GEOMETRIC REPRESENTATION THEORY AND p-ADIC GEOMETRY

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Abstract. We discuss the number theoretic origins of the Langlands program, its geometrization and categorification over function fields, and more recently over p-adic fields by Fargues–Scholze. We conclude by describing some of our own contributions to the emerging field and possible future directions.

1. Reciprocity laws

One of the most celebrated theorems of Gauss [Gau86] is the quadratic reciprocity law. It states that for two distinct odd primes $p \neq q$, there is an equality

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

where the Legendre symbol on the left equals 1 if p has a square root modulo q, and -1 otherwise. This law has the mesmerizing effect of determining the existence of square roots of a prime p modulo another prime q, provided one understands the case of q modulo p, which would be a completely different problem a priori.

There are several non-trivial proofs of this fact, but we are going to regard it as a special case of a very general reciprocity law using the language of algebraic number theory. Consider a finite field extension $F/\mathbb{Q}$ and let $O_F \subset F$ be the subring of algebraic integers. While this is no longer a unique factorization domain, the ideals $I \subset O_F$ do factor uniquely into a product of prime ideals $\mathfrak{p}$, i.e., such that the quotient $O_F/\mathfrak{p}$ is an integral domain. In algebraic number theory, the goal is to gather as much information as possible on the prime ideal decomposition of pO_F for any finite extension $F/\mathbb{Q}$. This can lead to the solution of certain Diophantine equations, e.g., decompose $p = m^2 + n^2$ with p prime inside $\mathbb{Z}[i]$.

We say that the prime p is unramified with respect to $F/\mathbb{Q}$ if pO_F decomposes as a product of different primes. In this case, there is a conjugacy class of Frobenius $\varphi_p \in \mathrm{Gal}_{F/\mathbb{Q}}$ whose elements reduce to $x \mapsto x^p$ modulo some $\mathfrak{p}$ above p. The Frobenius plays a decisive role in the entirety of this article. For instance, the Legendre symbols can be rewritten in terms of the value of φ_p in $\mathrm{Gal}_{\mathbb{Q}(\sqrt{\pm q})/\mathbb{Q}} \simeq \{\pm 1\}$. Emil Artin [Art27] found a formulation of reciprocity encompassing all previously known examples.

Theorem 1.1 ([Art27]). *For any abelian Galois extension $F/\mathbb{Q}$ and every character $\rho\colon \mathrm{Gal}_{F/\mathbb{Q}} \to \mathbb{C}^\times$, there exists a Dirichlet character $\chi_\rho\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for some N such that $\rho(\varphi_p) = \chi_\rho(p)$.*

The original proof of this result was analytic in nature rather than algebraic, and had to do with the density of split primes. This is also not so surprising if we think in terms of L-functions. Dirichlet [LD69] proved the existence of infinitely many primes in arithmetic progressions by studying the L-function given by the Euler product

$$L(s,\chi) = \prod_{p \nmid N} (1-\chi(p)p^{-s})^{-1}$$

that converges on the right half-plane $\mathrm{Re}(s) > 1$ and then meromorphically continued to the entire complex plane. In a similar fashion, we have the Artin L-function $L(s,\rho)$ for $\mathrm{Re}(s) \gg 0$ attached to a n-dimensional representation $\rho\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$, whose unramified portion is given by the characteristic polynomial of $\rho(\varphi_p)$ evaluated at p^{-s}. Artin reciprocity can be formulated in terms of the equality $L(s,\chi_\rho) = L(s,\rho)$.

Let us now mention the ring of *adèles*. Hensel [Hen01] introduced the ring $\mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ of p-adic integers, a discrete valuation ring and a profinite set. Its fraction field $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$ is called the field of p-adic numbers. We define the finite adèles $\mathbb{A}^\infty := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$, and the full adèles $\mathbb{A} := \mathbb{A}^\infty \oplus \mathbb{R}$ by including the real place. This is a locally compact abelian group and $\mathbb{Q}$ embeds diagonally as a discrete closed subgroup. The ring $\mathbb{A}$ plays a crucial role in number theory, as it encapsulates the *local-global principle*, i.e., the idea that one should compare global questions over $\mathbb{Q}$ to local questions over $\mathbb{A}$.

Chevalley [Che40] rephrased Artin reciprocity in terms of $\mathbb{A}$ and proved it algebraically. First, one defines a dense injection $\mathbb{Q}_p^\times \to \mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}}$ into the profinite Galois group of the maximal abelian extension. The image of p equals φ_p on the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}}$ and the identity on the maximal cyclotomic extension $\mathbb{Q}_p^{\mathrm{cyc}}$. The local maps can be explicitly constructed via Galois cohomology and Lubin–Tate theory [LT65], and then are assembled into a global isomorphism

$$\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} \simeq \mathrm{Gal}_{\mathbb{Q}}^{\mathrm{ab}}$$

of topological groups. This is the adèlic formulation of Artin reciprocity.

