The existence and properties of rectangular structures

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Part 1. Fraïssé theory

Definition

A class of finite structures $\mathcal{K}$ is a *Fraïssé class*, if satisfies the following:

1. Has countably many isomorphism types,
2. Is closed under taking substructures,
3. Satisfies the Joint Embedding Property,
4. Satisfies the Amalgamation Property.

Theorem (Fraïssé, 1954)

If $\mathcal{K}$ is a Fraïssé class, then there exists a unique countable structure $\mathbb{K}$, called the Fraïssé limit of $\mathcal{K}$, satisfying the following:

1. *The class of finite substructures of $\mathbb{K}$ coincides with $\mathcal{K}$ (universality),*
2. *For each $a \in \mathcal{K}$, each embedding $i : a \to \mathbb{K}$, and $a \subseteq b \in \mathcal{K}$, there exists an extension of i to b, i.e.*

$$\bar{i} : b \to \mathbb{K},$$

such that

$$\bar{i} \restriction a = i.$$

Proposition

If $\mathcal{K}$ is the class of finite models of a first order theory $\mathcal{T}$, then $\mathbb{K}$ is the universal object if size ω.

Proposition

If $\mathcal{K}$ is a Fraïssé class with the Fraïssé limit $\mathbb{K}$, then each isomorphism between finite substructures of $\mathbb{K}$ extends to an automorphism of $\mathbb{K}$ ([ultra]homogeneity).

Examples:

1. The order of rationals $(\mathbb{Q}, \leq)$. (linear orders)
2. The countable universal, homogeneous partial order. (partial orders)
3. The countable atomless Boolean algebra. (Boolean algebras)
4. The random graph. (graphs)
5. The random tournament. (tournaments)
6. The Hall's locally finite group. (finite groups)
7. The rational Urysohn space. (finite rational metric spaces)

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Corollary

Uniqueness of the above objects.

For our purposes we focus on cases where:

1. Structures are relational.
2. The class $\mathcal{K}$ satisfies the Strong Amalgamation Property.

Definition

For a class $\mathcal{K}$ we define the forcing

$$\mathrm{Fn}(S, \mathcal{K}, \omega) = \{\mathbb{A} \in \mathcal{K} \mid A \in [S]^{<\omega}\},$$

where A denotes the universe of $\mathbb{A}$.
The order is the reversed inclusion of substructures.

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Familiar case: take as $\mathcal{K}$ the class of finite sets a together with mappings $p : a \to 2$.

Let $\mathcal{LO}$ be the class of (finite) linear orders. What does $\mathrm{Fn}(\omega, \mathcal{LO}, \omega)$ do?

- It adds a Cohen real.
- It adds a very complicated ordering $(\omega, \leq_G)$, isomorphic to $(\mathbb{Q}, \leq)$.
- Every infinite ground model set $S \subseteq \omega$ will be dense in $(\omega, \leq_G)$.

More generally:

Proposition

If S is countable, this is just a version of the Cohen forcing, that adds an isomorphic copy of the Fraïssé limit of $\mathcal{K}$, living on S.

Proof.

A density argument. □

Proposition

For any set S, $\mathrm{Fn}(S, \mathcal{K}, \omega)$ is Knaster.

Remark: This is due to the Strong Amalgamation Property. Otherwise, it may collapse ω_1.

Theorem (K. 2021)

Assume S is uncountable, and $\mathcal{K}$ is any of the following classes: graphs, linear orders, partial orders, tournaments, rational metric spaces. Then

$$\mathrm{Fn}(S, \mathcal{K}, \omega) \Vdash \text{“ the generic structure on } S \text{ is rigid”}.$$

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Theorem (K. 2021)

If $|S| > \omega_1$, and CH holds, then $\mathrm{Fn}(S, \mathcal{LO}, \omega_1)$ forces that the generic linear order is rigid in any ccc extension of the universe.

Part 2. Meanwhile, in a mostly unrelated part of mathematics...

A set $A \subseteq \mathbb{R}$ is *increasing* if it is uncountable and the following holds: For each sequence of pairwise disjoint tuples $\{(a^\xi_1, \ldots, a^\xi_k) \mid \xi < \omega_1\} \subseteq A^k$ there exists $\xi < \eta < \omega_1$ such that for all $i, j = 1, \ldots, k$

$$a^\xi_i < a^\eta_i \iff a^\xi_j < a^\eta_j.$$

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Proposition

If A is increasing, then A is not isomorphic to $-A = \{-a |\ a \in A\}$.

Theorem (Avraham-Shelah, 1981)

The existence of an increasing set is consistent with $\mathrm{ZFC} + \mathrm{MA} + 2^\omega = \omega_2$.

We start with a model $V \models \mathrm{CH}$, and an increasing set A.

