Combinatorial Sets of Reals, II

Induced logarithmic measures and coherent systems

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Question

Is it consistent that $\aleph_1 < \mathfrak{b} < \mathfrak{s}$?

Recall Shelah's creature poset $\mathbb{Q}$.

Definition

1. A family of pure conditions $\mathscr{C}$ is centered if whenever $X, Y \in \mathscr{C}$ there is $R \in \mathscr{C}$ which is their common extension.
2. If $\mathscr{C}$ is a family of pure conditions, then $\mathbb{Q}(\mathscr{C})$ is the suborder of $\mathbb{Q}$ consisting of all $(u, T) \in \mathbb{Q}$ such that $\exists R \in \mathscr{C}(R \leq T)$.

Remark

We work exclusively with centered families $\mathscr{C}$ which are closed with respect to final segments. Note that any two conditions of $\mathbb{Q}(\mathscr{C})$ are compatible as conditions in $\mathbb{Q}(\mathscr{C})$ iff they are compatible in $\mathbb{Q}$.

Definition (Induced logarithmic measure)

Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family. Then P induces a logarithmic measure h on $[\omega]^{<\omega}$ defined recursively on the $|s|$ for $s \in [\omega]^{<\omega}$ as follows:

1. $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$
2. $h(e) > 0$ iff $e \in P$ and $|e| > 1$
3. for $l \geq 1$, $h(e) \geq l+1$ iff $|e| > 1$ and whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq l$ or $h(e_1) \geq l$.

Then $h(e) = l$ if l is maximal for which $h(e) \geq l$. The elements of P are called positive sets and h is said to be induced by P.

Theorem (V.F., J. Steprans, 2008)

Let κ be a regular uncountable cardinal, $\mathrm{cov}(\mathscr{M}) = \kappa$,

$$\mathscr{H} \subseteq {}^{\omega}\omega$$

be an unbounded, $\leq^*$-directed family of cardinality κ. Assume that

$$\forall \lambda < \kappa (2^\lambda \leq \kappa).$$

Then, there is a centered family

$$\mathscr{C}_{\mathscr{H}}$$

of pure conditions, such that $|\mathscr{C}_{\mathscr{H}}| = \kappa$ and such that

1. $\Vdash_{\mathbb{Q}(\mathscr{C}_{\mathscr{H}})}$ "$\mathscr{H}$ is unbounded"
2. $\mathbb{Q}(\mathscr{C}_{\mathscr{H}})$ adds a real not split by the ground model reals.

Theorem (V.F., J. Sterpans, 2008)

(GCH) Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

1. Can we do better? Is it consistent that there is an arbitrarily large spread between $\mathfrak{b}$ and $\mathfrak{s}$?
2. The techniques leading to the above result, heavily use the fact that the cardinality of the unbounded family is κ!
3. So, we do need a new approach. But, before that, an observation:

Lemma (V.F., B. Irrgang, 2010)

Let $\mathscr{C}$ be a centered family of pure conditions in $\mathbb{Q}$. Then $\mathbb{Q}(\mathscr{C})$ is densely embedded in $\mathbb{M}(\mathscr{F}_{\mathscr{C}})$, where

$$\mathscr{F}_{\mathscr{C}} = \{X \in [\omega]^{\omega} : \exists T \in \mathscr{C}(\mathrm{int}(T) \subseteq X)\}.$$

Proof

The mapping $(u, T) \mapsto (u, \mathrm{int}(T))$ is a dense embedding. □

Corollary

Let κ be a regular uncountable cardinal, $\text{cov}(\mathscr{M}) = \kappa$, $\mathscr{H} \subseteq {}^\omega\omega$ is an unbounded, $\leq^*$ directed family of cardinality κ. Assume that $\forall\lambda < \kappa(2^\lambda \leq \kappa)$. Then, there is an ultrafilter $\mathscr{U}_\mathscr{H}$ such that

$$\Vdash_{\mathbb{M}(\mathscr{U}_\mathscr{H})} \text{“}\mathscr{H} \text{ is unbounded”}.$$

Canjarness

1. There are earlier examples of ultrafilters, such that the relativized Mathias forcing, preserves the unboundedness of a given family: In Blass-Shelah consistency proof of $\mathfrak{u} = \kappa < \mathfrak{d} = \lambda$ from 1989, one can find the construction of a special ultrafilter $\mathscr{U}_\mathscr{H}$, associated to a set of Cohen reals $\mathscr{H}$, such that $\mathbb{M}(\mathscr{U}_\mathscr{H})$ preserves the unboundedness of $\mathscr{H}$.
2. In both instances, $|\mathscr{H}| = \mathfrak{c}$.

