Renormalisation of enhanced quartic tensor field theories

Reiko Toriumi

Okinawa Institute of Science and Technology (OIST) Graduate University

joint work with Joseph Ben Geloun
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From perturbative to non-perturbative QFT,
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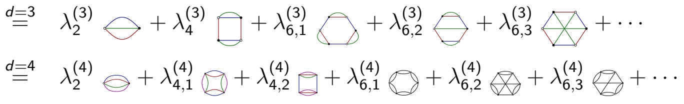
Tensor field theories

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\; e^{-(S^{\text{kinetic}} + S^{\text{interaction}})}\,,$$

where ϕ, $\bar{\phi}$ are order-d tensor fields $\phi : G^d \to \mathbb{C}$, and

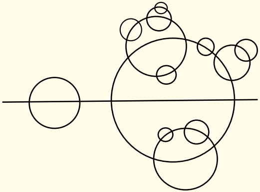
$$\begin{aligned} S^{\text{kinetic}}[\phi,\bar{\phi}] &= \mu \mathrm{Tr}_2(\phi^2) + \mathrm{Tr}_2(\bar{\phi}\cdot K \cdot \phi) \\ S^{\text{interaction}}[\phi,\bar{\phi}] &= \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \mathrm{Tr}_{2n_{\mathcal{B}}}(\bar{\phi}^{n_{\mathcal{B}}} \cdot \mathcal{V}_{\mathcal{B}} \cdot \phi^{n_{\mathcal{B}}}) \end{aligned}$$



- After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs.
- $(d+1)$-edge-colored graphs are dual to simplicial triangulations of piecewise linear (PL) d-dimensional pseudo-manifolds [Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986].
- Relevant for quantum gravity in dimensions $d \geq 3$.

tensor models

Melons dominate and they are branched polymers.



[V. Bonzom, R. Gurau, A. Riello, V. Rivasseau "Critical behavior of colored tensor models in the large N limit," Nucl. Phys. B 853, 174 (2011)]

[R. Gurau, J Ryan "Melons are branched polymers," Annales Henri Poincare 15, no. 11, 2085 (2014).]

Enhanced tensor models

[V.Bonzom, T. Delepouve, V. Rivasseau "Enhancing non-melonic triangulations: A tensor model mixing melonic and planar maps," Nucl. Phys. B 895, 161 (2015)]

Introduced a non-melonic interaction (necklace) properly scaled in N along with a melonic interaction, and recovered the string suceptibility exponent of pure 2D gravity $\gamma = -1/2$, $\gamma = 1/2$ (trees/branched polymers), and $\gamma = 1/3$ (a proliferation of baby universes).

Tensor field theory models

- Consider a field theory defined by a complex field $\phi : (U(1)^D)^{\times d} \to \mathbb{C}$.
- The Fourier transform of ϕ yields an order d complex tensor $\phi_{\mathbf{P}}$, with $\mathbf{P} = (p_1, p_2, \ldots, p_d)$ a multi-index, where $p_1, p_2, \ldots, p_d$ are also multi-indices $p_s = (p_{s,1}, p_{s,2}, \ldots, p_{s,D})\,, \;\; p_{s,i} \in \mathbb{Z}\,.$
- $\bar{\phi}_{\mathbf{P}}$ denotes its complex conjugate.

The action

$$S[\bar{\phi}, \phi] = S^{\text{kinetic}}[\bar{\phi}, \phi] + S^{\text{int}}[\bar{\phi}, \phi]\,,$$

is given by convolutions of tensors

$$S^{\text{kinetic}}[\bar{\phi}, \phi] = \mathrm{Tr}_2(\bar{\phi} \cdot \mathbf{K} \cdot \phi) + \mu\, \mathrm{Tr}_2(\phi^2)$$

with

$$\mathrm{Tr}_2(\phi^2) = \sum_{\mathbf{P}} \bar{\phi}_{\mathbf{P}} \phi_{\mathbf{P}}\,,$$

$$\mathrm{Tr}_2(\bar{\phi} \cdot \mathbf{K} \cdot \phi) = \sum_{\mathbf{P}, \mathbf{P}'} \bar{\phi}_{\mathbf{P}}\, \mathbf{K}(\mathbf{P}; \mathbf{P}')\, \phi_{\mathbf{P}'}\,,$$

where the kinetic term kernel can be simply given by

$$\mathbf{K}(\mathbf{P};\mathbf{P}') = \boldsymbol{\delta}_{\mathbf{P};\mathbf{P}'}\mathbf{P}^{2b}\,,$$

with $\boldsymbol{\delta}_{\mathbf{P};\mathbf{P}'} = \prod_{s=1}^{d}\prod_{i=1}^{D}\delta_{p_{s,i},p'_{s,i}}$, $\mathbf{P}^{2\xi} = \sum_{s=1}^{d}\sum_{i=1}^{D}|p_{s,i}|^{2b}$.

Then, denote $\mathrm{Tr}_2(\bar{\phi}\cdot\mathbf{K}\cdot\phi) = \mathrm{Tr}_2(p^{2b}\phi^2)\,.$

Remark

In ordinary QFT, the restriction $b \leq 1$ ensures the Osterwalder-Schrader positivity axiom, however, here a priori we have no such restriction but we still restrict b to be a positive real number.

Our enhanced quartic models

$(D, d, a, b) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+$. order-$d$ tensor field $\phi : (U(1)^D)^{\times d} \to \mathbb{C}$

- model $+$

$$S_+^{\text{int}}[\bar\phi, \phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_+}{2}\,\mathrm{Tr}_4(p^{2a}\,\phi^4) + Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$$

$$S_+^{\text{kin}}[\bar\phi, \phi] = Z_b \mathrm{Tr}_2(p^{2b}\phi^2) + \mu \mathrm{Tr}_2(\phi^2)\,,$$

- model $\times$

$$S_\times^{\text{int}}[\bar\phi, \phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_\times}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) + \sum_{\xi=a,2a} Z_\xi \mathrm{Tr}_2(p^{2\xi}\phi^2)$$

$$S_\times^{\text{kin}}[\bar\phi, \phi] = Z_b \mathrm{Tr}_2(p^{2b}\phi^2) + \mu \mathrm{Tr}_2(\phi^2)$$

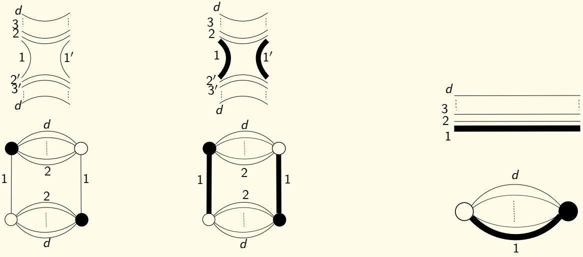
where

$$\mathrm{Tr}_4(\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \phi_{12\ldots d}\, \bar\phi_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar\phi_{12'3'\ldots d'} + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,,$$

$$\mathrm{Tr}_4(p^{2a}\,\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} |p_1|^{2a} \phi_{12\ldots d}\, \bar\phi_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar\phi_{12'3'\ldots d'} + \mathrm{Sym}(1 \to 2 \to \cdots \to d),$$

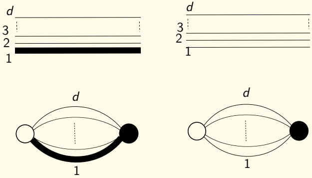
$$\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \left(|p_1|^{2a}|p'_1|^{2a}\right) \phi_{12\ldots d}\, \bar\phi_{1'23\ldots d}\, \phi_{1'2'3'\ldots d'}\, \bar\phi_{12'3'\ldots d'} + \mathrm{Sym}\,(1 \to 2 \to \cdots \to d)\,.$$

Enhanced model $\times$

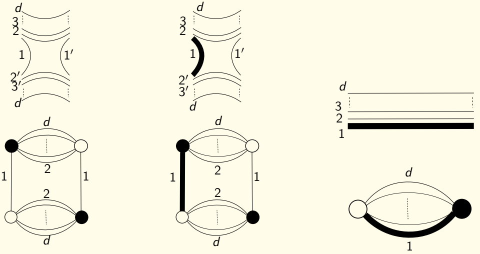


$$S_{\times}^{\mathrm{int}}[\bar{\phi},\phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_{\times}}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) + \sum_{\xi=a,2a} Z_{\xi}\,\mathrm{Tr}_2(p^{2\xi}\phi^2)$$

$$S_{\times}^{\mathrm{kinetic}}[\bar{\phi},\phi] = Z_b\mathrm{Tr}_2(p^{2b}\phi^2) + \mu\mathrm{Tr}_2(\phi^2)\,,$$

