# Exercises for the Lectures on L-theory
# September 2020
# 1. Exercise Sheet

**Exercise 1.1:** Show that $M$-valued quadratic forms in the sense of Lecture 1 are equivalently described by

(a) an $M$-valued symmetric form $b\colon P \otimes P \to M$, and

(b) a function $q\colon P \to M_{C_2}$

satisfying the following compatibility:

- $q(rx) = (r \otimes r) \cdot q(x)$,
- $q(x+y) - q(x) - q(y) = [b(x,y)]$,
- $b(x,x) = q(x) + \sigma(q(x))$.

In particular, convince yourself that the $(r \otimes r) \cdot q(x)$ is a well-defined expression.

**Exercise 1.2:** Let $P \in \mathrm{Proj}(R)$. Show that $\mathrm{hyp}(P)$ is canonically an $M$-valued quadratic form.

**Exercise 1.3:**

(1) Show that there are symmetric forms which are metabolic but not hyperbolic.

(2) Let $(P,b,q)$ be a quadratic form on $P$. Show that if $(P,b,q)$ admits a quadratic Lagrangian $L$, then $(P,b,q) \cong \mathrm{hyp}(L)$.

Hint: For (2), I recommend to do the exercise again after Lecture 2.

**Exercise 1.4:** Show that $\mathrm{W}^{\mathrm{s}}(R;M)$ and $\mathrm{W}^{\mathrm{q}}(R;M)$ are abelian groups. Show moreover that the symmetrization map induces a group homomorphisms

$$\mathrm{W}^{\mathrm{q}}(R;M) \longrightarrow \mathrm{W}^{\mathrm{s}}(R;M).$$

**Exercise 1.5:**

(1) Let $K$ be a field with $\mathrm{char}(K) \neq 2$. Show that $\mathrm{W}^{-\mathrm{s}}(K) = 0$.

(2) Let $K$ be a field such that the map $K^\times \to K^\times$, sending $x$ to $x^2$, is surjective. Show that $\mathrm{W}^{\mathrm{s}}(K) \cong \mathbb{Z}/2$.

**Exercise 1.6:** Let $K$ be a field with $\mathrm{char}(K) \neq 2$. Let $(W,b)$ be a unimodular symmetric bilinear form with $b(x,x) = 0$ for all $x \in W$. Show that $W = 0$.

**Exercise 1.7:** Let $P \in \mathrm{Proj}(R)$. Show that

(1) $\Omega^{\geq 0}_M(P) = \mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{C_2}$,

(2) $\Omega^{\geq 2}_M(P) = \mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{C_2}$, and

(3) $\Omega^{\mathrm{b}}_{\mathbb{Z}}(\mathbb{Z}) = B(C_2)$, where $B(G)$ is the Burnside ring of a finite group $G$.

Deduce that $\Omega^{\geq 0}_M = \Omega^{\mathrm{gs}}_M$ and $\Omega^{\geq 2}_M = \Omega^{\mathrm{gq}}_M$.

**Exercise 1.8:** Show that

(1) $\mathrm{L}_0(\mathcal{C}, \Omega)$ is an abelian group.

(2) The relation

$$(X, q) \sim (X', q') \Leftrightarrow (X \oplus X', q \oplus -q') \text{ is metabolic}$$

is a congruence relation, i.e. an equivalence relation compatible with $\oplus$ on $\mathrm{Pn}(\mathcal{C}, \Omega)$.

(3) Show that $\mathrm{L}_0(\mathcal{C}, \Omega)$ as defined in Lecture 1 is isomorphic to the $\mathrm{L}_0$-group defined in Yontan's first lecture.

**Exercise 1.9:** Show that $\pi_0(\mathrm{L}(\mathcal{C}, \Omega)$ is isomorphic to $\mathrm{L}_0(\mathcal{C}, \Omega)$. Deduce that $\mathrm{L}(\mathcal{C}, \Omega)$ is a grouplike $\mathbb{E}_\infty$-space and hence a connective spectrum.

**Exercise 1.10:** Show that the geometric realization of the simplicial space

$$[n] \longmapsto \mathrm{Fm}(\rho_n(\mathcal{C}, \Omega))$$

is contractible. Hint: Use extension by 0's to define an extra degeneracy.