Definition: Hechler's poset for adding a dominating real, $\mathbb{D}$:

The poset consists of all pairs $(s, f) \in {}^{<\omega}\omega \times {}^{\omega}\omega$ such that $(s_1, f_1) \leq (s_2, f_2)$ iff

- s_2 is an initial segment of s_1
- for all $i \in \mathrm{dom}(s_1) \backslash \mathrm{dom}(s_2)$, $s_1(i) \geq f_2(i)$;
- for all $i \in \omega$, $f_2(i) \leq f_1(i)$.

If only elements of a given family $\mathscr{F} \subseteq {}^{\omega}\omega$ are allowed as second coordinates in the the above definition, we speak about restricted Hechler forcing, denoted $\mathbb{D}^{\mathscr{F}}$ of $\mathbb{D}(\mathscr{F})$.

Definition: Hechler's poset for adding a mad family, $\mathbb{H}(\gamma)$:

Let γ be an ordinal. Then, $\mathbb{H}(\gamma)$ is the poset of all finite partial functions

$$p : \gamma \times \omega \to 2$$

such that $\mathrm{dom}(p) = F_p \times n_p$ where $F_p \in [\gamma]^{<\omega}$, $n_p \in \omega$.

The order is given by $q \leq p$ if

1. $p \subseteq q$
2. $|q^{-1}(1) \cap F^p \times \{i\}| \leq 1$ for all $i \in n_q \backslash n_p$.

The complete embedding property

Lemma

Let G_I be $\mathbb{H}(I)$-generic, $\mathscr{A}_I = \{A_i\}_{i \in I}$, $A_i = \{n : \exists p \in G_I p(i,n) = 1\}$ for $i \in I$.

Then $\mathscr{A}_I = \{A_i\}_{i \in I}$ is almost disjoint.

1. If I is uncountable, then $\mathscr{A}_I$ is maximal almost disjoint.
2. If $J \subset I$ then $\mathbb{H}(J) \lessdot \mathbb{H}(I)$ and the quotient is 'good'.

Lemma: Diagonalization

Let $I = J \cup \{i\}$, where $i \notin J$. Then $G_I = G_J * G(i)$:

1. $\mathscr{A}_I = \mathscr{A}_J \cup \{A_i\}$ is almost disjoint
2. If $X \in V[G_J] \cap ([\omega]^\omega \setminus \mathscr{I}(\mathscr{A}_J))$, then $X \cap A_i$ is infinite.

Definition: Elimination of Intruders

Let $M \subseteq N$ be models, $\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$ and let $A \in N \cap [\omega]^\omega$ such that $\mathscr{B} \cup \{A\}$ is almost disjoint. We say that

$$A \text{ diagonalizes } \mathscr{B} \text{ over } M$$

if for every

$$X \in M \cap ([\omega]^\omega \setminus \mathscr{I}(\mathscr{B})),$$

where $\mathscr{I}(\mathscr{B})$ denotes the ideal generated by $\mathscr{B}$, we have

$$|A \cap X| = \infty.$$

Remark

- Alternatively, we say that A eliminates $\mathscr{B}$-intruders over M.
- Thus, A_i diagonalizes $\mathscr{A}_I$ over $V[G_J]$.

