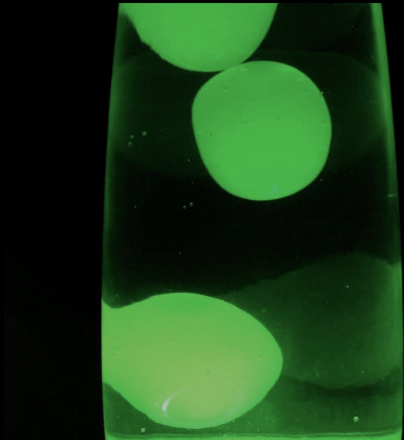

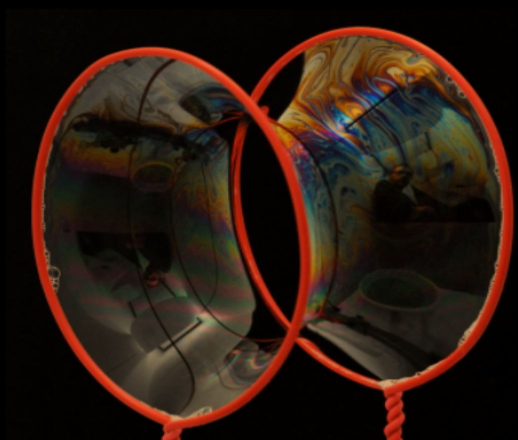

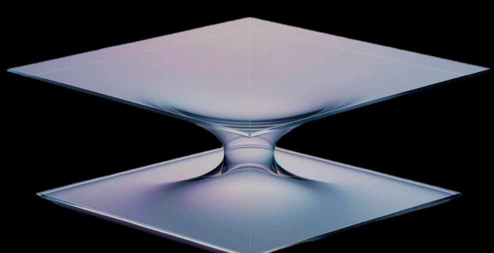

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Stable phase transitions: open questions and new results

Mid Term Conference Mathematics Münster
26.03.2024

Joaquim Serra

ETH zürich

OUTLINE OF THE TALK

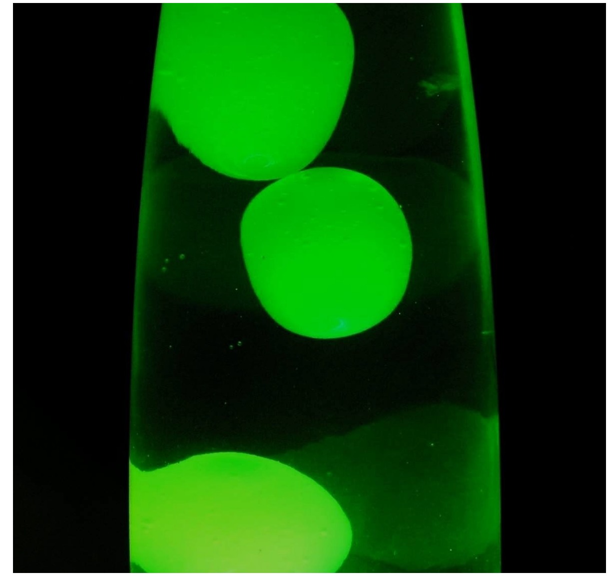
1. **Motivation**

2. Phase-field models and Allen-Cahn

3. Known results open questions on stable equilibria

4. New results: free boundary Allen-Cahn and one phase Alt-Caffarelli

From surface tension to minimal surfaces



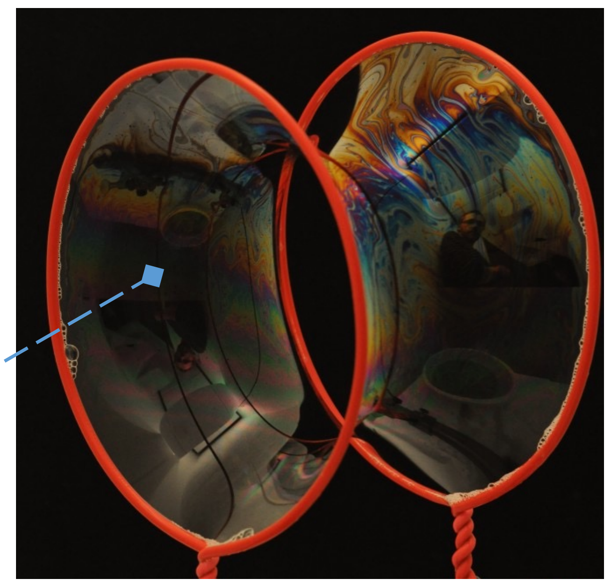
Due to forces akin to **surface tension**, some **physical phenomena** exhibit **interfaces** that tend to **minimize** their **surface area at macroscopic scales**

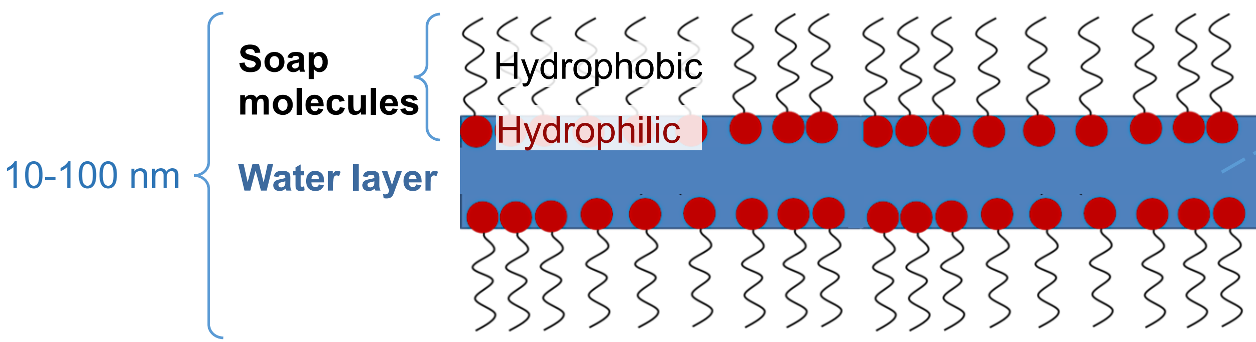
But **while Surface area** is **scaling invariant …**

$$A(rS) = r^2 A(S) \quad S \text{ surface and } r > 0$$

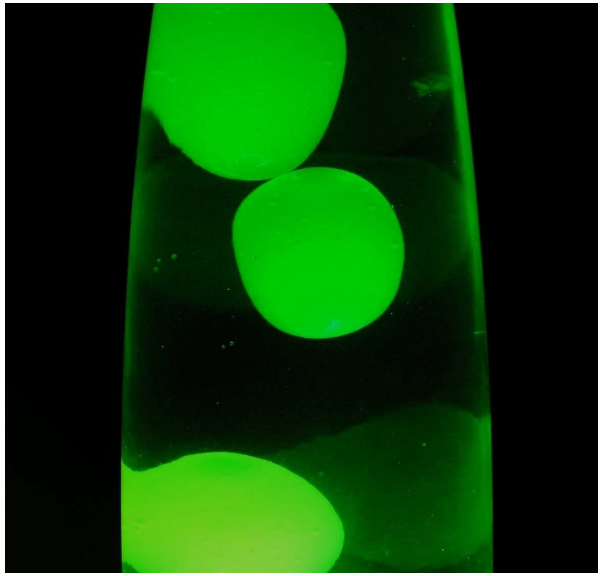
… actual physical energies are not!

(e.g., soap film molecule's size: ~5 nm)



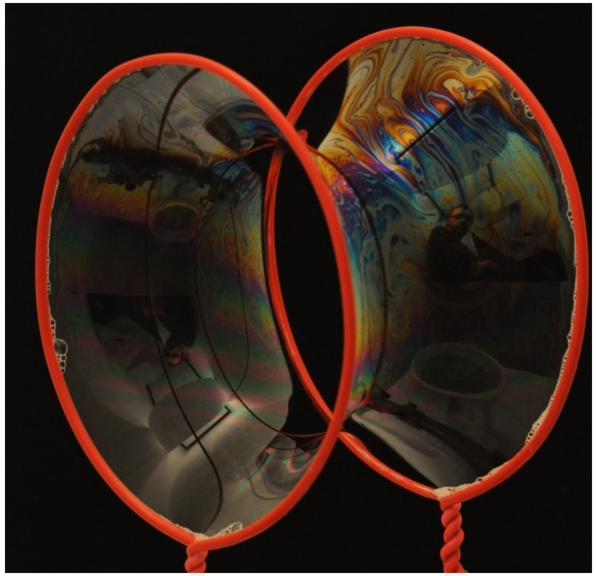


Absolute energy-minimizing configurations are known to behave like minimal surfaces at macroscopic scales



- **Regularity theory** for **energy-minimizing** minimal surfaces has been **successfully extended** to several important **scale-dependent models**

- Notable instance: **macroscopic regularity theory for Allen-Cahn** by **Savin** (in Ann. Math. 2009)



- A key consequence of the macroscopic regularity theory is that **absolute minimizing configurations** for the scale-dependent model **really behave like area-minimizing** surfaces at macroscopic scales

However, **what happens if we replace *absolute* minimizing** configurations **with stable equilibria**?