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2. Elliptic curves and modular forms

In the previous section, we saw how the adèle units relate to the abelian Galois group of $\mathbb{Q}$. Note that $\mathbb{A}^\times$ equals the $\mathbb{A}$-valued points of the algebraic group $\mathbb{G}_m$. At the same time, $\mathrm{Gal}_{\mathbb{Q}}^{\mathrm{ab}}$ captures the information afforded by characters $\chi\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$. In this section, we discuss what happens when $\mathbb{G}_m$ is replaced by GL_2.

Consider the upper half space $\mathscr{H} = \{\tau \in \mathbb{C} : \mathrm{im}(\mathbb{C}) > 0\}$. The real Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathscr{H}$ by Möbius transformations:

$$\gamma(\tau) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

so the point i has stabilizer given by the maximal compact subgroup $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$. This identifies $\mathscr{H}$ with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, the simplest example of a *Hermitian symmetric space*. At the same time, we still have an action of the arithmetic group

$\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{D} := \mathscr{H}$ and we define the *modular curve* $X_{\Gamma(1)} := \Gamma(1)\backslash\mathcal{D}$. Besides, one can write

$$X_{\Gamma(1)} = \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/(\mathrm{SL}_2(\hat{\mathbb{Z}}) \times \mathrm{SO}_2(\mathbb{R}))$$

so the modular curve also fits into the previous framework of adelic uniformazation.

However, there is more to this story and it turns out that the points of the modular curve are *elliptic curves* themselves, i.e., genus 1 smooth curves in $\mathbb{P}^2$. This means $X_{\Gamma(1)}$ is a *moduli space* of a class of algebraic varieties, a recurrent theme in algebraic geometry. Indeed, for every τ, we associate a complex torus $\mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau := (\mathbb{Z} + \tau\mathbb{Z})$. The homothety class of Λ_τ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, so our assignment descends to a bijection between $X_{\Gamma(1)}$ and the set of complex tori. Now, Weierstrass defined a certain series $\wp_\tau(z)$ converging everywhere on $\mathbb{C}$ except Λ_τ, and proved that it satisfies the functional equation $\wp'_\tau(z)^2 = 4\wp_\tau(z)^3 - g_2\wp_\tau(z) - g_3$. The coefficients $g_2 = 60G_4$ and $g_3 = 140G_6$ are rescaled Eisenstein series $G_{2n}(\tau) := \sum_{0\neq\lambda\in\Lambda_\tau} \lambda^{-2n}$ converging when $n \geq 3$, some of the most famous examples of *modular forms*. We define an elliptic curve $E_\tau = \{(\wp_\tau(z), \wp'_\tau(z)), z \in \mathbb{C}/\Lambda_\tau\}$, realizing $X_{\Gamma(1)}$ as a moduli space of elliptic curves.

The advantage of realizing $X_{\Gamma(1)}$ as a moduli space of elliptic curves is that this definition extends to an algebraic variety over $\mathbb{Q}$. One can obtain variants of the modular curve by replacing $\Gamma(1)$ by deeper level $\Gamma \subset \Gamma(1)$, e.g., the congruence subgroups $\Gamma(n) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$. The resulting map $X_{\Gamma(n)} := \Gamma(n)\backslash\mathcal{D} \to X_{\Gamma(1)}$ is a finite étale cover and admits a $\mathbb{Q}(\zeta_n)$-realization by adding isomorphisms of the n-torsion $E[n] := \ker([n] : E \to E)$ with $(\mathbb{Z}/n\mathbb{Z})^2$. Modular curves can be compactified to X^*_Γ by adding cusps in $\mathbb{P}^1_{\mathbb{Q}}$, and Deligne–Rapoport [DR73] studied their integral models $\mathcal{X}_\Gamma$, i.e., certain schemes over $\mathbb{Z}[\zeta_n]$, whose generic fiber recovers X_Γ.

Until now, we have only described the underlying geometry of the automorphic representations of GL_2. The main players in the automorphic representation theory are certain Γ-equivariant functions investigated by Hecke [Hec27], called *modular forms*. Concretely, a modular form of weight k and level Γ is a function $f\colon \mathscr{H} \to \mathbb{C}$ such that $(c\tau + d)^k f = f \circ \gamma$ for every $\gamma \in \Gamma$, admitting a Fourier expansion near the cusps at infinity. We give some examples below so the reader can get a better feeling. We have already seen the Eisenstein series $G_n(\tau)$, which are modular forms of level $\Gamma(1)$ and weight n. Jacobi defined the theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ converging when $\tau \in \mathscr{H}$ by the Poisson summation formula, a modular form of level $\Gamma(2)$ and weight $1/2$. Furthermore we have the discriminant of the Weierstrass equation $\Delta(\tau) = e^{2\pi i\tau} \prod_{n\geq1}(1 - e^{2n\pi i\tau})^{24}$, a modular form of level $\Gamma(1)$ and weight 12. Its Fourier coefficients $\tau(n)$ were conjectured by Ramanujan to satisfy $|\tau(p)| < 2p^{11/2}$. The study of modular forms has lead to many remarkable arithmetic identities.