Persistent (!?) Elimination of Intruders

Let $M \subseteq N$ be models,

$$\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega, \; A \in N \cap [\omega]^\omega$$

and suppose A eliminates $\mathscr{B}$-intruders over M. Let

$$\bar{\mathbb{P}}_s = \langle \mathbb{P}^s_\alpha : \alpha \leq \lambda \rangle,$$

where $s \in \{0,1\}$ be FS iterations in M, N (for $s = 0$ and $s = 1$ respectively) such that $\mathbb{P}^0_\alpha \lessdot \mathbb{P}^1_\alpha$ for each α. Then:

1. $\mathscr{B} \cup \{A\}$ is almost disjoint throughout(!). However:
2. Is it necessarily the case that

$$A \text{ eliminates } \mathscr{B}\text{-intruders over } M^{\mathbb{P}_\alpha}$$

   for each $\alpha \leq \lambda$?

Definition (Strong diagonalizaiton)

Let $M \subseteq N$ be models of set theory, $\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$ and let $A \in N \cap [\omega]^\omega$. Then $(*^{M}_{\mathscr{B}}\ ^{N}_{A})$ holds if for every

$$h : \omega \times [\gamma]^{<\omega} \to \omega$$

$h \in M$ and every $m \in \omega$ there are $n \geq m$ and $F \in [\gamma]^{<\omega}$ such that

$$[n, h(n,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A.$$

We say that A strongly diagonalizes $\mathscr{B}$ over M.

Lemma

If $(*^{M}_{\mathscr{B}}\ ^{N}_{A})$, then A eliminates $\mathscr{B}$-intruders.

Lemma (Persistent elimination of intruders)

Let $\bar{\mathbb{P}}_s = \langle \mathbb{P}_{s,n} : n \leq \omega \rangle$, $s \in \{0, 1\}$ be FS iterations such that

$$\mathbb{P}_{0,n} \lessdot \mathbb{P}_{1,n} \text{ for all } n.$$

Let $V_{s,n} = V^{\mathbb{P}_{s,n}}$. Let

$$\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq V_{0,0} \cap [\omega]^\omega,\ A \in V_{1,0} \cap [\omega]^\omega.$$

If A strongly diagonalizes $\mathscr{B}$ over $V_{0,n}$ for each n, then

$$A \text{ strongly diagonalizes } \mathscr{B} \text{ over } V_{0,\omega}.$$

Lemma (Strong diagonalization)

Let $G_{\gamma+1}$ be $\mathbb{H}(\gamma+1)$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{H}(\gamma)$ and $A_\gamma = \{A_\alpha\}_{\alpha<\gamma}$, where

$$A_\alpha = \{i : \exists p \in G_{\gamma+1} p(\alpha, i) = 1\},$$

for $\alpha \leq \gamma$. Then

$$(*^{V[G_\gamma]}_{\mathscr{A}_\gamma} \quad {}^{V[G_{\gamma+1}]}_{A_\gamma})$$

holds.

Lemma (Strong diagonalization and ultrafilters)

Let $M \subseteq N$ be models, $\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that

$$A \text{ strongly diagonalizes } \mathscr{B} \text{ over } M.$$

Let $\mathscr{U}$ be an ultrafiler in M. Then $\exists$ an ultrafilter $\mathscr{V}$ in N such that $\mathscr{U} \subseteq \mathscr{V}$ and

1. every maximal antichain of $\mathbb{M}(\mathscr{U})$ which belongs to M is a maximal antichain of $\mathbb{M}(\mathscr{V})$ in N,
2. for every $\mathbb{M}(\mathscr{V})$-generic filter G over N, which by item (1) is $\mathbb{M}(\mathscr{U})$-generic over N, the set

$$A \text{ strongly diagonalizes } \mathscr{B} \text{ over } M[G].$$

Lemma (... more strong diagonalizaiton)

Let $M \subseteq N$ be models, $\mathbb{P} \in M$ a poset, G a $\mathbb{P}$-generic filter over N. Let

$$\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega,\ A \in N \cap [\omega]^\omega.$$

If A strongly diagonalizes $\mathscr{B}$ over M, then

$$A \text{ strongly diagonalizes } \mathscr{B} \text{ over } M[G].$$

Assume GCH and let $\kappa < \lambda$ be regular uncountable cardinals. Let

$$f : \{\eta < \lambda : \eta \equiv 1 \mod 2\} \to \kappa$$

be an onto mapping such that

$$\forall \alpha < \kappa, f^{-1}(\alpha) \text{ is cofinal in } \lambda.$$

Recursively define a system of finite support iterations

$$\langle \langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda \rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda \rangle \rangle$$

as follows:

- For all α, ζ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$.