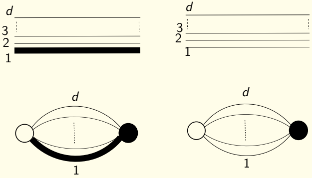


Enhanced model +



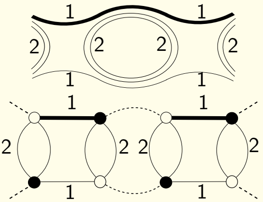
$$S_+^{\rm int}[\bar\phi,\phi] = \frac{\lambda}{2}\,{\rm Tr}_4(\phi^4) + \frac{\lambda_+}{2}\,{\rm Tr}_4(p^{2a}\,\phi^4) + Z_a {\rm Tr}_2(p^2\phi^2)\,,$$

$$S_+^{\rm kinetic}[\bar\phi,\phi] = Z_b {\rm Tr}_2(p^{2b}\phi^2) + \mu {\rm Tr}_2(\phi^2)\,,$$



for illustration $d = 3$,

A melonic graph



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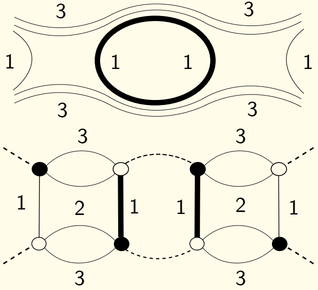
A non-melonic Feynman graph



Power counting theorems

The amplitude of a Feynman graph $\mathcal{G}(\mathcal{V},\mathcal{L})$ with a set of vertices $\mathcal{V}$ and a set of propagator lines $\mathcal{L}$, in perturbation theory:

$$A_{\mathcal{G}}(\{p_{\rm ext}\}) = \sum_{\mathbf{P}_v} \prod_{l\in\mathcal{L}} C_{\bullet,l}(\mathbf{P}_v,\mathbf{P}'_{v'}) \prod_{v\in\mathcal{V}} (-\mathbf{V}_v(\mathbf{P}_v))$$

where $C_{\bullet,l}$ is a propagator with line index l, $\mathbf{V}_v(\mathbf{P}_v)$ is a given vertex weight that contains a coupling constant but also a momentum weight if the vertex v is enhanced. Superficial degrees of divergence are given by,

- model $+$

$$\begin{aligned}\omega_{\rm d;+}(\mathcal{G}) = &-\tfrac{2D}{(d-1)!}(\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G}))-D(C_{\partial\mathcal{G}}-1)\\ &-\tfrac{1}{2}\left[(D\,(d-1)-2b)N_{\rm ext}-2D\,(d-1)\right]\\ &+\tfrac{1}{2}\left[-2D\,(d-1)+(D\,(d-1)-2b)n\right]\cdot V\\ &+2a\rho_{+}+2a\rho_{2;a}+2b\rho_{2;b}\,.\end{aligned}$$

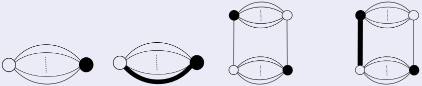
- model $\times$

$$\begin{aligned}\omega_{\rm d;\times}(\mathcal{G}) \;=\; &-\tfrac{2D}{(d-1)!}(\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G}))-D(C_{\partial\mathcal{G}}-1)\\ &-\tfrac{1}{2}\left[(D\,(d-1)-2b)N_{\rm ext}-2D\,(d-1)\right]\\ &+\tfrac{1}{2}\left[-2D\,(d-1)+(D\,(d-1)-2b)n\right]\cdot V+2a\rho_{\times}+\textstyle\sum_{\xi=a,2a,b}2\xi\rho_{2;\xi}\,.\end{aligned}$$

Power counting theorem for model +

Proposition (List of primitively divergent graphs for the model +)

The $p^{2a}\phi^4$-model + with parameters $a = \frac{1}{2}D(d-2), b = \frac{1}{2}D(d-\frac{3}{2})$ for two integers $d > 2$ and $D > 0$, has primitively divergent graphs



$\mathrm{class}_{\mathcal{G}}$		N_{ext}	V_2	$V_{2;a}$	V_4	ρ_+	$\omega_{\mathrm{d};+}(\mathcal{G})$
	(4-pt λ)	*4*	*0*	*0*	*0*	$V_{+;4}$	*0*
I	*(mass)*	*2*	*0*	*0*	*0*	$V_{+;4}$	$D/2$
II	*(2-pt Z_a)*	*2*	*0*	*0*	*0*	$V_{+;4}-1$	$D/2$
III	*(mass)*	*2*	*0*	*0*	*1*	$V_{+;4}$	$D/2$
IV	*(mass)*	*2*	*0*	*1*	*0*	$V_{+;4}$	0
V	*(2-pt Z_a)*	*2*	*0*	*1*	*0*	$V_{+;4}-1$	0
VI	*(mass)*	*2*	*0*	*1*	*1*	$V_{+;4}$	0

List of primitively divergent graphs of the $p^{2a}\phi^4$-model +.

Power counting theorem for model +

order-d tensor field $\phi : (U(1)^D)^{\times d} \to \mathbb{C}$

Theorem

The $p^{2a}\phi^4$ model + with parameters $a = \frac{1}{2}D(d-2), b = \frac{1}{2}D(d-\frac{3}{2})$ for arbitrary order $d \geq 3$ and dimension $D > 0$ is just-renormalisable at all orders of perturbation theory.

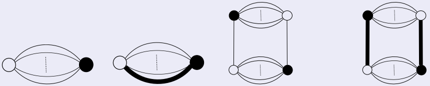
	$d=3$	$d=4$
$D=1$	$a=\frac{1}{2}$, $b=\frac{3}{4}$	$a=1$, $b=\frac{5}{4}$
$D=2$	$a=1$, $b=\frac{3}{2}$	$a=2$, $b=\frac{5}{2}$
$D=3$	$a=\frac{3}{2}$, $b=\frac{9}{4}$	$a=3$, $b=\frac{15}{4}$
$D=4$	$a=2$, $b=3$	$a=4$, $b=5$

Values of a and b for potentially just-renormalisable theories ($\omega_{\mathrm{d};+}(\mathcal{G})|_{N_{\mathrm{ext}}\geq 6} < 0$ and $\omega_{\mathrm{d};+}(\mathcal{G})$ is independent of numbers of vertices) with $\omega_{\mathrm{d};+}(\mathcal{G}^{\mathrm{non-melon}})|_{N_{\mathrm{ext}}=4} = 0$ with $d \leq 4$ and $D \leq 4$.

Power counting theorem for model $\times$

Proposition (List of primitively divergent graphs for the model $\times$)

The $p^{2a}\phi^4$-model $\times$ with parameters $D=1, d=3, a=\frac{1}{2}, b=1$, has the following primitively divergent graphs which obey



$\mathrm{class}_{\mathcal{G}}$		N_{ext}	V_2	$V_{2;a}$	V_4	$\rho_\times$	$\omega_{\mathrm{d};\times}(\mathcal{G})$
I	*(2-pt Z_a)*	*2*	*0*	*0*	*0*	$2V_{\times;4}-1$	0
II	*(2-pt Z_{2a})*	*2*	*0*	*0*	*0*	$2V_{\times;4}-2$	0
III	*(mass)*	*2*	*0*	*0*	*1*	$2V_{\times;4}$	0

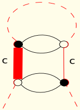
List of primitively divergent graphs of the $p^{2a}\phi^4$-model $\times$.

Theorem

The $p^{2a}\phi^4$ model $\times$ with parameters $D=1, d=3, a=\frac{1}{2}, b=1$ is renormalisable at all orders of perturbation.

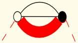
model $+$

- enhanced melonic move



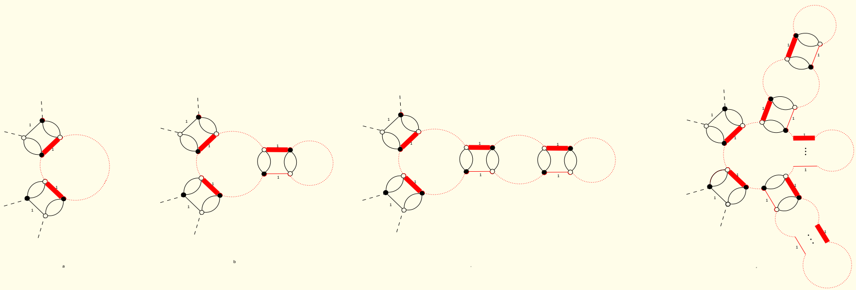
An enhanced melonic insertion has $\triangle\omega_{d;+} = 0$.

- enhanced dipole move

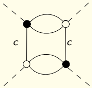


An enhanced d-dipole insertion has $\triangle\omega_{d;+} = -\frac{D}{2}$.