Hecke defined the L-series $L(s, f) = \sum_{n\geq1} a_n n^{-s}$ attached to a *cusp* form f, i.e., vanishing at the cusps, of weight k and level Γ, where the sequence a_n are the Fourier coefficients of f, and proved it admits a meromorphic continuation to $\mathbb{C}$ with poles at $s = 0$ and $s = k$. Moreover, when $a_1 = 1$, then $L(s, f)$ has an Euler product with terms equal to $(1 - a_p p^{-s} + p^{k-1-2s})^{-1}$ for almost all p. Deligne [Del71a] associated 2-dimensional complex Galois representations ρ_f to modular forms f of weight $k \geq 2$ by using the étale cohomology of modular curves, and proved the Ramanujan conjecture with the help of his proof in [Del74] of the Weil conjectures.

On the other hand, an elliptic curve over $\mathbb{Q}$ has an associated L-function $L(s,E)$. Indeed, consider the Galois representation $\rho_E\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ deduced via scalar extension from the natural action on the Tate $\mathbb{Z}_\ell$-module $T_\ell(E) := \lim_n E[\ell^n] \simeq \mathbb{Z}_\ell^2$ for a fixed prime ℓ, and set $L(s,E) := L(s,\rho_E)$. This L-function has an Euler product expansion with terms of the form $(1-a_p p^{-s} + p^{1-2s})^{-1}$ for almost all p, with a_p giving the trace of φ_p. This is not a coincidence:

Theorem 2.1 ([BCDT01]). *Elliptic curves over $\mathbb{Q}$ are modular.*

The statement means that for each such E, there exists some normalized cusp form f of weight 2 such that the associated representations $\rho_E \simeq \rho_f$ or equivalently L-functions $L(s,E) = L(s,f)$ coincide. Originally, this result was conjectured by Taniyama–Shimura [ST61]. Nowadays it is known as the *modularity theorem* and was proved by Breuil–Conrad–Diamond–Taylor [BCDT01] building on work of Wiles [Wil95] and Taylor–Wiles [TW95] for semistable curves. The work of Wiles received widespread attention because Ribet [Rib90] had previously observed it implies Fermat's last theorem.

3. Shimura varieties and Langlands

The previous two sections handled the Langlands program for GL_n with $n \leq 2$. Now, we address the much more demanding case of GL_n for arbitrary n. We being by explaining how to replace elliptic and modular curves. An *abelian variety* A is a geometrically connected projective algebraic group over a field. Elliptic curves are abelian varieties of dimension 1, with group law given by $P+Q+R=0$ for any collinear triple. Over $\mathbb{C}$, an abelian variety is given by a n-dimensional torus $A_Z := \mathbb{C}^n/\Lambda_Z$ with $\Lambda_Z := \mathbb{Z}^n + Z\mathbb{Z}^n$, where Z belongs to the *Siegel upper half space* $\mathcal{D} := \mathscr{H}_n$. The elements of $\mathcal{D}$ are symmetric matrices $Z \in M_n(\mathbb{C})$ with positive definite imaginary part. Note that A_Z carries a polarization λ_Z (automatic for elliptic E), i.e., an alternating form on the lattice Λ_Z. Again, we have an identification $\mathcal{D} \simeq \mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n(\mathbb{R})$ via Möbius transformations, where $\mathrm{U}_n(\mathbb{R}) \subset \mathrm{Sp}_{2n}(\mathbb{R})$ is the real unitary subgroup of the real symplectic group. For any congruence subgroup $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$, the quotient $X_\Gamma := \Gamma\backslash\mathcal{D}$ is thus a moduli space of polarized abelian varieties with level structure, defined over a finite extension of $\mathbb{Q}$.

It is not a fluke that the symplectic group Sp_{2n} appeared above instead of SL_n: unfortunately, the automorphic quotients of SL_n for $n > 2$ are never complex manifolds. To remedy this, we need to work with more general groups. Let G be a reductive group over $\mathbb{Q}$, i.e., a linear algebraic $\mathbb{Q}$-group whose maximal smooth unipotent connected subgroup vanishes. This includes semi-simple groups like Sp_{2n} and SL_n, but also general linear groups GL_n, unitary groups $\mathrm{U}_{n,F/\mathbb{Q}}$, orthogonal groups O_n, etc. Let $\mathcal{D} = G(\mathbb{R})/Z_G(\mathbb{R})^+K_\infty$ be the Riemannian symmetric space obtained by quotienting out the connected component of the center $Z_G(\mathbb{R})$, and a compact real Lie subgroup $K_\infty \subset G(\mathbb{R})$ with maximal compact Lie subalgebra. Finally, we set