- If $\zeta = 0$ then for all $\alpha \leq \kappa$, let $\mathbb{P}_{\alpha,0}$ be Hechler's poset for adding an a.d. family $\mathscr{A}_\alpha = \{A_\beta\}_{\beta<\alpha}$. Note that for $\alpha \geq \omega_1$, $\mathscr{A}_\alpha$ is maximal almost disjoint in $V_{\alpha,0}$.

- If $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 2$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}(\dot{\mathscr{U}}_{\alpha,\eta})$ where $\dot{\mathscr{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$,

$$\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathscr{U}}_{\alpha,\eta} \subseteq \dot{\mathscr{U}}_{\beta,\eta}.$$

- If $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 2$, then
  - if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion.
  - If $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.

- If ζ is a limit, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$.

Furthermore, we guarantee that the construction satisfy the following properties:

1. $\forall \zeta \leq \lambda$ and $\forall \alpha < \beta \leq \kappa$,
$$\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}.$$

2. For all $\zeta \leq \lambda$, $\forall \alpha < \kappa$ the strong elimination of intruders property
$$(*\,{}^{V_{\alpha,\zeta}}_{\mathscr{A}_\alpha}\ {}^{V_{\alpha+1,\zeta}}_{A_{\alpha+1}})$$
implying that A_α eliminates $\mathscr{A}_\alpha$-intruders over $V_{\alpha,\zeta}$ for each $\zeta \leq \lambda$.

Lemma

The construction satisfies that for all $\alpha < \beta \leq \kappa$, and all $\zeta < \eta \leq \lambda$,

$$\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\eta}.$$

Lemma

For each $\zeta \leq \lambda$:

1. For every $p \in \mathbb{P}_{\kappa,\zeta}$ there is $\alpha < \kappa$ such that $p \in \mathbb{P}_{\alpha,\zeta}$.
2. For every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\dot{f}$ there is $\alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.

Theorem (V. F., J. Brendle, 2010)

$$V_{\kappa,\lambda} \vDash \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda.$$

Proof: $\mathfrak{a} \leq \kappa$

- We will show that family $\{A_\alpha\}_{\alpha<\kappa}$ remains maximal in $V_{\kappa,\lambda}$.
- Otherwise $\exists B \in V_{\kappa,\lambda} \cap [\omega]^\omega$ such that

$$\forall \alpha < \kappa |B \cap A_\alpha| < \omega.$$

However there is $\alpha < \kappa$ such that

$$B \in V_{\alpha,\lambda} \cap [\omega]^\omega.$$

- Note that $B \notin \mathscr{I}(\mathscr{A}_\alpha)$. Then strong elimination of intruders

$$(*\begin{smallmatrix} V_{\alpha,\lambda} & V_{\alpha+1,\lambda} \\ \mathscr{A}_\alpha & A_{\alpha+1} \end{smallmatrix})$$

holds and so $|B \cap A_{\alpha+1}| = \infty$, which is a contradiction.
- Thus, $\mathfrak{a} \leq \kappa$.

Proof: $\kappa \leq \mathfrak{b}$ and so $\mathfrak{b} = \mathfrak{a} = \kappa$

- Let $B \subseteq V_{\kappa,\lambda} \cap {}^{\omega}\omega$ be of cardinality $< \kappa$. Then there are $\alpha < \kappa$, $\zeta < \lambda$ such that $B \subseteq V_{\alpha,\zeta}$.
- Since $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in λ, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$.
- Then $\mathbb{P}_{\alpha+1,\zeta'+1}$ adds a real dominating $V_{\alpha,\zeta'} \cap {}^{\omega}\omega$, and so in particular $V_{\alpha,\zeta} \cap {}^{\omega}\omega$.
- Thus B is not unbounded.
- Therefore in $V_{\kappa,\lambda}$, we have that $\mathfrak{b} \geq \kappa$. However $\mathfrak{b} \leq \mathfrak{a}$ and so, in $V_{\kappa,\lambda}$ we have $\mathfrak{b} = \mathfrak{a} = \kappa$.

