THE CLASSIFICATION OF PURELY INFINITE C*-ALGEBRAS USING KASPAROV'S THEORY

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After Eberhard Kirchberg's death in August, 2022, his wife Sommai agreed to make this manuscript accessible to the mathematical public. The manuscript contains a wealth of deep results, ideas, and techniques, but it is not completely finished. The manuscript is in the form in which we found it on his computer and we did not make any changes.

The plan is that Jamie Gabe and Mikael Rørdam will add an explanatory text and that the manuscript will be published via the University Library of the University of Münster. In the meantime it will be available at:

https://ivv5hpp.uni-muenster.de/u/echters/ekneu1.pdf

For the time being, please refer to this manuscript as

Eberhard Kirchberg, *The Classification of Purely Infinite C*-Algebras Using Kasparov's Theory*, manuscript available at https://ivv5hpp.uni-muenster.de/u/echters/ekneu1.pdf

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The Classification of Purely Infinite C*-Algebras Using Kasparov's Theory

(preliminary version: 27.05.2022)

Eberhard Kirchberg

Contents

Attempt to give a very short overview (abstract)

First we consider a simple special case and show that KK-equivalent *simple* purely infinite separable unital nuclear C^*-algebras A and B are isomorphic if the group isomorphism from $K_0(A)$ onto $K_0(B)$, that are induced by the KK-equivalence, maps the K_0-class $[1_A]$ of the unit element of A in $K_0(A)$ to the class $[1_B] \in K_0(B)$.

We call them *pi-sun algebras*, because they are automatically simple by the requirement of the algebraic property that for each non-zero $a \in A$ there are elements $c, d \in A$ with $cad = 1_A$. It yields that pi-sun algebras are isomorphic if their K_*-groups are isomorphic by an isomorphism that maps the K_0-classes of their unit elements into each other and if both are KK-equivalent to Abelian C^*-algebras, i.e., if the Universal Coefficient Theorem applies to both of them.

Comment on this.

Here are the first question:
Is KK-equivalence to commutative C^*-algebras really needed?
Is each p.i.s.u.n. algebra A isomorphic to $A \otimes \mathcal{O}_\infty$?
Is $A \otimes \mathcal{O}_2$ isomorphic to $\mathcal{O}_2$ if A is a separarable unital nuclear simple C*-algebra?
Give references for or against it (with cite or refs)
Where are examples or counter-examples?

The considerations are based on the here applicable Cuntz–Kasparov isomorphism $KK(A, B) \cong \mathrm{Ext}(A, SB)$ and the homotopy invariance of $KK(A, B)$, that will be used to show that the natural group-morphism from the Rørdam groups $R(A, B)$ – a sort of "unsuspended" and "nuclear" E-theory – into $\mathrm{Ext}(A, SB)$ is an isomorphism. The last step for this proof of the functorial equivalence will be done in Chapter 8.

M. Rørdam was/sounded not happy with my interpretation
of -- from him defined groups $R(A, B)$.
Perhaps my interpretation is different
and I should call them $\mathbb{GK}(A, B)$.

A similar partly classification is given later for *non-simple* separable stable nuclear *strongly* purely infinite C^*-algebras. Those algebras absorb the Cuntz-algebra $\mathcal{O}_\infty$ tensorial, i.e., a separable nuclear C^*-algebra A is isomorphic to $A \otimes \mathcal{O}_\infty$, if and only if, A strongly purely infinite (cf. Definition 1.2.2), that is, if A satisfies a generalized Weyl–von-Neumann–Voiculescu type theorem for weakly residually nuclear maps (cf. Chapters 5 and 10).

The basic study of properties of tensorial self-absorbing unital separable C^*-algebras will be postponed to Chapter 11 because it has another nature, e.g. it is not really related to pure infiniteness.

We remove in Chapter 12 by study of some properties of ultrapowers some technical additional assumptions that we added in the other chapters to make the some of the proofs, e.g. in Chapter 6, more transparent: We contain a proof that $\mathrm{Prim}(A)$ for separable non-simple C^*-algebras A has always a natural property that we call "Abelian factorization" property 1.2.4. It leads to a rather long study over the interrelations between general Dini spaces (i.e., second countable locally quasi-compact point-complete T_0-spaces), "coherent" Dini spaces and second countable locally compact Hausdorff spaces and is not only related to functional analysis.

The in Chapter 6 made additional assumption – that the considered separable stable σ-unital nuclear C^*-algebras have some property that is (formally) weaker than "regular" factorization through a l.c. Hausdorff space for their prime ideal space – can here (in this book) only be reduced in Chapter 12 where we use that every separable C^*-subalgebra of an ultra-power (or more generally in that what we call "corona algebras") is contained in a bigger separable C^*-algebras that contains a nice Abelian C^*-algebra with the much stronger property that it not only separates the ideals but is also "regular" in the sense of our Definition 1.2.9. (or B.4.1 ????)..

This is a much stronger property than regular factorization. And regular factorization is usually stronger than having an Abelian C^*-subalgebra that separates the ideals of a stable separable C^*-algebra (with the additional property that it can be done in a way that the norm functions of its elements exhausts all Dini functions on $\mathrm{Prim}(A)$).

A special consequence of our classification results is the following:

If A and B are separable stable nuclear C^*-algebras and if there is a homeomorphism γ from the T_0-space $X := \mathrm{Prim}(A)$ of primitive ideals in A onto $\mathrm{Prim}(B)$, then we prove the existence of an isomorphism ψ from $A \otimes \mathcal{O}_2$ onto $B \otimes \mathcal{O}_2$, such that ψ induces γ in the sense $\psi(J \otimes \mathcal{O}_2) = \gamma(J) \otimes \mathcal{O}_2$ for $J \in \mathrm{Prim}(A)$. The ψ with this property is unique up to approximate unitary equivalence.

We generalize Kasparov's equivariant functor $\mathcal{R}\,\mathrm{KK}^G(X;\, .,.)$ (here only in the case where the group G is trivial) to some kind of generalized "actions" of locally quasi-compact T_0-spaces X on C^*-algebras (cf. Definitions 1.2.6 and 1.2.10). We denote this functor by $\mathrm{KK}(X;\, A, B)$. In fact, we show that Kasparov's KK-functor extends in a natural way to a functor $\mathrm{KK}(\mathcal{C};\, A, B)$ from the category of matrix operator convex cones (m.o.c. cone) $\mathcal{C} \subset \mathrm{CP}(A, B)$ into the abelian groups. In this technical approach the algebras itself play then only the role of indices and we have to study only the category of m.o.c. cones, and then to translate it back to the category of C^*-algebras ...

The *main result* of this considerations is the following:
Suppose that A_1 and A_2 are strongly purely infinite, separable, stable and nuclear,

and that there is given an homeomorphism ψ from $X := \mathrm{Prim}(A_1)$ onto $\mathrm{Prim}(A_2)$ and that A and B are $\mathrm{KK}(X; ., .)$-equivalent in a ψ-equivariant manner. Then there exist ψ-equivariant *-monomorphisms $h_1 : A_1 \to A_2$ and $h_2 : A_2 \to A_1$ that induces the given $\mathrm{KK}(X, ., .)$-equivalence, and that $h_2 \circ h_1$ and $h_1 \circ h_2$ unitarily homotopic to the identity maps of A_1 respectively A_2. It implies the existence of an isomorphism $\gamma : A_1 \to A_2$ from A_1 onto A_2, such that γ is approximately unitary equivalent to h_1 and γ^{-1} is approximately unitary equivalent to h_2. (But it is at present still not clear if γ itself can be chosen such that γ is unitarily homotopic to the given h_1.)

On the way of the proof we obtain also several results that are basic for our main results, e.g. the following:

1. Every exact separable C^*-algebra is a C^*-subalgebra of $\mathcal{O}_2$ (Theorem A). More generally we show the following “embedding theorem”:

If a (not necessarily simple) C^*-algebra D is separable, stable and is “strongly purely infinite” in the sense of Definition 1.2.2, and if $X := \mathrm{Prim}(D)$ acts by a map $\Psi_A : \mathbb{O}(X) \to \mathcal{I}(A)$ on a separable stable exact C^*-algebra A lower semi-continuously and upper monotone continuously, then there is a *-monomorphism $h_0 : A \to D$ such that $h_0 \oplus h_0$ is unitarily equivalent to h_0 and that h_0 induces the given action Ψ_A. The h_0 is unique up to unitary homotopy (Theorem K), moreover, h_0 can be realized by a non-degenerate *-monomorphism, if and only if, the action Ψ_A of X on A is non-degenerate.

If D is a non-degenerate stable C^*-subalgebra of a C^*-algebra B, then B is stable, D defines an upper semi-continuous action Ψ_B of X on B, and h_0 defines an element of the semigroup of unitary equivalence classes $[\mathrm{Hom}_{\mathrm{nuc}}(X;\, A, B)]$ of Ψ_A–Ψ_B–residually nuclear *-homomorphisms from A to B.

We use the above h_0 (a generator of some m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ to describe the nuclear KK-groups $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ as classes of unitarily homotopic Ψ-residually nuclear homomorphisms modulo stabilization with $[h_0]$, cf. Theorems B and M:

$$[\mathrm{Hom}_{\mathrm{nuc}}(X;\, A, B)] + [h_0] \cong \mathrm{KK}_{\mathrm{nuc}}(X; A, B)\,.$$

2. The spatial tensor product of two simple C^*-algebras is purely infinite if one of them is not stably finite and the other is non-elementary, cf. Theorem E. More generally the minimal C^*-tensor products $A \otimes B$ of any C^*-algebra A with a strongly purely infinite C^*-algebra B is strongly purely infinite.

3. The UCT holds for all separable nuclear C^*-algebras, if and only if, up to isomorphisms there is only one pi-sun algebra A with $\mathrm{K}_*(A) = 0$, cf. Corollary J ([1]).

??? The question is here: Is every separable nuclear C^*-algebra B KK-equivalent to some pi-sun C^*-algebra A. One can first tensor A with $\mathcal{O}_\infty \otimes \mathbb{K}$ ... (Seems to be that ??: $A \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ is KK–equivalent to A – but here should no

[1] This is a very important observation, that should direct all the further research to the study of all pi-sun algebras A with $\mathrm{K}_*(A) = 0$. Nobody started it yet (January 2022). Why?

commutative C*-algebra be involved by constructing the KK-equivalence ... The KK-equivalence of $\mathcal{O}_2$ with $C_0((0,1])$ should be directly constructed. Same with $\mathcal{O}_\infty$ with C([0, 1]) or with $\mathbb{C}$ ...)

If A is stable, i.e., $A \cong A \otimes \mathbb{K}$ then one can repeat the constructions given by using a suitable unital mono-morphism

$$\mathcal{M}(A \otimes \mathbb{K}) \mapsto 1_{\mathcal{M}(A)} \otimes \mathcal{L}(\mathbb{K}) \subseteq \mathcal{M}(A \otimes \mathbb{K})\,.$$

This operation will be infinitely repeated and the resulting inductive limit (restricted to the tower of images of $A \otimes c_0(\mathbb{K})$ should be KK-equivalent to $A \otimes c_0(\mathbb{K})$ and the obvious endomorphism given by the forward shift $(k_1, k_2, \ldots) \mapsto (0, k_1, k_2)$ on c_0 respectively by the natural endomorphism of the above described inductive limit.

This should be KK-equivalent to $A \otimes c_0(\mathbb{K})$, and the natural shift endomorphism S on c_0 should produce the isomorphism $S \rtimes c_0 \cong Kp$ using $c_0(\mathbb{K}) \cong c_0 \otimes \mathbb{K}$ and $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ KK-equivalence of $A \cong A \otimes \mathbb{K}$ with the above defined inductive limit that is a stable pi-sun C*-algebra.

4. Hausdorff distances of C^*-convex hulls of elements a, b and distance between the generalized Gelfand-transformations of a and b are the same, cf. Corollary 3.10.12 and Remark 3.11.3.

5. Results on ideal spaces and approximate 1-step-innerness of residually nuclear maps are given in Chapters 2, 3 and 12.

6. Generalized Weyl–von-Neumann–Voiculescu type theorems and asymptotic analogs of them are proven in Chapters 5 and 7.

7. The non-negative Dini functions on Prim(A), cf. Definition 12.2.5, are the generalized Gelfand transforms of elements of A if A is separable and $A \cong A \otimes \mathcal{O}_\infty$ (Proposition 12.2.6 – a very special case of [**447**] –).

CHAPTER 1

Introduction and main results

Following topics should be extracted and published in more detail (also as beamer presentations, and with lists of its definitions of properties in an elementary way).

1.) "Squeezing Property" implies K_1-injectivity. (See Chp. 4. for definitions of this properties!)

Here a question remains:

When K_1-injectivity implies the Squeezing Property?

At least in unital separable nuclear case ?? (Would be nice !)

(But have no idea until now: Jan 2022 !!!)

2.) Precise version of a quasi-trace state on a type-I C^*-algebra that is not a 2-quasi-trace.

See end of Chapter 1 (= Introduction), and the more details in a (still in work) section of the appendices.

Here following questions remain:

What happens if a separable nuclear C^*-algebra has no type-I sub-quotient? Are then all quasi-traces "trivial" or 2-quasi-traces?

3.) Forgotten!!! Try to remember!

Something important ...? Also related to topics in this book, e.g.:

When (all?) pi-sun algebras A satisfy the UCT?

(Has it to do with the existence/non-existence of ?? Cartan sub-algebras ??)

Seems to be studied in new papers (since 2016 –2021)!

For example: Is every separable nuclear C*-algebra B KK-equivalent to a pi-sun C*-algebra?

Not clear what he is thinking here.

Possible ideas? Check it!: $B \otimes \mathcal{O}_\infty$ and $B \otimes \mathcal{K}$ are KK-equivalent to B ? (Seems to be.)

Here $\mathcal{K}$ (– also denoted by $\mathcal{K}(\ell_2)$ –) means the algebra of compact operators on $\ell_2(\mathbb{N})$.

Then the stable separable nuclear C*-algebra $C := B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is strongly purely infinite and absorbs $\mathcal{O}_\infty$ tensorial. It should be KK-equivalent to B. (Seems to work!)

Notice that $\mathcal{L}(\ell_2) = \mathcal{M}(\mathcal{K})$ is unitally contained in $\mathcal{M}(C)$, because $\mathcal{M}(C)$ contains a natural unital copy of

$$\mathcal{M}(B) \otimes \mathcal{O}_\infty \otimes \mathcal{M}(\mathcal{K}) .$$

Therefore, there exists an injective non-degenerate endomorphism ρ of C into $1 \otimes 1 \otimes \mathcal{M}(\mathcal{K}) \subset \mathcal{M}(C)$. Here "non-degenerate" means that each of the products $\rho(C)C$ and $C\rho(C)$ are dense in C.

This implies that ρ extends naturally to a unital endomorphism

$$\mathcal{M}(\rho)\colon \mathcal{M}(C) \mapsto \mathcal{M}(C)$$

.

The inductive limit with help of iterations of this unital endomorphism $\mathcal{M}(\rho)$ produces a (huge) unital C*-algebra E with an "distinguished" endomorphism $\lambda\colon E \mapsto E$.

Moreover, there is an injective C*-algebra morphism $\eta\colon C \to ??? \subset E$ such that $\eta(C)$ is an ideal of E with the property $\eta(C) + \lambda(E) = E$

(??? check it ! It is very important)

and $\eta(C) \cap \lambda(E) = \{0\}$. ???

Now define inductive $D_1 := \eta(C)$, $D_2 := \eta(C) + \rho(C) = C \subseteq \mathcal{M}(C)$,

$\ldots$, $D_3 := C + ...??$

Is it at least the case if B satisfies the UCT? It asks then the weaker question only:

Is at least every separable nuclear C*-algebra B K-equivalent to a pi-sun algebra (in general)?

Can a positive answer for the special case $K_*(A) = 0$ give also a positive answer for the general case?

E.g. that $K_*(A) = 0$ implies $A \cong \mathcal{O}_2$???

Case of $K_*(A) = (\mathbb{Z}, 0)$, as $A \cong \mathcal{O}_\infty$???

Are there relations to the QWEP conjecture?

By definition, a C*-algebra B has the QWEP if B is a quotient C*-algebra $B \cong C/J$ of a C*-algebra C that has the "weak expectation property" (WEP).

A C*-algebra $C \subseteq \mathcal{L}(H)$ has the property WEP, if and only if, there exists a completely positive contraction $V\colon \mathcal{L}(H) \to C^{**}$ into the bi-dual C^{**} of C such that $V(c) = c$ for all $c \in C$.

Perhaps it could follow from (????):

$$C^*(F_\infty) \otimes^{\max} \mathcal{L}(H) = C^*(F_\infty) \otimes^{\min} \mathcal{L}(H)$$

where F_∞ is the free group on countably many (or more!) generators.

In particular, for every C*-representation $d\colon C \to \mathcal{L}(K)$ on a Hilbert space K there exists completely positive contraction $W\colon \mathcal{L}(K) \to d(C)'' \subseteq \mathcal{L}(K)$ with the property that $W(d(c)) = d(c)$ for all $c \in C$. This can be used also as definition, because one can take here a sufficiently universal representation d of C.

(So far the answer to the QWEP question is open for exact C*-algebras.)

All nuclear separable C*-algebras B are WEP algebras, because their bi-duals B^{**} are always injective von-Neumann algebras $B^{**} \cong N = N'' \subseteq \mathcal{L}(H)$ (because $\mathcal{L}(H)$ is injective for each Hilbert space H).

The QWEP conjecture and its possible answer is very important, because is could be that we need some very selective and constructive definition of ultra-filters and approximation methods to get a coherent picture of the desired results with sensible methods.

One can reduce this question a bit if every simple, separable, nuclear, unital, strongly purely infinite C*-algebra A contains a UCT-class member $1_A \in B \subseteq A$ with all other properties the same as A.

This is clearly the case.

Are all separable typ I C*-algebras A in the UCT-class? (Seems to be !)

This must be known.

Is the (separable nuclear) UCT-class closed under extensions, tensor-products and inductive limits? (Seems to be! But with which necessary restrictions?)

Also under crossed products – by actions of amenable groups G on A ?

What about crossed products by an endomorphism? I.e., by an injective action of the natural numbers $\mathbb{N}$

I am not sure what he refers to here.

Look to the Rørdam - Property !! ??Which one?

Preview.

Later add (???) on weak semi-projectivity
of pi-sun (??) algebras and
kernel of η: $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{K}_*(A))$ in UCT case (important!)
and
classification of tensorial self-absorbing algebras in UCT class

Def. of ''pi-sun'' is on 3 places -- 2 with motivation
Is ''pi-sun'' good notation? ??

A long time ago George Elliott [**258**] did conjecture that *simple* separable amenable C^*-algebras can be classified by invariants that are, in some sense, K-theoretic in nature ([1]).

In fact the status of the conjecture of G. Elliott can now described very roughly as a classification up to isomorphisms of those simple separable nuclear C^*-algebras A with following properties:

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[1] A C^*-algebra A is "amenable", if and only if, A is "nuclear", cf. [**340**] and [**159**]. We rarely use the terminology "amenable" for nuclear C^*-algebras, because we work with matrix operator-convex cones $\mathcal{C}$ of nuclear maps, but not directly with the (co-) homological properties of amenability. But it would be interesting to see if some direct (!) applications of the (co-) homological definition of amenability give additional new insight for answers to the questions and problems that we study here, give new insides of open questions ...

(a) that absorb the Jiang-Su algebra $\mathcal{Z}$ tensorial, in the sense that $A \cong A \otimes \mathcal{Z}$ (see Definition A.3.1 of the Jiang-Su algebra in Section 3) of Appendix A, and

(b) where A is KK-equivalent to some separable commutative C^*-algebra C.

The properties (a,b) are satisfied e.g. for the (Joachim) Cuntz algebra $\mathcal{O}_2$: for (a) because, $\mathcal{O}_2$ absorbs any unital simple nuclear separable C^*-algebra B tensorial, i.e. by isomorphisms $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes B$ for any unital simple nuclear separable C^*-algebra B, – especially if we take the Jiang-Su algebra $\mathcal{Z}$ as B. The property (b) is satisfied with $C := \mathrm{C}_0(0,1]$, because $\mathrm{K}_*(\mathrm{C}_0(0,1]) = \{0\} = \mathrm{K}_*(\mathcal{O}_2)$., – an observation of J. Cuntz. That will be shown later in this book in any detail as partial result of some more general observations.

A basic desire is an explicit answer to the following question: Let A a pi-sun C^*-algebra, i.e., a purely infinite separable unital nuclear C^*-algebra with the property that $\{0\} = \mathrm{K}_*(A)$. Is A isomorphic to $\mathcal{O}_2$?

(Because all separable exact C^*-algebras are C^*-subalgebras of $\mathcal{O}_2$, we can suppose the additional assumption that A is a unital C^*-subalgebra of $\mathcal{O}_2$.)

The algebra $C := \mathrm{C}[0,1]$ of continuous functions on the interval $[0,1]$, and the unital simple C*-algebras $\mathcal{Z}$ (the *Jiang-Su algebra*) and $\mathcal{O}_\infty$ (the *Cuntz algebra* generated by an infinite sequence $s_1, s_2, \ldots$ of isometries with mutually orthogonal ranges) are all KK-equivalent to the algebra complex numbers $\mathbb{C}$ (considered as a unital C*-algebra).

This is Kirchberg's fundamental theorem.

Give here precise references for the observation that $\mathcal{O}_2 \cong B \otimes \mathcal{O}_2$ for unital separable simple and nuclear $C*$-algebras B etc. !!!
Where is the KK-equivalence to commutative $C*$-algebras shown??
Give reference!!!

We say in Section 4 of this introduction more about his formulations and the now (almost) complete state of the proof of the Conjecture of George Elliott. Except in the until now only partially understood case of stably projection-less algebras there is a still remaining questions that is equivalent to the below stated question **(Q2)** that is equivalent to the question if all separable amenable C^*-algebras are KK-equivalent to Abelian C^*-algebras and, to some more general questions that are “in spirit similar” to our question **(Q1)**, namely what additional structure has to be imposed to make sure that a given amenable simple separable stably infinite C^*-algebra can be classified by the so far accepted invariants of simple C^*-algebras, among them e.g. the corona factorization property (CFP) and real rank zero together, or absence of certain types of “infinitesimal” sequences in its Cuntz-semigroup.

Give ref.s to next??
if invariants of K_*-theoretic nature (including the pairing with trace cones) together and with “local approximation” allow to detect if a given separable simple amenable C^*-algebras A tensorial absorbs the Jiang-Su algebra $\mathcal{Z}$. It leads for all separable,

purely infinite and nuclear C^*-algebras A to the important conclusion that A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if A is isomorphic to $A \otimes \mathcal{Z}$.

This ????? second ?????

open question is for the stably infinite amenable simple separable C^*-algebras if they are KK-equivalent to an Abelian C^*-algebra $\mathrm{C}_0(X)$:
Give a direct construction of a pi-sun C^*-algebra that is KK-equivalent to $\mathrm{C}_0(X)$ for a given finite polyhedron X. They should meet transparently in the ?????

in the study of the generators and relations of there K_*-theories ?????

Is there any example of ?????

Here are some special questions:

Let A denote a separable C*-algebra ...

The History of this circle of questions:

All of the work on this book was mainly motivated and inspired by the original conjectures of G. Elliott and pioneering ideas of J. Cuntz on this circle of questions. The "easy" beginning sections of the chapters in this book are concerned with an alternative proof of the Elliott conjecture but only in the case of *simple purely infinite separable nuclear* C^**-algebras* A *in the UCT class.* The original definition of purely infinite (simple) algebras given by J. Cuntz [**172**, p. 186] – stated there only in the unital case –, can be reformulated for general unital rings A as: *For each non-zero* $a \in A$*, there are* $c, d \in A$ *with* $cad = 1$. (See proof of [**172**, prop. 1.6] and [**169**, 1.3, 3.4].) Clearly, this definition implies automatically that A is simple. But one has also to require then that A not $\cong \mathbb{C} \cdot 1_A$.

It is easy to see that $K_*(\mathcal{L}(\ell_2)) = 0$, because $B := \mathcal{L}(\ell_2)$ has real rank zero $M_n(B) \cong B$, $(1_B \oplus p) \cong 1_B$ for each projection $p \in B$, and for each unitary in $u \in B$ one finds $q \in B$ and $\xi \in \mathbb{C}$ such that $|\xi| = 1$ and $\|(1-q)u(1-q) + \xi \cdot p - u\| < 1/9$.

(Moreover the unitary group of $B := \mathcal{L}(\ell_2)$ is "globally" contractible to 1, in the sense that ???? WHAT ????)

The elements are all exponentials $u = exp(ih)$ of self-adjoint operators $h^* = h \in B$, because each of them is contained in commutative W*-subalgebra of B.

It follows local contractibility.

(But gives it global contractibility??)

But the Calkin-Algebra $Q(\ell_2) := \mathcal{L}(\ell_2)/\mathbb{K}(\ell_2)$ with $\ell_2 := \ell_2(\mathbb{N})$ is also a simple C*-algebra with K-theory, that is $K_0(Q) = 0$ and $K_1(Q) = \mathbb{Z}$ K-theory, because again $Q(\ell_2) \cong M_n(Q(\ell_2))$ and non-zero projections are equivalent (thus all are stably equivalent), numbers in $K_1(Q)$ are the "indices" (e.g. of the Toeplitz operators on ℓ_2.

Moreover, $Q(\ell_2)$ contains infinitely many ... pair-wise non-isomorphic ... separable simple unital C*-subalgebras with trivial K-theory. (Can they be exact ?

Are there continuously many – or at least countably many – unital, non-nuclear and simple C*-subalgebras of $\mathcal{O}_2$ not pairwise isomorphic?)

If one tensors the Cuntz algebra $\mathcal{O}_2$ with any separable simple exact and unital C*-algebra B, then $A := B \otimes \mathcal{O}_2$ has again trivial K-theory $K_*(A) = 0$.

Thus, only in case of pi-sun C*-algebras A (i.e. purely infinite, separable, unital and nuclear C*-algebras) there is a chance that $K_*(A) = 0$ implies that A is isomorphic to the Cuntz algebra $\mathcal{O}_2$.)

The below given Definition 1.2.1 of purely infinite C^*-algebras is equivalent to this algebraic definition of J. Cuntz if A is a *unital and simple C^*-algebra*, cf. Proposition 2.2.1: It turns out that a *not necessarily simple or unital* C^*-algebra A is *purely infinite* (more precisely: is 1-purely infinite) in the sense of the below given Definition 1.2.1 if every non-zero element $a \in A_+$ is properly infinite, i.e., if there exists a sequence $d_n \in M_2(A)$ (depending on a, and the d_n can be chosen with zeros on the second line: $(d_n)_{21} = 0 = (d_n)_{22}$) with $\lim_n d_n^*(a \oplus 0)d_n = a \oplus a$, ([2]). This is property pi-1, that is a special case of property pi-n defined in Definition **??**

and the, – in a non-trivial manner equivalent – Definition 1.2.1 that is a special case of 1-purely infinite algebras, (i.e., pi(1)-algebras) of the weakly purely infinite algebras defined in Definition 2.0.4.

See Proposition 2.12.12 for the general equivalences **??**.

The name and definition of *nuclear* C^*-algebras is "inspired" – beside is very different – by the description of "nuclear" locally convex vector spaces V by uniqueness of the topology given by any compatible locally convex structure on algebraic tensor products $V \odot W$ with other locally convex vector spaces W, given by A. Grothendieck [**336**]. The C^*-algebra A is *nuclear*, if and only if, there is *only one* C^*-norm on the algebraic tensor product $A \odot B$ for every C^*-algebra B, cf. [**766**] (where it was called " property T " in 1964), and [**509**].

It has been later observed that the C^*-algebra A is nuclear if and only if there exists a *net* $\{V_\tau\}_{\tau \in T}$ of completely positive maps $V_\tau : A \to A$ of finite linear rank that converges in point-norm topology to id_A, (see [**426**] as outlined in our Remark 3.1.2(iii,f) that does *not* factorize through the work of A. Connes on von Neumann algebras, – or consult the paper of Choi and Effros [**145**, thm. 3.1] that applies results of A. Connes on von Neumann algebras).

A. Connes has shown that separable C^*-algebras A are nuclear if they are *amenable* in the sense of Banach algebra theory, cf. [**159**, cor. 2]. Later, U. Haagerup proved that each nuclear C^*-algebra A is amenable in this sense, cf. [**340**, thm.3.1]. We do never use the amenability of nuclear C^*-algebras in our considerations, and read throughout this book "amenable" as synonym of "nuclear".

[2]Here and some other places the notation $x \oplus y$ denotes the "diagonal" 2×2-matrix $[x_{jk}]$ with entries $x_{11} := x$, $x_{12} := 0$, $x_{21} := 0$ and $x_{22} := y$.

We say that a separable C^*-algebra A *is in the UCT class*, if A satisfies the conditions of the Universal Coefficient Theorem of Rosenberg and Schochet [**699**], cf. [**73**, thm. 23.1.1]. An observation of G. Skandalis says that the UCT for a separable C^*-algebra A is equivalent to: *There exists a locally compact metric space X such that A is* KK*-equivalent to* $\mathrm{C}_0(X)$ ([3]).

If a separable C^*-algebra A is KK-equivalent to an Abelian C^*-algebra, then A is also KK-equivalent to an algebra $\mathrm{C}_0(X)$, with X a locally compact *Polish* space that is *locally* (!) a finite CW-complex of dimension ≤ 3. This is because for every countably generated $\mathbb{Z}_2$-graded Abelian group $G = G_0 \oplus G_1$ there is a locally compact Polish space X that is *locally* an at most 3-dimensional CW-complex with $\mathrm{K}^*(X) \cong G$ for those X, cf. Lemma B.11.1 or proof of [**73**, cor.23.10.3]. If G is finitely generated then one can take for X a usual (= global) CW-complex of dimension ≤ 3.

CHECK (and replace if necessary) "equivariant" by some accepted expression ...

It seems likely that every separable nuclear C^*-algebra C is KK-equivalent to a pi-sun algebra A. Then only 2 questions remain: Does there exists a separable, simple, and nuclear C that is not in the UCT class ...

Other question ???

What about $K_*(A) = 0$ for general separable nuclear A ?

Can the UCT-class – in a sense – distinguish between $\mathcal{O}_\infty$ and the Jiang-Su algebra? ...

The proof of the Elliott Conjecture that is given in this book for the very special class of purely infinite, simple, nuclear and separable algebras uses some methods that will be stepwise generalized later to the case of non-simple *strongly* purely infinite (s.p.i.) C^*-algebras cf. Definition 1.2.2. One of the remaining problems for the applicability of the later described ideal-system equivariant classification of separable nuclear s.p.i. C^*-algebras has – among others – to do with the question under which conditional properties a purely infinite nuclear C^*-algebras A becomes strongly purely infinite (which is then in case of separable nuclear A equivalent to $A \cong A \otimes \mathcal{O}_\infty$ or to $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ denotes the Jiang-Su algebra).

Give reference!! where this is proved here.

Conj.: ?????

(OK!)

If A is separable then $A_c := (A' \cap A_\omega)/\mathrm{Ann}(A)$ is always a unital C^*-algebra.

Here $A \subseteq A_\omega$ is natural embedded via $a \in A \mapsto \pi_\omega(a, a, \ldots)$ into $A_\omega := \pi_\omega(\ell_\infty(A))$ for some ultra-filter ω (= non-trivial character of the commutative C^*-algebra $\ell_\infty(\mathbb{C})$).

(seems that there are counterexamples for next !!)

[3] See [**73**, thm. 23.10.5], cf. also below reasoning for Corollary J.

If A_c has no character then A_c contains a unital copy of the Jiang-Su algebra $\mathcal{Z}$ and $A \cong A \otimes \mathcal{Z}$??? ???? It seems one needs much stronger assumptions ...

At least if A is s.p.i., separable and exact, or ?????

The question is:
Let B a separable unital C^*-algebra without characters. Does B contain a *nuclear* C^*-subalgebra D with $1_B \in D$ and without any character?

Moreover: Does $C := B \otimes_{\max} B \otimes_{\max} \cdots$ contain a copy of $\mathcal{Z}$?

Seems not to be the case ? Need stronger assumptions.

But does C contain unitally a copy of a *unital nuclear* D without characters?

Let D nuclear and unital, but without characters, does $D \otimes_{\min} D \otimes_{\min} \cdots$ contain the Jiang-Su algebra $\mathcal{Z}$?

Seems to have a negative answer ...?

This generalization explores *all aspects* of the technics that we introduce for the study of simple nuclear C^*-algebras and requires some additional study of the m.o.c. cones ...

`Give reference res.nuc. c.p. maps HERE !!` of the later in Definition ????

See the definition of residually nuclear c.p. maps in Definition 3.10.1 defined "residually nuclear" completely positive maps.

The Definition 3.5.9 – here considered in the special case of the ideal lattices of the considered algebras.

The residually nuclear maps generalize fiber-wise nuclear $\mathrm{C}(X)$-modular maps between $\mathrm{C}(X)$-C^*-algebras and generalize asymptotic $\mathrm{C}(X)$-algebras. But a suitable equivariant sort of the UCT and related families of representatives for sufficiently general UCT classes for the classification of non-simple amenable C^*-algebras is not in sight yet. But in case of separable nuclear C^*-algebras A with non-Hausdorff primitive ideal space $\mathrm{Prim}(A)$ of finite *decomposition* dimension ([4]) there could be a chance to find classifying representatives, cf. more in Chapter 8.

This monograph is essentially self-contained and requires only very basis knowledge on C^*-algebras, very elementary facts of K_*-theory and the classical results of J. Cuntz [**169**], [**172**] on his algebras $\mathcal{O}_n$ ([5]).

The proofs of Theorems A and B use that every unital separable exact C^*-algebra is a sub-quotient of the CAR-algebra M_{2^∞} (first shown in cf. [**437**] and

[4] Notice that the decomposition dimension and the covering dimension are very different for non-Hausdorff topological spaces in general!

[5] See Sect. 1 of Appendix A. We need only the defining relations and that $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are simple, purely infinite, separable, unital and nuclear, with $\mathrm{K}_*(\mathcal{O}_2) = 0$, $\mathrm{K}_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $\mathrm{K}_1(\mathcal{O}_\infty) = 0$. And that they really are KK-equivalent to the commutative C^*-algebras $\mathrm{C}_0(0,1]$ and $\mathrm{C}[0,1]$), or the algebra of one point: $\mathbb{C}$ in case of $\mathcal{O}_\infty$.

[**438**]). The ideal-equivariant generalizations in Theorem 6.3.1 is more general than Theorem A but uses similar ideas in its proof. But the point is that those ideas have to be generalized before in an ideal system equivariant manner. Its proof contains implicitly the old result on sub-quotients of M_{2^∞} in [**438**], that is a kind of "inverse" of Glimm's characterization of non-type-I C^*-algebras as a very special case.

We refer the reader to [**73**], [**692**] and [**816**] for the needed basic K-theory, and to [**73**] for some of the later used ideas of KK-theory, but we give in Chapter 4 proofs of those facts in K-theory that will be used for an interpretation of KK-groups as K_*-groups of generated by certain relative commutators.

We call a *simple*, purely infinite, separable, unital and nuclear C^*-algebra A a **pi-sun algebra** ([6]), cf. Chapters 2 and 3 for equivalent definitions and properties, for example, Part(vi) of Proposition 2.2.1 is another reason for our *decision* not to mention the *simplicity* of A in "pi-sun".

The class of *simple* **purely infinite, separable, unital and nuclear** C^*-algebras (in short: **pi-sun** algebras) include the simple Cuntz-Krieger algebras associated to Markov chains [**183**], [**174**] or – more generally – some simple graph-algebras and the Ruelle algebras associated to hyperbolic homeomorphisms of compact spaces [**653**] as well as some C^*-algebras associated to boundary actions of certain groups and to a class of groupoids [**19**], [**507**]. The crossed product C^*-algebra $\mathrm{C}(X) \rtimes \Gamma$ defined by the action of a lattice $\Gamma \subset G$ on the Fuerstenberg boundary X of G is a pi-sun algebra if G is a real connected semi-simple Lie group G without compact factors and with trivial center, [**19**]. Sometimes invariants of pi-sun algebras reflect geometric properties of the underlying dynamical systems, for example the K-theory groups of the Cuntz-Krieger algebras $\mathcal{O}_A$ are the Bowen–Franks invariants of flow equivalence for the related matrix A ([**102**], [**183**]). Many, but not all, Cuntz-Pimsner algebras of Hilbert bi-modules over commutative C^*-algebras, and certain classes of crossed products by "properly outer" actions are strongly purely infinite ([7]).

All pi-sun algebras A in the UCT-class are stably isomorphic to (generalized) Toeplitz C^*-algebras of a suitable Hilbert $\mathrm{C}_0(X, \mathbb{K})$-bi-module $\mathcal{H}$ where X is a locally compact Polish space with $\mathrm{K}^*(X) = \mathrm{K}_*(A)$. This is one of the many possible ways to show that the K-theory invariants are really exhausted by the pi-sun algebras in the UCT-class up to stable equivalence.

To provide the reader just here with at least one example in each stable isomorphism class we give *here* a description of suitable bi-modules $\mathcal{H}$ for given locally compact X (see Chapter 11 for more details):

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[6] We do not mention the "simplicity" in the name "pi-sun" because the – very first – original algebraic definition of pure infiniteness of *unital* algebras A was: *For* $0 \neq a \in A$ *there are* $b, c \in A$ *with* $bac = 1$. Non-simple p.i. algebras require anyway a much more elaborate classification theory!

[7] Results of [**359**] show that the example of Rørdam [**687**] of a unital simple finite, but stably infinite (!), algebra $\mathcal{R}$ is a full corner of a Cuntz-Pimsner algebra given by a suitable Hilbert bi-module $\mathcal{H}(A, h)$ over $A := \mathrm{C}(S_2 \times S_2 \times \cdots, \mathbb{K})$. Thus, $\mathcal{R}$ is KK-equivalent to A.

Let $D\colon \mathrm{C}_0(X,\mathbb{K}) \to \mathcal{L}(\ell_2) = \mathcal{M}(\mathbb{K})$ be a faithful non-degenerate *-representation that is unitarily equivalent to its infinite repeat $\delta_\infty \circ D \cong D \oplus D \oplus \cdots$. Then $\mathcal{H} := \mathrm{C}_0(X,\mathbb{K})$ becomes a Hilbert $\mathrm{C}_0(X,\mathbb{K})$-bi-module with left action ϕ of $\mathrm{C}_0(X,\mathbb{K})$ on $\mathcal{H}$ given by

$$\phi(f)g := (1 \otimes D(f))g \quad \text{for} \quad f, g \in \mathrm{C}_0(X,\mathbb{K})\,.$$

The corresponding generalized Toeplitz-Pimsner C^*-algebra $\mathcal{T}(\mathcal{H})$ is simple, separable, stable and nuclear, cf. Corollary 2.18.7 ([8]). It is KK-equivalent to $\mathrm{C}_0(X,\mathbb{K})$ (and, hence, to A) by results of Pimsner [**633**, thm.4.4, cor.3.14]. Using Voiculescu's generalization of the Weyl–von-Neumann theorem, cf. [**43**] – or our further going generalizations of it in Proposition 5.4.1, one can show that $\mathcal{T}(\mathcal{H})$ is moreover purely infinite (cf. Corollary 2.18.14). Our Theorem B implies that $A\otimes\mathbb{K}$ is isomorphic to $\mathcal{T}(\mathcal{H})$ if A is a pi-sun algebra that is KK-equivalent to $\mathrm{C}_0(X)$. Since $\mathcal{T}(\mathcal{H})$ is stable and simple, it is the same as the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$. The latter implies that $A \otimes \mathbb{K}$ *is the crossed product* $B \rtimes \mathbb{Z}$ *of some type-I algebra* B *by a suitable automorphism of* B.

In this way, we can see that the possible K_*-invariants are exhausted by the above examples, because they are exhausted by the $\mathrm{K}^*(X)$-groups of locally compact Polish spaces X, cf. Lemma B.11.1. Other constructions of pi-sun algebras with given K_*-invariants have been used by Elliott and Rørdam [**269**]. But all this examples do not say that it is their only appearance. Usually it is not easy to verify from some given informations whether an algebra is a (simple) pi-sun algebra in the UCT class or not. The present status is that there is no general method to detect from defining relations on generators if the universal C^*-algebra defined by this relations is KK-equivalent to a commutative C^*-algebra or not.

See more about the relations between our interpretation of the classification conjecture and its formulations by G. Elliott in Section 4, cf. also [**686**, sec. 2.2].

We consider here at first the special case where A *is simple, separable, nuclear,* $T^+(A) = \{0\}$ *and the real rank of* A *is zero.* Here we denote by $T^+(A)$ the cone of non-negative semi-finite lower semi-continuous traces $\tau\colon (A\otimes\mathbb{K})_+ \to [0,\infty]$.

It turns out that our considerations of the special case of (simple) pi-sun C^*-algebras imply that the general Conjecture of Elliott – here restricted to the particular case of pi-sun algebras – is equivalent to the below stated questions **(Q1)** and **(Q2)**. The notion of simple purely infinite algebras was introduced in [**172**, p. 186] and will be discussed in Chapter 2 in detail. We prefer to work with the below given Definition 1.2.1.

By [**269**], or by Theorem I below, the possible Elliott invariants are exhausted by C^*-algebras in the class of those *simple purely infinite separable nuclear unital* C^*-algebras (here called **pi-sun algebra**) that are KK-equivalent to commutative C^*-algebras, i.e., that are in the "UCT-class".

[8] Since D is unitarily equivalent to its infinite repeat $\delta_\infty \circ D$, the algebra $\mathcal{T}(\mathcal{H})$ is equal to the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{H})$.

Summing up we get: A complete proof of the Conjecture of Elliott in the case of simple separable nuclear C^*-algebras A and B of real rank zero and with $T^+(A) = T^+(B) = 0$ would imply in particular that the following two long-standing open Questions **(Q1)** and **(Q2)** could be answered affirmative:

(Q1) *Is every simple stably infinite unital* nuclear *C^*-algebra A of* real rank zero *also purely infinite?*
This means: *Is every non-zero projection $p \in A$ infinite in A if A has "real rank zero", A is simple, is nuclear, is separable, and A contains at least one infinite projection $q \neq 0$?*

(Q2) *Is $\mathcal{O}_2$ up to isomorphisms the only (simple) pi-sun C^*-algebra A with* $\mathrm{K}_*(A) = 0$ *?*

Here $\mathcal{O}_2$ denotes the *Cuntz algebra on two isometries as generators* (i.e. $s_1, s_2 \in \mathcal{O}_2$, with $s_1^* s_1 = s_2^* s_2 = 1 = s_1 s_1^* + s_2 s_2^*$ cf. Section 1 of Appendix A or [**169**].

By Theorem E(ii) below, the Question **(Q1)** has a positive answer (also if A has not real rank zero) in the special case where $A \cong B \otimes C$ and where the algebras B and C both are *not "elementary"*, i.e., both are not isomorphic to the compact operators $\mathbb{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ of infinite or finite dimension. Implicitly, our results imply also that a simple *unital* separable nuclear C^*-algebra A is purely infinite, if and only if, the relative commutant $A' \cap A_\omega$ of A in its ultrapower A_ω is *not* stably finite (cf. [**448**, rem. 2.13] for details), and this is the case, if and only if, this relative commutant is purely infinite and simple ([9]). Then, one has $A \cong \mathcal{O}_2$, if and only if, $0 = [1] \in \mathrm{K}_0(A' \cap A_\omega)$ (cf. Corollary G).

But even if it turns out that Question **(Q1)** has a negative answer, we could add to the invariants the pre-ordered "local" Cuntz semigroup $\mathrm{CS}(A)$, cf. Definition 2.4.3, or [**171**], [**175**] for the unital case, as an *almost* K-theoretic invariant. Then we can describe the *purely infinite* algebras in the following way (cf. Corollary 2.4.6):
A *simple* C^*-algebra A is purely infinite – or is isomorphic to $\mathbb{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ –, if and only if, the "local" Cuntz semigroup $\mathrm{CS}(A)$ of A does not contain an "infinitesimal sequence" $[a_n] \in \mathrm{CS}(A)$ in the sense of Definition 2.4.4 of infinitesimal sequences in pre-ordered semi-groups.

Another possible "weak additional assumption" for an answer of question **(Q1)** is the assumption that A has the corona factorization property (CFP) – that we discuss in Section 26 of Appendix A.

If a simple separable C^*-algebra A has real rank zero and satisfies the (CFP) then A is stably finite or is purely infinite.

If $B := A \otimes \mathbb{K}(\ell_2)$ contains an infinite projection P, then this projection is properly infinite by simplicity of B. More generally if $P \in B$ is a projection and

[9] More generally: If a C^*-algebra A has the property that $F(A) := (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$ and A are both separable, then A is simple and *nuclear*, and then only the three cases can occur: $A = 0$, or A and $F(A)$ are both purely infinite, or where $A \otimes \mathbb{K} \cong \mathbb{K}$ and $F(A) \cong \mathbb{C}$, cf. [**448**].

$0 \neq b = PbP$ is an infinite element, then b and P are properly infinite in B by simplicity of B

$\Leftarrow$ Where is this proved?

How the (CFP) should manage that this strong alternative happens?

If one of the (non-zero) projections q is infinite, then q is automatic properly infinite by simplicity of A.

It is sufficient then to consider qAq because then there exists a unital morphism of the Cuntz algebra $\mathcal{O}_\infty$ into qAq. To get that A is purely infinite, one needs that every non-zero projection $0 \neq p \leq q$ is infinite (otherwise no projection in pAp is infinite!).

It needs that each non-zero projection p is infinite, because then p is automatic properly infinite by simplicity of A. This follows from the general property that the set of elements $a \in A$ with $a \oplus p \precsim p$ is a closed ideal of A.

But the assumptions give only that all non-zero projections p have the property that there exists an individual number $n(p) \in (N)$ such that $1_{n(p)} \otimes p$ is infinite in $M_{n(p)} \otimes A$. – It is then there in $M_{n(p)} \otimes A$ properly infinite.

Until here we have not used the ... ?????

The nuclearity of A is not considered in all combinations of above listed additional assumptions to get A purely infinite: There could be some weaker additional assumption in case of nuclear A, not found or conjectured yet (2022).

The below given Corollary J shows that Question **(Q2)** is equivalent to the following important open question **(Q2*)**, and that positive answers to **(Q1)** and **(Q2)** together imply a proof of the Conjecture of Elliott in the case $T^+(A) = \{0\}$ and amenable separable A with real rank zero .

(Q2*) *Does every separable* nuclear *C^*-algebra satisfy the Universal Coefficient Theorem?*

Here the Universal Coefficient Theorem (=: UCT) in the sense of Rosenberg and Schochet is considered, [**699**]. We refer the reader to [**73**, sec. 23]. A separable C^*-algebra satisfies the UCT, if and only if, it is KK-equivalent to a commutative C^*-algebra (cf. [**73**, thm: 23.10.5], [**726**, prop. 3.5], [**699**, cor. 7.5]). It suffices to consider in question **(Q2)** simple purely infinite A with $\mathrm{K}_*(A) = 0$ that are fixpoint-algebras A of circle actions on $\mathcal{O}_2$, cf. Corollary J.

We use a careful explained and conceptual approach for proofs in case of simple algebras, that is not the shortest possible way in this special case, but illustrate more involved generalizations that allow later to generalize some of the results to non-simple algebras: the results allow to classify all separable stable nuclear C^*-algebras up to tensor products with the Cuntz algebra $\mathcal{O}_\infty$ by its KK-equivalence classes for some generalized ideal-system equivariant KK-theory. We consider the non-simple case and the needed definitions in the second part of this introduction.

In Chapter 12 we give the proofs of lemmas and propositions which are used to extend the ideas for simple algebras to the non-simple case.

If we consider all from this general viewpoint, then, in Chapters 2 and 3, we establish the implication { purely infinite $\Rightarrow$ "strongly" purely infinite } in the particular case where $\mathrm{Prim}(A)$ consists only of one point. In Chapters 5 and 7 we study the consequences of this implication. Generalizations of results of Chapters 2 and 3 to the case of non-simple algebras are in parts joint works with E. Blanchard and M. Rørdam. Detailed proofs did appear in separate papers, ([**462**], [**463**], [**92**], [**93**]), cf. remarks in Chapters 2 and 3.

Our exposition is completely self-contained and needs only basic knowledge on C^*-algebras, as e.g. functional calculus and some basic observations of KK-theory from textbooks, e.g. [**73**]. The use of some ideas from [**172**] will be explained in Chapter 4 and Appendix A.

Notice that this *very detailed introduction contains all the proofs of the Corollaries C, D, F, G, H, J, L and N.*

Only the very key Theorems A, B, E, I, K, M and O will be proved in the chapters of this book, together with some other independent results that could be initial for some future research.

1. The case of simple purely infinite algebras

We consider first the case of simple C^*-algebras. The spatial tensor product of C^*-algebras A and B by $A \otimes B$ it is the completion of the algebraic tensor product $A \odot B$ with respect to the minimal C^*-norm on it, cf. [**766**]. C^*-algebras A are called **exact** if the functor $B \mapsto A \otimes B$ (here with spatial = minimal tensor product) is an exact functor on the category of C^*-algebras (see [**428**]-[**438**], [**810**] and Chapters 3 and 6).

Obviously every nuclear C^*-algebra A is exact, by the definition of nuclearity via uniqueness of the C^*-norm on the algebraic tensor product $A \odot B$ for each C^*-algebra B ([**766**], [**509**]).

The following Theorems A, B and E in the case of simple algebras (and Theorems K and M in the non-simple cases) are the fundamental results of this book. ...

THEOREM A. *Let A denote a unital separable C^*-algebra.*

(i) *A is exact, if and only if, A is isomorphic to a C^*-subalgebra of $\mathcal{O}_2$.*
(ii) *A is nuclear, if and only if, A is isomorphic to the range of a unital conditional expectation from $\mathcal{O}_2$ onto a C^*-subalgebra of $\mathcal{O}_2$.*

Clearly Part (i) remains true if A is not unital because A is exact if and only if its unitization $\widetilde{A}$ is exact.