Divergent graphs for 4-pt coupling λ (model $+$)

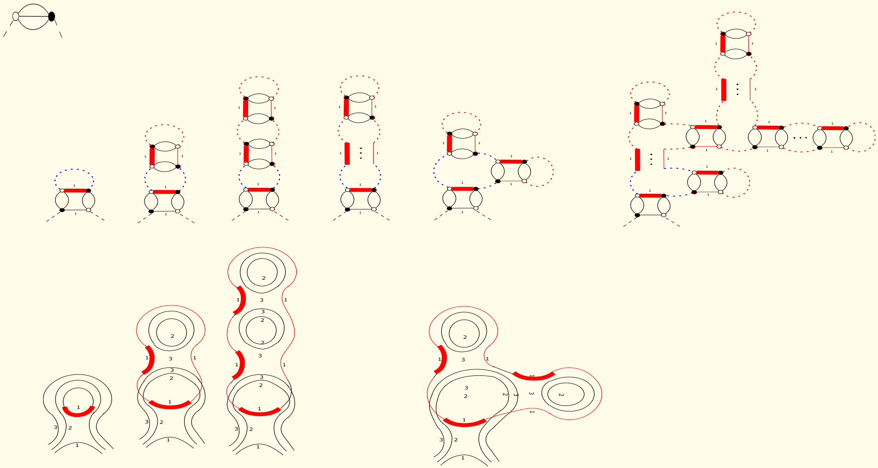


4-pt primitively divergent graphs with $\omega_{d;+} = 0$. Renormalise 4-pt coupling $\lambda \, \mathrm{Tr}_4(\phi^4)$.



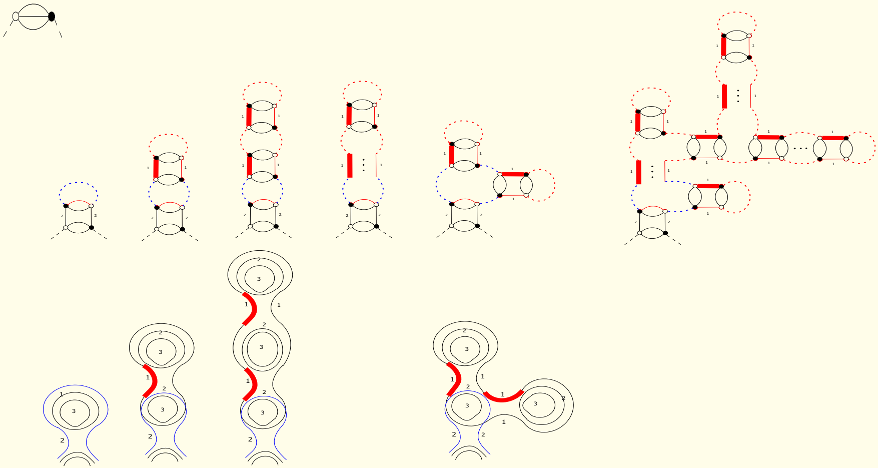
their boundary graph

Divergent graphs for mass (model +)



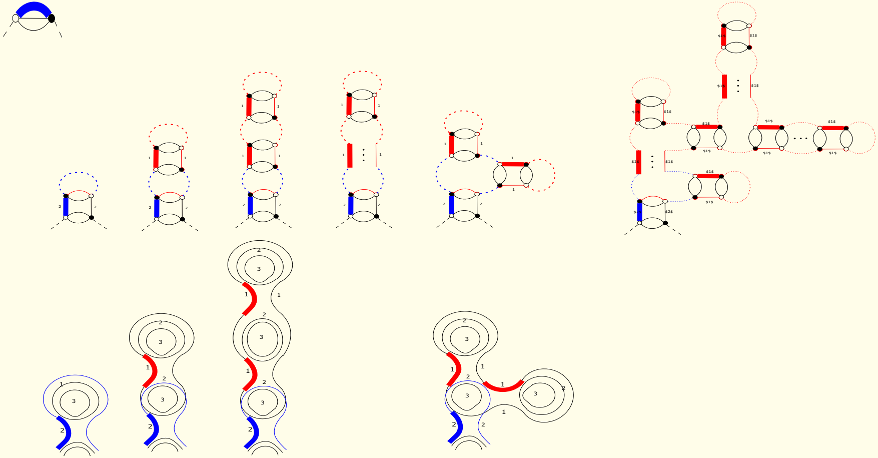
Renormalise mass $\mu \mathrm{Tr}_2(\phi^2)$. 2-pt primitively divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class I. We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either color 1 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then $\omega_{d;+} = 0$ and they belong to the class IV and renormalise mass.

Divergent graphs for mass (model +)



Renormalise mass $\mu \mathrm{Tr}_2(\phi^2)$. 2-pt primitively divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class III. We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either colors 1 or 3 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then, $\omega_{d;+} = 0$ and they belong to the class VI and renormalise mass.

Divergent graphs for 2-pt coupling Z_a (model $+$)



Renormalise $Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$. 2-pt primitively divergent graphs with $\omega_{d;+} = \frac{D}{2}$. Class II. We can insert *one* d-dipole anywhere on a propagator; one d-dipole with either colors 1 or 3 enhanced on a blue dotted propagator, or one d-dipole with any color 1, 2, or 3 enhanced on a red dotted propagator. Then, $\omega_{d;+} = 0$ and they belong to the class V and renormalise $Z_a \mathrm{Tr}_2(p^{2a}\phi^2)$.

Effective Action via multiscale analysis

We slice our covariance in a discrete sum of contributions, each corresponsing to an energy sector (scale),

$$C(\mathbf{P};\mathbf{P}') = \tilde{C}(\mathbf{P})\,\boldsymbol{\delta}_{\mathbf{P},\mathbf{P}'}\,, \qquad \tilde{C}(\mathbf{P}) = \frac{1}{\mathbf{P}^{2b}+\mu} = \sum_{i=0}^{\infty} \tilde{C}_i(\mathbf{P})\,,$$

with $M > 1$ positive real number, in Schwinger parametrisation,

$$\tilde{C}_i(\mathbf{P}) = \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\, e^{-\alpha(\mathbf{P}^{2b}+\mu)}\,, \quad \tilde{C}_0(\mathbf{P}) = \int_1^{\infty} d\alpha\, e^{-\alpha(\mathbf{P}^{2b}+\mu)}\,.$$

(UV: big i, small α)

→ Integrate out the fields at high scales (UV) $> i$ and include their effects in the effective action W^i.

$$Z = \int d\nu_{C_{\leq i}}(\bar{\phi}_{\leq i}, \phi_{\leq i})\, e^{-W^i(\bar{\phi}_{\leq i},\, \phi_{\leq i})}\,, \quad \text{where} \quad C_{\leq i}(\mathbf{P};\mathbf{P}') = \boldsymbol{\delta}_{\mathbf{P},\mathbf{P}'} \sum_{j\leq i} \tilde{C}_j(\mathbf{P})\,.$$

→ Integrate out another layer down to scale $i-1$. Decompose $C_{\leq i} = C_i + C_{\leq i-1}$ and the corresponding fields $\phi_{\leq i} = \psi_i + \phi_{\leq i-1}$ ($\bar{\phi}_{\leq i} = \bar{\psi}_i + \bar{\phi}_{\leq i-1}$).

$$Z = \int d\nu_{C_{\leq i-1}}(\bar{\phi}_{\leq i-1}, \phi_{\leq i-1}) e^{-W^{i-1}(\bar{\phi}_{\leq i-1},\, \phi_{\leq i-1})}\,,$$

Effective Action

where the effective action at scale $i-1$ is given by

$$-W^{i-1}(\bar{\phi}_{\leq i-1},\, \phi_{\leq i-1}) = \log \int d\nu_{C_i}(\bar{\psi}_i, \psi_i) e^{-W^i(\bar{\psi}_i + \bar{\phi}_{\leq i-1},\, \psi_i + \phi_{\leq i-1})} .$$

If the theory is renormalisable, one can assert the effective action at any scale i takes the same form as the interaction action, therefore

$$-W^{i-1}(\bar{\phi}_{\leq i-1}, \phi_{\leq i-1}) = \mathrm{Tr}_2(\bar{\phi}_{\leq i-1} \cdot \Sigma \cdot \phi_{\leq i-1}) + \frac{1}{2}\mathrm{Tr}_4(\phi^4_{\leq i-1} \cdot \Gamma_4) + R(\phi_{\leq i-1}) \,,$$

- $\Sigma(\{p\})$ is the sum over all amputated 1PI 2-pt graphs,
- $\Gamma_4(\{p\})$ is the sum of 1PI 4-pt graphs following the pattern of $\mathrm{Tr}_4(\phi^4)$, and
- $R(\phi_{\leq i-1})$ is the rest of the terms containing 1PR graphs (they do not contribute to the iteration process) and the finite terms.