Proof: $\mathfrak{s} = \lambda$

To see that in $V_{\kappa,\lambda}$, $\mathfrak{s} = \lambda$, note that if

$$S \subseteq V_{\kappa,\lambda} \cap [\omega]^\omega$$

is of cardinality $< \lambda$, then there is $\zeta < \lambda$ such that

$$\zeta = \eta + 1, \zeta \equiv 1 \mod 2$$

and

$$S \subseteq V_{\kappa,\lambda}.$$

Then $\mathbb{M}(\mathscr{U}_{\kappa,\eta})$ adds a real not split by S and so S is not splitting. □

Observation (Strongly $\mathscr{H}$-Canjar)

Note that if μ is a cardinal such that

$$\kappa < \mu \leq \lambda$$

in the above construction, then in $V_{\kappa,\mu+1}$ there is an ultrafilter

$$\mathscr{U} = \mathscr{U}_{\kappa,\mu+1}$$

such that

$$\mathbb{M}(\mathscr{U}) \text{ preserves the unboundedness of } \mathscr{H} \subseteq {}^{\omega}\omega,$$

where $|\mathscr{H}| = \kappa < \mu \leq \mathfrak{c}$ (!)

Questions:

1. Is it consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$?
2. Is it consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?

Maximal Eventually Different Families

Definition

A family $\mathscr{E} \subseteq {}^\omega\omega$ is eventually different(abbreviated e.d.) if for any two distinct $f, g \in \mathscr{E}$ there is $n \in \mathbb{N}$ such that

$$\forall m > n(f(m) \neq g(m)).$$

We write $f \neq^* g$. An e.d. family is maximal if it is not properly contained in any other e.d. family.

We denote such maximal families MED, their minimal cardinality $\mathfrak{a}_e$. For $f, g \in {}^\omega\omega$ if it is not the case that f, g are e.d., we write $f =^\infty g$.

Maximal cofinitary groups

Definition

- A group $\mathscr{G} \leq S_\infty$ is cofinitary if its elements are pairwise eventually different.
- A cofinitary group is maximal if it is not properly contained in any other cofinitary group.
- We denote such groups with MCG and their minimal cardinality $\mathfrak{a}_g$.

It is clear that MED and MCG are close relatives to maximal almost disjoint families and so $\mathfrak{a}_g$, $\mathfrak{a}_e$ are close relatives of $\mathfrak{a}$, the minimal cardinality of an infinite maximal almost disjoint subfamily of $[\omega]^\omega$.

To what extent are those distinct?

$\mathrm{non}(\mathscr{M})$ and $\mathfrak{a}$ are independent, while $\mathrm{non}(\mathscr{M}) \leq \mathfrak{a}_g, \mathfrak{a}_e$.

Comparing those combinatorial notions with respect to their projective complexity provides further clear distinctions:

- (A. Mathias) There are no analytic MAD families.
- (H. Horowitz, S. Shelah) There are Borel MED and Borel MCG.

MCG

- (Gao, Zhang) In L there is a MCG with a co-analytic generating set.
- (Kastermans) In L then there is a co-analytic MCG.
- (Horowitz, Shelah) There is a Borel MCG.

Question

What can we say about the existence of such nicely definable combinatorial sets of reals in models of large continuum?

Cohen forcing

Theorem (F., Schrittesser, Törnquist)

Assume $V = L$. Then there is a co-analytic MCG which is indestructible by Cohen forcing.

Corollary

The existence of a Π^1_1 MCG of cardinality $\aleph_1$ is consistent with $\mathfrak{c}$ begin arbitrarily large.

Our construction is inspired by the forcing method...