The embedding of unital separable exact A into $\mathcal{O}_2$ as expressed in Part (i) can be always taken as unital *-monomorphism from A into $\mathcal{O}_2$, because $\mathrm{K}_*(\mathcal{O}_2) = 0$

implies that every non-zero projection $p \neq 1$ of $\mathcal{O}_2$ is the range of an isometry, cf. [**172**] and Lemma 4.2.6(ii).

A slight modification of the arguments in the proof of Corollary C (below) shows that Theorem B(i)-(iii) implies that any two unital monomorphisms $h, k\colon A \to \mathcal{O}_2$ are unitarily homotopic in the sense of the below given definition. In particular, they are approximately unitarily equivalent.

We prove Theorem A in Chapter 6. The idea of the proof is the following: Our Weyl–von-Neumann–Voiculescu type theorem for p.i. algebras (see Chapter 5)

give exact cite for our WvNV-theorem in Chp.5 !!!

implies that sub-quotients of $\mathcal{O}_2$ are isomorphic to subalgebras of $\mathcal{O}_2$. We apply Glimm's theorem to the anti-liminal algebra $\mathcal{O}_2$, and then we use the characterization of separable exact algebras as sub-quotients of the CAR algebra M_{2^∞}, [**438**]. Certainly, one could also use directly the natural inclusion of M_{2^∞} into $\mathcal{O}_2$. But then one does not get the additional result that the embedding of unital *nuclear* A into $\mathcal{O}_2$ can be chosen such that the conditional expectation in Part (ii) of Theorem A, is an *extreme point* of the linear contractions of $\mathcal{O}_2$, cf. Remark 6.2.2. It implies e.g. that there are unital separable nuclear C^*-algebras A and unital embeddings of A into $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes 1 \subset \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ that can not be "conjugate" in $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathbb{K}$ by using automorphisms of $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathbb{K}$, though even all unital C^*-monomorphisms of A into $\mathcal{O}_2$ are unitarily homotopic (in the sense of Definition 5.0.1 in Chapter 5).

Theorem A will be used to formulate and prove the below stated Theorem B, which we start to explain now:

Suppose that A is a separable unital exact C^*-algebra. Theorem A shows the existence of unital *-monomorphisms $h_1\colon A \to \mathcal{O}_2$ and $h_2\colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$, because $\mathcal{O}_2 \otimes \mathcal{O}_2$ is nuclear. We define a unital *-monomorphism $h_0^u\colon A \to \mathcal{O}_2$ by $h_0^u(a) := h_2(h_1(a) \otimes 1)$. If B is a unital C^*-algebra which contains a (fixed) unital copy of $\mathcal{O}_2$, then h_0^u defines a unital nuclear *-monomorphism from A into B (which we fix from now on).

We define $h_0\colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ by $h_0 := h_0^u \otimes \mathrm{id}_{\mathbb{K}}$.

Let D, E be C^*-algebras and $h\colon D \to E$, $k\colon D \to E$ C^*-morphisms. We say that h and k are **unitarily equivalent** if there is a unitary u in the multiplier algebra $\mathcal{M}(E)$ of E such that $u^*k(\cdot)u = h$, and we say that h and k are **unitarily homotopic** ([10]) if there exists a strongly continuous map $t \mapsto U(t)$ from $\mathbb{R}_+ = [0, \infty)$ into the unitary operators in $\mathcal{M}(E) \subseteq \mathcal{L}(E)$ such that $\lim_{t\to\infty} U(t)^*k(b)U(t) = h(b)$ for every $b \in D$. (Note that this does *not* imply that h and k are homotopic: unitarily equivalent morphisms are unitarily homotopic, but unital inner automorphisms are not necessarily homotopic to the identity map.) If h and k are unitarily homotopic, then h is homotopic to $U(0)^*k(\cdot)U(0)$ in the usual sense. But we can conclude homotopy from unitary homotopy in the case

[10] Compare with the more general Definition 5.0.1 in the case where $h, k\colon D \to \mathcal{M}(E)$.

where the unitary group of $\mathcal{M}(E)$ is connected, e.g., if E is stable and σ-unital, if E is an AF-algebra, or if E is purely infinite and simple and $\mathrm{K}_1(E) = 0$, cf. [**172**], e.g. in case $E = \mathcal{O}_\mathrm{n}$, $n = 2, 3, \ldots, \infty$. Under this very special circumstance the unitary homotopy becomes considerably *stronger* than homotopy. But homotopy implies unitary homotopy in some cases, see Corollary D and the remarks below Corollary L and Theorem M.

Let $[\mathrm{Hom}_{\mathrm{nuc}}(D, E)]$ denotes the classes of unitarily equivalent nuclear *-homomorphisms from D into E. If $\mathcal{M}(E)$ contains a unital copy $C^*(s,t) \subseteq E$ of $\mathcal{O}_2$, then one can define the **Cuntz addition** $[h] + [k] := [h \oplus_{\mathcal{O}_2} k]$ of unitary equivalence classes of morphisms $h, k \in \mathrm{Hom}_{\mathrm{nuc}}(D, E)$ by:

$$h \oplus_{\mathcal{O}_2} k(d) := \; sh(d)s^* + tk(d)t^* \qquad \text{for} \quad d \in D\,,$$

where s and t denote the canonical generators of the copy of $\mathcal{O}_2$ in E. The definition is independent – up to unitary equivalence – of the choice of the representatives h and k of the unitary equivalence classes $[h]$ and $[k]$ and of the chosen unital copy of $\mathcal{O}_2$. $\mathrm{Hom}_{\mathrm{nuc}}(D, E)$ becomes with Cuntz addition a commutative semigroup. All that follows from the observation of Cuntz [**172**] that $t_i = us_i$ where $u = \sum t_i s_i^*$ is unitary if s_i and t_i for $(i = 1, ..., n)$ are canonical generators of unital copies of $\mathcal{O}_\mathrm{n}$ (i.e., $t_i^* t_i = s_i^* s_i = 1$, $\sum t_i t_i^* = \sum s_i s_i^* = 1$).

The Cuntz addition coincides with the natural addition of morphisms if E is *stable*, i.e., if $E \cong F \otimes \mathbb{K}$ for some C^*-algebra F, because the "natural" embedding $\mathbb{K} \oplus \mathbb{K} \subset M_2(\mathbb{K}) \cong \mathbb{K}$ is defined by a selection of a unital copy of $\mathcal{O}_2$ in $\mathcal{L}(H) = \mathcal{M}(\mathbb{K})$, and this copy is so natural or artificial as the Cuntz addition is in general: It becomes "natural" or "well-defined" only after *passage to unitary equivalence* classes. It is useful to remind this phenomenon if one seeks for generalizations of this constructions to the case where one wants to get some sort of "equivariant" isomorphisms up to conjugacy for actions of groups, quantum-groups, (or specific sub-systems of ideals) by suitable generalizations of our approach, because then one can only expect some sort of outer co-cycle conjugacy, sometimes only by 2-co-cycles.

The natural map $\mathrm{Hom}(A, B) \ni h \mapsto [h - 0] \in \mathrm{KK}(A, B)$ is given by the Kasparov module $(h, B, 0)$, i.e., $\phi := h$, $E := B$ and $F := 0$. Or some other equivalent constructions, as e.g. by the natural isomorphism $\mathrm{KK}(A, B) \cong \mathrm{Ext}(SA, B)$ and the mapping cone construction.

We compare the semi-group of unitary equivalence classes of nuclear morphisms in $\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$ with nuclear KK-theory and nuclear Ext-theory in Chapters 8 and 9, using Theorem 4.4.6 and other basics from Chapter 4. A natural semigroup morphism

$$\alpha\colon\; \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B \otimes \mathbb{K}) \to \mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B)$$

is defined as follows:

We define a nuclear *-monomorphism

$$\psi_h \colon A \otimes \mathbb{K} \to \mathcal{M}(SB \otimes \mathbb{K})/(SB \otimes \mathbb{K})$$

(i.e., a representative of an element of $\mathrm{Ext}_{\mathrm{nuc}}(A\otimes\mathbb{K}, SB)$) for a given representative $h\colon A\otimes\mathbb{K} \to B\otimes\mathbb{K}$ of $[h] \in \mathrm{Hom}_{\mathrm{nuc}}(A\otimes\mathbb{K}, B\otimes\mathbb{K})$. It gives the Busby invariant of an extension $[h-0]$ in $\mathrm{Ext}_{\mathrm{nuc}}(A\otimes\mathbb{K}, SB)$, which is just the mapping cone construction. It can be described with help of the natural monomorphisms and isomorphisms

$$B \otimes \mathbb{K} \subset \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}) := \mathrm{C_b}([0,\infty), B \otimes \mathbb{K})/\, \mathrm{C_0}([0,\infty), B \otimes \mathbb{K})$$

and ???

Is this the correct definition?

Or have we to take here $0 \oplus (B \otimes \mathbb{K})$? **??**

Or can we use here that $B \otimes \mathbb{K} \subset \mathrm{C_b}([0,\infty), B \otimes \mathbb{K})$?

Compare with image of $[(h, B, 0)] \in \mathrm{KK}(A, B)$
by the map $\mathrm{KK}(A, B) \to \mathrm{Ext}(A, SB)$!!!

Or explore here that $B \otimes \mathbb{K} \subset \mathrm{C_b}([0,\infty), B \otimes \mathbb{K})$?

$$(B \otimes \mathbb{K}) \oplus (B \otimes \mathbb{K}) \subset \mathrm{Q}(\mathbb{R}_-, B \otimes \mathbb{K}) \oplus \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}) \cong \mathrm{Q}(\mathbb{R}, B \otimes \mathbb{K})$$

and $\mathrm{Q}(\mathbb{R}, B \otimes \mathbb{K}) \subset \mathcal{M}(SB \otimes \mathbb{K})/SB \otimes \mathbb{K}$, where we use the notation

$$\mathrm{Q}(X, D) := \mathrm{C_b}(X, D)/\, \mathrm{C_0}(X, D)$$

for a locally compact space X and a C^*-algebra D. Then ψ_h is defined by

$$\psi_h(a) := (0, h(a)) \in (B \otimes \mathbb{K}) \oplus (B \otimes \mathbb{K}) \subset \mathrm{Q}(\mathbb{R}, B \otimes \mathbb{K}).$$

The corresponding extension of $A \otimes \mathbb{K}$ by $SB \otimes \mathbb{K}$ is the well-known *mapping cone* C_h of h, see [**73**, p. 236/237].

We denote by $[h-0]$ the class of h in $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$.

`Next red nearly a black-out??`
`One has to add the universal trivial extension`
$H_0 \colon A \otimes \mathbb{K} \to \mathcal{M}(SB)/SB$ `and gets` $[h-0] = [k-0]$
`if and only if`
$H_0 \oplus h$ `and` $H_0 \oplus k$ `are unitary equivalent` $\mathrm{mod}\, SB$ `!!`

We have that $[h-0] = [k-0]$ in $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$ if h and k are stably unitarily equivalent by unitaries in $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B \otimes \mathbb{K}))/\, \mathrm{C_0}(\mathbb{R}_+, B \otimes \mathbb{K})$, because, for unital B, every unitary of $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B \otimes \mathbb{K}))/\, \mathrm{C_0}(\mathbb{R}_+, B \otimes \mathbb{K})$ lifts to a unitary in $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B \otimes \mathbb{K}))$, but those unitaries are given by strictly continuous maps from $\mathbb{R}_+$ into the unitaries of $\mathcal{M}(B \otimes \mathbb{K})$ ([11]).

[11] In this particular case the unitary equivalence can be given by an operator-norm continuous path $t \mapsto \exp(ih_1(t)) \cdot \ldots \cdot \exp(ih_n(t))$ with continuous maps $t \mapsto h_k(t) = h_k(t)^* \in B \otimes \mathbb{K}$.

In particular, $\alpha([h]) := [h - 0]$ is well-defined, and the natural isomorphism $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B)$ maps the element $[h - 0]$ into the usual KK-difference of h and the zero morphism, which is given by the $\mathrm{KK}_{\mathrm{nuc}}$-class of the Kasparov module $(B, h, 0)$.

The key result for the classification of pi-sun algebras is the following Theorem B, which combines three essentially different "theorems" as Parts (i)+(ii), (iii) and (iv).

THEOREM B. *Let A be a separable unital exact C^*-algebra and suppose that B is unital and contains a copy of $\mathcal{O}_2$ unitally. Let $h_0\colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ the up to unitary homotopy unique *-monomorphism that is defined by Theorem A.*

(i) *The map $\alpha\colon [h] \mapsto [h - 0]$ (given by the mapping cone construction by h) is a semigroup epimorphism from the semigroup of unitary equivalence classes $[h]$ of nuclear C^*-morphisms $h \in \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$ onto $\mathrm{KK}_{\mathrm{nuc}}(A, B)$, and*

(ii) `Needs to prove first homotopy invariance of` $\mathrm{R}(h_0\,;\, A, B)$`?`
`The decomposition argument works now well ! ??`
$[h - 0] = [k - 0]$ in $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ if and only if $h \oplus h_0$ and $k \oplus h_0$ are unitarily homotopic.

(iii) *If moreover B is purely infinite and simple, then $h \oplus h_0$ is unitarily homotopic to h for* all **-monomorphisms $h \in \mathrm{Hom}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$.*

(iv) `Check again proof of part (iv)!`
`Does it work for stable` D `and` E `? ??`
*Let D and E separable C^*-algebras and $h\colon D \to E$, $k\colon E \to D$ *-monomorphisms such that $h \circ k$ is unitarily homotopic to id_E and $k \circ h$ is unitarily homotopic to id_D.*

Then there exists an isomorphism φ from D onto E that is unitarily homotopic to h.

`Can only prove -- so far -- that:`

Then there exists an isomorphism φ from D onto E that is approximately unitary equivalent to h.

Notice here that $\mathrm{KK}(A, B) = \mathrm{KK}_{\mathrm{nuc}}(A, B)$ if one of A or B is nuclear.

The proof of Parts (i) and (ii) of Theorem B is given in Chapter 9. We prove Part (iii) in Chapter 7 and Part (iv) in Chapter 10.

One can use the *result* of Theorem B to relax the above given very special definition of the "zero-element" h_0 as follows.

A C^*-algebra morphism $h\colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ is unitarily homotopic to h_0, if and only if, there exist *-monomorphisms $k_1\colon A \otimes \mathbb{K} \to \mathcal{O}_2 \otimes \mathbb{K}$ and $k_2\colon \mathcal{O}_2 \otimes \mathbb{K} \to B \otimes \mathbb{K}$ such that

$h = k_2 k_1$?????? (or only h unitarily homotopic to $k_2 k_1$??) **??**

and $B \otimes \mathbb{K}$ is the closed ideal generated by $k_2(\mathcal{O}_2 \otimes \mathbb{K})$. This can be seen from Theorem B and following general observations (1)-(3) using [**172**]:

(1) If $\mathcal{O}_2 \cong C^*(s,t) \subseteq B$ is a given copy of $\mathcal{O}_2$ in B with $1_B \in \mathcal{O}_2$, then every (non-zero) C^*-morphism $h\colon \mathcal{O}_2 \otimes \mathbb{K} \to B \otimes \mathbb{K}$ that generates $B \otimes \mathbb{K}$ as the closed ideal of $B \otimes \mathbb{K}$ generated by the image of h, i.e., the C^*-morphism h satisfies $(B \otimes \mathbb{K})h(\mathcal{O}_2 \otimes \mathbb{K})(B \otimes \mathbb{K}) = B \otimes \mathbb{K}$, is "unitary homotopic" in the multiplier algebra $\mathcal{M}(B \otimes \mathbb{K})$ – in the sense of Definition 5.0.1 – to a C^*-morphism $k \otimes \mathrm{id}_{\mathbb{K}}\colon \mathcal{O}_2 \otimes \mathbb{K} \to B \otimes \mathbb{K}$, where $k\colon \mathcal{O}_2 \to B$ is a unital C^*-morphism.

This new copy $k(\mathcal{O}_2)$ of $\mathcal{O}_2$ can be different from the given copy $C^*(s,t)$ and k is not necessarily homotopic in $\mathrm{Hom}(\mathcal{O}_2, B)$ to the C^*-morphism $h\colon \mathcal{O}_2 \to B$ defined by $h(s) := s$ and $h(t) := t$, because it can happen that the class $[k(s)s^* + k(t)t^*]$ is non-zero in $\mathrm{K}_1(B)$ ([12]).

(2) The *statements* (i) and (ii) of Theorem B are independent of a particular selection of a unital copy of $\mathcal{O}_2$ in B.

(3) $h_0\colon A \otimes \mathbb{K} \to \mathcal{O}_2 \otimes \mathbb{K}$ is unitarily homotopic to $k \otimes \mathrm{id}_{\mathbb{K}}$ by Parts (i,ii,iii) of Theorem B because $\mathrm{KK}(A, \mathcal{O}_2) = 0$, cf. [**172**].

It requires that Thm. B is correct with ????

But this can be shown also by elementary considerations form $\mathrm{K}_*(\mathcal{O}_2) = 0$, the K_1-injectivity of $\mathcal{O}_2$ and of $\mathcal{D}_2 := \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$, if one uses both of [**172**] and the Elliott-Rørdam isomorphism $\mathcal{O}_2 \cong \mathcal{D}_2$.

The proof of Parts (i) and (ii) of Theorem B is given in Chapter 9, using Chapters 7, 8 and ????. It can be outlined as follows:

We extend α to a semigroup morphism from $\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))$ into $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B)$ in the obvious way.

(It is formally almost the same definition as for the mapping cone construction!)

We use basic observations in Chapter 4 to recognize that

$$[h_0] + \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))$$

is naturally isomorphic to the Grothendieck group $\mathrm{R}(A, B)$ of the semi-group of unitary equivalence classes of morphisms in $\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))$.

```
in between: there is a new way to apply directly Theorem 4.4.6,
using Section 5 of Chapter 8!
proof property (CD) first?
```

It requires some careful technical observations and in essence an independent proof for homotopy invariance of $\mathrm{R}(A, B)$ with respect to the second variable B

In fact is not clear if the older variant does that,
except by N.Ch. Phillips argument,
seems not to work by application of Theorem 4.4.6,
because $[k \oplus H_0] = [H_0]$ in $\mathrm{Ext}(A, SB)$
does (perhaps) not automatically imply that $k \oplus h_0$
and h_0 are stably asymptotically homotopic.

[12] This is one of the reasons that we pass to stable algebras

--> This is now at the end of Chp. 8 fixed.

Is it still open if the ''algebraic'' KK
is really homotopy invariant?.
Needs that evaluations at $t \in [0,1]$ define
isomorphisms $\mathrm{KK}([0,1]\mathcal{C};\, A, \mathrm{C}([0,1], B)) \to \mathrm{KK}(\mathcal{C};\, A, B)$.
Here, the operator homotopy seems to be enough?

to show that α induces an isomorphism of $\mathrm{R}(A, B)$ and $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$. This yields, in particular, a proof of Theorem B(ii).

??? Next needs first to show the injectivity of $G(h_0;\, A, B) \to G(H_0;\, A, B)$ to get the homotopy invariance !!!

The natural isomorphism $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B)$ and the homotopy invariance of $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ imply that, for a nuclear asymptotic C^*-morphism k from $A \otimes \mathbb{K}$ into $B \otimes \mathbb{K}$, i.e., , for $k \in \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))$, $h_0 \oplus k$ is unitarily equivalent to all its "up/down-scales", given by homeomorphisms of $\mathbb{R}_+$.

But we show this homotopy invariance directly and use the resulting scaling invariance much earlier in the way of proving that α induces an isomorphism from the continuous Rørdam groups $\mathrm{R}(A, B)$ onto $\mathrm{KK}_{\mathrm{nuc}}(A, B)$.

next? Senseless? Correction?

It means that the powerful Theorem B can be used

(and are used here ?) as a reminder of several results that can be deduced from Theorem B but can be also proved independently by other methods.

The homotopy invariance implies, for

$$k\colon A \otimes \mathbb{K} \to \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K})\,,$$

the existence of $h \in \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$ that is unitary equivalent to $h_0 \oplus k$ in

$$\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))\,.$$

Does it prove finally Part (i) of Theorem B ???

The absorption result (iii) of Theorem B is an asymptotic version (given in Chapter 9) of the generalized Weyl–von-Neumann–Voiculescu theorem (cf. Chapter 5), and Part (iv) of Theorem B is a "continuous" version of a result of Elliott, cf. [**256**], [**678**], and Chapter 10.

Notice that our proof of Part (iv) is more general, because it needs only assumptions on D and E that are independent and weaker than all those required for A and B in Theorem B. In particular, exactness, amenability (nuclearity), stability or pure infiniteness are all not used.

Theorem B leads to the below given classification results Corollaries C and H.

If A and B are pi-sun algebras, then we find pi-sun algebras C and D, such that C and D both contain $\mathcal{O}_2$ unitally, and $A\otimes\mathbb{K}\cong C\otimes\mathbb{K}$, $B\otimes\mathbb{K}\cong D\otimes\mathbb{K}$, cf. J. Cuntz [**172**]. If E or F is nuclear and separable, we have $\operatorname{Hom}(E,F)=\operatorname{Hom}_{\mathrm{nuc}}(E,F)$ and $\mathrm{KK}(E,F)=\mathrm{KK}_{\mathrm{nuc}}(E,F)$. Thus, by Theorem B, for $z\in\mathrm{KK}(A,B)$, there is a *-monomorphism $h\in\operatorname{Hom}(A\otimes\mathbb{K},B\otimes\mathbb{K})$ such that $z=[h-0]$ and that $h(A\otimes\mathbb{K})$ contains a strictly positive element of $B\otimes\mathbb{K}$.

If moreover $z=[h-0]$ is a KK-equivalence and $w=[k-0]\in\mathrm{KK}(B,A)$ is the KK-inverse of z then $[\mathrm{id}_A-0]=z\otimes_B w=[kh-0]$ in $\mathrm{KK}(A,A)$ and $[id_B-0]=[hk-0]$ in $\mathrm{KK}(B,B)$. By Theorem B(ii)+(iii), hk is unitarily homotopic to $\mathrm{id}_{B\otimes\mathbb{K}}$ and kh is unitarily homotopic to $\mathrm{id}_{A\otimes\mathbb{K}}$. `Depends from validity of OLD B(iv)`

By Theorem B(iv), there exists an isomorphism λ from $A\otimes\mathbb{K}$ onto $B\otimes\mathbb{K}$ which is unitarily homotopic to h.

Let $\gamma\colon\mathrm{KK}(A,B)\to\operatorname{Hom}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ be the natural functor from $\mathrm{KK}(\cdot,\cdot)$ into $\operatorname{Hom}(\mathrm{K}_*(\cdot),\mathrm{K}_*(\cdot))$ given by $\gamma(z)\colon[x]\in\mathrm{K}_*(A)\mapsto[x]\otimes_A z\in\mathrm{K}_*(B)$, see [**73**, sec. 23.1]. Then $\gamma([h-0])=\mathrm{K}_*(h)$ if $h\in\operatorname{Hom}(A\otimes\mathbb{K},B\otimes\mathbb{K})$, cf. Chapter 8 for more details.

But $[1_A]=[1_A\otimes p_{11}]\in\mathrm{K}_0(A)$ and $\mathrm{K}_0(h)([1_A\otimes p_{11}])=[h(1_A\otimes p_{11})]\in\mathrm{K}_0(B)$, where p_{jk} here denotes the natural basis of $\bigcup_n M_n\subseteq\mathbb{K}$ with $p_{jk}^*=p_{kj}$ and $p_{jk}p_{\ell m}=\delta_{k,\ell}p_{jm}$ i.e., p_{11} is the "upper left corner minimal non-zero projection" in $\mathbb{K}$. Thus, if $z=[h-0]$ and $\gamma(z)$ maps $[1_A]$ to $[1_B]$, then $[h(1_A\otimes p_{11})]=[1_B\otimes p_{11}]$, and, by [**172**], there exist a unitary $v\in\mathcal{M}(B\otimes\mathbb{K})$ with $v^*h(1_A\otimes p_{11})v=1_B\otimes p_{11}$. $b\in B\mapsto b\otimes p_{11}\in B\otimes\mathbb{K}$ is an isomorphism from B onto $(1\otimes p_{11})(B\otimes\mathbb{K})(1\otimes p_{11})$. Let μ be the inverse of this isomorphism. Then $\varphi(a):=\mu(v^*h(a\otimes p_{11})v)$ is a unital *-monomorphism from A into B with $[\varphi-0]=[h-0]=z$ in $\mathrm{KK}(A,B)$, because composition with $a\mapsto a\otimes p_{11}$ (respectively with μ and $v^*(\cdot)v$) induce the identity on $\mathrm{KK}(A,B)$, cf. [**73**, chap. VIII].

If h is moreover unitarily homotopic to λ by $u(t)\in\mathcal{M}(B\otimes\mathbb{K})$ and λ is an isomorphism from $A\otimes\mathbb{K}$ onto $B\otimes\mathbb{K}$, then we find as above a unitary $w\in\mathcal{M}(B\otimes\mathbb{K})$ such that $w^*\lambda(1_A\otimes p_{11})w=1_B\otimes p_{11}$ and $\psi(a):=\mu(w^*\lambda(a\otimes p_{11})w)$ is an isomorphism from A onto B. $t\to u_1(t):=v^*u(t)w$ defines a unitary homotopy from $v^*h(\cdot)v$ to $w^*\lambda(\cdot)w$. In particular, $\lim_{t\to\infty}u_1(t)^*(1_B\otimes p_{11})u_1(t)=1_B\otimes p_{11}$. Thus there exists $t_0>0$ such that a small perturbation of $b(t)=\mu(1_B\otimes p_{11}u_1(t)1_B\otimes p_{11})$ for $t>t_0$ defines a unitary homotopy from φ to the isomorphism ψ.

The above arguments prove that Theorem B has the following corollary:

Corollary C. *Suppose that A and B are pi-sun algebras.*

(i) *Every $z\in\mathrm{KK}(A,B)$ such that $\gamma(z)$ maps $[1_A]\in\mathrm{K}_0(A)$ to $[1_B]\in\mathrm{K}_0(B)$ is given by a* unital *C^*-morphism $\varphi\colon A\to B$, such that $z=[\varphi-0]$.*
Also here is to check: $z-[\varphi-0]$ is perhaps only modulo the kernel $\mathrm{KK}(A,B)\to\operatorname{Hom}(\mathrm{K}_0(A),\mathrm{K}_0(B))$ determined!

(ii) *`Check proof of C(ii) again!` ??*

If z is moreover a KK*-equivalence and $\varphi\colon A \to B$ is a unital *-monomorphism with $z = [\varphi - 0]$,*

then there are an isomorphism ψ from A onto B and a continuous map $t \mapsto u(t)$ from $\mathbb{R}_+ = [0, \infty)$ into the unitaries of B, such that

$$\psi(a) = \lim_{t\to\infty} u(t)^* \varphi(a) u(t) \quad \text{for all } a \in A\,.$$

Thus, $z = [\psi - 0]$.

In particular, $\mathrm{Aut}(A) \ni \varphi \mapsto [\varphi - 0] \in \mathrm{KK}(A, A)$ *maps onto the invertible elements $z \in \mathrm{KK}(A, A)$ with $\gamma(z)([1_A]) = [1_A]$.*

Two automorphisms of A have the same image in $\mathrm{KK}(A, A)$*, if and only if, they are unitarily homotopic.*

The KK-equivalence of unital endomorphisms φ and ψ gives
that $\psi(1) = \varphi(1)$ and that
$\phi_0 \oplus h_0$ and $\psi_0 \oplus h_0$ on A^{st} are unitarily homotopic.
Now ''absorption'' , i.e., $\phi_0 \oplus h_0$ unitary homotopic to ϕ_0.

Corollary C implies the following Corollary D, because KK is homotopy invariant (*without assuming UCT !*), and homotopy equivalences fixes the K_0-class of the unit element. Here we consider general homotopy (in the usual sense), and not our above defined *unitary* homotopy (cf. also Definition 5.0.1).

Corollary D. *Homotopy equivalent pi-sun algebras are isomorphic.*

Next depends from old version of Cor. C(ii) ??

The isomorphism can be chosen such that it is unitarily homotopic to the C-morphism that defines the homotopy equivalence.*

One of the problems is:
Let $\alpha\colon A \to A$ and $\beta\colon A \to A$ such that
$\alpha \circ \beta$ and $\beta \circ \alpha$ are unitarily homotopic to id_A.
Is α unitarily homotopic to an automorphism of A ?
Really? Is that enough?

Notice that here A is "stably infinite" if $A \otimes \mathbb{K}$ contains a full properly infinite projection. In case of simple A this is equivalent to the existence of a (non-zero) infinite projection in $A \otimes \mathbb{K}$. The following theorem shows that finite but not stably finite simple C^*-algebras must be necessarily "tensorial prime". Its proof is given in Section 3 of Chapter 2.

Theorem E. *Suppose that A and B are simple C^*-algebras, that are not isomorphic to the algebra of compact operators on a Hilbert space.*

(i) *If A or B is stably infinite, then the spatial tensor product $A \otimes B$ is purely infinite (and simple).*
(ii) *If $A \otimes B$ is nuclear and is stably infinite, i.e., $A \otimes B \otimes \mathbb{K}$ contains an infinite projection, then $A \otimes B$ is purely infinite.*

In particular, $A \otimes B$ is purely infinite if B is pi-sun and A is simple, e.g. $A \otimes \mathcal{O}_n$ is pi-sun for $n = 2, \ldots, \infty$ and each simple, separable, unital and nuclear C^*-algebra A ([13]).

Notice here, that the identity map id of $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is homotopic to $\mathrm{id} \oplus \mathrm{id}$, because this is true for $\mathcal{O}_2$, cf. [**172**]. Thus, $\mathrm{KK}(A \otimes \mathcal{O}_2, A \otimes \mathcal{O}_2) = 0$ and $\lambda\colon a \in \mathcal{O}_2 \mapsto 1_A \otimes a \in A \otimes \mathcal{O}_2$ must necessarily define a KK-equivalence $z = [\lambda - 0] \in \mathrm{KK}(\mathcal{O}_2, A \otimes \mathcal{O}_2)$. By Theorem E and Corollary C, λ is unitarily homotopic to an isomorphism from $\mathcal{O}_2$ onto $A \otimes \mathcal{O}_2$. In the same way one can see that for *pi-sun* A with $\mathrm{KK}(A, A) = 0$ the only element of $\mathrm{KK}(\mathcal{O}_2, A) = 0$ is given by an isomorphism from $\mathcal{O}_2$ onto A. (Here we have used that $\mathcal{O}_2$ is pi-sun, cf. [**172**].)

The map $\varphi(a) = a \otimes 1$ is a unital monomorphism from A into $A \otimes \mathcal{O}_\infty$, and $z = [\varphi - 0]$ is just the Kasparov tensor product of $w = [id_A - 0]$ with the KK-equivalence $[\varphi_1 - 0]$ given by $\varphi_1\colon t \in \mathbb{C} \mapsto t \cdot 1 \in \mathcal{O}_\infty$. Thus, z is a KK-equivalence (– or use [**73**, 17.8.5 and 17.8.6] twice, or see the related arguments used in Chapter 8 for the generalization to $\mathrm{KK}(X; \cdot, \cdot)$ and $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$ –). So we get from $\varphi(1_A) = 1_A \otimes 1_{\mathcal{O}_\infty}$ and from Corollary C that *φ is unitarily homotopic to an isomorphism from A onto $A \otimes \mathcal{O}_\infty$*. Thus, we obtain the following corollary of Theorems B, E and Corollary C:

Corollary F. *If A is a separable unital nuclear simple C^*-algebra, then:*

(i) *$A \otimes \mathcal{O}_\infty$ is a pi-sun algebra.*
(ii) *$A \otimes \mathcal{O}_\infty \cong A$ if and only if A is pi-sun.*
(iii) *$A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.*
(iv) *If A is pi-sun, then $\mathrm{KK}(A, A) = 0$ if and only if $A \cong \mathcal{O}_2$.*

Perhaps proof more generally:

`Give first ref's to:` $\mathcal{O}_n$ `is pi-sun!`

(ii,1) The C^*-algebra $A \otimes D$ is strongly p.i. in sense of Definition 1.2.2 for every non-zero C^*-algebra A and for every (non-zero) strongly p.i. C^*-algebra D.

In particular this applies to $D := \mathcal{O}_\infty$.

(ii,2) For every nuclear separable strongly p.i. C^*-algebra A there exists an isomorphism φ from $A \otimes \mathcal{O}_\infty$ onto A.

(ii,3) If a C^*-algebra A is isomorphic to $A \otimes \mathcal{O}_\infty$, then there exists an (other) isomorphism φ from $A \otimes \mathcal{O}_\infty$ onto A such that the endomorphism $a \in A \mapsto \varphi(a \otimes 1) \in A$ is unitarily homotopic to the identity map id_A on A.

(In particular, the isomorphism $\varphi^{-1}\colon A \to A \otimes \mathcal{O}_\infty$ has the property that $\varphi^{-1}(J) = J \otimes \mathcal{O}_\infty$ for each closed ideal J of A.)

Way of proof:

It is enough to show:

---

[13] If $B = \mathcal{O}_n$ then one can allow here also $A = M_k$, $k = 1, 2, \ldots$, or $A = \mathbb{K}$ by [**172**], cf. refhere in book appendix A???

(1) $A \otimes D$ is strongly p.i.

[should be in Chp. 2 ready?]

(2) $\mathcal{O}_\infty$ is strongly p.i., because pi-sun, and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$,

[should use Chp. 2,4, 11 ???]

(3) Separable nuclear s.p.i. C^*-algebra A has property: $1_{F(A)}$ is properly infinite in $F(A) := A' \cap A_\omega$.

[should be first in Chp. 2 ?]

(4) $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots \subseteq F(A)$ (unitally) implies for separable A that $A \cong A \otimes \mathcal{O}_\infty$

[Should be first in Chp. 11 complete?

Needs that $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ has approximately inner flip and that $x \in \mathcal{O}_\infty \to x \otimes 1 \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ is approximately unitary equivalent to an isomorphism from $\mathcal{O}_\infty$ onto $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$.]

(5) Prove all of (ii,3) for $A = \mathcal{O}_\infty$ and then for $A \otimes \mathcal{O}_\infty$ (– In place of $A \cong A \otimes \mathcal{O}_\infty$). Then use the isomorphism of $A \otimes \mathcal{O}_\infty$ onto A.

M. Rørdam [**678**] gave a proof for the isomorphism $A \cong A \otimes \mathcal{O}_2$ if A is unital, separable and contains an approximately central sequence of unital copies of $\mathcal{O}_2$. A modification of an idea of G. Elliott leads us to a more general result from [**438**, cor. 2.4(II)] (see also [**786**]) at the end of Chapter 10:
There is an isomorphism $A \cong A \otimes B \otimes B \otimes \cdots$ if A contains an approximately central sequence of unital copies of B in A, and if the C^*-monomorphisms $b \mapsto b \otimes 1_B$ and $b \mapsto 1_B \otimes b$ are approximately unitarily equivalent in $B \otimes B$. Notice that then B is automatically nuclear and simple by an argument from Effros and Rosenberg [**243**], cf. Corollary 10.3.5. In the case of non-unital separable A, one has to require that B has an approximately inner flip, and that a copy of B is unitally contained in $F(A) := (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$, then one gets $A \cong A \otimes B$ if A is stable or contains an approximate unit consisting of projections. We get the equivalence of $B \subseteq F(A)$ and of $A \cong A \otimes B$ for (general) separable A if B has the additional property that B is unitally contained in the ultrapower $\mathcal{E}(B,B)_\omega$ of tensorial join algebra

$$\mathcal{E}(B,B) := \{ f \in \mathrm{C}([0,1], B \otimes B)\,;\; f(0) \in B \otimes 1\,,\; f(1) \in B \otimes B \,\}\,.$$

See [**448**, thm. 4.5(3)]. The property follows from the K_1-injectivity of B, and is therefore satisfied for all C^*-algebras B that are purely infinite or that absorb the Jiang-Su algebra tensorial by

Rørdam ?????? citation ? ... Def. of K_1-injectivity of B ?

But the latter property is now known for all separable C^*-algebras B that absorb tensorial, – i.e., in sense $B \otimes E \cong B$ –, a tensorial self-absorbing separable C^*-algebra E, because all this E absorb the Jiang-Su algebra $\mathcal{Z}$ tensorial by a result of W. Winter in [**831**].

Is the K_1-bijectivity not secure by $\mathcal{Z}$ absorption from work of M. Rørdam?
Check Ref's again:

This property is now established for all tensorial self-absorbing separable E with approximately inner flip that allow in addition to apply the Künneth Theorem, e.g. if they are in the UCT-class. This is because W. Winter [**831**] observed that all those absorb the Jiang-Su algebra $\mathcal{Z}$ tensorial.

The tensorial absorption $B \otimes \mathcal{Z} \cong B$ of $\mathcal{Z}$ implies K_1-injectivity, and allows to deduce that $\mathcal{U}_0(B) = \mathcal{U}(B)$ if B is tensorial self-absorbing, separable and is in the Künneth Theorem class.

On the other hand, – by Corollary F(iii) –, we have that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$. Thus $\mathcal{O}_2$ contains an approximately central sequence of unital copies of $\mathcal{O}_2$. Again by Corollary F(iii), this together implies the following corollary.

Corollary G. *Let A be a simple, separable, unital and nuclear C^*-algebra. Then $A \cong \mathcal{O}_2$, if and only if, A contains an approximately central sequence $\psi_n \colon \mathcal{O}_2 \to A$ (of copies of $\mathcal{O}_2$) unitally.*

With other words: *$A \cong \mathcal{O}_2$, if and only if, A is simple, separable, unital and nuclear, and $A' \cap A_\omega$ contains a copy of $\mathcal{O}_2$ unitally.*

Recall that the natural group morphism $\gamma\colon \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ is an epimorphism if A satisfies the UCT. Moreover, if A and B both satisfy the UCT and $\gamma(z)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ is bijective, then $z \in \mathrm{KK}(A, B)$ is a KK-equivalence, cf. [**73**, prop. 23.10.1]. Therefore, we obtain from Corollary C the following.

Corollary H. *Suppose that A and B are pi-sun algebras, and that A satisfies the universal coefficient theorem* (UCT) *for* $\mathrm{KK}(A, \cdot)$.

(i) *If $\sigma_* = (\sigma_0, \sigma_1)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ is an element of* $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ *such that $\sigma_0([1_A]) = [1_B]$ in $\mathrm{K}(B)$, then there is a unital *-monomorphism $\tau\colon A \to B$ such that $\mathrm{K}_*(\tau) = \sigma_*$.*
(ii) *If B also is in the* UCT*-class and σ_* is an* isomorphism *from $\mathrm{K}_*(A)$ onto $\mathrm{K}_*(B)$ and $\tau\colon A \to B$ is a unital *-morphism with $\mathrm{K}_*(\tau) = \sigma_*$, then there is a* *-isomorphism *$\psi\colon A \to B$ from A* onto *B*
NEXT has to be checked again: ??
such that ψ is unitarily homotopic to τ .

Check proof again!
At least ψ and τ are approximately unitary equivalent!?
In particular, $\mathrm{K}_*(\psi) = \sigma_*$.
(iii) *If* $\mathrm{Ext}^1(\mathrm{K}_*(A), \mathrm{K}_*(B)) = 0$ *, then all unital *-monomorphisms $\tau\colon A \to B$ with $\mathrm{K}_*(\tau) = \sigma_*$ are unitarily homotopic.*

We list here some immediate consequences of Corollary H:

$$\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$$

by an isomorphism that is *unitarily homotopic* to

$$\tau\colon a \in \mathcal{O}_\infty \mapsto a \otimes 1 \otimes 1 \otimes \cdots \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots \,,$$

because both algebras are pi-sun with UCT and K_*-groups $\mathbb{Z}[1]$ and 0 (i.e., here $\mathrm{K}_*(\tau)$ is a K_*-isomorphism and $\mathrm{Ext}^1 = 0$).

But we outline also alternative
proofs of the latter conclusions, by proving first
the homotopy invariance of a general version of Rørdam groups.

In the beginning we gave explicit examples of pi-sun algebras A in the UCT-class with prescribed $\mathrm{K}_*(A)$ using Cuntz-Pimsner algebras, and we give some alternative construction further below. It follows that Corollary H(i) implies:
*If B is pi-sun and $G_*^{(n)} \subseteq \mathrm{K}_*(B)$ is an increasing sequence of finitely generated graded subgroups with $[1_B] \in G_0^{(n)}$ and $\bigcup_n G_*^{(n)} = \mathrm{K}_*(B)$, then there is a sequence (A_n) of pi-sun algebras in the UCT-class, unital *-morphisms $\tau_n\colon A_n \to A_{n+1}$ and a unital *-morphism $\eta\colon A \to B$ for $A := \mathrm{indlim}_n A_n$ with $\mathrm{K}_*(A_n) = G_*^{(n)}$, $\mathrm{K}_*(\tau_n)$ the inclusion $G_*^{(n)} \hookrightarrow G_*^{(n+1)}$ and $\mathrm{K}_*(\eta \circ \tau_n)$ is the inclusion $G_*^{(n)} \hookrightarrow \mathrm{K}_*(B)$ In particular, $\mathrm{K}_*(\eta)$ is an isomorphism from $\mathrm{K}_*(A)$ onto $\mathrm{K}_*(B)$.*

Since A is in the UCT-class, Corollary H(ii) now implies:

Next works also if Corollary H(ii)
has to be corrected to weaker version,
where KK-equivalences are not all expressible by isomorphisms.

*If B is in the UCT-class, then the above *-morphism $\eta\colon A \to B$ can be chosen as an isomorphism from A onto B.*

We denote by $\mathrm{Aut}(\mathrm{K}_*(A), [1])$ the automorphisms of the $\mathbb{Z}_2$-graded Abelian group $\mathrm{K}_*(A)$ that fix the element $[1_A] \in \mathrm{K}_0(A)$.

Then we get from Corollary H:
Suppose that A is pi-sun and satisfies the UCT.

This is partly still in question ?? Or oK?

The composition $\gamma \circ \alpha$ of the map α given in Theorem B(i) with the natural map $\gamma\colon$ $\mathrm{KK}(A, A) \to \mathrm{End}(\mathrm{K}_*(A))$ defines an epimorphism *???????*

$$\gamma \circ \alpha\colon\ \mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{K}_*(A), [1])\,.$$

If $\mathrm{Ext}^1(\mathrm{K}_*(A), \mathrm{K}_*(A)) = 0$, then the kernel of this epimorphism ????????

Must check how the kernel of

$$\mathrm{KK}(\cdot,\cdot) \to \mathrm{Hom}_*(\mathrm{K}_*(\cdot), \mathrm{K}_*(\cdot))$$

looks like?

is equal to the set of those automorphisms of A that are unitarily homotopic to id *of A.*

Every pi-sun algebra A in the UCT-class is anti-isomorphic to itself. Moreover, if $[1_A] = 0$ in $\mathrm{K}_0(A)$, then there is a kind of "co-multiplication"-like unital monomorphism $A \otimes A \to A$. It comes from a – not necessarily unital – commutative algebra $C \subset A$ that is KK-equivalent to A (via this embedding), and from the diagonal map $\widehat{C} \ni \chi \mapsto (\chi, \chi) \in \widehat{C} \times \widehat{C}$. It is co-commutative and co-associative.

A first application of the above general results is the following:
Let A_m be finite direct sums of algebras of form $C(X, M_k(\mathcal{O}_\mathrm{n}))$, where X are compact metric spaces and $k = 1, 2, \ldots$, $n = 2, 3, \ldots$. Let $h_m\colon A_m \to A_{m+1}$ be unital C^*-morphisms and $B := \mathrm{indlim}(h_m\colon A_m \to A_{m+1})$. It is easy to see that $B \cong B \otimes \mathcal{O}_\infty$, because above we have seen that $A_n \cong A_n \otimes \mathcal{O}_\infty$ and $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$. In particular B is purely infinite if B is simple (see Corollary 3.2.23 for a more general result). The algebras A_n and therefore B are in the UCT class. Thus Corollary H classifies the simple one's of those inductive limits up to isomorphism.

To see which invariants are exhausted in given classes of metric spaces X and values k, m, we need to calculate the K_*-invariants, because the corresponding realizations by C^*-morphisms can be obtained from Theorem B using the UCT ([14]). The below described construction (before Theorem I stated below), gives a simpler way to produce pi-sun examples representing given KK-classes of nuclear separable C^*-algebras. This construction can be generalized to the category of C(X)-algebras (see end of Chapter 11).

An separable C^*-algebra A satisfies the UCT if and only if, A is in the "bootstrap category" or "bootstrap class" N (see [**699**] or [**73**, def. 22.3.4] for a definition), if and only if, A is KK-equivalent to a commutative C^*-algebra by an observation of G. Skandalis [**724**].

G. Elliott and M. Rørdam have shown that, given any pair of countable commutative groups G_0 and G_1 and a distinguished element $g_0 \in G_0$, there exists a simple purely infinite separable unital nuclear C^*-algebra A in the (little) bootstrap class N_ℓ such that $(\mathrm{K}_0(A), [1_A]) \cong (G_0, g_0)$ and $\mathrm{K}_1(A) \cong G_1$, cf. [**269**, thm. 5.6], and [**679**, thms. 3.1, 3.6]. Our Corollary H says that such A is unique if we consider simple pi-sun algebras in the UCT class. In particular, simple pi-sun algebras in the bootstrap class N are automatically in "small" bootstrap class N_ℓ.

A more general result by a rather simple method is given by the following Theorem I if we combine this with [**73**, cor.23.10.3] or Lemma B.11.1.

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[14] The K_*-groups and its group homomorphisms for certain "elementary" CW-complexes X can be calculated in good cases with help of the Künneth theorem, 6-term exact sequence, Bott periodicity and Mayer-Vietoris sequence. The calculation of $\mathrm{K}^*(X)$ for a arbitrary CW complex X is not straight-forward, except one knows explicitly the involved elements of $\mathrm{Ext}(\mathrm{K}^{*+1}(X_n), \mathrm{K}^*(\mathbb{R}^n))$ and of $\mathrm{Hom}(\mathrm{K}^*(X_n), \mathrm{K}^*(\mathbb{R}^n))$ that correspond to attaching n-cells to the $(n-1)$-dimensional CW-subcomplex X_n, ...etc.

We describe now an almost functorial construction of simple separable exact purely infinite algebras $P(A)$, such that $P(A)$ is KK-equivalent to a given separable exact C^*-algebra A

Check: Now on new places?!
Check again the exact case! Check references!!! ([15]).

Let A be a separable *exact* C^*-algebra and let ${\mathcal{O}_\infty}^{\mathrm{st}} := (1-s_1s_1^*)\mathcal{O}_\infty(1-s_1s_1^*)$, where $s_1, s_2, \ldots$ are the canonical generators of $\mathcal{O}_\infty$. Then ${\mathcal{O}_\infty}^{\mathrm{st}}$ contains a copy of $\mathcal{O}_2$ unitally, cf. [**172**] or Lemma 4.2.6(iii). We let $A_t := A \otimes {\mathcal{O}_\infty}^{\mathrm{st}}$ if A is unital, and define A_t as the natural unital split extension of $\mathcal{O}_2 \cong 1_{\mathcal{M}(A)} \otimes \mathcal{O}_2$ by $A \otimes {\mathcal{O}_\infty}^{\mathrm{st}}$ if A is not unital, i.e., we have then a unital split-exact sequence

$$0 \to A \otimes {\mathcal{O}_\infty}^{\mathrm{st}} \to A_t \to 1 \otimes \mathcal{O}_2 \to 0 \,.$$

The algebra A_t is a unital separable exact C^*-subalgebra of $\widetilde{A} \otimes {\mathcal{O}_\infty}^{\mathrm{st}}$ and contains a copy of $\mathcal{O}_2$ unitally. The injective C^*-morphism $\eta_0 \colon a \in A \mapsto (a \otimes (s_2 s_2^*)) \in A_t$ defines a KK-equivalence of A with A_t. Clearly, the algebra A_t is nuclear if and only if A is nuclear.

The Theorem A provides a unital *-monomorphism $h_0^u \colon A_t \to \mathcal{O}_2 \subset A_t$, e.g. as described before Theorem B, and then define a unital *-monomorphism for $a \in A_t$ with help of isometries $t_1, t_2 \in \mathcal{O}_2 \subset {\mathcal{O}_\infty}^{\mathrm{st}} \subseteq A_t$ by

$$h(a) := t_1 \, a \, t_1^* + t_2 \, h_0^u(a) \, t_2^* \,,$$

that is here expressed using the Cuntz-addition – discussed in Chapter 4 ([16]) – of the identity map and h_0

$$\mathrm{id}_{A_t} \oplus_{t_1,t_2} h_0^u \colon \; A_t \to A_t \,.$$

Now simply let $P(A) := \mathrm{indlim}(h \colon A_t \to A_t)$ the "stationary" inductive limit with natural defining monomorphisms $h_n^\infty \colon A_t \to P(A)$ given by $h_{n+1}^\infty h = h_n^\infty$, and let $\eta(a) = h_1^\infty(\eta_0(a)) \in P(A)$ the associated natural unital embedding from A_t into $P(A)$.

The author has outlined the proof of the following easy theorem at the

Check again: ??

Conference on Operator Algebras and Quantum Field Theory (in Rome, Italy) July 1996.

THEOREM I. *Let A be a separable exact C^*-algebra. Then:*

(i) *The algebra $P(A)$ is a* simple, *separable and* exact *C^*-algebra that contains a copy of $\mathcal{O}_2$ unitally.*
(ii) *$P(A) \cong P(A) \otimes \mathcal{O}_\infty$. In particular, $P(A)$ is* purely infinite.

[15] See [**810**] or Chapters 3 and 6 on the here needed definition and basic properties of exact C^*-algebras.

[16] The "Cuntz sum" defined in Chapter 4 is the base of our there discussed version of KK-groups that is "constructive" in some sense.

