Determinacy, Partition Properties, and Combinatorics II

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Proving Partition Properties

We first show the easiest version of the partition relations.

Theorem

$\omega_1 \to (\omega_1)^{<\omega_1}$.

Remark

The same proof shows that if $\boldsymbol{\Gamma}$ is a $\boldsymbol{\Pi}^1_1$-like class ($\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and pwo($\boldsymbol{\Gamma}$)) then $\delta \to (\delta)^{<\omega_1}$.

Remark

The proof uses the Martin framework for showing partition properties from AD.

Proof: Fix $\epsilon < \omega_1$, fix a bijection $\pi \colon \omega \cdot \epsilon \to \omega$.

For $x \in \omega^\omega$, x codes a partial $\omega \cdot \epsilon$ sequence f_x by: for $\alpha < \omega \cdot \epsilon$,

$$f_x(\alpha) = |(x)_{\pi(\alpha)}|.$$

If f_x has domain $\omega \cdot \epsilon$ and is increasing, we let $F_x \colon \epsilon \to \omega_1$ be the function it induces: $F_x(\alpha) = \sup f_x \restriction \omega \cdot (\alpha + 1)$.

Given the partition $\mathcal{P}\colon [\omega_1]^{\epsilon}_{*} \to 2$, play the following game.

I plays out x, II plays out y.

If there is a least $\alpha < \omega \cdot \epsilon$ such that either $x_{\pi(\alpha)} \notin \mathrm{WO}$ or $y_{\pi(\alpha)} \notin \mathrm{WO}$ then I wins iff $x_{\pi(\alpha)} \in \mathrm{WO}$.

Otherwise, $f_x, f_y \colon \omega \cdot \epsilon \to \omega_1$ are defined. Let for $\alpha < \epsilon$,

$$F(\alpha) = \max\{F_x(\alpha), F_y(\alpha)\}.$$

Then I wins iff $\mathcal{P}(F) = 0$.

Suppose w.l.o.g. that II has a winning strategy τ.

For $\alpha < \omega \cdot \epsilon$ and $\beta < \omega_1$, Let

$$R_{\alpha,\beta} = \{x \colon \forall \alpha' \leq \alpha \; f_x(\alpha') \leq \beta\}$$

Here $f_x(\alpha') \leq \beta$ means $x_{\pi(\alpha')} \in \mathrm{WO}$ and $|x_{\pi(\alpha')}| \leq \beta$.

Easily, $R_{\alpha,\beta} \in \boldsymbol{\Delta}^1_1$. So, $\tau[R_{\alpha,\beta}] \in \boldsymbol{\Sigma}^1_1$.

By the payoff condition on the game,
$\tau[R_{\alpha,\beta}] \subseteq R_\alpha = \{y \colon \forall \alpha' \leq \alpha \; y_{\pi(\alpha')} \in \mathrm{WO}\}$.

A $\mathbf{\Sigma}^1_1$ subset of R_α codes a bounded set of ordinals.

Let $g(\alpha,\beta) = \sup\{|y_{\pi(\alpha)}\colon y \in \tau[R_{\alpha,\beta}]\}$.

Let C be c.u.b. and closed under g.

Then C' is homogeneous for $\mathcal{P}$:

Fix $F\colon \epsilon \to C'$ of the correct type and let $f\colon \omega \cdot \epsilon \to C$ induce F, i.e., $F(\alpha) = \sup\{f(\alpha')\colon \alpha' < \omega \cdot (\alpha + 1)\}$.

Fix x coding f.

Since $x \in R_{\alpha',f(\alpha')}$ for all $\alpha' < \omega \cdot \epsilon$, $f_y(\alpha') < g(\alpha', f(\alpha')) < f(\alpha' + 1)$.

So, $F_x = F_y = F$, and as τ is winning for II, $\mathcal{P}(F) = 1$.

We abstract the above argument into a definition.