Definition

The forcing $\mathbb{P}$ satisfies A-cc if for each sequence of pairwise disjoint k-tuples $\{(p_\xi, a_1^\xi, \ldots, a_k^\xi) | \xi < \omega_1\}$ there exist $\xi < \eta < \omega_1$ such that:

- p_ξ and p_η are compatible,
- for all $i, j = 1, \ldots, k$

$$a_i^\xi < a_i^\eta \iff a_j^\xi < a_j^\eta.$$

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1. A finite support iteration of A-cc forcings is A-cc.
2. If $\mathbb{P}$ is A-cc, then $\mathbb{P} \Vdash$ "A is increasing.".

The conclusion is that

$$\mathrm{ZFC} + \mathrm{MA}(A\text{-cc}) + \text{“}2^{\omega} = \omega_2\text{”} + \text{“}A \text{ is increasing.”}$$

is consistent. But in fact, $\mathrm{MA}(A\text{-cc})$ implies MA.

Theorem (Avraham-Shelah, 1981)

It is consistent with ZFC *that there exists an uncountable set of reals* A *with the property that each uncountable partial function* $f \subseteq A \times A$ *is non-decreasing on some uncountable set.*

Proof.

Let $V \models \mathrm{ZFC} + \mathrm{MA}_{\omega_1} + A$ " is increasing". Fix an uncountable partial function $f \subseteq A \times A$.

- If f is constant on an uncountable set, there is nothing to do.
- Otherwise, we can assume that f is 1-1. We apply MA_{ω_1} to the partial order

$$\{E \in [\mathrm{dom}(f)]^{<\omega} \mid \quad f \restriction E \text{ is increasing.}\}.$$

□

The funny observation:

Proposition

$$\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega) \Vdash \text{“}(\omega_1, \dot{\leq}) \text{ is increasing”}.$$

Proof:
Take a sequence of names $\{(\dot{\alpha}_1^\xi, \ldots, \dot{\alpha}_k^\xi)\}_{\xi<\omega_1}$ for pairwise disjoint k-tuples from ω_1. Fix a condition $p \in \mathbb{P} = \mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$. For every $\xi < \omega_1$ we may find a condition $p_\xi \leq p$ such that p_ξ decides $\{\alpha_1^\xi, \ldots, \alpha_k^\xi\}$ and the generic order on this set. We may assume that $\{\alpha_1^\xi, \ldots, \alpha_k^\xi\} \subseteq p_\xi = (p_\xi, \leq_\xi)$.

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We choose a suitable Δ-system $\{p_\xi |\ \xi \in S\}$, $S \subseteq \omega_1$, with a root $R = (R, \leq_R)$. We may also guarantee that for each $i = 1, \ldots, k$ all extensions of the form $(R, \leq_R) \subseteq (R \cup \{\alpha_i^\xi\}, \leq_\xi)$ are pairwise isomorphic. For any $\xi \neq \eta \in S$ we can amalgamate p_ξ and p_η, so that we obtain $(q, \leq_q) \supseteq (p_\xi, \leq_\xi), (p_\eta, \leq_\eta)$, and

$$a_i^\xi <_q a_i^\eta,$$

for $i = 1, \ldots, k$.

Can we apply this idea to other structures added by forcings $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$?

Definition

A metric space (X, d) is *rectangular* if it is uncountable, and for every family of tuples $\{(x_1^\xi, \dots, x_n^\xi) | \ \xi < \omega_1\} \subseteq X^k$ there exists $\xi < \eta < \omega_1$ such that $(x_1^\xi, \dots, x_n^\xi) \circledast (x_1^\eta, \dots, x_n^\eta)$.

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Where $(x_1, \ldots, x_n) \circledast (y_1, \ldots, y_n)$ stands for the conjunction of the following axioms

A1 $\forall\, i, j = 1, \ldots, n\ (d(x_i, y_i) = d(x_j, y_j))$

A2 $\forall\, i, j = 1, \ldots, n\ (d(x_i, x_j) = d(y_i, y_j))$

A3 $\forall\, i, j = 1, \ldots, n\ (x_i \neq x_j \implies d(x_i, x_j) = d(x_i, y_j))$

Proposition

Let Metr denote the class of all rational metric spaces. $\mathrm{Fn}(\omega_1, \mathrm{Metr}, \omega) \Vdash$ "$(\omega_1, \dot{d})$ is rectangular".

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We denote the generic space $(\mathcal{M}, d)$. It contains a dense isomorphic copy of the rational Urysohn space.

Definition

A forcing $\mathbb{P}$ satisfies $(\mathcal{M}, d)$-cc if for each sequence of pairwise disjoint k-tuples $\{(p_\xi, x_1^\xi, \ldots, x_k^\xi) | \xi < \omega_1\} \subseteq (\mathcal{M}, d)^k$ there exist $\xi < \eta < \omega_1$ such that:

- p_ξ and p_η are compatible,
- for all $i, j = 1, \ldots, k$

$$(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta).$$

Like the last time, we have

Proposition

- A finite support iteration of $(\mathcal{M}, d)$-cc forcings is $(\mathcal{M}, d)$-cc.
- If $\mathbb{P}$ satisfies $(\mathcal{M}, d)$-cc, then $\mathbb{P} \Vdash$ "$(\mathcal{M}, d)$ is rectangular".