(iii) *The monomorphism* $\eta\colon A \to P(A)$ *defines a* KK*-equivalence* $[\eta - 0]$ *between* A *and* $P(A)$*.*
(iv) *There is a conditional expectation from the hereditary* C^**-subalgebra* $D \subseteq P(A)$*, that is generated by* $\eta(A)$*, onto* $\eta(A) \cong A$*.*
(v) $P(A)$ *is nuclear, if and only if,* A *is nuclear.*

A proof of a more general result for continuous fields is given in Chapter 11. We surround the discontinuity of the functor $\mathrm{KK}(X\,; P(A), (\,\cdot\,))$ with respect to inductive limits, and use the continuity of $\mathrm{KK}(X; (\,\cdot\,), P(A))$ in the sense of [**73**, thms. 21.5.2, 19.7.1] and some considerations of E-theoretic nature, the X-equivariant continuous version of the

??? Effros-Rørdam ??? or Elliott-Rørdam ??

groups considered in Chapter 7.

`Check next red and over-next again!`
`check correct refs to Chapter 11`
`Are so many footnotes necessary?` **??**

In fact, we prove in Chapter 11 a more general result for continuous fields $(A_x)_{x\in X}$ of exact C^*-algebras A_x on a compact metric space X with the additional property that the C^*-algebra $\mathcal{A}$ of continuous sections is exact ([17]):
We take again $:= \mathcal{A}\otimes \mathcal{O}_\infty{}^{\mathrm{st}}$ if $\mathcal{A}$ is unital, and define $\mathcal{A}_t$ as the natural unital split extension of

$$\mathrm{C}(X, \mathcal{O}_2) \cong [\mathrm{C}(X)1_{\mathcal{M}(\mathcal{A})}] \otimes \mathcal{O}_2$$

by $\mathcal{A}\otimes \mathcal{O}_\infty{}^{\mathrm{st}}$ if $\mathcal{A}$ is not unital.

`Not clear if the` $\mathrm{C}(X)$`-algebra` $\mathcal{A}_t$ `is again continuous and has unital fibers that are the minimal unitizations of fibers of` $\mathcal{A}$
`Check construction and its continuity.`
`Compare with fibre-wise construction!`

Then we can use the sub-triviality theorem of E. Blanchard [**89**] ([18]), to get a unital $\mathrm{C}(X)$-module *-monomorphism $h_0^u\colon \mathcal{A} \to \mathrm{C}(X, \mathcal{O}_2) \subseteq \mathcal{A}$. We define again the Cuntz sum $h := \mathrm{id}_{\mathcal{A}_t} \oplus h_0^u$ and let then $\mathcal{P}(\mathcal{A}) := \mathrm{indlim}(h\colon \mathcal{A}_t \to \mathcal{A}_t)$ – the stationary inductive limit. All the operations are compatible with the above defined (fiber-wise) constructions of $(A_x)_t$ and $P(A_x)$ for the fibers A_x of $(A_x)_{x\in X}$. It turns out that $(P(A_x))_{x\in X}$ is a continuous field of C^*-algebras, and that $\mathcal{P}(\mathcal{A})$ is the algebra of continuous sections of this field. By construction, $\mathcal{P}(\mathcal{A})$ is exact, and is $\mathrm{C}(X)$-module isomorphic to $\mathcal{P}(\mathcal{A}) \otimes \mathcal{O}_\infty$. Further, the *-monomorphisms $\eta_x\colon A_x \to P(A_x)$ define a $\mathrm{C}(X)$-module monomorphism η from $\mathcal{A}$ into $\mathcal{P}(\mathcal{A})$.

---

[17] By [**471**], this is equivalent to exactness of all A_x and the continuity of the field $(A_x \otimes B)_{x\in X}$ for each separable C^*-algebra B. It follows e.g. from nuclearity of the fibers A_x and the continuity of the field.

[18] Theorem A implies a proof of the sub-triviality. The sub-triviality itself is a special case of Theorem K below.

We show in Chapter 11 a more general result:
The monomorphism η *defines a* $\mathrm{KK}(X;\,\cdot,\cdot)$*-equivalence between* $\mathcal{A}$ *and* $\mathcal{P}(\mathcal{A})$. ([19]).

At the end of Chapter 11 we outline a different construction that leads to $\mathrm{KK}(X;\,\cdot,\cdot)$-equivalent exact (respectively nuclear) *stable* algebras $P(A)$ with primitive ideal space isomorphic to a T_0 space X that acts monotone-continuously on A (cf. the definitions below and Theorems 11.4.1 and O):
One starts with a suitable "universal" Hilbert bimodule and builds the corresponding Cuntz-Pimsner algebra. The universality of the construction has the advantage that G-actions of locally compact groups G on $(A_x)_{x\in X}$ lead in a natural way to G-actions on $(P(A_x)\otimes\mathbb{K})_{x\in X}$ ([20]).

Furthermore, we modify in Chapter 11 the above described (non-universal !) construction of $A \Rightarrow P(A)$ in a way that we get (cf. Theorem 11.0.7):
Suppose that A *is a unital separable nuclear* C^**-algebra,* G *is a discrete countable* exact *group, and that* $\alpha\colon G\to\mathrm{Aut}(A)$ *is group-morphism. Then there exist a pi-sun algebra* B*, a group-morphism* $\beta\colon G\to\mathrm{Aut}(B)$ *and a* G*-equivariant* unital *monomorphism* $\eta\colon A\to B$ *that defines a* KK^G*-equivalence* $[\eta]\in\mathrm{KK}^G(A,B)$.

The latter construction plays nicely together with permutation actions of G on multiple tensor products.

`Next is wrong! ??`

$\mathcal{P}_\infty$ is isomorphic to the crossed product $({\mathcal{O}_\infty}^{\mathrm{st}}\otimes{\mathcal{O}_\infty}^{\mathrm{st}})\rtimes\mathbb{Z}_2$ *by the flip automorphism* $a\otimes b\mapsto b\otimes a$ *on* ${\mathcal{O}_\infty}^{\mathrm{st}}\otimes{\mathcal{O}_\infty}^{\mathrm{st}}$.

Here $\mathcal{P}_\infty$ denotes the unique pi-sun algebra in the UCT-class with $\mathrm{K}_0(\mathcal{P}_\infty)=\{0\}$ and $\mathrm{K}_1(\mathcal{P}_\infty)=\mathbb{Z}$.

(But it is likely that the $\mathbb{Z}_2$ crossed product by the flip on $\mathcal{P}_\infty\otimes\mathcal{P}_\infty$ is isomorphic to ${\mathcal{O}_\infty}^{\mathrm{st}}$.)

The flip automorphism is approximately inner on ${\mathcal{O}_\infty}^{\mathrm{st}}\otimes{\mathcal{O}_\infty}^{\mathrm{st}}$, because the flip on $\mathcal{O}_\infty\otimes\mathcal{O}_\infty$ is approximately inner:
(U_n) a sequence of unitaries that approximately implement the flip.

Then $[U_n,(1-s_1s_1^*)\otimes(1-s_1s_1^*)]\to 0$ in $\mathcal{O}_\infty\otimes\mathcal{O}_\infty$. Small perturbations of the U_n produces unitaries V_n that commute with $(1-s_1s_1^*)\otimes(1-s_1s_1^*)$.

The $W_n := V_n\cdot((1-s_1s_1^*)\otimes(1-s_1s_1^*))$ are unitaries in ${\mathcal{O}_\infty}^{\mathrm{st}}\otimes{\mathcal{O}_\infty}^{\mathrm{st}}$ that approximately implement the flip on ${\mathcal{O}_\infty}^{\mathrm{st}}\otimes{\mathcal{O}_\infty}^{\mathrm{st}}$.

Let us now consider an other *application of Theorem I*:
If A is a separable *nuclear* C^*-algebra, then there is a pi-sun algebra $B=P(A)$ which is KK-equivalent to A and contains a unital copy of $\mathcal{O}_2$.

---

[19] The bi-functor $\mathrm{KK}(X;\mathcal{A},\mathcal{B})$ is, for compact metric spaces X, the same as the Kasparov functor $\mathcal{R}\,\mathrm{KK}^G(X;\,\cdot,\cdot)$, with $G=$ the trivial group.

[20] At least in the case where the fibers A_x are nuclear, because then, by Theorem M(iii), there are $\mathrm{C}(X)$-module isomorphisms of $\mathcal{P}(\mathcal{A})\otimes\mathbb{K}$ onto any other p.i. stable separable nuclear $\mathrm{C}(X)$-algebra $\mathcal{B}$ that is $\mathrm{KK}(X;\,\cdot,\cdot)$-equivalent to $\mathcal{A}$ and has simple fibers B_x.

In particular $\mathrm{K}_*(B) \cong \mathrm{K}_*(A)$. We can find a separable *commutative* C^*-algebra D with $\mathrm{K}_*(D) \cong \mathrm{K}_*(A)$, cf. [**73**, cor. 23.10.3], [**715**] or Lemma B.11.1.

We replace D by a KK-equivalent pi-sun algebra E, which contains a unital copy of $\mathcal{O}_2$. Then E satisfies the UCT and there is an isomorphism λ from $\mathrm{K}_*(E)$ onto $\mathrm{K}_*(B)$. By the UCT for E, there exists $z \in \mathrm{KK}(E, B)$ with $\gamma(z) = \lambda$, and by Corollary C there exists a unital monomorphism φ from E into B such that $z = [\varphi - 0]$, and thus $\lambda = \mathrm{K}_*(\varphi)$. But then

$$\lambda \colon \mathrm{K}_*(E) \to \mathrm{K}_{(*-1) mod 2}(SB) = \mathrm{K}_*(B)$$

is the connecting map for the six term exact sequence for the mapping cone C_φ and therefore $\mathrm{K}_*(C_\varphi) = 0$. C_φ is nuclear because C_φ is an extension of nuclear C^*-algebras.

If $\mathrm{KK}(C_\varphi, C_\varphi) = 0$, then $\mathrm{KK}(C_\varphi, F) = 0$ for every (σ-unital) C^*-algebra F. By the six term exact sequence for KK one gets that B and E are KK-equivalent, i.e., that A is KK-equivalent to a commutative C^*-algebra.

Hence, *if the nuclear separable C^*-algebra A is* not KK*-equivalent to a commutative C^*-algebra, then* $\mathrm{KK}(C_\varphi, C_\varphi)$ *is not trivial, and the pi-sun algebra $N := P(C_\varphi)$ satisfies* $\mathrm{K}_*(N) = 0$ *but* $\mathrm{KK}(N, N)$ *is non-zero.*

We can go a bit further to a more special class of algebras that is sufficient to decide the UCT-problem by following observations ([21]):
There is a circle action on $\mathcal{O}_2$ with simple and purely infinite fix-point algebra B_0 in the UCT-class with $\mathrm{K}_0(B_0) = \mathbb{Z}^2$, $\mathrm{K}_1(B_0) = 0$ *and* $0 = [1] \in \mathrm{K}_0(B_0)$.

Indeed: Take $B_0 := P(\mathbb{C} \oplus \mathbb{C})$ in Theorem I. It is a pi-sun algebra that is KK-equivalent to $\mathbb{C} \oplus \mathbb{C}$ and contains a copy of $\mathcal{O}_2$ unitally. By Corollary F(ii), $B_0 \cong B_0 \otimes \mathcal{O}_\infty$ by an isomorphism that is unitarily homotopic to $b \mapsto b \otimes 1$, and, – by Corollary H(ii) –, there is an *automorphism* ψ of B_0 such that

(i) $\psi_0(m, n) = (n, n + m)$ for $(n, m) \in \mathbb{Z}^2 \cong \mathrm{K}_0(B_0)$ (hence $\mathrm{id} - \psi_0$ has determinant $= -1$),
(ii) ψ^n is strictly outer (in the sense of [**578**]) for every $n \neq 0$, because $\psi_0^n \neq 0$ for all $n \neq 0$.
(iii) $B_0 \rtimes_\psi \mathbb{Z} \cong (B_0 \rtimes_\psi \mathbb{Z}) \otimes \mathcal{O}_\infty$

In part (iii) we could also use that $B_0 \cong B_0 \otimes \mathcal{O}_\infty$ and then change ψ to $\mathrm{id}_{\mathcal{O}_\infty} \otimes \psi$ to omit the proof of pure infiniteness for $B_0 \rtimes_\psi \mathbb{Z}$. One can see, e.g. by [**578**] or by Proposition 2.18.1, that $C := B_0 \rtimes_\psi \mathbb{Z}$ is *simple*, because B_0 is simple and ψ satisfies (ii). It is known that C is nuclear, because B_0 is nuclear and $\mathbb{Z}$ is amenable, and that C is in the UCT-class because B_0 is in the UCT-class (cf. [**73**, 22.3.5g]). It follows that C is a pi-sun algebra in the UCT-class with $\mathrm{K}_*(C) = 0$, by Corollary F(ii) and by the Pimsner-Voiculescu sequence [**73**, thm.10.2.1]. Then B_0 is the fixpoint algebra of the (dual) $\mathbb{R}/\mathbb{Z}$-action on C, and $C \cong \mathcal{O}_2$ by Corollary H(ii).

---

[21] Build the tensor product of the example [**73**, 23.15.12] of Blackadar with $\mathcal{O}_\infty$ to get a version of N.

Now we consider any separable nuclear C^*-algebra D and let $B_0 := P(\mathbb{C} \oplus \mathbb{C})$ and $A := P(D) \otimes B_0$. The natural embedding η of $D \oplus D \cong D \otimes (\mathbb{C} \oplus \mathbb{C})$ into $P(D) \otimes P(\mathbb{C} \oplus \mathbb{C})$ defines a KK-equivalence $\eta \colon (D \oplus D) \to A$, because KK-equivalences are compatible with tensor products if one of the factors are nuclear, cf. [**389**, lem. 2.1.27]. The algebra A is the fixpoint algebra of a circle action on $P(D) \otimes C \cong \mathcal{O}_2$, by Corollary F(iii).

Together with Corollary F(iv) and [**73**, cor.23.10.8] the above reductions (to the case of pi-sun A with $\mathrm{K}_*(A) = 0$ and to the fix-point algebra of a circle action) on $\mathcal{O}_2$ prove the following.

COROLLARY J. *For every separable nuclear C^*-algebra D there is a pi-sun algebra A with $[1_A] = 0$ in $\mathrm{K}_0(A)$ and a *-monomorphism $\eta \colon D \oplus D \hookrightarrow A$ such that*

(i) *$[\eta]$ is a KK-equivalence of $D \oplus D$ and A.*
(ii) *A is a fix-point algebra of a circle action on $\mathcal{O}_2$.*

In particular, every separable nuclear C^*-algebra is KK-equivalent to a commutative C^*-algebra, if and only if, every pi-sun algebra A with $\mathrm{K}_*(A) = 0$, that is isomorphic to a fixpoint algebra of a circle action on $\mathcal{O}_2$, is also isomorphic to $\mathcal{O}_2$ itself.

Thus, Questions (Q2) and (Q2*) have the same answer, and we can limit (Q2) to the case where A is the fix-point algebra of a circle action on $\mathcal{O}_2$.

`revise/check next text blue/red text`

One can even step a bit further: If we consider $P(A^{op} \oplus A) \otimes B_0$ in place of A, then we can see by the above arguments that the possible existence of a *separable and nuclear* example B that does *not* satisfy the UCT implies the existence of pi-sun A with $\mathrm{K}_*(A) = 0$ that is anti-isomorphic to itself, that does *not* satisfy the UCT, and A is the fix-point algebra of a circle action on $\mathcal{O}_2$. The reason is that Thus, the non-existence of an anti-automorphism does not necessarily produce a counterexample for the UCT in the class of amenable separable C^*-algebras.

If one considers the class of arbitrary (not necessary simple) separable nuclear stable C^*-algebras that have trivial KK-theory (e.g. contractible algebras) then every separable nuclear C^*-algebra is KK-equivalent to a fixed point algebra of a $\mathbb{Z}_2$-action of one of them. But one has to consider also *inner* actions, e.g. given by conjugation by a symmetry in the multiplier algebra.

On the other hand one can say: If the fix-point algebra of the flip automorphism $\Pi(a \otimes b) := b \otimes a$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ (respectively on ${\mathcal{O}_\infty}^{st} \otimes {\mathcal{O}_\infty}^{st}$) is isomorphic to $\mathcal{O}_2$ (respectively to ${\mathcal{O}_\infty}^{st}$) then the Cuntz standard form A^{st} is isomorphic to the fixed point-algebra of a $\mathbb{Z}_2$-action on $\mathcal{O}_2$.

The reason is, that this conjectures together imply the conjecture that the fix-point algebra of of the flip on $\mathcal{P}_\infty \otimes \mathcal{P}_\infty \cong {\mathcal{O}_\infty}^{st}$ is isomorphic to $\mathcal{O}_2$. The flip is an order-2 isomorphism of ${\mathcal{O}_\infty}^{st}$ that changes the signs of elements in $\mathbb{Z} \cong \mathrm{K}(\mathcal{O}_\infty)$.

Similar arguments show that the Künneth Theorem on tensor products holds for all separable nuclear C^*-algebras, if and only if, $\mathrm{K}_*(A \otimes A) = 0$ for all pi-sun

A with $\mathrm{K}_*(A) = 0$ that is the fix-point algebra of a circle action on $\mathcal{O}_2$ and is anti-isomorphic to itself ([22]).

If one has such a C^*-algebra A, then one can consider $C := P(A \otimes \mathcal{O}_\infty)$ and ask if ?????? $C := \mathcal{O}_\infty{}^{\mathrm{st}}$.

In particular, let $D := pCp$ for a projection $p \in C$ with $1 = [p] \in \mathrm{K}_0(C) \cong \mathbb{Z}$. Is D tensorial self-absorbing ?

Here we say that a unital separable D is a **tensorial self-absorbing** algebra if $D \ncong \mathbb{C}$ and the unital monomorphism $D \ni d \mapsto d \otimes 1 \in D \otimes D$ is approximately unitarily equivalent to an isomorphism from D onto $D \otimes D$ (see [**448**, sec.4.def.4.1]).

We mention this, because it shows that the UCT-problem and related problems have connections with the question of the classification of the tensorial self-absorbing algebras A – without assuming before that A is in the UCT-class !

A special case of the UCT problem is the following interesting and long standing open question:
Let A a separable unital tensorial self-absorbing algebra that has the property that M_n is not unitally contained in A for every $n > 1$. Is $A \cong \mathcal{O}_\infty$ or $A \cong \mathcal{Z}$? Here $\mathcal{Z}$ denotes the Jiang-Su algebra. The answer is positive in the UCT-class. But so far one has no idea what happens without UCT.

Is M_n also not unitally (!) contained in $A \otimes \mathcal{O}_\infty$? There exist no projection in $A \otimes \mathcal{O}_\infty$ with $n[p] = [1]$ in $\mathrm{K}_0(A \otimes \mathcal{O}_\infty) \cong \mathrm{K}_0(A)$ ...

2. Towards a classification of non-simple purely infinite algebras

In the period from 1995-2015 the author has studied methods that allow to generalize the classification of pi-sun algebras to a classification of non-simple nuclear strongly purely infinite stable separable C^*-algebras.

`Compare the 2 below given blue discussions of [442, lem.3.8]`

A report on results and outline of the proofs has been published in [**442**]. But the important "abstract" Lemma [**442**, lem.3.8] was not proved there and the author has to regret for his claim "It is a straight-forward proof" (Es ist ein Gradeaus-Beweis). Indeed, the author had overseen that the list of conditions in the technical Lemma [**442**, lem. 3.8] is incomplete:
The surjectivity of the there considered natural group morphism of in Chapter 4 defined groups $G(h_0\,;\; A, E)$ to $G(H_0\,;\; A, E)$ is easy to derive from the listed properties, but the there proposed injectivity is unclear. Therefore we have revised the Lemma [**442**, lem. 3.8] in a very detailed technical Chapter 4 and have reformulated it as Theorem 4.4.6. It is in a sense a general tool for comparing sorts of *unsuspended but stable* E-theories with naturally corresponding KK-theories. This Theorem lists additional conditions, each of them equivalent to the injectivity of

[22] It suffices to check $\mathrm{K}_0(A \otimes A) = 0$, because one can replace A by $P(A \oplus (A \otimes \mathcal{P}_\infty)) \otimes B_0$.

the canonical epimorphism from $G(h_0\,;\; A, E)$ onto $G(H_0\,;\; A, E)$. An explicit (certainly very abstract !) example where the natural map is not injective only under the assumptions of the old Lemma [**442**, lem. 3.8] has not been found so far – but still could exist for some example that fits the general "axiomatic" assumptions of [**442**, lem. 3.8] ([23]).

The Lemma [**442**, lem.3.8] is here worked out in more detail and extended to Theorem 4.4.6 by adding a condition (DC), and we give a detailed proof of it in Chapter 4. But we have used/misused Chapter 4 to give also a very detailed account of sorts of calculations in K-theory of properly infinite unital C^*-algebras and of $\mathrm{K}_*(h_0(A)' \cap E)$ for "KK-theory-defining" morphism $h_0 \colon A \to E$, where $h_0(A)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally as our minimal permanent assumption. It leads to groups $G(h_0\,;\; A, E)$ that must be understood in detail by the reader before he can understand the proofs in several other parts of this book entirely, e.g. the groups $\mathrm{Ext}(\mathcal{C}\,;\; A, B)$ in Chapter 5.

Fortunately we can generalize Kasparov's proof (and the way over the cobordism variant defined by J. Cuntz and G. Skandalis) of the identities $\mathrm{KK}_p(A, B) = \mathrm{KK}_c(A, B) = \mathrm{KK}_h(A, B)$, i.e., the proof of the homotopy invariance of KK-theory, to get that our cone-depending groups $\mathrm{KK}(\mathcal{C};\, A, B)$ are homotopy invariant. This $\mathcal{C}$-related result will be used to prove general decomposition theorems for strictly continuous paths of "deriving" unitaries, compare Section 5 in Chapter 8 for details. This decomposition results allow to proof that $\mathrm{Ext}(S\mathcal{C}\,;\; A, SB)$ – which is naturally isomorphic to $\mathrm{KK}(\mathcal{C}\,;\; A, B)$ – satisfies the assumptions of Theorem 4.4.6 in Chapter 4, in particular the – long time disputed – decomposition property (DC). It implies that the natural group *epimorphism* from the m.o.c. cone $\mathcal{C}$-equivariant unsuspended E-theory – the generalized Rørdam groups $\mathrm{R}(\mathcal{C};\, A, B)$ – onto the group $\mathrm{KK}(\mathcal{C};\, A, B)$ is an *isomorphism*. This is our base for isomorphism results, including the case of pi-sun A, B and $\mathcal{C} = \mathrm{CP}(A, B)$.

Has next blue part to be changed? Technics works?
Phillips method works?
Relation to KK-theory clear?

Independently from this result that completes of [**442**, lem. 3.8] we have in Section 3 of Chapter 9 also outlined a generalization of the arguments of N. Ch. Phillips in a way that they allow to show the homotopy invariance of m.o.c. cone $\mathcal{C}$-related unsuspended stable E theories – like our continuous variants $\mathrm{R}(\mathcal{C};\, A, B)$ of a kind groups that M. Rørdam and G. Elliott used in their proofs for the classification of certain Cuntz-Krieger algebras.

It is based on the homotopy invariance of the *Grothendieck group* of those kinds of semigroups – but that are not homotopy invariant as semigroups itself (sic !) – defined as unsuspended, but stable, variants of E-theory that satisfy asymptotically some additional relations, who's proof is essentially inspired by ideas

[23] A specific counter-example would be interesting for finding the *minimal necessary* assumptions for the axioms that make the surjective map from $G(h_0\,;\; A, E)$ onto $G(H_0\,;\; A, E)$ injective.

of N.Ch. Phillips in [**627**]. But we do not know if this can be used *to prove* in this special case the injectivity of $G(h_0\,;\ A, E) \to G(H_0\,;\ A, E)$, i.e., if two asymptotic morphisms $h_1 \colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ and $h_2 \colon A \to \mathrm{Q}(R_+, B)$ are "stably homotopic" if h_1 and h_2 define the same element of $\mathrm{Ext}(S\mathcal{C}\,;\ A, SB)$ – with $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ generated by h_0.

`Above and below should be solved now at the end of Chapter 8 !!!`

It is also not clear how to produce an asymptotic homotopy between $k \oplus h_0$ and h_0 if there is strictly continuous path of unitaries $t \in [0, \infty) \mapsto U(t) \in \mathcal{M}(B)$ with $U(0) = 1$ and

$$t \in [0, \infty) \to U(t)^*(k \oplus H_0(a))U(t) - H_0(a) \in B\,,$$

(where $H_0 := \delta_\infty \circ h_0 = h_0 \oplus h_0 \oplus \cdots$) and

$$\lim_{t \to \infty} \| U(t)^*(k \oplus H_0(a))U(t) - H_0(a) \| = 0\,.$$

But please without (sic !) using the decomposition argument (DC) that we prove in Section 5 of Chapter 8, because this proves anyway the full Theorem 4.4.6 in Chapter 4, and factorize through the whole – most abstract – study of $\mathcal{C}$-related KK-theory in Chapter 8 to get the general (unrestricted) homotopy invariance of all this functors. It is a generalization of non-trivial observation of G. Kasparov (plus arguments of J. Cuntz and G. Skandalis).

But above needs the still not proven facts that $\mathrm{KK}(\mathcal{C}\,;\ A, B)$ is homotopy invariant with respect to B, and that it is isomorphic to the kernel of

$$\mathrm{K}_1\big(\pi_B(H_{\mathcal{C}}(A))' \cap \mathrm{Q}^{\mathrm{s}}(B)\big) \ \to \mathrm{K}_1\big(\mathrm{Q}^{\mathrm{s}}(B)\big)\,.$$

If one wants to conclude as in [**442**] then one needs as an additional information, e.g. the homotopy invariance of the Rørdam groups – a very special case of unsuspended but stable ideal-equivariant $\mathcal{E}$-theory introduced and studied in Chapters 7 and 9.

`But need also that the kernel of R-groups to KK`
`is` $\mathcal{E}$`-homotopy equivalent of` h_0`.  ?????`

But there is then needed an extra condition, ????? related to Kasparov's arguments ????, for the homotopy invariance of the groups $\mathrm{KK}(\mathcal{C};\,\cdot,\cdot)$ and $\mathrm{Ext}(\mathcal{C};\,\cdot,\cdot)$.

But finally the classification is given by a generalized Kasparov theory. In particular, $A \otimes \mathcal{O}_2 \cong B \otimes \mathcal{O}_2$ if A and B are separable stable nuclear C^*-algebras and have topologically isomorphic primitive ideal spaces.

Certainly, there is the still open question about the *explicit calculation* of our "generalized KK-invariants" given by

$$\mathrm{KK}_{\mathrm{nuc}}(X;\,\cdot,\cdot) := \mathrm{KK}(\mathcal{C}_X;\,\cdot,\cdot)$$

for the m.o.c. cone $\mathcal{C}_X := \mathrm{CP}^X_{nuc}$ of the X-equivariant residually nuclear c.p. maps, i.e., nuclear maps that are residually nuclear with respect to a given "action" of X. Notice that this is the open Question **(Q2)** in the special case where X is a point. But one can define generalizations of the UCT-class, e.g. with help of the "tautological" epimorphism and the 6-term exact sequence, cf. the proof of Theorem O in Chapter 12 for some simple example. Again, the non-trivial question is, which nuclear C^*-algebras are in this class.

Compare Theorem O for an example of some related difficulties in question.

Now for applications of the classification the most natural question is:

When does a separable nuclear C^*-algebra A absorb $\mathcal{O}_\infty$ tensorial, i.e., when $A \otimes \mathcal{O}_\infty \cong A$? (If such an isomorphism exist then there exists an other that is moreover an ideal-system preserving *-isomorphism.)

There are several necessary and sufficient conditions for A absorbing $\mathcal{O}_\infty$. We show – finally at the end of Chapter 10 and independently from the classification results and only from general properties of *strong pure infiniteness* and from the observation that $\mathcal{O}_\infty \cong \mathcal{D}_\infty := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$ by an isomorphism that is unitarily homotopic to the embedding $a \in \mathcal{O}_\infty \mapsto a \otimes 1 \otimes \cdots \in \mathcal{D}_\infty$ –, that *a separable nuclear A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if A is "strongly" purely infinite in the sense of the below given Definition 1.2.2* ([24]). The following definition is essentially the same as [**462**, def. 4.1].

Definition 1.2.1. We call a C^*-algebra A **purely infinite** if A has following properties (i) and (ii):

(i) A has no character, i.e., every irreducible representation of A has dimension > 1, and

(ii) For every $a \in A_+$ and every positive c in the closed ideal generated by a, and every $\varepsilon > 0$, there exists an element $d := d(a, c, \varepsilon) \in A$ such that

$$\| c - d^* a d \| < \varepsilon . \tag{2.1}$$

([**462**, def. 2.3] is a Defination of MvN-equivalence, with reference to G.K. Pedersen [**621**] that it is an equivalence relation on A_+.)

A C^*-algebra A is p.i. if and only if A has the – formally stronger – property that every nonzero $c \in A_+$ is properly infinite, cf. [**462**, lem. 4.2], or Corollary 2.5.6.

`To be checked:` or more directly:

because an element $a \in A$ is properly infinite in A, if and only if, $\pi_J(a)$ is infinite in A/J for *every* closed ideal $J \lhd A$ with $a \notin J$, cf. Part (v) of Lemma 2.5.3, and since properties (i) and (ii) of the definition pass to $\pi_J(a)$ for all quotients A/J of A with closed ideals $J \not\ni a$.

Implies ????

Definition 1.2.1 has not much to do with that for "pi-1" or "pi(1)" because Def. 1.2.1 includes simplicity ????

[24] In the case of stable or unital A we get this also from Theorem M in the same way as we can deduce $A \cong A \otimes \mathcal{O}_\infty$ from Theorem B in the case of pi-sun algebras A.

NO! So as it is now, it is exactly the same as pi(1) (up to that it is not required that $\ell_\infty(A)$ has no character!! But this can be shown in this case, because pi(1) and pi-1 are equivalent properties.

It is still an open question – also for *separable nuclear* non-simple C^*-algebras A – if purely infinite C^*-algebras are "strongly" purely infinite in sense of the following Definition 1.2.2. Our ideal-system equivariant $\mathrm{KK}(X;\cdot,\cdot)$-classification works only for separable nuclear C^*-algebras that are "strongly" purely infinite C^*-algebras in the sense of the following Definition 1.2.2.

DEFINITION 1.2.2. The non-zero C^*-algebra A is **strongly purely infinite** if A satisfies the following condition:
For every $a, b \in A_+$ and $\varepsilon > 0$ there exist elements $d := d(\varepsilon, a, b), e := e(\varepsilon, a, b) \in A$ such that

$$\| a^2 - d^* a^2 d \| < \varepsilon , \quad \|b^2 - e^* b^2 e\| < \varepsilon \ \text{ and } \ \|d^* abe\| < \varepsilon . \tag{2.2}$$

(We abridge 'purely infinite' by 'p.i.' and 'strongly purely infinite' by 's.p.i.')

It is easy to see that every non-zero *strongly* p.i. algebra is a purely infinite C^*-algebras in sense of Definition 1.2.1, indeed: Let $a := b := c^{1/2}$ in Definition 1.2.2 for $c \in A_+$, then Definition 1.2.2 shows that each non-zero $c \in A_+$ is a properly infinite element of A in the sense of [**462**, def. 3.2], cf. also Section 1 of Chapter 2.

If we use the isomorphism of C and $C \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cdots$, then we can see that $A := B \otimes C$ is strongly p.i. for every (simple) pi-sun algebra C and every (non-zero) C^*-algebra B. We show moreover that $A := B \otimes \mathcal{Z}$ is s.p.i. if A (2-quasi-) traceless and $\mathcal{Z}$ denotes the Jiang-Su algebra, cf. also [**690**] in case of exact B.

If a C^*-algebra A has the WvN-property of Definition 1.2.3, then A has the property that for every self-adjoint $h^* = h \in \mathcal{M}(A \otimes \mathbb{K})_\omega$ there is a *-monomorphism $\lambda_h \colon C^*(h) \otimes \mathcal{O}_\infty \to \mathcal{M}(A \otimes \mathbb{K})_\omega$ with $\lambda(h \otimes 1) = h$. The latter implies that $\mathcal{M}(A)$ is purely infinite. (It is still an open question if the pure infiniteness of $\mathcal{M}(A)$ implies that A is strongly p.i., except some special cases as
e.g. ??? Even in case that ???? .)

M. Rørdam and the author have shown in [**463**] that strongly purely infinite C^*-algebras have the WvN-property, see Remarks 2.15.12 and 3.11.4 in Chapters 2 and 3 for an outline of the proof (including correction of typos in [**463**]).

It is still an *open question*, whether or not p.i. (or at least the existence of the above described *-monomorphisms λ_h in sufficiently general position) implies "strongly" p.i. in general.

By [**93**, thm.4.17], a *sufficient* condition for A being strongly p.i. is that every non-zero quotient A/J of A is p.i. in the sense of the definition of J. Cuntz [**172**, p. 186] (cf. Introduction to Chapter 2 and compare also [**169**, thm. 1.13, thm. 1.4] in the special cases of $A = \mathcal{O}_n$ for $n = 2, \ldots, \infty$), i.e., for every closed ideal $J \subseteq A$ holds that every non-zero hereditary C^*-subalgebra D of A/J contains a (non-zero) properly infinite projection $p \in D$.

In particular, if A has real rank zero, then A is strongly p.i., if and only if, each projection of A is infinite.

The algebra $\mathrm{C}_0((0,1],\mathcal{O}_2)$ is strongly p.i. but is not purely infinite in the sense of J. Cuntz. The unitization of $\mathcal{O}_2 \otimes \mathbb{K}$ has real rank zero and is purely infinite in the sense of J. Cuntz but is not p.i. in the sense of our Definition.

We list the needed results on purely infinite and strongly purely infinite non-simple C^*-algebras in Chapters 2 and 3 and outline the ideas for their proofs. This results are mainly joint work with M. Rørdam and E. Blanchard and we refer sometimes for details of the proofs to [**462**], [**463**] and [**93**].

`Begin of discussion of ''WvN-property'':`
`1) Has to be revised at several places.`

`2) Sort mentioned results for chapters, and formulate them there.`

`3) Where are Defs. of: ''residually nuclear maps'' ?.`
`In case of` $A \subseteq B$ `and` $V\colon A \to B$ `residually nuclear c.p. contraction ?.`
V `is ''1-step approximately inner'' c.f. Definition 3.10.1? Not a good place!`

We prefer to work in this book with the *WvN-property* of the following Definition 1.2.3 and, sometimes, even with additional assumptions as listed in Remark 3.11.1, because the WvN-property is, in a conceptional sense, a starting point for most of the proofs, and the strong pure infiniteness is a basic property for all of our constructions and its applications. It turns out in Chapter 12 as a result of ideal-system equivariant $\mathrm{KK}(X;\cdot,\cdot)$-classification that this two properties are equivalent.

DEFINITION 1.2.3. A C^*-algebra B has the **Weyl–von-Neumann property** – in short *WvN-property* – if for every σ-unital hereditary C^*-subalgebra $D \subseteq B$ and every C^*-subalgebra C of the multiplier algebra $\mathcal{M}(D\otimes\mathbb{K})$ of $D\otimes\mathbb{K}$, every *residually nuclear* completely positive contraction $V\colon C \to D \otimes \mathbb{K}$ can be approximated in point-norm by maps $C \ni c \mapsto d^*cd$ with contractions $d \in D \otimes \mathbb{K}$ ([25]).

We say that a C^*-algebra B has **residually nuclear separation** if, for every separable C^*-subalgebra $C \subseteq B$, every $a \in C_+$ and every $\varepsilon > 0$, there exists a residually nuclear contraction $V\colon C \to B$ such that $\|V(a) - a\| < \varepsilon$.

Notice that we can modify the residually nuclear contraction $V\colon C \to B$ in Definition 1.2.3 that it has in addition the property $V(C) \subseteq CBC$.

`Next should be part of a Proposition/ Theorem -- to be cited`

The WvN-property implies that B is purely infinite in the sense of Definition 1.2.1, that equivalently says that all elements of B_+ are properly infinite in B, i.e.,

[25] It implies that the c.p.c. V is "1-step approximately inner" in the sense of Definition 3.10.1.

B satisfies the equivalent properties **pi**(1) of Definition 2.0.4 and pi-1 of Definition **??**, cf. also [**462**, def. 3.2, def. 4.1, thm. 4.16].

`Give ref.s/not.cite for def.s and proofs`

`Shift details to other places?!!`

Moreover the WvN-property implies that for every separable C^*-subalgebra $D \subseteq B$ with the property that the inclusion map $\iota_D \colon d \to d$ is nuclear (which implies automatically that D is exact), there is a C^*-morphism $h\colon (D\otimes\mathbb{K})\otimes\mathcal{O}_\infty \to (B\otimes\mathbb{K})_\omega$ such that $h((d\otimes k)\otimes 1) = d\otimes k$ for $d \in D$, $k \in \mathbb{K}$ under the natural inclusion $B\otimes\mathbb{K} \subset (B\otimes\mathbb{K})_\omega$ of $B\otimes\mathbb{K}$ into the ultrapower $(B\otimes\mathbb{K})_\omega$. ([26]).

Indeed, let $C := D\otimes\mathbb{K}$, $s_1, s_2 \in \mathcal{M}(D\otimes\mathbb{K})$ isometries with $s_j^* s_k = \delta_{j,k}1$ the map $c \mapsto s_1 c s_1^* + s_2 c s_2^*$ is residually nuclear, thus is 1-step approximately inner in $B\otimes\mathbb{K}$ if B has the WvN-property.

And it shows that each element of $(B\otimes\mathbb{K})_+$ is properly infinite if B has the WvN-property.

`It seems (! open question?) that the WvN property also implies that` $\pi_B(D)' \cap \mathcal{M}(B)/B$ `contains a unital copy of` $\mathcal{O}_\infty$.
`Try to factorize the cone of` D `over` $\ell_\infty(B\otimes\mathbb{K})/c_0(B\otimes\mathbb{K})$ `???`

One can show ([27]) that a C^*-algebra B is strongly p.i. if B has both of the WvN-property and the residually nuclear separation property of Definition 1.2.3.

It is obvious that the WvN-property passes to hereditary C^*-subalgebras and is invariant under stabilization, hence is an invariant of Morita equivalence. The definition of the WvN-property says only something that one can limit the algebras C in the definition of the WvN-property to the case of separable C.

Moreover it is not difficult to see

`BY DEFINITION ??`

that B has the WvN-property if B_ω has the WvN-property.

Clearly the identity map $\mathrm{id}_B =: V$ on nuclear C^*-algebras B defines residually nuclear separation for each separable C^*-subalgebra C of B. Thus, for nuclear C^*-algebras B the WvN-property and strong pure infiniteness are the same.

Studies of primitive ideal spaces of nuclear C^*-algebras in combination with [**359**] and [**456**] lead to the observation that all stable separable C^*-algebras A have the (formally) stronger property of Abelian factorization, which obviously implies residually nuclear separation:

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[26] We define the ultrapower A_ω of A by $A_\omega := \ell_\infty(A)/c_\omega(A)$ where $\omega \in \gamma(\mathbb{N}) := \beta(\mathbb{N}) \setminus \mathbb{N}$ is a point in the non-elementary part of the Stone-Čech compactification of $\mathbb{N}$.
Other authors use the notation A^ω for this algebra and then A_ω for the central sequence algebra $A' \cap A^\omega$.

[27] E.g. by ideas in the proof of [**463**, cor. 7.22] – that contains unfortunately some typos, see Chapters 2/3?? for corrected versions.

DEFINITION 1.2.4. A stable separable C^*-algebra A has the **Abelian factorization property** if there exists a locally compact Polish space P and non-degenerate *-monomorphisms $h\colon \mathrm{C}_0(P,\mathbb{K}) \to \mathcal{M}(A)$ and $k\colon A \to \mathcal{M}(\mathrm{C}_0(P,\mathbb{K}))$ such that the "C^*-correspondence" $\gamma := \mathcal{M}(h)\circ k\colon A \to \mathcal{M}(A)$ (– a special kind of Hilbert A-module with a left-sided action by A –) has the property

$$J = \gamma^{-1}\mathcal{M}(A,J) \quad \text{for all} \quad J \in \mathcal{I}(A)\,.$$

All simple stable C^*-algebras A have Abelian factorization by the trivial reason that one can take here for $P := \{p\}$ a point, i.e., $\mathrm{C}_0(P,\mathbb{K}) = \mathbb{K}$ and use that $A\otimes\mathbb{K} \cong A$ if A is stable, $h(T) := 1\otimes T \in \mathcal{M}(A\otimes\mathbb{K}) \cong \mathcal{M}(A)$ and let $k\colon A \to \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$ any non-degenerate *-representation of A on ℓ_2.

It turns out that, if A has the Abelian factorization property, then the l.c. Polish space P and h can be chosen always such that P is 1-dimensional ([28]).

Therefore strong pure infiniteness and the WvN-property (?????? ref?) are now known to be equivalent for all C^*-algebras, because both have to be checked only on suitable sufficiently large separable C^*-subalgebras with one of this properties.

A proof of the Abelian factorization (see **??**) Definition in all generality uses a detailed study of the topology of second countable locally quasi-compact sober (= "point-complete") T_0 spaces (that we call "Dini spaces") and their "Dini functions", using the very last conclusions in Chapter 12. Therefore we have not included a proof here in the book. And we avoid to use Abelian factorization here directly. Instead we use methods that are still fairly elaborate but use formally weaker assumptions than the "residual factorization property" of A, (cf. Definition 1.2.3).

But in the special case of separable *exact* C^*-algebras, one can derive *residually nuclear separation*, defined in Definition 1.2.3 with help of Theorem K by the following construction:

Use Theorem K with $A\otimes\mathcal{O}_2\otimes\mathbb{K}$ in place of both A and B and with the natural "action" of $\mathrm{Prim}(A) = \mathrm{Prim}(B)$ on A for the proof of residually nuclear separation of separable exact A:

The resulting residually nuclear endomorphism h_0 has the property that its restrictions to commutative C^*-subalgebras C of A approximately dominates the inclusion map $C \hookrightarrow A$ in the sense of Definition 3.10.1. Thus, the point-norm closed matricial operator-convex cone generated by h_0 defines "residually nuclear separation" for A as defined in Definition 1.2.3. This argument works so simple only for separable exact C^*-algebras A. And, unfortunately, needs ultra-power constructions that we want not to bore the reader at the main parts of this books. So we make sometimes additional assumptions that we all replace in Chapter 12 by embeddings into suitable separable C^*-algebras in the ultrapower that have the required properties. Then we observe that this what we can do in the ultrapower

[28] But P is not something like a one-dimensional polyhedron. It is only some projective limit of such polyhedra.

"in one step" can be done approximately in the given algebra itself. Such a "fitting together" and approximation arguments in the earlier chapters can be applied to all studied cases in a controlled way.

Alternatively, mone can use a deeper result for separable exact C^*-algebras A: It says that $A \otimes \mathcal{O}_2$ contains a *regular* Abelian C^*-subalgebra C in the sense of Definition B.4.1. It implies that A has *Abelian separation* (in sense of: give reference here!) and follows also from Corollary 12.3.1 and applications of Theorem K and Corollary L in the remark following Corollary 12.3.1, and the corresponding result for nuclear C^*-algebras in [**464**] and [**359**]. But we should not get lazy and put non-trivial stuff simply in the assumptions: Applicability or Non-applicability depends mainly from the difficulty to observe the assumptions of some derived useful applications and conclusions (in the theorems). The study of assumptions and its interrelations, weaker properties and inducing properties is what a result makes a true invention.

Thus, we get finally that *an* exact *C^*-algebra A is strongly p.i., if and only if, A has the WvN-property.* But we can not use this simple (but not obvious) identity of two properties before the very end of Chapter 12!

But the reader should not forget that one first has to give a proof of Theorem K to get a base e.g. for the *deduction* of the permanently used and needed WvN-property, that we must derive from strong pure infiniteness to reach our applications.

We go along an alternative way as follows:

A C^*-algebra B is s.p.i. if and only if its ultra-power B_ω is s.p.i., cf. Proposition 2.16.8. A C^*-algebra B is weakly purely infinite if and only its ultra-power B_ω has "residually nuclear separation" in the sense of Definition 1.2.3 (and in a very "local" way), see also [**463**, prop. 7.13] ([29]).

`Next should be a Prop./Thm. in Chp.3 ?? Which section?`

More precisely:

If B is weakly purely infinite (i.e., B is n-pi for some $n \in \mathbb{N}$) and $A \subseteq B_\omega$ is a separable C^*-subalgebra of B_ω, $C \subseteq A$ an Abelian C^*-subalgebra of A then there exists a separable C^*-subalgebra $D \subseteq B_\omega$, an Abelian C^*-subalgebra $C_1 \subseteq D$ and a conditional expectation $E\colon D \to C_1$ such that $A \subseteq D$, $C \subseteq C_1$, and $E(D \cap J) \subseteq C_1 \cap J$ for each closed ideal of $J \lhd B_\omega$.

Moreover, one can find such a system $(D, E\colon D \to C_1 \subset D, D)$ for each given separable $A \subset B_\omega$ with this properties and with the additional property that E is an – inside D – approximately inner c.p. map. In particular, C_1 is a "regular" C^*-subalgebra of D.

---

[29] This property of B_ω is "natural" because B is *completely characterless* – i.e., no σ-unital hereditary C^*-algebra of B_ω has a non-zero character – if B weakly p.i.

Conversely, if B_ω has no irreducible representations that contain the compact operators in its image, and if in B_ω such systems $(D, E\colon D \to C_1)$ exists for each given $C \subseteq A \subset B_\omega$, then B is weakly purely infinite.

Since B is strongly p.i. if and only if B_ω is strongly p.i., it follows that B_ω has the WvN-property if B is strongly p.i.

In this way we obtain that *a C^*-algebra A is s.p.i. in the sense of Definition 1.2.2, if and only if, A_ω has the WvN-property of Definition 1.2.3.*

That the WvN-property of A_ω induces the WvN-property of A, can be shown also directly from the definition of the WvN-property. ???? Really ???? `Idea forgotten?` **??**

Thus, *every s.p.i. C^*-algebra A has the WvN-property.*

The converse direction – that the WvN-property of A implies that A is s.p.i.– is for the final proofs of theTheorems **??** and ????? important. ????

We need "nuclear separation" to establish the realizations of actions Ψ by the related cones $\mathcal{C}_\Psi$ of Ψ-equivariant residually nuclear maps.

Which proofs use "nuclear separation" and where is it defined?

Chp.2: residual nuclear separation and WvN-property imply strong p.i.

But for simple C^*-algebras there are many formally weaker – but for them equivalent – properties.

Chp.3: residually nuclear maps build the "minimal" ideal system invariant cone (the later needed property is that this cone is non-degenerate for separable stable C^*-algebras).

Chp.6: "nuclear separation" is – needed / required – to proof the "embedding theorem" (give references to def's and theorems !!!)

(Starting alone from any non-degenerate l.s.c. action of $\mathrm{Prim}(B)$ on separable exact A that is monotone upper s.c., it can be proved only fully after one has proven with help of this special case in chp.12 that the nuclear separation of the action can be proved with help of results in Chp. 7, 9 and 12).

Chp.12: // B separable and B stable, σ-unital, and $B \otimes \mathcal{O}_\infty \cong B$, show existence of separable stable C^*-subalgebras D with $B \subseteq D \subset \mathrm{Q}(\mathbb{R}_+, B)$, D stable, σ-unital, and $D \otimes \mathcal{O}_\infty \cong D$, with a regular Abelian subalgebra C (that separates the ideals of D).

Then finding for $A :=???$ c.p. maps $V\colon A \to C \subset D$ (that realize the action of $\mathrm{Prim}(B) = pi_B(\mathrm{Prim}(\mathrm{Q}(\mathbb{R}_+, B)))$ on A and with their help a nuclear embedding $k_1\colon A \otimes \mathcal{O}_2 \to D$ suitable (existing by the main embedding theorem) and realizing the action of $\mathrm{Prim}(D)$ on A. But $h_1 := k_1((\cdot) \otimes 1)$ is re-scaling invariant by Chapter 7. Thus, unitary equivalent to a "defining" nuclear embedding $A \to \mathcal{O}_2 \to B \otimes \mathcal{O}_\infty$.

In particular this implies that the given action of $\mathrm{Prim}(B)$ on A has nuclear separation.

This argument works only for separable exact C^*-algebras. Because for non-separable or non-exact C^*-algebras there does not exist an “embedding theorem”. (Into $\mathcal{O}_2$?)

The WvN-property of A implies always that A is p.i.

Since WvN-property implies p.i., it is also part (??) of the open problem if p.i. implies s.p.i.

It is unknown if p.i. implies s.p.i.

It is not unlikely that an elementary observation exists that shows that A with WvN-property is always s.p.i.

The results of this book allow to proof that for all separable stable A the natural action of $X := \mathrm{Prim}(A)$ on A – given by the natural isomorphism $\mathbb{O}X \cong \mathcal{I}(A)$ – has “residually nuclear” separation (i.e., this action is “induced” from the m.o.c. cone $\mathrm{CP}_{\mathrm{rn}}(A, A)$ of residually nuclear maps from A to A), but to carry the proof out one needs the possibly stronger – and not in this book proved – “Abelian separation” property for separable C^*-algebras.

```
 important???// This can be used for ????????
The QUESTION is/was:
Is there a simple way to derive directly
for separable *-algebras A from WvN-property of A
the STRONG pure infiniteness of A?
Or, -- equivalently -- from WvN-property of A
the WvN-property of A_ω.
```

The residual nuclear separation follows in case of separable exact C^*-algebras from Corollary L, it implies directly that every separable exact C^*-algebra A has the stronger property that $A \otimes \mathcal{O}_2$ contains a “regular” Abelian C^*-subalgebra that separates the ideals of A.

The existence of a regular Abelian C^*-subalgebra implies in particular the existence of an “Abelian factorization” for $A \otimes \mathbb{K}$, which is (formally) stronger than “residually nuclear separation”.

A more easy way to prove that the WvN-property implies strong pure infiniteness is not in sight yet.

