

# Taming unsummable series

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Quantum field theorists often resort to **perturbative** expansions in a reasonably small parameter. Yet these may have **no** radius of convergence. Summation of **sign-alternating** asymptotic series is sometimes achievable by a **Borel transform**, which may be improved by **Padé approximants**. I shall give an example that yields to these methods. The less tractable **sign-constant** case occurs in condensed matter problems. Here, **Michael Borinsky** and I have developed a **trans-series**. This is a formal expansion in powers of  $x$ ,  $\exp(-1/x)$  and  $\log(x)$ , with coefficients whose **asymptotic** growth is determined by further terms in the trans-series, and so on, *ad infinitum*. This is referred to as **resurgence**. [arXiv:2202.01513]

1. Two sources of **divergent series**
2. **Dyson-Schwinger** equation and **asymptotic** expansion
3. **Padé-Borel** summation with **alternating** signs
4. **Trans-series** and **resurgent** hyperasymptotic expansions

## Wilhem Killing (*with help from Wolfgang Hein at Siegen*)

1. Born 10 May **1847**, Burbach, near *Siegen*. Died 11 February **1923**, *Münster*.
2. **1865–1867**: studied at the Königlich Theologische und Philosophische Akademie in **Münster**.
3. **1867–1872** student of Helmholtz, Kummer and Weierstrass in **Berlin**.
4. Taught in Berlin **1873–1878**, Brilon **1878–1882**, Braunsberg **1882-1892**.
5. **1892–1919** Professor at Royal Academy/WWU, **Münster**. Rektor **1897/8**.
6. *Einführung in die Grundlagen der Geometrie*, 1. Bd. **1893**, 2. Bd. **1898**.
7. **1894**: Élie **Cartan**, doctoral thesis, *Sur la structure des groupes de transformations finis et continus*: “Le présent travail a pour but d’**exposer** et de **compléter en certains points** les recherches de M. **Killing**, en y introduisant toute la rigueur desirable.”

# 1 Two sources of divergent series

**Too many diagrams:** *An example of a divergent perturbation expansion in field theory* by Angus **Hurst** in 1951 considers the **super-renormalizable** case of  $\phi^3$  theory in  $D = 4$  space-time dimensions. The **number** of primitive  $n$ -loop diagrams increases **factorially** with  $n$ . Each diagram gives a **positive** integral over Feynman parameters, bounded from below **geometrically**.

**A renormalon:** *On high order estimates in QED* by Benny **Lautrup** in 1977 considers the magnetic moment of the electron. At  $n$  loops there is a **single** gauge-invariant diagram whose contribution grows **factorially** with  $n$ .

**Imaginary coupling:** *Divergence of perturbation theory in quantum electrodynamics* by Freeman **Dyson** in 1952 remarks that convergence in  $\alpha = e^2/(4\pi)$ , would make sense of a **non-unitary** theory with **imaginary** coupling giving  $\alpha < 0$  and a nightmarish world in which electrons **repel** positrons.

**Recent work:** *Instantons or renormalons? A comment on  $\phi^4$  theory in the  $MS$  scheme* by Gerald **Dunne** and Max **Meynig** in 2022 uses recent perturbative results to investigate a suggestion by Gerard 't **Hooft** that renormalons do **not** contribute to **anomalous dimensions** in a **minimal subtraction** scheme.

## 2 Dyson-Schwinger equation and asymptotic expansion

Consider the perturbation expansion generated by a single divergent diagram, via the **non-linear** Dyson-Schwinger equation

$$\text{shaded circle} = \text{white circle} + \text{shaded circle with loop} + \text{shaded circle with two shaded circles} + \dots$$

contributing to the **self-energy** term  $\Sigma$  in the inverse propagator  $q^2(1 - \Sigma)$ , for a **massless scalar** particle with a  $\phi^3$  interaction, in the **critical** space-time dimension  $D = 6$ , for which the coupling constant is **dimensionless**.

The dependence of  $\Sigma$  on the external momentum  $q$  comes **solely** from **renormalization**. At  $n$  loops, we get a contribution that is a **polynomial** of degree  $n$  in the **logarithm**  $\log(q^2/\mu^2)$ , multiplied by  $a^n$  where  $a = \lambda^2/(4\pi)^3$ ,  $\lambda$  is the coupling constant and  $\mu$  is the **renormalization scale**.