Definition

Let $\lambda \leq \kappa$, where $\lambda \in \mathrm{On}$, κ a cardinal. We say κ is λ-reasonable if there is a non-selfdual pointclass $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$ and a map ϕ with domain ω^ω satisfying:

1. $\phi(x) \subseteq \lambda \times \kappa$.
2. $\forall f \colon \lambda \to \kappa \; \exists x \in \omega^\omega \; \phi(x) = f$.
3. $\forall \alpha < \lambda \; \forall \beta < \kappa \; R_{\alpha,\beta} \in \boldsymbol{\Delta}$, where $x \in R_{\alpha,\beta} \leftrightarrow \phi(x)(\alpha,\beta) \wedge (\phi(x)(\alpha,\beta') \to \beta' = \beta)$.
4. Suppose $\alpha < \lambda$, $A \in \exists^{\omega^\omega} \boldsymbol{\Delta}$, and $A \subseteq R_\alpha = \{x \colon \exists \beta < \kappa \; x \in R_{\alpha,\beta}\}$. Then $\exists \beta_0 < \kappa \; \forall x \in A \; \exists \beta < \beta_0 \; \phi(x)(\alpha,\beta)$.

Theorem (Martin)

Suppose κ is $\omega \cdot \lambda$ reasonable. Then $\kappa \rightarrow \kappa^{\lambda}$.

Proof: Exactly as in the previous proof.

We now show the strong partition relation at ω_1.

Theorem (Martin)

$\omega_1 \to (\omega_1)^{\omega_1}$.

Proof (J): We show there is a coding of the functions $f : \omega_1 \to \omega_1$ witnessing that ω_1 is ω_1-reasonable.

The main step is to analyze the measure on ω_1, and then convert this to an analysis of the subsets of ω_1 via an argument of Kunen.

The (cub) partition relation $\kappa \to (\kappa)^2$ gives that the ω-cofinal c.u.b. filter of κ is a normal measure W_1^1 on κ.

Let W_1^n denote the n-fold product of W_1^1.

Theorem ($\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)

Let μ be a measure on ω_1. Then μ is equivalent to W_1^n for some n (or to a principal measure).

Proof: Assume μ is non-principal.

Let $f_1 \colon \omega_1 \to \omega_1$ represent the least equivalence class such that f_1 is almost everywhere non-constant, and monotonically increasing.

Let $\nu_1 = f_1(\mu)$. Then $\nu_1 = W^1_1$. Fix a μ measure one set A_1 on which f_1 is monotonically increasing.

Let $g_1(\beta) = \sup\{\alpha \in A_1 \colon f_1(\alpha) \leq \alpha\}$.

Let x_1 be such that $\forall^* \beta_1 \; g_1(\beta_1) < |T_{x_1} \restriction \beta_1|$.

For μ almost all α, let $r_1(\alpha)$ be such that

$$\alpha = |T_{x_1} \restriction f_1(\alpha)(r_1(\alpha))|.$$

Now we proceed with the measure $r_1(\mu)$.

Consider the case r_1 not constant almost everywhere. Note that a.e. $r_1(\alpha) < f_1(\alpha)$.

Let f_2 represent the least μ equivalence class such that f_2 is not a.e. constant, and is a.e. monotonically increasing with respect to r_1.

That is, there is a μ measure one set A such that if α, α' are in A, $f_1(\alpha) = f_1(\alpha')$, and $r_1(\alpha) \leq r_1(\alpha')$, then $f_2(\alpha) \leq f_2(\alpha')$.

Note that there does not exist a c.u.b. $C \subseteq \omega_1$ and a μ measure one set A such that for all $\beta \in C$, $\{f_2(\alpha) \colon f_1(\alpha) = \beta \wedge \alpha \in A\}$ is bounded below $f_1(\alpha)$. [Otherwise r_2 is constant μ almost everywhere.]

Claim

We have $f_2(\mu) = W^1_1$.

For suppose $C \subseteq \omega_1$ is c.u.b. and $\forall^*_\mu \alpha \; f_2(\alpha) \notin C$.

Let $f_2' = \ell_C \circ f_2$ where $\ell(\gamma)$ is the largest element of $C \geq \gamma$.

Then for μ almost all α we have $f_2'(\alpha) < f_2(\alpha)$ and f_2' is monotonically increasing "on the f_1 blocks" with respect to r_1. Also, f_2' is not constant μ almost everywhere. This contradicts the definition of f_2.

