)$ is the second largest eigenvalue of the normalized adjacency matrix of a regular graph G.

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Definition

Let $G = (V, E)$ be a finite 3-regular graph and $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space. We will denote by $\gamma(G, d_{\mathcal{M}}^2)$ the least constant $\gamma \in (0, \infty)$ such that every function $f : V \to \mathcal{M}$ satisfies the Poincaré inequality

$$\frac{1}{|V|^2} \sum_{x,y \in V} d_{\mathcal{M}}\big(f(x), f(y)\big)^2 \leq \frac{\gamma}{|V|} \sum_{\{x,y\} \in E} d_{\mathcal{M}}\big(f(x), f(y)\big)^2.$$

We say that a sequence of 3-regular graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ as $n \to \infty$ is an expander graph sequence with respect to $\mathcal{M}$ if we have $\sup_{n \geq 1} \gamma(G_n, d_{\mathcal{M}}^2) < \infty$.

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Suppose that $G_n = (V_n, E_n)$ is a sequence of finite 3-regular graphs which are expanders with respect to a metric space $(\mathcal{M}, d_{\mathcal{M}})$. Then $\{(G_n, d_{G_n})\}_{n\geq 1}$ do not embed equi-coarsely into $\mathcal{M}$.

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The nonlinear spectral gap inequality is one of the most important coarse invariants available in the literature.

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Proof. Suppose that f is such an embedding. Then

$$\frac{1}{|V_n|} \sum_{\{x,y\} \in E_n} d_{\mathcal{M}}\big(f(x), f(y)\big)^2 \leq \frac{\|f\|_{\mathrm{Lip}}^2}{|V_n|} \sum_{\{x,y\} \in E_n} d_{G_n}(x,y)^2 \lesssim \|f\|_{\mathrm{Lip}}^2$$

and (since at least 1% of the pairs of vertices in V_n are $\log n / 10^{10}$ apart),

$$\frac{1}{|V_n|^2} \sum_{x,y \in V_n} d_{\mathcal{M}}\big(f(x), f(y)\big)^2 \geq \frac{1}{|V_n|^2 \|f^{-1}\|_{\mathrm{Lip}}^2} \sum_{x,y \in V_n} d_{G_n}(x,y)^2 \gtrsim \frac{(\log n)^2}{\|f^{-1}\|_{\mathrm{Lip}}^2}.$$

Nonlinear spectral gaps and embeddings (continued)

The importance of the logarithmic bound in Matoušek's observation stems from the following imporant embedding theorem of Bourgain.

Theorem (Bourgain's embedding theorem, 1985)

Any finite metric space $(\mathcal{M}, d_{\mathcal{M}})$ *embeds into* ℓ_2 *with distortion at most a constant multiple of* $\log|\mathcal{M}|$.

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In particular, Matoušek's observation along with the existence of classical expanders implies that the distortion bound in Bourgain's theorem is asymptotically sharp as a function on $|\mathcal{M}|$.

The parallelogram identity

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It is well known that **Hilbert spaces** (i.e. Euclidean geometry) are characterised among all normed spaces by the parallelogram identity

$$\forall\, x, y \in \mathscr{H}, \quad \|x-y\|_{\mathscr{H}}^2 + \|x+y\|_{\mathscr{H}}^2 = 2\|x\|_{\mathscr{H}}^2 + 2\|y\|_{\mathscr{H}}^2$$

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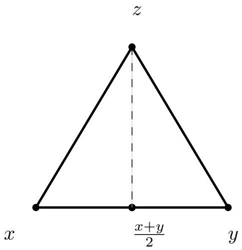

Equivalently, if $x, y, z \in \mathcal{H}$ then

$$\left\| z - \frac{x+y}{2} \right\|_{\mathcal{H}}^2 = \frac{\|z-x\|_{\mathcal{H}}^2}{2} + \frac{\|z-y\|_{\mathcal{H}}^2}{2} - \frac{\|x-y\|_{\mathcal{H}}^2}{4}.$$

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Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\{x, y, z\}$ be a triangle in $\mathcal{M}$. Let m be any metric midpoint of x, y, i.e. a point of $\mathcal{M}$ such that $d_{\mathcal{M}}(x, m) = d_{\mathcal{M}}(m, y) = \frac{1}{2} d_{\mathcal{M}}(x, y)$.