Definition: Coding a real into a group element

Let σ be a partial function from $\mathbb{N}$ to $\mathbb{N}$. Then

1. σ codes a finite string $t \in 2^l$ with parameter $m \in \mathbb{N}$ iff
$$(\forall k < l)\sigma^k(m) = t(k) \mod 2.$$
2. σ exactly codes t with parameter m iff
$$\text{it codes } t \text{ and } \sigma^l(m) \text{ is undefined.}$$
3. σ codes $z \in 2^{\mathbb{N}}$ with parameter m iff
$$(\forall k \in \mathbb{N})\sigma^k(m) = z(k) \mod 2.$$

To summarize

1. The existence of a co-analytic MCG of cardinality $\aleph_1$ is consistent with $\mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$.
2. The existence of a co-analytic MED of cardinality $\aleph_1$ is consistent with $\mathfrak{a}_e = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$.

How to obtain a model in which there is a co-analytic MED family of cardinality $\aleph_1$ and $\mathfrak{d} < \mathfrak{c}$?

Theorem (F., Schrittesser)

In the constructible universe L there is a co-analytic MED which remains maximal after countable support iterations or countable support products of Sacks forcing.

To summarize

The existence of a co-analytic MED family of cardinality $\aleph_1$ is consistent with

$$\mathfrak{a}_e = \mathfrak{d} = \aleph_1 < \mathfrak{c}.$$

Definition

A forcing notion $\mathbb{P}$ has the property ned iff for every countable $\mathscr{F}_0 \subseteq {}^\omega\omega$ and every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^\omega\omega$ such that

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is e.d. from } \check{\mathscr{F}}_0,$$

there are $h \in {}^\omega\omega$ which is e.d. from $\mathscr{F}_0$ and $p \in \mathbb{P}$ with

$$p \Vdash_{\mathbb{P}} \check{h} =^\infty \dot{f}.$$

Theorem

Sacks forcing, as well as its countable support products and iterations have property ned.

Theorem

Suppose $\mathscr{E}$ is a Σ^1_2 MED family. Then, there is a Π^1_1 MED family $\mathscr{E}'$ such that for any forcing $\mathbb{P}$, if $\mathscr{E}$ is $\mathbb{P}$-indestructible, then so is $\mathscr{E}'$.

Tightness

Observations

- If X is a set of functions, then $\bigcup X \subseteq \omega^2$.
- Similarly if $T \subseteq \omega^{<\omega}$ is a tree then $\bigcup T \subseteq \omega^2$.

Definition

Let $X \subseteq {}^{\omega}\omega$, $T \subseteq {}^{<\omega}\omega$ be a tree. We say that X almost covers T if

$$\bigcup T \subseteq^* \bigcup X.$$

The tree ideal generated by $\mathscr{E}$

Definition (F., C. Switzer)

1. The tree ideal generated by $\mathscr{E}$, denotes $\mathscr{I}_{tr}(\mathscr{E})$, is the set of all trees $T \subseteq \omega^{<\omega}$ so that there are

$$t \in T \text{ and a finite } X \subseteq \mathscr{E}$$

so that

$$\bigcup T_t \subseteq^* \bigcup X.$$

2. A tree $T \subseteq \omega^{<\omega}$ is said to be in $\mathscr{I}_{tr}(\mathscr{E})^+$ if for each $t \in T$ it is not the case that $\bigcup T_t$ can be almost covered by a finite $X \subseteq \mathscr{E}$.

Tight eventually different families

Definition

Let $T \subseteq \omega^{<\omega}$ be a tree, $g \in {}^{\omega}\omega$. We say that g densely diagonalizes T, if for every $t \in T$ there is a branch h through t in T such that $h =^{\infty} g$.

Definition

An eventually different family $\mathscr{E}$ is tight if for any $\{T_n\}_{n\in\omega} \subseteq \mathscr{I}_{tr}(\mathscr{E})^{+}$ there is a single $g \in \mathscr{E}$ which densely diagonalizes all the T_n's.

Observations

- If $\mathscr{E}$ is a tight eventually different family, then it is maximal.
- MA(σ-linked) implies that every e.d. family $\mathscr{E}_0$, $|\mathscr{E}_0| < \mathfrak{c}$ is contained in a tight e.d. family.
- CH implies that tight eventually different families exist.

Moreover...

tight eventually different families are never analytic, which is a strong distinction with the Borel MED family of Horowitz-Shelah.

