A new approach to $O(N)$ breaking interfaces and boundaries

Sabine Harribey

Work in progress with Igor Klebanov and Zimo Sun

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NORDITA
The Nordic Institute for Theoretical Physics

Boundary CFTs and defects

- Critical $O(N)$ models: thoroughly studied class of $3D$ CTFs
  $\rightarrow$ ϵ expansion, bootstrap, $1/N$ expansion
- Introduce boundaries or defects [Diehl, cond-mat/9610143]
- Interface: codimension 1 defect
- Boundary: codimension 1 defect where the extra dimension is integrated only over half-space
- *Extraordinary*: break $O(N)$ symmetry to $O(N-1)$

Surface defects

- RG analysis and $1/N$ expansion: $3D$ extraordinary log universality class [Metlitski 2009.05119]
- Recently: D dimensional bulk and defects of codimension $D-2$ [Trepanier; Giombi, Liu; Raviv-Moshe, Zhong]
  - $\rightarrow$ Quadratic defects
  - $\rightarrow$ $4-\epsilon$ and $6-\epsilon$ expansion
  - $\rightarrow$ Ordinary and extraordinary defects
  - $\rightarrow$ $1/N$ expansion: singular for $D \rightarrow 3$

Surface defects

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- Recently: D dimensional bulk and defects of codimension $D-2$ [Trepanier; Giombi, Liu; Raviv-Moshe, Zhong]
  - $\rightarrow$ Quadratic defects
  - $\rightarrow$ $4-\epsilon$ and $6-\epsilon$ expansion
  - $\rightarrow$ Ordinary and extraordinary defects
  - $\rightarrow$ $1/N$ expansion: singular for $D \rightarrow 3$
- Here: $4-\epsilon$ expansion with defects of codimension 1
- Cubic interactions on the defect are then marginal

Free vs interacting bulk

Schematically

$$S = \int d^{d+1}x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \lambda_n \phi^n \right] + \int d^d x \lambda_p \phi^p$$

- UV dimension of the field $\Delta_\phi = \frac{d-1}{2}$
- The defect interaction is then marginal for $d_c = \frac{p}{p-2}$
- In $d_c + 1 = \frac{2p-2}{p-2}$, the operator $\phi^{2(p-1)}$ is marginal in the bulk

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Dimension d_c	Bulk	Defect
3	$n = 4$	$p = 3$
2	$n = 6$	$p = 4$[1]
5/3	$n = 8$	$p = 5$

1 [Söderberg Rousu 2304.05786]

Outline

1. Multi-scalar model
2. Applications
   - $N = 1$
   - $O(N-1)$ vector model
   - Symplectic fermions
3. Boundary with cubic interactions

Model

$$S[\phi] = \int d^{d+1}x \left[\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + \frac{\lambda^{(4)}_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d\right] + \int d^d x \left[\frac{\lambda_{abc}}{3!}\,\phi_a\phi_b\phi_c\right]$$

- Indices take values from 1 to $\mathcal{N}$
- $\lambda^{(4)}_{abcd}$ and λ_{abc} fully symmetric tensors

  $\Rightarrow$ In general $\binom{\mathcal{N}+3}{4}$ and $\binom{\mathcal{N}+2}{3}$ couplings respectively

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  $\Rightarrow$ In general $\binom{\mathcal{N}+3}{4}$ and $\binom{\mathcal{N}+2}{3}$ couplings respectively
- Non-trivial propagators on the interface

Propagator on the defect

Fourier transform the free propagator along the defect directions:

$$\langle \phi_I(\mathrm{p}_1, y_1)\phi_J(\mathrm{p}_2, y_2)\rangle = \delta_{IJ}\frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{\frac{d+1}{2}}}\int d^d\mathrm{x}_1\, d^d\mathrm{x}_2 \frac{e^{i\mathrm{p}_1\cdot\mathrm{x}_1+i\mathrm{p}_2\cdot\mathrm{x}_2}}{\left(\mathrm{x}_{12}^2+y_{12}^2\right)^{\frac{d-1}{2}}}$$

$$= (2\pi)^d\delta^d(\mathrm{p}_1+\mathrm{p}_2)\,\delta_{IJ}\frac{e^{-|\mathrm{p}_1||y_{12}|}}{2|\mathrm{p}_1|}$$

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- Interface-to-bulk propagator: $y_1 = 0$

$$K_{IJ}(\mathrm{p}, y) = \frac{e^{-|\mathrm{p}||y|}}{2|\mathrm{p}|}\delta_{IJ}$$

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$$K_{IJ}(\mathrm{p}, y) = \frac{e^{-|\mathrm{p}||y|}}{2|\mathrm{p}|}\delta_{IJ}$$