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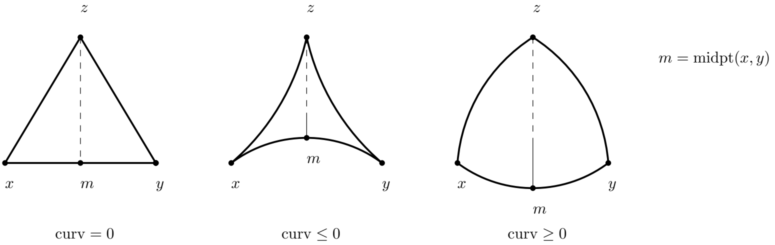

One would expect that

$$d_{\mathrm{curv}\leq 0}(z, m) \leq d_{\mathrm{curv}=0}(z, m) \leq d_{\mathrm{curv}\geq 0}(z, m).$$

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Theorem

A complete, simply connected Riemannian manifold $(\mathcal{M}, g)$ *has nonpositive sectional curvature if and only if for every* $x, y, z \in \mathcal{M}$ *and every metric midpoint* m *of* x, y*, we have*

$$d_{\mathcal{M}}(z, m)^2 \leq \frac{1}{2} d_{\mathcal{M}}(z, x)^2 + \frac{1}{2} d_{\mathcal{M}}(z, y)^2 - \frac{1}{4} d_{\mathcal{M}}(x, y)^2.$$

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Similarly, $(\mathcal{M}, g)$ has nonnegative sectional curvature if and only if the reverse inequality holds true for every such quadruple $\{x, y, z, m\}$ in $\mathcal{M}$.

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A complete geodesic metric space (X, d_X) is said to be nonpositively curved (or simply a *Hadamard space*) if for every $x, y, z \in X$ and every metric midpoint m of x, y, we have

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Similarly, (X, d_X) is said to be nonnegatively curved if the reverse inequality holds true for every such quadruple $\{x, y, z, m\}$ in X.

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- Euclidean cones over metric spaces of curvature bounded above by 1 (Berestovskiĭ, 1983)

Questions of interest

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The isometric structure of Hadamard spaces is (in a way) well understood: there are many satisfactory criteria to characterise those *geodesic* metric spaces which are isometric to a Hadamard space.

However, it remains a challenging open problem (Gromov, 1999) to obtain an intrinsic characterisation of those metric spaces which admit a bi-Lipschitz (or even isometric) embedding into a Hadamard space.

Expanders with respect to Hadamard spaces

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Mendel and Naor (2015). There exists a Hadamard space (X, d_X) which admits a sequence of expanders, yet almost every 3-regular graph is *not* a good expander with respect to X.

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- For arbitrarily large n, there exists an n-point metric space $\mathcal{M}_n$ such that any embedding of $\mathcal{M}_n$ into a Hadamard space incurs distortion at least a constant multiple of $\log n$, thus matching Bourgain's bound.

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We will refer to the latter problem as Bourgain's embedding problem in nonpositive curvature.

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These results have been extended to treat target spaces which behave like Hilbert spaces, including nonpositively curved Riemannian manifolds, spaces of bounded singularities and uniformly convex Banach spaces.

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In fact, it follows from work of Zolotov (2017) that every metric space admits a coarse embedding in a nonnegatively curved manifold and the sharp moduli are known to be $\omega(t), \Omega(t) \asymp \sqrt{t}$.

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This result along with the Andoni–Naor–Neiman theorem highlight a qualitative distinction between nonpositive and nonnegative curvature. The large scale structure of nonpositively curved spaces turns out to be better behaved that the one of nonnegatively curved ones, contrary to the intuition from classical Riemannian geometry.

Metric invariants and the Ribe program

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In particular, intuition stemming from classical results in Banach space theory is often fundamental in designing metric invariants which are used to prove nonembeddability of metric spaces into others.

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Definition (K. Ball)

A metric space (X, d_X) has Markov type 2 with constant $M \in [1, \infty)$ if for every $n \in \mathbb{N}$, every stationary reversible Markov chain $\{Z_t\}_{t=0}^{\infty}$ with values in $\{1, \ldots, n\}$ and every function $f : \{1, \ldots, n\} \to (X, d_X)$, we have

$$(*) \qquad \forall\, t \geq 1, \quad \mathbb{E} d_X\big(f(Z_t), f(Z_0)\big)^2 \leq M^2 t \mathbb{E} d_X\big(f(Z_1), f(Z_0)\big)^2.$$

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Clearly having Markov type 2 is a bi-Lipschitz invariant and it can be shown (K. Ball, 1993) that Hilbert spaces have Markov type 2 with constant 1. Therefore, metric spaces which do not satisfy $(*)$ do not bi-Lipschitzly embed into Hilbert space. For instance, this can be inferred for any metric space which contains the family $(\{0,1\}^n, \|\cdot\|_1)_{n=1}^{\infty}$.

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Remark. We have shown (E.–Mendel–Naor, 2021+) that nonnegatively curved spaces also have metric cotype 2.

Sharp metric cotype implies coarse nonuniversality

We will show that the grids $[m]_\infty^n := (\{1, \ldots, m\}^n, \|\cdot\|_\infty)$ do not equi-coarsely embed into any Hadamard space. In particular, ℓ_∞ (or c_0) does not coarsely embed into any Hadamard space. By abuse of notation, we will identify $\{1, \ldots, 2m\}$ with $\mathbb{Z}_{2m}$.