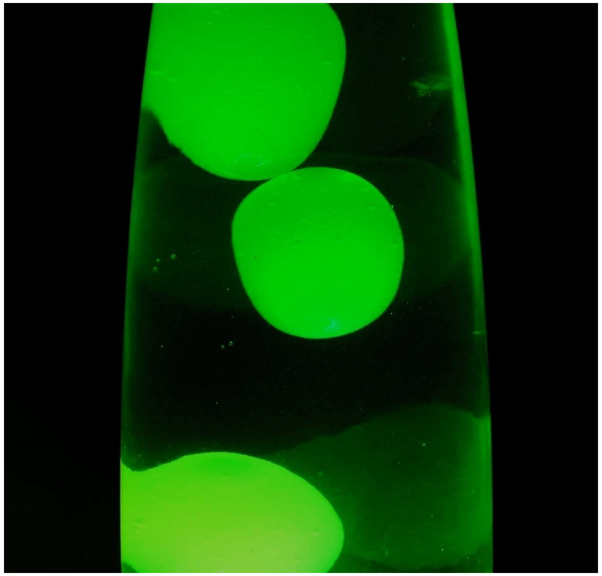
- **Stable equilibria** are defined as those minimizing the energy among sufficiently small perturbations

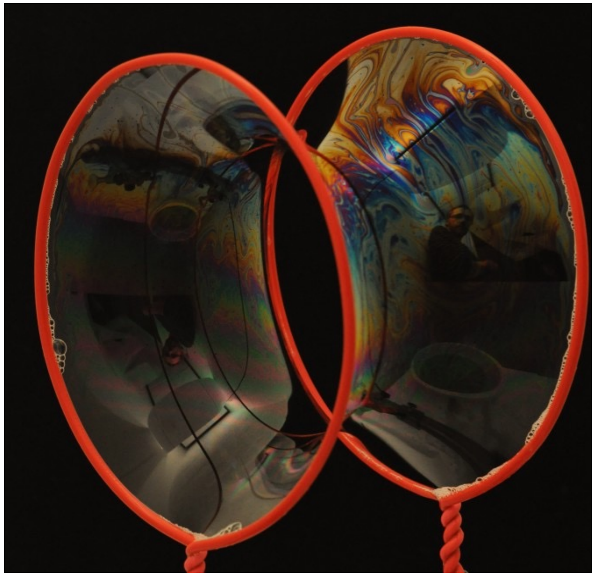
- They encompass the observable states in nature, as unstable equilibria spontaneously decay to more stable lower energy states

- As we will see**, efforts to establish a regularity theory** akin to that of minimal surfaces **for stable equilibria in scale-dependent models** present **profound mathematical challenges**

- **It is** then arguably **premature to assert that** all **stable configurations** within such models necessarily **resemble minimal surfaces at macroscopic scales**

- Addressing this discrepancy emerges as a pressing **open question in the Calculus of Variations**





OUTLINE OF TALK

1. Motivation

2. **Phase-field models and Allen-Cahn**

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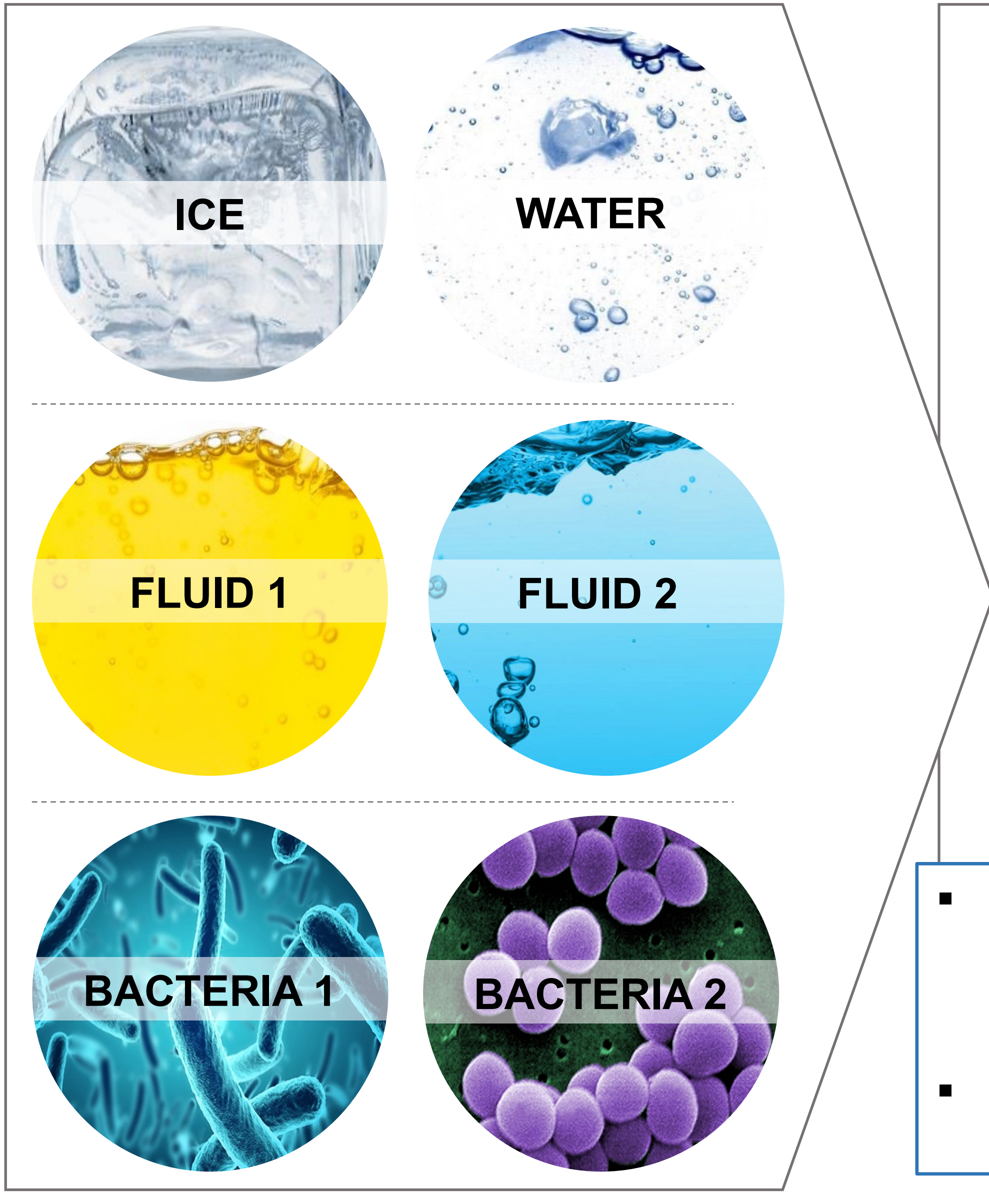
4. New results: free boundary Allen-Cahn and one phase Alt-Caffarelli

Phase transition models: two distinct states coexist within a defined spatial region, leading to the emergence of an interface

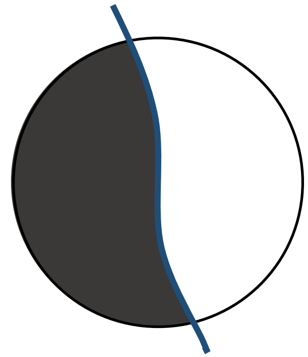


- Intimately connected with **free boundaries** and **minimal surfaces**, **phase transitions** are characterized by the notable presence of **'interfaces'**
- Interfaces often increase the system's energy, prompting them to minimize their surface area

Phase field models aim to describe the phase transitions using solutions to a PDE: the Euler-Lagrange equation associated to suitable energy functional

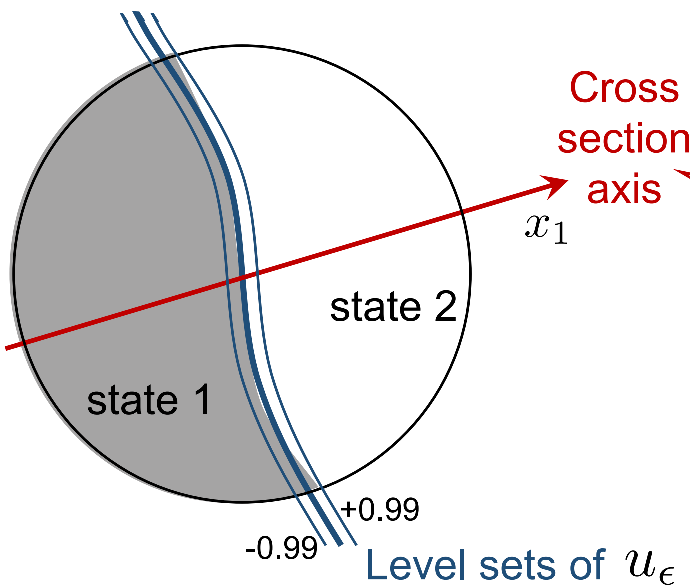
The "black-or-white" or "± 1" model …

■	state 1	□	state 2
Black = -1		**White = +1**	



$\Longleftrightarrow \quad u_0 : \mathbb{R}^n \to \{-1, +1\}$

…is replaced by a "smoothed model" $u_\epsilon : \mathbb{R}^n \to [-1, 1]$ depending on a small parameter $\epsilon > 0$
Instead of sharp interface, transition from -0.99 to +0.99 occurs on a 'fat surface' with thickness ϵ