$$X_K := G(\mathbb{Q})\backslash(\mathcal{D} \times G(\mathbb{A}^\infty))/K$$

for any compact open subgroup $K \subset G(\mathbb{A}^\infty)$. One can show that this equals the disjoint union of quotients $\Gamma\backslash\mathcal{D}$ by arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$, so it is a real orbifold. If K is sufficiently small, X_K is a real manifold. After Shimura [Shi63] worked out a wide variety of examples, Deligne [Del71b] introduced the notion of a Shimura datum, where $\mathcal{D}$ arises as the conjugacy class of homomorphisms $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_\mathbb{R}$, and satisfies a series of axioms to ensure that $\mathcal{D}$ is a disjoint union of *Hermitian* symmetric spaces. For Shimura data and neat K, Baily–Borel [BB66] proved that the X_K are quasi-projective smooth $\mathbb{C}$-varieties, by constructing minimal compactifications of $\Gamma\backslash\mathcal{D}$. These X_K are called *Shimura varieties* and vastly generalize moduli of polarized abelian varieties. Deligne [Del79] proved that, for almost all $(G, \mathcal{D})$ with G classical, they descend to a finite extension $E/\mathbb{Q}$, and the general case was handled independently by Borovoi [Bor83] and Milne [Mil83]. Shimura varieties play a distinguished role in the Langlands program, because their étale cohomology relates to both automorphic and Galois representations.

Finally, we are ready to discuss the notion of an automorphic representation of GL_n, or even of a general reductive group G over $\mathbb{Q}$. We consider the Hilbert space $L^2([G])$ of square-integrable functions on the automorphic space $[G] := G(\mathbb{Q})A_G(\mathbb{Q})\backslash G(\mathbb{A})$ with its natural Radon measure. Here, A_G is the maximal $\mathbb{Q}$-split central torus of G, and one kills it to ensure the finite volume of $[G]$. An *automorphic representation* of $G(\mathbb{A})$ is an irreducible unitary $G(\mathbb{A})$-representation appearing as a subquotient of $L^2([G])$. At the same time, we define *automorphic forms* as K-invariant smooth functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ of moderate growth with finite translates under K_∞ and the center of $U(\mathrm{Lie}(G_\mathbb{R}))$. For GL_2, we recover modular forms up to a twist and furthermore, automorphic representations can be described in terms of automorphic forms.

Our final ingredient for stating the global Langlands correspondence (GLC) is the notion of automorphic L-functions. Langlands [Lan70] was studying the *constant terms* of Eisenstein series, i.e., their values at the boundary of $\Gamma\backslash\mathcal{D}$, when he was led to the L-function of an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ for all n. Indeed, the Satake isomorphism encountered in the next section associates a dominant coweight $\mu(\pi_p)$ of $\mathrm{GL}_n(\mathbb{C})$ to the p-primary part π_p of the automorphic representation π. For unramified p, one sets $L(s, \pi_p)$ to be the value at p^{-s} of the characteristic polynomial of $\mu(\pi_p)$. More care is needed to define $L(s, \pi_\infty)$ and $L(s, \pi_p)$ for ramified p, and show that the Euler product $L(s, \pi) := L(s, \pi_\infty) \prod_p L(s, \pi_p)$ admits a meromorphic continuation to $\mathbb{C}$.

Conjecture 3.1 ([Lan70])**.** *The Artin L-function $L(s, \rho)$ of an irreducible Galois representation $\rho\colon \mathrm{Gal}_\mathbb{Q} \to \mathrm{GL}_n(\mathbb{C})$ coincides with the L-function $L(s, \pi_\rho)$ of an automorphic representation π_ρ of $\mathrm{GL}_n(\mathbb{A})$.*

This is a crude version of the GLC that does not pin down the automorphic representation π_ρ. Advances in the theory of Shimura varieties and p-adic Hodge theory have led to a better understanding of how the GLC should look like, and we refer to [BG14] for a modern treatment of the GLC in this direction. Over the p-adic field $\mathbb{Q}_p$, we get the local Langlands correspondence (LLC) with a much clearer formulation. The LLC predicts a bijection.between isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ on the automorphic side and $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of semisimple L-parameters $\mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}_{\mathbb{Q}_p}$. These so-called L-parameters are certain group homomorphisms from the Weil–Deligne group $\mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$, such that the image of the Frobenius φ is semisimple. This conjecture was proved by Harris–Taylor [HT01] using global ingredients coming from the cohomology of Shimura varieties, whose decomposition allowed them to relate automorphic and Galois representations. For general reductive groups other than GL_n, the situation is more complicated as one does not expect a bijection anymore, but rather a map with finite fibers called L-packets. This is a matter of current intense investigation.