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Let $m,n \in \mathbb{N}$ with $m = \sqrt{n}$ and consider a mapping $f : \{1,\ldots,2m\}^n \to X$ such that

$$\forall\, x,y \in [m]_\infty^n, \quad \omega\big(\|x-y\|_\infty\big) \leq d_X\big(f(x),f(y)\big) \leq \Omega\big(\|x-y\|_\infty\big),$$

where $\lim_{t\to\infty}\omega(t) = \infty$.

Sharp metric cotype implies coarse nonembeddability

Recall that

$$\sum_{i=1}^{n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X\big(f(x+me_i), f(x)\big)^2 \lesssim \frac{m^2}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X\big(f(x+\varepsilon), f(x)\big)^2.$$

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$$\mathrm{RHS} \leq \frac{m^2(2m)^n}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \Omega(\|\varepsilon\|_\infty)^2 = n(2m)^n \Omega(1)^2$$

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Therefore

$$\omega\big(\sqrt{n}\big) \lesssim \Omega(1) < \infty$$

which is a contradiction.

Barycenter maps

Barycenter maps

We will need some terminology. For a set X, denote by $\mathscr{P}_X^{<\infty}$ the set of all finitely supported probability measures on X. A map $\mathscr{B} : \mathscr{P}_X^{<\infty} \to X$ is called a *barycenter map* if $\mathscr{B}(\delta_x) = x$ for every $x \in X$.

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Example: If X is a vector space, then we set

$$\mathscr{B}(\mu) = \int_X x \, \mathrm{d}\mu(x) = \sum_{i=1}^{N} \mu(\{x_i\}) x_i,$$

where $\{x_1, \ldots, x_N\}$ is the support of μ.

Conditional barycenters

Conditional barycenters

Let X be a set equipped with a barycenter map $\mathscr{B} : \mathscr{P}_X^{<\infty} \to X$, Ω be a finite set and $\mu : 2^\Omega \to [0,1]$ be a probability measure with full support, i.e. $\mu(\{\omega\}) > 0$ for every $\omega \in \Omega$. Fix a random variable $Z : \Omega \to X$.

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For a σ-algebra $\mathscr{F} \subseteq 2^\Omega$, the *$\mu$-conditional barycenter* of Z is the function $\mathscr{B}_\mu(Z|\mathscr{F}) : \Omega \to X$ given by

$$\mathscr{B}_\mu(Z|\mathscr{F})(\omega) = \mathscr{B}\Big(\frac{1}{\mu\big(\mathscr{F}(\omega)\big)} \sum_{a \in \mathscr{F}(\omega)} \mu(a)\delta_{Z(a)}\Big),$$

where $\mathscr{F}(\omega)$ is the cluster of the partition inducing $\mathscr{F}$ which contains ω.

Barycentric martingales

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Let X be a set equipped with a barycenter map, Ω be a finite set, $\mu : 2^{\Omega} \to [0,1]$ be a probability measure with full support and $\{\mathscr{F}_i\}_{i=0}^{n}$ a filtration on Ω. A sequence of functions $\{Z_i : \Omega \to X\}_{i=0}^{n}$ is called a μ-martingale with respect to the filtration $\{\mathscr{F}_i\}_{i=0}^{n}$

$$\forall\ i \in \{1, \ldots, n\}, \quad \mathscr{B}_\mu\big(Z_i | \mathscr{F}_{i-1}\big) = Z_{i-1}.$$

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Warning! The tower property

$$\forall\ 0 \le i < j \le n, \quad \mathscr{B}_{\mu}\big(Z_j | \mathscr{F}_i\big) = Z_i$$

fails in general.

Barycenters in Hadamard spaces

We will need the following classical fact.

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Let (X, d_X) be a Hadamard space. Then, the function $\mathscr{B} : \mathscr{P}_X^{<\infty} \to X$ given by

$$\mathscr{B}(\mu) = \text{the point } z \in X \text{ which minimises } \int_X d_X(a,z)^2 \, \mathrm{d}\mu(a)$$

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is a well defined barycenter map which additionally satisfies the inequality

$$(*) \quad d_X\big(z, \mathscr{B}(\mu)\big)^2 + \int_X d_X\big(a, \mathscr{B}(\mu)\big)^2 \,\mathrm{d}\mu(a) \le \int_X d_X(a,z)^2 \,\mathrm{d}\mu(a)$$

for every $z \in X$ and $\mu \in \mathscr{P}_X^{<\infty}$.

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Remark: For $\mu = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$, inequality $(*)$ is the definition of nonpositively curved metric spaces.

