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High-dimensional cohomology of arithmetic Chevalley groups

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Summary

Understanding the cohomology of arithmetic groups such as $\mathrm{SL}_n(\mathbb{Z})$ or $\mathrm{Sp}_{2n}(\mathbb{Z})$ is a fundamental problem, which connects many areas of mathematics. It is motivated by questions in number theory, has applications in algebraic K-theory, and is closely related to the cohomology of moduli spaces such as $\mathcal{A}_g$. However, computing these invariants is notoriously difficult – even in the simplest case: the rational cohomology. Low cohomological degrees are accessible by classical homological stability techniques and computer calculations, but little is known in high degrees. In this project, we study these high-degree rational cohomology groups for a well-behaved class of arithmetic groups, namely Chevalley groups over number rings, such as

$$\mathcal{G}(\mathcal{O}) \in \{\mathrm{SL}_{n+1}(\mathcal{O}), \mathbb{P}\mathrm{GL}_{n+1}(\mathcal{O}), \mathrm{Spin}_{2n+1}(\mathcal{O}), \mathrm{Sp}_{2n}(\mathcal{O}), \mathrm{SO}_{2n}(\mathcal{O})\}$$

for $\mathcal{O}$ the ring of integers of a number field $\mathbb{K}$. In particular, we investigate the following generalization of a conjecture by Church–Farb–Putman.

Vanishing conjecture in high degrees

$H^{\mathrm{vcd}-i}(\mathcal{G}_n(\mathcal{O});\mathbb{Q}) = 0$ if $i \leq n-1$ and $\mathcal{O}$ is Euclidean, i.e. the rational cohomology of $\mathcal{G}_n(\mathcal{O})$ is as in Figure 1.

State of the art: What is known ...

... for the case $\mathcal{G}_n(\mathcal{O}) = \mathrm{SL}_{n+1}(\mathbb{Z})$?

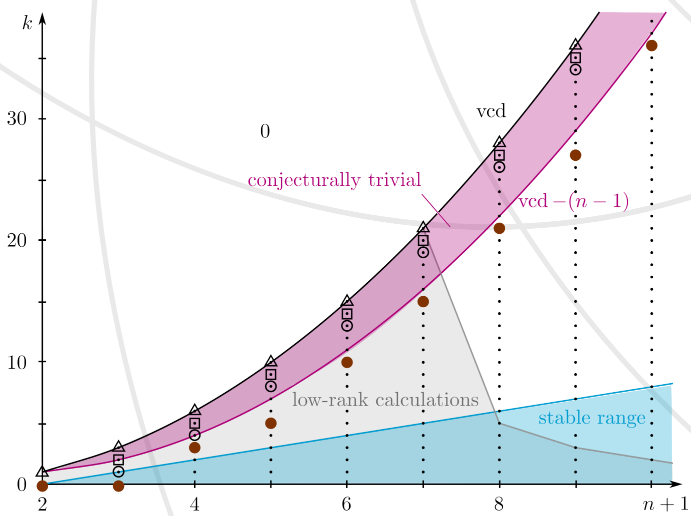


Figure 1: $H^k(\mathrm{SL}_{n+1}(\mathbb{Z});\mathbb{Q})$. Results in magenta see Table 1; remaining results due to Borel, Brown, Dutour Sikirić, Elbaz-Vincent, Gangl, Kupers, Lee, Martinet, Soulé, Szczarba.

... for other arithmetic Chevalley groups $\mathcal{G}_n(\mathcal{O})$?