4. Geometric Langlands

It is well known that there is an analogy between number fields and function fields, by which we mean a field $K(X)$ of rational functions on a geometrically connected proper smooth curve X over $\mathbb{F}_p$. Indeed, if one considers the affine spectrum of $\mathbb{Z}$, this is a 1-dimensional scheme and its closed points are given by its primes, just like the places of $K(X)$ correspond to closed points of X. From the point of view of algebraic geometry, it is however much easier to work with curves over $\mathbb{F}_p$. Besides, one also gets a canonical Frobenius homomorphism φ.

Let G be a split connected reductive group over the curve X in the sense of Chevalley [Che55], see also [GD63] (for the unfamiliar reader, this includes classical groups such as GL_n, Sp_{2n}, SO_n, but unitary groups like SU_n are non-split). Consider the moduli stack Bun_G classifying G-bundles on the curve X. This is a smooth Artin stack over $\mathbb{F}_p$ and we can write

$$\mathrm{Bun}_G(\mathbb{F}_p) = G(\mathbb{F}_p)\backslash G(\mathbb{A}_X)/G(\mathbb{O}_X) \tag{4.1}$$

where the right side resembles the automorphic side of Artin reciprocity. Indeed, in this setting, we define automorphic forms as sections of constant $\bar{\mathbb{Q}}_\ell$-sheaves on this space for some prime $\ell \neq p$. Besides the stack of G-bundles, there are other important stacks to consider, such as the *Hecke stack* denoted by Hk_G which classifies modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ of G-bundles along a point of the curve $x \in X$; or the stack of G-shtukas Sht_G that is given as the pullback of the Frobenius graph of Bun_G along the natural map $\mathrm{Hk}_G \to \mathrm{Bun}_G$. Shtuka stacks play a similar role to Shimura varieties in the number field case.

Originally, Drinfeld [Dri74] came up with the notion of vector bundles equipped with a meromorphic φ-semilinear structure, which he named shtukas (Russian for "thing, stuff") and applied it in [Dri80] to prove the GLC when GL_2 over global function fields. These techniques were then further developed by Laumon–Rapoport–Stuhler [LRS93] to prove the local Langlands correspondence (LLC) for local fields in characteristic p. After that, Laurent Lafforgue [Laf02] used shtukas to prove the GLC for GL_n over X, and finally his brother Vincent Lafforgue [Laf18] figured out the automorphic to Galois direction for arbitrary G over X.

Theorem 4.1 ([Laf18])**.** *There is a natural map $\pi \to \sigma_\pi$ from automorphic representations of $G(\mathbb{A}_X)$ to semisimple Langlands parameters, i.e., $G^\vee(\bar{\mathbb{Q}}_\ell)$-conjugacy classes of continuous 1-cocycles $\mathrm{Gal}_{K(X)} \to G^\vee(\bar{\mathbb{Q}}_\ell)$.*

Several ingredients go into the proof of this theorem, the most original one being the construction of the so-called excursion operators. Another fundamental ingredient that we want to address is the *geometric Satake equivalence*. Let O_x be the complete local ring of X at x and F_x be its fraction field. The classical Satake isomorphism following [Sat63] identifies the spherical Hecke algebra $\mathcal{H}_G := \mathbb{C}[G(O_x)\backslash G(F_x)/G(O_x)]$ of G with the Weyl invariants $\mathcal{H}_T^W = \mathbb{C}[X_*(T)]^W$ of the Hecke algebra of a maximal torus $T \subset G$. Up to passing to ℓ-adic coefficients, we can identify $\mathcal{H}_G$ with the Grothendieck group K_0 of étale $\bar{\mathbb{Q}}_\ell$-sheaves on the fiber $\mathrm{Hk}_{G,x}$ of the Hecke stack at the point x and $\mathcal{H}_T^W$

with that of representations of the L-group ${}^L G = G^\vee \rtimes \mathrm{Gal}_{F_x}$. This observation can be upgraded to an equivalence of categories:

Theorem 4.2 ([Gin95])**.** *There is a natural symmetric monoidal equivalence of abelian categories between the category $\mathcal{P}(\mathrm{Hk}_{G,x})$ of perverse $\bar{\mathbb{Q}}_\ell$-sheaves on the Hecke stack and the category $\mathrm{Rep}({}^L G)$ of representations of the L-group of G.*

Recall that for a smooth variety over $\mathbb{F}_p$, Poincaré duality holds for étale cohomology by [SGA73b]. However, if we work with non-singular varieties, then this is no longer the case and $\mathrm{Hk}_{G,x}$ is very far from smooth. It admits a pro-smooth cover by the affine Grassmannian $\mathrm{Gr}_{G,x}$ which is an ind-scheme. Its closed $G(O_x)$-equivariant subvarieties $\mathrm{Gr}_{G,x,\leq\mu}$ are called Schubert varieties and indexed by dominant coweights μ of G. These are very rarely smooth, but are always normal and Cohen–Macaulay if $\pi_1(G)$ is p-torsion free by a theorem of Faltings [Fal03] (if $\pi_1(G)$ has p-torsion, pathologies occur by [HLR18]), see [Lou23] for a new proof via distribution $\mathbb{F}_p$-algebras.