If we use **momentum-space subtraction**, so that  $\Sigma$  vanishes at  $q^2 = \mu^2$ , the dependence on momentum is completely determined by the **anomalous dimension**, with

$$\gamma(a) = -q^2 \left. \frac{d\Sigma}{dq^2} \right|_{q^2=\mu^2} \quad \text{giving} \quad \frac{d \log(1 - \Sigma)}{d \log q^2} = \gamma \left( \frac{a}{(1 - \Sigma)^2} \right).$$

## How many $n$ -loop Feynman diagrams for this problem?

The number of distinct diagrams at  $n$  loops is the number  $T_n$  of **rooted trees** with  $n$  nodes, which gives the sequence

1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381,

up to 16 loops. The **iterated** structure: **tree = root + branches**, with every branch being itself a tree, gives the asymptotic growth

$$T_n = \frac{b}{n^{3/2}} c^n (1 + O(1/n))$$

$b = 0.43992401257102530404090339143454476479808540794011 \dots$

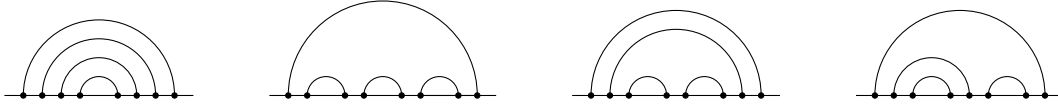
$c = 2.95576528565199497471481752412319458837549230466359 \dots$

At **250 loops** the number of Feynman diagrams is

T\_250=517763755754613310897899496398412372256908589980657316  
271041790137801884375338813698141647334732891545098109934676

This is **not** the main source of the problem. If the contribution of each diagram was bounded, there would be a **finite** radius of convergence for the perturbation expansion. The divergence of the series comes from **renormalization**, which makes the  $n$ -loop term grow **factorially**. This is called a **renormalon** singularity.

At 4 loops, we have a **rainbow**, a **chain** and two more interesting diagrams:



The sum of rainbows converges. Chains can be summed by Borel transformation.

$$\begin{aligned} \gamma_{\text{rainbow}} &= \frac{3 - \sqrt{5 + 4\sqrt{1+a}}}{2} = -\frac{a}{6} + 11 \frac{a^2}{6^3} - 206 \frac{a^3}{6^5} + 4711 \frac{a^4}{6^7} + O(a^5) \\ \gamma_{\text{chain}} &= - \int_0^\infty \frac{6 \exp(-6z/a) dz}{(z+1)(z+2)(z+3)} = -\frac{a}{6} + 11 \frac{a^2}{6^3} - 170 \frac{a^3}{6^5} + 3450 \frac{a^4}{6^7} + O(a^5) \\ \gamma &\sim \sum_{n>0} G_n \frac{(-a)^n}{6^{2n-1}} = -\frac{a}{6} + 11 \frac{a^2}{6^3} - 376 \frac{a^3}{6^5} + \mathbf{20241} \frac{a^4}{6^7} + O(a^5) \end{aligned}$$

with **large integers**  $G_n$  in the **alternating** asymptotic series for  $\gamma$ . Note that  $G_4 = 20241 > 4711 + 3450$ , because of two further diagrams, above. In one we have a chain inside a double rainbow. In the other, a double rainbow is chained with the primitive divergence. This interplay is coded by **rooted trees**.

At **500 loops** I determined the **integer** coefficient

G500=206261451966080541451119356265266407905816117576895601520616328670543304097  
62369668214104674763068056454522518617422020409397336434904863988900797769773644  
47129884863324773181376863120291798830884688213932683869821267125662274428136514  
68974978228592824043044373847281757207937081063432528806815509319762088807291996  
54549245884853496719417048678199825379018355919198123075612308008976364608893906  
00835837012056033720017238115336850340799075684336975651857656078799282745256216  
85768456030809283727097722850488278232311177219444745322287340871435443707536590  
64304859950724683157717734493071321199539578218428617617722892100276682781401203  
04983974209704793621909710059353724523231635766062166284812903992269403282699432  
81718327508638643305481989940132234093616573076862094588977827344981584305605437  
66475002382217933275761312682929603923397580260987048907414858143897114762331252  
08694985337972553885925402003826420205441859988844001867088083850782378303677991  
14077650584544145709672328391394562704209221732180879565868213522109303655045186  
92714017665002971967455255310508358729281544729403249398746232320441525286283859  
23093041626365262630100048817481274793707664791767175677240144896307853488347045  
21622394885797995125083750860330519417878429051836575477220881369445751634601965  
33191009573619480068718080810533581305996863996579338522874547127421808710757882  
86996199556804886954946559116947132125235605586627322129268965041445488085748194  
82341875039156647569797757032552836429751077302524927736861138479038542006096835  
73747720303607608007740173613335602076396299832459826245418033598839559699294537  
37336134624690115674194793212055897162647586497730033948880084738561472545509216