- Interface propagator: $y_1 = y_2 = 0$

$$G_{IJ}(\mathrm{p}) = \frac{\delta_{IJ}}{2|\mathrm{p}|}$$

Divergences and regularization

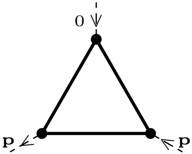
Power counting in $d = 3$:

- Two-point graphs: power divergent
- Three-point graphs: log divergent
- Wave function renormalization: not modified by the boundary couplings $\rightarrow$ neglected at one-loop

Choice of scheme:

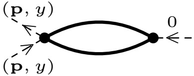
- Dimensional regularization $d = 3 - \epsilon$
- BPHZ subtraction with symmetric external momenta

Graph with only cubic couplings



T

Graph with both cubic and quartic couplings



B

$$\Gamma^{(3)}_{\mathsf{abc}} = \lambda_{\mathsf{abc}} + \lambda_{\mathsf{ade}}\lambda_{\mathsf{bdf}}\lambda_{\mathsf{cef}}\, \mu^{-\epsilon} T - \frac{1}{2}\big(\lambda^{(4)}_{\mathsf{abef}}\lambda_{\mathsf{efc}} + 2 \text{ terms}\big)\, \mu^{-\epsilon} B$$

- Why "+ 2 terms"? To conserve permutation symmetry
- T: computed in momentum space with interface propagators
- B: have to use a mixed representation

$$I_B = \mu^{\epsilon} \int \frac{d^d \mathsf{k}}{(2\pi)^d} \int_{\mathbb{R}} dy \left(\frac{1}{2|\mathsf{k}|}\right)^2 e^{-2(|\mathsf{p}|+|\mathsf{k}|)|y|}$$
$$= \frac{\mu^{\epsilon}}{4} \int \frac{d^d \mathsf{k}}{(2\pi)^d} \frac{1}{|\mathsf{k}|^2(|\mathsf{p}| + |\mathsf{k}|)} = \frac{2}{(4\pi)^2\, \epsilon} + \cdots$$

Beta functions

Running coupling $\leftrightarrow$ dimensionless three-point function

$$g_{\mathsf{abc}} = \mu^{-\epsilon/2} \Gamma^{(3)}_{\mathsf{abc}}$$

Beta functions: scale derivative of the running coupling

$$\beta^{(3)}_{\mathsf{abc}} = \mu \partial_\mu g_{\mathsf{abc}}$$

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Method:

- Derive the bare expansion with respect to μ
- Invert the bare expansion to obtain the renormalized series
- Substitute the bare coupling by its expression in terms of the renormalized one

One-loop beta functions

- Beta function for the cubic couplings:

$$\beta^{(3)}_{\mathsf{abc}} = -\frac{\epsilon}{2}\tilde{g}_{\mathsf{abc}} - \frac{1}{4}\tilde{g}_{\mathsf{ade}}\tilde{g}_{\mathsf{bdf}}\tilde{g}_{\mathsf{cef}} + \left(\tilde{g}^{(4)}_{\mathsf{abef}}\tilde{g}_{\mathsf{efc}} + 2\ \mathrm{terms}\right)$$

  where we rescaled the coupling as $g_{\mathsf{abc}} = (4\pi)^{\frac{d}{4}}\Gamma(\frac{d}{2})^{1/2}\tilde{g}_{\mathsf{abc}}$

- Theory in the bulk not modified by the defect:

$$\beta^{(4)}_{\mathsf{abcd}} = -\epsilon\tilde{g}^{(4)}_{\mathsf{abcd}} + \left(\tilde{g}^{(4)}_{\mathsf{abef}}\tilde{g}^{(4)}_{\mathsf{efcd}} + 2\ \mathrm{terms}\right)$$

  where we rescaled the coupling as $g^{(4)}_{\mathsf{abc}} = (4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})\tilde{g}^{(4)}_{\mathsf{abc}}$

$N = 1$

$$S = \int d^{d+1}x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda_4}{4!} \phi^4 \right] + \int d^d x \frac{\lambda_3}{3!} \phi^3$$

Only one cubic coupling:

$$\beta_4 = -\epsilon g_4 + 3 g_4^2$$
$$\beta_3 = -\frac{\epsilon}{2} g_3 - \frac{g_3^3}{4} + 3 g_4 g_3$$

