Postdoc Paper Talk

Welcome to the Postdoc Paper Talk of Mathematics. Münster. In this series, postdocs from the Cluster of Excellence talk about their research. We would like to give an insight into the research of our more than 150 early career scientists.

In this interview from November 2024, we will learn about the research of Ksenia Fedosova. Ksenia completed her doctorate at the University of Bonn, then worked as a postdoctoral researcher at Chalmers University of Technology and the University of Freiburg, studying the field of global analysis and spectral geometry. Since 2023, Ksenia has been a postdoc with us at Mathematics Münster in the research group of Professor Ursula Ludwig.

Professor Christopher Deninger, professor of Mathematics at the University of Münster, talks with Ksenia about her paper recently published in the *Proceedings of the National Academy of Sciences*.

Ksenia Fedosova, Kim Klinger-Logan, Danylo Radchenko (2024): Convolution identities for divisor sums and modular forms. Proceedings of the National Academy of Sciences (PNAS), Vol. 121, No. 44
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Introduction to Ksenia's paper

Christopher Deninger: Hi, Ksenia. So very nice to interview you. Maybe you could tell our audience a little bit more about what fields you are researching and what your core topics are.

Ksenia Fedosova: I'm working in the field of a global analysis mixed with number theory and mathematical physics. Global analysis is a field about differential equations on manifolds and in particular spectral information about these differential equations. Or if we say this in simpler words: "If we punch a manifold, at which frequencies will it scream back at you?".

CD: Resonating... Can you hear the shape of the drum?

KF: Here we are not hearing shapes. For this particular topic, we're not going to be hearing shapes of the drum. We are going to be hearing the shape of one very particular orbifold, which is beloved by number theorists, differential geometers and string theorists as well. We are going to be talking about the the modular surface.

CD: Oh, that sounds lovely. I'm principally a number theorist. And on the other hand, I'm very interested in the spectral geometry and I'm very curious. So I know that you have recently published in the prestigious journal, the "Proceedings of the National Academy of Sciences"with coauthors. And I even know the title of the article, because you just told me: "Convolution of divisor functions and modular forms". I'm curious. That's a good occasion to learn what is in the article. What is it all about? And maybe also who are your coauthors?

KF: The coauthors are Kim Klinger-Logan, she's from the University of Kansas and the other one is Danylo Radchenko. He is from Lille. So the paper has a motivation in the mathematical physics and it should help us understand the scattering process of gravitons. It means we have two gravitons coming in, two gravitons coming out. We want to study what is happening in this black box in the process in between. After some complicated calculation of Feynman diagrams and so on, we are coming to something which is more down to earth, namely to our lovely modular surface. And in particular to a partial differential equation on top of it. So it is a Laplace type equation on an unknown function f with an inhomogeneous part, which is a product of two Eisenstein series.

$$(\Delta - r(r+1))f(z) = E_a(z)E_b(z) \tag{1}$$

where f is an unknown function, $E_a(z), E_b(z)$ are non-holomorphic Eisenstein series with $r \in \mathbb{N},\ a, b \in \mathbb{N}_0 + \frac{1}{2}$ and

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \tag{2}$$

is the Laplace operator on a modular surface.

Eisenstein series are almost eigenfunctions of the Laplace operator on this surface, but not really because they are not square integrable.
This is a problem which is very down to earth. A partial differential equation problem arising in string theory, surprisingly. But then in a different paper, which is a pre paper to this one, we started with Kim Klinger-Logan the problem of figuring out what this function could be for various values of our parameters.

Properties of the equation

CD: Does someone tell you that this equation actually has a solution?

KF: We know by the work of Kim Klinger-Logan, her PhD thesis, that for some mild restrictions on the parameters there always exists a solution. One can do this with the help of spectral methods. And then later on, in a joint paper with her, for certain values for r, a and b we have found a particular solution of this problem. We have about 6000 different values of r,a and b for which we found the solution.

CD: But do we always know that there are solutions?

KF: Not at the moment. It is a work in progress. We hope to be able to.

CD: Are you also interested in the multiplicity of the solution space or the dimension? Or are you happy with one solution, or is there a reason that it is always one dimensional?

KF: That is a very good question. So what do we do with this equation? We do what Analysis II teaches us. From partial differential equation, we create a system of ordinary differential equations.