$u_\epsilon : \mathbb{R}^n \to (-1, 1) \qquad \epsilon > 0$ small parameter

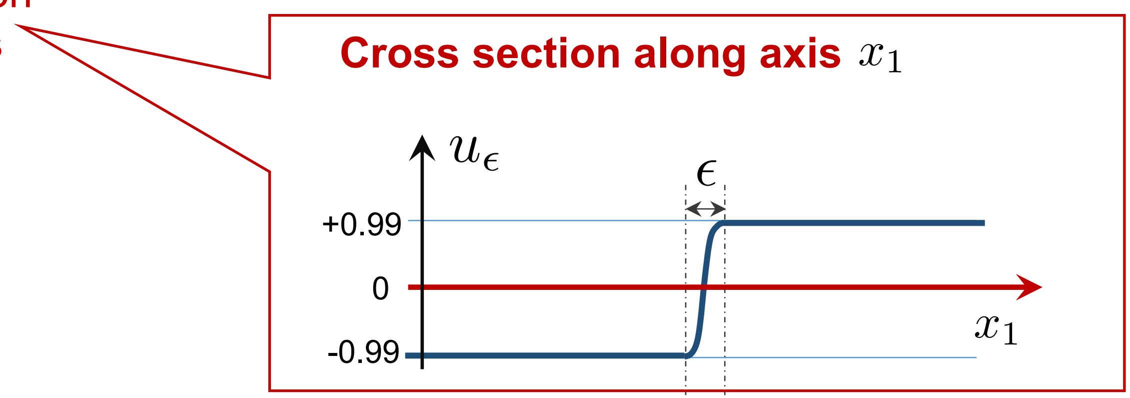
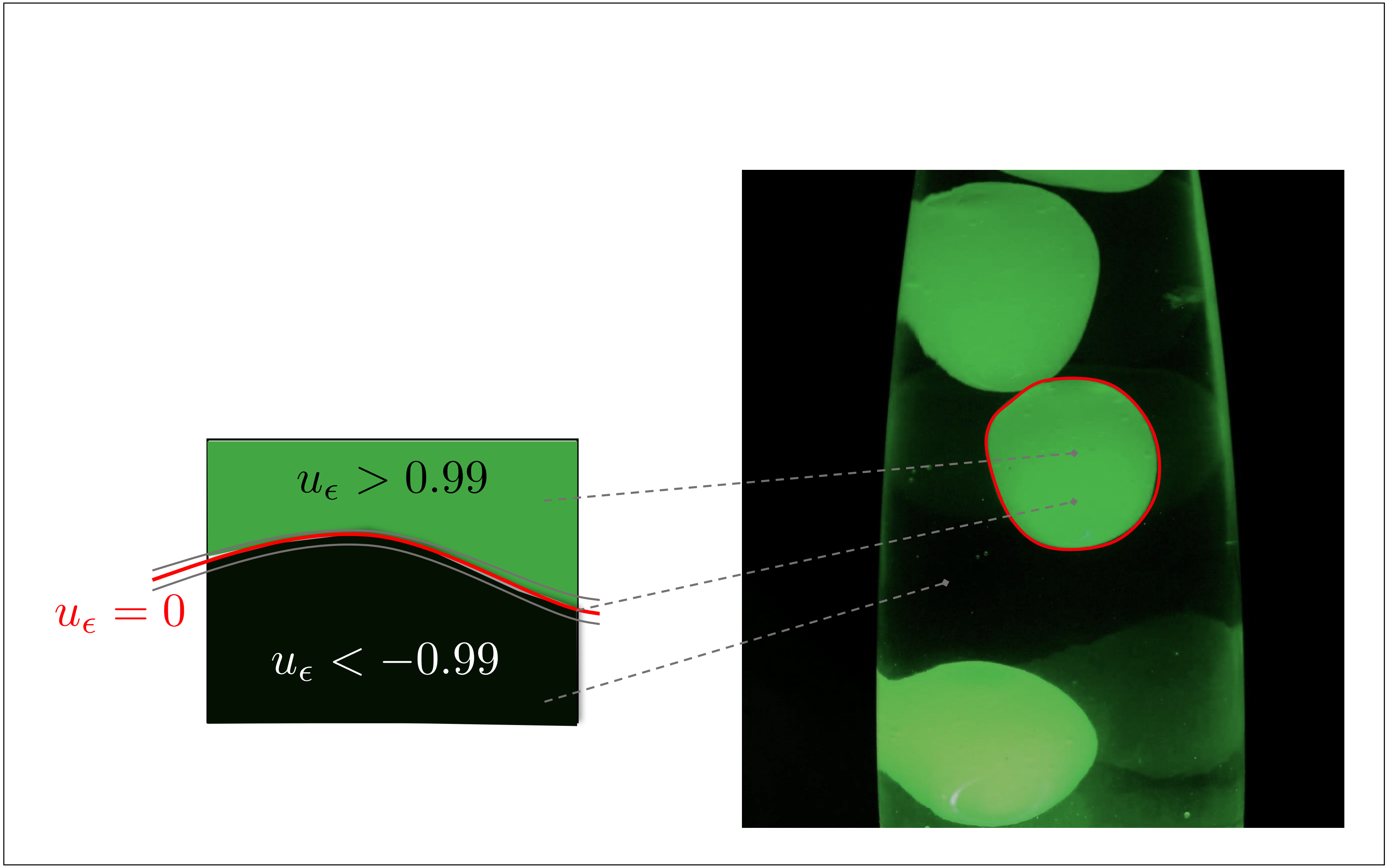


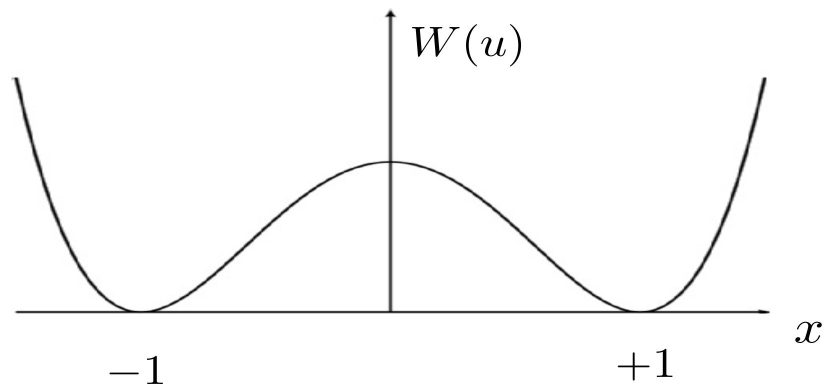
Illustration: phase field model for a binary fluid



Allen-Cahn, a pragmatic phenomenological model, enjoys notable popularity in Physics, Calculus of Variations, Geometric analysis, …

$$E_\epsilon(u) := \int \epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} W(u) dx$$

Double-well potential

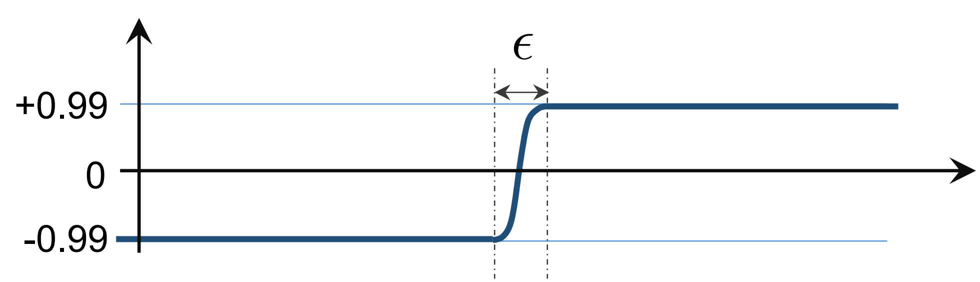


Typical choices of double-well potential:

$$W(u) = \frac{1}{4}(1-u^2)^2 \qquad W(u) = \cos(\pi u/2)$$

- The **potential term** strongly penalizes intermediate states, not close to ± 1
- In equilibria, transitions are expected to occur on "fat surfaces" with thickness epsilon
- In one dimension, solutions to the Euler-Lagrange equation $\phi'' = \frac{1}{\epsilon^2}(\phi - \phi^2)$ are of the following type:

$$\phi(t) = \tanh\left(\frac{t-b}{\sqrt{2}\,\epsilon}\right)$$



For infinitesimal epsilon, energy minimizers of Allen-Cahn become minimal surfaces

Theorem (Modica-Mortola, 1977)