Fortunately, Goresky–MacPherson [GM83] discovered in the topological setting, later rephrased by Beilinson–Bernstein–Deligne–Gabber [BBDG18] in the algebraic setting, that constant sheaves shifted by the dimension along a smooth stratification of X can be glued to complexes on X (called a perverse *sheaf* nonetheless). One gets an abelian full subcategory $\mathcal{P}(X) \subset \mathcal{D}(X)$ of the derived category of sheaves, which satisfies a form of Poincaré duality even for non-smooth X. The geometric Satake equivalence furnishes a plethora of perverse sheaves (one calls them also *Satake sheaves*) on the Hecke stack, which are used as the convolution kernels of geometric Hecke operators. It is also known with $\bar{\mathbb{Z}}_\ell$- and $\bar{\mathbb{F}}_\ell$-coefficients thanks to the work of Mirković–Vilonen [MV07].

5. Geometrization for p-adic fields

In this section, we want to explain some of the emerging story over p-adic fields due to Fargues–Scholze [FS21], which takes a lot of inspiration from global function fields. It is already a daunting task to work in the local p-adic field situation, so from now on we will forget about global number fields.

The first problem that we encounter is that we do not really have a decent curve, or at least the curve that we would normally have, i.e. the affine spectrum of $\mathbb{Z}_p$, is not that rich geometrically. Even the number field setting is not useful because we lack a canonical Frobenius to move things around... The idea here comes in a sense from the theory of Witt vectors [Wit37]. They allow us to lift perfect $\mathbb{F}_p$-algebras to mixed characteristic. Scholze [Sch12] defined a tilting functor that passes from mixed characteristic perfectoid rings to perfect $\mathbb{F}_p$-algebras. While tilting is a functor, there is a myriad of untilts and classifying them yields the Fargues–Fontaine curve [FF18].

More precisely, a *perfectoid Tate ring* is a pair (R, R^+) consisting of a subring $R^+ \subset R = R^+[1/\varpi]$ equipped with the ϖ-adic topology such that ϖ^p divides p and the Frobenius $\varphi\colon R^+/\varpi \to R^+/\varpi^p$ is an isomorphism. The tilting functor of [Sch12] takes (R, R^+) to the perfect Tate ring $(R^\flat, R^{\flat+})$, where $R^{\flat+}$ is the limit of R^+/p along φ. Kedlaya–Liu [KL15] proved that every untilt $(S^\sharp, S^{\sharp+})$ of a perfect Tate ring (S, S^+) can be uniquely obtained as the quotient $S^{\sharp+} = W(S^+)/\xi$ with $\xi = p + [\varpi]\alpha$, where $[\varpi]$ is the Teichmüller lift. This leads us to define the absolute curve Y over $\mathrm{Spa}(\mathbb{Q}_p)$ whose (S, S^+)-valued points are given by the non-vanishing locus of $p[\varpi]$ in the affinoid adic space $\mathrm{Spa}(W(S^+))$. The adic Fargues–Fontaine curve $X := Y/\varphi^{\mathbb{Z}}$ is the quotient by the totally discontinuous action of the Frobenius φ. Here, we have to use the theory of adic spaces due to Huber [Hub96] which captures analytic features in a better fashion; this is related also to the recently developed notion of analytic rings and stacks by Clausen–Scholze [CS19].

Now, one can define Bun_G in this setting again as the stack of G-torsors on X. Its geometric points are in bijection with Kottwitz's set $B(G)$ classifying φ-conjugacy classes in $G(W(\bar{\mathbb{F}}_p)[1/p])$ by a theorem of Fargues [Far20], with topology explicitly described via the combinatorics of Newton polygons as shown by Viehmann [Vie21]. Scholze [Sch17] developed a formalism of étale cohomology for perfectoid stacks, and [FS21] proves that $\mathcal{D}(\mathrm{Bun}_G)$ captures the derived categories of smooth representations of the inner forms J_b of the Levi subgroups of G attached to $b \in B(G)$, glued in a yet mysterious way.

One can also define the Hecke stacks Hk_G, affine Grassmannians Gr_G and shtuka stacks Sht_G in this setup. These are stacks on perfectoids with a natural map to the mirror curve Div^1_X of the Fargues–Fontaine curve X. Using the concept of universally locally acyclic sheaves, [FS21] proved the geometric Satake equivalence for Hk_G, but the L-group is given by the semi-direct product $G^\vee \rtimes \mathrm{W}_{\mathbb{Q}_p}$ with the Weil group. Most of the formal arguments in [Laf18] concerning excursion operators can be repeated to yield the automorphic to Galois direction of the LLC for p-adic fields.