Effective 2-pt function (model $+$)

Expand the 2-pt function contribution,

$$\Sigma(\{p\}) = \Sigma(\{0\}) + \sum_c |p_c|^{2b} \partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0} + \sum_c |p_c|^{2a} \partial_{|p_c|^{2a}} \Sigma\big|_{\{p\}=0} + \cdots$$

- mass renormalisation $\Sigma(\{0\})$ is divergent with $\omega_{d;+} = D/2$ (classes I and III) and $\omega_{d;+} = 0$ (classes IV and VI).
- $\partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0} = 0$.
- $\partial_{|p_c|^{2a}} \Sigma|_{\{p\}=0} \equiv \Gamma_2^{(c)}(\{0\})$ is divergent.
  $|p_c|^{2a} \Gamma_2^{(c)}(\{p\})$ is the sum of all amputated 1PI 2pt-functions following the pattern of $\mathrm{Tr}_2(p_c^{2a}\phi^2)$ on their boundary graphs as dictated by the classes of II ($\omega_{d;+} = D/2$) and V ($\omega_{d;+} = 0$).
- $\cdots$ are finite.

$a = \frac{1}{2}D(d-2), b = \frac{1}{2}D(d-\frac{3}{2})$

Effective 4-pt function (model +)

Similarly, expand the 4-point function contribution,

$$\Gamma_4(\{p\}) = \sum_c \left\{ \Gamma_4^{(c)}(\{0\}) + |p_c|^{2a} \partial_{|p_c|^{2a}} \Gamma_4^{(c)} \Big|_{\{p\}=0} + |p_c|^{2b} \partial_{|p_c|^{2b}} \Gamma_4^{(c)} \Big|_{\{p\}=0} \right\} + \cdots ,$$

- $\sum_c \Gamma_4^{(c)}(\{0\}) \equiv \Gamma_4(\{0\})$ is the sum of all amputated 1PI 4pt-functions following the pattern of $\mathrm{Tr}_4(\phi^4)$ on their boundary graphs, and is divergent ($\omega_{d;+} = 0$).
- $\partial_{|p_c|^{2a}} \Gamma_4^{(c)} \Big|_{\{p\}=0} \equiv \Gamma_{4;+}^{(c)}(\{0\})$ are all amputated 1PI 4pt-functions following the pattern of $\mathrm{Tr}_{4;c}(p^{2a}\phi^4)$ having a boundary with external $|p|^{2a}$-enhancement. In fact, there is only the leading order $\mathcal{O}(\lambda_+)$ contribution in $\Gamma_{4;+}^{(c)}(\{0\})$ and there are no contributions from higher orders in perturbation theory in λ_+.
- $\partial_{|p_c|^{2b}} \Gamma_4^{(c)} \Big|_{\{p\}=0}$ is finite.
- $\cdots$ are finite.

Effective Gaussian measure (model +)

The effective Gaussian measure is given by

$$d\nu_{\tilde{C}^{i-1}(\phi_{\leq i-1})} \exp\left[\Sigma_{i-1}(\{0\})\mathrm{Tr}_2(\phi^2_{\leq i-1}) + \sum_c (\partial_{|p_c|^{2b}}\Sigma|_{\{p\}=0})_{i-1}\mathrm{Tr}_2(p_c^{2b}\phi^2_{\leq i-1})\right],$$

with actually $\partial_{|p_c|^{2b}}\Sigma\big|_{\{p\}=0} = 0$. The new covariance for the above Gaussian measure,

$$\frac{1}{Z_{b,\,i-1}}\int_{M^{-2b(i-1)}}^{M^{-2b(i-2)}} d\alpha\, e^{-\alpha(|p|^{2b}+\mu_{\mathrm{ren},i-1})} = \frac{1}{Z_{b,\,i-1}}\tilde{C}^{i-1}(p)\,,$$

- the wave function renormalisation $Z_{b,\,i-1} \equiv 1 + (\partial_{|p_c|^{2b}}\Sigma\big|_{\{p\}=0})_{i-1}$.
- the renormalised mass $\mu_{\mathrm{ren},i-1} = \frac{1}{Z_{b,\,i-1}}(\mu_{i-1} - \Sigma_{i-1}(\{0\}))$.

Then, the effective theory for $\phi_{\leq i-1}$ can be written as

$$d\nu_{\frac{1}{Z_{b,\,i-1}}\tilde{C}^{i-1}}(\phi_{\leq i-1})$$
$$\exp\Big[\sum_c \Gamma^{(c)}_{2,\,i-1}(\{0\})\mathrm{Tr}_2(p_c^{2a}\phi^2_{\leq i-1}) + \sum_c \frac{\Gamma^{(c)}_{4,\,i-1}(\{0\})}{2}\mathrm{Tr}_4(\phi^4_{\leq i-1})$$
$$+ \sum_c \frac{\Gamma^{(c)}_{4;+,\,i-1}(\{0\})}{2}\mathrm{Tr}_4(p_c^{2a}\phi^4_{\leq i-1}) + \tilde{R}(\phi_{\leq i-1})\Big]\,.$$

Effective action (model +)

With a field rescaling $\phi_{\leq i-1} \to \sqrt{Z_{b,\, i-1}}\phi_{\leq i-1}$ (which in our specific case, there is no actual rescaling because $Z_{b,\, i-1} = 1$ and trivial), the effective theory for $\phi_{\leq i-1}$ can be recast:

$$
\begin{aligned}
& d\nu_{\tilde{C}_{i-\mathbf{1}}}(\phi_{\leq i-1}) \\
& \exp\Big[\sum_c \frac{\Gamma^{(c)}_{2,i-1}(\{0\})}{Z_{b,i-1}} \mathrm{Tr}_{2;c}(p^{2a}\phi^2_{\leq i-1}) + \sum_c \frac{\Gamma^{(c)}_{4,i-1}(\{0\})}{2Z^2_{b,i-1}} \mathrm{Tr}_{4;c}(\phi^4_{\leq i-1}) \\
& + \sum_c \frac{\Gamma^{(c)}_{4;+,i-1}(\{0\})}{2Z^2_{b,i-1}} \mathrm{Tr}_{4;c}(p^{2a}\phi^4_{\leq i-1}) + \tilde{R}(\sqrt{Z_{b,\, i-1}}\phi_{\leq i-1}) \Big] \,.
\end{aligned}
$$

Now we can identify the effective couplings at scale $i-1$,

$$
Z_{a,i-1} = -\frac{\Gamma^{(c)}_{2,i-1}(\{0\})}{Z_{b,i-1}} \,, \quad \lambda^{(c)}_{i-1} = -\frac{\Gamma^{(c)}_{4,i-1}(\{0\})}{Z^2_{b,i-1}} \,, \quad \lambda^{(c)}_{+;\, i-1} \;\; = \;\; -\frac{\Gamma^{(c)}_{4;+,i-1}(\{0\})}{Z^2_{b,i-1}} \,.
$$

Renormalisation of model $+$

Note that in our case, $(\partial_{|p_c|^{2b}}\Sigma\big|_{\{p\}=0})_{i-1} = 0$ therefore, throughout, we actually had

$$\begin{aligned}
Z_{b,\,i-1} &= 1\,, \\
\mu_{\mathrm{ren},i-1} &= \mu_{i-1} - \Sigma_{i-1}(\{0\})\,, \\
Z_{a,i-1} &= -\Gamma^{(c)}_{2,i-1}(\{0\})\,, \\
\lambda^{(c)}_{i-1} &= -\Gamma^{(c)}_{4,i-1}(\{0\})\,, \\
\lambda^{(c)}_{+;\,i-1} &= -\Gamma^{(c)}_{4;+,i-1}(\{0\})\,.
\end{aligned}$$

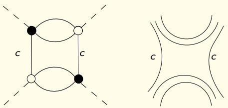
For the model $+$, the β-functions can be computed for generic parameters $a = (d-2)/2$, and $b = (d-3/2)/2$ and $d > 2$ but with fixed group dimension $D = 1$.

β-function of 4-pt coupling λ (model +)

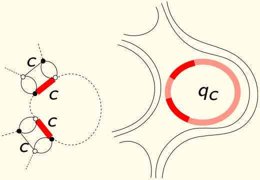
$$\Gamma_4^{(c)}(\{p\}) = \sum_{\mathcal{G}^{(c)}_{\mathbf{4},\iota}} K_{\mathcal{G}^{(c)}_{\mathbf{4},\iota}} \; S_{\mathcal{G}^{(c)}_{\mathbf{4},\iota}}(\{p\}) \,,$$

where $K_{\mathcal{G}^{(c)}_{\mathbf{4},\iota}}$ is a combinatorial factor and $S_{\mathcal{G}^{(c)}_{\mathbf{4},\iota}}(\{p\})$ is a formal amplitude sum.