Despite the rather involved result of existence of “Abelian factorization” it remains the more special and interesting *question*, whether $A \otimes \mathcal{O}_2$ contains a *regular* Abelian C^*-subalgebra in the sense of Definition B.4.1 *for every separable* C^*-algebra A. By Corollary 12.3.1 and [**464**, thm. 6.11] the latter is the case, if and only if, $A \otimes \mathcal{O}_2$ contains a regular exact C^*-subalgebra.

```
End discussion of ‘‘WvN-property’’. Still not well explained !
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In special cases we obtain that “p.i.” implies “strongly p.i.”, e.g., for algebras of real rank zero, for algebras with Hausdorff primitive ideal space, or for separable

C^*-algebras with linearly ordered lattice of closed ideals. In particular, all *simple* purely infinite algebras are strongly purely infinite.

Why we consider different definitions of pure infiniteness?

One can only work with the property of strong pure infiniteness, but the others are usually easier to verify. We can then try to check additional properties of the studied algebras that allow to conclude strong pure infiniteness e.g. from (1-) pure infiniteness. But this is also an open problem for separable nuclear C^*-algebras B. There are only some sufficient conditions as e.g. B has Hausdorff primitive ideal space,

(of finite dimension ?),

$\mathcal{I}(B)$ is linearly ordered ideal lattice, and to "s.p.i." equivalent properties, as e.g. $B \cong B \otimes \mathcal{O}_\infty$ if B is nuclear. *Simple and exact* strongly purely infinite separable C^*-algebras B do not absorb $\mathcal{O}_\infty$ tensorial in general, as e.g. the example of the (strongly) purely infinite algebra $B := C^*_{reg}(F_2) \otimes \mathcal{R}$ shows, where $\mathcal{R}$ is the finite but not stably finite simple nuclear C^*-algebra of M. Rørdam [**687**] and $C^*_{reg}(F_2)$ is the (exact) regular group C^*-algebra of $C^*_{reg}(F_2)$. M. Rørdam itself denoted his algebra $\mathcal{R}$ by W.

At first we give some definitions concerning non-Hausdorff spaces, which are needed for suitable generalizations of the main results for simple p.i. algebras to non-simple p.i. algebras.

DEFINITION 1.2.5. A topological T_0-space X is **sober** if *every* prime closed subset of X is the closure of a point ([30]).

Here a closed subset $F \subseteq X$ is **prime**, if F is not the union $F_1 \cap F_2$ of two closed subsets F_1, F_2 of F that are both different from F.

Since, by results of J. Dixmier

citation? See also book of Pedersen ...

for a separable C^*-algebras A, closed prime ideals of A are primitive, the primitive ideal space $\mathrm{Prim}(A)$ of a separable C^*-algebra is a "sober" T_0-space (we call it "point-complete") with a countable base of its topology. The countable base can be described by the supports of the lower semi-continuous functions $J \in \mathrm{Prim}(A) \mapsto \|\pi_J(b_{n,k})\|$, where $b_{n,k} = (a_n - 1/k)_+$ for a sequence $a_1, a_2, \ldots$ which is dense in $\{a \in A_+ : \|a\| = 1\}$ and $n, k \in \mathbb{N}$.

In 2001 N. Weaver [**815**] published an example of non-separable prime C^*-algebras that is not non-primitive. Later T. Katsura [**411**] constructed examples of non-separable AF-algebras that are prime but not non-primitive.

[30] It is also called **point-complete** or **tidy**. Sober l.c. T_0-spaces are sometimes also called *spectral space* , but some authors define spectral spaces as point-complete T_0-spaces that have a base of the topology consisting of compact open subsets.

But it seems that one needs some set theory axiom that is stronger than the Axiom of choice?
But weaker than the Continuum Hypothesis??

DEFINITION 1.2.6. Let X be a sober T_0-space that is second countable, i.e., has a topology with a countable base.

We say that X **acts** on a C^*-algebra A, if there is given a monotone map Ψ from the lattice $\mathbb{O}(X)$ of open subsets Z of X to the closed ideals $\mathcal{I}(A)$ of a C^*-algebra A with $\Psi(\emptyset) = 0$ and $\Psi(X) = A$.

The action Ψ of X on A will be called **upper semi-continuous** , if, moreover,

(i) $\Psi(Z_1 \cup Z_2) = \Psi(Z_1) + \Psi(Z_2)$ for every pair of open subsets Z_1 and Z_2 of X, and
(ii) $\Psi(\bigcup Z_n)$ is the closure of $\bigcup \Psi(Z_n)$ if $Z_1 \subseteq Z_2 \subseteq \cdots$ is an increasing sequence of open subsets of X, i.e., Ψ is monotone.

The action Ψ is called **lower semi-continuous** if (instead of (i) and (ii)):

(iii) $\Psi(Z_1 \cap Z_2) = \Psi(Z_1) \cap \Psi(Z_2)$ for every pair of open subsets Z_1 and Z_2 of X, and
(iv) $\Psi(K) = \bigcap \Psi(Z_n)$, where K is the *interior* of $\bigcap Z_n$ in X, if $Z_1 \supset Z_2 \supset \cdots$ is a decreasing sequence of open subsets of X.

If the monotone map Ψ satisfies (i)–(iv), then we say that X acts on A **continuously**. (Or: Ψ **is continuous**).

It is **monotone continuous** if it satisfies only (ii)–(iv).

An example of a continuous action is the action of $Y := \text{Prim}(A)$ on A given by the one-to-one correspondence between the open subsets of $\text{Prim}(A)$ and closed ideals $\mathcal{I}(A)$ of A:

$$Z \in \mathbb{O}(Y) \mapsto \Psi_A(Z) := \bigcap \{J : J \in \text{Prim}(A) \setminus Z\} .$$

Note that $\Psi_A(\emptyset) = 0$ and that here we define $\Psi_A(Y) := A$.

If nothing is said about an action of $\text{Prim}(A)$ *on* A*, then this natural correspondence is used.*

Therefore, the monotonous maps Ψ from X to the closed ideals of A are in one-to-one correspondence to the monotonous maps from the open subsets of X into the open subsets of the primitive ideal space $Y = \text{Prim}(A)$ of A.

Check cite of Fell60 [296]!

By [**559**] and [**296**], if A is separable, then $\mathcal{I}(A)$ is naturally order anti-isomorphic (respectively the family $\mathbb{F}(\text{Prim}(A))$ of closed subsets of $\text{Prim}(A)$ is order isomorphic) to a closed subset Z_A of the Hilbert cube $[0, 1]^\infty$ with the coordinate-wise order: Take the map

$$J \in \mathcal{I}(A) \mapsto (\|\pi_J(c_n)\|)_{n=1}^\infty ,$$

where $c_1, c_2, \ldots$ is a dense sequence in $\{a \in A_+ \colon \|a\| = 1\}$. The induced Hausdorff topology on $\mathbb{F}(\mathrm{Prim}(A))$ is the **Fell-Vietoris topology**. The one-to-one map $F \in \mathbb{F}(\mathrm{Prim}(A)) \mapsto \mathrm{Prim}(A) \setminus F \in \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{I}(A)$ defines on $\mathbb{O}(\mathrm{Prim}(A))$ a topology that is equal to the so-called **Lawson topology**.

Since the Dini-functions $\varphi\colon X \to [0,1]$ on the Dini spaces X (= point-complete second countable l.c. T_0-spaces) are naturally related to the ''way-below'' (= ''well-below'') continuous monotone functions on the continuous lattice $\mathbb{O}(X)$ into $[-\infty, \infty]$ if we take e.g. the strictly order-monotone homeomorphism $\psi\colon t \in [0,1] \mapsto [-\infty, +\infty]$ given by $\psi(t) := \tan(\pi(t-1/2))$.

Countably generated Continuous lattices Ω are in 1-1-correspondence with $\mathbb{O}(X)$ with -- up to homeomorphisms unique -- Dini space X.

There is a Theorem in lattice theory that gives the same embedding in a different formulation (Graetzer [332], Lattice Theory , Thm. 47 (of Larson), see also [321, thm. IV-3.20].

The set $Z_A \cong \mathcal{I}(A)$ contains zero $(0,0,\ldots) \leftrightarrow A$ and Z_A is preserved under forming of component-wise maxima: If $\alpha, \beta \in Z_A$, then $\max(\alpha,\beta) \in Z_A$. The points of $\mathrm{Prim}(A)$ can be found by considering the prime elements of Z_A with respect to the continuous "multiplication" given by $(\alpha,\beta) \mapsto \max(\alpha,\beta)$.

If we identify $\mathrm{Prim}(A)$ with this subset of prime elements in $Z_A \cong \mathcal{I}(A)$, then the closure of a set S in $\mathrm{Prim}(A) \subseteq Z_A$ is given by the hull-kernel operation $hk(S) := \{\alpha \in \mathrm{Prim}(A) \colon \alpha \leq \max S\}$.

A simple set-theoretical argument shows, that, for second countable sober (point-complete) T_0-spaces X, the above defined upper semicontinuity of a faithful and monotonous map $\Psi\colon \mathbb{O}(X) \to \mathcal{I}(A)$ which satisfies moreover (iii) and the non-degeneracy $\Psi(\emptyset) = 0$ and $\Psi(X) = A$, is equivalent to the existence of a (unique) continuous map p from $\mathrm{Prim}(A)$ into X such that $\Psi(Z)$ is the ideal corresponding to $p^{-1}(Z)$.

If Ψ is given by p^{-1}, then properties (i)–(iii), $\Psi(X) = A$ and $\Psi(\emptyset) = 0$ hold. The action Ψ is faithful, if and only if, $p(\mathrm{Prim}(A)) \cap (U \setminus V) \neq \emptyset$ for all open subsets $V \subset U \subseteq X$ with $V \neq U$. Then, furthermore, (iv) holds for Ψ, if and only if, p is "pseudo-open" (which is the same as "open" only if X is Hausdorff, see [**359**] for details).

Let us consider the *special case* where X is a locally compact *Hausdorff* space and Ψ is faithful and satisfies (i)–(iii) and $\Psi(\emptyset) = 0$, $\Psi(X) = A$. By Dauns-Hofmann theorem ([**402**, exercise 10.5.84], [**616**, cor. 4.4.8]) one gets that there is a unique C^*-morphism α from $\mathrm{C}_0(X)$ into the center of $\mathcal{M}(A)$ such that $\Psi(Z) = \alpha(\mathrm{C}_0(Z))A$

for every open subset Z of X. It is then easy to see, that A is C^*-bundle with base space X in the sense of [**471**] ([31]), if and only if, Ψ is non-degenerate and satisfies conditions (i)–(iv). Thus, the faithful continuous actions of locally compact Hausdorff spaces X on a C^*-algebra A are in natural one-to-one correspondence to the structures of C^*-bundles on A in the sense of [**471**]. It says that *our general theory contains those results for C^*-algebra bundles.*

In the case of continuous bundles, E. Blanchard [**89**] has found a special case of the below given "trivialization" Theorem K. The remaining points are, to give necessary and sufficient conditions on the bundle under which it is exact (this was done in [**471**]), and to give other conditions under which it is (strongly) purely infinite. See Chapter 2 for partial answers and related problems.

A C^*-algebra B defines on its multiplier algebra $\mathcal{M}(B)$ or, more generally, on the C^*-algebra $\mathcal{L}(E)$ of B-linear operators with bounded adjoints on a Hilbert B-module E, an action Ψ^{up} of $X = \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B)$ as follows:

$$\Psi^{\mathrm{up}}\colon J \in \mathcal{I}(B) \mapsto \mathcal{M}(E, J) \in \mathcal{I}(\mathcal{L}(E))$$

where $\mathcal{M}(E, J) := \{T \in \mathcal{L}(E)\,;\; TE \subset EJ\}$. In the case where $E = B$ (considered as left B-module), we have $\mathcal{M}(B) = \mathcal{L}(E)$ and $\mathcal{M}(B, J)$ is the kernel of the natural C^*-morphism $\mathcal{M}(B) \to \mathcal{M}(B/J)$. Ψ^{up} satisfies (i), (iii) and (iv) if B is σ-unital and E is countably generated over B. But Ψ^{up} does not satisfy (ii) in general, e.g. in the case $B = c_0 \otimes \mathbb{K}$. If λ is an isometric B-module isomorphism from E_1 onto E_2, then $\lambda(\mathcal{M}(E_1, J)) = \mathcal{M}(E_2, J)$ for every closed ideal J of B.

How could the realization of a given action of $\mathrm{Prim}(B)$ on A look like?

Above we have defined the action $\Psi^{\mathrm{up}}\colon \mathcal{I}(B) \to \mathcal{I}(\mathcal{M}(B))$. In a more general setting we have:

Definition 1.2.7. Let $A \subseteq \mathcal{M}(B)$. One has two natural maps $\Psi^{A,B}_{\mathrm{down}}$ and $\Psi^{\mathrm{up}}_{B,A}$ from the open subsets $\mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{I}(A)$ of $\mathrm{Prim}(A)$ into $\mathcal{I}(B)$ (respectively from $\mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B)$ into $\mathcal{I}(A)$), given by:

$$\Psi^{A,B}_{\mathrm{down}}\colon K \in \mathcal{I}(A) \mapsto \overline{BKB} \in \mathcal{I}(B)\,,$$

respectively by

$$\Psi^{\mathrm{up}}_{B,A}\colon J \in \mathcal{I}(B) \mapsto A \cap \mathcal{M}(B, J) \in \mathcal{I}(A)\,.$$

Here $\overline{BKB}$ means the closure of the linear span of elements bac, where $a \in K$, $b, c \in B$

The corresponding actions of $\mathrm{Prim}(A)$ on B and of $\mathrm{Prim}(B)$ on A are increasing maps, the one corresponding to $\Psi^{A,B}_{\mathrm{down}}$ satisfies (i) and (ii) (but it is in general not faithful and has neither (iii) nor (iv)), and the one corresponding to $\Psi^{\mathrm{up}}_{B,A}$ fulfills (iii) and (iv) (but it is in general not faithful and has neither property (i) nor (ii)).

[31] I.e., A is the C^*-algebra of continuous sections in a continuous field of C^*-algebras $\{A_x\}_{x\in X}$, equipped with its $\mathrm{C}_0(X)$-module structure.

If $A \subseteq \mathcal{M}(B)$ satisfies $A \cap (I + J) = A \cap I + A \cap J$ for all closed ideals I, J of $\mathcal{M}(B)$ and if B is σ-unital, then $\Psi^{\mathrm{up}}_{B,A}$ satisfies (i).

In view of Theorem K, it is *important* that $\Psi^{\mathrm{up}}_{B,A}$ satisfies (ii) if $A \subseteq B$.

There are examples which show that

$$J \in \mathcal{I}(B) \mapsto \mathcal{M}(B, J) + B \in \mathcal{I}(\mathcal{M}(B)/B)$$

is in general neither lower semi-continuous nor upper semi-continuous, but satisfies (i) and (iii) if B is σ-unital. This causes a striking technical difficulty for proofs of generalized Weyl-von Neumann-Voiculescu theorems, cf. Chapter 5.

The results and methods of this monograph allow to prove
(but not here in this book because it requires in detail several
another hundert pages)
that each separable stable C^*-algebra A has
''Abelian factorization''.
This implies a proof of the n.c. Michael selection Conj.
and a proof that the MvN-property implies *strong* pure infiniteness

It remains the question if, for every separable C^*-algebra A,
the $\mathcal{O}_2$-stabilization $A \otimes \mathcal{O}_2$ contains a
''regular abelian C^*-algebra'' C in sense of Definition 1.2.9

The *non-commutative Michael selection Conjecture* 12.3.2 says that for separable stable C^*-algebras A and B and lower semi-continuous actions $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ of $\mathrm{Prim}(B)$ on A there is a C^*-morphism $h \colon A \to \mathcal{M}(B)$ such that $\Psi = h^{-1}\Psi^{\mathrm{up}}_{B,h(A)}$, i.e., that $\Psi(J) = h^{-1}\mathcal{M}(B, J)$ for $J \in \mathcal{I}(B)$. Moreover, h should be weakly Ψ-residually nuclear in the sense of Definition 1.2.8. A proof of Conjecture 12.3.2 would imply that C^*-algebras with WvN-property in Definition 1.2.3 are strongly purely infinite, because Conjecture 12.3.2 implies the conjecture that every C^*-algebra has *residually nuclear separation* in sense of Definition 1.2.3.

Theorem 12.1.8 and the below stated Theorem K are partial results in the direction of the non-commutative Michael selection Conjecture 12.3.2. Theorem 12.1.8 (and Proposition 12.2.15) will be used for the proof of Theorem K.

DEFINITION 1.2.8. Suppose that X is a point-complete (also called "spectral", "sober" etc.) T_0-space with a countable base of its topology. Let Ψ_A and Ψ_B monotonous actions of X on C^*-algebras A and B. A completely positive map $V \colon A \to B$ is called **Ψ-equivariant** (or, more precisely, Ψ_A–Ψ_B-equivariant) if $V(\Psi_A(Z)) \subseteq \Psi_B(Z)$ for every open subset Z of X.

V is **Ψ-residually nuclear** (or, more precisely Ψ_A–Ψ_B-residually nuclear) if it is Ψ-equivariant and the class maps $[V] \colon A/\Psi_A(Z) \to B/\Psi_B(Z)$ are nuclear for every open subset Z of X.

A completely positive map T from A into $\mathcal{M}(B)$ will be called **weakly Ψ-equivariant** (respectively **weakly Ψ-residually nuclear**) if $V_b := b^*T(\cdot)b$ is Ψ-equivariant (respectively is Ψ-residually nuclear) for every $b \in B$.

We say that V is **residually nuclear** (respectively that T is **weakly residually nuclear**) in the special situation where $X = \mathrm{Prim}(B)$, $A \subseteq \mathcal{M}(B)$, Ψ_B is natural (i.e., $\Psi_B := \mathrm{id}_{\mathcal{I}(B)}$), $\Psi_A = \Psi^{\mathrm{up}}_{B,A}$ and V is Ψ-residually nuclear (respectively T is weakly Ψ-residually nuclear).

If A is exact, then V is Ψ-residually nuclear if V is Ψ-equivariant *and* is nuclear. But this is not true for arbitrary separable A (and arbitrary actions). If $A \subseteq B$ and $V : A \to B$

In Chapter 12 we finish the proof of the following embedding (or "sub-triviality") Theorem K which generalizes Theorem A and E. Blanchard's sub-triviality theorem for exact C^*-algebra bundles in [**89**].

THEOREM K (Embedding Theorem). *Suppose that A and B are separable and stable C^*-algebras, such that A is exact and B is strongly purely infinite.*

Let $\Psi_A : \mathbb{O}(X) \cong \mathcal{I}(B) \to \mathcal{I}(A)$ be a lower semi-continuous action of $X :=$ $\mathrm{Prim}(B)$ *on A which satisfies the "monotone upper semi-continuity" condition (ii) of Definition 1.2.6, together with the non-degeneracy condition*

$$\Psi_A(B) = \Psi_A(X) = A \text{ and } \Psi_A(\emptyset) = 0 .$$

*Then there exists a Ψ-residually nuclear *-monomorphism $h : A \otimes \mathcal{O}_2 \hookrightarrow B$ such that, for $h_0 := h((\cdot) \otimes 1)$ and every closed ideal J of B,*

$$h_0(\Psi_A(J)) = J \cap h_0(A) .$$

Moreover, h_0 is unitarily homotopic in the sense of Definition 5.0.1 to $h_0 \oplus h_0$.

*Every nuclear *-monomorphism h_1 from A into B, which is unitarily homotopic to $h_1 \oplus h_1$ and satisfies $h_1(\Psi_A(J)) = J \cap h_1(A)$ for $J \in \mathcal{I}(B)$, is unitarily homotopic to h_0.*

If $\Psi_A(J) = A$ always implies that $J = B$, then h can be found such that $h_0(A)B$ is dense in B.

A part of the proof of this Embedding Theorem is given in Chapter 6. It uses an additional assumption of a property of B: The existence of a "regular Abelian C^*-subalgebra" in $B \otimes \mathcal{O}_2$ in sense of Definition 1.2.9. From this property we derive the "Abelian factorization" property and the (for some arguments in our proof needed) "minimal" assumption of the "residual nuclear separation" property of B that we must require for the proof in Chapter 6. After that we can free the proof of the embedding theorem from the extra assumption of existence of "residually nuclear separation" by showing in Chapter 12 that we can find in the asymptotic corona $\mathrm{Q}(\mathbb{R}_+, B)$ of B a separable strongly purely infinite subalgebra F that contains B, contains a regular abelian C^*-subalgebra and has a monotone continuous action and upper s.c. action of $\mathcal{I}(A)$ on F that is compatible under rescaling automorphisms of $\mathrm{Q}(\mathbb{R}_+, B)$. The uniqueness up to unitary equivalence of the embedding of $A \otimes \mathcal{O}_2$ in $\mathrm{Q}(\mathbb{R}_+, B)$ gives that this is unitarily equivalent to an embedding from $A \otimes \mathcal{O}_2$ in B that fits into the given action of $\mathrm{Prim}(B)$ on A.

The results of this book (all together) lead to a proof that any separable stable C^*-algebra B has an Abelian factorization. The existence of regular Abelian C^*-subalgebras in $B \otimes \mathcal{O}_2$ follows for separable *exact* B from Corollary L given below. But despite of the not in this book proven existence of "Abelian factorization" for separable stable B, the *Question* about the existence of a regular Abelian C^*-subalgebra in $C^*_{max}(F_\infty) \otimes \mathcal{O}_2$ is still open. (If it exists, then $B \otimes \mathcal{O}_2$ contains a regular Abelian C^*-subalgebra $A_B \subseteq B \otimes \mathcal{O}_2$ for every separable C^*-algebra B.)

Why that? Give reason here, or reference to reasons given on other places!

Definition 1.2.9. We call an Abelian C^*-subalgebra $C \subseteq B$ of a C^*-algebra B **regular** in B if

(i) $(J + I) \cap C = (J \cap C) + (I \cap C)$ for all $I, J \in \mathcal{I}(B)$, and
(ii) C separates the lattice of ideals of B, i.e., $I \cap C = J \cap C$ implies $I = J$.

The additional assumption of the existence of a *regular* Abelian C^*-subalgebra C in $B \otimes \mathcal{O}_2$ will be removed finally in Chapter 12 by showing that the asymptotic corona $\mathrm{Q}(\mathbb{R}_+, B)$ contains a separable strongly purely infinite C^*-subalgebra $B_1 \subset \mathrm{Q}(\mathbb{R}_+, B)$ with the properties that $B \subset B_1$ and that $B_1 \otimes \mathcal{O}_2$ contains a regular Abelian C^*-subalgebra $C \subset B_1 \otimes \mathcal{O}_2$. The action Ψ of $\mathrm{Prim}(B)$ on A can be extended to an action Ψ_1 of $\mathrm{Prim}\, B_1$ on A by letting $\Psi_1(K) := \Psi(K \cap B)$ for $K \in \mathcal{I}(B_1)$. This extended action is invariant under "rescaling" (i.e., parameter change).

The uniqueness of the Ψ-preserving nuclear *-monomorphisms from $A \otimes \mathcal{O}_2$ in $\mathrm{Q}(\mathbb{R}_+, B)$ up to unitary homotopy (cf. Chapters 5, 7 and 9) leads to a "constant" nuclear embedding of $A \otimes \mathcal{O}_2$ into B itself that realizes the action Ψ.

With other words, under the above assumptions on A, B and Ψ_A, the action Ψ_A of $\mathrm{Prim}(B)$ on A can be realized as $h_0^{-1}\Psi^{\mathrm{up}}_{B,h_0(A)}$ for some non-degenerate nuclear *-monomorphism $h_0 \colon A \to B$ with $h_0 \oplus h_0$ unitarily homotopic to h_0. Moreover, then the nuclear *-monomorphism h_0 is uniquely determined, up to unitary homotopy, by Ψ_A and the property that $h_0 \oplus h_0$ is unitarily homotopic to h_0.

The existence and uniqueness of h_0 up unitary homotopy immediately implies the following Corollary L of Theorem K:

Corollary L. *Suppose that A and B are separable stable nuclear C^*-algebras and that γ is a topological isomorphism from* $\mathrm{Prim}(A)$ *onto* $\mathrm{Prim}(B)$.

*Then there is a *-isomorphism φ from $A \otimes \mathcal{O}_2$ onto $B \otimes \mathcal{O}_2$ that "realizes" γ, in the sense that $\varphi(J \otimes \mathcal{O}_2) = \gamma(J) \otimes \mathcal{O}_2$ for primitive ideals J of A.*

The isomorphism φ with this property is unique up to unitary homotopy.

MOVE TO PROOF of?

Notice, that the uniqueness up to unitary homotopy quoted in Corollary L is an "elementary" fact because $[\varphi_1 \oplus \varphi_2] = [\varphi_k]$ in $\mathrm{R}(\mathrm{Prim}(A); A \otimes \mathcal{O}_2, B \otimes \mathcal{O}_2)$ for $k \in \{1, 2\}$ for any two γ-equivariant φ_1, φ_2.

Another easy consequence of Theorem K is Corollary 12.3.1, that implies that *for every separable **exact** C^*-algebra A there exists a unique separable stable **nuclear** C^*-algebra B with the same primitive ideal space* $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ *and with* $B \cong B \otimes \mathcal{O}_2$.

Such an algebra B and the homeomorphism γ from $\mathrm{Prim}(A)$ onto $\mathrm{Prim}(B)$ are unique up to algebraical and topological isomorphisms.

By Theorem K and with this properties the C^*-algebra B is unique up to isomorphisms that fix the given homeomorphism γ from $\mathrm{Prim}(B)$ onto $\mathrm{Prim}(A)$. There are γ-equivariant non-degenerate C^*-morphisms from B into A and from B into A.

We are now going to use the Theorem K and the there defined $h_0 \colon A \to B$ in the same way as we have used above Theorem A to obtain Theorem B.

For that we need to generalize Kasparov's $\mathcal{R}\,\mathrm{KK}^G(X;\,\cdot,\cdot)$ functor in some direction.

Definition 1.2.10. Let X be a point-complete ([32]) T_0-space with countable basis, and suppose that Ψ_A and Ψ_B are monotonous actions of X on A and B.

By $\mathrm{Hom}(X;\,A,B)$ – or more precisely by $\mathrm{Hom}(\Psi;\,A,B)$ – we denote the Ψ-equivariant C^*-morphisms from A to B. If the multiplier algebra of B contains a unital copy of $\mathcal{O}_2$, then the unitary equivalence classes $[h]$ of elements $h \in \mathrm{Hom}(X;\,A,B)$ are preserved under Cuntz addition and they constitute in a natural way an abelian semigroup.

By $\mathrm{Hom}_{\mathrm{nuc}}(X;\,A,B)$ we denote the Ψ-residually nuclear C^*-morphisms.

We define a **Ψ-equivariant Kasparov module** as a Kasparov (A,B)-module $\mathcal{E} = (E,\phi,F)$ (see e.g. Definition 8.2.1) which has the additional property that ϕ is Ψ_A-$\Psi^{\mathrm{up}} \circ \Psi_B$-equivariant. Or equivalently, for each $e \in E$, the completely positive map

$$V_e \colon a \in A \mapsto \langle \phi(a)e, e\rangle \in B$$

satisfies $V_e(\Psi_A(U)) \subseteq \Psi_B(U)$ for all open subsets $U \in \mathbb{O}(X)$. The class of Ψ-equivariant Kasparov (A,B)-modules is closed under direct sums of Kasparov modules and under unitary equivalence.

Let $\sim_{scp}$ denote the equivalence relation in the (A,B)-Kasparov modules given by unitary isomorphisms and compact perturbations (of F, cf. Section 1 of Chapter 8). The equivalence classes build a commutative semigroup $\mathbb{E}(A,B)/\sim_{scp}$, such that $\mathrm{KK}(A,B) := \mathrm{Gr}(\mathbb{E}(A,B)/\sim_{scp})$.

If $(E_1,\phi_1,F_1) \sim_{scp} (E,\phi,F)$ and (E,ϕ,F) is Ψ-equivariant, then (E_1,ϕ_1,F_1) is so.

We define the **Ψ-equivariant KK-group** $\mathrm{KK}(X;\,A,B)$ as the Grothendieck group of the sub-semigroup $\mathbb{E}(X;\,A,B)/\sim_{scp}$ of the $\sim_{scp}$ classes of Ψ-equivariant

[32] Also called "*spectral*" or "*sober*".

(A, B)-Kasparov modules in $\mathbb{E}(A, B)/\sim_{scp}$. The induced equivalence-relation on the Ψ-equivariant (A, B)-Kasparov modules implies the equivalence $\sim_c$ of "cobordism" or "homology" (but is build *inside* the semigroup of unitary equivalence classes of Ψ-equivariant Kasparov modules, cf. Chapter 8).

Give precise references and def's !!!

A Ψ-equivariant Kasparov module $\mathcal{E} = (E, \phi, F)$ is called **Ψ-nuclear** if for every $e \in E$ the completely positive map $a \mapsto \langle \phi(a)e, e \rangle$ from A to B is *Ψ-residually nuclear*. If a Kasparov module is $\sim_{scp}$ equivalent to a Ψ-nuclear Kasparov module, then it is itself Ψ-nuclear. The Ψ-nuclear $\sim_{scp}$ classes are closed (i.e. "invariant") under direct sums and unitary equivalence.

Thus, the $\sim_{scp}$ equivalence classes of Ψ-nuclear Kasparov (A, B)-modules build a sub-semigroup $\mathbb{E}_{\text{nuc}}(X;\, A, B)/\sim_{scp}$ of the semigroup $\mathbb{E}(A, B)/\sim_{scp}$ of $\sim_{scp}$ equivalence classes of Kasparov modules.

We define the **Ψ-nuclear** KK-group $\text{KK}_{\text{nuc}}(X;\, A, B)$ as the *Grothendieck group* of the sub-semigroup $\mathbb{E}_{\text{nuc}}(X;\, A, B)/\sim_{scp}$ of the semi-group $\mathbb{E}(A, B)/\sim_{scp}$.

More generally, we consider in Chapters 8 and 9 the groups $\text{KK}(\mathcal{C};\, A, B) := \text{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp})$ where $\mathcal{C} \subseteq \text{CP}(A, B)$ is a point-norm closed operator-convex cone of completely positive maps, and $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp} \subseteq \mathbb{E}(A, B)/\sim_{scp}$ denotes the semigroup of $\sim_{scp}$-equivalence classes of $\mathcal{C}$-compatible Kasparov modules (E, ϕ, F) (i.e., of the modules with the property that for every $e \in E$ the completely positive map $a \mapsto \langle \phi(a)e, e \rangle$ is in $\mathcal{C}$). Notice that almost nothing is changed in Kasparov's theory except that we restrict the constructions to more special classes of A-B-modules that produce automatically the desired equi-variances.

In this more systematic approach the old non-equivariant theory is still contained and can be seen by considering $\text{KK}(A, B) = \text{KK}(\mathcal{C};\, A, B)$ for $\mathcal{C} := \text{CP}(A, B)$, and $\text{KK}(X;\, A, B) = \text{KK}(\mathcal{C};\, A, B)$ for the m.o.c. cone $\mathcal{C}$ of with respect to $\mathbb{O}(X)$-action equivariant elements in $\text{CP}(A, B)$ ...

An adaptation of the proofs in usual KK-theory shows that $\text{KK}(X;\, \cdot, \cdot)$, $\text{KK}_{\text{nuc}}(X;\, \cdot, \cdot)$ and the most general $\text{KK}(\mathcal{C};\, A, B)$ behave analogously to KK:

There is a natural semigroup morphism

$$[h] \in [\text{Hom}(X;\, A, B)] \mapsto [h - 0] := [\mathcal{E}_h] \in \text{KK}(X;\, A, B),$$

for stable separable A and B, where the Kasparov B-module $\mathcal{E}_h$ is given by $\mathcal{E}_h := (B, h, 0)$ and $[h]$ denotes the unitary equivalence classes by unitaries in $\mathcal{M}(B)$.

The usual KK-groups can be rediscovered from our general approach by $\text{KK}(A, B) = \text{KK}(\{\text{point}\};\, A, B)$.

One can prove the existence of Kasparov products $x \otimes_B y \in \text{KK}(X;\, A, C)$ for $x \in \text{KK}(X;\, A, B)$ and $y \in \text{KK}(X;\, B, C)$ in the same way as for ordinary KK-theory. There is a natural bi-additive map

$$\text{KK}(X;\, A, B) \times \text{KK}(C, D) \to \text{KK}(X;\, A \otimes C, B \otimes D),$$

that is induced by tensor products of modules (if C and D are nuclear), where the action of X on the tensor products, e.g. on $A\otimes C$, is given by

$$\mathbb{O}(X)\ni Z\mapsto \Psi_A(Z)\otimes C\in\mathcal{I}(A\otimes C)\,.$$

If we fix X, then $\mathrm{KK}(X;\,A,B)$ is a bi-functor in the category of Ψ-equivariant C^*-morphisms. The bi-functor $\mathrm{KK}(X;\,\cdot,\cdot)$ is *homotopy invariant* for Ψ-equivariant homotopy, has Bott periodicity in each variable, and has six-term exact sequences in each variable on Ψ-equivariant (and Ψ-equivariant semi-split) short exact sequences of separable algebras,

where we suppose that the actions Ψ of X on A and B satisfy some regularity conditions expressed by the related m.o.c. cone $\mathcal{C}(X,\Psi_A,\Psi_B;A,B)$.

`Exactly which ''regularity conditions'',`
`Where mentioned?`

If X is Hausdorff and the actions on A and B are continuous, then $\mathrm{KK}(X;\,A,B)$ is the same as Kasparov's $\mathcal{R}\,\mathrm{KK}^G(X;\,A,B)$ for the trivial group $G=\{e\}$, even if differently defined.

The functor $\mathrm{KK}_{\mathrm{nuc}}(X;\,A,B)$ has similar properties, but $\mathrm{KK}(X;\,\cdot,\cdot)$-equivalence does not necessarily imply $\mathrm{KK}_{\mathrm{nuc}}(X;\,\cdot,\cdot)$-equivalence. If A or B is nuclear, then the natural morphism from $\mathrm{KK}_{\mathrm{nuc}}(X;\,A,B)$ into $\mathrm{KK}(X;\,A,B)$ is an isomorphism.

Consider e.g. $X=\{0,1\}$ with topology $\mathbb{O}(X)=\{\emptyset,\{0\},\{0,1\}\}$. Then the elements of $\mathrm{KK}(X;\,A,B)$ give invariants for transformations between extension $0\to I\to A\to A/I\to 0$ into $0\to J\to B\to B/J\to 0$, if the Busby invariants of the extensions dominate the zero representation (cf. Section 8 of Chapter 5).

We define in Chapter 5 weakly nuclear extension groups $\mathrm{Ext}_{\mathrm{nuc}}(X;\,A,B)$ – and more general $\mathrm{Ext}(\mathcal{C}\,;\,A,B)$ for countably generated operator convex cones $\mathcal{C}\subseteq\mathrm{CP}(A,B)$. The analog of Kasparov's proof for the isomorphism $\mathrm{Ext}_{\mathrm{nuc}}(A,SB)$ and $\mathrm{KK}_{\mathrm{nuc}}(A,B)$ shows also that $\mathrm{Ext}_{\mathrm{nuc}}(X;\,A,SB)$ is naturally isomorphic to $\mathrm{KK}_{\mathrm{nuc}}(X;\,A,B)$, see Chapter 8.

If B is σ-unital and stable, then a kind of Weyl–von-Neumann–Voiculescu theorem for the class of weakly residually nuclear maps $V\colon C\subseteq\mathcal{M}(B)\to\mathcal{M}(B)$ holds if and only if B has the **WvN-property** of Definition 1.2.3.

Moreover, B_ω (respectively $\mathrm{Q}(\mathbb{R}_+,B)$) has the WvN-property, if and only if, B is "strongly purely infinite" in the sense of Definition 1.2.2. Therefore all proofs, mapping cone constructions and difference constructions carry over to this new category, cf. Chapters 5, 7, 8 and the proofs of Theorems B and M in Chapter 9.

Now we make, for the next Theorem M, the following assumptions (1)-(3) on A, B, $N\subseteq B$, $X:=\mathrm{Prim}(N)$ and the actions of X on A and B:

(1) *Suppose that A is separable, stable and exact, B is σ-unital, and N is a strongly purely infinite separable stable C^*-subalgebra of B such that NB is dense in B.*
(2) *Let $X:=\mathrm{Prim}(N)$, and let X act on B by $\Psi_B:=\Psi^{N,B}_{\mathrm{down}}$.*

(3) *Further, suppose that X acts lower semi-continuously on A, such that, moreover, $\Psi_A(X) = A$, $\Psi_A(\emptyset) = 0$, $\Psi_A(Z) = A$ implies $Z = X$, and such that Ψ_A satisfies (ii) of Definition 1.2.6,* i.e., the action Ψ_A of A on A is non-degenerate and monotone continuous.

We can apply Theorem K to the action of $\mathrm{Prim}(N)$ on A, and get a non-degenerate nuclear *-monomorphism $h_0 \colon A \to N \subseteq B$, such that h_0 is unitarily homotopic to $h_0 \oplus h_0$, and $h_0(\Psi_A(J)) = h_0(A) \cap J$ for $J \in \mathcal{I}(N)$. Note, that h_0 is determined by this properties up to unitary homotopy as a nuclear C^*-morphism from A into N.

Is the following intermediate action not contained
in the general type of $\mathcal{C}$- or Ψ-equivariant actions?

Theorem M. *Suppose that A, B, $N \subseteq B$, X, and the actions Ψ_B, Ψ_A satisfy the above assumptions* (1)-(3).

*Let $h_0 \colon A \to N \subseteq B$ be the *-monomorphism given by Theorem K, let $[h] \in [\mathrm{Hom}_{\mathrm{nuc}}(X;\, A, B)]$ denote the unitary equivalence class of h, and let $[h - 0] \in \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ be the difference construction for $h \in \mathrm{Hom}_{\mathrm{nuc}}(X;\, A, B)$.*

(i) *The map*

$$\alpha \colon\, [\mathrm{Hom}_{\mathrm{nuc}}(X;\, A, B)] \ni [h] \;\mapsto\; [h - 0] \in \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$$

is a semigroup epimorphism.

(ii) *$[h - 0] = [k - 0]$ in $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ if and only if $h \oplus h_0$ and $k \oplus h_0$ are unitarily homotopic.*

(iii) *If, moreover, B is itself strongly purely infinite and Ψ_B is an isomorphism, i.e., comes from an isomorphism of $\mathrm{Prim}(N)$ onto $\mathrm{Prim}(B)$, then $h \oplus h_0$ is unitarily homotopic to h for every *-monomorphisms $h \colon A \hookrightarrow B$ which satisfies $h(\Psi_A(Z)) = h(A) \cap \Psi_B(Z)$ for open subsets $Z \subseteq X$.*

Notice that Ψ_B is an isomorphism, if and only if, Ψ_B defines an isomorphism of $\mathrm{Prim}(N)$ onto $\mathrm{Prim}(B)$ – and is defined by such an isomorphism.

The proof of Parts (i) and (ii) will be given together with the proof of Parts (i) and (ii) of Theorem B. But we assume there the existence of the nuclear *-monomorphism $h \colon A \otimes \mathcal{O}_2 \hookrightarrow N$ with the properties listed in Theorem K for N (in place of B). Theorem K will be proven independently in Chapter 12, based on a partial result in Chapter 6. The important absorption result (iii) of Theorem M is shown in Chapter 7.

$\mathrm{KK}(X;\, \cdot, \cdot)$ classifies all separable stable nuclear C^*-algebras up to $\mathcal{O}_\infty$-stable isomorphisms by the following generalization of Corollary C.

Corollary N. *Suppose that A and B are separable, stable and nuclear C^*-algebras, and that there is a topological isomorphism γ from $\mathrm{Prim}(A)$ onto $\mathrm{Prim}(B)$. Let Ψ_B denote the continuous action of $X := \mathrm{Prim}(A)$ on B defined by γ.*

If $z \in \mathrm{KK}(X;\, A, B)$ is a $\mathrm{KK}(X;\,\cdot,\cdot)$-equivalence between A and B, then there is a Ψ-equivariant isomorphism φ from $A \otimes \mathcal{O}_\infty$ onto $B \otimes \mathcal{O}_\infty$

*It seems only true that there are X-equivariant *-monomorphisms $h\colon A \to B$ and $k\colon B \to A$ such that $k \circ h$ is unitarily homotopic to id_A and $h \circ k$ is unitarily homotopic to id_B such that $[\varphi - 0] = z$.*

There is only one isomorphism φ with this property up to unitary homotopy.

The proof follows from Theorem M in the same way as the proof of Corollary C follows from Theorem B (see the arguments above Corollary C).

As in the case of simple nuclear algebras, the C^*-algebras A and B absorb $\mathcal{O}_\infty$ tensorial if A and B are moreover strongly p.i.

BEGIN: discussion of homotopy invariance.
The example in detail should be moved to some suitable chapter.

It should be noted that homotopy invariance of the isomorphism classes of strongly purely infinite separable stable nuclear C^*-algebras does not follow, even if we assume in addition that the ideal structure is preserved on the way of the homotopy and can be "corrected" to an ideal system fixing path of full endomorphisms by adding the non-degenerate "zero" endomorphism h_0 to this path.

This can be illustrated transparently by separable nuclear s.p.i. algebras that ??????

For example, consider the half-open interval $X_0 := (0,1]_{\mathrm{lsc}}$ with the "lower semi-continuity"-describing T_0-topology given by the intervals $(r,1]$ $(r \in [0,1))$ and $\emptyset$ as open sets, see also [**359**, thm. 1.4].

By [**559**] (see also [**689**]), there is a canonical way to produce a separable nuclear C^*-algebra C with $\mathrm{Prim}(C) = X_0 = (0,1]_{\mathrm{lsc}}$ that is "locally purely infinite" in a rather weak sense of Definition 2.0.3.

It implies by Corollary 2.6.6 that C is strongly p.i. , which implies $C \cong C \otimes \mathcal{O}_\infty$ by Theorem **??**.

There exist separable nuclear C^*-algebras D with $\mathrm{Prim}(D) := (0,1]_{\mathrm{lsc}}$ that are not purely infinite. Since C is separable, purely infinite and has no unital quotient it is stable. Thus, there is a unital copy of $C^*(s_1, s_2) \cong \mathcal{O}_2$ in its multiplier algebra $\mathcal{M}(C)$. This shows $C \cong C \otimes \mathcal{O}_\infty \cong C \otimes \mathcal{O}_\infty \otimes \mathbb{K}$, and that there exists an ideal system preserving non-degenerate *-monomorphism $h_0\colon C \otimes \mathcal{O}_2 \cong C \otimes \mathcal{O}_2 \otimes \mathbb{K} \to C$, by using a non-degenerate C^*-morphism $\mathcal{O}_2 \otimes \mathbb{K} \to \mathcal{O}_\infty \otimes \mathbb{K}$.

Question: Are all approximate to id unitary equivalent isomorphisms of separable $\mathbb{K} \otimes A \otimes \mathcal{O}_2$ unitary homotopic to id?
Here the question could be how near the unitaries $U_n^* U_{n+1}$ of such approximate unitary equivalence defining unitaries can be taken to the unitary group $\mathcal{U}(A \otimes \mathcal{O}_2 + 1_{\mathcal{M}(A)} \otimes \mathcal{O}_2)$.

This group of isomorphisms is not studied well until yet.

We can build the Cuntz sum $\mathrm{id}_C \oplus_{s_1,s_2} h_0 \colon C \to C$ that is unitarily homotopic to id_C because id_C asymptotically dominates h_0 by Corollary **??**.

This unitary homotopy – in sense of Definition 5.0.1 – defines a point-norm continuous path $\psi_t \colon C \to C$ with $\psi_0 = \mathrm{id}$ and $\psi_t(c) = u(t)^* c u(t)$ and $\psi_1(c) = \mathrm{id}_C \oplus h_0$. Moreover, since C is σ-unital and stable, there is a norm-continuous path of unitaries $t \in [0,1) \mapsto u(t) \in \mathcal{M}(C)$ such that $\lim_{t\to 1} u(t)^* s_2 c s_2^* u(t) = c$ for all $c \in C$, i.e., id_C and $0 \oplus_{s_1,s_2} \mathrm{id}_C$ are unitarily homotopic, in particular $0 \oplus_{s_1,s_2} h_0$ is ideal system preserving homotopic to h_0.

If we combine the above defined paths of endomorphisms then we find a *-monomorphism from $\varphi \colon C \to \mathrm{C}([0,1], C)$ that is ideal system preserving, thus defines an element of $\mathrm{KK}(\mathrm{Prim}(C)\,; C, \mathrm{C}([0,1], C))$ and satisfies $\pi_0 \circ \varphi = \mathrm{id}_C$ and $\pi_1 \circ \varphi = h_0$. If we apply the above arguments in a bit more general situation (with obvious generalization), we get the following corollary of the classification:

COROLLARY 1.2.11. *If A is a separable nuclear strongly purely infinite C^*-algebra that admits a ideal system preserving zero homotopy, then A is stably isomorphic to $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$, i.e., $A \otimes \mathbb{K} \cong A \otimes \mathcal{O}_2 \otimes \mathbb{K}$.*

It follows that the homeomorphism class of the primitive ideal space is the only invariant of the isomorphism class of $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$. And it is not difficult to see that $\mathrm{Prim}(A)$ must by homeomorphic to $(0,1]_{\mathrm{lsc}} \times \mathrm{Prim}(A)$.

Indeed, let $a_1, a_2, \ldots$ a dense sequence in the positive contractions of A and let $b := \sum_n 2^{-n} a_n$. Then, clearly, $\|\pi_I(b)\| \leq \|\pi_J(b)\|$ for $J \subseteq I$. The equation $\|\pi_I(b)\| = \|\pi_J(b)\|$ implies that $J = I$.

`Give a transparent proof:`

The latter, because in case $J \neq I$ (and $J \subset I$) there exists a pure state ρ on A with $\rho(J) = \{0\}$ but $\rho(I) \neq \{0\}$ ...

(All non-empty open sets are prime and there is a countable sequence of open subsets V_n of $\mathrm{Prim}(A)$ that build a base of the topology of $\mathrm{Prim}(A)$. If we take for each of this a strictly positive contraction $a_n \in J_n$ for the corresponding ideal and let $b := \sum_n 2^{-1} a_n$. If $\|\pi_{J_k}(b)\| = \|\pi_{J_\ell}(b)\|$ and $J_k \subseteq J_\ell$ ...????

`Check cite/ref and explain`

A modification of the construction in ???

[**816**, ????] **??** ????

`It seems not to be the right class`
`Perhaps Rørdam's paper [689] seems to be nearer to the idea.`
`reference to his Israel J. Math. paper?`

allows to construct $C \cong C \otimes \mathcal{O}_\infty$ in a manner, that it has an explicit ideal-preserving zero-homotopy, this is a C^*-morphism $\gamma \colon C \to \mathrm{C}((0,1], C)$ with $\gamma_t \cong \pi_t \circ \gamma \colon J(s,1] \to J(ts,1]$,

where we ?????????????? and ???????????

Notice that $J(t,1] \subset J(s,1]$ and $\bigcap_{t\in(0,1]} J(t,1] = \{0\}$, i.e., there is a point-norm continuous path $\{\gamma_t\,;\ t\in[0,1]\}$ of endomorphisms $\gamma_t\colon C\to C$ such that $\gamma_1 = \mathrm{id}$, $\gamma_0 = 0$ and $\gamma_t(J)\subseteq J$ for every closed ideal J of C and every $t\in[0,1]$, cf. [**464**, prop. 6.1].

`Give the path explicit ?!!`

`Check cite !! ??`

It follows that the (non-zero) strongly purely infinite algebra $A := C \oplus (\mathcal{O}_2 \otimes \mathbb{K})$ is homotopic to $B := \mathcal{O}_2 \otimes \mathbb{K}$ in a Ψ-equivariant manner,

but the algebras have primitive ideal spaces that are not homotopic.

`Check: Are` $Z := X_0 \uplus \{-1\}$ `with ''open''` $\{-1\}$ `and the space` $\{-1\}$ `NOT homotopic??`
`Let` $\psi(t)(s) := t\cdot s$ `and` $\psi(t)(-1) = -1$.
`Is each` $\psi(t)$ `continuous on` Z?
`Is` $\psi(t)$ `globally continuous?`
`Is` X_0 `via` $\psi(t)$ `homotopic to` $\emptyset$?

Clearly, this phenomenon can not appear if A and B are replaced by *simple* (p.i.) C^*-algebras.

Now let, more generally, C an arbitrary strongly purely infinite stable separable nuclear C^*-algebra (in particular $C\cong C\otimes\mathcal{O}_\infty\otimes\mathbb{K}$), and let $\gamma_t\colon C\to C$ an ideal system preserving homotopy between id and 0 (like in the special case above). Then we can (Cuntz-)add to the morphisms γ_t the mono-morphism h_0, and get that $\gamma_t\oplus h_0\colon C\to C$ defines a Ψ-equivariant homotopy from $\mathrm{id}_C\oplus h_0$ to the monomorphism $0\oplus h_0$. In this special case it is equivalent to the property that $(\gamma_t\oplus h_0)(J)$ generates J for each $J\in\mathcal{I}(C)$.

They must define the same element of $\mathrm{KK}(X; C, C)$ by homotopy invariance of $\mathrm{KK}(X;\cdot,\cdot)$. Both satisfy the assumptions of Part (iii) of Theorem M. It follows, that $h_0\colon C\to C$ is unitarily homotopic to id_C, which implies that

$$C\ni c\mapsto c\otimes 1\in C\otimes\mathcal{O}_2$$

is an $\mathrm{KK}(X_0,\cdot,\cdot)$-equivalence. Thus, $C\ni c\mapsto c\otimes 1\in C\otimes\mathcal{O}_2$ is approximately unitary equivalent to an isomorphism from C onto $C\otimes\mathcal{O}_2$. This shows:

If a strongly purely infinite separable nuclear C^*-algebra C admits an ideal-system preserving zero homotopy, then C absorbs $\mathcal{O}_2$ tensorial, i.e., $C\cong C\otimes\mathcal{O}_2$.

See the paper [**464**] for a more detailed study of this class of nuclear C^*-algebras:
It turns out that they are all AH-algebras with building blocks $M_n(\mathrm{C}_0(P,p_\infty))$ where (P,p_∞) are certain pointed one-dimensional finite CW-complexes P with each point connected to $p_\infty\in P$.

`Give precise conditions on` P!

The converse does not hold, e.g. $\mathcal{O}_2 \otimes \mathbb{K}$ tensorial absorbs $\mathcal{O}_2$ but is not homotopic to zero: No (non-zero) simple C^*-algebra A is zero-homotopic, [**175**], because $A \otimes \mathcal{O}_2$ has real rank zero for simple A, cf. Theorem E.

It follows that one has necessarily to consider a sort of more special Ψ-equivariant homotopy, that ensure that also the corresponding primitive ideal spaces are *homeomorphic*.

DEFINITION 1.2.12. Suppose that $\Psi_A\colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B\colon \mathbb{O}(X) \to \mathcal{I}(B)$ are actions of a T_0 space X on A respectively B.

Let $k_0, k_1\colon A \to B$ be Ψ_A-Ψ_B–equivariant C^*-morphisms (respectively Ψ-residually nuclear C^*-morphisms, respectively $k_0, k_1 \in \mathcal{C} \subseteq \mathrm{CP}(A,B)$).

We say that k_0 and k_1 are **Ψ-homotopic** ,

if ????? and ??? `how to avoid singularities?`,

if there is is a point-norm continuous path $[0,1] \ni t \mapsto \gamma_t$ in the Ψ_A-Ψ_B–equivariant C^*-morphisms (respectively Ψ-residually nuclear C^*-morphisms, respectively $\gamma_t \in \mathcal{C}$ is $\mathcal{C}$-compatible for $\mathcal{C} \subseteq \mathrm{CP}(A,B)$) $\gamma_t\colon A \to B$ such that $\gamma_0 = k_0$ and $\gamma_1 = k_1$.

The family $\{\gamma_t\}$ is an **equivariant homotopy**, respectively is a **$\mathcal{C}$-compatible homotopy**.

The Ψ-equivariant homotopy γ_t between k_0 and k_1 will be called **full** if, for each $U \in \mathbb{O}(X)$, $\gamma_t(\Psi_A(U))$ generates for every $t \in [0,1]$ the same ideal $\Phi(U) := \overline{Bk_0(\Psi_A(U))B} \subseteq \Psi_B(U)$ of B.

`Check homotopy statements`
`above and from here to next label !   : ??`

The homotopy invariance of $\mathrm{KK}(X;\, A,B)$ and of $\mathrm{KK}_{\mathrm{nuc}}(X;\, A,B)$ ensures that Ψ-homotopic k_0 and k_1 have the same $\mathrm{KK}(X;\, A,B)$-class, respectively same $\mathrm{KK}_{\mathrm{nuc}}(X;\, A,B)$-class.

On the other hand, we can find (by Corollary N) a Ψ-equivariant isomorphism from $A \otimes \mathcal{O}_\infty$ onto $B \otimes \mathcal{O}_\infty$, if we consider only those **homotopy equivalences** which are given by $h\colon A \to B$, $k\colon B \to A$, $p(t)\colon A \to A$ and $q(t)\colon B \to B$, which satisfy $kh = p(0)$, $p(1) = \mathrm{id}_A$, $p(t)(A) \cap J = p(t)(J)$ and $Ap(t)(J)A$ dense in J for every $J \in \mathcal{I}A$ and every $t \in [0,1]$, and $hk = q(0)$, $q(1) = \mathrm{id}_B$, $q(t)(B) \cap J = q(t)(J)$ and $Bq(t)(J)B$ is dense in J for every $J \in \mathcal{I}B$ and every $t \in [0,1]$. It means that we require that $p(t)$ induces the identity map of the lattice $\mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{I}(A)$ for each $t \in [0,1]$ (and same for $q(t)$ and $\mathbb{O}(\mathrm{Prim}(B))$).

The latter (strong) non-degeneracy condition for an Ψ-equivariant homotopy can be arrived by replacing h by $h \oplus h_0^{A,B}$ (respectively k by $k \oplus h_0^{B,A}$, p_t by $p_t \oplus h_0$ and q_t by $q_t \oplus h_0$).

This strong conditions on the homotopy imply in particular that $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are homeomorphic. If then $h\colon A \to B$ and $k\colon B \to A$ define an equivariant homotopy with respect to the natural actions of $\mathrm{Prim}(B)$ (respectively of $\mathrm{Prim}(A)$)

of the *above stronger type*, then $[h] \in \mathrm{KK}(\mathrm{Prim}\, B;\, A, B)$ is a $\mathrm{KK}(\mathrm{Prim}\, B;\, \cdot, \cdot)$-equivalence, and $h \oplus h_0$ is unitarily homotopic to an isomorphism from A onto B ([33]).

Check above formula! And next statement on ??????

Thus, *a strongly p.i. separable stable nuclear C^*-algebra A is isomorphic to $A \otimes \mathcal{O}_2$ if it is homotopic to zero in a* Prim(A)*-equivariant manner.*

(For example, $C \cong C \otimes \mathcal{O}_2$ if C is the above considered algebra with ideal system preserving zero homotopy). The *converse does* not hold as the example $A := \mathcal{O}_2 \otimes \mathbb{K}$ shows. But one has, for s.p.i. separable nuclear stable algebras A, that $A \otimes \mathcal{O}_2 \cong A$ if and only if $\mathrm{KK}(\mathrm{Prim}(A);\, A, A) = 0$. (Use almost verbatim the same arguments as for Part (iv) of Corollary F.)

Partly mentioned further above? What? ??

END: discussion of homotopy invariance.

Corollary L tells us nothing about the structure of the primitive ideal spaces of separable nuclear C^*-algebras A. A pure topological description of the primitive ideal spaces of nuclear C^*-algebras A is given in [**359**], that uses results of [**464**] and of Chapters 3 and 12.

It would be desirable to know under which conditions on A the algebra $D = A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is the inductive limit of C^*-algebras $\mathrm{C}_0(Y_n, \mathcal{O}_2 \otimes \mathbb{K})$ for Polish l.c. spaces Y_n. This is the case, e.g. , if $\mathcal{I}(A)$ is linearly ordered (by inclusion of ideals).

Unfortunately there are also examples where D can not be expressed as such an inductive limit, e.g. , if $A := \{f \in \mathrm{C}([0,1], M_2) \colon f(1) \in \Delta\}$ where Δ denotes the subalgebra of diagonal matrices in M_2. (In fact, for the last example A, the algebra D can not be any inductive limit of C^*-algebras with Hausdorff primitive ideal spaces, cf. Remark B.12.1.)

All strongly purely infinite separable nuclear C^*-algebras A that admit an ideal-system preserving homotopy beween id_A and 0 satisfy $A \cong A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and are inductive limits of algebras $A_1 \subseteq A_2 \subseteq \cdots$ with A_n isomorphic to $\mathrm{C}_0(\Gamma, p) \otimes M_{k_n}$, where Γ is a finite connected graph and $\mathrm{C}_0(\Gamma, p)$ denotes the algebra of continuous functions on Γ that vanish at a distinguished point $p \in \Gamma$, cf. [**464**].

It is still an *open question* whether, for every separable stable nuclear C^*-algebra B, the C^*-algebra $D := B \otimes \mathcal{O}_2$ is an inductive limit of C^*-subalgebras of D that are isomorphic to C^*-algebras $A \otimes \mathcal{O}_2$, where A is of type I.
One can show that D is a crossed product of an inductive limit of algebras of the form $\mathrm{C}_0(P_n \backslash \{q_n\}, M_n)$ with $\mathbb{Z}$, where P_n denotes a connected finite one-dimensional polyhedron and q_n is a point of P_n, cf. [**464**].

It implies that for every nuclear separable C^*-algebras A the algebra $A \otimes \mathcal{O}_2$ contains an Abelian C^*-subalgebra $C \subset A \otimes \mathcal{O}_2$ that is "regular" in the sense of Definition 1.2.9.

[33] Notice here that $h_0^{A,B} \circ h_0^{B,A}$ is unitarily homotopic to $h_0^{A,A}$.

It follows that for the T_0 space $Y := \mathrm{Prim}(A)$ of a separable exact C^*-algebra A there exist a locally compact Polish space X ([34]) and a continuous map $\psi\colon X \to Y$ that is "pseudo-open" and "pseudo-surjective" in the sense of Definitions **??** and **??**. In fact, then $C := \mathrm{C}(X)$ can be embedded by Theorem **??** into $A \otimes \mathcal{O}_2$ such that $\varphi\colon C \to A \otimes \mathcal{O}_2$ induces ψ^{-1}.

Conversely, for any locally quasi-compact sober T_0 space Y with the property that such a continuous map $\psi\colon X \to Y$ exists for some locally compact Polish space X there exists a separable nuclear C^*-algebra A such that $\mathrm{Prim}(A) \cong Y$ by [**359**, thm. 1.4].

`Next must be changed drastically, because of New results ??`

Which New results?? Give Ref's or Cite !!!

Let us consider an other aspect of the the "UCT":

M. Dadarlat published in [**193**, sec. 3] an example (based on an observation in [**369**]) of a continuous field $(A_x)_{x\in Q}$ of C^*-algebras A_x over the **Hilbert cube** $Q := [0,1]^\infty$ that has fibers $A_x \cong \mathcal{O}_2$ and that its C^*-algebra of continuous sections $\mathcal{A}$ has non-trivial $\mathrm{K}_1(\mathcal{A}) \neq 0$. If one tensors with $\mathcal{P}_\infty$ then one gets an example with $A_x \cong \mathcal{O}_2$ but $\mathrm{K}_0(\mathcal{A}) \neq 0$.

In particular, $(A_x)_{x\in Q}$ can't be locally trivial. One can see (e.g. from the construction of Dadarlat) that the nuclear algebra $\mathcal{A}$ is strongly purely infinite (i.e., absorbs $\mathcal{O}_\infty$ tensorial) and satisfies the UCT.

Let $(A_x)_{x\in Q}$ an arbitrary continuous field over $Q := [0,1]^\infty$ with all fibers isomorphic to $\mathcal{O}_2$.

Is the C^*-algebra $\mathcal{A}$ of continuous sections in the field $(A_x)_{x\in Q}$ always in the UCT-class?

This kind of algebras $\mathcal{A}$ are all "locally" purely infinite in the sense of Definition 2.0.3. Are this algebras $\mathcal{A}$ purely infinite (weakly purely infinite, strongly purely infinite)? Counterexamples? What are reasonable invariants to distinguish them? It leads to the question of classification of the isomorphisms from $\mathcal{O}_2 \otimes \mathcal{O}_2$ onto $\mathcal{O}_2$ by a suitable "classifying space".

`Next comes some old stuff. Partly to be removed!! ??`

Let Y a (not-necessarily Hausdorff) sober T_0-space. We use for an action $\Psi\colon \mathbb{O}(Y) \to \mathcal{I}(A)$ of a topological space Y on A the notation $A|Z := \Psi(U)/\Psi(U) \cap \Psi(U \setminus Z)$ if Z is a closed subset of an open subset $U \subseteq X$. We say that X acts on A *continuously* if Ψ is a lattice *monomorphism* and is both "upper semi-continuous" and "lower semi-continuous" in the sense of Definitions **??**.

[34] i.e., X is a locally compact second countable Hausdorff space, or equivalently expressed: X is homeomorphic to a separable complete metric space that is in addition locally compact. The complete metric is not uniquely determined by this property of X.

THEOREM O. *Let X a second countable locally quasi-compact point-complete T_0 space, and let G a second countable l.c. group that acts continuously on X by $\mu\colon G \to \mathrm{Homeo}(X)$.*

Then holds:
If X acts continuously on a stable separable nuclear C-algebra A and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action on A with $\alpha(g)(\Psi(U)) = \Psi(\mu(g)(U))$ for all $g \in G$, then there exists a unique separable stable purely infinite nuclear C*-algebra B with a continuous action $\beta\colon G \to \mathrm{Aut}(B)$, an isomorphism $\lambda\colon X \to \mathrm{Prim}(B)$ from X onto $\mathrm{Prim}(B)$, an X-equivariant monomorphism $\varphi\colon A \to B$ and a new action $\alpha'\colon G \to \mathrm{Aut}(A)$, such that*

α' is (exterior equivalent ?? or weakly?? outer) 1-cocycle conjugate to α ?

only 2-cocycle conjugate ????

??

$\varphi \circ \alpha'(g) = \beta(g) \circ \beta(g)$

$\beta(g)(B|\lambda(U)) = B|\lambda(\mu(g)(U))$

$[\varphi] \in \mathrm{KK}(X; A, B)$ *is a* $\mathrm{KK}(X; \cdot, \cdot)$*-equivalence.*

B is determined up to unitary homotopy and X-equivariant isomorphisms.

Moreover, TFAE:

(i) *If A is nuclear and separable,* $\mathrm{Prim}(A) \cong X$, *and* $A/J \cong (A/J) \otimes \mathcal{O}_2$ *for every* primitive *ideal, then* $A \cong A \otimes \mathcal{O}_2$.
(ii) *If X acts on separable nuclear C*-algebras A and B continuously, and if $\psi\colon A \to B$ is an X-equivariant *-monomorphism such that, for each $x \in X$, the induced morphism $A|\overline{\{x\}} \to B|\overline{\{x\}}$ defines a* $\mathrm{KK}(\overline{\{x\}}; A|\overline{\{x\}}, B|\overline{\{x\}})$ *equivalence,*
then ψ defines a $\mathrm{KK}(X; A, B)$ equivalence.
(iii) *more?* $\dim(X) < \infty$*?*
(iv) *more?*

In the meantime Dadarlat [**192**] has shown that $A \cong \mathrm{C}_0(X, \mathcal{O}_2)$ (respectively $A \cong \mathrm{C}_0(X, \mathcal{O}_\infty)$) if A is separable and unital, $X := \mathrm{Prim}(A)$ is a a *finite-dimensional* Hausdorff space and if every primitive quotient of A is isomorphic to $\mathcal{O}_2$ (respectively $\mathcal{O}_\infty$).

Is some local triviality needed?

He shows also that $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are the only pi-sun algebras B with the property $A \cong \mathrm{C}(X, B)$ for all compact metric spaces X of finite dimension, and all $\mathrm{C}(X)$-algebras A with fibers $A_x \cong B$.

(We did not check if here only the UCT class has been studied !!!)

It follows that X satisfies (i) of Theorem O if X has a decomposition series X_γ with $X_{\gamma+1} \setminus X_\gamma$ is a finite-dimensional Hausdorff space, `Def. of strong UCT-class for` $\mathrm{KK}(X;\cdot,\cdot)$ given by suitable actions of X on Polish l.c. spaces Y or on separable type one C^*-algebras.

HERE we should also list the new defined basic properties for the classification and its terminology.

Riesz decomposition property, "strongly self-absorbing" C*-algebras, Completely positive maps of order zero, QDQ vs. UCT, Decomposition rank (of subhomogenuous C*-algs), pure C*-algebras, Nuclear dimension, Covering dimension for nuclear C*-algebras, locally finite decomposition rank, Controlled KK-theory, property A, Watatani Index for C*-algebras, slice maps, nonc. Weyl - von Neumann theorem, Elliott invariant, Flat dimension growth for C*-algebras, Nonstable K-Theory, corona factorisation property,

3. Background and some basic ideas in proofs

The conceptional background of our proofs of Parts (i) and (ii) of Theorems B and M can be described as follows:

If we have "actions" $\Psi_A\colon \mathbb{O}(X)\to\mathcal{I}(A)$ and $\Psi_B\colon \mathbb{O}(X)\to\mathcal{I}(B)$ of spaces X on C^*-algebras A and B, then we can select suitable matrix operator-convex cones $\mathcal{C}\subseteq \mathrm{CP}(A,B)$ of c.p. maps V that are equivariant with respect to the actions, i.e., $V(\Psi_A(U))\subseteq \Psi_B(U)$ for $U\in\mathbb{O}(X)$, and satisfy like-wise additional conditions, e.g. that V is residually nuclear for the ideal systems selected via the actions Ψ_A and Ψ_B.

This procedure is almost equivalent to a functorial selections of subsemigroups of the semigroups of unitary equivalence classes of Kasparov A-B-modules.

We define in Chapters 5 and 8 groups $\mathrm{Ext}(\mathcal{C};\,A,B)$ and $\mathrm{KK}(\mathcal{C};\,A,B)$, depending on non-degenerate operator-convex cones $\mathcal{C}\subseteq\mathrm{CP}(A,B)$ in a functorial way, such that e.g. $\mathrm{KK}(A,B)=\mathrm{KK}(\mathrm{CP}(A,B);\,A,B)$ or $\mathrm{KK}_{\mathrm{nuc}}(A,B)=\mathrm{KK}(\mathrm{CP}_{nuc}(A,B);\,A,B)$ if A is separable and B is σ-unital (see Chapters 5, 8, 9). Our extension groups $\mathrm{Ext}(\mathcal{C};\,A,B)$ will be defined by relations that are similar to relations given for $\mathrm{KK}^1(A,B)\cong\mathrm{Ext}(A,B)$ for trivially graded stable σ-unital A and B, the difference is that the generalized morphisms (ψ,P) of the Cuntz-Kasparov picture – cf. [**73**, 17.6.4] – are required to fulfill the extra condition that $b^*\psi(\cdot)b\in\mathcal{C}$ for each $b\in B$. We prove and apply that

$$\mathrm{KK}(\mathcal{C};\,A,B)\cong\mathrm{Ext}(\mathcal{C}(\mathbb{R});\,A,\mathrm{C}_0(\mathbb{R},B))$$

to bring $\mathcal{C}$-compatible asymptotic morphisms and KK-groups together.

The functorial construction $(\mathcal{C};\,A,B)\mapsto\mathrm{KK}(\mathcal{C};\,A,B)$ is homotopy invariant inside the – chosen/allowed – matricial operator-convex cones (defined and partly studied in Chapter 3) and there is a natural isomorphism $\mathrm{Ext}(\mathcal{C}(\mathbb{R});\,A,\mathrm{C}_0(\mathbb{R},B))\cong\mathrm{KK}(\mathcal{C};\,A,B)$, where $\mathcal{C}(\mathbb{R})\subseteq\mathrm{CP}(A,\mathrm{C}_0(\mathbb{R},B))$ denotes the operator-convex cone of c.p. maps $V\colon A\to\mathrm{C}_0(\mathbb{R},B)$ with $V(\cdot)(t)\in\mathcal{C}$ for all

$t \in \mathbb{R}$. Moreover, $\mathrm{Ext}(\mathcal{C}(\mathbb{R}); A, \mathrm{C}_0(\mathbb{R}, B))$ *is naturally isomorphic to the kernel of*

$$\mathrm{K}_1\left(\pi_B(H_0(A))' \cap \mathrm{Q}^s(B)\right) \to \mathrm{K}_1\left(\mathrm{Q}^s(B)\right) .$$

(Here $H_0 \colon A \to \mathcal{M}(B)$ is a non-degenerate *-morphism with infinite repeat $\delta_\infty \circ H_0$ unitarily equivalent H_0, such that the c.p. maps $A \ni a \mapsto b^* H_0(a) b \in B$ are dense in $\mathcal{C}$.)

The results will be used in Chapter 9 to prove with help of Theorem A (respectively Theorem K) the following result that is more general than Parts (i) and (ii) of Theorems B and M:
Let A and B stable C^*-algebras, where A is separable and B is σ-unital, and let $h_0 \colon A \hookrightarrow B$ a non-degenerate *-monomorphism such that $h_0 \oplus h_0 = h_0$. Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ denote the point-norm closed matrix operator-convex cone generated by h_0. Then,
the difference construction $h \mapsto [(B, h, 0)] =: [h - 0]$ *defines an additive map from* $\mathrm{Hom}(A, B) \cap \mathcal{C}$ onto $\mathrm{KK}(\mathcal{C}; A, B)$ *such that* $[h - 0] = [k - 0]$ *if and only if* $h \oplus h_0$ *and* $k \oplus h_0$ *are unitarily homotopic.*
Here the addition on $\mathrm{Hom}(A, B)$ is given (up unitary equivalence in $\mathcal{M}(B)$) by the Cuntz addition with a copy of $\mathcal{O}_2$ that unitally contained in $\mathcal{M}(B)$.

If we use the non-degenerate monomorphism $h_0 \colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$, from Theorems A and K, then we get Parts (i) and (ii) of Theorems B and M. With this general result in hand, the hard work reduces to the proof of Theorems A and K.

On the way to the proofs of Theorems A and K we obtain and use (among others) a sufficient criteria for *existence of an X-equivariant lift*:

`Next must be a Prop/Thm in Chap. 5. Check!` ?? Suppose that A and B are stable C^*-algebras, where A is separable B is σ-unital, and that $H_0 \colon A \hookrightarrow \mathcal{M}(B)$ and $\varphi \colon A \hookrightarrow \mathrm{Q}(B) = \mathcal{M}(B)/B$ are nuclear *-monomorphisms with the properties that $H_0(A)B$ is dense in B and, for every $a \in A$, $\pi_B(H_0(a))$ and $\varphi(a)$ generate the same closed ideal of $\mathrm{Q}(B)$. Then, *there is a unitary* $U \in \mathcal{M}(B)$ *such that* $\varphi = \pi_B(U^* H_0(\cdot) U)$, *if one of the following conditions (i) or (ii) is satisfied:*

(i) There exists a C^*-algebra C such that $B \cong C \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$, or
(ii) $\pi_B^{-1}(\varphi(A))$ is *stable*, and there are C^*-morphisms $H \colon A \otimes \mathcal{O}_2 \hookrightarrow \mathcal{M}(B)$ and $\Phi \colon A \otimes \mathcal{O}_2 \hookrightarrow \mathrm{Q}(B) = \mathcal{M}(B)/B$ with $H_0(a) = H(a \otimes 1)$ and $\varphi(a) = \Phi(a \otimes 1)$ for $a \in A$.

It has to be emphasized that much of the machinery needed for this work was already present in the literature (at least implicitly), especially done in the works of J. Cuntz, J. Glimm, G. Kasparov and D. Voiculescu until mid of the eighties of last century (cf. Chapters 2, 4. 5 and 8). Our viewpoints are inspired by early work of Elliott and Rørdam on the classification of Cuntz-Krieger algebras.

Our proofs of Theorem A and of Theorem B need old results of J. Cuntz concerning the realization of K_0 and K_1 for purely infinite C^*-algebras in [**172**], our characterization of exact C^*-algebras [**438**], some basic results of Kasparov theory [**405**], and improvements of Kasparov's Proof by Cuntz and Skandalis and others, see [**73**] and the "nuclear" variant of Kasparov theory given by Skandalis [**726**].

Additional background basic material can be found in textbooks [**73**], [**616**], [**810**], and in the articles [**172**] and [**726**].

Generalizations of results from Glimm's early work are important technical tools. This gives a way of approximating nuclear maps by maps of the form $x \mapsto c^* x c$ (under certain additional assumptions). It leads us in Chapter 5 to a generalized Weyl–von-Neumann–Voiculescu theorem, by modifying and using ideas of Kasparov [**404**].

The results of Chapters 2, 3, 4, 5 and 7 are fundamental and each is crucial for the proofs in the following Chapters. Chapter 6 contains the proof of Theorem A. Its method will be explored to prove Theorem 6.3.1, which is a special case of Theorem K, but Theorem K without any extra technical assumption will be finally proved in Chapter 12 with help of a sort non-commutative variant of the Michael selection theorem.

4. On J. Elliott's Classification Programm and Conjectures

Section to be rewritten/shorten:
Short cut, and outline some new results.
But only as summary e.g. following ICM lectures?
Don't forget the stable projection-less case
My (in-adequate?) remarks could be moved in Appendix A or B?

The below formulated Elliott classification conjecture is now verified for all simple separable nuclear C^*-algebras A that tensorial absorb the Jiang-Su algebra $\mathcal{Z}$ and satisfy the Universal Coefficient Theorem (UCT) for its KK-theory. The latter means that there exists a separable commutative C^*-algebras C that is KK-equivalent to A.

The classification of the "elementary" cases, e.g. $\mathrm{K}_*(A) = 0$ could perhaps also replace the UCT by other asymptotic equivalence..???

But it gives not an essentially simpler proof for the above considered case of (simple) pi-sun C^*-algebras. And the methods used in the case where e.g. A is stably projection-less is a bit away from the methods developed and used here. In particular, stably projection-less simple nuclear C^*-algebras have automatic one or more quasi-traces (cf. E.K. paper cited ???), and are in a sense the opposite of the pi-sun C^*-algebras.

The complete data are the ordered K-theory, – with K_0 pointed and scaled by a possible unit element –, and the cone of l.s.c. traces and its pairing with K_0.

It seems that the case of stably projection-less C^*-algebras is not completely studied, e.g. it could be that most of them are mostly crossed products of stable purely infinite algebras by some $\mathbb{R}$-action.

An inspiring role for this development of this final results was the idea of "decomposition rank" and to use the "balancing" role of tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, that spells out many in early study unexpected difficulties. The cases of AF-algebras, special study on AH-algebras and ASH-algebras did pave the way, also the in this book considered special case of nuclear purely infinite algebras related to graphs did play an inspiring role on the way to the success until now.

Partial results on stably projection-less nuclear C^*-algebras do also exist, but have until now no systematic description of all possible cases with given trace cone, or has not appeared until now. For example, stably projection-less C^*-algebras A remain stably projection-less if we tensor them with the Jiang-Su algebra $\mathcal{Z}$ (or other ASH-algebras). All simple separable nuclear stably projection-less C^*-algebras have densely defined (e.g. on the positive part of the Pedersen algebra) additive traces that play a role for the classification.

It should be emphasized that the, up to tensorial $\mathcal{Z}$ stably isomorphic, classes of simple nuclear C^*-algebras A can have different $\mathrm{K}_0(A)_+$, e.g. the embedding $a \mapsto a \otimes 1$ defines a natural isomorphism $\mathrm{K}_*(A) \cong \mathrm{K}_*(A \otimes \mathcal{Z})$ but the ordered "cones" $\mathrm{K}_*(A)_+$ and $\mathrm{K}_*(A\otimes\mathcal{Z})_+$ of the stable equivalence classes of idempotents in $(A\otimes\mathbb{K})_+$ (respectively in $(A\otimes\mathcal{Z}\otimes\mathbb{K})_+$) are in some cases different. This can happen also for simple ASH-algebras A, because there are examples of unital simple ASH-algebras where the ordered semigroup $\mathrm{K}_0(A \otimes \mathbb{K})_+$ is not "weakly unperforated", but it is known (!!!!! find citation / reference !!!!!) that $\mathrm{K}_0(A \otimes \mathcal{Z} \otimes \mathbb{K})_+$ is weakly unperforated for every C^*-algebra A.

It has to do – and in the "locally" purely infinite case that is our minimal assumption – with properties of the "large" version of the Cuntz semi-group $\mathrm{Cu}(A)$, that reduces to the set of Dini-functions on $\mathrm{Prim}(A)$ in the cases that $A \cong A \otimes \mathcal{O}_\infty$ as it happens in the case that we consider.

We restate a *special case* of the original Classification Conjecture of G. Elliott in a way that allows the reader to see, how our viewpoints and the results of N.Ch. Phillips and the author in the special case of separable simple unital amenable C^*-algebras are related to it, and which questions – namely $(Q1)$ and $(Q2)$ concerning the classification of simple purely infinite algebras remain open.

It is possible that one has answers if one requires additional properties, as e.g. the CFT property or infinitesimally properties.

Before we begin with this we want to acknowledge with congratulation the important almost final result for the classification of unital simple C^*-algebras:

If A_1, A_2 are a separable unital simple nuclear C^*-algebra that are KK-equivalent to separable Abelian C^*-algebra C_1, C_2 and tensorial absorb the

Jiang-Su algebra $\mathcal{Z}$ in the sense $A_k \otimes \mathcal{Z} \cong A_k$, then A_1 and A_2 are isomorphic if and only if A_1 and A_2 the same Elliott invariant (as defined below **??**).

Since A and $A\otimes\mathcal{O}_\infty$ are isomorphic for all strongly purely infinite separable nuclear algebras (and since for simple A pure infiniteness and strong pure infiniteness are the same) $A \cong A \otimes \mathcal{Z}$ follows from $\mathcal{O}_\infty \otimes \mathcal{Z} = \mathcal{O}_\infty$.

(But I do not know if also for the stably projection-less separable simple nuclear C^*-algebras A with $A \cong A \otimes \mathcal{Z}$ the classification in the UCT class becomes now complete.

An interesting question is if they all come from crossed products $A \rtimes \mathbb{R}$ of stable pi-sun algebras A by an action of the real numbers $\mathbb{R}$, or as crossed product by an action of the semigroup $\mathbb{R}_+ = [0, \infty)$, or by $(0, \infty)$.

`Next is a bit out of discussion and perhaps not related.`

We define a Z_2-graded group $\mathrm{K}_*(A) := \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$, and a pre-order on $\mathrm{K}_0(A)$ by the subsemigroup $\mathrm{K}_0(A)^+ := \{[p] \in \mathrm{K}_0(A) \colon p \in \mathrm{Proj}(A \otimes \mathbb{K})\}$.

Let A be a C^*-algebra and let $T^+(A)$ denote the locally compact cone of non-negative (additive) traces on the Pedersen ideal of $A\otimes\mathbb{K}$. There is a natural pairing $\langle \cdot, \cdot \rangle$ between $\mathrm{K}_0(A)$ and $T^+(A)$, defined by $\langle z, \tau \rangle := \tau(p - q) \in \mathbb{R}$ for $z \in \mathrm{K}_0(A)$ and $\tau \in T^+(A)$.

`Is next correct?` Here p, q are projections in $\widetilde{A} \otimes \mathbb{K}$ such that $z = [p] - [q]$, $p - q$ is in the Pedersen ideal of $A \otimes \mathbb{K}$, and $\widetilde{A}$ is the unitization of A.

Suppose that A and B are simple, separable and nuclear C^*-algebras, and that $\sigma_0 \colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ and $\sigma_1 \colon \mathrm{K}_1(A) \to \mathrm{K}_1(B)$ are group isomorphisms from the K-groups of A onto the K-groups of B such that $\sigma_0(\mathrm{K}_0(A)^+) = \mathrm{K}_0(B)^+$. Let $\sigma^T \colon T^+(B) \to T^+(A)$ be a continuous additive isomorphism from $T^+(B)$ onto $T^+(A)$, such that they have the coherence properties $\langle z, \sigma^T(\tau)\rangle = \langle \sigma_0(z), \tau\rangle$ for all $z \in \mathrm{K}_0(A)$ and $\tau \in T^+(B)$.