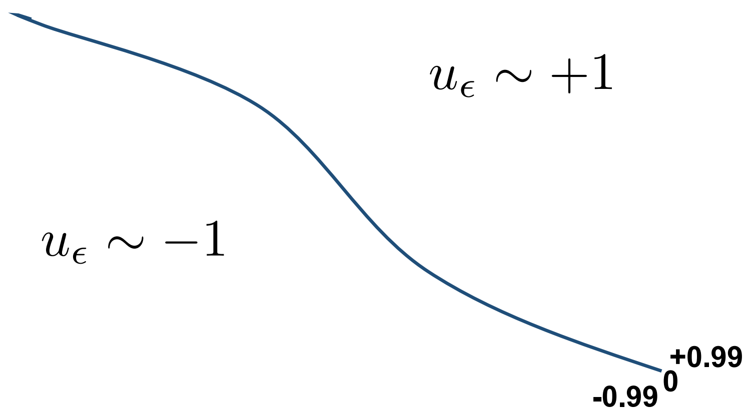
u_{ϵ_k} minimizers of E_{ϵ_k}

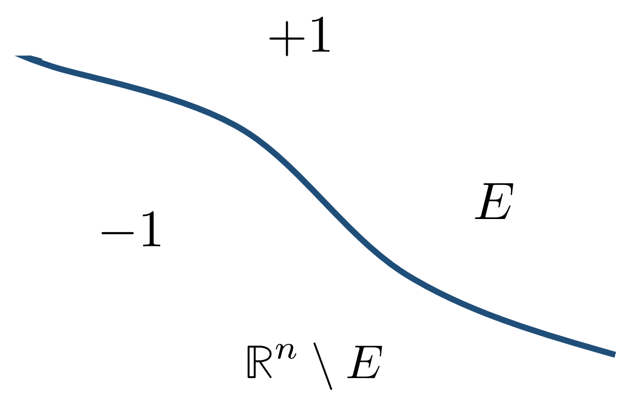
$\epsilon_k \downarrow 0$

➡

- $u_{\epsilon_k} \xrightarrow{L^1_{\text{loc}}} \mathbf{1}_E - \mathbf{1}_{\mathbb{R}^n \setminus E}$
- $E \subset \mathbb{R}^n$ is a **minimizer of the perimeter** (in every compact subset)
- ∂E is a **minimal surface**

ALLEN-CAHN $\xrightarrow{\epsilon \downarrow 0}$ MINIMAL SURFACES





Moreover, the convergence is smooth and quantifiable in low dimensions

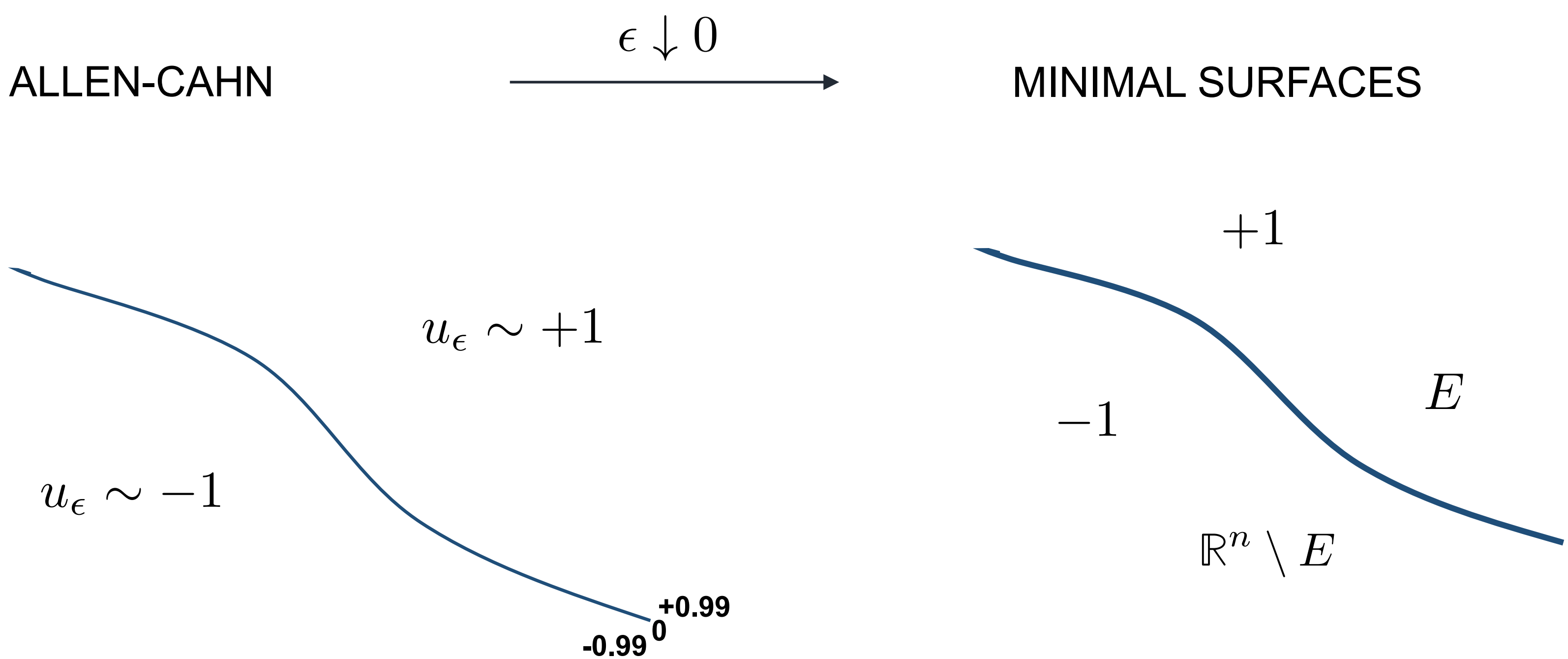
Theorem (Savin, 2009; Wang-Wei 2019)

u_{ϵ_k} minimizers of E_{ϵ_k}

$\epsilon_k \downarrow 0$

➡

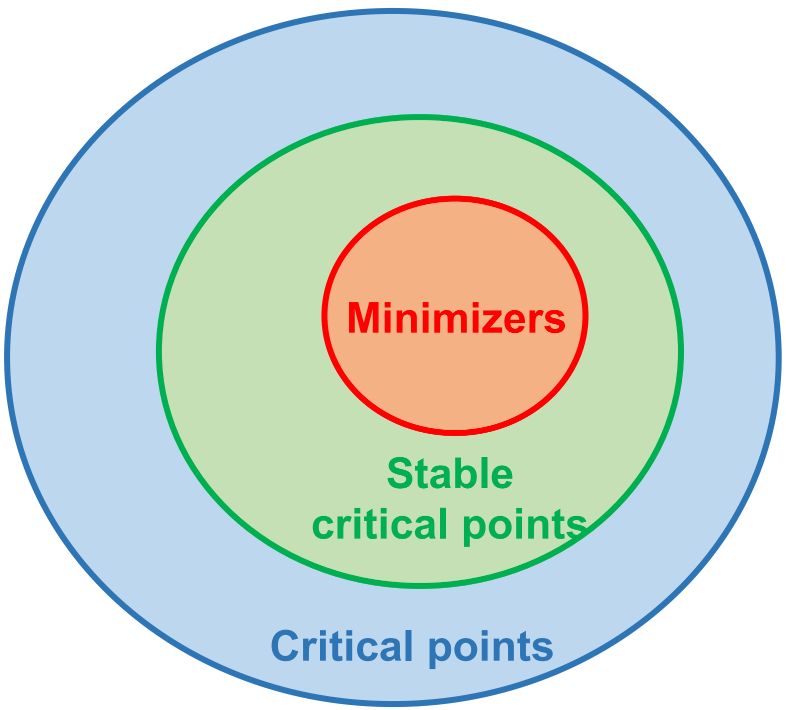
For $n \leq 7$ the 0-level sets $u_{\epsilon_k} = 0$ are smooth hypersurfaces with uniform $C^{2,\alpha}$ bounds, and their mean curvature converges to zero with precise estimates in terms of ϵ

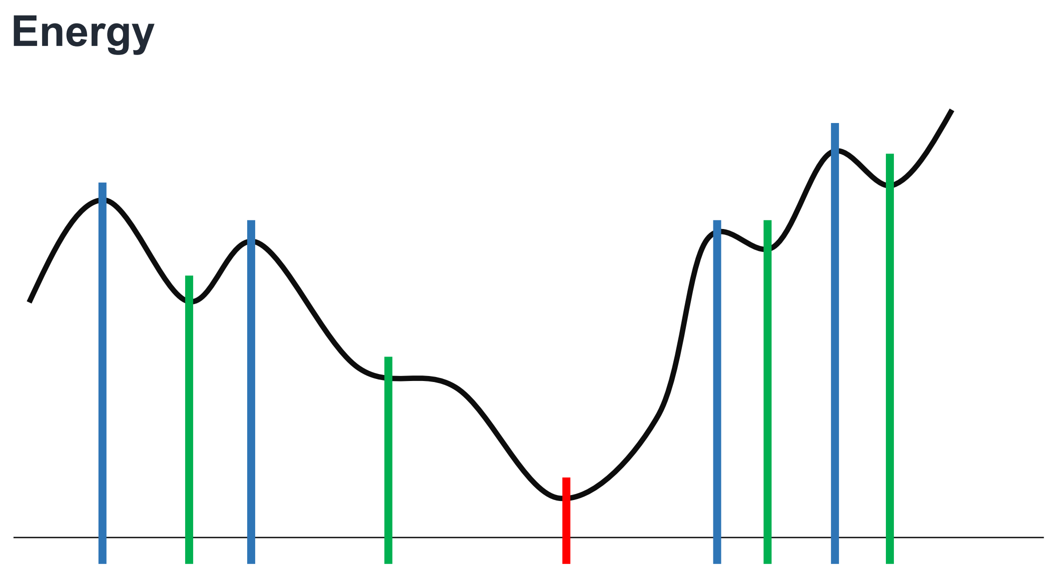


Analogous to real functions of real variables, non-convex functionals like Allen-Cahn can exhibit several types of ‘equilibria’ or critical points