Besides, the Galois side of the LLC can be geometrized via the stack $Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)$ of L-parameters studied by Zhu [Zhu20] and Dat–Helm–Kurinczuk–Moss [DHKM20], which classifies continuous 1-cocycles $\varphi\colon \mathrm{W}_{\mathbb{Q}_p} \to G^\vee(\bar{\mathbb{Q}}_\ell)$. The Langlands philosophy combined with the previous geometrization efforts suggests that one should find a correspondence between derived categories of sheaves on the automorphic space and the Galois space.

Conjecture 5.1 ([FS21])**.** *There is an equivalence* $\mathcal{D}(\mathrm{Bun}_G)^\omega \simeq \mathcal{D}^b_{\mathrm{coh}}([G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)])$ *of derived categories.*

On the left of the equivalence, we consider the full subcategory of compact objects inside $\mathcal{D}(\mathrm{Bun}_G)$, whereas on the right of the equivalence we consider the category of bounded complexes of coherent sheaves on the stack $[G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)]$. Similar versions of this conjecture have recently appeared also by Hellmann [Hel23] and Zhu [Zhu20] and it can be made much more explicit as follows. In [FS21], the Hecke action on $\mathcal{D}(\mathrm{Bun}_G)$ given by the Satake sheaves is extended to a full action by the category $\mathrm{Perf}([G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)])$ of perfect complexes on the stack of L-parameters. One can therefore ask that the equivalence above respects the spectral action. Moreover, it is expected that the inverse equivalence maps the structure sheaf $\mathcal{O}$ to a Whittaker sheaf $\mathcal{W}_\psi$, i.e., obtained via compact induction from a Whittaker datum ψ on a maximal unipotent subgroup $U(\mathbb{Q}_p)$.

6. Sheaves on integral models

In this section, we discuss further developments related to integral $\mathbb{Z}_p$-models and their reduction to $\mathbb{F}_p$. The curve Y admits an obvious integral model $\mathcal{Y}$ by including characteristic p untilts, but it is no longer natural to consider its Frobenius quotient, because the action is not free anymore. Let $\mathcal{G}$ be a parahoric $\mathbb{Z}_p$-model in the sense of Bruhat–Tits [BT84] of our connected reductive $\mathbb{Q}_p$-group G: this notion means that

$\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ behaves for many purposes like a parabolic subgroup, e.g. take the pullback of an actual parabolic subgroup along the reduction map $G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$. Then, one can still define the notions of Hecke stacks $\mathrm{Hk}_{\mathcal{G}}$ and shtuka stacks $\mathrm{Sht}_{\mathcal{G}}$, and affine Grassmannians $\mathrm{Gr}_{\mathcal{G}}$. The bounds by coweights μ extend to $\mathbb{Z}_p$ by taking closures, and one of our contributions revolved around understanding this procedure in detail.

Theorem 6.1 ([AGLR22, GL22])**.** *The v-sheaf $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ is normal with special fiber equal to the μ-admissible locus. If μ is minuscule, then it is representable by a unique normal, Cohen–Macaulay, flat $\mathbb{Z}_p$-scheme with reduced special fiber.*

Part of this had been previously conjectured by Scholze–Weinstein [SW20] and much of the motivation stemmed from the arithmetic of Shimura varieties, where the minuscule integral Schubert varieties appear as local models for controlling the singularities, see the book of Rapoport–Zink [RZ96]. These were studied extensively in the last decades, most notably by Pappas, Rapoport, and Zhu [PR08, PZ13], but the approach in [AGLR22] was the first to actually provide a complete and functorial theory.

One important ingredient in [AGLR22] is the formalism of kimberlites due to Gleason [Gle22], which are v-sheaves of formal nature with a scheme-theoretic reduction whose complement is a diamond, and admitting a specialization map between the underlying topological spaces. In [AGLR22] we proved that $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ can be recovered from its $\mathbb{Z}_p$-fibers and the specialization map between them. Note that the special fiber of the unbounded Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ was first defined by Zhu [Zhu17] and then Bhatt–Scholze [BS17] proved it is an ind-perfect scheme. The μ-admissible locus in the theorem goes back to Kottwitz–Rapoport [KR00] and equals the union of the $\mathcal{G}(\mathbb{Z}_p)$-orbit closures in $\mathrm{Gr}_{\mathcal{G},\mathbb{F}_p}$ of the Weyl conjugates of μ. We are able to pinpoint the specialization map for minuscule μ by using a convolution analogue of the Iwasawa decomposition.