If σ_1, σ_2 and σ_T are given, then a variant of the **Conjecture of Elliott** `Cite source for Elliott invariant !!!` can be reformulated in an equivalent way as follows:

There exists an isomorphism ψ from $A \otimes \mathbb{K}$ onto $B \otimes \mathbb{K}$ such that

$$\sigma_* = \mathrm{K}_*(\psi) \qquad \text{and} \qquad \sigma^T(\tau) = \tau \circ \psi \quad \forall\, \tau \in T^+(B)\,.$$

If A and B are unital and $\sigma_0([1]) = [1]$ then one *requires* in addition that the isomorphism σ can be chosen such that, $\psi(1 \otimes p_{11}) = 1 \otimes p_{11}$.

If one of the algebras A or B is not known to be of real rank zero, then sometimes additional invariants have been introduced to verify the Elliott conjecture even only in very special cases. The now most prominent one is tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, cf. [**391**], because $A\otimes\mathcal{Z}$ and A have "almost" the same Elliott invariants. This excludes also the cases where one of A or B is "elementary", i.e., is stably isomorphic to the compact operators $\mathbb{K}$, e.g. to distinguish $\mathbb{C}$ and the Jiang-Su algebra $\mathcal{Z}$, or where one of the algebras is unital and finite but is not stably finite, or [**687**].

And ??????????????

??

([35]),

extra requirements on the stable rank, on the real rank, or the decomposition rank ... weakly unperforated perforated $\mathrm{K}_0(A)_+$ etc. ??? .

`It happens if one tensors by Jiang-Su algebra as shown by Rørdam ‘‘strongly` K_1`-surjective’’ [690, def. 6.1]????.`

An example of a property that is perhaps not encoded in the Elliott invariants is the *local* K_1*-surjectivity*:
We say that a *simple* C^*-algebra A is **locally K_1-surjective** , if for every *non-zero* projection $p \in A \otimes \mathbb{K}$ and every $x \in \mathrm{K}_1(A)$ there exists a projection $q \in A \otimes \mathbb{K}$ such that

(i) $[p] = [q]$ in $\mathrm{K}_0(A)$, and
(ii) x is in the image of the natural C^*-morphism from the unitary group of $q(A \otimes \mathbb{K})q$ into $\mathrm{K}_1(A)$.

This condition is trivially satisfied for all stably projection-less C^*-algebras A. The property of local K_1-surjectivity has an explanation with help of the graded K_*-groups: There is a natural isomorphism

$$\mathrm{K}_*(A) := \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \cong \mathrm{K}_0(\mathrm{C}(S^1, A)),$$

cf. [**73**, sec.9.4.1], and one can use this isomorphism to introduce an pre-order structure $\mathrm{K}_*(A)_+$ on $\mathrm{K}_*(A)$ by the sub-semigroup $\mathrm{K}_0(\mathrm{C}(S^1, A))^+$ of $\mathrm{K}_0(\mathrm{C}(S^1, A))$.

One can see that this semigroup consists of the elements $([p], [u]) \in \mathrm{K}_*(A)$, where $p \in A \otimes \mathbb{K}$ is a projection and $u \in \mathcal{U}(p(A \otimes \mathbb{K})p)$ is a unitary (that has to be extended to a unitary in the unitization $(A \otimes \mathbb{K})^{\sim}$ by $u \mapsto u + (1 - p)$).

Indeed, if $P = \{p(e^{2\pi it}) \in M_n(A)\,;\ t \in [0, 1]\}$ is a projection in $\mathrm{C}(S^1, M_n(A)) \subseteq \mathrm{C}(S^1, A \otimes \mathbb{K})$ then there exists a continuous path $t \in [0, 1] \to v(t) \in \mathcal{U}(M_n(A + \mathbb{C} \cdot 1))$ with $v(0) = 1$ and $v(t)^* p(1) v(t) = p(e^{2\pi it})$.

Then $u := pv(1)p \in \mathcal{U}(p(A \otimes \mathbb{K})p)$ for $p := p(1)$. The element $[p] \in \mathrm{K}_0(A)$ is the image of $[P]$ under the map $[\lambda]_0\colon \mathrm{K}_0(\mathrm{C}(S^1, A)) \to \mathrm{K}_0(A)$ induced by $\lambda\colon f \in \mathrm{C}(S^1, A \otimes \mathbb{K})) \mapsto f(1) \in A \otimes \mathbb{K}$. This attaches to P a pair (p, u) with $u \in \mathcal{U}(p(A \otimes \mathbb{K})p)$ and $p = \lambda(P)$.

[35] The semigroup of all Murray–von-Neumann equivalence classes of projections in $\mathcal{M}(A \otimes \mathbb{K})$, or the infimum of the dimensions of the maximal ideal spaces of the family of Abelian Kadison-Singer-type C^*-subalgebras of A (i.e., with unique pure states extension property), or comparison properties (certain sub-semigroups of) the Cuntz semigroup of $A \otimes \mathbb{K}$.

Let $([q],[w]) \in \mathrm{K}_*(A)$ with $q \in A\otimes\mathbb{K}$ and $w \in \mathcal{U}(q(A\otimes\mathbb{K})q)$. Then $w\oplus_{s,t}(w^*) \in \mathcal{U}_0((q\oplus_{s,t}q)(A\otimes\mathbb{K})(q\oplus_{s,t}q))$ and if $w(t) \in \mathcal{U}_0((q\oplus_{s,t}q)(A\otimes\mathbb{K})(q\oplus_{s,t}q))$ is a path with $w(0) = q\oplus_{s,t} q$ and $w(1) = w\oplus_{s,t} w^*$, then $P(e^{2\pi it}) := w(t)(q\oplus_{s,t}0)w(t)^*$ is a projection in $(q\oplus_{s,t}q)(A\otimes\mathbb{K})(q\oplus_{s,t}q)$ with $p := \lambda(P) = q\oplus_{s,t}0$ and $pw(1)p = w\oplus_{s,t}0 \in \mathcal{U}(p(A\otimes\mathbb{K})p)$. more details???? **??**

All simple stably infinite C^*-algebras A are locally K_1-surjective, and for them holds $\mathrm{K}_0(A)^+ = \mathrm{K}_0(A)$ and $\mathrm{K}_*(A)^+ = \mathrm{K}_*(A)$, because for each projection $p \in A\otimes\mathbb{K}$ there exists a full and properly infinite projection $q \in A\otimes\mathbb{K}$ with $[q] = [p]$ in $\mathrm{K}_0(A)$, cf. Lemma 4.2.6(ii) (compare also classical work of J.Cuntz [**172**] !). Then there are natural isomorphisms $\mathrm{K}_1(A) \cong \mathrm{K}_1(A\otimes\mathbb{K}) \cong \mathrm{K}_1(E)$ for the unital C^*-algebra $E := q(A\otimes\mathbb{K})q$, and E has a properly infinite unit q, which implies that E is K_1-surjective, i.e., that $u \in \mathcal{U}(E) \to [u] \in \mathrm{K}_1(E)$ is surjective, cf. Lemma 4.2.6(v).

If A is simple and *stably finite*, then it is easy to see that A is locally K_1-surjective, if and only if,

$$\mathrm{K}_*(A)^+ = \{(0,0)\} \cup \big((\mathrm{K}_0(A)^+ \setminus \{0\}) \oplus \mathrm{K}_1(A)\big)\,,$$

where $\emptyset\oplus\mathrm{K}_1(A) := \emptyset$, e.g. stably projection-less A are locally K_1-surjective because $\mathrm{K}_*(A)^+ = 0\oplus 0 = \{(0,0)\}$.

Simple unital *locally* K_1-surjective algebras A *with cancellation property* (for the semi-group $V(A) = [\mathcal{P}_\infty(A)]$) are necessarily K_1-*surjective* in the sense that $u \in \mathcal{U}(A) \to [u] \in \mathrm{K}_1(A)$ is a surjective map.

Unital simple C^*-algebras A with *stable rank one* are locally K_1-surjective ([36]). Unital simple C^*-algebras that are inductive limits of homogeneous algebras have in general stable rank > 1, cf. Villardsen [**797**], [**795**].

QUESTION 1.4.1. Can methods of [**797**] and [**795**] help to construct a unital simple ASH algebra that is not locally K_1-surjective?

Even for simple nuclear C^*-algebras of real rank zero it is not known if they are locally K_1-surjective (but they are K_1-injective, cf. [**525**, cor.4.2.10]).

The Conjecture of Elliott has been confirmed in many cases, but always under strong restrictions on A and B, e.g. that they are inductive limits of certain special sub-homogenous algebras A_n and B_n ([37]), or, in some very special cases, where A and B are stably isomorphic to (cocycle) crossed products of such inductive limits by outer actions of $\mathbb{Z}$, $\mathbb{Z}_n$ or by $\mathbb{R}$. Until now, on has not found a method to deduce this required properties of A and B from conditions of K-theoretic nature alone, or from "general defining relations" of A and B given by non-commutative polynomials.

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[36] Apply Gauss algorithm to invertible matrices in $M_n(pAp)$ that have invertible entries.

[37] The primitive ideal spaces $\mathrm{Prim}(A_n)$ and $\mathrm{Prim}(B_n)$ contain only Hausdorff subspaces X with "small" dimension in comparison with the *minimal* dimension of its irreducible representations

Stably projection-less simple exact C^*-algebras A have always non-zero *additive* traces (cf. e.g. [**441**]), i.e., $T^+(A) \neq 0$. There exist inductive limits A of sub-homogenous algebras such that A is stably projection-less *and* $\mathrm{K}_*(A) = 0$. It is then not difficult to see that they satisfy $T^+(A) \neq 0$. The existence of such algebras A can be seen from [**259**, thm. 3.2, thm. 5.2], that shows that the possible Elliott invariants $(G_0^+, G_0, G_1, T^+, \langle \cdot, \cdot \rangle)$, with the additional conditions $T^+ \neq 0$ and that $\langle z, \tau \rangle > 0$ (for all $\tau \in T^+$) implies $z \in G_0^+$, are exhausted by the invariants of simple inductive limits A of sub-homogenous algebras A_n that are extensions of sums of sub-homogenous algebras with at most 2-dimensional Hausdorff subspaces of primitive ideal spaces ([38]).

Let us consider from now on only the the Elliott invariant $(\mathrm{K}_*(A)^+ \subseteq \mathrm{K}_*(A))$ of *simple separable nuclear C^*-algebras A with* $T^+(A) = 0$:

A combination of works of Blackadar, Cuntz [**78**] and Haagerup [**342**] shows that, for *simple and exact* C^*-algebras A, the equation $T^+(A) = 0$ holds, if and only if, A is not stably finite (cf. B.4.5 or [**441**]). If, in addition, A is separable then there is a simple unital C^*-algebra B with properly infinite unit (in fact we find then B that contains a copy of $\mathcal{O}_2$ unitally) such that $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Thus, if $T^+(A) = 0$, then $\mathrm{K}_0(A)^+ = \mathrm{K}_0(A)$ by [**172**], *and* $A \otimes \mathbb{K}$ contains a properly infinite projection, which implies e.g. that $A \otimes \mathbb{K}$ can not be the inductive limit of type-I C^*-algebras ([39]). Every *simple* inductive limit A of type-I C^*-algebras has a non-zero positive trace on its minimal dense ideal $\mathrm{Ped}(A)$. More generally, a simple C^*-algebra A without a densely defined 2-quasi-trace can't be a subalgebra of an ultra-product

$\prod_\omega (B_1, B_2, \ldots)$ of a sequence $B_1, B_2, \ldots$ of type-I algebras, which involves a striking difference between the class of stable infinite nuclear simple C^*-algebras and all classes of algebras that can be described as inductive limits of type-I C^*-algebras. This sorts of inductive limits contain strictly all other sorts of algebras that have been successfully classified up to isomorphisms by its Elliott invariant until so far.

If $\mathrm{K}_0(A) \neq 0$ *and* $\mathrm{K}_0(A)_+ = \mathrm{K}_0(A)$, or if $\mathrm{K}_0(A) = 0$ *and* $A \otimes \mathbb{K}$ contains a non-zero projection, then A is stably infinite and $T^+(A) = 0$. So we see that the equation $T^+(A) = 0$ for simple nuclear A describes the stably infinite algebras among the simple nuclear C^*-algebras A. For them, *the invariants of Elliott reduce only to* K_0 *and* K_1. M. Rørdam has constructed in [**687**] a simple, separable, unital, and nuclear C^*-algebra $\mathcal{R}$ that is finite but is stably infinite.

[38] They are defined by a generalized mapping torus construction A_n (a special difference construction) for pairs of morphisms between sums of sub-homogenous algebras with primitive ideal spaces such that the Hausdorff parts have dimension ≤ 2. See [**273**, appendix] for an alternative proof, where the building blocks A_n are sub-homogenous algebras with at most 1-dimensional Hausdorff subspaces of its primitive ideal spaces.

[39] Moreover, there is a finite subset X of A such that $1/2 \leq \max\{\mathrm{dist}(x, B)\,;\ x \in X\}$ for *every* type-I C^*-subalgebra B of A^{**}.

In particular, $\mathcal{R}$ is not purely infinite in the sense of Definition 1.2.1, given below. The algebra $\mathcal{R}$ is stably isomorphic to the crossed product of a certain type I C^*-algebra C by an endomorphism λ of C, thus, $\mathcal{R}$ is stably isomorphic to a crossed product $D \rtimes_\mu \mathbb{Z}$ by an automorphism μ of $D := \lim_{n\to\infty} \lambda_n : C_n \to C_{n+1}$ with $C_n := C$, and $\lambda_n := \lambda$. The algebra $\mathcal{R}$ has *not* real rank zero, but contains "small" projections, cf. [**691**] and $\mathrm{K}_*(\mathcal{R}) = \mathrm{K}_*(\mathrm{C}(X))$ for $X := S^2 \times S^2 \times \cdots$ where S^2 means here the 2-dimensional sphere. In fact it is stably isomorphic to the generalized Fock-Toeplitz algebra (and to the Cuntz-Pimsner algebra) of some Hilbert $\mathrm{C}(X, \mathbb{K})$–bi-module.

Its K_*-groups are both isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots$, i.e., $\mathrm{K}_*(\mathcal{R}) \cong \mathrm{K}^*(S^1 \times S^1 \times \cdots)$.

Let us say some words about the *general* classification problem for non-p.i. algebras and its connection with the Rørdam groups $\mathrm{R}(\mathcal{C};\, A, B)$ that we introduce in Chapter 7.

`to be filled in: overview! and Rordam groups` ??

`update:` (Now 2021 ???): The classification covers now all – unital? – separable simple nuclear C^*-algebras that absorb the Jiang-Su algebra tensorial and are in the UCT-class?)

Some of the proofs of Elliott's Conjecture work well for inductive limits $A := \mathrm{indlim}_n A_n$ of sub-homogenous algebras A_n, e.g. provided that A is of *real rank zero* and that there is some bound $\infty > \Gamma \geq D_n/R_n$ for the ratio between the supremum D_n of the dimensions of the Hausdorff subspaces of $\mathrm{Prim}(A_n)$ and the infimum R_n of the dimensions of the irreducible representations of the A_n's, or that A is simple and the A_n's are extensions of finite dimensional algebras C_n by suspensions SB_n of finite dimensional algebras B_n, i.e., if there are exact sequences $0 \to SB_n \to A_n \to C_n \to 0$.

No description of this additional assumptions in terms of the K-theory has been found yet.

E.g. it is even an open problem whether a simple separable unital nuclear C^*-algebra A of real rank zero with unique trace state and with the pre-ordered K- and KK-groups of the CAR algebra M_{2^∞} is stably isomorphic to M_{2^∞}. In particular suppose that $A \otimes \mathcal{O}_\infty \cong M_{2^\infty} \otimes \mathcal{O}_\infty$ and that $(\mathrm{K}_0(A), [1_A], \mathrm{K}_0(A)_+)$ and the Elliott invariant of the UHF-algebra M_{2^∞} are order isomorphic by an isomorphism that is compatible with the – by $\mathcal{O}_\infty$-stable isomorphism – given isomorphism of $\mathrm{K}_0(A)$ with $\mathrm{K}_0(M_{2^\infty})$.

Question: Is A stably isomorphic to M_{2^∞} ?

This should be the case if A is in the UCT-class ... ???

The true problem is here if such A must automatic tensorial absorb the Jiang-Su algebra $\mathcal{Z}$. Or at least if $F(A) := A' \cap A_\omega$ has no character ... ??????

It is even not clear if there exists a simple unital nuclear C^*-algebra C of real rank zero with the Elliott invariant $(\mathbb{Z}, \mathbb{Z}_+, 1)$ and unique trace state, but (!) such that $F(C)$ has a character.

(I want to see how it is constructed as an inductive limit of finite-dimensional C^*-algebras A_n by completely positive contractions, because all separable nuclear C^*-algebras A are such inductive limits up to completely positive and completely isometric isomorphisms.)

Forgive me, if I want to see pi-sun algebra with trivial K-theory that is not isomorphic to $\mathcal{O}_2$, which is simply generated by two isometries $s_1, s_2 \in \mathcal{L}(\ell_2(\mathbb{N}))$ with

$$1 = s_1^* s_1 = s_2^* s_2 = s_1 s_1^* + s_2 s_2^* .$$

Perhaps it is e.g. the case if C has not the "corona factorization property" (CFP).

One can see here that ??????? CFP is very necessary and excludes cases that have not CFP (or satisfy not the UCT ?). Therefore we write here all at a constructive and elementary level that shows where we suddenly this additional properties UCT or CFP need.

Moreover it is even unknown whether pre-ordered KK-invariants and trace invariants are enough to characterize the CAR algebra among the simple separable nuclear quasi-diagonal C^*-algebras in the UCT-class with real rank zero and stable rank one. (Please don't misunderstand me: Here we should start with data from the ordered K_*-theory and *not* from the methods to construct them from inductive limits of (many step) extensions of stabilized sub-homogenous algebras combined with semi-direct products by semigroups or quantum groups. The question should be: how many pairwise stably non-isomorphic simple nuclear C^*-algebras have the same invariants, and how look the amenable C^*-algebras in those classes of partially ordered K_*-theory that are NOT in (some sort of) the UCT-class.

Is "seems" that the (non-zero) separable, unital, nuclear (= amenable), purely infinite, and simple C^*-algebras A with $A \cong A \otimes \mathcal{Z}$ are all contained in the UCT-class.

But no proof or counter examples have been found yet (end of 2021?). Notice here that all pi-sun A satisfy $A \cong A \otimes \mathcal{O}_\infty$ (give references here) and that $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{Z}$.

Next has to be edited !!! `update:`

The latter follows (not simply) from the fact that $\mathcal{O}_\infty$, $\mathcal{Z}$ and $\mathcal{O}_\infty \otimes \mathcal{Z}$ are all nuclear, simple, separable, unital and are in the UCT class, and that $\mathcal{O}_\infty$ and $\mathcal{O}_\infty \otimes \mathcal{Z}$ are strongly purely infinite (because are p.i. and simple).

In the case of simple separable nuclear projection-less stable algebras there are only some interesting examples studied so far.

One of it is given by a crossed product of $\mathcal{O}_2$ with an irreducible circle action, ????? In the case of a unital C^*-algebra without non-trivial projections there is e.g. the open question if the Jiang-Su algebra $\mathcal{Z}$, [**391**], is the only tensorial self-absorbing separable unital C^*-algebra $A \neq \mathbb{C}$ in the UCT-class that has the same Elliott invariant as $\mathbb{C}$.

Since W. Winter has shown that all separable tensorial self-absorbing unital C^*-algebras with approximately inner flip absorb the Jiang-Su algebra $\mathcal{Z}$, and since the class of $\mathcal{Z}$-absorbing unital simple nuclear stably finite C^*-algebras in the UCT-class are classified by its Elliott invariant it follows that tensorial $\mathcal{Z}$-absorbing and a kind of ordered $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, 1_A)$ is then a complete invariant.

The big open problem is: Can we escape from the very serious additional assumption that A is in the UCT-class for the classification of pi-sun algebras A?

More generally is $A \otimes \mathcal{O}_\infty$ (or $A \otimes \mathcal{Z}$) for all amenable separable A in the UCT class?

Is the UCT property equivalent to the Corona factorization property?

"Sudden" Question:

Consider the universal C^*-algebra $A_n = C^*(a, d_k, 1, s, t)$ given for $a \geq 0$, $d_1, \ldots, d_n, 1, s, t$ by the relations $\sum_k d_k^* a d_k = 1$, $s^* t = 0$, $s^* s = 1$, $t^* t = 1$.

(It implies that A_n contains a copy of $\mathcal{O}_\infty$. Let $t_1, t_2, \ldots$ natural generators build from s and t.)

$X := \sum_k t_k a^{1/2} d_k$ satisfies $X^* X = 1$.

Is $a \otimes 1$ properly infinite in $A_n \otimes \mathcal{Z}$?

I.e., does there exist $g \in A_n \otimes \mathcal{Z}$ with $g^*(a \otimes 1) g = 1 \otimes 1$?

It is enough to find $h \in A \otimes \mathcal{Z}$, and non-zero $b \in \mathcal{Z}_+$ with $h^*(a \otimes 1)h = 1 \otimes b$, because $\mathcal{Z}$ is simple, $\mathcal{O}_\infty \otimes \mathcal{Z} \cong \mathcal{O}_\infty$ and $1_A \in \mathcal{O}_\infty \subseteq A$.

Take elements b and $x_1, \ldots, x_n \in \mathcal{Z}$ with $x_k^* x_\ell = \delta_{k,\ell} b$ (e.g. coming from an n-homogenous element $\psi\colon \mathrm{C}_0(0,1] \otimes M_n \to \mathcal{Z}$ $b = \psi(f_0 \otimes p_{11})$).

Then $g := \sum_k d_k \otimes x_k$ has this property.

In [**467**] it is shown that the (CFP) is valid if $F(A) = (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$ has no character.

Theorem 4.3. (of [**467**]):

Let A be a unital separable C^*-algebra such that the central sequence algebra $F(A)$ has no characters. Then A has the Strong Corona Factorization Property (s CFP ? SCFP ?).

"Strong" Corona Factorization Property for semigroups, cf. citation [18, Definition 2.12] (in [**467**] ??? what is 18??):

For every $x', x, y_1, y_2, y_3, \ldots$ in $\mathrm{Cu}(A)$ and $m \in \mathbb{N}$ such that $x' << x$ and $x \leq m y_n$ in $\mathrm{Cu}(A)$ for all $n \geq 1$, there exists $k \geq 1$ such that $x' \leq y_1 + y_2 + \ldots + y_k$ in $\mathrm{Cu}(A)$.

The (strong) Corona Factorization Property can therefore be viewed as a weak comparability property for $\mathrm{Cu}(A)$.

It is not known how the Rørdam semi-groups $\mathrm{SR}(A, B)$ of Chapter 7 play together with (versions of) the Cuntz semigroups, even not in the important case, where A and B tensorial absorb the Jiang-Su algebra $\mathcal{Z}$.

This shows that we can't present here an idea for answering classification questions in full generality if other types of algebras or additional structure like group actions have to be studied. We develop in this book a method to reduce some classification problems to questions on the realization of KK-theory with help of generalized Weyl–von-Neumann–Voiculescu theorems. The main tools are continuous modifications of the Rørdam groups $\mathrm{R}(A, B)$, as defined in Chapter 7. They play a role in several generalizations, e.g. to the non-simple case as described above, or others, as mentioned in the remarks of Chapter 7. Some of the known stable invariants for simple separable nuclear C^*-algebras are the same for $\mathrm{R}(\cdot, \cdot)$-equivalent C^*-algebras A and B (e.g. they are KK-equivalent, $F(A)$ and $F(B)$ contain, up to isomorphisms, the same simple separable C^*-subalgebras, the stable rank, the real rank and decomposition rank are the same). A part of further study on classification could be the study of the consequences of $\mathrm{R}(\cdot, \cdot)$-equivalence. Partial results in this direction are Theorems B and M and its Corollaries C and N above. Above where? Above where? Give references !!

5. The way to here. Acknowledgements

This book is a very expanded and generalized version of the preprint of December 1994 (3rd draft) with the same title “Classification of purely infinite C*-algebras using Kasparov's theory”. In the original 1994-preprint a simpler “discrete” asymptotic theory using $\mathrm{EK}_\omega(A, B)$ has been taken to eliminate some technical difficulties arising in the continuous case. Now we have replaced the discrete asymptotic methods (working with sequences) by their technically more involved continuous versions (working with paths going to infinity). This more elaborate theory allows to see more clearly where we have homotopy-invariant results. Now they are all integrated into our newest version that rigorously uses matricial operator-convex cones, because, up to the – not here – discussed group- or quantum-group equivariance.

A pre-version (of some sections concerning basics on purely infinite simple algebras) partly has been written in 1998 by postdoc students of the Fields Institute year on C^*-algebras. It was based on a seminar which was organized and held by postdoc students, but was supported by several lectures, consultations, e-mails and outlines from the author. The sections concerning simple algebras in the new presentation reflects a compromise between their detailed and elementary approach and the authors new insights, that changed during more then three decades considerably.

More details and background are given. In fact, the exposition is sometimes very detailed. But here we are running into a problem: half of the technical lemmas are folklore trivialities for those readers coming from the theory of exact C^*-algebras, tensor products and operator spaces. The other half is trivial for readers who have a clear understanding how different variants of KK-theories and K-theories can be constructed, and how they can be compared. But both is presented here at an elementary level, – so far as it is really used for the proof of our theorems. Only textbook material is omitted or is only sketched ([40]).

The old and sketchy proof of the natural isomorphism of the Grothendieck group $\mathrm{R}(A, B)$ of $[\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K}))]_u$ with $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ from the Appendix A of the Dec 94 preprint is now explained in a more transparent manner, by an axiomatic study of the semigroups of morphisms which are dominated by a fixed morphism (see Theorem 4.4.6, Proposition 7.2.13 and Chapter 9). But, up to using a technical framework that divide the arguments into smaller steps and offer more details, we use the same idea as in the proof outlined in the Appendix A of the Dec 94 preprint. Now it becomes visible that the ideas in the preprint apply also to the semigroups $\mathrm{SR}(X;\, A, B) := [\mathrm{Hom}_{\mathrm{nuc}}(X;\, A, \mathrm{Q}(\mathbb{R}_+, B))]$ and to the even more general semigroups $\mathrm{SR}(\mathcal{C};\, A, B)$ for (non-simple) stable separable exact A and arbitrary σ-unital stable strongly purely infinite B, and, more generally, to the semigroups $\mathrm{SR}(\mathcal{C};\, A, B)$ for in a "functorial way" selected matrix operator-convex cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$. But one needs to use the homotopy invariance of the generalized m.o.c. cone equivariant Kasparov groups $\mathrm{KK}(\mathcal{C};\, A, B)$ to get the basic ingredient for the proof that the natural group epimorphism from the $\mathcal{C}$-controlled unsuspended-but-stable E-theory groups $\mathrm{R}(\mathcal{C};\, A, B)$ onto $\mathrm{KK}(\mathcal{C};\, A, B)$ is also injective.

Notice that E. Blanchard [**89**] has extended Theorem A to C^*-algebra bundles. He uses Theorem A and Rørdam's proof of $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ (cf. [**678**]). This was done independent by N. Ch. Phillips [**?**] and the author with other methods and applications.

Also, there is an alternative proof of Part (i) of Theorem A and Parts (ii) and (iii) of Corollary F, given by N. Ch. Phillips and the author [**?**], that does *not* use a generalized Weyl–von-Neumann–Voiculescu theorem, that we, here in this book, prove and use.

N. Ch. Phillips [**627**] gave a shorter proof of Corollary C based on the observations given here in Parts (ii) and (iii) of Corollary F above (as in the beginning of the authors Dec 94 preprint). His proof uses $\mathcal{E}$-theory and deep results of M. Dadarlat, N. Higson, A. Loring, H. Lin and M. Rørdam.

One can proceed in the non-simple case along a way similar to the approach of N.Ch. Phillips with a generalized ideal lattice-equivariant version of E-theory in the non-simple case, after one has proved Theorem K (= non-simple variant of

[40] Some knowledge on Cuntz-Krieger algebras is useful for considering examples. But the most basic examples are discussed here in the Appendix A.

Theorem A). In fact, some results of N. Higson, J. Cuntz and others indicate that every sort of an KK-theory is functorial equivalent to a “liftable” (or “semi-split”) variant of some E-theory. We are quite happy to find a convincing proof in our case of $\mathrm{KK}(\mathcal{C};\, A, B)$ and the related stable but unsuspended version of E-theory, here realized by the generalization $\mathrm{R}(\mathcal{C};\, A, B)$ of Rørdam's sequence groups.

This can be made precise in analogy with our natural isomorphisms

$$\mathrm{R}(\mathcal{C};\, A, B) \cong \mathrm{Ext}(\mathcal{C};\, A, SB) \cong \mathrm{KK}(\mathcal{C};\, A, B)\,,$$

where A and B are stable and separable and $\mathcal{C} = \mathcal{C}(h_0)$ for some non-degenerate *-morphism $h_0\colon A \to B$ that is unitarily homotopic to $h_0 \oplus h_0$. That means: We show that in the considered cases there is a bijection between certain m.o.c. cones and such morphisms h_0.

If one imposes into the *definitions* directly some sort of homotopy-equivalence then the above considered isomorphisms are “almost evident”. Then the isomorphisms are much easier to derive than our “constructive” results in Chapters 8 and 9 that give really unitary homotopy rather than general homotopy equivalence. But general homotopy equivalences are useless for our purpose. In fact, – at least in view of our applications –, the great invention of Kasparov was the observation that the groups $\mathrm{Ext}(A, B)$ with its fairly *algebraic* definition – see Chapter 5 – are naturally isomorphic to the homotopy invariant groups $\mathrm{KK}(A, SB)$, cf. Chapters 5 and 8 for more details and Section 5 for the most important implication of this homotopy invariance. We need a more “constructive” proof of the isomorphism $\mathrm{Ext}(\mathcal{C};\, A, SB) \cong \mathrm{KK}(\mathcal{C};\, A, B)$ than those proofs given usually, because it becomes an important ingredient of our proof of the isomorphism $\mathrm{R}(\mathcal{C};\, A, B) \cong \mathrm{Ext}(\mathcal{C};\, A, SB)$ at the very end of Chapter 9. The reader also should observe that some technical preparation for the proofs of above isomorphisms contain new results, that we have formulated and established in much more generality than actually used here in the book for proofs of the main topics.

The author has outlined proofs of Theorem A(i), Corollary F(iii) and Corollary G (and of the result that a pi-sun algebra has a central sequence of unital copies of $\mathcal{O}_\infty$) in his talk on the ICM-satellite conference in Geneva in August 1994. The idea of the proofs of Theorem I and Corollary J has been outlined in the authors talk in Rome, July 1996, at the conference on Operator Algebras and Quantum Field Theory.

In 1995-1998 we have extended the results, except Corollary H, to continuous bundles of C^*-algebras with finite dimensional metrizable compact base, in the sense of [**471**]. One only has to replace $\mathrm{KK}_{\mathrm{nuc}}(\cdot,\cdot)$ by a “nuclear” version of the Kasparov functor $\mathcal{R}\,\mathrm{KK}^G(X;\,\cdot,\cdot)$ for the base space X (where here $G := \{e\}$ is the trivial group!), and the sub-triviality theorem of E. Blanchard [**89**] replaces Theorem A.

In the period from March to September 1998 we arrived at a complete classification of all stable separable nuclear C^*-algebras that tensorial absorb $\mathcal{O}_\infty$ up to ideal system preserving isomorphisms by their $\mathrm{KK}(X; \cdot, \cdot)$-class.

Theorem M has been presented in the authors talk in Copenhagen, August 1998. A talk about Theorem K was given in Münster, March 1999. An overview of the main results of this book was given in [**442**], including outlines of some proofs. In the meantime many new results have appeared, and our new exposition should allow the reader to see that several fundamental observations can be derived also by our approach. The more general viewpoint using operator-convex cones in Chapters 3, 5, 7, 8 and 9 has been worked out in 2001–2006, and was outlined in conference and workshop talks in 2007 and 2008. Among the applications are ??????

`list some applications of classification` ??????? **??**

The author likes to thank R. Bhat, S. Friese, P. Friis, T. Häberlen, H. Harnisch, H. Osaka and G. Vaillant. They have helped to prepare versions of parts of the early 1994 preprint and/or have worked out a more detailed version of parts of it.

Inspiring, helpful and sometimes (with any right !) critical remarks came from C. Anantharaman-Delaroche, D. Bisch, B. Blackadar, E. Blanchard, U. Bunke, J. Cuntz, M. Dadarlat, G. Elliott, U. Haag, G. Kasparov, M. Rørdam, V. Voiculescu, A. Wassermann, S. Wassermann, J. Weidner and W. Werner. They also have suggested improvements that I tried to follow on.

J. Cuntz, J. Elliott and M. Rørdam have supported the work of the author on this monograph several times.

6. To do: Unify NOTATION in all Chapters!

Use $\approx$ (Cuntz equiv.) and $\sim_{MvN}$ (Murray–von-Neumann equiv.)

`Adjust in all chapters of the book the use of` $\sim$ `and` $\approx$

And the same with $[a]_\sim$, $\langle a\rangle_\sim$, $[a]_\approx$, $\langle a\rangle_\approx$.

Put it in the Index list. Decide where a reminder is necessary... **??**.

CHAPTER 2

Basics on purely infinite C^*-algebras

We consider here several properties that cause pure infiniteness of simple or non-simple C^*-algebras. It requires to discuss first the relations between the different definitions of infiniteness and the related terminology for C^*-algebras. The three Sections 2, 3 and 4 consider mainly the case of simple algebras. But the later considered pure infiniteness of non-simple algebras play also for the classification of simple C^*-algebras an important role. It causes a partly study of the huge variety of definitions of sorts of "infiniteness" for non-simple C^*-algebras or of elements in them. Then some of the later used permanence properties will be studied here in this chapter.

The last three Sections 16, 17 and 18 study properties of strong purely infinite C^*-algebras in sense of Definition 1.2.2, and some constructions that lead to strong p.i. C^*-algebras. Before and in-between there is the study of general observations needed in the proofs of results in later chapters, where sometimes only locally some sort of infiniteness plays a role for the applications.

The inventor – and pioneer of the study – of "purely infinite simple C^*-algebras" was Joachim Cuntz. He introduced in [**172**, p. 186] a notion of pure infiniteness in case of *simple* C^*-algebras using his observation in below given Lemma 2.1.6 for simple C^*-algebras:
A C^*-algebra A is said to be **purely infinite** – in the sense of J. Cuntz – if *every non-zero hereditary C^*-subalgebra D of A contains an infinite projection $p \in D$,* $p \neq 0$, i.e., a non-zero projection p that is *Murray–von-Neumann equivalent* to a proper sub-projection of itself in the sense of following definition:

Definition 2.0.1. Elements $a, b \in A_+$ are **Murray–von-Neumann equivalent** (short: **MvN-equivalent**) in A if there exists $d \in A$ with $d^*d = a$ and $dd^* = b$. The relation will be denoted by $a \sim_{MvN} b$ or simply by $a \sim b$.

The MvN-equivalence class of $a \in A_+$ will be denoted by $[a]_{MvN}$ – or simply by $[a]$ if there is no danger to be mixed up with the considerably "bigger" class $[a]_{\approx}$ of all $b \in A_+$ that are Cuntz–equivalent to a (cf. Definition 2.5.1).

Basic examples of purely infinite *simple* algebras are the Cuntz algebras $\mathcal{O}_{\mathrm{n}}$, $n = 2, 3, \ldots,$ that are the universal unital C^*-algebras generated by elements $s_1, \ldots, s_n$ with defining relations $s_j^* s_k = \delta_{j,k} 1$ for $j, k \in \{1, 2, 3, \ldots, n\}$ and $\sum_{k=1}^{n} s_k s_k^* = 1$, respectively the Cuntz algebra $\mathcal{O}_\infty$ generated by a sequence of elements $s_1, s_2, \ldots$ with relations $s_j^* s_k = \delta_{j,k} 1$ for $j, k \in \mathbb{N}$, cf. [**169**]. In particular, J. Cuntz studied and confirmed in early work some related conjectures of J. Dixmier (1964) in [**214**] concerning $\mathcal{O}_2$.

Some properties of $\mathcal{O}_\infty$ and $\mathcal{O}_2$ did play a fundamental role for the proofs of the classification results for pi-sun algebras both in the approach of N. Ch. Phillips and in the the approach of the author of this book. Therefore, we list some properties of Cuntz-algebras $\mathcal{O}_{\rm n}$ and $\mathcal{O}_\infty$ in Section 1 of Appendix A and outline there elementary proofs for this properties, – but only so far as they do not follow immediately from some of the very basic observations in Chapters 2, 3 and 4. Moreover, we give in Section 1 of Appendix A some additional informations on the algebras $\mathcal{O}_{\rm n}$ that will be used often in proofs, applications and examples.

Our "long" Proposition 2.2.1 says that on the class of *simple* C^*-algebras pure infiniteness in the sense of J. Cuntz is equivalent to each of the local, weak or strong pure infiniteness defined below, and to many other properties that are useful for several applications useful. But there are examples of amenable (= nuclear) separable unital simple C^*-algebras that are (quasi-) trace-less but are not purely infinite in the sense of any here given and considered definitions, cf. the example of M. Rørdam [**687**] ([1]).

It has been shown in [**462**, prop. 4.7], [**463**, cor. 6.9] and [**93**, thm. 4.17], – cf. also our Proposition 2.6.5 –, that in case of C^*-algebras A of *real rank zero* all of our below given definitions of pure infiniteness of simple C^*-algebras A coincide.

Unfortunately, this is not the case for non-simple A. ????

`Next blue/red text has to be verified, is ''under observation":`
More generally, equivalence of *all* definitions of pure infiniteness (– except possibly *strong* pure infiniteness! –) for C^*-algebras holds for all C^*-algebras with the property that

for every closed ideal $J \neq A$ of A and every non-zero hereditary C^*-subalgebra D of A/J there exists a non-zero projection $p \in D$,

i.e., where A is "rich of projections" in the sense of our Definition 2.6.1: If C^*-algebras A with this property has our "weakest" property of infiniteness, – i.e. the "local pure infiniteness" (l.p.i) of Definition 2.0.3 – then it turns out that A is purely infinite in the sense of Definition 1.2.1.

This class of C^*-algebras A contains the class given by the property that

every non-zero hereditary C^*-subalgebra $D \subseteq A/J$ of any quotient A/J of A contains a non-zero infinite projection $p \in D$,

– i.e., those where all quotients are purely infinite in the sense of the above quoted

---

[1] Since a long time it seems to be still (January 2022) an open question if there exists a simple nuclear C^*-algebra of "real rank zero" that is stably infinite but is not purely infinite, cf. Question (Q1) in Chapter 1. This cited famous example of M. Rørdam has *not* this properties. Such an example (of real rank zero), – if it exists –, can be modified to produce a unital simple algebra A with real rank zero, that contains no infinite projection, but $M_2(A)$ contains a copy of $\mathcal{O}_2$ unitally. Even if this exists, then the question is: Can such kind of A be a *nuclear* C*-algebra with all this properties?

definition of J. Cuntz in [**172**] ([2]) –,
are purely infinite in sense of Definition 1.2.1.

We do not know if purely infinite C^*-algebras A, that are "rich of projections", in sense of our Definition 2.6.1, are moreover *strongly purely infinite* in sense of Definition 1.2.2, – our so far strongest infiniteness property of a C^*-algebra that is "rich of protections".

`We need only a kind of semi-splitting`
`(in cases where we can reduce all arguments to the separable case`
`by considering suitable` $C*$`-subalgebras` E `if possible):`
An element $b \in E_+$ with $\overline{bEb} \subseteq A$ with $b \approx (b \oplus b)$ and a non-zero element $c \in E_+$ orthogonal to b, i.e., $cb = 0$ and $\pi_A(c) \neq 0$.
Can only work if $A(??)$ is not "essential"???...
But it could be that all non-zero projections of cEc are contained in cAc ...?
No progress ... !!!
To find b, c one could take a strictly positive element of $a \in A_+$, i.e., $A = \overline{aEa}$.
Try to use the projectivity of $C_0((0,1], M_2)$, and consider $M_2 \subseteq E/A$ ...
It gives positive contractions $c, d \in E_+$ with $cd = 0$, $\pi_A(c) = p_{11}$, $\pi_A(d) = p_{22}$.
Then try to modify $a + d$ ...

NO PROGRESS !!!

One *question* is, if in case of separable σ-unital A, where A is an ideal of E, that is "rich of projections" and is purely infinite ($\Leftarrow$ HERE: A or E ?), has the property that an exact sequence $0 \to A \to_\eta E \to_\pi \mathcal{O}_2 \to 0$ must split, i.e., that there exists a C^*-morphism $\phi \colon \mathcal{O}_2 \to E$ with $\pi \circ \phi = \mathrm{id}_{\mathcal{O}_2}$ for the quotient epimorphism $\pi \colon E \to \mathcal{O}_2$ with kernel $\eta(A)$ for the inclusion morphism $\eta \colon A \to E$ onto an ideal of E if E has real rank zero.

(Questions: Is this splitting necessary for E being s.p.i.? Or is "semi-splitness" enough?)

Because then it would be the same as our Definition 2.6.1 (but with property l.p.i. added as additional property) if we could show that each copy of $\mathcal{O}_2$ can be lifted for a strongly p.i. ideal with many projections. Here we can reduce all to the separable case, and build later inductive limits of them.

In general the question is:

Does there exists a separable p.i. C^*-algebra A such that *every* non-zero hereditary C^*-subalgebra D of A is *not* strongly p.i.?

There exists D that contains may nonzero n-homogenous C^*-subalgebra $E_n := F_n \otimes M_n$ with $E_{n+1} \subseteq E_n$.

(The injections ??????? .)

If such example *not* exists, then every p.i. algebra is s.p.i. !!!

[2]... defined there mainly for the study of simple Cuntz-Krieger algebras!

(Where are the proofs of this?)

Thus, it suffices to consider suitable separable C^*-subalgebras of ultrapowers of p.i. algebras ... ???

But if A has real rank zero then local pure infiniteness of A implies strong pure infiniteness of A, because it is a special case of the following result:

If A satisfies the stronger condition that, – for every hereditary C^*-subalgebra $D \subseteq A$ and pure state ρ on A with $\rho|D \neq 0$ – , there exists a projection $p \in D$ with $\rho(p) \neq 0$ ([3]), then A is strongly purely infinite in sense of Definition 1.2.2 if A is locally purely infinite in sense of Definition 2.0.3 .

So far we don't have seen any example of a purely infinite separable C^*-algebra A that satisfies the properties in Definition 2.6.1 of C^*-algebras with "many projections", but is not strongly purely infinite.

`Check, compare and relate this blue part to above to the written text. Remove repeats in blue parts!!!`

`The two next discussed cases seem not to be equivalent! Check again!`

`Merge below and above text about special cases!`
`Look more to applications than to`
`problems with very weak definitions!`

`THIS is somewhere in the proof:`

It implies immediately that every $\pi_J(a) \in A/J$ is infinite for each $a \in A_+$ and closed ideal J of A with $a \notin J$, if one takes $D := \overline{\pi_J(a)(A/J)\pi_J(a)}$. Thus, each non-zero element of A is infinite.

Suppose that each non-zero element of A_+ is infinite, is then each non-zero element of A_+ properly infinite?

??? lem:2.infinite.elem ??? cf. Lemma **??**(v) and discussion of case of algebras that are "rich of projections" ...

Definition 2.6.1

Propositions on generalized Cuntz def. **??** and **??**

[prop:2.Cases.where..Cuntz.pi..implies..l.p.i]

[prop:2.cases.where..l.p.i..implies..p.i.?]

`Refs.: proof of cases where l.p.i implies p.i.`
`e.g. tensor ‘‘q-traceless’’ with Jiang-Su algebra`
`linear ordered lattice of ideals leads to s.p.i.`
`being ‘‘rich of projections’’ ...` ???????

... then A is moreover strongly purely infinite in sense of Definition 1.2.2. (Really???)

---

[3] Equivalently expressed: For every $a \in A_+$ with $\|a\| = 1$ and pure state ρ on A with $\rho(a) = 1$ there exists a projection $p \in \overline{aAa}$ with $\rho(p) > 1/2$.

It seems that this class of C^*-algebras (with real rank zero or at least with many projections) and the class of C^*-algebras A with the property that the ideal-lattice of A is linear ordered, are the only cases where we can show so-far that the properties (l.p.i.) and (s.p.i) coincide.

Then the permanence properties for the class of strong purely infinite C^*-algebras allows to define from this classes other classes of C^*-algebras where (l.p.i.) and (s.p.i) coincide.

But until now (Jan 2022) we have seen no example of a C^*-algebra A that is locally or weakly purely infinite but is not strongly purely infinite ... but, unfortunately, some proofs and most interesting applications require strong pure infiniteness. END OF BLUE DISPUT TEXT. ????

On the other hand, the algebra $\mathcal{O}_2 \otimes \mathbb{K} + \mathbb{C} \cdot 1$ (with unit $1 := 1_{\mathcal{O}_2} \otimes id_{\ell_2}$) has real rank zero and is purely infinite in the sense of the above mentioned definition of J. Cuntz [**172**, p. 186], but is even not locally p.i. in the weakest sense: the "local pure infiniteness" of Definition 2.0.3, because its unit element 1 is not *properly* infinite (inside the multiplier algebra $\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ of $\mathcal{O}_2 \otimes \mathbb{K}$. The algebra $\mathrm{C}_0((0,1], \mathcal{O}_2)$ is strongly p.i. in the sense of our Definition 1.2.2, but is not purely infinite in sense of the definition in [**172**], because it is stably projection-less.

Therefore, it is necessary to underline that we use or consider the original definition of J. Cuntz *only in case of simple* C^*-algebras A:

The Definition 1.2.2 of *strongly* purely infinite algebras is better applicable to non-simple C^*-algebras, and has nice functorial properties, e.g. passes to quotients, extensions (see Section 17) and to tensorproducts with exact C^*-algebras ([4]).

One can even work with the below given (possibly weaker) definitions, that are both equivalent to strong pure infiniteness in particular cases, e.g. where A has *real rank zero*, or is *simple*, or has a *Hausdorff* primitive ideal space $\mathrm{Prim}(A)$ *of finite dimension*, or where the lattice $\mathcal{I}(A)$ of closed ideals of A is *linearly ordered*.

What about examples or counter examples?

But here is good news for readers that feel disturbed by the "zoological garden" of definitions of pure infiniteness of non-simple C^*-algebras A and their still incomplete list of permanence properties:

If one tensors a C^*-algebra $A \neq \{0\}$ with the Jiang-Su algebra $\mathcal{Z}$ then all our definitions of pure infiniteness become the same, i.e., $A \otimes \mathcal{Z}$ is "locally purely infinite" in sense of Definition 2.0.3 if and only if $A \otimes \mathcal{Z}$ is *strongly purely infinite* as defined in Definition 1.2.2.

It implies immediately that $A \otimes \mathcal{O}_\infty$ is s.p.i. for every C^*-algebra $A \neq \{0\}$, because $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{Z}$.

[4] Despite we haven't seen so far a, – necessarily non-simple –, example of a weakly p.i. C^*-algebra that is not strongly p.i. It is related to the more general open question if non-zero elements have good approximate Glimm halving in non-simple residually anti-liminary C^*-algebras, cf. Section 7.

Give cite or ref for isomorphism $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{Z}$!

(Must show KK-equivalence ??? of $\mathcal{Z}$ to $\mathbb{C}$ or $\mathcal{O}_\infty$)

Moreover this is even the case if $A \otimes \mathcal{Z}$ has no non-trivial 2-quasi-trace. Here "trivial" means that the 2-quasi-trace takes only the values $\{0, +\infty\}$ on $(A \otimes \mathcal{Z})_+$.

(All quasi-traces on C*-algebras A are automatically 2-quasi-traces if A is both simple and exact ... ?

"Prime" C*-algebras with non-trivial l.s.c. 1-quasi-traces can exist. The full free group C*-algebra $C^*(F_\infty)$ has all sorts of quasi-traces, among them those that are not 2-quasi-traces.)

Moreover $A \otimes \mathcal{Z}$ is strongly purely infinite if each lower semi-continuous 2-quasi-trace $\tau \colon A_+ \to [0, \infty]$ takes only the values 0 and $+\infty$, i.e., if each τ is "trivial" in the following sense:

`Compare and choose from the kernel descriptions!!`

A lower semi-continuous 2-quasi-trace $\tau \colon A_+ \to [0, +\infty]$ is called *trivial* if τ takes only values in $\{0, +\infty\}$, and is defined by its closed kernel ideal $\tau^{-1}(0) =: J \subseteq A$.

"Trivial" lower semi-continuous 2-quasi-traces $\tau \colon A_+ \to \{0, +\infty\}$ correspond naturally to closed ideals J_τ defined by $(J_\tau)_+ := \tau^{-1}(0)$, and, conversely, τ_J is defined from a closed ideals $J \lhd A$ by $\tau_J(a) := 0$ if $a \in J_+$ and by $\tau_J(a) := +\infty$ for $a \in A_+ \setminus J$.

If A is an exact C^*-algebra, then $A \otimes \mathcal{Z}$ is s.p.i., if and only if, A_+ has no non-trivial additive trace.

Why we need here that A is exact? Has to do with the additivity of of 2-quasi-traces (are they natural extendable to $M_2(A)_+$ etc. ...) ???

Give exact citation !!!

Corollary 3.12 of the authors Abel Proc. paper – submitted in 2005! (or 2004?) – says moreover:

$A \otimes \mathcal{Z}$ is s.p.i. if every l.s.c. 2-quasi-trace on A_+ is trivial.

(It is (= seems to be) clear that $A \otimes \mathcal{Z}$ has no non-trivial 2-quasi-trace if A has no non-trivial 2-quasi-trace.

Thus, the point in the proof is: If $A \otimes \mathcal{Z}$ has no non-trivial 2-quasi-trace, then $A \otimes \mathcal{Z}$ is s.p.i.

(All s.p.i. C*-algebras have no traces ... A bit imprecise expressed:

The traces on B_+ a C*-algebra B can be extended to and calculated from the "open" projections in W*-algebra $(B \otimes \mathbb{K})^{**}$. The natural order and equivalence will be respected ... If they are approximated from below by stable projections then only the values 0 and $+\infty$ can appear.)

The point seems to be that we have to consider the corresponding "dimension functions" on the open projections. They must be trivial in sense that only 0 and $+\infty$ can appear.

Definition: Pedersen Ideal := The ideal of B that is generated by the elements $(b - \varepsilon \cdot 1)_+$ with $b \in B_+$ and $\varepsilon \in (0, \|b\|)$ and $1 \in \mathcal{M}(B)$.

Does this arguments use that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$?)

This has been proven by M. Rørdam, first in the special case of nuclear C^*-algebras A in [**690**, thm.5.2], – where one can use the result of U. Haagerup [**342**] that all 2-quasi-traces are additive on exact C^*-algebras – and later in [**690**, thm.5.?2?] for all A. We give a different proof in Section **??**, that is a modification of the authors original proof in ?? Abel proc...??? ... ??????

?????? Kirchberg ???

More important for our more special applications is that the tensor products $A \otimes \mathcal{O}_\infty$ of any non-zero C^*-algebra A with the Cuntz algebra $\mathcal{O}_\infty$ is strongly purely infinite (= s.p.i.) in sense of Definition 1.2.2.

Only in the very special case of separable *nuclear* C^*-algebras A we have always the strong conclusion: Separable nuclear C^*-algebras A are strongly purely infinite, if and only if, $A \cong A \otimes \mathcal{O}_\infty$. For general exact and separable C^*-algebras A this seems to be `Give reference to this isomorphism, it is likely in Chapter 11?`

EXAMPLE 2.0.2. `!!! Adjust my notation for` $\mathcal{R}$ `by notion` W`,`
`because M. Rørdam called it` W `in [467].`
`Then change it also on other places,` e.g. `in the Introduction Chapter 1`
`Also in the Introduction where it is mentioned!!`
`Where is written that this` A `has the property`
`(CFP) (= Corona Factorization property ) and where that`
$F(A)$ `has a character ?`
`Where is` $F(A)$ `defined ???? Give Ref's !!!`
`No-Where! But` W `and` $C^*_{red}(F_2)$ `have the properties`
`that` $F(W)$ `and` $F(C^*_{red}(F_2))$ `have characters,`
`It is not known if` $F(A)$ `has a character.`
`There is a claim that some conjectures related to (CFP) would be`
`destroyed if` $F(A)$ `has a character.`

Let $W := \mathcal{R}$ denote the unital simple separable nuclear C^*-algebra W with $\mathrm{K}_*(W) \cong \mathrm{K}_*(S^2 \times S^2 \times \cdots)$ constructed by M. Rørdam in [**687**] that is finite but is not stable finite, where S^2 denotes here the 2-dimensional unit sphere in $\mathbb{R}^3$.

Where is the existence of a character on $F(W)$ proven? Or a general theorem that says: $F(A)$ without character implies that A is not stably finite ...

The simple, separable, unital and *exact* C^*-algebra $A := C^*_{red}(F_2) \otimes \mathcal{R}$ is purely infinite by Theorem E. Hence, it is also strongly purely infinite by simplicity of this

A. But this C^*-algebra A does not absorb tensorial any amenable C^*-algebra $\neq \mathbb{C}$. ([5]).

Moreover, the algebras $B := C^*_{red}(F_2)$ and $C := \mathcal{R}$ ($= W$) have both the property that the unital C^*-algebra $F(B)$ and $F(C)$ have a character.

(Where are W and $\mathcal{R}$ defined, and described. In a paper of Rørdam. The W seems to be)

In case of $C^*_{red}(F_2)$ this follows from the classical observation that the commutant $vN(F_2)' \cap vN(F_2)^\omega$ of the von-Neumann algebra $vN(F_2)$ in its von-Neumann algebra ultrapower $vN(F_2)^\omega$ has a character, because it is simply $\mathbb{C} \cdot 1$.

Notice that $vN(F_2)^\omega$ is the adjoint of the Banach space $(vN(F_2)_*)_\omega$ (in our terminology). There is a natural unital C^*-morphism from $C^*_{red}(F_2)_\omega$ into the von-Neumann algebra $vN(F_2)^\omega$. It maps $C^*_{red}(F_2)' \cap C^*_{red}(F_2)_\omega$ into $vN(F_2)' \cap vN(F_2)^\omega$, because $\|[a,u]\| + \|[a,v]\| \geq \|[a,u]\|_2 + \|[a,v]\|_2$ and the unit ball of $C^*_{red}(F_2)$ is dense in the unit ball of $vN(F_2)$, – with respect to the ℓ_2-norm on $C^*_{red}(F_2) \subset \ell_2(F_2)$.

`This seems TO BE WRONG and has not been claimed in [467]!!:` A has perhaps the (then rather surprising) property that the central sequence algebra $F(A) := A' \cap A_\omega$ has a character, see [**467**, ex. 4.6].

NO? that above is still not proven! But is a question how near an operator $\sum_{g \in X} L_g \otimes c_g$ is to $1 \otimes \mathcal{R}$ (now denoted by $1 \otimes W$?) if it commutes almost with the generators of F_2 .

??? Next also still to check:
This example could show also that simple separable unital exact C^*-algebras A with the Corona Factorization Property (CFP)

Give reference to Def. of (CFP) !!!

exist ??? with the property that the invariant

$$F(A) := A' \cap A_\omega / \operatorname{Ann}(A, A_\omega)$$

has a character ???

Seems not clear if this example has a character!, All simple purely infinite exact C^*-algebras should have always the (CFP)? Or not?? Some answer ???

All σ-unital *simple* purely infinite C^*-algebras A are unital or stable.

Citation of Prop./Cor.?

This will be shown for separable A, but carries automatically over to σ-unital simple purely infinite C^*-algebras A, because they can be considered as an limit of an directed net $\{A_\tau\}$ of separable simple purely infinite C^*-subalgebras of A that contain a strictly positive element of A.

[5] It is not shown there that $A \not\cong A \otimes M_n$, but that follows for this A from the fact that there exist non-zero $[p] \in \mathrm{K}_0(A)$ with the property that there does not exist $[q] \in \mathrm{K}_0(A)$ with $[p] = n[q]$, i.e., $[p]$ is not "divisible" inside $\mathrm{K}_0(A)$ by some $n \in \mathbb{N}$.

If A is then stable, then $\mathcal{M}(A)/A$ is simple and purely infinite. This could/should imply the *corona factorization property* (CFP) ???

Above was not proven! Ref. to Def?

If a simple separable and exact C^*-algebra A has the property that $F(A)$ has *not* a character then one has always that (simple ?) A has property (CFP).

?? Above and below the same ??

But it is also known that if A is separable and $F(A)$ has no character then A has corona factorization property (CFP), cf. [**467**, thm. 4.3].

Our study starts with a variety of helpful weaker properties that are related to pure infiniteness but are often easier to verify than pure infiniteness itself, e.g. for C^*-bundles (i.e., C^*-algebras of bounded continuous sections in continuous fields of C^*-algebras), or for C^*-algebras with linearly ordered ideal lattice.

The property that an element $a \in A$ with $a \neq 0$ is "infinite" in A has been defined by J. Cuntz by the requirement that there exists non-zero element $b \in A$ such that

$$b \oplus a \precsim a\,.$$

See Definitions 2.0.3 **??** and **??**.

Among the almost "weakest" definition of pure infiniteness of C*-algebras is that of "local pure infiniteness":

Check blue text, refs and labels of l.p.i
Check/COMPARE other places of Def. ?? ??? with next blue.
Move equivalent formulations to places where they are used.

The local stability is perhaps a very rare property! Have to find other types of elements to work with.

DEFINITION 2.0.3. A non-zero C^*-algebra A is **locally purely infinite** (**l.p.i.**) if, for every non-zero $a \in A_+$ and pure state ρ on A with $\rho(a) > 0$, there exists a stable C^*-subalgebra $D \subseteq \overline{aAa}$ with $\rho(D) \neq \{0\}$.

Later the Lemma 2.7.15 says that we can replace in Definition 2.0.3 the "pure state" simply by "state", and get therefore a formally stronger definition than that formulated in Definition 2.0.3.

(That definition could be "incorrect" for a suitable version of "local pure infiniteness", because "infinite" is not necessarily "stable" ... Moreover, it should be called "locally stable" !!! But a stable C^*-algebra D is isomorphic to $D \otimes \mathbb{K}(\ell_2)$, i.e. $D \cong D \otimes \mathbb{K}$ and contains a σ-unital stable separable C^*-subalgebra $E \subseteq D$ with the property that there exists a non-zero epimorphism from $F := \mathrm{C}_0(0,1] \otimes \mathbb{K}$ onto E. But the diagonal compact operator $f := (1, 1/2, 1/4, \ldots, 2^{-n} \ldots)$ is properly infinite operator in $\mathbb{K}$, in the sense that there exist sequences of elements (a_n) and (b_n) such that

$$f \otimes 1_2 = \lim_n [a_n, b_n]^* [f a_n, f b_n]$$

in $M_2(\mathbb{K}) \cong \mathbb{K}$ This implies that F and E contain a strictly positive contraction that is properly infinite.

Or is it somewhere shown that for p.i. elements $a \in A_+$ the algebra $\overline{aAa}$ contains a non-zero stable C^*-subalgebra $D \subseteq \overline{aAa}$?

If a is "infinite", then (perhaps ??????) some element $e \in \overline{aAa}_+$ with $\|e\| = 1$ exists that is infinite in sense $(e-t)_+ \oplus (e-t)_+ \precsim (e-t)_+$ for suitable $t \in (0,1)$.

Perhaps ... this implies that $\overline{aAa}_+$ contains a (non-zero) stable C^*-algebra?

$\Leftarrow$ Would be important and interesting to know!

But stability $D \cong D \otimes \mathbb{K}$ of D implies natural pure infiniteness of each strictly positive elements $a \in D_+$, i.e. with $D = \overline{aDa}$.

Suppose that B is "purely infinite" ... (says that $b \oplus b \precsim b$ for all $b \in B_+$)

... or suppose something weaker ???, this could be that every element of B is "infinite" in the sense, that – for each non-zero $b \in B_+$ – there exits a non-zero $c_b \in B_+$ with $c_b \oplus b \precsim b$ ... ??? It is known that those elements c_b build the positive part $J(b)_+$ of some closed ideal $J(b)$ of B. And one knows that the element $\pi_{J(b)}(b) := b + J(b)$ of $B/J(b)$ is finite in $B/J(b)$ ... (Give References ! the strict finiteness of).

It implies: If every non-zero positive element of every quotient of B/J is infinite, then each element $b \in B_+$ is properly infinite, i.e. $b \oplus b \precsim b$.

Moreover, suppose that B is σ-unital, non-unital, and each (non-zero) quotient B/J has no unit element.

Is B then itself stable? (Stability Question: $B \otimes \mathbb{K} \cong B$?)

Look to Rordam paper ... ???

Ref.? Big question: to what kind of "purely infinite" C^*-algebras.!!)

" Zhang's Dichotomy ":

A σ-unital, purely infinite (!!!), simple C*-algebra is either unital or stable. (Reference/Cite ???).

What kind of definition of "pure infiniteness" is used for "Zhang's Dichotomy"?

[**688**, prop. 5.4] Rordam, Japan 2004 (or 2003),

Definition 6.10.? of this Rordam paper (defines Property: "regular")

A C*-algebra I is called "regular" if every full, hereditary sub-C*-algebra of I, that has no unital quotients and no bounded traces, is stable.

(E.K.: $I := \mathbb{K}$ – of $\ell_2(\mathbb{N})$ – is not regular ????) Every non-zero hereditary C*-subalgebra D of $\mathbb{K}$ is "full" in $\mathbb{K}$? because each nonzero element x of $\mathbb{K}_+$ generates $\mathbb{K}$ as the closed ideal of $\mathbb{K}$ that is generated by x. The D is given by a positive compact operator (by taking one of its strictly positive contractions in it). Alternatively it is given as $P\mathbb{K}P$ for some orthogonal projection P on $\ell_2(\mathbb{N})$. $P\mathbb{K}P$

has finite dimension, if and only if, $P\ell_2(\mathbb{N})$ is finite-dimensional (and has a faithful trace). Otherwise $P\mathbb{K}P$ is stable.

Thus, each $D := P\mathbb{K}(\ell_2)P$ satisfies the definition of a regular hereditary sub-C*-algebra of $\mathbb{K}(\ell_2)$ (given by M. Rørdam for all hereditary C*-subalgebras of $\mathbb{K}(\ell_2)$.

(Check this again !!!)

[**688**] Proposition 6.12. (of Rordam [**688**] ???) Let $0 -- > I -- > A -- > B -- > 0$ be a short exact sequence of separable C*-algebras and suppose that I is regular. Then A is stable, if and only if, I and B are stable.

Question 6.5. of Rordam [**688**]:

Does every (separable) C*-algebra A have a greatest stable ideal (i.e., a stable ideal that contains all other stable ideals)?

(Is a the closure of an upward directed family of stable ideals of A a stable ideal of A?)

Equivalently: Let A be the sum of two stable ideals $A = J + I$. Is A stable?

Is $\mathrm{C}_0(0,1]\otimes\mathbb{K}$ ($= C_0((0,1],\mathbb{K})$??) semi-projective ?.

`compare next blue with Lemma 2.7.15!`

If each stable C^*-subalgebra $D \subseteq \overline{aAa}$ satisfies $\rho(D) = \{0\}$, then ρ is zero on the hereditary C^*-subalgebra E of $\overline{aAa}$ that is generated by all stable elements in $\overline{aAa}$.

DEFINITION (cite/ref: where first def.?) of "stable" elements $b \in A_+$: $\overline{bAb}$ is stable, i.e. $\overline{bAb} \cong \overline{bAb} \otimes \mathbb{K}$.

Equivalently: $b \in A_+$ is a stable element, if and only if, there exists a C^*-homomorphism $\psi\colon \mathrm{C}_0(0,1]\otimes\mathbb{K} \to A$ such that $b = \psi(f_0 \otimes (t_1, t_2, \ldots))$, for $(t_1, t_2, \ldots) \in c_0 \subset \mathbb{K}$ with $1 \geq t_1 \geq t_2 \geq \cdots$, and all $t_n > 0$.

(First one has to find an isomorphism $\overline{bAb} \cong \overline{bAb} \otimes \mathbb{K}$ and then ???? use ???? that b is there approximately equivalent to $b \otimes c_0$, (or simply replace b by $b \otimes c_0$ ...).

But E is invariant under automorphisms of D, in particular E is a closed ideal of D, i.e., in general the hereditary C^*-subalgebra E of D generated by the convex combinations of all positive "stable" elements in D is an ideal of D. The E must be equal to D if A is locally purely infinite in sense of Definition 2.0.3.

If a hereditary C^*-subalgebra E of D is invariant under conjugation by certain unitary elements in the multiplier algebra $\mathcal{M}(D)$ ... ?

It follows that ??????

The following Definition is equivalent to Definition [**93**, def. 1.2].

DEFINITION 2.0.4. We call a C^*-algebra A **n-purely infinite** (or: A is **pi(n)**) for some fixed $n \in \mathbb{N}$ if A satisfies that

(i) for every nonzero $a \in A_+$, positive b in the closed ideal $J(a)$ of A generated by a, and every $\varepsilon > 0$, there exists $d_1, \ldots, d_n \in A$ such that

$$\| b - \sum_{1 \leq k \leq n} d_k^* a d_k \| < \varepsilon ,$$

and

(ii) every quotient C^*-algebra $\neq 0$ of $\ell_\infty(A)$ has (linear) dimension $> n^2$.

The Part (ii) says here that $\ell_\infty(A)$ has no irreducible *-representation on a Hilbert space $\mathbb{H}$ of dimension $k \leq n$.

Why we use here in Part (ii) of Definition 2.0.4 the huge $\ell_\infty(A)$ in place of A? Is there an example where (ii) for A (in place of $\ell_\infty(A)$) is not enough?

Is not clear if we can prove that $\ell_\infty(A)$ has no irreducible representation on a Hilbert space of dimension $k \leq n$ if A has no irreducible representation on a Hilbert space of Dimension $k \leq n$ ...

Use projectivity of $M_k(\mathrm{C}_0(0,1])$ ...

But there are more general and precise observations needed ...

... because this "proof" gives only a sequence of (needed) positive functions

$$f_m \colon (0,1] \to (0,1]$$

with $f_m(1/m, 1] = 1$ and $f_{m+1} f_m = f_m$, and that have the property that a convex combination a_k of $n+1$-homogenous contractions in $\ell_\infty(A)$ exist that are ... (!!! check hand-written papers for an idea of proof ... but seems not complete ...).

If we can use only the following weaker requirement in Part (ii*) of Definition 2.0.5 in place of Part (ii) of Definition 2.0.4 then it is not clear if the two Definitions 2.0.4 and 2.0.5 are really equivalent, or if Definition 2.0.4 is strictly stronger than Definition 2.0.5 ...

Definition 2.0.5. Let A a C*-algebra. Then A is called (n-step) "locally purely infinite" (or "A is pi(n)") if

(i*) Each element $a \in A_+$ is n-step infinite in the sense:
For each elements $c_1, \ldots, c_{n+1} \in A$ and $\varepsilon > 0$ there exist $d_1, \ldots, d_n \in A$ such that

$$\| \sum_{1 \leq j \leq n+1} c_j^* a c_j - \sum_{1 \leq k \leq n} d_k^* a d_k \| < \varepsilon ,$$

and

(ii*) A has no irreducible representation on a Hilbert space $\mathbb{H}$ of dimension $k \leq n$.

We say that A is **weakly purely infinite** (for short **w.p.i.**) if A is pi(n) for some $n \in \mathbb{N} = \{1, 2, \ldots\}$.

The property (i*) in Definition 2.0.5 implies that for a C*-algebra A with this property implies that the property (ii*) is equivalent to each of the following properties (ii*,1)–(ii*,4?) for its elements and irreducible representations:

(ii*,1) No hereditary C*-subalgebra D of A has a (non-zero) character.

(ii*,2) No hereditary C*-subalgebra D of A has a (non-zero) finite-dimensional quotient D/J. (Here one can take a closed ideal I of A and consider $J := I \cap D$, because for each closed ideal J of a hereditary C^*-subalgebra D of A the closed ideal J of A generated by I, i.e., the closure I of the linear span of $A \cdot J \cdot A$, satisfies $J = I \cap D$, because $DAJAD \subseteq I$.)

(ii*,3) No irreducible representation $\rho(A)$ of A contains a (non-zero) compact operator in its image.

(ii*,4) No closed ideal J of A has an irreducible representation $\rho\colon J \mapsto \mathcal{L}(\mathcal{H})$ with $\rho(J) = \mathbb{K}(\mathcal{H})$.

It is not difficult to see that the Part (i) of Definition 2.0.4 and

??????

are equivalent to Part (i*) of Definition 2.0.5.

It follows that A is *purely infinite* in the sense of Definition 1.2.1, if and only if, A is pi(1) in the sense of Definition 2.0.4, but pi(1) follows from the formally weaker requirements (i) for $n := 1$ and (ii*) that A has no character (instead to require moreover that $\ell_\infty(A)$ has no character !).

The conjecture (!) is that the same holds for the requirement (ii) for pi(n), i.e., that it is enough to require that

(ii*) A (itself) has no irreducible representation of dimension $\leq n$.

But until now (2022) no general argument for a proof of this has been found. But properties (i) and (ii*) imply together that all (non-zero) n-homogenous elements of A are properly infinite, cf. Proposition ??? **??**. And this implies that all non-zero n-homogenous elements of $\ell_\infty(A)$ and of A_ω are properly infinite...

But this argument does not exclude the existence of irreducible representations of dimension $< n$ on A_ω or $\mathcal{M}(A)$... ??? Examples ?? for this phenomenon ??

Part (ii) says that $\ell_\infty(A)$ has no irreducible representation $d\colon \ell_\infty(A) \to \mathcal{L}(\mathcal{H})$ with $\mathrm{Dim}(\mathcal{H}) \leq n$. Since $\mathcal{M}(A)$ is a unital C^*-subalgebra of $\ell_\infty(\mathcal{M}(A))$ and $\ell_\infty(A)$ is an essential ideal of $\ell_\infty(\mathcal{M}(A))$, it would be stronger to require that $\mathcal{M}(A)$ has no irreducible representation of dimension $\leq n$. It is not known, even in the case where A is separable, if $\mathcal{M}(A)$ can have an irreducible representation of dimension $\leq n$, but its sequence algebra $\ell_\infty(A)$ has no irreducible representation of dimension $\leq n$. (But $\ell_\infty(\mathcal{M}(A))$ has then a C^*-epimorphism onto $\ell_\infty(M_k)$ for some $k \leq n$.)

It has to do with the more general question if $\ell_\infty(A)$ is again residual antiliminary if a C^*-algebra A is residually antiliminary, cf. the discussions below Definition 2.7.2 of residual antiliminary C^*-algebras.

The Definition 2.0.4 is an equivalent formulation of [**93**, def. 1.2]. It was used in [**93**] for some results on pure infiniteness of C^*-bundles (:= algebras of continuous sections vanishing at ∞ in continuous fields of C^*-algebra).

An alternative description of weak pure infiniteness is given in the following definition of a perhaps stronger property "pi-n" defined in [**463**, def. 4.3].

`This is mentioned below a second time (with footnote?). Compare it and remove one of them:`

The pi(n) C^*-algebras have the formally stronger property pi-m of Definition 2.0.6 for suitable (but still unknown) $m \in \mathbb{N}$ with $n \leq m$.

DEFINITION 2.0.6. We say that a C^*-algebra A has **property pi-n** (or: A is pi-n) if, for each non-zero $a \in A$, the n-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$, if considered as a diagonal matrix in $M_n(A)$.

If A has the property of Definition **??** then no (non-zero) hereditary C^*-subalgebra D of A has a character and any n-homogenous positive contraction $a \in A$ is properly infinite in A.

The Definition 1.2.1 and the above given Definitions 2.0.3, 2.0.4, and **??** of local or weak pure infiniteness coincide in case of simple C^*-algebras, cf. Proposition 2.2.1.

In Section 12 of this chapter it is shown that property pi-n implies property pi(n) on non-simple A.

Conversely, if A has property pi(m) for some m then there exists $n \geq m$ such that A has property pi-n, but no general function $f\colon \mathbb{N} \to \mathbb{N}$ is known yet (Febr.2022) with the property that $n \leq f(m)$. Moreover, there exists, for every factorial state ρ on A, a closed ideal J of A with the properties

(i) $\rho|J \neq 0$,
(ii) J is "essential" in A, i.e., $a \in A$ and $a \cdot J = \{0\}$ implies $a = 0$, and
(iii) the ideal J has property pi-n (if A has property pi(n)).

Moreover, if A has property pi(m) then there exists $n \geq m$ such that A has property pi-n. But until now (Febr 2022 ?) no estimating function $f(m) \in \mathbb{N}$ is known with the property that A is pi-$f(m)$ and $f(m) \geq m$ if A has property pi(m). It seems that such estimating functions $m \to f(m)$ depend from topological properties of Prim(A).

We give an overview about the known results and open questions concerning non-simple (locally, weakly or strongly) purely infinite algebras in Section 11 of this chapter – with some parts of the needed proofs postponed to later sections and chapters, because there are then the needed observations available for the reader.

By Definition 1.2.1 in the Introduction Chapter 1, a C^*-algebra A is *purely infinite*, if and only if, A has a property that is formally weaker than the Property pi(1) of Definition 2.0.4: But it suffices to require that A has no non-zero character

– instead to require the additional property that $\ell_\infty(A)$ has no character as it is extra required in Definition 2.0.4.