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# 2. Exercise Sheet

**Exercise 2.1:** Show that the underlying object of the surgery output associated to the surgery datum

$$(S \to 0, \eta \in \Omega Ϙ(S))$$

(this is a surgery datum on the Poincaré object $(0, 0 \in Ϙ(0) = 0)$) is given by

$$\mathrm{cofib}(S \xrightarrow{\eta_\#} \Omega \mathrm{D}_Ϙ(S)).$$

**Exercise 2.2:** Let $(X, q) \in \mathrm{Pn}(\mathcal{D}^p(R), Ϙ)$, where $Ϙ$ is a Poincaré structure sending $\mathrm{Proj}(R)$ to Eilenberg–Mac Lane spectra with only homotopy in degree 0. Let $k$ be the maximal number $i$ such that $\pi_{-i}(X) \neq 0$ and assume that $k > 0$. Show that

(1) $Ϙ(P[-k])$ is $k$-connective (this has nothing to do with $k > 0$),

(2) The surgery output $(X', q')$ of a map $f\colon P[-k] \to X$ inducing a surjection on $\pi_k$ and any nullhomotopy of $f^*(q)$ satisfies that $X'$ is $(-k+1)$-connective.

Note that by (1), the existence of a null homotopy of $f^*(q)$ is implied by the assumption $k > 0$.

**Exercise 2.3:** Let $R$ and $M$ be as always. Assume $Ϙ$ is an $M$-compatible Poincaré structure sending $\mathrm{Proj}(R)$ to EM spectra in degree 0. Let $(X, q)$ be a 0-connective Poincaré object and $(L \to X, \eta)$ a Lagrangian. Show that if $L$ is $(-1)$-connective, then it can be represented by a chain complex of finitely generated projectives

$$0 \longrightarrow Q \longrightarrow N \longrightarrow 0$$

with $Q$ sitting in homological degree 0.

**Exercise 2.4:** Let $R$, $M$, and $Ϙ$ be as in Exercise 2.3, and let $(f\colon L \to X, \eta)$ be a Lagrangian for $(P[0], q)$ with $P \in \mathrm{Proj}(R)$. In other words, $(f\colon L \to P[0], \eta) \in \mathrm{Pn}(\mathrm{Met}(\mathcal{D}^p(R), Ϙ))$. Let $k$ be the maximal number $i$ such that $\pi_{-i}(L) \neq 0$, assume $k > 0$, and assume given a map $Q[-k] \to L$ inducing a surjection on $\pi_{-k}$ with $Q \in \mathrm{Proj}(R)$. Consider the diagram

$$\begin{array}{ccc} Q[-k] & \longrightarrow & L \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & P[0] \end{array}$$

as a morphism in $\mathrm{Met}(\mathcal{D}^p(R), Ϙ)$, from the left vertical map to the right vertical map. Show that

(1) The morphism just described admits a refinement of a surgery datum on the Poincaré object $(f\colon L \to X, \eta)$ in $\mathrm{Met}(\mathcal{D}^p(R), Ϙ)$.

(2) The surgery output is given by a Poincaré object $(f'\colon L' \to X, \eta')$ with $L'$ $(-k+1)$-connective.

**Exercise 2.5:** Let $R$ and $M$ be as always and $\Qoppa$ an $M$-compatible Poincaré structure. Show that

(1) $\Qoppa$ is $m$-quadratic if and only if $L_{\Qoppa}(X)$ is $(m+k)$-connective whenever $\mathrm{D}_{\Qoppa}(X)$ is $k$-connective, and

(2) $\Qoppa$ is $r$-symmetric if and only if $\mathrm{fib}(\Qoppa(X) \to \Qoppa^s_M(X))$ is $(-r-k)$-truncated whenever $X$ is $k$-connective.

**Exercise 2.6:** Show that the functor $\Omega\colon \mathcal{D}^p(R) \to \mathcal{D}^p(R)$ extends to an equivalence of Poincaré $\infty$-categories

$$(\mathcal{D}^p(R), \Qoppa_M^{\geq m}) \xrightarrow{\simeq} (\mathcal{D}^p(R), \Omega^2 \Qoppa_{-M}^{\geq(m+1)}).$$

Deduce that $\mathrm{L}(\mathcal{D}^p(R), \Qoppa^q_M)$ and $\mathrm{L}(\mathcal{D}^p(R), \Qoppa^s_M)$ are 4-periodic, and 2-periodic $R$ is an $\mathbb{F}_2$-algebra (i.e. $2 = 0$ in $R$).