with 1675 decimal digits. The number of diagrams has *merely* 231 digits.

This was achieved in work with **Dirk Kreimer** that resulted in a **third-order** differential equation

$$8a^3\gamma \{ \gamma^2\gamma''' + 4\gamma\gamma'\gamma'' + (\gamma')^3 \} + 4a^2\gamma \{ 2\gamma(\gamma - 3)\gamma'' + (\gamma - 6)(\gamma')^2 \} + 2a\gamma(2\gamma^2 + 6\gamma + 11)\gamma' - \gamma(\gamma + 1)(\gamma + 2)(\gamma + 3) = a$$

with **quartic** non-linearity.

Our interest in this problem came from his discovery of the **Hopf algebra** of the iterated subtraction of subdivergences, whose utility we illustrated in this example, with a single primitive divergence leading to **undecorated** rooted trees.

For the corresponding diagrams in **Yukawa** theory, in its critical dimension  $D = 4$ , we found a **first-order** equation with merely **quadratic** non-linearity, which we solved using the **complementary error function**, thereby achieving explicit **all-orders** results for both the anomalous dimension and the self energy. The expansion coefficients in this simpler case enumerate **connected cord diagrams**.

We also investigated the  $D = 4$  and  $D = 6$  examples in the more cumbersome **minimal subtraction** scheme, where one retains finite parts of  $\Sigma$  at  $q^2 = \mu^2$ . Here one encounters unwieldy products of **zeta values** with weights that increase linearly with the loop-number. Recently **Paul-Hermann Balduf** has shown how to absorb these into a rescaling of  $\mu$  that can be expanded in the coupling  $a$ .



### 3 Padé-Borel summation with alternating signs

We sought to resum the factorially divergent alternating series by an Ansatz

$$\gamma(a) = -\frac{a}{6\Gamma(\beta)} \int_0^\infty B(ax/3) \exp(-x)x^{\beta-1} dx, \quad B(z) = \frac{N(z)}{D(z)}.$$

The expansion coefficients of the **Borel transform**  $B(z) = 1 + O(z)$  are obtained from those of  $\gamma(a)/a$  by dividing the latter by factorially increasing factors, producing a function expected to have a **finite** radius of convergence in the Borel variable  $z$ , with singularities on the **negative**  $z$ -axis, as for the sum of chains.

The **Padé** trick is to convert the expansion of  $B$ , up to  $n$  loops, into a **ratio**  $N/D$  of polynomials of degrees close to  $n/2$ . Then one can **check** how well this method reproduces  $G_{n+1}$ . We found that this works rather well with  $\beta \approx 3$ .

We fitted the first 29 values of  $G_n$  with a ratio of polynomials of degree 14 and found a pole, coming from the denominator  $D(z)$ , at  $z = -0.994$ . The other 13 poles occurred further to the left, with  $\Re z < -1$ . This method reproduced the first 15 decimal digits of  $G_{30}$ . Gerald Dunne has recently shown that this method works even **better** with  $\beta = 35/12$ , for reasons that I shall now explain.

## 4 Trans-series and resurgent hyperasymptotics

There is an old and rather loose argument, going back to **Freeman Dyson** in 1952, that we should **not expect** realistic field theories to give convergent expansions in the **square** of a coupling constant. If they did, we could get sensible answers for a pathological non-unitary theory with an **imaginary** coupling constant, such as an electrodynamics in which electrons repel positrons.

There is an amusing **converse** of this suggestion. If you find an expansion that is Borel summable, then study it at **imaginary coupling**, where  $\phi^3$  theory gives the **Yang-Lee edge singularity** in condensed matter physics, using PT symmetry.