We give later in Proposition **??** a list of order between infiniteness.

In fact, – without (!) some sort of infiniteness –, there *could be* examples of separable (non-simple?) C^*-algebras A that have no irreducible representations of finite dimension but that $\ell_\infty(A)$ has a non-zero character. Then $\mathcal{M}(A)/A$ has a character ??? and, for all $b \in A$, the sequence $(b, b, \ldots) \in \ell_\infty(A)$ is in the kernel ideal of this representation.

It is related to questions about the ultrapowers of states on A and finite dimensional quotients of $\mathcal{M}(A)/A$.

`What about` $A := M_{2^\infty} \otimes \mathbb{K}$`?`
`Why` $\mathcal{M}(A)/A$ `should have a "character"?` Reference ?? The question if this A has no character is equivalent to the questions: Has $\mathcal{M}(A)$ a character ?? Has $\mathcal{M}(A)$ a trace?

It turns out that A is purely infinite in sense of Definition 1.2.1, if and only if, each non-zero element $a \in A$ is properly infinite in the sense of Definition 2.1.1. But this says that A has Property pi-1 of Definition 2.0.6. See Corollary 2.5.6 for a proof of this equivalences based on a more engaged discussion of the variety of notions of “infinity” of elements and of pure infiniteness of C^*-algebras. It is still not known (2022 ?) what happens in case $n = 2$, e.g. even in case of $\mathrm{C}([0,1], A)$ for non-simple properly infinite C^*-algebras A. Up to some special cases, it is not known (Jan 2022 ?) if Property pi-1 implies that A is *strongly purely infinite* in sense of Definition 1.2.2, except in some very particular cases, e.g. if A contains “many projections” in a sense perhaps equivalent to A having “real rank zero”, or where the lattice of closed ideals of A is linear ordered ...

Recall that a non-zero element $a \in A_+$ is called *n-homogenous* if there exists a (non-zero) C*-algebra morphism $\psi : \mathrm{C}_0((0,1]) \otimes M_n \mapsto A$ with $a = \psi(f_0 \otimes 1_n)$, where $f_0(t) := t \in (0,1]$. With other words: There exists a non-zero positive contraction $b \in A_+$ (e.g. with $b := \psi(f_0 \otimes e_{1,1})$ and elements $c_2, \ldots, c_n \in A$ with $c_j^* c_j = b$, $c_j^* b = 0$ and $c_j^* c_k = 0$ for $j \neq k$, $j, k \in \{2, \ldots, n\}$. Then the n-homogenous $a \in A$ becomes $a = b + c_2(c_2^*) + \cdots + c_n(c_n^*)$.

The comparison between the definitions should be reached with the formulations with help of the J. Cuntz semigroup and by absence of traces ! No (non-zero) hereditary quasi-traces

Let us finish all this possible ideas concerning pure infiniteness of C*-algebras A by the following (until here weakest) definition of pure infiniteness:

We say that a C^*-algebra A has **property pi-n** (or: A is pi-n) if, for each non-zero $a \in A$, the n-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$, if considered as a diagonal matrix in $M_n(A)$.

There is a relation between the two definitions:

Above pi-n says that $a \otimes 1_n$ is s.p.i. in $M_n(A)$ for all $a \in A$. It is not obvious that this implies that $a = b + c_2(c_2^*) + \cdots + c_n(c_n^*)$ (like in Definition 2.0.7) is properly infinite. But the diagonal matrices with diagonals $(a, 0, 0, ..., 0)$, $(b, c_2c_2^*, ..., c_n(c_n^*))$ and $(b, b, ..., b)$ are equivalent in $M_n(A)$. The diagonal matrix $(b, b, ..., b) = b \oplus b \oplus \cdots \oplus b$ is p.i. in $M_n(A)$ if A has the property pi-n of Definition 2.0.6. This matrix is MvN-equivalent to $(a, 0, ..., 0) = a \oplus 0 \oplus \cdots \oplus 0$ in $M_n(A)$. Thus, the Property pi-n of Definition 2.0.6 implies the formally weaker Property "pi-n" of Definition 2.0.7.

It is now the question: What happens with the others: pi-n, pi(n), ...?

Definition 2.0.7. We say that a C^*-algebra A is "*locally n-purely infinite*" if

(i) No (non-zero) hereditary C^*-subalgebra D of A has a (non-zero) character.
(ii) Each (non-zero) n-homogenous positive contraction $a \in A$, $a := b + c_2(c_2^*) + \cdots + c_n(c_n^*)$ with $c_k^* c_j = \delta_{j,k} b$, has the property that $a \oplus c \precsim a$ for each $c \in J_+$, where $J \subseteq A$ is the closed ideal generated by b (that contains the element a).

In particular Part(ii) implies that any n-homogenous positive contraction $a \in A$ is properly infinite in A. And Part(i) is equivalent to the property that no irreducible representation of A has a non-zero compact operator in its image.

Below part of old text ?? Begin of change?

property pi-n (or: A is pi-n) if, for each non-zero $a \in A$, the n-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$.

We define in the next section a formally (much?) weaker property of pure infiniteness on C^*-algebras in Definition **??**. weak.p.i. ? Each ?????

1. Infinite elements, factorization and transitivity

This section collects basic definitions concerning infiniteness and some needed technical lemmata that are interesting in itself and play a role in several poofs in the other chapters. But some of those proofs will be postponed and then given or outlined in the Appendices A and B, simply to run not too far away from the main topic of this chapter: A small portion of many ideas for non-commutative versions of “infiniteness”.

We recall first our later often used definitions of *infinite*, *properly infinite*, *stable*, *n-infinite*, *n-properly infinite* and *n-homogenous* positive elements of a C^*-algebra A, and some equivalents of it. Notice that we have changed and refined them for our applications, because those given by other authors have only applications to very special types of C^*-algebras. See e.g. some remarks below and in other chapters on this differences.

New way:

All the variants of infiniteness imply the last n-homogenous variant.

(THAT is not clear now! And, if it is so, then the reachability in n-steps shows only that all $(n+1)$-homogenous elements are properly infinite. It requires the additional assumption that no hereditary C*-subalgebra D of A has a character.

But it is, in case of non-simple separable C*-algebras A, not enough to show that all non-zero elements $a \in A_+$ are infinite – but for each pure state ρ on $D := \overline{aAa}$ there exists an in D non-zero "infinite" element $d \in D_+$ with $\rho(d) > 0$.)

It is important to consider those definitions that carry over to suitable separable C*-sub-algebras of A ...

The minimal requirement should be that for every hereditary C*-subalgebra $D \subseteq A$ and every pure state ρ on D there exists an infinite positive contraction $d \in D$ with $\rho(d) > 2/3$. "Infinite" means here that there exists non-zero $e \in D_+$ with $e \oplus d \precsim_D d$ ...

No!!! We must require that the ideal J of D generated by the elements $e \in D_+$ with the property $e \oplus d \precsim_D d$ contains d, i.e., $d \oplus d \precsim_D d$ and d is properly infinite. This means that we have to find $e \in D_+$ with $e \oplus d \precsim_D d$ and $\rho(e) > 0$.

How is this related to:

1) No hereditary C*-subalgebra $D \subseteq A$ has a character

(i.e. equivalently (!): no irreducible representation ρ of A contains the compact operators in its image.),

and

2) There exists (general and fixed) $n \in N$ with the property that, for each $a \in A_+$, $b \in A_+$ with b in the closed ideal J_a of A generated by a, and $\varepsilon \in (0, \|a\|)$, there exists $d_1, \ldots, d_n \in A$ with $\sum_{k=1}^{n} d_k^* a d_k = (b - \varepsilon)_+$.

(Can we here replace this by $n := f(m)$ for a suitable function f and the requirement that $a \otimes 1_m$ is properly infinite in $M_m(A)$?)

!!! But it is not clear if we must require (below, later or here) that there exists suitable $e \in D_+$ with $\rho(e) > 1/2$ and $e \oplus d \precsim_D d$. !!!

If we require here that there exists $e \in D_+$ with $\rho(e) \neq 0$ and $e \oplus d \precsim_D d$, then this would show that d itself must satisfy $d \oplus d \precsim_D d$, i.e. that d is p.i. (and not only infinite).

The point in the case of separable C*-algebras A is that in each non-zero hereditary C*-subalgebra D of a quotient A/J there exists a properly infinite element in D. This generates "maximal" properly infinite elements in the quotients hereditary C*-sub-algebras of A.

If one has then an extension theorem (that is yet not available) for (rather special) extensions of separable C*-algebras with p.i. strictly positive elements. Then one can try to find out if the extension has also a p.i. strictly positive element.

The problem is then to show that every separable subset X of A is contained in a separable C*-sub-algebra B of A, that is again suitably generated by infinite elements ..., has again this (given) local pure infiniteness ... :

In particular it says: $c \oplus b \precsim_B b$ if $b, c \in B \subseteq A$ and $c \oplus b \precsim_A b$.

Requirements:

(1) No non-zero hereditary C*-sub-algebra D of A has a character. (This is equal to the property that for every irreducible representation ϕ of A does not intersect the algebra of compact operators.)

(2) if D is a non-zero σ-unital hereditary C*-sub-algebra of A, and $J \neq A$ a closed ideal of A, such that D is not contained in J, then (the requirement is that) $E := D/(D \cap J) \subseteq A/J$ contains an infinite element, i.e. there exists non-zero elements $f, g \in E_+$ with $g \oplus f \precsim_E f$.

But we need the stronger property: For every $a \in A_+$ and every closed ideal J of A with $a \notin J$ there exists a non-zero element $b \in (A/J)_+$ with $b \oplus \pi_J(a) \precsim_{A/J} \pi_J(a)$.

It says a is contained in the closed ideal J_a of A defined by: $e \in J_a$, if and only if, $e \oplus a \precsim_A a$. The element $\pi_{J_a}(a)$ is always finite in A/J_a.

Thus $a \in A$ is properly infinite

(or is "purely infinite" ??)

in A, i.e., $a \oplus a \precsim_A a$) if and only if, for every closed ideal J of A $\pi_J(a)$ is infinite in A/J or $a \in J$ (i.e., that $\pi_J(a) = 0$).

(This relation in $M_2(E)$ for $E := D/(D \cap J)$ is here the same as the relation $g \oplus f \precsim_{A/J} f$ in $M_2(A/J)$. One should require more precise that ?????

The point is:

So far, we can all reduce to the separable case (– depending much from the chosen definition of "locally infinite" elements! –), then we can reduce our study to separable A, and then for given $d \in A_+$ and let $D := \overline{dAd}$ there exists a "maximal" closed ideal $J := \overline{eAe} = \overline{eDe}$ of D with the property that $e \in D_+$ is

(1.) strictly positive in J and

(2.) e is properly infinite in D,

This maximality comes in the case of separable D from the fact that there exists an extremal closed ideal J of D such that J the strictly positive elements $e \in J$ of J are all properly infinite, i.e. $e \oplus e \precsim_J e$.

This existing ($\approx$-) maximal p.i. contraction $e \in D_+$ should turn out (by a suitable proof) to be strictly positive in D, i.e. $J = \overline{eAe}$ and $e \oplus e \precsim_J e$.

The proof is likely indirect:

(1.) We have to show that for each closed ideal of J of D with $D \neq J$ that each non-zero σ-unital hereditary sub-algebra E of D/J exists a (non-zero) infinite element $e \in E_+$.

that is ????

We could pass to $F := \{a \in D; \pi_J(a) \in E\}$. Then $J \subseteq F \subseteq D$ and F is hereditary in D and in A. Moreover, $J \subseteq F$ is a closed ideal of F.

If J is σ-unital then F is again σ-unital (as D by definition).

Since one can all reduce to the separable case (by preserving $\precsim$ with some care !) we can always suppose and use that all is σ-unital.

It is necessary and sufficient to consider the case of hereditary C*-sub-algebras D of separable C*-algebras A ... Each closed ideal J of D is the intersection $J = I \cap D$ of a closed ideal I of A with D.

This implies that ?????.

and D/J contains again a properly infinite element $f \in D/J$ ... ...)

All can be reduced to a suitable separable C*-sub-algebra B of A that contains a given separable subspace X of A and satisfies that $b_1 \precsim_{M_n(B)} b_2$ for $b_1, b_2 in M_n(B)$ if and only if $b_1 \precsim_{M_n(A)} b_2$.

(THAT IS NOT obvious ...)

All are compatible with inductive limits (of this separable subspaces).

In this sense:

It is likely that many p.i.-elements appear in the separable case:

There, in hereditary $D \subseteq A$, are (inside D) extremal (!) positive p.i. elements $d \in D_+$, in the sense that for every p.i. element $e \in D_+$ with $d \in \overline{eDe}$ holds that $e \precsim_D d$ (and automatically $d \precsim_D e$). The separability of A (and then that of D) secure that the closed ideal $J \supseteq D$ of D generated by d has the property that a strictly positive contraction of $f \in J_+$ has the property that $d \approx f$. It implies that there are elements $a_n, b_n \in A$ with $a_n^* d a_n = (f - 2^{-n})_+$ (by proper infiniteness of d) and $b_n^* f b_n = (d - 2^{-n})_+$. It shows that for the corresponding open projections $p, q \in A^{**}$ that define D and J are equal, i.e. we get that $D = J$.

(In fact then $J := \overline{dDd}$ is for such $d \in D_+$ is always an ideal of D by the maximality and this could give then also uniqueness! ???).

(NEXT: try to use this to show that "p.i." implies "s.p.i." ??? Find it in the appendix ??? !!!!!!)

SOME TECHNICAL CONSIDERATIONS:

Let $a, b \in A_+$ with $\|a\| \leq 1$ and $\|b\| \leq 1$, then straight calculation shows that

$$2(a - (a - b)_+) = (a + b) - \sqrt{(a - b)^2}.$$

Notice here that $\sqrt{(a - b)^2} = |a - b|$.

$a - (a - b)_+$ and $b - (b - a)_+$ have norms $\leq 2\|ab\|$

Pedersen book: Proposition 1.3.8.: If $0 < \beta \leq 1$ the function $t \mapsto t^\beta$ is operator monotone for each $t \in [0, \infty)$.

I.e. in a C*-algebra A, then $a, b \in A_+$, $0 < \beta \leq 1$, $a \leq b$ implies $a^\beta \leq b^\beta$.

(we need case $\beta = 1/2$).

Does it follow that $\|b^\beta - a^\beta\| \leq \|b - a\|^\beta$, at least for $\beta = 1/2$?

(cite here Pedersen book ...?)

The relevant formula for the needed decompositions by the estimate for contractions $a, b \in A_+$ is given by some estimate like (?? upper next is not really checked !)

$$\|(a + b) - |a - b|\| \leq 2\|ab\|,$$

or $\leq (2\|ab\|)^{1/2}$ and the (correct !) equation:

$$2(a - (a - b)_+) = (a + b) - \sqrt{(a - b)^2}.$$

Notice here that $\sqrt{(a - b)^2} = |a - b|$.

Needs study of operator-monotone function:

Something like $\|x - y\| \leq \|x^2 - y^2\|^{1/2}$ for $x, y \in A_+$??? or equivalently $\|a^{1/2} - b^{1/2}\| \leq \|a - b\|^{1/2}$ for $a, b \in A_+$ by operator-monotony of the map $a \in A_+ \mapsto a^{1/2}$.

SEE APPENDIX "C" !!! for more.

DEFINITION 2.1.1. An element $a \in A_+$ is **infinite** if there exists *non-zero* $b \in A_+$ such that, for every $\varepsilon > 0$, there exist $d_1, d_2 \in A$ (depending on ε) with

$$\|d_1^* a d_1 - a\| < \varepsilon, \quad \|d_2^* a d_2 - b\| < \varepsilon \quad \text{and} \quad \|d_1^* a d_2\| < \varepsilon\,. \tag{1.1}$$

Expressed by use of the Cuntz semigroup as: $a \oplus b \precsim_A a$ in $\mathcal{C}\sqcap(A)$.

A non-positive element a in $\mathrm{Ped}(A)$ is called "infinite" if $a^* a$ is infinite.

The element $a \in A_+$ is **properly infinite** if we can take $b := a$ in the second of the Inequalities (1.1).

Thus $a \in A$ and $a \oplus a \precsim_A a$ says that a is a properly infinite element of A.

We call a non-zero element $a \in A$ **stable** if the σ-unital hereditary C^*-subalgebra $\overline{a^* A a}$ is a stable C^*-subalgebra of A.

Notice here that $\overline{a^* A a}$ is stable, if and only if, $\overline{a A a^*}$ is stable. Following Lemma shows that Property pi-n of Definition **??** implies Property pi(n) of Definition 2.0.4.

LEMMA 2.1.2. *Let A denote a C^*-algebra and let $a \in A$ a non-zero element in A. Suppose that there exist $\ell \in \mathbb{N}$ sequences of elements $D_\ell, E_\ell \in M_{n+1}(A)$ with*

$$\lim_{\ell \to \infty} D_\ell \,\mathrm{diag}(a, \ldots, a, 0) E_\ell = \mathrm{diag}(a, \ldots, a, a)\,. \tag{1.2}$$

Then $1_n \otimes a$ is properly infinite in $M_n(A) \cong M_n \otimes A$, and, for each element $b \in J(a)$ in the closed ideal $J(a)$ of A generated by a and $\varepsilon > 0$, there exists $f_1, \ldots, f_n, g_1, \ldots, g_n$ with $\|b - \sum_{k=1}^n f_k a g_k\| < \varepsilon$. If a and b are positive then one can find here the elements $f_k, g_k \in A$ with $f_k = g_k^*$.

Let $a, b_1, b_2, \ldots \in A_+$ *contractions with the properties that* $(a-\varepsilon)_{\otimes}1_n$ *and* $(b_k - \varepsilon)_+ \otimes 1_n$ *are properly infinite in* $M_n(A)$ *or zero for each* $\varepsilon \in (0,1]$*.*

Then $b := (b_1, b_2, \ldots) \in \ell_\infty(A)$*, (respectively* $b := (a, a, \ldots)$*) has the property that* $h(b) \notin \mathbb{K}(\mathcal{H})$ *for every irreducible representation* $h \colon \ell_\infty(A) \to \mathcal{L}(\mathcal{H})$*, i.e.,* $h(b)$ *is not compact if* $h(b) \neq 0$*.*

In particular, if $a \otimes 1_n$ *is properly infinite in* $M_n(A)$ *for each non-zero element* $a \in A_+$*, then* $\ell_\infty(A)$ *has no irreducible representations that contains compact operators in its image.*

Especially $a \in A$*, respectively* $(a, a, \ldots)$ *and* $(b_1, b_2, \ldots)$ *in* $\ell_\infty(A)$ *are contained in the kernel of all irreducible representations of finite dimension of* A*, respectively of* $\ell_\infty(A)$ *.*

PROOF. The equations 1.2 say that $1_{n+1} \otimes a \precsim 1_n \otimes a$ in the $\precsim$-terminology of J. Cuntz, or, even more abstractly expressed, that $(n+1)[a] \leq n[a]$, in the pre-ordered "large" Cuntz semi-group $\mathrm{Cu}(A)$, cf. Section 5 in Appendix A, our definitions in Section

section 2.infinitesimal **??**

and Definition 2.5.1.

We can repeat the approximation by suitable choice of *new* operators $D_{\ell,m}, E_{\ell,m} \in M_{n+m}(A)$ beginning with $D_{\ell,1} := D_\ell$, $E_{\ell,1} := E_\ell$ and the suitable products $D_{\ell,m} := \mathrm{diag}(1_m, D_\ell) \in M_{m+k}(A)$ and $E_{\ell,m} := \mathrm{diag}(1_m, D_\ell) \in M_{m+k}(A)$ to get that $1_{n+m+1} \otimes a$ is the limit of $D_{\ell,m}((1_{n+m} \otimes a) \oplus 0)E_{\ell,m}$ for $\ell \to \infty$.

The following uses that generally $\mathrm{diag}(h(b), \ldots, h(b)) \cong (h \otimes \mathrm{id}_n)(b \otimes 1_n)$ is properly infinite in $C \otimes M_n$ if $h(b) \neq 0$ and $b \otimes 1_n$ is properly infinite in $B \otimes M_n$, for each given C^*-morphism $h \colon B \to C$.

Notice here that positive compact operators $c \in \mathbb{K}(\mathcal{H})_+ \subseteq \mathcal{L}(\mathcal{H})$ have the property that $((c \otimes 1_n) - \varepsilon)_+ = (c - \varepsilon)_+ \otimes 1_n$ is finite or zero for each $\varepsilon > 0$, – because they have finite rank in $\mathbb{K}(\mathcal{H}) \otimes M_n \cong \mathbb{K}(\mathcal{H} \otimes \ell_2(n))$. In particular, $c = 0$ if $c \geq 0$, $c \otimes 1_n \in \mathbb{K}(\mathcal{H}) \otimes M_n$ and $((c \otimes 1_n) - \varepsilon)_+ = (c - \varepsilon)_+ \otimes 1_n$ is properly infinite or zero for each $\varepsilon > 0$.

Let $B := \ell_\infty(A)$ and $b := (b_1, b_2, \ldots)$ or $b := (a, a, \ldots)$ in B_+, respectively $B := A$ and $b := a \in A$, with the property that $(b - \varepsilon)_+ \otimes 1_n$ is zero or properly infinite in $B \otimes M_n$.

Suppose that there exists irreducible representation $h \colon B \to \mathcal{L}(\mathcal{H})$ such that $h(b)$ is a non-zero compact operator on $\mathcal{H}$.

Let $h \colon B \to \mathcal{L}(\mathcal{H})$ an irreducible representation, and suppose that $h(b)$ is compact, i.e., $h(b) \in \mathbb{K}(\mathcal{H})$. Then, for $\iota := \mathrm{id}_{M_n}$, $(h \otimes \iota)(b \otimes 1_n) \in \mathbb{K}(\mathcal{H}) \otimes 1_n \subseteq \mathbb{K}(\mathcal{H} \otimes_2 \ell_2(n))$ and $(h \otimes \iota)((b - \varepsilon)_+ \otimes 1_n) = ((h(b) \otimes 1_n) - \varepsilon)_+$. Thus, $(h \otimes \iota)((b - \varepsilon)_+ \otimes 1_n)$ has non-zero finite rank in the Hilbert space $\mathcal{H} \otimes_2 \ell_2(n)$ for $\varepsilon \in (0, \|b\|)$, or $((h(b) \otimes 1_n) - \varepsilon)_+ = 0$. But, by assumption, $(b - \varepsilon)_+ \otimes 1_n$ is properly infinite if $\varepsilon \in (0, \|b\|)$. It follows that $((h(b)\otimes 1_n) - \varepsilon)_+$ is properly infinite or zero, and that $((h(b)\otimes 1_n) - \varepsilon)_+$ has finite rank. Thus, $((h(b)\otimes 1_n) - \varepsilon)_+ = 0$ for every $\varepsilon > 0$, i.e., $h(b) = 0$. It says $h(b) = 0$ if $h(b) \in \mathbb{K}(\mathcal{H})$.

Thus, the images of irreducible representations of A and $\ell_\infty(A)$ can not contain compact operators. In particular, they have no irreducible representations that contain non-zero compact images in its images. □

If we use Lemma 2.1.9, then we can see that an element $a \in A_+$ is *infinite* if there exists non-zero $b \in A_+$ with the property that for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ and $d := d(a, b, \varepsilon, \delta) \in M_2(A)$ such that

$$d^*\big((a-\delta)_+ \oplus 0\big)d \;=\; (a-\varepsilon)_+ \oplus (b-\varepsilon)_+\,.$$

Clearly the element $a \in A_+$ is *properly infinite* if we find for each $\varepsilon > 0$ some $\delta > 0$ and $d := d(a, \varepsilon) \in M_2(A)$ such that

$$d^*\big((a-\delta)_+ \oplus 0\big)d \;=\; (a-\varepsilon)_+ \oplus (a-\varepsilon)_+\,.$$

It is not difficult to obtain from Definition 2.1.1 that positive $a \in A_+$ is *properly* infinite *inside* the hereditary C^*-subalgebra $D := \overline{aAa}$ of A if and only if a is properly infinite in A, cf. the arguments in the proof of Proposition 2.2.5(i).

It implies that every stable element is properly infinite. But (non-zero) properly infinite *projections* are obviously not stable.

Sometimes it is useful to allow also $a = 0$ to be "properly infinite" (respectively "stable") in the formulation of some results, but notice that 0 is not an infinite element by *our* Definition 2.1.1.

But the Definition 2.1.1 of (non-properly infinite but) infinite elements allows to conclude only that a is infinite in A if (and only if) a^*a is infinite inside the hereditary C^*-subalgebra $D := \overline{a^*Aa} = \overline{(a^*a)A(a^*a)}$. Therefore $a \in A$ is infinite in A if and only a is infinite inside the closed ideal $J(a) := \overline{\mathrm{span}(AaA)}$ of A generated by a, cf. the "compression" arguments in the proof of Proposition 2.2.5(i).

By Lemma 2.5.3(v), a non-zero $a \in A_+$ is properly infinite if (and only if) $\pi_J(a)$ is zero or infinite in A/J for every closed ideal J of A. It implies e.g. that all non-zero elements in A_+ are properly infinite in A if A is purely infinite in the sense of Definition 1.2.1, which is, almost verbatim, equal to the below considered property pi(1) in Definition 2.0.4, and this turns later out to be equal to property pi-1 of Definition 2.0.6, and says that each non-zero $a \in A_+$ is properly infinite.

But the "property pi-1" defined in Definition 2.0.7 seems to be different. because it requires that a is 2-homogenous. ???

We ask here for some care, because different definitions of infiniteness of elements in some publications do not imply infiniteness in the sense of our Definition 2.1.1, e.g. H. Lin and Sh. Zhang gave in [**532**, def. 1.1] the following definition for positive elements a in the Pedersen ideal $\mathrm{Ped}(A)$:

An element $a \in \mathrm{Ped}(A)_+$ is called *infinite* in [**532**], if there are nonzero positive elements b and c in $\mathrm{Ped}(A)_+$ such that $bc = cb = 0$, and there exist sequences (x_n) and (y_n) in A such that $b+c = \lim_n x_n^* c x_n$, and $c = \lim_n y_n^* a y_n$. In our terminology, this property of $a \in \mathrm{Ped}(A)_+$ is equivalent to $c \precsim a$ and is slightly stronger than $b \in I_A(c)$, i.e., c is infinite and a majorize c. And, in our terminology, this implies that $[a] \geq [c]$ in $\mathrm{Cu}(A)$ and that c is infinite. Lemma 2.1.6 shows that in case of *simple* C^*-algebras A the definition [**532**, def. 1.1] of infinite elements is equivalent to our notion of *infinite elements* given in our Definition 2.1.1. The existence of a nonzero element $b \in A_+$ with $bc = 0$ and $b + c \precsim c$ inside A is not *equivalent* to infiniteness of c if the C^*-algebra A is not simple. But it is a *sufficient* condition for being infinite in non-simple C^*-algebras, cf. Lemma 2.5.4.

This definitions of proper infiniteness are not the same in case of non-simple C^*-algebras A:

Let A here denote the unitization of $\mathrm{C}_0(0,1] \otimes \mathcal{O}_2$: Then $1 \otimes 1 \geq f_0 \otimes 1$ and the element $f_0 \otimes 1 \sim_{MvN} f_0 \otimes (s_1 s_1^*)$ is properly infinite in A in the sense of definition [**532**, def. 1.1], because $f_0 \otimes (s_1 s_1^*)$ is orthogonal to the element $f_0 \otimes (s_2 s_2^*)$. But $1 \otimes 1$ is finite, because a non-unitary isometry can not be homotopic to 1 in $\mathcal{O}_2$ inside the class of isometries. We don't know if the above cited definition of Lin and Zhang of infinite elements are equivalent to our definition at least for non-simple A with *real rank zero*, (but didn't see a counterexample). Consider the unitization of $\mathrm{C}_0(X, \mathcal{O}_2)$ for $X := C \cap (0,1]$, where $C \subset [0,1]$ is the Cantor subset. The unit is infinite but is not properly infinite: It can't be properly infinite because restriction to a zero-sequence in X shows this by observing that $(1,1,\ldots)$ is not properly infinite in $c_0(\mathcal{O}_2) + \mathbb{C} \cdot (1,1,\ldots) \subset \ell_\infty(\mathcal{O}_2)$.

Let $A \subset \ell_\infty(\mathcal{O}_2)$ denote the C^*-algebra generated by $c_0(\mathcal{O}_2)$ and the unit element $(1,1,\ldots) \in \ell_\infty(\mathcal{O}_2)$, then each element of A is infinite, but $(1,1,\ldots)$ is not *properly* infinite in A. This A has real rank zero and each projection is infinite, but only the elements of $c_0(\mathcal{O}_2)$ are properly infinite.

But for *projections* a, b, c in a C^*-algebra A of real rank zero the projection a is infinite in sense of our Definition 2.1.1 if one restricts the definition of Lin and Zhang only to projections $a, b, c \in A$. It says then only that each non-zero projection a that satisfies their conditions has the property that $a = p + q$ with p an infinite projection.

We have to consider also several non-simple C^*-algebras with non-zero real rank if we study general purely infinite C^*-algebras.

Some conditions allow only to see that diagonal elements $\mathrm{diag}(a, a, \ldots, a) = a \otimes 1_n \in M_n(A) \cong A \otimes M_n$ are infinite or properly infinite for some unknown (!) fixed $n \in \mathbb{N}$, e.g. this is the case where one can show only that $\ell_\infty(A)$ has no non-trivial quasi-trace. We are forced to variate and refine the Definition 2.1.1 in the following manner:

DEFINITION 2.1.3. An element $a \in A_+$ is **n-infinite** (respectively is **n-properly infinite**), if $a \otimes 1_n$ is infinite (respectively is properly infinite) in $A \otimes M_n$.

An element $a \in A_+$ is **n-stable** if the hereditary C^*-subalgebra $D := \overline{(a \otimes 1_n)(A \otimes M_n)(a \otimes 1_n)}$ of $A \otimes M_n$ contains a stable hereditary C^*-subalgebra E that is *full* in D.

Notice that here n-stability of $a \in A_+$ only means that the stable $E \subseteq D$ generates the same ideal of $A \otimes M_n$ as D does.
Important warning: 1-stable elements $a \in A_+$ are not necessarily stable elements in the sense of Definition 2.1.1.

If each element $a \in A_+$ is n-infinite (or n-properly infinite), then at least every n-homogenous element $a \in A_+$ is infinite (respectively n-properly infinite) in A, and it is always contained in the kernel of every irreducible representation of A of dimension $< n$. It is also contained in the kernel of Of all irreducible representations od dimension n if a is n-properly infinite.

We remind the definition of n-homogenous positive elements:

DEFINITION 2.1.4. An element $a \in A_+$ is called **n-homogenous**, if there exists a C^*-morphism $\rho\colon \mathrm{C}_0(0,1] \otimes M_n \to A$ with $\|a\| \cdot \rho(f_0 \otimes 1_n) = a$ for the generator $f_0(t) = t$ of $\mathrm{C}_0(0,1]$.

The proof of Proposition A.8.4. or [**538**, prop. 2.7], give the following equivalent description of an n-homogenous element $a \in A_+$: There exist – *not uniquely determined* – contractions $z_2, \ldots, z_n \in A$ with properties $z_k z_j^* = 0$ for $j \neq k$, $z_j z_k = 0$ for $j, k = 2, \ldots, n$ and $z_j^* z_j = z_2^* z_2$ for $j = 3, \ldots, n$, such that $a = \|a\| \cdot (z_2^* z_2 + \sum_{k=2}^n z_k z_k^*)$.

A C^*-morphism $\rho\colon \mathrm{C}_0(0,1] \otimes M_n \to A$ with $\|a\| \cdot \rho(f_0 \otimes 1_n) = a$ is then defined by $\{z_2, \ldots, z_n\}$ and the assignments

$$\rho\colon\ f_0 \otimes p_{k,1} \ \mapsto\ z_k\,.$$

By Remark 2.1.16, a C^*-algebra A contains an n-homogenous element $a \in A_+$ with $\|a\| = 1$ if A is not a k-sub-homogenous C^*-algebra for some $k < n$.

Recall that C^*-algebra A is **k-sub-homogenous** if A is isomorphic to a C^*-subalgebra of $M_k(C)$ for some *commutative* C^*-algebra C. This happens, if and only, if all irreducible representations of A have dimension $\leq k$. All *separable* k-sub-homogenous C^*-algebras are C^*-subalgebras of $M_k \otimes \ell_\infty = \ell_\infty(M_k)$.

Our notions of n-homogenous and of n-stable positive elements have nothing in common, because n-stable elements $a \in A_+$ have the property that $a \otimes 1_n$ is properly infinite in $M_n(A)$, but n-homogenous $a \in A_+$ have not necessarily this property, as e.g. $a := 1_n \in M_n$.

Sometimes we use following important property of n-homogenous elements: If $a \in A_+$ is n-homogenous, then $a \otimes p_{11} \in A \otimes M_n$ is MvN-equivalent to $b \otimes 1_n$ in $M_n(A) \cong A \otimes M_n$ for suitable $b \in \overline{aAa}_+$ that commutes with a in A.

We allow only in some exceptional cases for all the above given definitions of properties of elements $a \in M_n(A)$ that $a = 0$, then just to simplify the formulation of conditions or of results.

Remark 2.1.5. Some authors prefer to use (completely positive) "order-zero morphisms" $\psi \colon C \to A$ from C to A instead of C^*-algebra homomorphisms $h \colon \mathrm{C}_0((0,1], C) \to A$. Those ψ are often used in cases where $C = M_n$ (and more generally if C is of finite dimension).

The order-zero morphisms ψ are defined as completely positive linear maps $\psi \colon C \to A$ with $\|\psi\| \leq 1$ and the property that $\psi(c)\psi(d) = 0$ for all $c, d \in C_+$ with $cd = 0$. In particular, $\psi(c) := h(f_0 \otimes c)$ is an order-zero morphism if $h \colon \mathrm{C}_0((0,1], C) \to A$ is a C^*-algebra homomorphism.

We modify in Section 9 of Appendix A arguments in the proofs of [**838**, thm.2.3] of M. Wolff and of W. Winter and J. Zacharias in [**836**], where they show that for each completely positive order-zero map $\psi \colon C \to A$ there exists a unique C^*-algebra homomorphism $h \colon \mathrm{C}_0((0,1], C) \to A$ with $h(f_0 \otimes c) = \psi(c)$ for all $c \in C$.

In fact we need for our recovery of h from ψ only the 2-positivity of the linear map ψ and the orthogonality property of ψ on C_+, i.e., that $\mathrm{id}_{M_2} \otimes \psi$ is positive and that $\psi(c)\psi(d) = 0$ for $c, d \in C_+$ for $cd = 0$. to obtain the latter, cf. Section 9 of Appendix A.

A crucial starting point for the study of simple purely infinite C^*-algebras was observed first by J. Cuntz. It is the nice observation stated in the following Lemma 2.1.6. Its proof should be an exercise, – e.g. by using an argument that is a bit more elementary than using properties of the general Cuntz relation $\precsim$ on elements of $A \otimes \mathbb{K}(\ell_2)$.

Lemma 2.1.6. *Suppose that A is a* simple *C^*-algebra. Let $a, b \in A_+$. If $a \leq b$ and a is infinite in A, then a and b are both* properly *infinite in A, and a is properly infinite inside the hereditary C^*-subalgebra $\overline{aAa}$ of A.*

This Lemma does not apply to non-simple C^*-algebras, cf. Example 2.5.13, but it is also a very special case of Lemma 2.5.3(i) on the Cuntz majorization $\precsim$ in arbitrary C^*-algebras.

Another often used fact is Part (ii) of the following Lemma 2.1.7 concerning stably generated ideals, and ideals generated by ideals of C^*-subalgebras. The first statement in each item plays a particular role `Consider blue changes! Give new proofs?// Consider also the remark before proof!// Compare with Lemma 2.7.3!!`

LEMMA 2.1.7. *Let A a C^*-algebra, $B \subseteq A$ a C^*-subalgebra, and let J the closed ideal of A that is generated by B, i.e., J is the closed linear span of the set of products $A \cdot B \cdot A$.*

(o) *The* set $B \cdot A \cdot B$ *of products $b_1 a b_2$ with $b_1, b_2 \in B$ and $a \in A$ is the same as the hereditary C^*-subalgebra $E := \overline{\text{span}(BAB)}$ of A generated by B, where* span$(\,\cdot\,)$ *denotes the linear span, i.e., no addition operations or approximations are needed to get* all *elements of the hereditary C^*-subalgebra E from the set of products $b_1 a b_2$.*

In particular, B is hereditary in A, if and only if, the set $B \cdot A \cdot B$ *is equal to B.*

(i) *If the unit $1_{\mathcal{M}(B)}$ is properly infinite in $\mathcal{M}(B)$, then the hereditary C^*-subalgebra $E = B \cdot A \cdot B$ of A contains a* stable *hereditary C^*-subalgebra D of A that generates the same closed ideal J of A as generated by B (also generated by E).*

(ii) *If the unit $1_{\mathcal{M}(B)}$ is properly infinite in $\mathcal{M}(B)$, and $a_1, a_2, \ldots \in J_+$ is any sequence in J, then there exists a sequence $d_1, d_2, \ldots \in A$ with the properties $d_j^* d_k = \delta_{jk} a_k$ and $d_k d_k^* \in D \subseteq E$.*

(iii) *In particular, if $p \in \mathcal{M}(A)$ is a non-zero* properly *infinite projection, then pAp contains a* stable *hereditary C^*-algebra D such that the ideal J generated by D contains pAp.*

(iv) *If, moreover, B is stable then $1_{\mathcal{M}(B)}$ is properly infinite and the hereditary C^*-algebra $E = BAB$ itself is stable.*

(v) *(Ideal-separation and ideals of hereditary C^*-subalgebras)*

Let $B \subseteq A$ a C^*-subalgebra of A and denote by $J_B := \overline{\text{Span}(ABA)}$ the closed ideal of A generated by B.

Consider the maps $\Phi \colon Y \mapsto \overline{\text{Span}(AYA)}$ from the family of all subsets $Y \subseteq A$ of A into the family closed ideals of A, and the map $\Psi \colon I \mapsto I \cap B$, from the family of closed ideals I of A into the closed ideals J of B.

Then, $\Phi(\Psi(I)) = I \cap J_B$ for each closed ideal I of A, if and only if, the elements of B separate the closed ideals of the hereditary C^*-subalgebra BAB of A.

If,– in addition –, each closed ideal of J of B is the intersection $J = I \cap B$ of a closed ideal $I \subseteq A$ with B then the restriction $\Phi | \mathcal{I}(B)$ of Φ to the family of closed ideals J of B is the inverse map of the map $I \mapsto B \cap I$, i.e., $\Psi(\Phi(J)) = J$ each for each closed ideal J of B.

Thus, then $J = B \cap \overline{\text{Span}(AJA)}$ for all ideals $J \in B$ and $I = \overline{\text{Span}(A(B \cap I)A)}$ for all closed ideals | of A.

In particular, for all hereditary C^*-subalgebras D of A and every closed ideal I of A, the closed ideal $D \cap I$ of D generates the closed ideal $I \cap J_D$ of A and each closed ideal K of D is the intersection $K = D \cap I$ of the closed ideal I of A generated by K.

PROOF. (o): Let $E := \overline{\text{span}(B \cdot A \cdot B)}$ the closed linear span of the elements $b_1 a b_2 = (b_2^* a^* b_1^*)^*$ with $a \in A$ and $b_1, b_2 \in B$.

Clearly, $E \subseteq A$ is a hereditary C^*-subalgebra of A, and E is a non-degenerate left and right B-module. Moreover for every countable subset X of E there exists a sequence of positive contractions $b_1, b_2, \ldots \in B_+$ such that $x = \lim_n b_n x = \lim_n x b_n$ for all $x \in X$. If we let $d := \sum_n 2^{-n} b_n$, then

$$X \subseteq \overline{d \cdot E \cdot d} \subseteq \overline{d \cdot A \cdot d} \subseteq E\,.$$

We can apply the factorization theorem of G.K. Pedersen [**621**, thm. 4.1] to the non-degenerate $C^*(d)$-bimodule $\overline{d \cdot E \cdot d} \subseteq E$, cf. also our special version in Theorem A.11.1. This allows to see that for each finite sequence $e_1, \ldots, e_n \in \overline{d \cdot E \cdot d} \subseteq E$ there exists a positive contraction $d_0 \in C^*(d)_+ \subseteq B_+$ and elements $f_1, \ldots, f_n \in E$ such that $e_k = d_0 f_k d_0$ for $k = 1, \ldots, n$. In particular, E is identical with the set $B \cdot A \cdot B$ of products $b_1 a b_2$ with $a \in A$ and $b_1, b_2 \in B$.

(i): The non-degenerate *-monomorphism defined by the inclusion $B \hookrightarrow E := B \cdot A \cdot B$ defines a unital *-monomorphism $\mathcal{M}(B) \hookrightarrow \mathcal{M}(E)$.

By assumptions, there are isometries $s, t \in \mathcal{M}(B) \subseteq \mathcal{M}(E)$ with orthogonal ranges. Then $s_1 := s$, $s_n := t^{n-1}s$ for $n = 2, 3, \ldots$ correspond to a unital C^*-morphism from $\mathcal{O}_\infty$ into $\mathcal{M}(E)$.

Thus, $\mathcal{M}(E)$ has a properly infinite unit, i.e., contains a copy of $\mathcal{O}_\infty := C^*(s_1, s_2, \ldots\,;\ s_j^* s_k = \delta_{jk} 1)$ unitally. The C^*-subalgebra D of E generated by $\{s_k e s_j^*\,;\ j, k \in \mathbb{N}, e \in E\}$ is a stable C^*-subalgebra of E because D is the closure of $\bigcup_{m,n} s_m E s_n^*$, which implies that $D \cong E \otimes \mathbb{K}$, e.g. via the isomorphism ψ given by $\psi(e \otimes p_{kj}) = s_k e s_j^*$ for the matrix units p_{jk} of $\mathbb{K}$.

The D is full in E and is hereditary in E, because it is the closure of the union of the increasing family of split corners $P_n E P_n$ of E with $P_n := s_1 s_1^* + \cdots + s_n s_n^*$, $s_1 E s_1^* \subset P_n E P_n \subset E$, and $s_1 e s_1^* \sim_{MvN} e$ inside E for each $e \in E_+$.

Hence, D, E and B generate the same closed ideal $J := \overline{\operatorname{span}(ABA)}$ of A.

(ii): By Part (i), D is isomorphic to $E \otimes \mathbb{K}$ via natural transformation with help of the isometries $s_1, s_2, \ldots \in \mathcal{M}(E)$. (But the s_n are not necessarily in $\mathcal{M}(D)$.)

If we use that $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ we get isometries $T_n \in \mathcal{M}(\mathbb{K})$ with $T_j^* T_k = \delta_{jk} 1$ such that $\sum_k T_k T_k^*$ strictly converges to the unit $1_{\mathcal{M}(\mathbb{K})}$. It follows that the isometries $S_k := 1 \otimes T_k$ are isometries in $\mathcal{M}(E) \otimes \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(E \otimes \mathbb{K})$ such that $\sum_k S_k S_k^*$ converges strictly to 1 in $\mathcal{M}(E \otimes \mathbb{K})$. The isomorphism ψ from D onto $E \otimes \mathbb{K}$ extends to a strictly continuous *-isomorphism $\mathcal{M}(\psi)$ from $\mathcal{M}(D)$ onto $\mathcal{M}(B \otimes \mathbb{K})$. Then the sequence $t_1, t_2, \ldots \in \mathcal{M}(D)$ of isometries $t_k := \mathcal{M}(\psi)^{-1}(S_k)$ has the property that $t_j^* t_k = \delta_{jk} 1$ and that $\sum_k t_k t_k^*$ converges in $\mathcal{M}(D)$ strictly to 1.

It is enough to show that for each $a \in J_+$ there exists an element $h \in J$ with $h^* h = a$ and $hh^* \in D$ to obtain the requested elements $d_n \in D$:
Indeed, we find for the given sequence $a_1, a_2, \ldots \in J_+$ a sequence $h_1, h_2, \ldots \in J_+$ with $h_k^* h_k = a_k$ and $h_k h_k^* \in D$. Then the elements $d_k := t_k h_k$ satisfy $d_j^* d_k = \delta_{jk} a_k$ and $d_k d_k^* \in D$.

Let $a \in J_+$ where $J := \overline{\operatorname{span}(A \cdot D \cdot A)}$, and let $\varepsilon > 0$.

Then D is full in J, and there are $e_1,\ldots,e_m \in D_+$ and $f_1,\ldots,f_m \in J$ with $\|a-b\| < \varepsilon$ for $b := \sum_j f_j^* e_j f_j$.

If we use Lemma 2.1.9, then we find $c \in A$ such that $g_1 := f_1c,\ldots,g_m := f_nc \in A$ with $\sum_j g_j^* e_j g_j = (a-\varepsilon)_+$. The element $x := \sum_j t_j(e_j)^{1/2} g_j \in A$ satisfies $x^*x = (a-\varepsilon)_+$ and $xx^* \in D$. The latter uses that D is hereditary and $t_je_jt_j^* \in D$. In particular $x \in J$. Let $\nu(t) := (t-\varepsilon-\delta)_+/(t-\varepsilon)_+$, the element $y := x(1-\nu(a))^{1/2}$ satisfies $y^*y = (a-\varepsilon)_+ - (a-\varepsilon-\delta)_+$.

In this way, we find $y_1, y_2, \ldots \in J$ with $y_ny_n^* \in D$, $y_1^*y_1 = (a-1/2)_+$, $y_n^*y_n = (a-2^{-n})_+ - (a-2^{2-n})_+$ for $n>1$. Then $h := \sum_n t_ny_n$ is an element of the requested type, because the series is absolutely convergent by

$$\|t_ny_n\| \leq \|y_n\| \leq \|(a-2^{-n})_+ - (a-2^{-(n-1)})_+\|^{1/2} \leq 2^{-n/2} \quad \text{for} \quad n>1,$$

$h^*h = \sum_n y_n^*y_n = a$ and $hh^* = \lim_{n\to\infty}(\sum_{j,k\leq n} t_jy_jy_k^*t_k^*) \in D$.

(iii): There are partial isometries $u, v \in A$ with $u^*u = v^*v = p$ and $uu^*+vv^* \leq p$. Let $B = C^*(u,v)$. Then $\mathcal{M}(B) = B$ and $1_{\mathcal{M}(B)} = p$ is properly infinite, $E = pAp$ and D is the closure of $\bigcup_{m,n} s_mEs_n^*$, where here $s_n := u^{n-1}v$ for $n = 1,2,\ldots$ and $u^0 := p$. Notice that $ps_np = s_n$ and $s_j^*s_k = \delta_{jk}p$.

(iv): The C^*-algebra B is stable, if and only if, its multiplier algebra $\mathcal{M}(B)$ contains a sequence $t_1, t_2, \ldots$ of isometries with mutually orthogonal ranges, such that the sum $\sum_n t_nt_n^*$ converges to $1_{\mathcal{M}(B)}$ in the strict topology of $\mathcal{M}(B)$, cf. Remark 5.1.1(8) for more about "stability".

The natural map $\iota\colon \mathcal{M}(B) \to \mathcal{M}(E)$ is injective, unital and strictly continuous for $E = BAB$. Therefore $s_1 := \iota(t_1)$, $s_2 := \iota(t_2)$, $\ldots \in \mathcal{M}(E)$ satisfy again $\sum_n s_ns_n^* = 1$. Thus, E is stable if B is a stable.

(v): The subsets $\Phi(Y) := \overline{\mathrm{Span}(AYA)}$ of A are the closed ideals of A generated by non-empty subsets Y of A, in particular $J_B := \Phi(B)$ is the smallest closed ideal of A with $B \subseteq J_B$. Notice that $\Phi(I\cap B) \subseteq I \cap J_B$ for all closed ideals of I of A, and $\Phi(\Phi(Y)) = \Phi(Y)$ for all non-empty subsets $Y \subseteq A$ of A.

Let I a closed ideal of A. Recall that $\Psi(B\cap I)$ is the closed ideal of A generated by $B\cap I$. Since $B\cap I$ is contained in the ideal $J_B\cap I$, it follows that

$$\Psi(B\cap I) \subseteq J_B \cap I.$$

And it implies that $B\cap \Psi(B\cap I) = B\cap I$, because $B\cap I \subseteq \Psi(B\cap I)$ and $B\cap(J_B\cap I) = B\cap I$ by $B\subseteq J_B$.

The above equations show that $B\cap(J_B\cap I) = B\cap\Psi(B\cap I)$, which is equivalent to $B\cap(D\cap I) = B\cap(D\cap\Psi(B\cap I))$ for the hereditary C^*-subalgebra $D := B\cdot A\cdot B$, cf. Part (i).

It follows: The C^*-subalgebra B of A separates the closed ideals of $D := B\cdot A\cdot B$, if and only if, B separates the ideals of J_B, if and only if, the ideal $J_B\cap I$ of J_B that is generated by $B\cap I$ for all closed ideals I of J_B, i.e., each closed ideal of J_B is determined by its intersection with B.

Thus, always $B \cap \Psi(I \cap B) = B \cap I$, or B does not separate the closed ideals of BAB, respectively of J_B.

It follows then that $I \cap B = \Psi(I \cap B) \cap B$ for all closed ideals I of A. Thus, if B is separating for the ideals of J_B (or – equivalently – for the ideals of $D := B \cdot A \cdot B$) then this says that $\Psi(I \cap B) = I \cap J_B$ for all closed ideals I of A.

But the ideals $I \cap B$ are not all ideals of B if B is not hereditary in A. There are examples of finite-dimensional C^*-algebras A where B can have much more closed ideals than the special ideals $I \cap B$ for ideals I of A. □

EXAMPLE 2.1.8. The stable C^*-subalgebra D defined in the proof of Lemma 2.1.7 is not necessarily equal to E, because $B := A := \mathcal{O}_\infty$ satisfy our assumptions, but $E := \mathcal{O}_\infty$ and $J = \mathcal{O}_\infty$ are not stable.

The behaviour of the isometries and elements constructed during the proof has to be considered with some care because they are usually not in $\mathcal{M}(A)$:
Consider the C^*-algebra $A := \mathcal{O}_\infty \otimes \mathbb{K}$, and a pure state ρ on $\mathcal{O}_2$. Let $L := \{c \in \mathcal{O}_2\,;\ \rho(c^*c) = 0\}$ the closed left-ideal corresponding to ρ and $B := (L^* \cap L) \otimes p_{11} \subset A$. Then B is a full stable hereditary C^*-subalgebra of A, and the algebras $B \subseteq A$ satisfy with $E = B = D$ and $J = A$ all the assumptions of Lemma 2.1.7, where here B itself is full, stable and hereditary, but the in the proof of Lemma 2.1.7(i) defined and used isometries and projections $s_n, P_n \in \mathcal{M}(B) \subseteq \mathcal{M}(E)$ are *not* elements of $\mathcal{M}(J) = \mathcal{M}(A)$. But there exists an element $X \in A$ with X^*X strictly positive in A and XX^* strictly positive in B that creates an almost canonical isomorphism from $\mathcal{M}(B) \subset B^{**} \subset A^{**}$ onto $\mathcal{M}(A) \subseteq A^{**}$ by the isometry $V \in A^{**}$ given by the polar decomposition $X = V(X^*X)^{1/2}$.

Beginning of transport back from Appendix A:
Replace e.g. all ‘‘Lemma lem:A.old.2.5 (iv)’’ by Remark 2.1.16

The following often used Lemma is [**463**, lem. 2.4(iii)]. We give here a different proof that is short, elementary and does not use results about operator monotone functions. The self-adjoint operator $a_+ - a_-$ is the polar decomposition of $a^* = a$.

LEMMA 2.1.9. *Suppose that $a \in \mathcal{M}(A)$, $b \in A$ and $\gamma \in [0, \infty)$ satisfy in $\mathcal{M}(A)$ that*

$$0 \leq b \leq a + \gamma \cdot 1 .$$

Then, for every $\varepsilon > \gamma$, there exists $d := d(\varepsilon) \in \overline{bC^*(a,b)b} \subseteq A$ with

$$\|d\| \leq 1 \qquad \text{and} \qquad d^*\, a_+\, d = (b - \varepsilon)_+ .$$

In particular, there exists $d \in A$ with $\|d\| \leq 1$ and $d^* a_+ d = (b - \varepsilon)_+$ if $a^* = a \in A$ and $b \in A_+$ satisfy $\|b - a\| < \varepsilon$.

PROOF. Let $\varphi(t) := \big((t-\gamma)_+^{-1}(t - (\varepsilon+\gamma)/2)_+\big)^{1/2}$ and $g := \varphi(b) a_+ \varphi(b)$.

The function $\varphi(t)$ is zero on $[0, \mu]$ for $\mu := (\varepsilon + \gamma)/2 > \gamma$ and is a non-negative continuous function on $[\mu, \infty)$. It is increasing because $\varphi(t)^2 = 1 - (\mu - \gamma)(t-\gamma)^{-1}$

for $t \in [\mu, \infty)$. This and the inequalities $0 < \gamma < \mu < \varepsilon$ imply $(t-\gamma)_- \cdot \varphi(t) = 0$, $\varphi(t) \in [0,1)$, $0 \le \varphi(b) \le \varphi(\|b\|) \cdot 1$. Recall that $(b-\varepsilon)_+ \le (b-\mu)_+$ for $0 \le \mu \le \varepsilon$.

Then $\|\varphi(b)\| < 1$, $\varphi(b) \cdot (b-\gamma)_- = 0$ and

$$(b-\varepsilon)_+ \le (b-\mu)_+ = \varphi(b)(b-\gamma)\varphi(b) \le g\ . \tag{1.3}$$

We define d as the norm-limit of the elements $d_n := \varphi(b)(g+1/n)^{-1/2}(b-\varepsilon)_+^{1/2}$:

The $d_n \in A$ are contractions and the sequence $(d_1, d_2, \ldots)$ converges in A, because $\|\varphi(b)\| \le 1$ and Inequality (1.3) imply $\|d_n^* d_n\| \le 1$ and, by monotony of the C^*-norms on the cone of positive operators, that

$$\|d_n - d_m\|^2 \le \| g(g+1/n)^{-1} - g(g+1/m)^{-1} \| \le 2/\min(n,m)\,.$$

It is obvious from the definition of the d_n that $(b-\varepsilon)_+ = \lim_n d_n^* a_+ d_n$, – and this convergence is independent from the above shown convergence for the sequence (d_n).

If $a^* = a \in A$ and $b \in A_+$ satisfy $\gamma := \|b-a\| < \varepsilon$, then $b \le a + \gamma$. □

Remark 2.1.10. It seems to be well-known folklore that matrices $a = [a_{j,k}] \in M_n(A)$ have operator norm estimates

$$\|a\| \le n \cdot \max\{\|a_{j,k}\|\,;\ 1 \le j,k \le n\}\,.$$

Moreover, one gets the better estimate

$$\|a\| \le (n-1) \cdot \max\{\|a_{j,k}\|\,;\ 1 \le j,k \le n\}\,.$$

if the "diagonal" sub-matrix $\mathrm{LR}(n) = \mathrm{diag}(a_{1,1}, a_{22} \ldots, a_{n,n})$ of $[a_{j,k}]$ has only zero entries,.

The estimate follows from the choice of n sub-matrices $\mathrm{LR}(\ell)$, $\ell \in \{1,\ldots,n\}$, each with at most n non-zero suitably chosen entries $a_{j,k}$, e.g. with $(k-j)\mathrm{mod}\ell$ for $\ell \in \{1,\ldots,n\}$ in $\mathbb{Z}/n\mathbb{Z}$, and the property that $\mathrm{LR}(\ell)^*\mathrm{LR}(\ell)$ is a diagonal matrix. Thus, the norm of $\mathrm{LR}(\ell)$ is given exactly by the maximum of the norms of the entries of $\mathrm{LR}(\ell)$. Then $\|a\| \le$ the sum of the norms of this sub-matrices.

See Section 18 in Appendix B for more details on such sub-matrices.

An immediate consequence of Lemma 2.1.9 is the following observation:

Lemma 2.1.11. *Let $e_1, e_2, \ldots, e_n \in A$ and $\gamma > (n-1) \cdot \|e_j^* e_k\|$ for all $j > k$. Then there exists a contraction $D = [d_{j,k}] \in M_n(A)$ such that $f_j^* f_k = \delta_{j,k}(e_k^* e_k - \gamma)_+$ for the elements $f_k := e_1 d_{1,k} + e_2 d_{2,k} + \ldots + e_n d_{n,k}$ with $k \in \{1,2,\ldots,n\}$.*

Proof. Consider the matrices $a, b \in M_n(A)_+$ defined by

$$a := [e_1,\ldots,e_n]^* \cdot [e_1,\ldots,e_n] \quad \text{and} \quad b := \mathrm{diag}(e_1^* e_1, \ldots, e_n^* e_n)\,.$$

The matrix $a - b \in M_n(A)$ has zero diagonal elements, Remark 2.1.10 applies and says that

$$\|a-b\| \le (n-1) \cdot \max\{\|e_j^* e_k\|\,;\ 1 \le j < k \le n\}\,.$$

Lemma 2.1.9 provides a matrix $D = [d_{jk}] \in M_n(A)$ that satisfies $D^*aD = (b-\gamma)_+$ and hast norm $\|D\| \leq 1$. $\square$

We need in Chapter 3 and other places the equivalences for a given element $h \in A_+$ listed in the following Lemma:

LEMMA 2.1.12. *Let A a C^*-algebra, $h \in A_+$ and $2 \leq q \in \mathbb{N}$.*

The following properties (a, i)–(a,iii) *of h are equivalent:*

(a, i) *For every $\varepsilon > 0$, there exist $e_1, e_2, \ldots, e_q \in A$ with $\|e_j^* e_k - \delta_{j,k} h\| < \varepsilon$ for $j, k \in \{1, 2, \ldots, q\}$.*

(a, ii) *For each $\gamma > 0$ and $\psi \in \mathrm{C}_0(0, \|h\|]_+$ with $\psi|[0,\gamma] = 0$ there are elements $e_1, \ldots, e_q \in A$ with $e_j^* e_k = \delta_{jk} \cdot \psi(h)$ for $j, k \in \{1, \ldots, q\}$.*

(a,iii) *Each element $g \in \overline{hAh}_+$ satisfies the property in part* (a, i) *– with g in place of h.*

The following, considerable stronger, properties (b, i)–(b,iv) *of $h \in A_+$ and the closed ideal $J := J(h) \subseteq A$, generated by h, are equivalent to each other:*

(b, i) *For every $a \in J_+$ and $\varepsilon > 0$ there exist elements $e_1, e_2 \in A$ with*

$$\|e_j^* e_k - \delta_{jk} a\| < \varepsilon \quad \text{for } j, k \in \{1, 2\}.$$

(b, ii) *For each $n > 1$ and $\varepsilon > 0$ there exist elements $e_1, \ldots, e_n \in A$ with*

$$e_j^* e_k = \delta_{jk}(h - \varepsilon)_+ \quad \text{for } j, k \in \{1, 2, \ldots, n\}.$$

(b,iii) *For every $a \in J_+$, $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $d_1, \ldots, d_n \in J$ such that*

$$d_j^* d_k = \delta_{jk}(a - \varepsilon)_+ \quad \text{for } j, k \in \{1, 2, \ldots, n\}.$$

(b,iv) *$A_\infty := \ell_\infty(A)/c_0(A)$ contains a stable hereditary C^*-subalgebra D such that the element $\Delta(h) := (h, h, \ldots) + c_0(A)$ is contained in the closed ideal of A_∞ generated by D.*

We do not require in part (a, i) that $e_j e_j^* \in \overline{hAh}$, e.g. the case $A = M_2$, $q = 2$ and $h = p_{11}$ shows that the properties (a, i) and (b, i) are very different.

The property (b, i) in Lemma 2.1.12 applies to all $a \in A_+$ e.g. if the unit element of $\mathcal{M}(A)$ is properly infinite, or if – more general – A contains an approximate unit consisting of positive contractions e_σ that are M-vN equivalent to two orthogonal pairs of contractions, e.g. there are $f_\sigma, g_\sigma \in A$ with $f_\sigma^* g_\sigma = 0$, $f_\sigma^* f_\sigma = e_\sigma$ and $g_\sigma^* g_\sigma = e_\sigma$.

PROOF. All items are equivalent if $h = 0$. It suffices to consider on all places the cases where $\|h\| = 1$, $\|g\| = 1$ respectively $\|a\| = 1$, because otherwise we can rescale the $\varepsilon > 0$, $\gamma > 0$, and change the function $\psi \in \mathrm{C}_0(0, \|h\|]_+$ in a suitable manner.

Equivalence of (a, i), (a,ii) *and* (a,iii):

(a,iii) $\Rightarrow$ (a, i): $h \in C^*(h^3)_+ \subseteq \overline{hAh}_+$.

(a, i) $\Rightarrow$ (a,iii): Let $\|h\| = 1 = \|a\|$, $g \in \overline{hAh}_+$ with $\|g\| = 1$, $\varepsilon \in (0, 1/2)$ and $\gamma := \varepsilon/7$. There exists $a \in A$ with $a^* = a$ and $\|g^{1/2} - h^{1/2}ah^{1/2}\| < \gamma$, because $\overline{hAh} = \overline{h^{1/2}Ah^{1/2}}$. Then $\|g - h^{1/2}ahah^{1/2}\| < (2+\gamma)\gamma < \varepsilon/2$.

Let $\mu := \varepsilon/(1 + 2\|a\|^2)$, then $\|ah^{1/2}\|^2 \cdot \mu \leq \mu\|a\|^2 < \varepsilon/2$.

By assumption there exists elements $e_1, e_2, \ldots, e_q \in A$ with $\|e_j^* e_k - \delta_{j,k} h\| < \mu$ for $j, k \in \{1, \ldots, q\}$. Let $f_k := e_k a h^{1/2}$. Then $\|f_j^* f_k - \delta_{j,k} h^{1/2}ahah^{1/2}\| < \mu \cdot \|ah^{1/2}\|^2 \leq \mu\|a\|^2 < \varepsilon/2$. Hence, $\|f_j^* f_k - \delta_{j,k} g\| < \varepsilon$.

(a,ii) $\Rightarrow$ (a, i): Let $\varepsilon > 0$ and take $\delta := \varepsilon/2$ and $\psi(t) := \max(0, t - \delta)_+$.

(a, i) $\Rightarrow$ (a,ii): It follows from the special case where $\psi := \varphi_\gamma$ for $\gamma > 0$ and $\varphi_\gamma(t) := \max(0, t - \gamma)$. Indeed, if $\psi \in \mathrm{C}_0(0, \|h\|]_+$ with and $\psi|[0, 2\gamma] = 0$ is given, and $d_1, \ldots, d_q \in A$ exists with $d_j^* d_k = \delta_{jk} \cdot \varphi_\gamma(h)$, then the elements $e_k := d_k a$ satisfy $e_j^* e_k = \delta_{jk} \cdot \psi(h)$ for $a := \lambda(h)$ with $\lambda(t) := \big(\max(0, t-\gamma)^{-1}\psi(t)\big)^{1/2}$.

In the special case $\varphi_\gamma(h) = (h - \gamma)_+$ we can apply Lemma 2.1.11 to the situation in Part (a, i):
Let $\mu := \gamma/(q+1)$, and find $f_1, \ldots, f_q \in A$ with $\|f_j^* f_k - \delta_{j,k} h\| < \mu$. By Lemma 2.1.9 there exist contractions $g_k \in A$ with $g_k^* f_k^* f_k g_k = (h - \mu)_+$. It follows that still $\|g_j^* f_j^* f_k g_k\| \leq \|f_j^* f_k\| < \mu$ for $j \neq k$. The inequality $q \cdot \mu < \gamma$ allows to apply Lemma 2.1.11 and obtain $e_1, \ldots, e_q \in A$ with $e_j^* e_k = \delta_{jk} \cdot (h - \gamma)_+$ for $j, k \in \{1, \ldots, q\}$.

Equivalence of (b, i)–(b, iv):

We use the equivalence of (a, i)–(a,iii) to prove equivalence of (b, i)–(b,iv).

The implications (b,iii) $\Rightarrow$ (b, i) and (b,iii) $\Rightarrow$ (b, ii) are obvious. We show the implications (b, i) $\Rightarrow$ (b,iii), (b,ii) $\Rightarrow$ (b,iv) and (b,iv) $\Rightarrow$ (b, i).

(b, i)$\Rightarrow$(b,iii): The equivalence of (a, i) and (a, ii), applied to $a \in J_+$ in place of h and $q := 2$, shows that the property in Part (b, i) is equivalent to the property that, for every $a \in J_+$ and every $\varepsilon > 0$, there exist elements $e_1, e_2 \in A$ with $e_j^* e_k = \delta_{j,k}(a - \varepsilon)_+$, for $j, k \in \{1, 2\}$. Notice that all $e_1, e_2 \in A$ with this property are automatically in J.

We proceed by induction over $n \geq 2$ and suppose that we have shown that for every $a \in A_+$ and every $\varepsilon > 0$ there exist elements $e_1, e_2, \ldots, e_n \in A$ – depending from a and ε – that satisfy the following equations 1.4:

$$e_j^* e_k = \delta_{j,k}(a - \varepsilon)_+ \qquad \text{for } j, k \in \{1, 2, \ldots, n\}. \tag{1.4}$$

The assumptions in (b, i) allow to find $n + 1$ elements $e_1, \ldots, e_n, e_{n+1} \in A$ that satisfy Equation (1.4):

Let $\varepsilon > 0$, $a \in J_+$ and $\gamma := \varepsilon/3$ and $2 \leq n \in \mathbb{N}$. There exists $d_1, d_2, \ldots, d_n \in A$ with with $d_j^* d_k = \delta_{j,k}(a - \gamma)_+$ by induction assumption. This implies that $d_k \in J$, $d_k d_k^* \in J_+$ and $d_j^* d_k = 0$ for $j, k \in \{1, 2, \ldots, n\}$. Let $b := \sum_{k=1}^n d_k d_k^* \in J_+$. $(b - \gamma)_+ = \sum_{k=1}^n (d_k d_k^* - \gamma)_+$ because $d_j^* d_k = 0$ for $j \neq k$.

By assumptions in Part (b, i), there exists $f_1, f_2 \in A$ with $f_j^* f_k = \delta_{j,k}(b - \gamma)_+$.

The orthogonality of the $f_k f_k^*$, uniqueness of polar decomposition allow to show with functional calculus that $(b-\gamma)_+^{1/2} d_k = (d_k d_k^* - \gamma)_+^{1/2} d_k = d_k (d_k^* d_k - \gamma)_+^{1/2} = d_k (a-2\gamma)_+^{1/2}$. Thus, $(f_\ell d_j)^* f_\ell d_k = \delta_{j,k}(a-2\gamma)_+(a-\gamma)_+$. Recall that $\varepsilon = 3\gamma$ and (well-) define a function $\varphi \in \mathrm{C}_0(0, \|a\|]_+$ by

$$\varphi(t) := \big((t-\varepsilon)_+ \,/\, ((t-2\gamma)(t-\gamma)) \big)^{1/2} .$$

Then $e_k := f_1 d_k \varphi(d_k^* d_k)$ for $k \in \{1, \ldots, n\}$ and $e_{n+1} := f_2 d_1 \varphi(b)$ satisfy $e_j^* e_k = \delta_{j,k}(b-\varepsilon)_+$ for $j,k \in \{1, \ldots, n, n+1\}$.

(b,ii)⇒(b,iv): By assumptions of Part (b,ii) we can find $e_{k,n} \in A$, $k \in \{1, \ldots, n\}$ with the property

$$e_{j,n}^* e_{k,n} = \delta_{j,k}(h - 2^{-n})_+ \quad \text{for } j,k \in \{1, \ldots, n\} .$$

Define $e_{k,n} := 0$ for $k > n$, and elements $R_k \in A_\infty := \ell_\infty(A)/c_0(A)$ by

$$R_k := (e_{k,1}, e_{k,2}, \ldots, e_{k,n}, e_{k,n+1}, \ldots) + c_0(A) .$$

Then $R_j^* R_k = \delta_{j,k} \cdot \Delta(h)$ for all $j,k \in \mathbb{N}$. Thus, the closed linear span of the elements $R_j R_k^* \in A_\infty$ for $j,k \in N$ generate a stable hereditary C^*-subalgebra D of A_∞ ([6]). The $\Delta(h)$ is contained in the closed ideal of A_∞ that is generated by D.

(b,iv)⇒(b, i): Let $D \subseteq A_\infty$ a stable hereditary C^*-subalgebra of $A_\infty := \ell_\infty(A)/c_0(A)$ and let $I \subseteq A_\infty$ the closed ideal of A_∞ that is generated by D. If $(h,h,\ldots) + c_0(A) =: \Delta(h) \in I$ then $\Delta(a) \in I$ for each $a \in J = \overline{\mathrm{span}(AhA)}$.

By Lemma 2.1.7(ii), there exist for each element $b \in I_+$ elements $c_1, c_2 \in I$ with $c_1^* c_2 = 0$ and $c_1^* c_1 = c_2^* c_2 = b$. If we apply this to $b := \Delta(a)$ and take, for $k \in \{1,2\}$, elements $(d_1^{(k)}, d_2^{(k)}, \ldots) \in \ell_\infty(A)$ with norms $= \|c_k\|$ and $(d_1^{(k)}, d_2^{(k)}, \ldots) + c_0(A) = c_k$ then

$$\lim_{n\to\infty} \|(d_n^{(j)})^* d_n^{(k)} - \delta_{j,k} b\| = 0 \quad \text{for } j,k \in \{1,\, 2\} .$$

Thus, the inequality in Part (b, i) is satisfied if we let $e_1 := d_n^{(1)}$ and $e_2 := d_n^{(2)}$ for suitable $n \in \mathbb{N}$. □

Remark 2.1.13. It is not clear if the equivalent properties in Parts (b,i)–(b,iv) of Lemma 2.1.12 are also equivalent to the following version of "local stability":

For every $\varepsilon > 0$ there exists a stable hereditary C^*-subalgebra D_ε of A such that $(h-\varepsilon)_+$ is in the closed ideal of A generated by D_ε. ([7])

Remark 2.1.14. The implication (b, ii) ⇒ (b,iii) in Lemma 2.1.12 was very indirectly obtained in the proof of Lemma 2.1.12 by the row of implications:
(b, ii) → (b,iv) → (b, i) → (b,iii).
Here is an alternative direct proof for the implication (b, ii) ⇒ (b,iii):

Let $a \in J_+$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Since a is a positive element in the ideal generated by h, there exist $m \in \mathbb{N}$ and elements $g_1, \ldots, g_m \in A$ and $\gamma \in (0, \varepsilon/2)$

[6] This D is isomorphic to $\mathbb{K} \otimes C^*(h)$

[7] It could be that this can be translated into a question about the existence of a full stable hereditary C^*-subalgebra in a certain hereditary C^*-subalgebra of an infinite *amalgamated free product*.

such that $\|a - \sum_{j=1}^{m} g_j^*(h-\gamma)_+ g_j\| < \varepsilon/2$. Let $q := mn$. By Part (b, ii) there exists q elements $e_{j,k} \in A$ with $e_{j,k}^* e_{j,k} = (h-\gamma)_+$ $(1 \le j \le m,\ 1 \le k \le n)$. The elements $f_k := \sum_{j=1}^{m} e_{j,k} g_j$ have the properties $f_\ell^* f_k = 0$ for $\ell \ne k$ and $\|a - f_k^* f_k\| < \gamma$ for $\ell, k \in \{1, \ldots, n\}$.

By Lemma 2.1.9 there exists contractions $d_k \in A$ with $d_k^* f_k^* f_k d_k = (a-\varepsilon)_+$. Thus, $e_j^* e_k = \delta_{j,k}(a-\varepsilon)_+$ for $e_k := f_k d_k$ and $j, k \in \{1, \ldots, n\}$.

`Some old citation changes/compare again:`

Proposition A.21.4 (`A.1.24 or:  A.old.2.4`)

Proposition A.21.4 (or: `K.2.4, A.old.2.4 also in old:  A.8.1`)
Only 1-cited in proof of Lemma 2.2.3.

Lemma A.6.1 (old A.1.9.)

Proposition A.21.4 and/or Lemma `lem:A.old.3.4c` = ?? Lemma 2.1.22

`(old A.1.24 (= lem:A.old.2.4) - old A.1.28)`

Lemma 2.1.15 `covert by new lem:2.advance.Kadison.transit exhausting old:  lem:A.old.3.4b or lem:2.6-1`

The Part (i) of the following Lemma 2.1.15 is one of the possible characterization of projections in the "socle" of bi-duals A^{**} ([8]). The observation that they are "closed" projections imply a very important and often used "excision" result.

Lemma 2.1.15 (Generalized Kadison transitivity and Excision).

Let A a C^*-algebra and $p \in A^{**}$ a non-zero projection.

(i) *The W^*-subalgebra $pA^{**}p$ of A^{**} has finite dimension, if and only if, there are pairwise inequivalent irreducible representations $D_k \colon A \to \mathcal{L}(\mathcal{H}_k)$ and projections $q_k \in \mathcal{L}(\mathcal{H}_k)$* of finite rank *such that for the natural normal (and unital) extensions of D_k to A^{**} satisfy*

$$(D_1 \oplus \ldots \oplus D_n)(pap) = q_1 D_1(a) q_1 \oplus \ldots \oplus q_n D_n(a) q_n$$

for $a \in A$ and that $(D_1 \oplus \cdots \oplus D_n)|pA^{**}p$ is faithful on $pA^{**}p$ ([9]).

(ii) (Generalized Kadison transitivity.) *If $p \in A^{**}$ is a projection such that $M := pA^{**}p$ is finite-dimensional, then the natural C^*-morphism π from $\{p\}' \cap A$ into $pA^{**}p$, given by $a \mapsto ap$, is an* epimorphism, *and there exists a C^*-morphism*

$$\psi \colon\ \mathrm{C}_0((0,1], M) \to \{p\}' \cap A$$

such that $\pi(\psi(f)) = \psi(f)p = f(1)$ for all $f \in \mathrm{C}_0((0,1], M)$.

(iii) (Generalized Excision Lemma.) *Suppose that $M := pA^{**}p$ is finite-dimensional, $\Omega \subseteq A$ is a compact subset of A and*

$$\psi \colon\ \mathrm{C}_0((0,1], M) \to \{p\}' \cap A$$

[8] "socle" is french, but the english translation "base" does not fit here.

[9] Our notation does not distinguish between a representation D of A and its normal extension to A^{**}, given by $\Pi \circ D^{**}$ via the natural normal *-epimorphism $\Pi \colon \mathcal{L}(\mathcal{H})^{**} \to \mathcal{L}(\mathcal{H})$.

is any C^**-morphism such that*

$$\pi(\psi(f)) = \psi(f)p = f(1) \qquad \textit{for all } \; f \in \mathrm{C}_0((0,1], M)\,. \tag{1.5}$$

Then there exists a positive contraction $g \in A_+$ *with* $gp = p$ *such that*

$$\lim_{n\to\infty} \|g^n\big(a - \psi(f_0 \otimes pap)\big)g^n\| = 0 \quad \textit{for all } \; a \in \Omega\,, \tag{1.6}$$

where $f_0(t) := t$*, i.e.,* $(f_0 \otimes pap)(t) = t \cdot pap \in pA^{**}p$ *for* $t \in [0,1]$*.*

We can Ω replace by the compact set $\Omega \cup \{\psi(f_0 \otimes c)\,;\; c \in M, \|c\| \le 1\}$, without change of other assumptions or conclusions in Part (iii).

Notice that $b - \psi(f_0 \otimes pbp) \in L^* + L$ for all $b \in \Omega$, because $L^* + L$ is the kernel of $A \ni a \mapsto pap$, where $L \subseteq A$ denotes the closed left-ideal

$$L := \{a \in A\,;\; p(a^*a)p = 0\,\}\,.$$

A suitable decomposition of Ω could help / can help / to give a proof of Parts (ii) and (iii). See the arguments for the special case where L is a left-kernel of a pure state on A.

Proof. (i): If one does not go the “rigorous” way mentioned in our Remark 2.1.19 and use instead e.g. G.K. Pedersen’s improved version [**616**, thm. 2.7.5] of the Kadison transitivity theorem, then we have to show before at least the (almost obvious?) existence of some normal unital representation $H\colon A^{**} \to \mathcal{L}(\mathcal{H})$ with the property that the restriction to $pA^{**}p$ is faithful and that $H(p)$ has finite rank in $\mathcal{L}(\mathcal{H})$, i.e., it is necessary to show first that H can be selected with the additional property $H(p) \in \mathbb{K}(\mathcal{H})$. The following is such an elementary argument that shows the existence of H with this for the application of [**616**, thm. 2.7.5] very needed properties:

Let $p^*p = p \in A^{**}$ nonzero. Suppose that $pA^{**}p$ has finite (linear) dimension. Then there are projections $q_1, \ldots, q_n$ in the center of $pA^{**}p$ such that $q_kA^{**}q_k = q_kpA^{**}p \cong M_{n_k}$ and $q_1 + \ldots + q_n = p$. We find “minimal” projections $0 \ne e_k \le q_k$ with $e_kA^{**}e_k = \mathbb{C} \cdot e_k$. It implies that central projections $Q \in Z(A^{**})$ can only satisfy one of $Qq_k = q_k$ or $Qq_k = 0$. Moreover, $Qe_k = 0$ implies $Qq_k = 0$. Let R_k the maximal central projection with $R_ke_k = 0$ and let $Q_k := 1 - R_k$. The $Q_k \in Z(A^{**})$ is the central support projection of e_k and $A^{**}Q_k$ is a type-I-factor. By definition of the Q_k holds $pQ_k = q_k$, $Q_kQ_j = 0$ for $k \ne j$ and $p(Q_1 + \ldots + Q_n) = p$. Thus, there are Hilbert spaces $\mathcal{H}_k$ and normal *-isomorphisms $d_k\colon A^{**}Q_k \to \mathcal{L}(\mathcal{H}_k)$ from $A^{**}Q_k$ onto $\mathcal{L}(\mathcal{H}_k)$. Let $D_k(a) := d_k(aQ_k)$ for $a \in A$. Here we identify A with its natural image in the W*-algebra A^{**}. Clearly the restrictions $D_k|A$ to A of the normal *-representations D_k of A^{**} are pairwise non-equivalent irreducible representations of A and the normalization D of the direct sum $(D_1|A) \oplus \cdots \oplus (D_n|A)$ of this irreducible representations is identical with

$$D_1 \oplus \cdots \oplus D_n\colon\; A^{**} \to \mathcal{L}(\mathcal{H}_1 \oplus_2 \cdots \oplus_2 \mathcal{H}_n)\,.$$

Moreover $D(p)$ is a projection of finite rank $\mathrm{Rk}(p) \le \mathrm{Dim}(pA^{**}p)$.

Conversely, suppose that $p \in A^{**}$ is a projection and that pairwise inequivalent irreducible representations $D_k \colon A \to \mathcal{L}(\mathcal{H}_k)$, $k = 1, \ldots, n$ and projections $q_k \in \mathcal{L}(\mathcal{H}_k)$ of finite rank exist such that, for $a \in A$ and the normal extensions of D_k to A^{**} (again denoted by D_k), holds

$$(D_1 \oplus \cdots \oplus D_n)(pap) = q_1 D_1(a) q_1 \oplus \cdots \oplus q_n D_n(a) q_n$$

and that $(D_1 \oplus \cdots \oplus D_n)|pA^{**}p$ is faithful on $pA^{**}p$. Now use that irreducible *-representations are always non-degenerate, direct sums of non-degenerate *-representations are non-degenerate, and that normalizations of non-degenerate *-representations are unital. It follows that the direct sum $D := D_1 \oplus \cdots \oplus D_n$ of the normal extensions of D_k to A^{**} is unital, satisfies $D(p) = q_1 \oplus \ldots \oplus q_n$, – i.e., $D(p)$ has finite rank – and $D|pA^{**}p$ is faithful. Thus, $pA^{**}p$ has finite linear dimension (equal to sum of squares of the ranks $\sum_k \mathrm{Rk}(q_k)^2$). This properties show that the above selected representation $D \colon A \to \mathcal{L}(\mathcal{H})$ satisfies all assumptions of [**616**, thm. 2.7.5].

(ii): With Part(i) in hand, we are in position to use [**616**, thm. 2.7.5] and get the desired *surjectivity* of the natural C^*-morphism $\pi \colon \{p\}' \cap A \mapsto pA^{**}p$ with $\pi(a) := ap = pap$ for $a \in \{p\}' \cap A$.

Alternatively, – but with some in Remark 2.1.18 given *necessary additional* observations –, one could use also [**400**, thm. 5.4.5] and its generalization by S. Sakai [**704**, thm. 1.21.16] to derive this surjectivity.

In fact we decide here to use Part(i) to obtain – from the assumption that $pA^{**}p$ is of finite dimension – the following special kind of *-representation D exists:
Let D a normal unital *-representation of A^{**} onto a von-Neumann algebra in some $\mathcal{L}(\mathcal{H})$ and $p \in A^{**}$ is a projection such that $D|pA^{**}p$ is faithful and $D(p)$ is of finite rank, i.e., $\mathrm{Dim}(D(p)\mathcal{L}(\mathcal{H})) < \infty$. Then there exists a central projection $Q \in Z(A^{**})$ with $Qp = p$ such that $D|QA^{**}Q$ is faithful (and again unital).

Thus, the natural C^*-morphism $(\{p\}' \cap A) \ni a \mapsto pa \in pA^{**}p$ is surjective by this observation and [**616**, thm. 2.7.5].

(See Remark 2.1.18 or Remark 2.1.19 for alternative proofs of this surjectivity.)

Let π denote the natural *-epimorphism from $A_0 := \{p\}' \cap A$ onto $M := pA^{**}p$, and let $J := \ker(\pi)$, and consider the evaluation map $\eta \colon CM \to M$ given by $f \mapsto f(1) =: \eta(f)$ from $CM := \mathrm{C}_0((0,1], M)$ onto M. By Proposition A.8.4, the cone $CM := \mathrm{C}_0((0,1], M)$ is projective. Thus, there exist a C^*-morphism $\psi \colon CM \to A_0 \subseteq A$ with $\pi \circ \psi = \eta$.

(iii): Recall that $M := pA^{**}p$ and $CM := \mathrm{C}_0((0,1], M)$. By Part(ii), there exists a C^*-morphism $\psi \colon CM \to A_0 := \{p\}' \cap A$ that satisfies $\pi(\psi(f)) = f(1) = pf(1)p$ for all $f \in CM$.

The separable case has to do with Proposition A.15.2 and the determination of the multiplicative domain of $\pi_{L^*+L} \colon A \to A/(L^* + L)$ considered in Theorem A.16.5(iii). ?????

The excision result needs Lemma A.21.1 on peak elements.

The non-separable case can then be considered by selecting a suitable separable C^*-subalgebra B of A that contains $\psi(CM)$ and the given separable subset. This can be done with help of the separable selection Lemma B.14.1 and Proposition A.15.2.

Later ??? bring ??? Proposition A.21.4 back to Chapter 2.???

Or refers in place of his proof to here???

To get more precise in calculations we discuss some topics ... with help of listed below/above.. ???

Here, $p \in A^{**}$ is a projection such that $M := pA^{**}p$ is a C^*-algebra of finite (linear) dimension, π denotes the restriction to A_0 of the completely positive contraction $V_p(a) := pap$ for $a \in A^{**} \supseteq A$.

The C^*-subalgebra B_0 of A_0 generated by $\psi(CM)$ is separable and $B_0/(B_0 \cap J) = A_0/J \cong M$ is unital, where $J \subset A_0$ is the kernel of π.

Obviously (why ???) $J = D := A \cap ((1-p)A^{**}(1-p))$ is a hereditary C^*-subalgebra of A and is equal to the kernel ideal of $\pi\colon a \in A_0 \mapsto pa = pap \in A^{**}$.

$L := A \cap (A^{**}(1-p))$, $A =? = L^* + L + A_0$, $D = L^* \cap L = J$, $A_0 = B_0 + D = \mathcal{N}(A, D)$.

By Proposition A.15.2, $L^* + L$ is a closed linear subspace of A and the natural map $A/(L^*+L) \to pA^{**}p$ induced by $\pi\colon A \ni a \mapsto pap \in pA^{**}p$ defines a completely isometric isomorphism from $(A/(L^*+L))^{**}$ onto $pA^{**}p =: M$. Since $\pi(\psi(CM)) = M$ It follows that $A = L^*+L+\psi(CM)$. Let A_1 denote the separable C^*-subalgebra of A generated by $\Omega \cup \psi(CM)$. By Lemma B.14.1, there exists a separable C^*-subalgebras B of A with the property that $A_1 \subseteq B$ and $B \cap (L^* + L) = (B \cap L)^* + (B \cap L)$ and the natural map from $B/\big((B\cap L)^* + (B \cap L)\big)$ to $A/(L^*+L)$ is completely isometric and completely positive.

Since $A/(L^*+L) \cong M$ and $\pi_{L^*+L} \circ \psi = \mathrm{id}_M$ it follows that we can repeat the above consideration in case for separable A now for B and $\Omega \cup \psi(CM)$.