**Exercise 2.7:** Show that the groups $\mathrm{L}^{a,b}_n(\mathcal{D}^p(R), \Qoppa)$ can be equivalently defined by the following constraints on Poincaré objects $(X, q)$ and Lagrangians $(L \to X, \eta)$:

(1) $X$ is $(\frac{-n-a}{2})$-connective,

(2) $L$ is $\lceil \frac{-n-1-b}{2} \rceil$-connective, and

(3) $\mathrm{fib}(L \to X)$ is $\lfloor \frac{-n-1-b}{2} \rfloor$-connective.

# Exercises for the Lectures on L-theory
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# 3. Exercise Sheet

**Exercise 3.1:** Show that the definition of $r$-symmetric D-compatible Poincaré structures on stable $\infty$-categories with $t$-structure of lecture 3 recovers the notion of $r$-symmetric $M$-compatible Poincaré structures on $\mathcal{D}^p(R)$ when $R$ is Noetherian and has finite global dimension.

**Exercise 3.2:** Using the theorem about the forget-control map on L-groups for stable $\infty$-categories with $t$-structures of lecture 3, prove the following theorem. Let $R$ be Noetherian of finite global dimension $d$ and let $M$ be an invertible module with involution over $R$. Let $Ϙ$ be an $M$-compatible, $r$-symmetric Poincaré structure. Then the map

$$\mathrm{L}_n(\mathcal{D}^p(R), Ϙ) \longrightarrow \mathrm{L}_n(\mathcal{D}^p(R), Ϙ^s_M)$$

is an isomorphism for $n \geq d - 2r + 3$.

**Exercise 3.3:** Under the assumption of the theorem on surgery for $r$-symmetric Poincaré structures of lecture 3, let $(X, q)$ be a Poincaré object for $\Omega^n Ϙ$. Let $W = \Omega^n \mathrm{D}_Ϙ \tau_{\leq -n-1} X$, equipped with the canonical map $W \to X$, making use of the equivalence $\Omega^n \mathrm{D}_Ϙ X \simeq X$ induced by $q$. Show that

(1) $W \in \mathcal{C}_{\geq -d+1}$ and that the map $W \to X$ refines to a surgery datum on $(X, q)$.

(2) The surgery trace is canonically equivalent to $\tau_{\geq -n}(X)$ and the surgery output $(X', q')$ satisfies $X' \in \mathcal{C}_{\geq -n}$.

**Exercise 3.4:** Let $R$ be a Noetherian ring of global dimension zero, i.e. semi simple, and let $M$ be an invertible module with involution on $R$. Show that

$$\mathrm{L}_{2n+1}(\mathcal{D}^p(R), Ϙ^q_M) = 0 = \mathrm{L}_{2n+1}(\mathcal{D}^p(R), Ϙ^s_M).$$

**Exercise 3.5:** Let $R$ be commutative, Noetherian of finite global dimension $d$. Let $M = R$ and $\sigma = \mathrm{id_R}$ be the canonical invertible module with involution. Assume that $R$ is 2-torsion free. Show that the map

$$\mathrm{L}_n(\mathcal{D}^p(R), Ϙ^{\mathrm{gq}}_R) \longrightarrow \mathrm{L}_n(\mathcal{D}^p(R), Ϙ^{\mathrm{gs}}_R)$$

is an isomorphism for $n \geq d + 1$.

**Exercise 3.6:** Let $R$ be a Dedekind ring whose fraction field has characteristic different from 2. Show that

$$\mathrm{L}_n(\mathcal{D}^p(R), Ϙ^{\mathrm{gs}}_R) \cong \begin{cases} \mathrm{L}_n(\mathcal{D}^p(R), Ϙ^q_R) & \text{for } n \leq -3 \\ \mathrm{L}_n(\mathcal{D}^p(R), Ϙ^s_R) & \text{for } n \geq -2 \end{cases}$$

# Exercises for the Lectures on L-theory
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# 4. Exercise Sheet

**Exercise 4.1:** Let $R$ be a Dedekind ring and $\mathfrak{p}$ a prime ideal. Show that $R/\mathfrak{p}$ is an object of $\mathcal{D}^p(R)$.