CD: How can you do that?

KF: There is a Laplace operator, a non-hyperbolic Laplacian. And the non-holomorphic Eisenstein series admit a Fourier expansion in the z-variable.

CD: And then you write out the q-expansion and try to find the coefficients.

KF: Yes. You write q-expansion and then on the right hand side you get a convolution.

CD: And then you get of course ordinary differential equations for all the coefficients.

KF: And in fact we already have one of one of our keywords here: convolution. We actually have the second keyword: divisor functions. Because if we look at the Fourier expansion of non-holomorphic Eisenstein series, we will have Bessel functions as a function of y, decorated by divisor functions. So divisor functions would be coefficients. Then we have this PDE and we have created some system of ODEs and this is going to be an inhomogeneous modified Bessel equation. And the right hand side would be a product of modified Bessel functions. So here I will not try to use that expression because I will mess up the signs most probably. But here, the n_1 part would correspond to the Fourier expansion of E_a

CD: Are there any singularities?

KF: It behaves weird. It behaves weird at zero and then at infinity.

CD: Are these regular singularities. Or irregular?

KF: I'm not so sure, but I don't think this is really important because we provide particular solutions.
So this ODE has a two dimensional space of solutions. Out of which we can kill one immediately, because physicists do not want to have solutions that grow too fast as $y \to \infty$.

CD: Number theorist don't want that, too.

KF: Very good. So now we can throw away one dimension and for the second dimension we can fix the solution precisely by demanding that in the original problem f would be automorphic. So the first degree of freedom is killed by the behavior at infinity and the second degree of freedom is killed by demanding automorphy.

CD: Automorphy for you means what?

KF: Automorphy means that $f(\gamma z) = f(z)$ for any $z \in \mathbb{H}$ and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$.

CD: Which is Γ-invariance.

KF: Yeah, Γ-Invariance. We do not want any extra factors in front of it.
Now the paper in question is, about how to choose an homogeneous solution to this PDE so that the result becomes automorphic and the whole idea of the approach is that by some physical consideration we have a fixed behavior at infinity. They have some restrictions on the zeroth Fourier terms from the number theoretical perspective and from a physical perspective, we do not want it to be growing exponentially. So the modified Bessel function of the first kind are not present anymore and the zeroth Fourier term is also fixed by some physical consideration, that I do not want to discuss too much. But now, if we have a function which grows at a certain rate when it goes to infinity and if it's automorphic, then it should exhibit some not so bad behavior as $y = 0$.

CD: Yeah, sure, by invariance...

KF: Yes. And then from this it turns out that we can write down explicitly the behavior as $y \to 0$ of this solution f and by some miracle, it turns out that this sum is connected to the behaviour at 0

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}\setminus\{0\}}} Q_d^{(2r_1,2r_2)}\left(\frac{n_2-n_1}{n_2+n_1}\right)\sigma_{r_1}(n_1)\,\sigma_{r_2}(n_2) \tag{3}$$

where σ_{r_1} and σ_{r_2} are divisor functions.

CD: And this entire sum is a convolution type?

KF: Yes, exactly.

CD: And the coefficients $Q_d^{(2r_1,2r_2)}$? What are they?

KF: It turned out, after about one year and a half of looking at it, that these Jacobi functions of the second kind.

CD: Okay.

KF: However, this is this is not really important for us. Because after I look at the paper, I could give you an integral representations for those, but they are not interesting to anyone. I can give you an example. I can tell you that $Q_d^{(0,0)}(1)$ has very easy form. It is an elementary function in n_1 and n_2, which only includes rational functions of n_1 and n_2 and logarithms. And so in fact, for any r_1 and r_2 as in the paper and the Jacobi function of the second kind will always be an elementary function in n_1 and n_2.

The physicists' conjectures

CD: So you have a relatively explicit expression. And what are you doing? Are you proving that this actually solves the equation?

KF: So we wanted our function as it goes to zero to be of not such a bad growth. And after we find the particular solution, the expansion as $y \to 0$, this particular solution looks like this convolution sum (3). It turns out that this sum appears as a coefficient in the y-expansion. However this solution is too singular. So what physicists were saying is, that maybe this whole expression just equals to zero because they didn't want the solution of the original PDE to have any homogeneous solution.