```
    Here possible end of proof,
if separable case is before completely shown.
Or is it below?
```

The crucial points of the proof are to show that the projection p is closed, and that the open support projection $q_D \in A^{**}$ of D is equal to $1-p$ and $J = D = L^* \cap L$.

Then $e \in A_+$ with $\|e\| = 1$, $e(1-e) \in D$, $ep = p$ and $e(1-e)$ strictly positive in D exists. By separability of D (if A is separable) and p is the unit of $M = pA^{**}p$.

Apply the Lemma A.21.1 to $E := \pi^{-1}(\mathbb{C} \cdot p) \subseteq A_0$ and J to get a positive contraction $g \in E$ with the properties that $\pi(g) = p$ and $g - g^2 \in J_+$ is strictly positive in J.

Then g is strictly positive in A and has the excision property of Equation (1.6).

(1) in case where A is separable, that $A = A_0 + DA + AD$ and that there exists $g \in (A_0)_+$ with $\|g\| = 1$, $\pi(g) = p$ and $g - g^2$ strictly positive in D. (Because then g has obviously the in (iii) proposed properties ????

$A_0 = \mathcal{N}(A, D) = D + \psi(CM)$???

(2) Reduction of the general case to the separable case if A is non-separable:

Find (!!!) separable C^*-subalgebra B of A with $\psi(CM) \cup \Omega \subseteq B$ such that $F := B \cap D$ satisfies $B = BF + FB + \psi(CM)$ and $B/(BF + FB) \cong M$ naturally.

(The point is to manage that $\mathcal{N}(B, F) \subseteq A_0$ and $BF = B \cap (AD)$.???)

Could it be that $\mathrm{dist}(b, BF) = \mathrm{dist}(b, AD)$ for all $b \in B$ is sufficient?

Notice that $\psi(CM) \cap D$ is an ideal of $\psi(CM)$.

Then apply above arguments to separable B (in place of A).

The argument works well if $D = J$ is σ-unital. Reduce to this case !!! ???