Orthogonality of martingale difference sequences revisited

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• One of the fundamental properties of real valued (or Hilbert space valued) martingales $\{Z_i\}_{i=0}^n$ is the identity

$$(1) \qquad \mathbb{E}\|Z_n\|_{\mathcal{H}}^2 = \mathbb{E}\|Z_0\|_{\mathcal{H}}^2 + \sum_{i=1}^{n} \mathbb{E}\|Z_i - Z_{i-1}\|_{\mathcal{H}}^2.$$

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• Applying the identity for a single random variable $Z_1 = Z$ and $Z_0 = \mathbb{E}Z$, (1) becomes

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• Applying the identity for a single random variable $Z_1 = Z$ and $Z_0 = \mathbb{E}Z$, (1) becomes

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• Finally, if $\mathbb{P}(Z = x) = \mathbb{P}(Z = y) = \frac{1}{2}$, then (2) is simply

$$(3) \qquad \frac{\|x\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}^2}{2} = \left\|\frac{x+y}{2}\right\|_{\mathcal{H}}^2 + \left\|\frac{x-y}{2}\right\|_{\mathcal{H}}^2,$$

the parallelogram identity!

Suborthogonality of Hadamard margingale differences

In Hadamard spaces (X, d_X), we have

$$(3') \qquad d_X\big(z, \mathscr{B}(x,y)\big)^2 \le \frac{1}{2}d_X(z,x)^2 + \frac{1}{2}d_X(z,y)^2 - \frac{1}{4}d_X(x,y)^2,$$

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Tensorising these inequalities gives the following.

Proposition (Mendel–Naor, 2013)

Every Hadamard space valued martingale $\{Z_i\}_{i=0}^n$ satisfies the inequality

$$(1') \qquad \mathbb{E}d_X(Z_0,z)^2 + \sum_{i=1}^n \mathbb{E}d_X(Z_i, Z_{i-1})^2 \leq \mathbb{E}d_X(Z_n,z)^2,$$

for every $z \in X$.

Proof of the main theorem

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Recall that we have to prove that for every Hadamard space (X, d_X), and $m \geq \sqrt{n}$ every function $f : \mathbb{Z}_{4m}^n \to X$ satisfies

$$\sum_{i=1}^{n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x+2me_i), f(x)\big)^2 \lesssim \frac{m^2}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x+\varepsilon), f(x)\big)^2.$$

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Idea: For fixed $x \in \mathbb{Z}_{4m}^n$, let $f_x : \{-1,1\}^n \to X$ be given by $f_x(\varepsilon) = f(x+\varepsilon)$. Let $\mathscr{F}_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i)$ be the usual filtration on $\{-1,1\}^n$ and for $h : \{-1,1\}^n \to X$, let $\mathsf{E}_n h = h$ and

$$\forall\ i \in \{0, 1, \ldots, n-1\}, \quad E_i h = \mathscr{B}_{\mathrm{unif}}\big(\mathsf{E}_{i+1} h | \mathscr{F}_i\big).$$

We will apply the martingale inequality for the $(4m)^n$ martingales $\big\{\{\mathsf{E}_i f_x\}_{i=0}^n\big\}_{x \in \mathbb{Z}_{4m}^n}$ and use the triangle inequality.

Quotients of the cube à la Khot and Naor

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Khot and Naor (2006) have shown that every function $f : \mathbb{F}_2^n \to \ell_2$ which is invariant under an $\mathbb{F}_2$-subspace V of $\mathbb{F}_2^n$ satisfies

$$(*)\quad \frac{1}{2^{2n}} \sum_{x,y\in\mathbb{F}_2^n} \big\|f(x)-f(y)\big\|_{\ell_2}^2 \lesssim \frac{\ell(V)^{-1}}{2^n} \sum_{i=1}^n \sum_{x\in\mathbb{F}_2^n} \big\|f(x+e_i)-f(x)\big\|_{\ell_2}^2,$$

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It is a classical fact in coding theory that for every $n \geq 1$ there exists such a code V_n with $\frac{n}{4} < \dim V_n < \frac{3n}{4}$ and $\ell(V_n) \asymp n$.

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Quotients of the cube à la Khot and Naor (continued)

The proof of $(*)$ relies on Parseval's identity along with the spectral observation that if a function is V-invariant, then all the Fourier coefficients in its Walsh expansion of order less than $\ell(V)$ vanish.

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Question

Can one prove a Khot–Naor-type inequality for Hadamard space valued functions on $\mathbb{F}_2^n$?

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Theorem (E.–Ivanisvili, 2020)

Let γ be the Gaussian measure on $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\mathbb{E}_\gamma[f(X)X^k] = 0$ for all $k \in \{0, 1, \ldots, d\}$, then for any $1 < p < \infty$

$$\mathbb{E}_\gamma\big|f(X)\big|^p \leq \frac{c_p}{\sqrt{d}}\mathbb{E}_\gamma\big|f'(X)\big|^p.$$

Thank you!