Fix a μ measure one set $A_2 \subseteq A_1$ on which f_2 is monotonically increasing on the f_1 blocks with respect to r_1.

Define g_2 by:

$$g_2(\beta_2, \beta_1) = \sup\{r_1(\alpha) \colon \alpha \in A_2 \wedge f_1(\alpha) = \beta_1 \wedge f_2(\alpha) = \beta_2\}.$$

Then for W_1^2 almost all (β_2, β_1), $g_2(\beta_2) < \beta_1$. This follows from the monotonicity of f_2 on the f_1 blocks and the fact that f_2 is not constant μ almost everywhere.

For W_1^2 almost all (β_2, β_1), $g_2(\beta_2, \beta_1)$ depends only on β_2.

Fix x_2 such that for W_1^1 almost all β_2, $g_2(\beta_2) < |T_{x_2} \restriction \beta_2|$.

This then defines r_2: $\forall^*_\mu \alpha$

$$\alpha = |T_{x_1} \restriction f_1(\alpha)(|T_{x_2} \restriction f_2(\alpha)(r_2(\alpha))|)|$$

Continuing, we define $f_1, \dots, f_n$, $g_1, \dots, g_n$ for some n, reals $x_1, \dots, x_n$, and $r_1, \dots, r_n$ such that r_n is constant almost everywhere, say equal to δ.

We then have: $\forall^*_\mu \alpha$

$$\alpha = |T_{x_1}(f_1(\alpha))(|T_{x_2}(f_2(\alpha))(\cdots(|T_{x_n}(f_n(\alpha)(\delta))|)\cdots)|)|$$

We also have that if $F(\alpha) = (f_1(\alpha), \dots, f_n(\alpha)$, then $F(\mu) = W_1^n$.

Let $G(\beta_1, \dots, \beta_n) = |T_{x_n} \restriction (\beta_n)(G_{n-1}(\beta_1, \dots, \beta_{n-1}))|$, where $G_k(\beta_1, \dots, \beta_k) = |T_{x_k} \restriction \beta_k(G_{k-1}(\beta_1, \dots, \beta_{k-1}))|$,

and $G_0(\emptyset) = \delta$.

We have defined a μ measure one set A_n on which F is one-to-one and $F(\mu) = W_1^n$.

This completes the analysis of measures on ω_1.

Coding at ω_1

Fact (AD)

(Martin) The cone filter is a measure on the set $\mathcal{D}$ of Turing degrees.

Definition

Θ is the supremum of the lengths of the pwos of $\mathbb{R}$.

Fact

(Kunen) Let $\lambda < \Theta$. Then every countably additive filter $\mathcal{F}$ on λ can be extended to a measure on λ.

Proof: Let $\pi \colon \omega^\omega \to \mathcal{P}(\lambda)$ be onto (coding lemma).

Let ν be the Martin measure on $\mathcal{D}$.

For $d \in \mathcal{D}$, let

$$f(d) = \min \bigcap \{\pi(x) \colon x \in d \wedge \pi(x) \in \mathcal{F}\}.$$

Let $\mu = f(\nu)$.

Fix the Kunen tree T at ω_1.

We say $\tau \in \omega^\omega$ is a code for a c.u.b. set if $\forall x \in \mathrm{WO}\ \tau(x) \in \mathrm{WO}$.

Let $C_x = \{\alpha < \omega_1 : \forall \gamma < \alpha\ |T_x \restriction \gamma| < \alpha\}$.

Fact

For every c.u.b. $C \subseteq \omega_1$ there is a code x such that $C_x \subseteq C$.

Definition

A set $S \subseteq \omega_1$ is simple if there is a c.u.b. code τ, an $\alpha_0 < \omega_1$, $x_1, \ldots, x_n$ with T_{x_i} wellfounded such that

$$S = \{\alpha \colon \exists \alpha_1 < \cdots < \alpha_n \in C_\tau \; \alpha = h_n(\alpha_1, \ldots, \alpha_n; \vec{x})\}$$

where

$$h_i(\alpha_1, \ldots, \alpha_i; \vec{x}) = |T_{x_i} \restriction \alpha_i(h_{i-1}(\alpha_1, \ldots, \alpha_{i-1}; \vec{x}))|$$

and

$$h_0(\vec{x}) = \alpha_0.$$

A code for the simple set S a real of the form $(x_0; x_1, \dots, x_n; \tau)$ where τ is a c.u.b. code, $x_0 \in \mathrm{WO}$, and T_{x_i} are wellfounded.