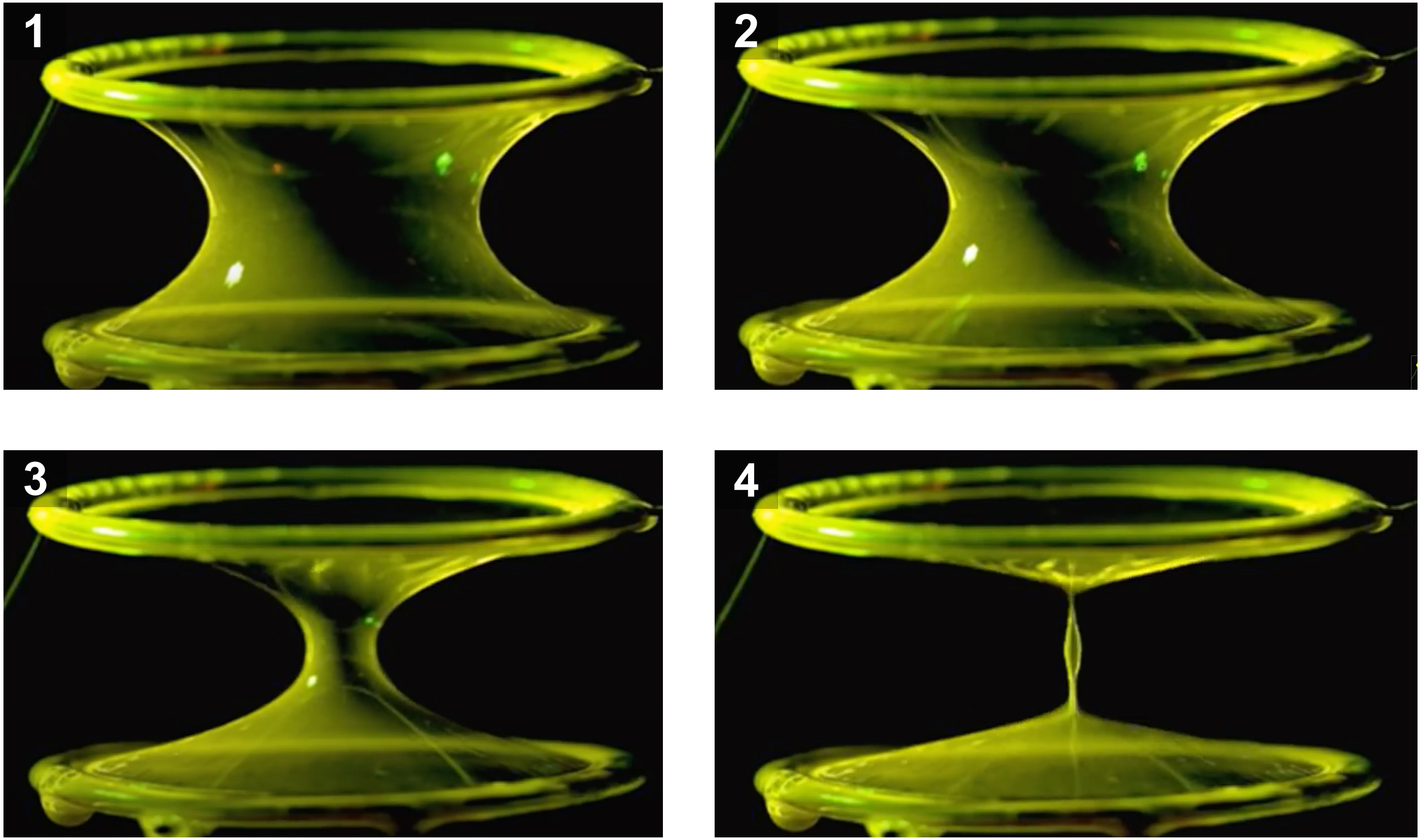
$u : \mathbb{R}^n \to \mathbb{R}$ is called…

- **Critical point** if $E(u + t\varphi) = E(u) + o(t)$ as $t \downarrow 0$ $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$
- **Stable critical point** if $E(u + t\varphi) \geq E(u) + o(t^2)$ as $t \downarrow 0$ $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$
- **(Absolute) minimizer** if $E(u + t\varphi) \geq E(u)$ $\forall t \in \mathbb{R}$ $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$





Stable equilibria (i.e., essentially local minima) are the **ones observable in nature**: minor disruptions make unstable states decay towards stable ones



Souce: University of Cambridge (youtube)

Natural question: can robust convergence of Allen-Cahn to minimal surfaces be generalized to stable critical points, at least in three spatial dimensions?

Connection to a celebrated **conjecture of De Giorgi** on monotone solutions of the Allen-Cahn equation (open in dimensions 4,5,6,7,8):

Regularity theory for stable critical points in $\mathbb{R}^n$ implies De Giorgi's conjecture in $\mathbb{R}^{n+1}$

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Very related question for minimal surfaces: **stable Bernstein-type result**

Assume $\Sigma \subset \mathbb{R}^n$ embedded minimal (hyper)surface is stable inside the ball of radius 2

In low dimensions, $n \leq 7$ is the **2nd fundamental form** of the surface **universally bounded** in the ball of radius 1?

Known to be **equivalent to Bernstein-type result**:
Must any complete stable minimal hypersurface be a hyperplane (up to dimension 7)?

$n = 3$ Do Carmo-Peng, Fischer Colbrie-Schoen, Pogorelov (1970's)

$n \geq 8$ Counterexample: Simons's cone (in Ann. Math. 1968) is minimizer by Bombieri-De Giorgi-Giusti (in Invent. Math. 1969)

Recent breakthroughs:

$n = 4$ Chodosh-Li (in Acta Math., 2023)

$n = 5$ Chodosh-Li-Minter-Stryker (preprint 2024)

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4. New results: free boundary Allen-Cahn and one phase Alt-Caffarelli

Natural question: can (robust) convergence of Allen-Cahn to minimal surfaces be generalized to stable critical points, at least in three spatial dimensions?

Partial results only in dimension 3 (essentially no results in higher dimensions):

- Ambrosio-Cabré (in JAMS, 2000). Dimension n=3, classification of entire stable solutions (epsilon =1) under energy growth assumption

IF: $u : \mathbb{R}^3 \to (-1,1)$ **stable** $\big(E_1(u) \big)$

with $\int_{B_R} \frac{|\nabla u|^2}{2} + W(u)dx \leq CR^2 \quad \forall R$

THEN: $\exists\ e \in \mathbb{S}^2 \qquad u(x) = \phi(e \cdot x)$

Natural question: can (robust) convergence of Allen-Cahn to minimal surfaces be generalized to stable critical points, at least in three spatial dimensions?

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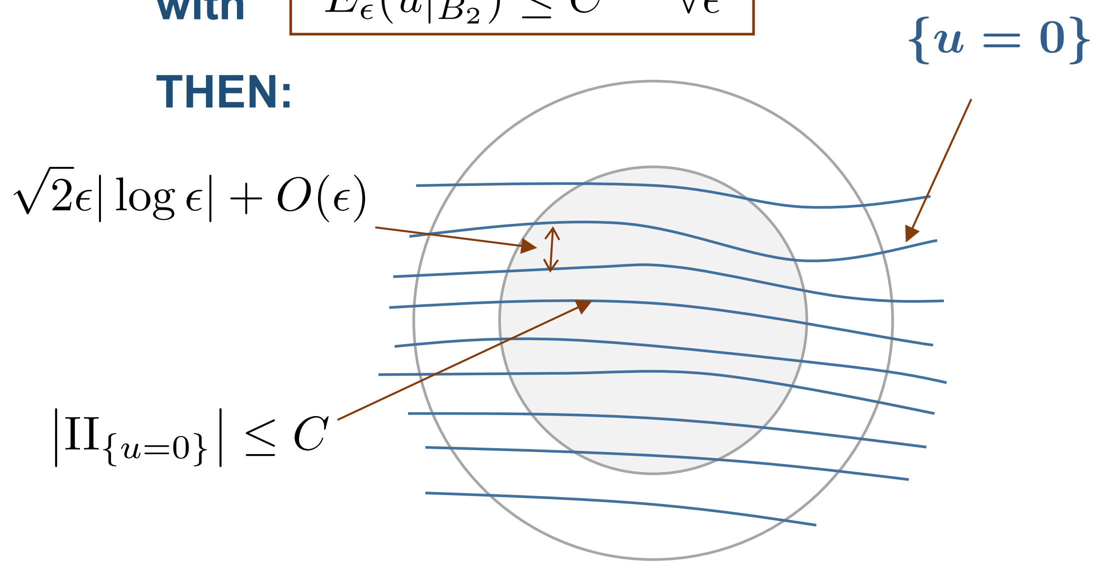
- Wang-Wei (in CPAM, 2019)
- Chodosh-Mantoulidis (in Ann. Math. 2020)

IF: $u_\epsilon : B_2 \subset \mathbb{R}^3 \to (-1,1)$ **stable**, $\big(E_\epsilon(u) \big)$

with $E_\epsilon(u|_{B_2}) \leq C \quad \forall \epsilon$

THEN:



- Precise sheet separation
- 2nd fundamental form of 0-level set uniformly bounded
- Mean curvature of 0-level set converging to zero with rate

Natural question: can (robust) convergence of Allen-Cahn to minimal surfaces be generalized to stable critical points, at least in three spatial dimensions?