So now I recast the Broadhurst-Kreimer problem, in the manner of Borinsky, Dunne and Meynig, by setting  $g(x) = \gamma(-3x)/x$ , to obtain an ODE that is economically written as

$$(g(x)P - 1)(g(x)P - 2)(g(x)P - 3)g(x) = -3, \quad P = x \left( 2x \frac{d}{dx} + 1 \right),$$

and has an **unsummable** formal perturbative solution

$$g_0(x) \sim \sum_{n=0}^{\infty} A_n x^n = \frac{1}{2} + \frac{11}{24}x + \frac{47}{36}x^2 + \frac{2249}{384}x^3 + \frac{356789}{10368}x^4 + \frac{60819625}{248832}x^5 + O(x^6).$$

The expansion coefficients behave as

$$A_n = S_1 \Gamma \left( n + \frac{35}{12} \right) \left( 1 - \frac{97}{48} \left( \frac{1}{n} \right) + O \left( \frac{1}{n^2} \right) \right),$$

at large  $n$ , with a **Stokes constant**

$$S_1 = 0.087595552909179124483795447421262990627388017406822 \dots$$

that can be determined, empirically, by considering a solution

$$g(x) = g_0(x) + \sigma_1 x^{-\beta} \exp(-1/x) h_1(x) + O(\sigma_1^2)$$

and retaining terms linear in  $\sigma_1$  in the ODE. This yields a **linear homogeneous** ODE for  $h_1(x)$ , which permits a solution that is **finite and regular** at  $x = 0$  if and **only if**  $\beta = \frac{35}{12}$ . Normalizing  $\sigma_1$  by setting  $h_1(0) = -1$ , we obtain the expansion of

$$h_1(x) \sim \sum_{k=0}^{\infty} B_k x^k = -1 + \frac{97}{48} x + \frac{53917}{13824} x^2 + \frac{3026443}{221184} x^3 + \frac{32035763261}{382205952} x^4 + O(x^5)$$

which gives the **first-instanton** correction to the perturbative solution, suppressed by  $\exp(-1/x)$ .

By developing the series  $A_n$  and  $B_k$ , I was able to determine **3000 digits** of  $S_1$  in

$$A_n \sim -S_1 \sum_{k \geq 0} \Gamma \left( n + \frac{35}{12} - k \right) B_k.$$

This is an example of **resurgence**: information about  $A_n$  resurges in  $B_k$ , and vice versa, because both  $A(x) = g_0(x)$  and  $B(x) = h_1(x)$  know about the **same** physics.

**Hyperasymptotic** expansions involve the study of how  $B_n$  behaves at large  $n$ , which involves another set of numbers  $C_k$ , at small  $k$ , and so on, and so on.

*Large A's need smaller B's, especially to guide them,  
and larger B's need smaller C's, and so ad infinitum.*

Hyperasymptotic investigation involves terms suppressed by  $\exp(-m/x)$ , with **action**  $m > 1$ . For this **third-order** ODE, there are **3 solutions** to the **linearized** problem, namely

$$g(x) = g_0(x) + \sigma_m \left( x^{-\frac{35}{12}} e^{-\frac{1}{x}} \right)^m h_m(x) + O(\sigma_m^2), \quad m \in \{1, 2, 3\},$$

with  $h_2/x^5 = C$  and  $h_3/x^5 = D$  finite and regular near the origin.

Then we use the linearized ODE to develop the expansions

$$C(x) = h_2(x)/x^5 = -1 + \frac{151}{24}x - \frac{63727}{3456}x^2 + \frac{7112963}{82944}x^3 - \frac{7975908763x}{23887872}x^4 + O(x^5),$$

$$D(x) = h_3(x)/x^5 = -1 + \frac{227}{48}x + \frac{1399}{4608}x^2 + \frac{814211}{73728}x^3 + \frac{3444654437}{42467328}x^4 + O(x^5).$$

Before presenting the **trans-series**, I remark on some of its general **features**.