Following an argument of Kunen we show:

Fact

Every $A \subseteq \omega_1$ is a countable union of simple sets.

Proof: Let $\mathcal{I}$ be the σ-ideal generated by the simple sets contained in A.

Assume toward a contradiction $\mathcal{I}$ is a proper ideal, and let μ be a measure on A extending the corresponding filter $\mathcal{F}$.

By the analysis of measures on ω_1, there are $x_1, \ldots, x_n$ with T_{x_i} wellfounded and an $\alpha_0 < \omega_1$ such that for all $B \subseteq \omega_1$ (assuming B is not bounded):

$$\mu(B) = 1 \leftrightarrow \exists \text{c.u.b. } C \subseteq \omega_1 \forall \beta_1 < \cdots < \beta_n \in C$$
$$h_n(\alpha_0; \beta_1, \ldots, \beta_n, x_1, \ldots, x_n) \in B.$$

Since $\mu(A) = 1$, we may fix a c.u.b. code τ, a $x_0 \in$ WO coding α_0, and the $x_1, \ldots, x_n$ above.

Let $S = S(x_0; x_1, \ldots, x_n; \tau)$ be the simple set given by these reals, so $S \subseteq A$.

Then $\mu(S) = 1$, but this contradicts $S \in \mathcal{I}$.

We now define the coding map ϕ. As a warm-up we first define a coding for subsets of ω_1, so $\phi(x) \subseteq \omega_1$.

View $x \in \omega^\omega$ as coding countably many $(x_0^i; x_1^i, \dots, x_{n_i}^i; \tau^i)$.

Set $\phi(x)(\alpha)$ iff $\exists i\ \alpha \in S(x^i) = S(x_0^i; x_1^i, \dots, x_{n_i}^i; \tau^i)$ iff

$$\exists i\ \exists \beta_1 < \cdots < \beta_{n_i} \in C_{\tau^i} \cap \alpha\ [|x_0^i| < \alpha \\ \wedge h(|x_0^i|; \beta_1, \dots, \beta_{n_i}; x_1^i, \dots, x_{n_i}^i) = \beta].$$

So, every $A \subseteq \omega_1$ is of the form $\phi(x)$ for some $x \in \omega^\omega$.

This is a $\mathbf{\Delta}^1_1$-coding of the subsets of ω_1:

For all $\alpha < \omega_1$, $\{x : \phi(x)(\alpha)\} \in \mathbf{\Delta}^1_1$.

We modify this coding to code functions from ω_1 to ω_1. So, $\phi(x) \subseteq \omega_1 \times \omega_1$.

It is not quite good enough to just regard $f : \omega_1 \to \omega_1$ as a subset of $\omega_1 \times \omega_1 \approx \omega_1$.

Suppose $f : \omega_1 \to \omega_1$ is increasing.

A simple subfunction $S \subseteq f$ is one where there is a c.u.b.code τ, $x_1, \dots, x_n$ with T_{x_i} wellfounded, and two $\gamma_0, \gamma_1 < \omega_1$ such that:

$$(\alpha, \beta) \in S \leftrightarrow \exists \max\{\gamma_0, \gamma_1\} < \beta_1 < \cdots < \beta_n < \alpha$$
$$[\beta_1, \dots, \beta_n \in C_\tau \wedge h(\gamma_0, \beta_1, \dots, \beta_n; \vec{x}) = \alpha$$
$$\wedge\, h(\gamma_1, \beta_1, \dots, \beta_n; \vec{x}) = \beta]$$

An argument similar to that for sets shows that every function f is a countable union of simple subfunctions.