Free bulk

- Stable purely imaginary fixed point
- Similar to standard Yang-Lee model

Fixed points and CFT data

- Usual Wilson-Fisher fixed point for the quartic coupling
- Two non-trivial fixed points for the cubic coupling

$$g_3^\star = \pm\sqrt{2\epsilon}$$

- Negative critical exponent $\omega = -\epsilon \rightarrow$ **unstable**
- Trivial fixed point is stable
- Dimension of quadratic operator

$$\Delta_2 = 2 - \frac{7\epsilon}{6} + \mathcal{O}(\epsilon)$$

$O(N-1)$ vector model

$$
\begin{aligned}
S[\phi] = &\int d^{d+1}x \left[\frac{1}{2}\partial_\mu\phi_I(x)\partial^\mu\phi_I(x) + \frac{\lambda_4}{4!}\left(\phi_I(x)\phi_I(x)\right)^2\right] \\
&+ \int d^d x\left[\frac{\lambda_1}{2}\,\phi_N(x)\phi_a(x)\phi_a(x) + \frac{\lambda_2}{3!}\phi_N^3\right]
\end{aligned}
$$

where I summed from 1 to N while a summed from 1 to $N-1$

$\Rightarrow$ Symmetry broken to $O(N-1)$

Beta functions obtained from the multi-scalar model by setting:

$$
\lambda^{(4)}_{\text{abcd}} = \frac{\lambda_4}{3}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)
$$

$$
\lambda_{\text{abc}} = \lambda_1\left(\delta_{aN}\delta_{bc} + \delta_{bN}\delta_{ac} + \delta_{cN}\delta_{ab}\right) + \lambda_2\delta_{aN}\delta_{bN}\delta_{cN}
$$

One-loop beta functions

$$\beta_{g_4} = -\epsilon g_4 + \frac{N+8}{3} g_4^2$$

$$\beta_{g_1} = -\frac{\epsilon}{2} g_1 - \frac{1}{4} g_1^2 \left(g_1 + g_2\right) + \frac{1}{3} g_4 \left((N+5) g_1 + g_2\right)$$

$$\beta_{g_2} = -\frac{\epsilon}{2} g_2 - \frac{1}{4} \left((N-1) g_1^3 + g_2^3\right) + g_4 \left((N-1) g_1 + 3 g_2\right)$$

Free bulk

- $N = 2$: two pairs of purely imaginary fixed points, one stable
- $N > 2$:
  - One pair of real unstable fixed points
  - Purely imaginary fixed points with $g_1 = 0$ and $g_2 = \pm i\sqrt{2\epsilon}$ and critical exponent $(\epsilon, -\frac{\epsilon}{2})$
  - Two pairs of complex conjugate fixed points: complex critical exponent but with positive real part

$\Rightarrow$ No unitarity

Interacting bulk

- $N = 2$: all fixed points real but unstable
- $N = 3, 4$: appearance of complex fixed points, unstable
- $N \geq 5$: one purely imaginary stable fixed point
- $N > N_{crit} \sim 7.1274 + \mathcal{O}(\epsilon)$: only complex fixed points

At the stable purely imaginary fixed points:

- Quadratic operators have real dimensions within unitarity bounds

Large-N behavior

Denoting $g_1 = 2i\sqrt{2\epsilon}x$, $g_2 = 2i\sqrt{2\epsilon}y$, the purely imaginary stable fixed points are given by:

$$x^\star = \pm\left(\frac{1}{N} + \dots\right) + \mathcal{O}(\epsilon)$$

$$y^\star = \mp\left(\frac{1}{2} - \frac{3}{2N} + \dots\right) + \mathcal{O}(\epsilon)$$

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Critical exponents

$$\omega_1 = \left(\frac{1}{2} - \frac{11}{N} + \dots\right)\epsilon + \mathcal{O}(\epsilon^2)$$

$$\omega_2 = \left(1 + \frac{6}{N} + \dots\right)\epsilon + \mathcal{O}(\epsilon^2)$$

Quadratic operators

- Two quadratic operators $\mathbb{O}_1 = \frac{\phi_i \phi_i}{\sqrt{N-1}}$ and $\mathbb{O}_2 = \phi_N^2$
- Compute dimensions at the purely imaginary stable fixed points at large N

$$\Delta_- = 2 - \frac{\epsilon}{2} - \frac{2\epsilon}{N}$$
$$\Delta_+ = 2 - \frac{5\epsilon}{N}$$

- Corresponding to the operators

$$\mathbb{O}_- = -\frac{1}{N^{1/2}}\mathbb{O}_1 + \mathbb{O}_2$$
$$\mathbb{O}_+ = \left(N^{1/2} - \frac{1}{2N^{1/2}}\right)\mathbb{O}_1 + \mathbb{O}_2$$