The sum over $\mathcal{G}^{(c)}_{4,\iota}$ runs over a list of 4pt-graphs obeying the multiscale power counting analysis. Up to one-loop, we have the following two graphs:



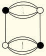
Zero-loop divergent graph at $d = 3$.

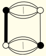


One-loop divergent graph, $n_4^{(c)}$ at $d = 3$ contributing to the flow of λ. $K_{n_{\mathbf{4}}^{(c)}} = 2$,

$$S_{n_{\mathbf{4}}^{(c)}}(\{\mathbf{p},\mathbf{p}'\}) = \frac{1}{2!}\left(\frac{-\lambda_+^{(c)}}{2}\right)^2 \sum_{q_c} \frac{|q_c|^{\mathbf{2}a}}{(|\mathbf{p}_{\check{c}}|^{\mathbf{2}b}+|q_c|^{\mathbf{2}b}+\mu)} \frac{|q_c|^{\mathbf{2}a}}{(|\mathbf{p}'_{\check{c}}|^{\mathbf{2}b}+|q_c|^{\mathbf{2}b}+\mu)}$$

β-functions of 4-pt couplings λ and λ_+ (model $+$)


λ


λ_+

The β-functions of the 4-pt couplings up to one-loop are

$$\lambda^{(c)}_{\rm ren} = \lambda^{(c)} - \frac{1}{4}(\lambda^{(c)}_+)^2 S_0 \,, \qquad S_0 = \sum_q \frac{|q|^{4a}}{(|q|^{2b}+\mu_i)^2} \,,$$

$$\lambda^{(c)}_{+;\rm ren} = \lambda^{(c)}_+ \,,$$

Set all couplings to $\lambda^{(c)} = \lambda$, and $\lambda^{(c)}_+ = \lambda_+$ to simplify,

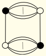
$$\lambda_{\rm ren} = \lambda - \frac{1}{4}(\lambda_+)^2 S_0 \,, \quad S_0 > 0$$

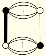
$$\lambda_{+,\rm ren} = \lambda_+ \,.$$

Observations

- λ_+ does not run ! and defines a fixed point at all orders of perturbation.
- λ and λ_+ never coincide and could not be set at equal value.
- $\lambda_{\rm ren} < \lambda$, i.e., λ increases in the UV. But it is not an ordinary Landau ghost.

β-functions of λ and λ_+ in multiscale analysis (model $+$)


λ


λ_+

In the multiscale analysis with discrete scale i, the system can be written as

$$\lambda_{i-1} = \lambda_i - \frac{1}{4}\lambda_{+,i}^2 \, S_{0,i} \, ,$$
$$\lambda_{+,i-1} = \lambda_{+,i} \, ,$$

where

$$\begin{aligned} S_{0,i} &= \sum_q |q|^{4a} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha e^{-\alpha(|q|^{2b}+\mu_i)} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha' e^{-\alpha'(|q|^{2b}+\mu_i)} \\ &= \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha' e^{-(\alpha+\alpha')\mu_i} \sum_q |q|^{4a} e^{-(\alpha+\alpha')|q|^{2b}} \, . \end{aligned}$$

Consider

$$\widetilde{S}_{0,i} = \sum_{q\in\mathbb{Z}} |q|^{4a} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha e^{-\alpha(|q|^{2b}+\mu_i)} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha' e^{-\alpha'(|q|^{2b}+\mu_i)}$$
$$= \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha' e^{-(\alpha+\alpha')\mu_i} \sum_{q\in\mathbb{Z}} |q|^{4a} e^{-(\alpha+\alpha')|q|^{2b}},$$

- Taylor expand $e^{-(\alpha+\alpha')\mu} = 1 + \mathcal{O}(\alpha+\alpha')$, (UV: big i, small α)
- Euler-Maclaurin formula

$$\textstyle\sum_{q\in\mathbb{Z}} |q|^{4a} e^{-(\alpha+\alpha')|q|^{2b}} = 2\sum_{q=1}^{\infty} q^{4a} e^{-(\alpha+\alpha')|q|^{2b}} = 2\int_1^{\infty} dq\, q^{4a} e^{-(\alpha+\alpha')|q|^{2b}} + R$$
$$= \frac{(\alpha+\alpha')^{\frac{-(4a+1)}{2b}}}{b}\Gamma\left(\frac{4a+1}{2b}, \alpha+\alpha'\right) + R = \frac{1}{b}\Gamma\left(\frac{4a+1}{2b}\right)(\alpha+\alpha')^{\frac{-(4a+1)}{2b}} + \mathcal{O}(1) + \mathcal{O}(\alpha+\alpha'),$$

to obtain

$$\widetilde{S}_{0,i} = \frac{1}{b}\log\frac{(M^{2b}+1)^2}{4M^{2b}} + \mathcal{O}(M^{-2bi}\log(M^{-2bi})),$$

where $\log\frac{(M^{2b}+1)^2}{4M^{2b}} > 0$ (Recall $M > 1$).

Revisiting the expression of the propagator in Schwinger parameterisation,

$$C(\mathbf{P};\mathbf{P}') = \tilde{C}(\mathbf{P})\,\delta_{\mathbf{P},\mathbf{P}'}\,, \quad \tilde{C}(\mathbf{P}) = \tfrac{1}{\mathbf{P}^{2b}+\mu} = \textstyle\sum_{i=0}^{\infty}\tilde{C}_i(\mathbf{P})\,, \quad \tilde{C}_i(\mathbf{P}) = \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\, e^{-\alpha(\mathbf{P}^{2b}+\mu)}\,,$$

we notice that α should have a dimension of $-2b$ in units of momentum scale k, i.e., $\alpha = k^{-2b}\,\tilde{\alpha}$, where $\tilde{\alpha}$ is dimensionless.
Perform the change of variables to let the dimensions be explicit in terms of a momentum scale k:

$$\begin{array}{ll} q = k\tilde{q}\,, & \tilde{q}\in\mathbb{Z} \\ \alpha = k^{-2b}\,\tilde{\alpha}\,. & \end{array}$$

We obtain in terms of dimensionless $\widetilde{S}_{0,i}$,

$$\begin{aligned} S_{0,i} &= k^{4a+1}k^{-4b}\widetilde{S}_{0,i} = \widetilde{S}_{0,i} \\ &= \frac{1}{b}\log\frac{(M^{2b}+1)^2}{4M^{2b}} + \mathcal{O}(M^{-2bi}\log(M^{-2bi}))\,, \end{aligned}$$

where $\log\frac{(M^{2b}+1)^2}{4M^{2b}} > 0$ (Recall $M > 1$).

β function of λ (model $+$)

In the multiscale formulation,

$$-(\lambda_{i-1} - \lambda_i) = \frac{\partial \lambda_i}{\partial i} \quad = \quad \frac{1}{4}\lambda_{+,i}^2 \, S_{0,i} \, .$$

We write the β-function for a given coupling g as $\beta_g(k) = k\partial_k g(k) = \partial_t g(t)$, where k is a momentum scale, and $t = \log(k/k_0)$. The momentum scale must be compared to the slice range as $k/k_0 \sim M^i$. Then, $t = \log(k/k_0) \sim i \log M$. $\lambda_+ = \lambda_{+,i}$ does not run.

$$\frac{\partial \lambda_i}{\partial((\log M)i)} = \partial_t \lambda(t) \quad = \quad \beta_\lambda \lambda_+^2$$

$$\beta_\lambda \quad = \quad \frac{1}{4b} \frac{\log \frac{(M^{2b}+1)^2}{4M^{2b}}}{\log(M)} > 0 \, .$$

Integrate both sides,

$$\lambda(t) = \beta_\lambda \, \lambda_+^2 \, (t - t_0) + \lambda(t_0) \, ,$$

where t_0 is some IR reference scale.

Running of λ and its discussion

$$\partial_t \lambda(t) = \beta_\lambda \lambda_+^2 \,, \qquad \beta_\lambda = \frac{1}{4b} \frac{\log \frac{(M^{2b}+1)^2}{4M^{2b}}}{\log(M)} > 0 \,.$$

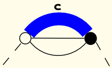
$$\lambda(t) = \beta_\lambda \, \lambda_+^2 \, (t - t_0) + \lambda(t_0) \,.$$

- There is no pole in the solution at first order (no Landau ghost).
- At large $t \geq t_0$ (UV), and for nonvanishing $\lambda_+ \neq 0$, since $\lambda(t) > \lambda(t_0)$, the bare coupling is supposedly not vanishing. Therefore, the model is not asymptotically free. This hints at an asymptotically safe model that only nonperturbative calculation can make rigorous.
- If $\lambda_+ = 0$, then $\lambda(t) = \lambda(t_0)$ and we have a fixed point. However, the enhancement disappears, both couplings λ and λ_+ do not flow. Note that the resulting model is not the usual quartic-tensor field theory model with only λ coupling and a different class of dominant graphs (melonic ones).