**Exercise 4.2:** Let $R$ be a Dedekind ring and let $\mathfrak{p} = (\pi)$ be a principal ideal. Let $p\colon R \to R/\mathfrak{p}$ be the projection, and let $p_*\colon \mathcal{D}(R) \to \mathcal{D}(R/\mathfrak{p})$ be the right adjoint of $p^*$. Show that the Bockstein homomorphism associated to the extension

$$R \xrightarrow{\pi} R \xrightarrow{p} R/\mathfrak{p}$$

induces (by adjunction) an equivalence $p_*(R) \simeq R/\mathfrak{p}[-1]$.

Bonus: Can you also show the equivalence $p_*(R) \simeq R/\mathfrak{p}[-1]$ for not necessarily principal prime ideals?

**Exercise 4.3:** Let $R$ be a Dedekind ring, $K = \mathrm{Frac}(R)$ a global field of characteristic $\neq 2$, and $d$ the number of dyadic primes of $R$. Show that

$$\mathrm{L}^s(R) \cong \begin{cases} \mathrm{W}^s(R) & \text{for } n \equiv 0(4) \\ (\mathbb{Z}/2)^d & \text{for } n \equiv 1(4) \\ 0 & \text{for } n \equiv 2(4) \\ \mathrm{Pic}(R)/2 & \text{for } n \equiv 3(4). \end{cases}$$

**Exercise 4.4:** Let $R$ be a commutative ring, and let $I$ be an ideal of $R$ which is contained in the Jacobson radical. Show that the map

$$\mathrm{L}^q_{2k}(R) \longrightarrow \mathrm{L}^q_{2k}(R/I)$$

is surjective.

**Exercise 4.5:** Let $R$ be a ring and let $R_2$ be its 2-completion. Assume that the 2-power torsion of $R$ is bounded. Show that the following is a pullback diagram.

$$\begin{array}{ccc} \mathrm{L}^q(R) & \longrightarrow & \mathrm{L}^q(R_2) \\ \downarrow & & \downarrow \\ \mathrm{L}^s(R) & \longrightarrow & \mathrm{L}^s(R_2) \end{array}$$

**Exercise 4.6:** Let $R$ be a Dedekind ring, $K = \mathrm{Frac}(R)$ a global field of characteristic $\neq 2$, $d$ the number of dyadic primes. Show that

$$\mathrm{L}^q_n(R) \cong \begin{cases} \mathrm{W}^q(R) & \text{for } n \equiv 0(4) \\ 0 & \text{for } n \equiv 1(4) \\ (\mathbb{Z}/2)^d & \text{for } n \equiv 2(4) \end{cases}$$

and that there is an extension

$$0 \longrightarrow \mathrm{coker}(\mathrm{W}^s(R) \oplus \mathrm{W}^q(R_2) \to \mathrm{W}^s(R_2)) \longrightarrow \mathrm{L}^q_3(R) \longrightarrow \mathrm{L}^s_3(R) \longrightarrow 0.$$

**Exercise 4.7:** Show that the signature of a unimodular, even symmetric form over $\mathbb{Z}$ is divisible by 8, and that there is a quadratic form of signature 8. Recall that $b$ is even if $b(x,x) \in 2\mathbb{Z}$ for all $x$. Hints:

(1) Show that an even form admits a characteristic element $c$, i.e. an element such that $b(x,x) \equiv b(x,c)(2)$ for all $x$.

(2) Show that the element $b(c,c) \in \mathbb{Z}/8$ is well-defined (i.e. independent of the choice of characteristic element $c$).

(3) Show that it suffices to prove that $b \oplus (1,-1)$ has signature divisible by 8.

(4) For this form, find a characteristic element $c$ with $b(c,c) = \mathrm{sgn}(b)$ and find $c'$ with $b(c',c') = 0$.

**Exercise 4.8:** Show that the map $\mathrm{W}^q(\mathbb{Z}) \to \mathrm{W}^q(\mathbb{Z}/2)$ is the zero map.

**Exercise 4.9:** Let $R$ be a Dedekind ring, $K = \mathrm{Frac}(R)$ a global field, and assume that the number of dyadic primes is at least 2. Show that

$$\mathrm{coker}(\mathrm{W}^s(R) \oplus \mathrm{W}^q(R_2) \to \mathrm{W}^s(R_2)) \neq 0.$$

**Exercise 4.10:** Show that dévissage fails for quadratic L-theory. Hint: Consider the Dedekind ring $\mathbb{Z}$ and the localisation $\mathbb{Z}[\frac{1}{2}]$.