- We let $X = f$, and analyze the measures on X.
- If μ is a measure on X, let $f_0 \colon X \to \omega_1$ represent the least equivalence class of a function which is not μ a.e. constant and monotonically increasing in the first argument (if $\alpha_1 \leq \alpha_2$, then $f_0(\alpha_1, \beta_1) \leq f_0(\alpha_2, \beta_2)$).
- $f_0(\mu) = W_1^1$ as before.
- Let $g_0(\delta) = \sup\{\max\{\alpha, \beta\} \colon (\alpha, \beta) \in X \wedge f_0(\alpha) \leq \delta\}$.
- The rest of the argument proceeds as before.

Coding at ω_1

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Every ultrafilter on a set X is countably additive (i.e., a measure).

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$$\mu(B) = 1 \leftrightarrow \exists \text{c.u.b. } C \subseteq \omega_1 \forall \beta_1 < \cdots < \beta_n \in C \\ h_n(\alpha_0; \beta_1, \dots, \beta_n, x_1, \dots, x_n) \in B.$$

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So, every $A \subseteq \omega_1$ is of the form $\phi(x)$ for some $x \in \omega^\omega$.

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Suppose $f\colon \omega_1 \to \omega_1$ is increasing.

A simple subfunction $S \subseteq f$ is one where there is a c.u.b.code τ, $x_1, \dots, x_n$ with T_{x_i} wellfounded, and two $\gamma_0, \gamma_1 < \omega_1$ such that:

$$
\begin{aligned}
(\alpha,\beta) \in S \leftrightarrow \exists \max\{\gamma_0,\gamma_1\} < \beta_1 < \cdots < \beta_n < \alpha \\
[\beta_1,\dots,\beta_n \in C_\tau \wedge h(\gamma_0,\beta_1,\dots,\beta_n;\vec{x}) = \alpha \\
\wedge\, h(\gamma_1,\beta_1,\dots,\beta_n;\vec{x}) = \beta]
\end{aligned}
$$

An argument similar to that for sets shows that every function f is a countable union of simple subfunctions.

- We let $X = f$, and analyze the measures on X.
- If μ is a measure on X, let $f_0 \colon X \to \omega_1$ represent the least equivalence class of a function which is not μ a.e. constant and monotonically increasing in the first argument (if $\alpha_1 \leq \alpha_2$, then $f_0(\alpha_1, \beta_1) \leq f_0(\alpha_2, \beta_2)$).
- $f_0(\mu) = W_1^1$ as before.
- Let $g_0(\delta) = \sup\{\max\{\alpha, \beta\} \colon (\alpha, \beta) \in X \wedge f_0(\alpha) \leq \delta\}$.
- The rest of the argument proceeds as before.

Non-partition results at ω_2

We give a new proof (Chan, J, Trang) of the Martin-Paris result that ω_2 does not have the strong partition property.

Theorem

$\omega_2 \not\to (\omega_2)^{\omega_2}$.

The new proof gives a specific partition witnessing the failure of the strong partition property.

Theorem (Chan, J, Trang)

Let $A \subseteq \omega_2$ and suppose there is a c.u.b. $C \subseteq \omega_2$ such that $A \cap C = cof_\omega \cap C$. Then $A \notin Ult_{W_1^1}$.

We use the following lemma.

Lemma (almost everywhere club uniformization)

Let $f \colon \omega_1 \to \mathcal{P}(\omega_1)$ with $\forall^* \alpha\ f(\alpha)$ contains a club. Then there is a club $C \subseteq \omega_1$ such that $\forall^* \alpha \in C \setminus \{\alpha + 1\} \subseteq f(\alpha)$.

Proof: Partition $f \colon \omega_1 \to \omega_1$ of the correct type according to whether $\overline{\mathrm{ran}(f)} \setminus \{f(0)\} \subseteq A_{f(0)}$. On the homogeneous side this must hold, say by C. Fix $f \colon \omega_1 \to C$ of the correct type. Then $\overline{\mathrm{ran}(f)}$ witnesses the Lemma.

Lemma

Assume $\kappa \to \kappa^\kappa$. Let μ be a normal measure on κ. Let $\delta = j_\mu(\kappa)$. Then if $D \subseteq \delta$ is c.u.b., there exists a c.u.b. $C \subseteq \kappa$ with $j_\mu(C) \subseteq D$.