Symplectic fermions

M pairs of symplectic fermions and one scalar:

$$S[\phi] = \int d^{d+1}x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\theta_a\partial^\mu\bar{\theta}_a + \frac{\lambda_4}{4!}\left(\phi^2 + 2\theta_a\bar{\theta}_a\right)^2\right] + \int d^d x \left[\lambda_1\phi\theta_a\bar{\theta}_a + \frac{\lambda_2}{3!}\phi^3\right]$$

- General M: $OSp(1|2M)$ symmetry broken to $Sp(2M)$
- $M = 1$: can preserve $OSp(1|2)$ if $\lambda_2 = 2\lambda_1$
- Beta functions obtained by setting $N \to 1 - 2M$

Fixed points

Quartic coupling: $g_4^\star = \frac{3\epsilon}{9-2M} \Rightarrow$ positive for $M \leq 4$

- $M = 1$
  - Four pairs of real fixed points
  - One preserves $OSp(1|2)$
  - Only the trivial fixed point is stable
- $2 \leq M \leq 4$
  - Complex non-trivial fixed points
  - Complex critical exponents
  - Only the trivial fixed point is stable

Boundary action and propagator

$$S[\phi] = \int_{y\geq 0} dyd^d x \left[\frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi_I + \frac{\lambda_4}{4!}\left(\phi_I\phi_I\right)^2\right] + \int d^d \mathrm{x}\left[\frac{\lambda_1}{2}\,\phi_N\phi_a\phi_a + \frac{\lambda_2}{3!}\phi_N^3\right]$$

- Neumann boundary conditions
- Two terms in the free bulk propagator

$$\langle\phi_I(\mathrm{x}_1, y_1)\phi_J(\mathrm{x}_2, y_2)\rangle_B = \delta_{IJ}\frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{\frac{d+1}{2}}}\left(\frac{1}{(\mathrm{x}_{12}^2 + (y_1 - y_2)^2)^{\frac{d-1}{2}}} + \frac{1}{(\mathrm{x}_{12}^2 + (y_1 + y_2)^2)^{\frac{d-1}{2}}}\right)$$

Propagators in momentum space

Boundary to bulk and boundary propagators

$$K_{IJ}^{(B)}(\mathrm{p},y) = \frac{e^{-|\mathrm{p}|y}}{|\mathrm{p}|}\delta_{IJ}, \quad G_{IJ}^{(B)}(\mathrm{p}) = \frac{1}{|\mathrm{p}|}$$

- Pure boundary graphs: factor 2 for each propagator
- Boundary-to-bulk graphs:
  - factor 2 for each propagator
  - extra dimension integrated only over half-space

- Not a simple rescaling of the coupling constants
- Qualitatively different from the interface case

Link with long-range models

- Non-trivial power of the Laplacian: $\phi_I \left(-\partial^2\right)^\zeta \phi_I$, $0 < \zeta < 1$
- Propagator $C(p) = \frac{1}{p^{2\zeta}}$
- Cubic interactions: marginal for $\zeta = \frac{d}{6}$, $d < 6$
- Study fixed points for $\zeta = \frac{d+\epsilon}{6}$
- Recover boundary model for $d = 3 - \epsilon$

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Long-range models $\Leftrightarrow$ Boundary interactions with free bulk

Beta functions and fixed points

After rescaling of the cubic couplings:

$$\beta_1 = -\frac{\epsilon}{2}\tilde{g}_1 - \frac{1}{4}\,\tilde{g}_1^2\,(\tilde{g}_1 + \tilde{g}_2) + \frac{2}{3}\,g_4\,((N+5)\tilde{g}_1 + \tilde{g}_2)$$

$$\beta_2 = -\frac{\epsilon}{2}\tilde{g}_2 - \frac{1}{4}\left((N-1)\tilde{g}_1^3 + \tilde{g}_2^3\right) + 2g_4\,((N-1)\tilde{g}_1 + 3\tilde{g}_2)$$

- $N = 1$: real unstable fixed points
- $2 \leq N \leq 16$: complex fixed points, only the trivial fixed point is stable
- $N \geq 17$: purely imaginary stable fixed points

Summary and outlook

- $N = 1$: **Real** fixed points but **unstable**
- Large N: One pair of purely imaginary stable fixed points
- Critical N: no real fixed points for $N > N_{crit} = 7.1274$
- **Stable** fixed points always **purely imaginary**

Summary and outlook

- $N = 1$: **Real** fixed points but **unstable**
- Large N: One pair of purely imaginary stable fixed points
- Critical N: no real fixed points for $N > N_{crit} = 7.1274$
- **Stable** fixed points always **purely imaginary**

- Unitarity? Extraordinary universality class?
- $\epsilon = 1$: compare with surface defect of [Krishnan, Metlitski 2301.05728]
- Monotonicity theorem