1. The terms suppressed by  $\exp(-2/x)$  involve  $\sigma_2$  and  $\sigma_1^2$ . The former are given by  $C$  and the latter are determined by an **inhomogeneous** linear ODE, whose solution is **ambiguous**, up to a multiple of the homogeneous solution  $h_2 = x^5C$ , since we can **shift**  $\sigma_2$  by a multiple of  $\sigma_1^2$ .
2. In the terms suppressed by  $\exp(-3/x)$  there a **second ambiguity**, since we can shift  $\sigma_3$  by a multiple of  $\sigma_1^3$ .
3. Ambiguities of inhomogeneous solutions occur at places in expansions where **logarithms** first arise. This happens when the **power** of  $x$  in an expansion is a multiple of **5**.
4. The **highest** power of  $\log(x)$ , in terms with **action**  $m$ , is  $\lfloor m/2 \rfloor$ .

The terms in the **trans-series** with action  $m \leq 4$  are of the form

$$g = \sum_{m \geq 0} g_m \left( x^{-\frac{35}{12}} e^{-\frac{1}{x}} \right)^m, \quad L = \frac{21265}{2304} x^5 \log(x),$$

$$g_0 = A, \quad g_1 = \sigma_1 B, \quad g_2 = \sigma_2 x^5 C + \sigma_1^2 (F + CL),$$

$$g_3 = \sigma_3 x^5 D + \sigma_1 \sigma_2 x^5 E + \sigma_1^3 (I + (D + E)L),$$

$$g_4 = \sigma_1 \sigma_3 x^5 G + \sigma_2^2 x^{10} H + \sigma_1^2 \sigma_2 x^5 (J + 2HL) + \sigma_1^4 (K + (G + J)L + HL^2).$$

Denoting the coefficients of  $x^n$  in functions by subscripts, we found that the choices

$$\frac{F_5}{2!} = \frac{I_5}{3!} = \frac{32642693907919}{36691771392}$$

**greatly simplify** of our system of hyperasymptotic expansions. Then

$$B_n \sim -2S_1 \sum_{k \geq 0} F_k \Gamma\left(n + \frac{35}{12} - k\right)$$

$$+ 4S_1 \sum_{k \geq 0} C_k \Gamma\left(n - \frac{25}{12} - k\right) \left( \frac{21265}{4608} \psi\left(n - \frac{25}{12} - k\right) + d_1 \right),$$

$$d_1 = -43.332634728250755924500717390319380703460728022278 \dots$$

with  $\psi(z) = \Gamma'(z)/\Gamma(z) = \log(z) + O(1/z)$ , shows the  $m = 1$  term, at large  $n$ , looking forward to  $m = 2$  terms, at small  $k$ .

For the asymptotic expansion of the **second-instanton** coefficients, we found

$$C_n \sim -S_1 \sum_{k \geq 0} E_k \Gamma(n + \frac{35}{12} - k) + S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n + \frac{25}{12} - k).$$

The first sum looks **forwards** to  $m = 3$  in the trans-series, where coefficients of

$$E(x) = -4 + \frac{371}{12}x - \frac{111785}{1152}x^2 + \frac{8206067}{18432}x^3 - \frac{18251431003}{10616832}x^4 + O(x^5)$$

appear. It does **not contain** the coefficients  $D_k$  of the **third instanton**, which **decouples** from the asymptotic expansion for the second instanton.

The second sum has **alternating** signs, looks **backwards** to  $m = 1$  and is **suppressed** by a factor of  $1/n^{5/6}$ . This can be understood using **alien calculus**. Likewise,

$$\begin{aligned} F_n \sim & -3S_1 \sum_{k \geq 0} I_k \Gamma(n + \frac{35}{12} - k) \\ & + 2S_1 \sum_{k \geq 0} (3D_k + 2E_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \\ & - 2S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right) \end{aligned}$$

looks forwards to  $I_k$ ,  $D_k$  and  $E_k$ , at  $m = 3$ , and backwards to  $B_k$  at,  $m = 1$ .