Arguably, the most crucial task in [AGLR22] is identifying the special fiber of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$. This requires working with the derived category $\mathcal{D}(\mathrm{Hk}_{\mathcal{G}})$ of étale $\bar{\mathbb{Q}}_\ell$-sheaves on the Hecke stack building on [Sch17]. The main tool for us is the functor of nearby cycles $R\Psi : \mathcal{D}(X_{\bar{\eta}}) \to \mathcal{D}(X_{\bar{s}})$ specializing between geometric fibers. Its origins lie in Morse theory: for a map $f : X \to \mathbb{D}$ with an isolated singularity at the origin, $R\Psi$ carries the cohomology classes of the non-singular fibers to the fiber at 0. In the case of schemes, Deligne defined $R\Psi$ for a map $f : X \to \mathbb{Z}_p$ in [SGA73a] by pushing sheaves forward along the absolute integral closure, and this definition of also works in the situation of [AGLR22]. Using constant terms functors, we could prove that universal locally acyclic sheaves are preserved under $R\Psi$ and their images are actually scheme-theoretic.

Theorem 6.2 ([AGLR22, ALWY23])**.** *The functor $R\Psi\colon \mathcal{P}(\mathrm{Hk}_{G,\mathbb{C}_p}) \to \mathcal{D}(\mathrm{Hk}_{\mathcal{G},\bar{\mathbb{F}}_p})$ lifts to the Drinfeld center and lands in $\mathcal{P}(\mathrm{Hk}_{\mathcal{G},\bar{\mathbb{F}}_p})$.*

This is the analogue of the main theorem of Gaitsgory [Gai01] in the function field case, but the p-adic setting complicates matters. For instance, nearby cycles of algebraic schemes preserve perversity by Artin vanishing, but this is no longer true for general v-sheaves. In order to prove it, we introduced Wakimoto sheaves in [ALWY23] at Iwahori level following Arkhipov–Bezrukavnikov [AB09], and used them in combination with geometric Satake to a filtration of $R\Psi$ by Wakimoto perverse sheaves. As for the centrality of $R\Psi$, we constructed the main isomorphisms in [AGLR22], but only verified the higher homotopy coherences in [ALWY23].

In [AGLR22], we applied nearby cycles and geometric Satake to show that the special fiber of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ coincides with the μ-admissible locus. Another application of the central sheaves is the normality of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ proved in [GL22] by a much simpler method than [Zhu14] in the function field case. We use the Wakimoto filtration to show connectedness for the analytic tubes of the closed points in the special fiber of $\mathrm{Gr}_{\mathcal{G},\mu}$ up to codimension 2. This reduces normality to a combinatorial S_2 property for the special fiber, which we verify in the function field case thanks to a dynamical argument of Le–Le Hung–Levin–Morra [LHLM22].

A natural continuation of the above is to extend the central functor to a Bezrukavnikov equivalence for p-adic fields in analogy with [Bez16] for Laurent series fields. It asserts that there is an equivalence of derived categories

$$\mathcal{D}_{\mathrm{coh}}([G^\vee \backslash \mathrm{St}_{G^\vee}]) \simeq \mathcal{D}(\mathrm{Hk}_{\mathcal{I},\bar{\mathbb{F}}_p}) \tag{6.1}$$

of coherent sheaves on the left and étale $\bar{\mathbb{Q}}_\ell$-sheaves on the left. Here, $\mathrm{St}_{G^\vee}$ is the dual Steinberg variety of triples and $\mathcal{I}$ is a Iwahori model of G. In [ALWY23], we followed [AB09] to construct roughly one half of the equivalence using the Springer resolution of the nilpotent cone $\mathcal{N}_{G^\vee}$ in place of the Steinberg variety. Unfortunately, there is one essential ingredient still missing in the p-adic setting beyond the GL_n case. We need to bound certain Hom spaces involving quasi-minuscule representations and the fastest way to do this is extending the monodromy operator on the image of $R\Psi$ to the entire category of perverse sheaves. This is classically done via rescaling uniformizers but cannot be performed in the p-adic setting. We hope to address this also in our future work.

The importance of the Bezrukavnikov equivalence lies in its usefulness for geometric Langlands. Recently, Zhu [Zhu20] proposed a different geometrization of local Langlands over $\mathbb{Q}_p$, where instead of Bun_G he considers the stack Isoc_G of *G-isocrystals*, which geometric points equal $B(G)$ but carries the opposite topology. In upcoming work of Hemo–Zhu, the Bezrukavnikov equivalence is used via the trace of Frobenius to produce a Langlands equivalence for tame representations with Isoc_G in place of Bun_G. We are thus led to anticipate the following.

Conjecture 6.3. *There is an equivalence $\mathcal{D}(\mathrm{Isoc}_G) \simeq \mathcal{D}(\mathrm{Bun}_G)$ of derived categories.*

Recently, Gleason–Ivanov [GI23] constructed a geometric correspondence between Isoc_G and Bun_G. In an ongoing project, we aim to tackle the conjecture above by using this geometric correspondence. Simultaneously, the Bezrukavnikov equivalence and our partial work towards it in [ALWY23] should be an essential ingredient in comparing the (tame) spectral action on the two sides.

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