Group $\mathcal{G}_n(\mathcal{O})$	$i=0$	$i=1$	$i=2$
$\mathrm{SL}_{n+1}(\mathbb{Z})$	Yes	Yes	**Yes**
$\mathrm{SL}_{n+1}(\mathcal{O})$ $\mathcal{O}$ Eucl.	Yes	Yes for $\mathcal{O} \in \{\mathcal{O}_{-1}, \mathcal{O}_{-3}\}$	?
$\mathrm{SL}_{n+1}(\mathcal{O})$ $\mathcal{O}$ not Eucl.	No if $\mathcal{O} \neq \mathcal{O}_{-19}$	?	?
$\mathrm{Sp}_{2n}(\mathbb{Z})$	**Yes**	**Yes**	?
$\mathrm{Sp}_{2n}(\mathcal{O})$ $\mathcal{O}$ not PID	**No**	?	?
$\mathcal{G}_n(\mathcal{O})$ $\mathcal{O}$ Eucl.	**Yes for $\mathtt{A}_n$, $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{D}_n$, $\mathtt{E}_6$, $\mathtt{E}_7$**	?	?

Table 1: Whether the vanishing conjecture is true. Results due to **Brück**, Church, Farb, Himes, Kupers, Lee, Miller, Patzt, Putman, Santos Rego, **Sroka**, Szczarba, Wilson, Yasaki.

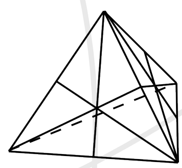
Research objectives

1. Show the following cases of the vanishing conjecture:
   (a) $\mathcal{G}_n(\mathcal{O}) = \mathrm{SL}_{n+1}(\mathbb{Z})$ and $i \leq 4$.
   (b) General $\mathcal{G}_n(\mathcal{O})$ and $i \leq 1$.
2. Use 1.(a) to show that $K_{12}(\mathbb{Z}) = 0$.

Understanding the Steinberg module of $\mathcal{G}_n(\mathcal{O})$

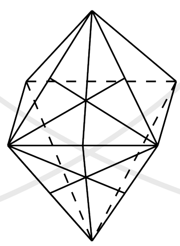
Work of Borel–Serre implies that $\mathcal{G}_n(\mathcal{O})$ satisfies a twisted version of Poincaré duality in rational cohomology,

$$H^{\mathrm{vcd}-i}(\mathcal{G}_n(\mathcal{O});\mathbb{Q}) \cong H_i(\mathcal{G}_n(\mathcal{O}); \mathrm{St}_{\mathcal{G}_n(\mathcal{O})} \otimes \mathbb{Q}).$$

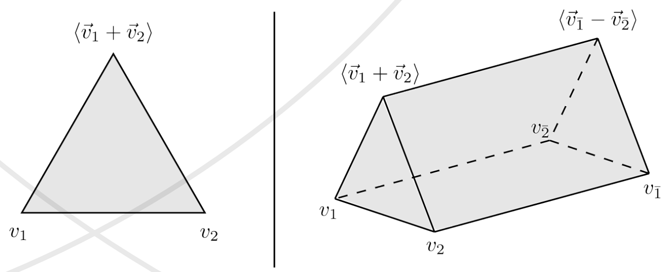


The dualizing module $\mathrm{St}_{\mathcal{G}_n(\mathcal{O})}$ is called the *Steinberg module*. The vanishing conjecture hence follows if there is a partial flat resolution

$$C_{n-1} \to C_{n-2} \to \cdots \to C_2 \to C_1 \to C_0 \to \mathrm{St}_{\mathcal{G}_n(\mathcal{O})} \otimes \mathbb{Q} \to 0$$



with the property that $(C_i)_{\mathcal{G}_n(\mathcal{O})} = 0$. Constructing it requires a good understanding of $\mathrm{St}_{\mathcal{G}_n(\mathcal{O})}$ in terms of a suitably generating set ($i = 0$), presentation ($i = 1$) and higher syzygies ($i > 1$). We plan to follow ideas of Church–Farb–Putman and use that $\mathrm{St}_{\mathcal{G}_n(\mathcal{O})}$ is the top-degree homology of a spherical building. This simplicial complex is built out of spheres, called apartments, whose combinatorics depends on $\mathcal{G}$. Our approach to the Research objectives is: Construct polyhedral complexes that encode information about the generators, relations and higher syzygies; prove that these are highly connected; use homological algebra to extract the desired partial resolution.



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