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}\setminus\{0\}}} Q_d^{(2r_1,2r_2)}\left(\frac{n_2-n_1}{n_2+n_1}\right) - \ldots = 0 \tag{4}$$

So they were conjecturing. So they came with one particular sum for parameters $a, b = \frac{3}{2}$ and $r = 3$ and conjectured that it is equal to 0.

CD: So some physical interpretation or expectations about their model tells them solutions of this type should be trivial. But from the explicit calculations it is not at all obvious that they are trivial. They are these very interesting, complicated convolutions.

KF: Exactly.

CD: ...involving some number theoretical functions like divisor functions. I guess the question is to prove that they vanish.

KF: So physicists were telling us: "Yeah, of course everything will work out for for all the values r, a and b the sum should be just equal to zero".
Of course I'm lying to you when I'm writing the sum is equal to zero, because one should extend appropriately to $n_1 = 0$ or $n_2 = 0$.

But this particular extension is not so bad. For example in the simplest case where we deal with this particular weight:

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}\setminus\{0\}}} \left(2 - \frac{n_2 - n_1}{n_2 + n_1} \cdot \log\left|\frac{n_2}{n_1}\right|\right) \sigma_0(n_1)\sigma_0(n_2) - \sigma_0(n)\left(2 - \log(4\pi^2 n)\right) = 0 \quad (5)$$

CD: And this is something that you calculate or this is something that's predicted?

KF: This is something that we proved. But it was predicted.

CD: So it's not really zero, but if you interpret the contribution from zero as this term, which you wrote down as the right hand side, it becomes zero?

KF: Yes. And this part is in a way an extension of this term to and $n_1 = n_2 = 0$.

CD: So this is kind of fascinating. And how do you do it?

KF: Oh, okay. So how do you prove that it's zero? So a wise men once told me that if you see...

CD: I'm curious. How do you prove it? So these are coefficients in some expansion, do you prove individually that the coefficients vanish or do you prove that by some a priori symmetry reasons, the entire function vanishes?

KF: Uh, I don't understand the question. I'm sorry.

CD: These are the coefficients of some function? Do you prove directly that the function vanishes or do you prove that the individual coefficients vanish? What's the strategy?

KF: So the way to think about this problem is, that this is the n-th Fourier coefficient of a holomorphic projection of certain weighted Eisenstein series. So once again, I think this is just a pure miracle. So the method of proving this thing is that in this case, I do not want say lucky guess, but you take some non-holomorphic modular function, you take its holomorphic projection and then there's a theorem that connects the Fourier series of the one via the Fourier series of another. And then it turns that this is a holomorphic projection of the product of two weighted non-holomorphic Eisenstein series with some very technically detailed chosen weight.

CD: Now you have proved that the projection is zero, yeah?

KF: The interesting part that it's not always zero. In the example that I have shown here, it is zero. But, I should have said that this is a Fourier coefficients of cusp form weight 4.

CD: This is of course extremely interesting. Cusp forms of weight one, by Langland, are connected to non-abelian extensions...

KF: So the physicists were telling us that this should vanish. But in fact they were they were looking at, the cusp forms of weight 8, so they couldn't see anything. But once we started to work with higher weights and bigger parameters we began to see some weird behavior in there in the remainder term that couldn't be blamed on various computer programs or, you know, my lack of programing skills. So, the in the end, fast forward, it turned out that the real part on the right hand side, should be the Fourier coefficient of a certain cusp form multiplied by values of it's L-functions at special points.

CD: Oh that's very nice. Are these, by the way, values of odd integers or even?

KF: It's even and odd. We kind of need both. We need both periods for this. And the real funny part was that these numbers in the L-functions depend on are the parameters a and b, but you know you can usually you have an $L6$, you can make an $L8$ out of this from Manin's theorem. And then the interesting part is, that it turns out the physicists were first extremely unhappy about the result, because they were thinking that their wonderful function should be not having a homogeneous part. But then they started to be happy, because it turns out that, instead of taking the solutions of this partial differential equations, they needed to take certain linear combinations of such. Because of this very particular right hand side, they know now how to combine these linear combinations and which coefficients should stand in front of those linear combinations.

CD: So very surprising result. And I must say that the intuition of this physicist was quite good, I mean, not good enough. You you had to correct it. But essentially they were on the right path, especially in certain weights, I guess.