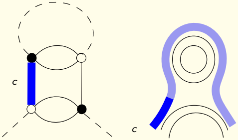
Renormalisation of 2-pt coupling Z_a (model +)

$$|p_c|^{2a}\Gamma_2^{(c)}(\{p\}) \quad = \quad \sum_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}} K_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}} S_{\mathcal{G}^{(c)}_{\mathbf{2};a;\iota}}(\{p\})\,,$$

where the sum is over all amputated 1PI 2-pt graphs at 1-loop whose boundaries are in the form of $\mathrm{Tr}_{2;(c)}(p^{2a}\phi^2)$. Up to the first order in perturbation theory, we have $\mathcal{G}^{(c)}_{2;a;\iota} \in \{z_a^{(c)}, m_e^{(c)}\}$,



$z_a^{(c)}$



$m_e^{(c)}$

$$Z^{(c)}_{a,\mathrm{ren}} \quad = \quad -\Gamma_2^{(c)}(\{0\}) = Z_a^{(c)} + \lambda_+^{(c)} \sum_{\{q_{\check{c}}\}} \frac{1}{(|\mathbf{q}_{\check{c}}|^{2b} + \mu)}\,.$$

Furthermore, set the couplings to be independent of colors.

Renormalisation of Z_a in multiscale analysis (model +)

Making explicit the dimensions ($q = k\tilde{q}\,,\ \tilde{q} \in \mathbb{Z}\,,\ \alpha = k^{-2b}\,\tilde{\alpha}$), we obtain the renormalisation group equation for Z_a in multiscale analysis as

$$Z_{a,i-1} = Z_{a,i} + k^{1/2}\lambda_{+,i}\,\widetilde{S}_{1,i} \quad \text{or} \quad -\left(Z_{a,i-1} - Z_{a,i}\right) = \frac{\partial Z_{a,i}}{\partial i} = -k^{1/2}\lambda_{+,i}\,\widetilde{S}_{1,i}\,,$$

where the dimensionless coefficient

$$\widetilde{S}_{1,i} = \sum_{\mathbf{q}\in\mathbb{Z}^{d-1}} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\ e^{-\alpha(|\mathbf{q}|^{2b}+\mu_i)} = \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha e^{-\alpha\mu_i} \left(\sum_{q\in\mathbb{Z}} e^{-\alpha|q|^{2b}}\right)^{d-1}$$
$$= \left(\frac{1}{b}\Gamma\left(\frac{1}{2b}\right)\right)^{d-1} (2d-3)M^{i/2}\left(1 - M^{-1/2}\right) + \mathcal{O}(M^{-i/2})\,,$$

and $\lambda_+ = \lambda_{+,i}$ does not run. So, with $t = \log(k/k_0)$ and $k/k_0 \sim M^i$,

$$\frac{\partial Z_{a,i}}{\partial((\log M)i)} = \partial_t Z_a(t) = -k^{1/2}\beta_{Z_a}\,\lambda_+\,,$$
$$\beta_{Z_a} = \frac{\widetilde{S}_{1,i}}{\log(M)} > 0\,.$$

β-function of 2-pt coupling Z_a (model +)

Introducing dimensionless quantities, $Z_a(t) = k^{1/2}\widetilde{Z}_a(t)$, the dimensionless RG equation can be written

$$\begin{aligned} \partial_t \widetilde{Z}_a(t) &= -\frac{1}{2}\widetilde{Z}_a(t) + k^{-1/2}\partial_t Z_a(t) \\ &= -\frac{1}{2}\widetilde{Z}_a(t) - \beta_{Z_a}\,\lambda_+\,, \end{aligned}$$

and λ_+ does not run. Integrate and

$$\widetilde{Z}_a(t) = c_1\, e^{-t/2} - 2\beta_{Z_a}\,\lambda_+\,, \qquad \beta_{Z_a} > 0\,.$$

Observation

- $\widetilde{Z}_a(t)$ decreases exponentially in the UV ($t \to \infty$) and suppressed up until it reaches a constant $-2\beta_{Z_a}\,\lambda_+$.
- In the IR ($t \to -\infty$), $\widetilde{Z}_a(t)$ blows up.

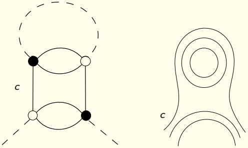
Renormalisation of self energy and mass (model +)

Compute the self energy,

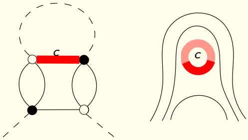
$$\Sigma_b(\{p\}) = \sum_{c=1}^{d} \sum_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}} K_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}} S_{\mathcal{G}^{(c)}_{\mathbf{2},\iota}}(\{p\}) \,,$$

where $\mathcal{G}^{(c)}_{2,\iota} \in \{m^{(c)}, n^{(c)}\}_{c=1,2,\dots,d}$ up to one loop.

$\Sigma_b(\{p\})$ *corresponds to the part* $\Sigma(\{0\}) + \sum_c |p_c|^{2b} \partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0}$ *of total self-energy function* $\Sigma(\{p\})$*. However,* $\partial_{|p_c|^{2b}} \Sigma\big|_{\{p\}=0} = 0$*, we only focus on the contribution* $\Sigma(\{0\})$*, namely the contribution to the mass renormalisation.*



The graph $m^{(c)}$ in the case $d = 3$. The degree of divergence $\omega_{d;+}(m^{(c)}) = \frac{D}{2}$.



The Feynman graph $n^{(c)}$ for $d = 3$. $\omega_{d;+}(n^{(c)}) = \frac{D}{2}$.

Renormalisation of mass (model +)

Impose color indepence, $\lambda^{(c)} = \lambda$, ${\lambda_+}^{(c)} = \lambda_+$,

$$\mu_{\rm ren} = \mu + d\Big(\lambda\, S_1 \;+\; \frac{1}{2}\lambda_+\, S_2\Big)\,,$$

$$S_1 = \sum_{\{q_1,\dots,q_{d-1}\}} \frac{1}{(|\mathbf{q}|^{2b}+\mu)}\,, \qquad S_2 = \sum_q \frac{|q|^{2a}}{(|q|^{2b}+\mu)}\,.$$

In the multiscale analysis,

$$S_{1,i} \;=\; \sum_{\mathbf{q}\in k\mathbb{Z}^{d-1}} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\; e^{-\alpha(|\mathbf{q}|^{2b}+\mu_i)} = k^{1/2}\widetilde{S}_{1,i}$$

$$S_{2,i} \;=\; \sum_{q\in k\mathbb{Z}} \int_{M^{-2bi}}^{M^{-2b(i-1)}} d\alpha\; |q|^{2a} e^{-\alpha(|q|^{2b}+\mu_i)} = k^{1/2}\widetilde{S}_{2,i}\,.$$

$$\widetilde{S}_{1,i} \;=\; \left(\Big(\tfrac{1}{b}\Gamma\left(\tfrac{1}{2b}\right)\Big)^{d-1}(2d-3)\Big(1-M^{-1/2}\Big)M^{i/2} + \mathcal{O}(M^{-i/2})\right)$$

$$\widetilde{S}_{2,i} \;=\; \left(4\,\Gamma\Big(\tfrac{2(d-1)}{2d-3}\Big)(1-M^{-1/2})M^{i/2} + \mathcal{O}\big(M^{-i(d-2)}\big)\right).$$

($D=1$ at any order d, $a=\frac{1}{2}D(d-2)$ and $b=\frac{1}{2}(d-\frac{3}{2})$.)

We obtain the β-function for the mass

$$
\begin{aligned}
-(\mu_{i-1} - \mu_i) &= \frac{\partial \mu_i}{\partial i} = -k^{1/2} d\Big(\widetilde{S}_{1,i}\,\lambda_i + \frac{1}{2}\widetilde{S}_{2,i}\,\lambda_{+,i}\Big)\,, \\
\frac{\partial \mu_i}{\partial((\log M)i)} &= \partial_t \mu = -k^{1/2}(\beta_{\mu,1}\,\lambda + \beta_{\mu,2}\,\lambda_+)\,, \\
\beta_{\mu,1} &= \frac{d}{\log M}\,\widetilde{S}_{1,i} > 0\,, \\
\beta_{\mu,2} &= \frac{d}{2\,\log M}\,\widetilde{S}_{2,i} > 0
\end{aligned}
$$

Following [Benedetti, Ben Geloun, Oriti, JHEP 1503, 084 (2015) [arXiv:1411.3180 [hep-th]]], the mass scaling dimension is determined by the maximal degree of divergence of the 2pt amplitudes. So, $\mu = k^{1/2}\widetilde{\mu}$, where $\widetilde{\mu}$ is dimensionless in scale, therefore, $\partial_t \mu = k\partial_k \mu = k^{1/2}(\frac{1}{2}\widetilde{\mu} + \partial_t \widetilde{\mu})$, i.e.,

$$
\partial_t \widetilde{\mu}(t) = -\frac{1}{2}\widetilde{\mu}(t) + k^{-1/2}\partial_t \mu(t)\,.
$$

Given that the coupling $\lambda_+ = \lambda_{+,i}$ does not run and that λ runs,

$$
\begin{aligned}
\partial_t \widetilde{\mu}(t) &= -\frac{1}{2}\widetilde{\mu}(t) - \beta_{\mu,1}\,\lambda(t) - \beta_{\mu,2}\,\lambda_+ \\
&= -\frac{1}{2}\widetilde{\mu}(t) - \beta_{\mu,1}\,\beta_\lambda\,\lambda_+^2\,t - \Big(\beta_{\mu,2}\,\lambda_+ - \beta_{\mu,1}\,\beta_\lambda\,\lambda_+^2\,t_0 + \beta_{\mu,1}\,\lambda(t_0)\Big)\,,
\end{aligned}
$$

where we recall $\beta_\lambda > 0$.