... and moreover:

1. tight eventually different families are Cohen indestructible;
2. in L there is a co-analytic tight e.d. family;
3. thus (once again!) $\mathfrak{a}_e$ has a co-analytic witness in a model of $\mathfrak{a}_e = \mathfrak{b} = \aleph_1 < \mathfrak{d} = \mathfrak{c}$.

Strong Preservation of Tightness

Definition: Strong preservation

Let $\mathbb{P}$ be a proper forcing notion and $\mathscr{E}$ a tight e.d. family. We say that $\mathbb{P}$ strongly preserves the tightness of $\mathscr{E}$ if for every sufficiently large θ and $M \prec H_\theta$ such that $p, \mathbb{P}, \mathscr{E}$ are elements of M,

if g densely diagonalizes every elements of $M \cap \mathscr{I}_T(\mathscr{E})^+$,

then there is an $(M, \mathbb{P})$-generic $q \leq p$ such that q forces that

g densely diagonalizes every element of $M[\dot{G}] \cap \mathscr{I}_T(\mathscr{E})^+$.

Such a q is called an $(M, \mathbb{P}, \mathscr{E}, g)$-generic condition.

Theorem

Suppose $\mathscr{E}$ is a tight e.d. family. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a countable support iteration of proper forcing notions such that for all α,

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } \check{\mathscr{E}},$$

then $\mathbb{P}_\gamma$ strongly preserves the tightness of $\mathscr{E}$.

Lemma

- Suppose $\mathbb{P}$ strongly preserves the tightness of $\mathscr{E}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, which strongly preserves the tightness of $\mathscr{E}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathscr{E}$.
- Moreover, if p is $(M, \mathbb{P}, \mathscr{E}, g)$-generic and forces $\dot{q}$ to be $(M[\dot{G}], \mathbb{P}, \mathscr{E}, g)$-generic then $(p, \dot{q})$ is $(M, \mathbb{P}, \mathscr{E}, g)$-generic.

Lemma

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ be a countable support iteration of proper forcing notions such that for all α,

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ densely preserves the tightness of } \check{\mathscr{E}},$$

θ sufficiently large and $M \prec H_\theta$ containing $\mathbb{P}_\gamma, \gamma, \mathscr{E}$. For each $\alpha \in M \cap \gamma$ and every $(M, \mathbb{P}_\alpha, \mathscr{E}, g)$-generic condition $p \in \mathbb{P}_\alpha$ the following holds:

If $\dot{q}$ is a $\mathbb{P}_\alpha$-name, $p \Vdash_\alpha \dot{q} \in \mathbb{P}_\gamma \cap M$ and $p \Vdash_\alpha \dot{q} \restriction \alpha \in \dot{G}_\alpha$, then there is an $(M, \mathbb{P}_\gamma, \mathscr{E}, g)$-generic condition $\bar{p} \in \mathbb{P}_\gamma$ so that

$$\bar{p} \restriction \alpha = p \text{ and } \bar{p} \Vdash_\gamma \dot{q} \in \dot{G}.$$

The notion of a tight eventually different family gives a uniform framework which applies to a long list of partial orders, including:

- Sacks,
- Miller rational perfect set forcing,
- Miller partition forcing,
- Infinitely often equal forcing,
- Shelah's poset for diagonalizing a maximal ideal

and gives rise to a MED family indestructible by the above posets.

Theorem (F., Switzer)

The following inequalities are all consistent and in each case there is a tight eventually different family and a tight eventually different set of permutations of cardinality $\aleph_1$, respectively.

1. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p < \mathfrak{d} = \mathfrak{a}_T = 2^{\aleph_0}$
2. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} < \mathfrak{a}_T = 2^{\aleph_0}$
3. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} = \mathfrak{u} < non(\mathcal{N}) = cof(\mathcal{N}) = 2^{\aleph_0}$.
4. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = cof(\mathcal{N}) < \mathfrak{u}$.

Moreover, if we work over the constructible universe, we can provide co-analytic witnesses of cardinality $\aleph_1$ to each of

$$\mathfrak{a}, \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{i}, \mathfrak{u}$$

in the above inequalities.

Thank you for your attention!