The new constants are

$$\begin{aligned} S_3 &= 2.1717853140590990211608601227903892302479464193027\dots \\ f_1 &= -40.903692509228515003814479126901354785263669553014\dots \end{aligned}$$

Two more were discovered in the backward looking terms of

$$\begin{aligned} I_n &\sim -4S_1 \sum_{k \geq 0} K_k \Gamma(n + \frac{35}{12} - k) \\ &+ 2S_1 \sum_{k \geq 0} (3G_k + 2J_k) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \\ &- 4S_3 \sum_{k \geq 0} F_k (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right) \\ &- 8S_3 \sum_{k \geq 0} C_k (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k), \\ Q(z) &= \left( \frac{21265}{4608} \right)^2 (\psi^2(z) + \psi'(z)) + 2c_1 \left( \frac{21265}{4608} \right) \psi(z) + c_2, \\ c_1 &= -41.031956764302710583921068101545509453704897898188\dots \\ c_2/c_1^2 &= 1.0002016472131992595822805380838324188011572304276\dots \end{aligned}$$

We believe that **6 constants suffice** for the complete description of resurgence.



**Conjecture:** The **trans-series** and its **resurgence** take the forms

$$\begin{aligned}
g(x) &= \sum_{m=0}^{\infty} \left( x^{-\frac{35}{12}} e^{-\frac{1}{x}} \right)^m \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-2i)/3 \rfloor} \sigma_1^{m-2i-3j} \widehat{\sigma}_2^i \widehat{\sigma}_3^j x^{5(i+j)} \sum_{n \geq 0} a_{i,j}^{(m)}(n) x^n, \\
\widehat{\sigma}_2 &= \sigma_2 + \frac{21265}{2304} \sigma_1^2 \log(x), \quad \widehat{\sigma}_3 = \sigma_3 + \frac{21265}{2304} \sigma_1^3 \log(x), \\
a_{i,j}^{(m)}(n) &\sim -(s+1) S_1 \sum_{k \geq 0} a_{i,j}^{(m+1)}(k) \Gamma(n + \frac{35}{12} - k) \\
&+ S_1 \sum_{k \geq 0} \left( 4(i+1) a_{i+1,j}^{(m+1)}(k) + 6(j+1) a_{i,j+1}^{(m+1)}(k) \right) \Gamma(n - \frac{25}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{25}{12} - k) + d_1 \right) \\
&+ \frac{1}{4} S_3 \sum_{k \geq 0} \left( 4(s+1) a_{i-1,j}^{(m-1)}(k) + 6(j+1) a_{i-2,j+1}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n + \frac{25}{12} - k) \\
&- 2(s-2i-1) S_3 \sum_{k \geq 0} a_{i,j}^{(m-1)}(k) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k) \left( \frac{21265}{4608} \psi(n - \frac{35}{12} - k) + f_1 \right) \\
&- S_3 \sum_{k \geq 0} \left( 8(i+1) a_{i+1,j}^{(m-1)}(k) + 6(j+1) a_{i,j+1}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n - \frac{95}{12} - k) Q(n - \frac{95}{12} - k) \\
&- (f_1 - c_1) S_3 \sum_{k \geq 0} \left( 2(i+1) a_{i+1,j-1}^{(m-1)}(k) + 6(i+j) a_{i,j}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n - \frac{35}{12} - k),
\end{aligned}$$

with  $s = m - 2i - 3j$  and  $Q(z) = \left( \frac{21265}{4608} \right)^2 (\psi^2(z) + \psi'(z)) + 2c_1 \left( \frac{21265}{4608} \right) \psi(z) + c_2$ .

## Comments and conclusions

1. The conjecture exhibits **17 resurgent terms**, all of which have been intensively tested at **high precision**, for all **actions**  $m \leq 8$ .
2. The **6 Stokes constants** have been determined to better than **1000 digits**.
3. Excellent **freeware**, from **Pari-GP** in Bordeaux, was vital to this enterprise.
4. First and second **derivatives** of  $\Gamma$  and **suppressions** by  $1/n^{5/6}$  make Richardson acceleration infeasible. I used systematic **matrix inversion**.
5. The presence of **logarithms** in trans-series has been ascribed to **resonant actions**. Michael Borinsky and I find this **misleading**. We showed that a closely analogous **second-order** problem is both resonant and **log-free**.
6. We have been guided by helpful advice from **Gerald Dunne** and encouraged by the programme and workshops on *Applicable Resurgent Asymptotics* hosted by the **Isaac Newton Institute** in Cambridge.
7. For physicists who wonder, as I did, why one might consider **imaginary** coupling, I remark that the idea goes back 70 years, to **Freeman Dyson**, who was a notable inquirer into both mathematics and quantum field theory.