KF: Yes. Absolutely.

CD: So this is really remarkable. And was it string theory or what branch of mathematical physics was it?

KF: String theory. And it was actually so bizarre because I was asking several physicists how they came up with this idea and a couple of them were telling me that it has something has to do with the AdS/CFT correspondence and so on and so on. But then, you get to know them better and then they said: we just substituted, because this expression looks too complicated. It should have been zero. And then the expression turned out to be zero...

CD: That's really crazy. But it is also a fascinating source of of conjectures, that no mathematician would ever come up with.

KF: I do not think that. I mean, figuring out these formulas outside of the setup, that one would have...

CD: And expecting any good answers. It's not something one would do.

KF: Not to my knowledge. And then we basically only knew that these were Jacobi functions of the second kind. Because they have this very specific, the best behaviour as the parameter goes to infinity. And we only knew that they would be the Jacobi functions, because we have considered the tons of examples that physicists have given to us. So I was extremely happy that it worked out in the end and that it wasn't a crazy mixture of values of functions.

CD: It could have been much more terrible. It looks like a real adventure. How long did it take you to figure this all out? To write this paper from start to end?

KF: You know, there are two more papers that were coming before this paper. One is by Stephen D. Miller, Kim Klinger-Logan and Danylo Radchenko, where in this particular case that was considered by physicists they came with a non-proof of the result which is from 2022. Then the joint result with Kim about these examples was in 2022. The non-proof of this result. With non-proof, I mean actually not really a proof, but trying to figure out what the boundary values would look like if we do some operations with derivatives outside of the domain of their convergence. I think this paper appeared on arXiv in December 2023.

CD: It did not take you so long to discover all these mysteries, but I guess it must have been a lot of calculations and computations.

KF: Oh, it is a lot. But I mean, on the other hand I'm also blessed with my extremely patient and absolutely brilliant collaborators.

Collaboration with Don Zagier

CD: Very good. Thank you. Now I think I have a pretty good understanding of what is in this paper and what it's all about. How it came about and the dramatic story of its discovery. I want to ask you something else. You once told me you collaborated with Don Zagier? What is the topic there? Is it related or is it something different?

KF: So, we allegedly have a more or less closed formula for this solution independent on values a, b, r.

CD: With Don?

KF: With Don, yes. What I mean by more or less closed formula, is that, it is a solution which is obtained recursively from easy defined functions, basically some Jacobi polynomials.

CD: I see. Did Don have some experience with this type of problems or was it completely new and easy to him?

KF: Oh, no. So when I was first giving a talk in 2024 about the first project that we did with Kim. Don Zagier was coming and saying: "No, this is too complicated". Then the next day, he's coming and saying: "I need to have exact solutions". The next day, he's saying: "I need, more solutions in order to make numerical calculations". And then, he disappears for five months and comes with 50 more files and say: "I think I found an abstraction to your to existence of the solution in a nice format". This abstraction is written with the help of Rankin–Cohen brackets.

CD: So that's kind of fantastic.

KF: This is kind of fantastic. I agree with you.

CD: Well, he's probably the biggest expert in the world for modular forms.

KF: Yeah. And then he saw the proof was saying: "I told you, I told you two years ago you should have done holomorphic projections all along". You see, in a way of attacking this problem, we had like a slightly different approach in mind as well. So we wanted to rewrite this in terms of trace formula. And the conjecture was that the right hand side, what we knew numerically it was corresponding so close to the hexagonal forms which are appearing

in many trace formula. So we had a conjecture that everything else would just vanish by some magic, but no.

CD: And did you follow this up?

KF: It is a project in progress. Because at the moment we only can deal with convolutions of even divisor sums. But it would be also interesting to take a look what's happening with the odd case. Because it is another type of our solution that appear in string theory. And they are known as modular graph functions.

CD: Never heard it...

KF: Weird stuff. But here instead of our Eisenstein series of half integers they have Eisenstein series of integers. Then in this case solution of the PDE becomes way simpler, because the right hand side is a combination of modified Bessel, which in this case can simplified to elementary functions. And then like the Fourier series follows and so on.

CD: Very good. Thanks a lot for these very enlightening, explanations. Really fascinating topics that you are working on. Fascinating results.

KF: Thank you.

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