β-function of mass renormalisation (model +)

We can solve this differential equation and obtain

$$
\begin{aligned}
\widetilde{\mu}(t) &= c_1 e^{-t/2} - 4\beta\, t + 4\beta + 2\gamma\,, \\
\beta &= \beta_{\mu,1}\, \beta_\lambda\, \lambda_+^2 > 0\,, \\
\gamma &= -\Big(\beta_{\mu,2}\, \lambda_+ - \beta_{\mu,1}\, \beta_\lambda\, \lambda_+^2\, t_0 + \beta_{\mu,1}\, \lambda(t_0)\Big)\,.
\end{aligned}
$$

where c_1 is an integration constant.

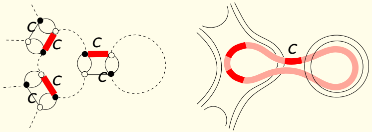
- In the UV ($t \to \infty$), the exponential term vanishes and the second linear term dominates. $\widetilde{\mu}(t) \sim -4\beta\, t$. $\beta > 0$ so the mass becomes negative and grows linearly. This is not the ordinary behavior of scalar field theory nor of tensor field theories.
- In the IR ($t \to -\infty$), the exponential term dominates $\widetilde{\mu}(t) \sim c_1 e^{-t/2}$.

Summary of perturbative renormalisation β-functions for model $+$

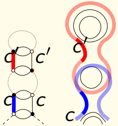
$\partial_t \, \lambda(t) = \beta_\lambda \lambda_+^2$	$\lambda(t) = \beta_\lambda \, \lambda_+^2 \, (t - t_0) + \lambda(t_0)$
$\partial_t \, \lambda_+ = 0$	$\lambda_+ = const.$
$\partial_t \, \widetilde{\mu}(t) = -\frac{1}{2} \widetilde{\mu}(t) - \beta_{\mu,1} \, \lambda(t) - \beta_{\mu,2} \, \lambda_+$	$\widetilde{\mu}(t) = c_1 e^{-t/2} - 4\beta \, t + 4\beta + 2\gamma$
$\partial_t \, \widetilde{Z}_a(t) = -\frac{1}{2} \widetilde{Z}_a(t) - \beta_{Z_a} \, \lambda_+$	$\widetilde{Z}_a(t) = c_2 \, e^{-t/2} - 2\beta_{Z_a} \, \lambda_+$

$$
\begin{aligned}
\beta_\lambda 0 &> 0 \\
\beta_{\mu,1} &< 0 \\
\beta_{\mu,2} &> 0 \\
\beta_{Z_a} &> 0 \\
\beta &= \beta_{\mu,1} \, \beta_\lambda \, \lambda_+^2 \; > \; 0 \\
\gamma &= -\Big(\beta_{\mu,2} \, \lambda_+ - \beta_{\mu,1} \, \beta_\lambda \, \lambda_+^2 \, t_0 + \beta_{\mu,1} \, \lambda(t_0) \Big)
\end{aligned}
$$

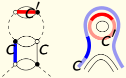
Higher order corrections for model +



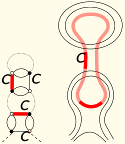
$\omega = 0$. 4-pt λ renorm.



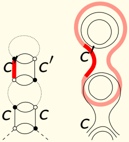
$\omega = D/2$, class II, 2-pt Z_a renorm.



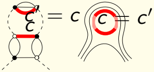
$\omega = 0$, class V, 2-pt Z_a renorm.



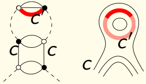
$\omega = D/2$, class I, mass renorm.



$\omega = D/2$, class III, mass renorm.



$\omega = 0$, class IV, mass renorm.



$\omega = 0$, class VI, mass renorm.

to all orders in perturbation (model +)

- λ_+. No diverging amplitudes contributing to the renormalisation of λ_+ at all orders in perturbation. *λ_+ is constant at all orders.*
- λ at an arbitrary n^{th} order.

$$\begin{aligned} \partial_t \lambda(t) &= P_n(\lambda_+)\,, \\ \lambda(t) &= P_n(\lambda_+)(t - t_0) + \lambda(t_0)\,, \end{aligned}$$

where $P_n(\lambda_+) = \beta_\lambda \lambda_+^2 + \ldots$.

- mass at arbitrary n^{th} order.

$$\begin{aligned} \partial_t \widetilde{\mu}(t) &= -\frac{1}{2}\widetilde{\mu}(t) + R_{1;n}(\lambda_+)\,\lambda(t) + R_{2;n}(\lambda_+) \\ &\quad + R_{3;n}(\lambda_+)\,t\,\widetilde{Z}_a(t) + R_{4;n}(\lambda_+)\,t\,\widetilde{Z}_a(t)\,\lambda(t)\,. \end{aligned}$$

$R_{i;n}$, with $i = 1, 2, 3, 4$ are polynomials in λ_+ and some constants.

to all orders in perturbation (model +)

- Z_a at an arbitrary $n^{\rm th}$ order.

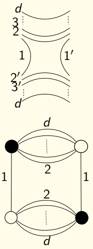
$$\partial_t Z_a(t) \quad = \quad k^{1/2} Q_{1;n}(\lambda_+) + \log(k/k_0)\, Z_a(t) Q_{2;n}(\lambda_+)\,,$$

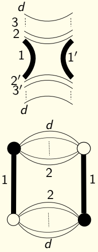
where $Q_{i;n}(\lambda_+)$, $i = 1, 2$ are polynomials in λ_+. Or in dimensionless quantities

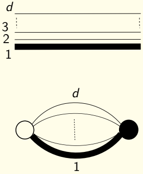
$$\begin{aligned} \partial_t \widetilde{Z}_a(t) &= \Big(t\, Q_{2;n}(\lambda_+) - \frac{1}{2}\Big)\widetilde{Z}_a(t) + Q_{1;n}(\lambda_+) \\ \widetilde{Z}_a(t) &= e^{\frac{Q_{2;n}t^2}{2} - \frac{t}{2}}\Big[c_1 + \sqrt{2}\,\frac{Q_{1;n}}{\sqrt{Q_{2;n}}}\, e^{\frac{1}{8Q_{2;n}}}\, \text{Erf'}\left(\frac{2Q_{2;n}t - 1}{2\sqrt{2}\sqrt{Q_{2;n}}}\right)\Big] \end{aligned}$$

where Erf' is the unnormalised incomplete error function $\text{Erf'}(z) = \int_0^z e^{-s^2} ds \sim \mathrm{Erf}' e^{-z^2}/z^2$, $\widetilde{Z}_a(t) \sim e^{\frac{Q_{2;n}t^2}{2} - \frac{t}{2}}$. Thus, $\widetilde{Z}_a(t)$ behaves the same in the UV ($t \to \infty$) and the IR ($t \to -\infty$), either can be suppressed or blows up depending on the sign of $Q_{2;n}$.