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- If $\mathbb{P}$ satisfies $(\mathcal{M}, d)$-cc, then $\mathbb{P} \Vdash$ "$(\mathcal{M}, d)$ is rectangular".

Theorem (K. 2021)

The theory

$$\mathrm{ZFC} + \mathrm{MA}((\mathcal{M}, d)\text{-}cc) + \text{“}2^\omega = \omega_2\text{”} + \text{“}(\mathcal{M}, d) \textit{ is rectangular”}$$

is consistent. Therefore also

$$\mathrm{ZFC} + \mathrm{MA} + \text{“}2^\omega = \omega_2\text{”} + \text{“}(\mathcal{M}, d) \textit{ is rectangular”}.$$

Theorem (K. 2021)

If is consistent with $\mathrm{ZFC} + \mathrm{MA} + \text{“}2^\omega = \omega_2\text{”}$ *that there exists an uncountable, separable metric space* $(\mathcal{M}, d)$ *such that any uncountable partial 1-1 function* $f \subseteq \mathcal{M} \times \mathcal{M}$*, is an isometry on an uncountable subspace.*

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Proof.

Apply MA_{ω_1} to the partial order

$$\mathbb{P}_f := \{E \in [\mathrm{dom}(f)]^{<\omega} | \quad f \restriction E \text{ is an isometry}\}.$$

If $(\mathcal{M}, d)$ is rectangular, then $\mathbb{P}_f$ is ccc. □

Part 3. Homogeneity

Theorem

MA_{ω_1} *implies that if there exists an* ω_1*-dense increasing linear order, it is unique to reversing the order. Moreover, it is (ultra)homogeneous.*

Definition

Let (X,d) be a metric space.

1. A set $D \subseteq X$ is a *saturating subset* if for any finite set $F \subseteq X$, any any point $x \in X$, there is $e \in D$ such that
$$d(f,x) = d(f,e)$$
for every $f \in F$.
2. (X,d) is *separably saturated* if it has a countable saturating subset.
3. (X,d) is hereditarily separably saturated (HSS) if for any countable $E \subseteq X$, $(X \setminus E, d)$ is separably saturated.

The idea behind it, is to have a metric analog of a separable linear order.

Proposition

$\mathrm{Fn}(\omega_1, \mathrm{Metr}, \omega)$ forces that the generic space is HSS.

Proof: By a density argument, every infinite ground model subset is a saturating subset.

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Theorem (K. 2021)

Assume MA_{ω_1}, and let (X, d) be any HSS rectangular rational metric space of size ω_1.

1. *If $Y \subseteq X$ is an uncountable HSS space, then X and Y are isometric.*
2. *(X, d) is ultrahomogeneous.*

From the previous part, we know that the existence of a HSS rectangular rational metric space is consistent with MA_{ω_1}.

Instead of rational metric spaces, we can look at metric space with distances in any given countable set $K \subseteq [0, \infty)$. In the case $K = \{0, 1, 2\}$ these are just graphs (even though in such class the AP might fail).

Proposition

Assume MA_{ω_1}. If G is a rectangular graph, then each uncountable partial 1-1 function $f \subseteq G \times G$ is a homomorphism on an uncountable set.

Definition

Let (G, E) be any graph.

1. A set $D \subseteq G$ is a *saturating subset* if for any finite, disjoint sets $A, B \subseteq G$, there is $d \in D$ such that
   1. $\forall\, a \in A \quad \{a, d\} \in E$,
   2. $\forall\, b \in B \quad \{b, d\} \notin E$.
2. (G, E) is *separably saturated* if it has a countable saturating subset.
3. (G, E) is hereditarily separably saturated (HSS) if for any countable $H \subseteq G$, $(G \setminus H, E)$ is separably saturated.

Proposition

$\mathrm{Fn}(\omega_1, \mathrm{Graph}, \omega)$ forces that the generic graph is HSS.

Proof: Again, any infinite ground model set will be a saturating set.

Theorem (K. 2021)

Assume MA_{ω_1}, *and let G be any HSS rectangular graph of size* ω_1.

1. *If H is another HSS rectangular graph of size* ω_1, *then* $G \simeq H$ *iff G and the complement of H contain no common uncountable subgraph.*
2. *If* $H \subseteq G$ *is an uncountable HSS subgraph, then* $H \simeq G$.
3. *G is the unique HSS rectangular graph of size* ω_1, *up to taking the complement.*
4. *G is ultrahomogeneous.*

That’s it! Thank you for attention. More on the topic can be found in the preprints

Cohen-like first order structures, to appear in Annals of Pure and Applied Logic, https://arxiv.org/abs/2009.03552

What would the rational Urysohn space and the random graph look like if they were uncountable?, https://arxiv.org/abs/2102.05590