Reduction to stable De Giorgi-type result

By a well-known scaling and compactness argument, the question boils down to:

Is every **stable** critical point $u : \mathbb{R}^n \to (-1, 1)$ of $\int \frac{|\nabla u|^2}{2} + W(u)dx$ $n \leq 7$

of the form $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$?

Counterexamples for $\boldsymbol{n \geq 8}$:

- Del Pino- Kowlaczyk-Wei (in Ann. Math 2011)
- Liu-Wang-Wei (JMPA, 2017)

Without assumptions on energy growth, only similar known result in dimensions >2 is for Peierls-Nabarro energy (crystal dislocations):

$$F_\epsilon(u) := \int |(-\Delta)^{1/4} u|^2 + \frac{1}{\epsilon} W(u)dx$$

Stable De Giorgi-type result by Figalli-Serra (in Invent. Math. 2020)

Natural question: can (robust) convergence of Allen-Cahn to minimal surfaces be generalized to stable critical points, at least in three spatial dimensions?

Open question we would like to address in coming years:

(A) Stable De-Giorgi for Allen-Cahn **without** energy growth assumptions in dimension 3

Is every **stable** critical point $u : \mathbb{R}^3 \to (-1,1)$ of $\int \frac{|\nabla u|^2}{2} + W(u)dx$

of the form $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^2$?

(B) Stable De-Giorgi for Allen-Cahn **with** energy growth assumptions in dimensions 4,5,6,7

Is every **stable** critical point $u : \mathbb{R}^n \to (-1,1)$ of $\int \frac{|\nabla u|^2}{2} + W(u)dx$ $n \leq 7$

satisfying $\int_{B_R} \frac{|\nabla u|^2}{2} + W(u)dx \leq CR^{n-1}$

of the form $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$?

Minimal (hyper)surface analog of (B): Complete stable minimal (hyper)surfaces with area growth assumption are flat in dimensions 3,4,5,6,7,8

- Schoen-Simon-Yau (in Acta Math. 1975)
- Schoen-Simon (in CPAM 1981)

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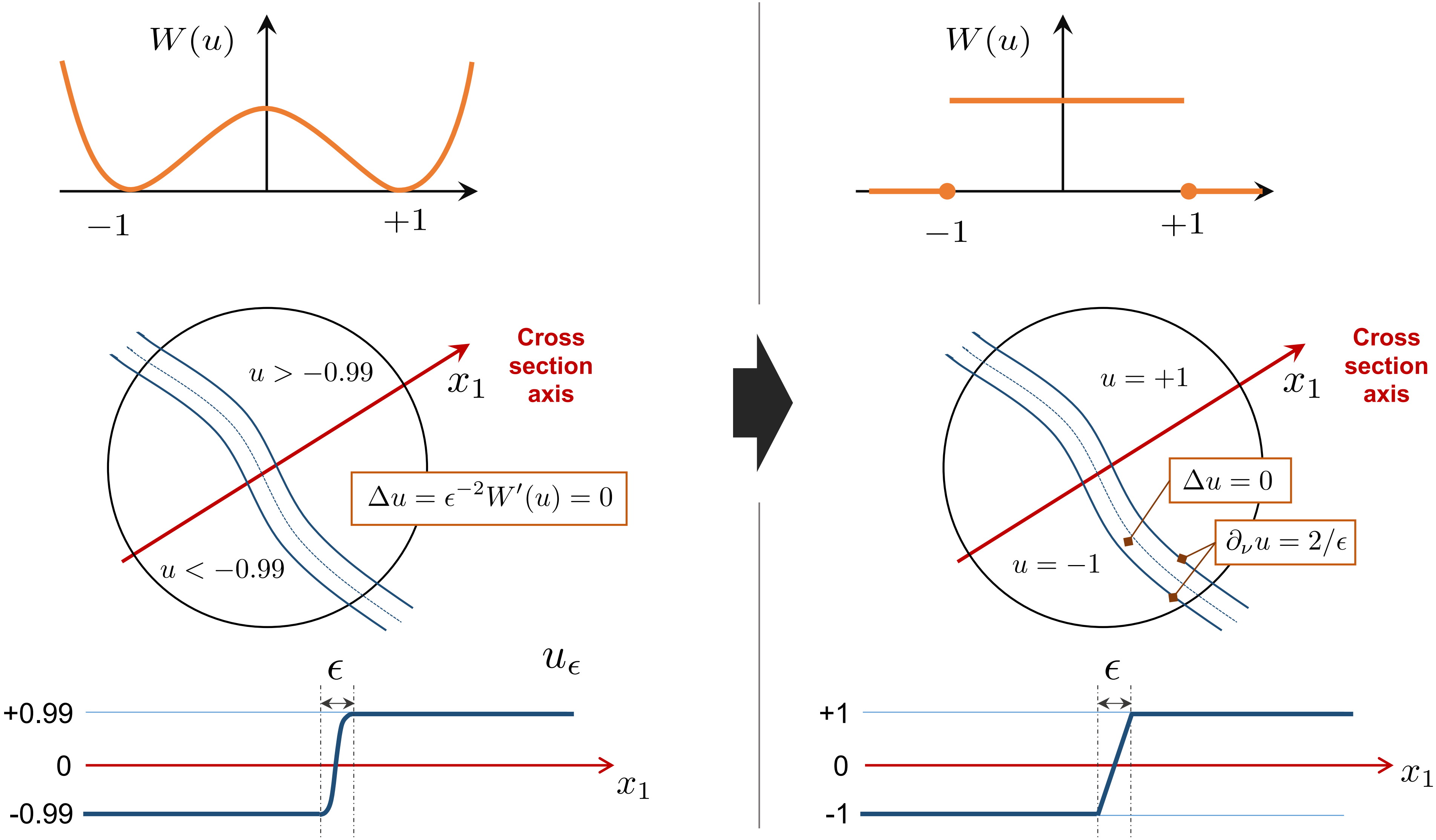
Free boundary Allen-Cahn (Jerison, Kamburov, Liu, Wang, Wei,…)

$$E_\epsilon(u) := \int \epsilon \frac{|\nabla u|^2}{2} + \frac{1}{\epsilon} W(u)\,dx$$

Recall potential term's purpose:
penalize intermediate states not close to ± 1



In forthcoming work (with Chan, Fernández Real, and Figalli) we establish the stable De Giorgi-type result for free boundary Allen-Cahn in 3D

THEOREM (Chan-Fernández Real-Figalli-Serra, 2024)

Every **stable** critical point $u : \mathbb{R}^3 \to (-1,1)$ of $\int \frac{|\nabla u|^2}{2} + W(u)dx$

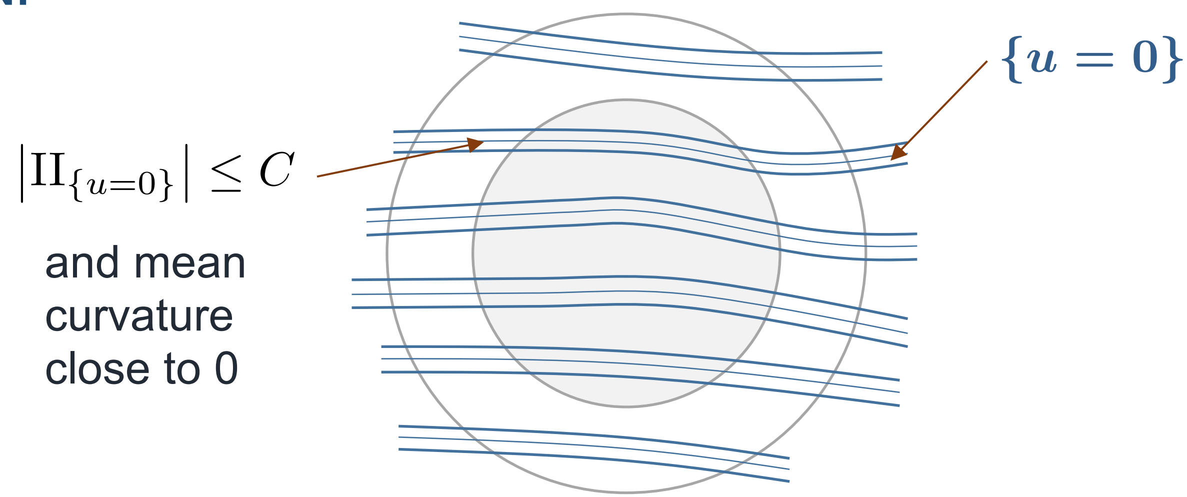
for $W(u) = \mathbf{1}_{(-1,1)}(u)$ must be of the form $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^2$