Proof: Partition $f, g \colon \kappa \to \kappa$ of the correct type with $f(\alpha) < g(\alpha) < f(\alpha + 1)$ according to whether $[g]_\mu > N_D([f]_\mu)$.

On the homogeneous side this holds. Say $C \subseteq \kappa$ is homogeneous for this side.

Then $j_\mu(C') \subseteq D$.

Proof of Theorem: Let $C \subseteq \omega_2$ be as in the Theorem, so $A \cap \mathrm{cof}_\omega = C \cap \mathrm{cof}_\omega$. Suppose $A = [F]_{W_1^1}$, where $F(\alpha) \subseteq \omega_1$.

Let $C_0 \subseteq \omega_1$ be such that $j_{W_1^1}(C_0) \subseteq C$.

Case 1. $\forall^* \alpha\ F(\alpha)$ contains a club.

By the Lemma, let $C_1 \subseteq \omega_1$ be such that $\forall^* \alpha\ C_1 \setminus \{\alpha + 1\} \subseteq F(\alpha)$.

Let $C_2 = C_0 \cap C_1$.

Fix $f : \omega_1 \to C_2$ such that $f(\alpha)$ has uniform cofinality α.

Then $[f]$ has cofinality ω_1 and is in $j(C_0) \subseteq C$.

So by the assumed property of A, $[f] \notin A$.

On the other hand, $\forall^* \alpha \; f(\alpha) \in C_1 \setminus \{\alpha + 1\} \subseteq F(\alpha)$. So, $[f] \in [F] = A$.

Case 2: $\forall^* \alpha \; F(\alpha)$ is disjoint from a club.

The argument is similar, but now taking $f \colon \omega_1 \to C_2$ such that $f(\alpha)$ has uniform cofinality ω.

Proof of $\omega_2 \not\to (\omega_2)^{\omega_2}$.

Consider the partition $\mathcal{P}\colon [\omega_1]_*^{\omega_1} \to \{0, 1\}$:

$$\mathcal{P}(f) = 1 \text{ iff } f \in \mathrm{Ult}_{W_1^1}.$$

Suppose $D \subseteq \omega_2$ were homogeneous for $\mathcal{P}$.

Let $C \subseteq \omega_1$ be c.u.b. with $j_{W_1^1}(C) \subseteq D$.

Case I: D is homogeneous for the $F \notin \mathrm{Ult}_{W^1_1}$ side.

Let $f\colon \omega_1 \to C$ be of the correct type. Then $F = j_{W^1_1}(f)\colon \omega_2 \to D$ is of the correct type. So, $F \notin \mathrm{Ult}_{W^1_1}$, a contradiction.

Case II: D is homogeneous for the $F \in \mathrm{Ult}_{W^1_1}$ side.

We show in this case that $\mathcal{P}(\omega_2) \subseteq \mathrm{Ult}_{W^1_1}$, a contradiction to a previous lemma.

Fix $H\colon \omega_2 \to D$ of the correct type.

Let $A \subseteq \omega_2$.

Define:

$$F(\alpha) = H(2 \cdot \alpha)$$

$$G(\alpha) = \begin{cases} H(2 \cdot \alpha) & \text{if } \alpha \in A \\ H(2 \cdot \alpha + 1) & \text{if } \alpha \notin A \end{cases}$$

By the homogeneity of D, both F, G are in $\mathrm{Ult}_{W_1^1}$.

Then easily $A \in \mathrm{Ult}_{W_1^1}$.

Note $\alpha \in A$ iff $F(\alpha) = G(\alpha)$.

Let $F = [f]_{W_1^1}$, $G = [g]_{W_1^1}$.

For any $\alpha = [k]_{W_1^1} < \omega_2$, $\alpha \in A$ iff $\forall^*_{W_1^1}\beta\ f(\beta)(k(\beta)) = g(\beta)(k(\beta))$.

So,

$$A = [\beta \mapsto \{\gamma < \omega_1 : f(\beta)(\gamma) = g(\beta)(\gamma)\}]_{W_1^1}.$$