Enhanced model $\times$

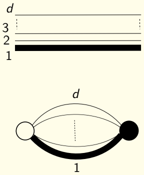


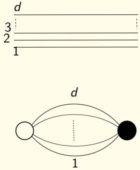




$$S_{\times}^{\mathrm{int}}[\bar{\phi},\phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\lambda_{\times}}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\phi^4) + \sum_{\xi=a,2a} Z_{\xi}\,\mathrm{Tr}_2(p^{2\xi}\phi^2)$$

$$S_{\times}^{\mathrm{kinetic}}[\bar{\phi},\phi] = Z_b\mathrm{Tr}_2(p^{2b}\phi^2) + \mu\mathrm{Tr}_2(\phi^2)\,,$$

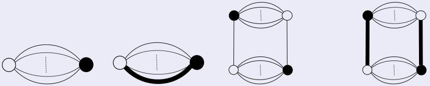




Power counting theorem for model $\times$

Proposition (List of primitively divergent graphs for the model $\times$)

The $p^{2a}\phi^4$-model $\times$ with parameters $D = 1, d = 3, a = \frac{1}{2}, b = 1$, has the following primitively divergent graphs which obey



$\mathrm{class}_{\mathcal{G}}$		N_{ext}	V_2	$V_{2;a}$	V_4	$\rho_\times$	$\omega_{\mathrm{d};\times}(\mathcal{G})$
I	*(2-pt Z_a)*	*2*	*0*	*0*	*0*	$2V_{\times;4} - 1$	0
II	*(2-pt Z_{2a})*	*2*	*0*	*0*	*0*	$2V_{\times;4} - 2$	0
III	*(mass)*	*2*	*0*	*0*	*1*	$2V_{\times;4}$	0

List of primitively divergent graphs of the $p^{2a}\phi^4$-model $\times$.

Theorem

The $p^{2a}\phi^4$ model $\times$ with parameters $D = 1, d = 3, a = \frac{1}{2}, b = 1$ is renormalisable at all orders of perturbation.

model $\times$

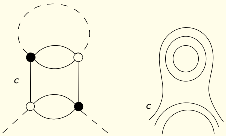
We find the effective couplings at scale $i-1$ to be related to scale i,

$$\begin{aligned} Z_{b,\,i-1} &= 1\,, \\ \mu_{\mathrm{ren},i-1} &= \mu_{i-1} - \Sigma_{i-1}(\{0\})\,, \\ Z_{a,i-1} &= -\Gamma^{(c)}_{2;a,i-1}(\{0\})\,, \\ Z_{2a,i-1} &= -\Gamma^{(c)}_{2;2a,i-1}(\{0\})\,, \end{aligned}$$

$d=3$, $D=1$, $a=\frac{1}{2}$, and $b=1$ so that the model is just-renormalisable.

Note that the mass does not have a scaling dimension, and that λ does not run.

Mass renormalisation (model $\times$)



Feynman graph that contributes to the mass renormalisation at one loop in perturbation theory. $\omega_{d;\times} = 0$. Class III.

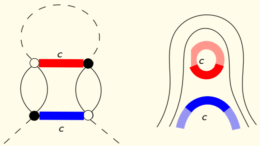
$$
\begin{aligned}
-(\mu_{i-1} - \mu_i) &= \frac{\partial \mu_i}{\partial i} = -d\, \widetilde{S}_{1,i}\, \lambda_i\,, \\
\frac{\partial \mu_i}{\partial((\log M) i)} &= \partial_t \mu(t) = -\beta_{\mu,1}\, \lambda\,, \\
\beta_{\mu,1} &= \frac{d}{\log M}\, \widetilde{S}_{1,i} = 2d\pi > 0\,.
\end{aligned}
$$

Fixing an initial condition at t_0, this integrates to give

$$\mu(t) = -(t - t_0)\beta_{\mu,1}\, \lambda + \mu(t_0)\,.$$

The mass in the model $\times$ grows linearly in t in its magnitude.

β-function of 2-pt coupling Z_a (model $\times$)



The graph contributes to the flow of Z_a, satisfies $\omega_{d;\times} = 0$ and belongs to the class I.

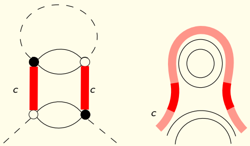
$$\begin{aligned} -(Z_{a,i-1} - Z_{a,i}) &= \frac{\partial Z_{a,i}}{\partial i} = -\widetilde{S}_{2,i}\,\lambda_{\times,i}\,, \\ \frac{\partial Z_{a,i}}{\partial((\log M)i)} &= \partial_t Z_a(t) = -\beta_{Z_a}\,\lambda_\times\,, \\ \beta_{Z_a} &= \frac{\widetilde{S}_{2,i}}{\log M} = 2 > 0\,, \end{aligned}$$

which integrates to

$$Z_a(t) = -(t - t_0)\,\beta_{Z_a}\,\lambda_\times + Z_a(t_0)\,.$$

Therefore, in exactly the same manner as the mass in this theory, the 2-pt coupling Z_a in the model $\times$, grows linearly in t in its magnitude.

β-function of 2-pt coupling Z_{2a} (model $\times$)



The graph that will contribute at 1-loop with $\omega_{d;\times} = 0$ in the class II.

$$
\begin{aligned}
-(Z_{2a,i-1} - Z_{2a,i}) &= \frac{\partial Z_{2a,i}}{\partial i} = -\,\widetilde{S}_{1,i}\,\lambda_{\times,i}\,, \\
\frac{\partial Z_{2a,i}}{\partial((\log M)i)} &= \partial_t Z_{2a}(t) = -\beta_{Z_{2a}}\,\lambda_\times\,, \\
\beta_{Z_{2a}} &= \frac{\widetilde{S}_{1,i}}{\log M} = 2\pi > 0\,,
\end{aligned}
$$

Then, at this order of perturbation, Z_{2a} yields also a linear function in the time scale t.

$$
Z_{2a}(t) = -(t-t_0)\,\beta_{Z_{2a}}\,\lambda_\times + Z_a(t_0)\,.
$$

The argument goes the same as the mass and the other 2-point coupling Z_a, i.e., the 2-pt coupling Z_{2a} in the model $\times$ grows linearly in t in its magnitude.

Summary of perturbative renormalisation β-functions for model $\times$

We give a summary of the 1-loop RG flow equations for the model $\times$ and their solutions.

$\partial_t \lambda(t) = 0$	$\lambda(t) = const.$
$\partial_t \lambda_\times(t) = 0$	$\lambda_\times(t) = const.$
$\partial_t \mu(t) = -\beta_{\mu,1}\lambda$	$\mu(t) = -2d\pi\lambda(t-t_0) + \mu(t_0)$
$\partial_t Z_a(t) = -\beta_{Z_a}\lambda_\times$	$Z_a(t) = -2\lambda_\times(t-t_0) + Z_a(t_0)$
$\partial_t Z_{2a}(t) = -\beta_{Z_{2a}}\lambda_\times$	$Z_{2a}(t) = -2\pi\lambda_\times(t-t_0) + Z_{2a}(t_0)$

$$\begin{aligned} \beta_{\mu,1} = 2d\pi &> 0\,, \\ \beta_{Z_a} = 2 &> 0\,, \\ \beta_{Z_{2a}} = 2\pi &> 0\,. \end{aligned}$$

The mass and the 2-point couplings Z_a and Z_{2a} in the model $\times$ all grow linearly in t in its magnitude.

to all orders in perturbation (model $\times$)

4-pt couplings λ and $\lambda_\times$ RG equation

The power counting theorem of the model $\times$ determines that at all orders in pertubation theory, there are no amplitudes which are divergent contributing to the renormalisation of 4-pt couplings λ and $\lambda_\times$. Hence, λ and $\lambda_\times$ of the model $\times$ are constant and do not flow with scale.

$$\begin{aligned} \partial_t \lambda &= 0\,, \quad \text{therefore} \quad \lambda(t) = const. \\ \partial_t \lambda_\times &= 0\,, \quad \text{therefore} \quad \lambda_\times(t) = const. \end{aligned}$$

Mass, 2-pt couplings Z_a and Z_{2a} RG equations

Observation of Proposition tells us that

- Mass renormalisation is decided by the class III, where only exactly one λ and a number of $\lambda_\times$ contribute.
- The Z_a renormalisation is decided by the class I, where only $\lambda_\times$ contributes.
- Only $\lambda_\times$ contributes to the renormalisation of Z_{2a}, as class II dictates.

Then, one can generalise the RG euqations for the first order to arbitrary n-th order,

$$\partial_t \mu(t) = \lambda\, P_n(\lambda_\times)\,, \quad \partial_t Z_a(t) = Q_n(\lambda_\times)\,, \quad \partial_t Z_{2a}(t) = R_n(\lambda_\times)\,,$$

where $P_n(\lambda_\times)$, $Q_n(\lambda_\times)$, and $R_n(\lambda_\times)$ are polynomials in $\lambda_\times$.
Solving the above system of equations,

$$\begin{aligned} \mu(t) &= (t-t_0)\lambda\, P_n(\lambda_\times) + \mu(t_0)\,, \\ Z_a(t) &= (t-t_0) Q_n(\lambda_\times) + Z_a(t_0)\,, \\ Z_{2a}(t) &= (t-t_0) R_n(\lambda_+) + Z_{2a}(t_0)\,. \end{aligned}$$

All couplings above grow linearly in t in their respective magnitudes.

Conclusions

- These models may not give rise to quantum gravity, but possibly a new kind of ϕ^4 models.
- Solve for higher orders. The models seem to be resummable.