COROLLARY:

IF: $u_\epsilon : B_2 \to (-1,1)$ **stable** critical point of $\int_{B_2} \epsilon|\nabla u|^2 + \frac{1}{\epsilon}\mathbf{1}_{(-1,1)}(u)\,dx$

in $B_2 \subset \mathbb{R}^3$ **(with no assumptions on energy)**

THEN:

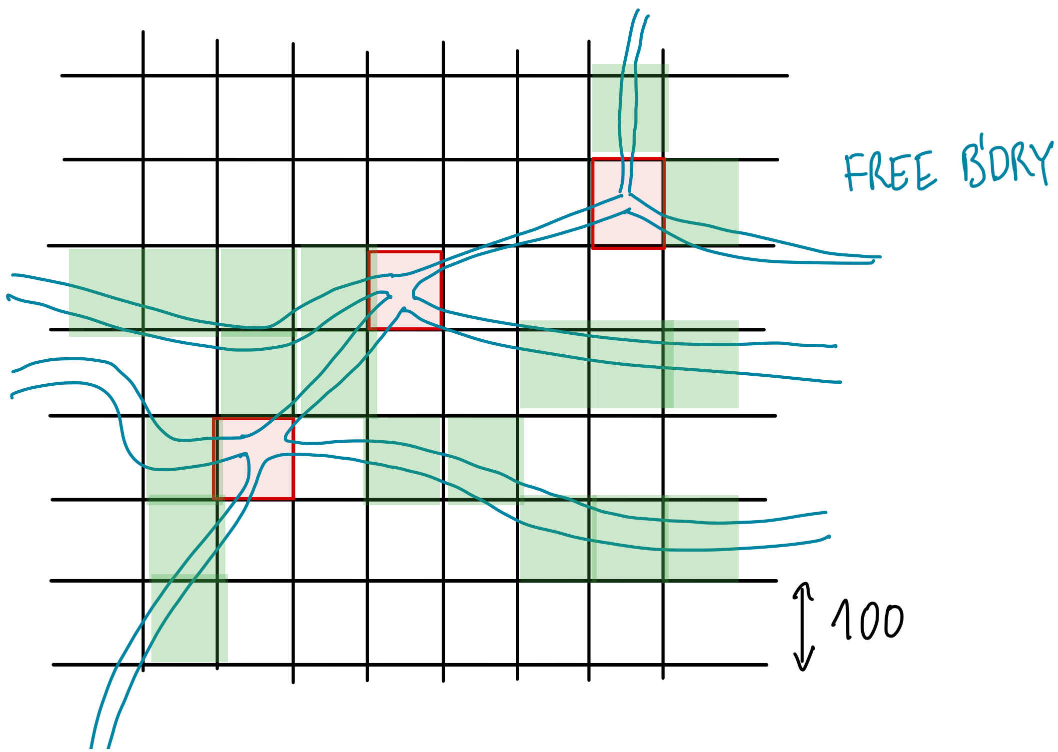


Rough idea of the proof

$u : \mathbb{R}^3 \to (-1,1)$ **stable** critical point of $\int \frac{|\nabla u|^2}{2} + W(u)dx$ $\quad W(u) = \mathbf{1}_{(-1,1)}(u)$

Stability condition
(Sternberg-Zumbrun) : $$\int_{\{|u|<1\}\cap B_{R(x_0)}} |D^2 u|^2 \le \frac{C}{R^2}\left|\{|u|<1\}\cap B_{2R(x_0)}\right|$$

Covering of $\{|u|<1\}$ **with 'good' and 'bad' cubes:** LHS (in cube) $\begin{cases} < \eta_\circ \\ \ge \eta_\circ \end{cases}$



- Each **'bad'** cubes contributes to the LHS of the stability, so we can bound their number (relatively to the energy)
- If we can show that in **'good'** cubes the zero-level set and free boundaries are approximately minimal surfaces, then we can try to modify the classical proof of area growth for stable minimal surfaces

But what could happen inside '**good**' cubes?

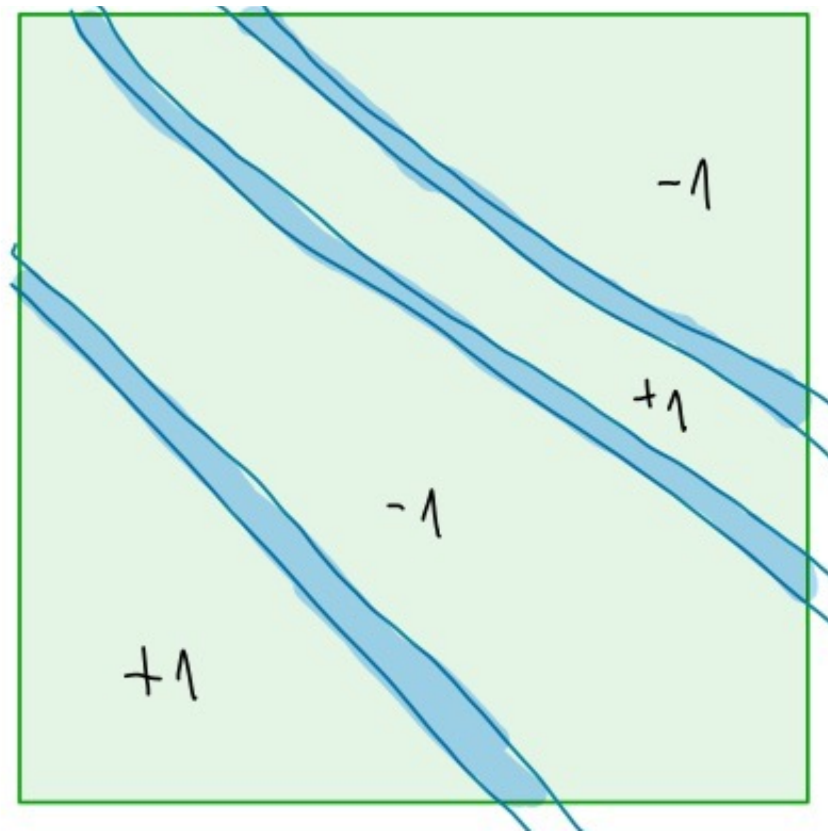
$$\int_{\{|u|<1\}\cap\text{cube}} |D^2u|^2 < \eta_\circ \ll 1$$

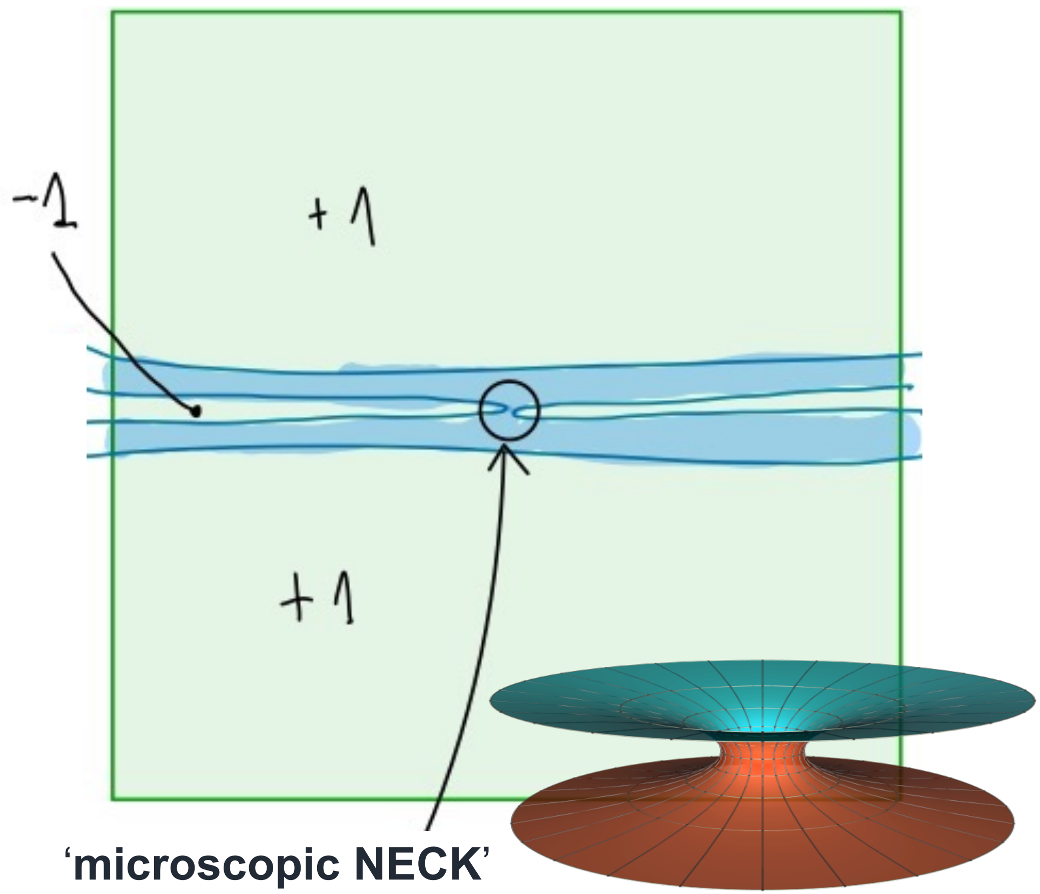
Desired scenario:

- Curvature of free boundary small
- Different components of $\{|u| < 1\}$ do **not** interact

A priori possible scenario:

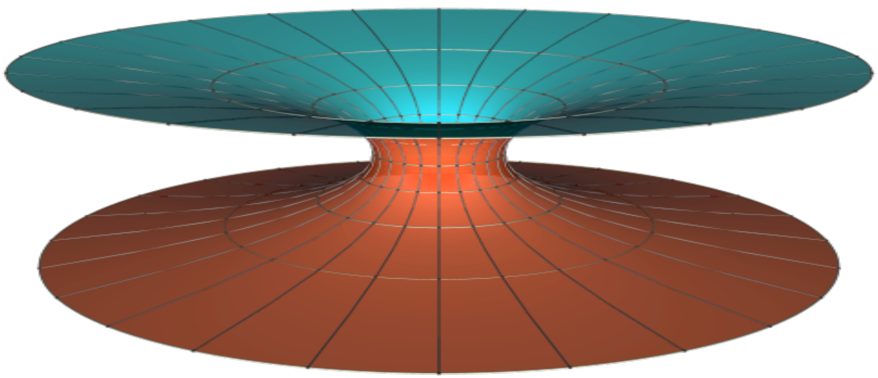
- Highly curved 'microscopic necks'
- Different components of $\{|u| < 1\}$ do interacting through the necks



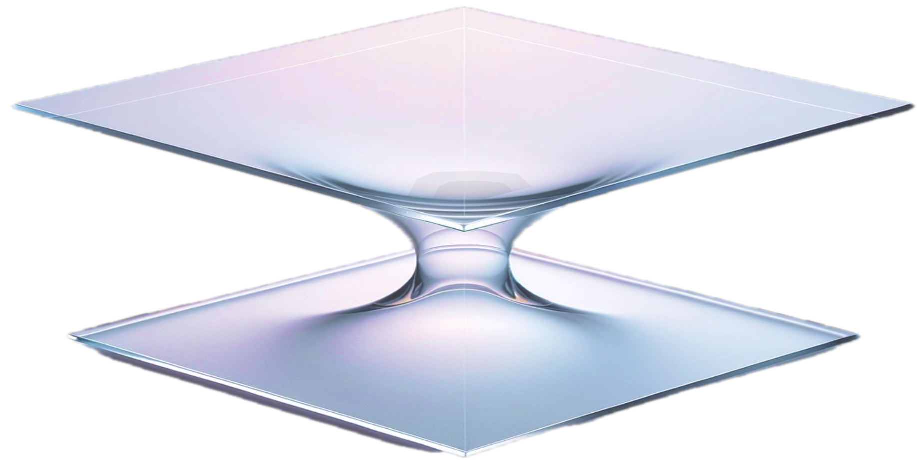


'microscopic NECK'

ILLUSTRATION OF NECK



Matthias Weber

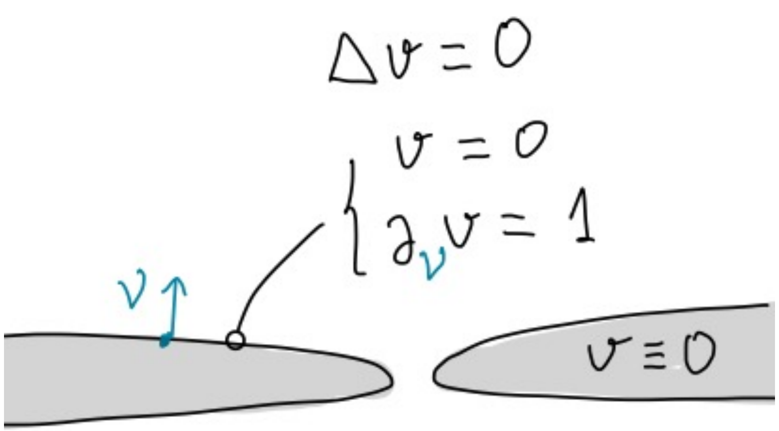


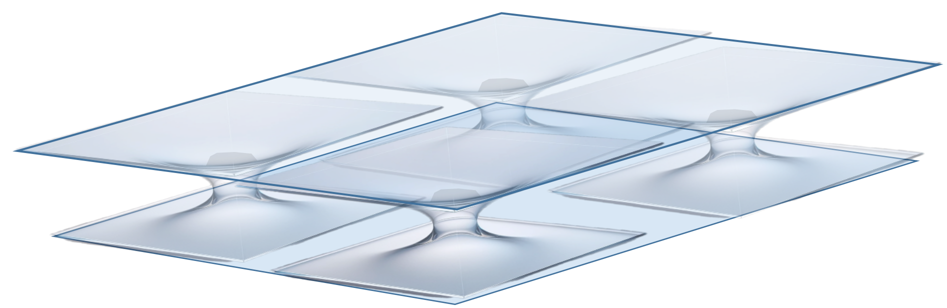
ChatGPT4.0

Zooming in at the neck we discover what are the possible models of necks

$$v_\varrho(x) := \frac{u(\varrho x) - 1}{\varrho} \qquad \left(\text{or} \quad v_\varrho(x) := \frac{-u(\varrho x) + 1}{\varrho} \right) \qquad \varrho \ll 1$$

We obtain **stable solutions** $v : \mathbb{R}^3 \to [0, +\infty)$ of the **Alt-Caffarelli problem:**





Explicit solutions to the PDE with necks can be found even in 2D
They are unstable, but they have finite Morse index!

In 3D, it is easy to conceive two planar like free boundaries connected by countably many necks with varied sizes… and ruling out such scenario using stability is hard!

In a forthcoming work (with Chan, Fernández Real, and Figalli) we prove that stable solutions of Alt-Caffarelli in 3D must have flat free boundaries

THEOREM (Chan-Fernández Real-Figalli-Serra, 2024)

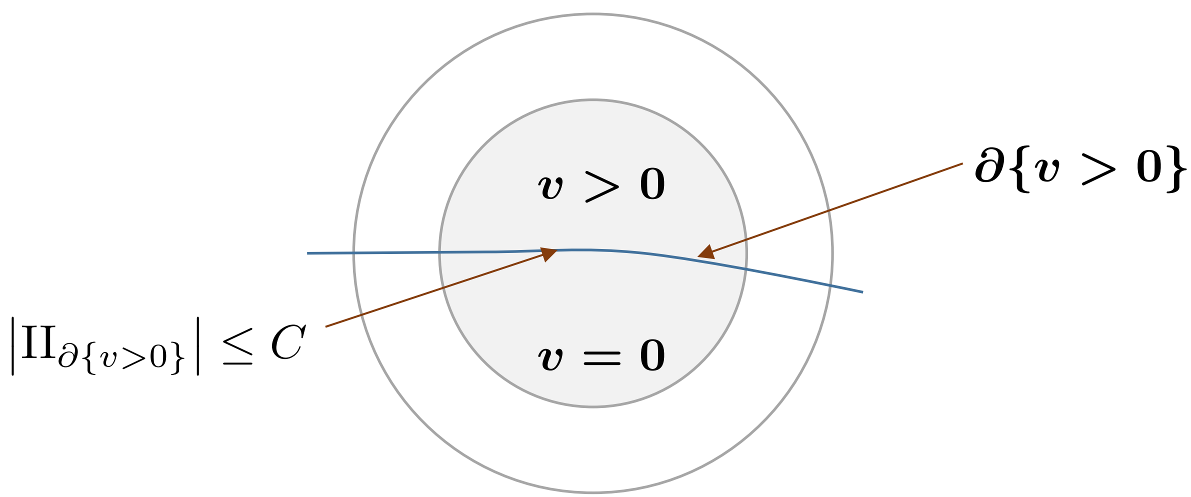
Every **stable** critical point $v : \mathbb{R}^3 \to [0, +\infty)$ of $\int |\nabla v|^2 + \mathbf{1}_{(0,+\infty)}(v)\, dx$

must be of either form: $v(x) = (e \cdot x - a)_+ \qquad e \in \mathbb{S}^2 \qquad a \in \mathbb{R}$

$v(x) = (e \cdot x - a)_+ + (e \cdot x - b)_- \qquad e \in \mathbb{S}^2 \qquad a, b \in \mathbb{R} \qquad a \geq b$

COROLLARY: **IF:** $v : B_2 \to [0, +\infty)$ **stable** critical point of Alt-Caffarelli in $B_2 \subset \mathbb{R}^3$

THEN: dimensional curvature estimates in B_1




Thank you for your attention