```
Next to be filled in / corrected !?
It is necessary to use characterizations of open and closed
projections -- as considered in Appendix A.
Take better all to there.
```

And $1 - p \in A^{**}$ is the "open" support projection of D, i.e., $D^{**} \cong \mathrm{wcl}(D) = (1 - p)A^{**}(1 - p)$.

We consider first the case where A is separable:

Cite here the Prop. ?? on closed/open projection!!!
Existence of contraction $e \in A_+$ with $pep = p$ has to be deduced!!!

There exists $e \in A_+$ with $pep = p$ and $\|e\| \leq 1$ by Part (ii), e.g. $e := \psi(f_0 \otimes 1_M)$.

Let $f \in D_+$ a strictly positive contraction for $D := L^* \cap L$, where $L := A \cap (A^{**}(1 - p))$.

Then $D = A \cap (1 - p)A^{**}(1 - p)$ is the kernel of the C^*-morphism $A_0 = \{p\}' \cap A \to pA^{**}p$, because $a \in \{p\}' \cap A$, if and only if, $a \in A \cap (pA^{**}p + (1 - p)A^{**}(1 - p))$. And $pap = 0$ if and only if $a \in (1 - p)A^{**}(1 - p))$.

We get for the above taken $e, f \in A_+$ that $e, f \in \{p\}' \cap A = \mathcal{N}(A, D)$, and that there exists $g \in (\{p\}' \in A)_+$ with $\|g\| \leq 1$, $pgp = p$ and $g(1 - g)$ is strictly positive in D.

Build g from e and f. !!!!!!

$a \mapsto pap = V(a)$ has kernel $A \cdot D + D \cdot A$???? $V(a - \psi(f_0 \otimes V(a))) = V(a) - pV(a) = 0$???

$g^n d, dg^n \to 0$ for all $d \in D$, because $g(1 - g)$ is strictly positive in D.

Way of arguments:

1) p is closed,

2) the open $1-p$ is the right support projection of the closed left ideal $L := A \cap A^{**}(1-p)$. (Thus $L^{**} = A^{**}(1-p)$.)

3) $pA^{**}p = pAp \cong A/(R+L)$ for $R := L^* := \{a^* ; \ a \in L\}$

4) $\{p\}' \cap A = \mathcal{N}(A, R \cap L)$

5) The completely positive map $V(a) := pap$ from A onto $pA^{**}p$ has kernel $L^* + L$?, i.e., $((1-p)A^{**} + A^{**}(1-p)) \cap A = L^* + L$, for $L := A \cap (A^{**}(1-p)) = \{a \in A ; \ ap = 0\}$ by p closed?

6) Cite here definition of $\mathcal{N}$?????

$A = L^* + L + \mathcal{N}(A, L^* \cap L)$

7) $D := L^* \cap L$ is ideal of $\mathcal{N}(A, D)$, $\mathcal{N}(A, D) = \{p\}' \cap A$.

8) ($M := pA^{**}p \neq \{0\}$ is finite-dimensional) $\Omega \subseteq A$ The compact subset Ω of A and the image of $\psi \colon \mathrm{C}_0((0,1], M) \to A$ generate a separable C^*-subalgebra B of A.

9) Can enlarge B to a separable C^*-subalgebra C of A such that $B \subseteq C$, $C \cap (R+L) = (C \cap R) + (C \cap L)$, $(C \cap D) \cdot C = L \cap C$, $\mathcal{N}(C, C \cap D) + C \cap (R+L) = C$,

$\{p\}' \cap C = \mathcal{N}(C, C \cap D)$

$(C \cap R) + (C \cap L)$ is always closed. See Appendices ??? behind Reduction to separable.

More ??

Now proceed as in case of separable A.

Fact list:

1. p is a closed projection with

$$p \in \mathrm{wcl}(\{p\}' \cap A) \cong \{p\}' \cap A^{**} = (1-p)A^{**}(1-p) + pA^{**}p$$

Needs that $pa^*ap = 0$ if and only if $a \in A^{**}(1-p)$.

π extends to the c.p. contraction $V \colon A \ni a \mapsto pap \in pA^{**}p$

i.e., $\{p\}' \cap A$ is the multiplicative domain of V.

1a.

$J :=$ kernel of π is the same as $D := A \cap (1-p)A^{**}(1-p)$, because $D \subseteq A_0$ and $D \subseteq J$. $a \in J \subseteq A_0$ implies $pa = ap = 0$, thus $a \in A$ and $a = (1-p)a(1-p)$.

and

1b.

$1 - p \in A^{**}$ is the open support projection of D,

???? Needs $pA^*p \cong (pA^{**}p)_*$ is $\sigma(A^*, A)$-closed ???

Unit ball of pA^*p is equal to unit ball of $X^\top$, for $X := \{a \in A ; \ v(a) = 0, \ v \in pA^*\}$???

because $J = D \subseteq (1-p)A^{**}(1-p)$ and the open support projection $q_D \in A^{**}$ of D is contained in the weak closure of of A_0 in A^{**} by $J = D$ and satisfies $q_D + p \leq 1$, i.e., $(1 - q_D) \geq p$. If $(1 - q_D) - p \neq 0$, then

$A_0/J \cong M = pA^{**}p$ via implies that $\mathrm{wcl}(A_0)/(q_D A^{**} q_D) \cong M$.

Needs $q_D A^{**} q_D = \mathrm{wcl}(D) \subseteq (1-p)A^{**}(1-p)$

The map $(1-q_D)A^{**}(1-q_D) \to pA^{**}p$ given by $(1-q_D)a(1-q_D) \mapsto pap$ is surjective, and is multiplicative for a in the weak closure of A_0 in A^{**}.

because p is a closed projection ????

why it is closed??? $pAp = pA^{**}p$ – confirmed in Part (ii).

1c.

The set $L := A \cdot D$ of products $a \cdot d$ is a closed left ideal of A, because the closed linear span of L is obviously contained in A and $L = L \cdot D$ by Cohen factorization.

(The arguments for this are similar to those in Part (o) in proof of Lemma 2.1.7.)

with $A/(L^* + L) \cong M$???

To be shown.

$D := L^* \cap L = L^* \cdot L$ is kernel of $\pi\colon \{p\}' \cap A \to M$. $D = ((1-p)A^{**}(1-p)) \cap A$? by $L = (A^{**}(1-p)) \cap A$

Needs: p closed, and $1 - p$ support of D.

$q \in A^{**}$ open then $qAq \subseteq A$

??? (If A is unital, then this is wrong in general, because then this implies $pinA$

$L = \overline{A \cdot D}$?

The C^*-morphism ϕ defines a linear map λ from A into $L^* + L$ given by $\lambda(a) := a - \phi(V(a))$.

Let $\{d_n\} \in D_+$ an approximate unit of D given by contractions

The rest is reduction to separable case.

We enlarge B_0 to the still separable C^*-algebra $B_1 := C^*(\Omega \cup B_0)$. The natural map $b \mapsto pbp$ is a completely positive contraction that maps the unit-ball of B_1 onto the unit ball of $pA^{**}p$ and maps $\phi(f_0 \otimes x)$ to $x \in pA^{**}p$.

But this is not enough to find in B_1 the desired positive contraction g. ....????

Let J the kernel ideal of $\{p\}' \cap A \to M := pA^{**}p = pAp$. Then one can show that $L := A \cdot J$ is a closed left ideal of A such that $L^{**} = A^{**}(1-p)$ (by noticing that p is a closed projection and $1-p$ is an open projection), that $\phi\colon A \ni a \mapsto pap \in M$ satisfies $\phi^{-1}(0) = L^* + L$ and defines a completely positive and completely isometric isomorphism from the C^*-system $A/(L^* + L)$ onto M.

If B is some separable C^*-subalgebra of A such that ϕ maps the closed unit-ball of B onto the closed unit-ball of M, then this does not imply that $B/(\phi^{-1}(0) \cap B)$ is completely isometric to M.

We must do a careful selection of a bigger separable C^*-algebra $B_e \supseteq B$ of A such that the kernel ideal I of $\pi|B_e \colon \{p\}' \cap B_e \to M$ defines a closed left-ideal $L_e := B_e \cdot I$ with the property that the natural map from $B_e/(L_e^* + L_e)$ onto M is completely positive and completely isometric.

The key for this inductive selection is the Lemma B.14.1.

We get from Lemma B.14.1 a separable C^*-subalgebra $B_e \supseteq B$ such that $B_e \subseteq \overline{BAB}$, $(B_e \cap L)^* + (B_e \cap L)$ is closed in B_e, that it is the same as $B_e \cap (L^* + L)$ and that the natural map from $B_e/(B_e \cap (L^* + L))$ to $A/(L^* + L) \cong pA^{**}p$ defines a surjective completely positive complete isometry from $(\{p\}' \cap B_e)/(J \cap B_e)$ onto $pA^{**}p$.

NOTICE 31.12. 2017:

If $B \subseteq A$ is a C^*-subalgebra of A and $L \subseteq A$ is a closed left ideal and $R \subseteq A$ is a closed left ideal, then $R + L$ is a closed subspace of A by Part (iv) of Proposition A.15.2.

This implies that $B \cap (R + L)$, $B \cap R$ and $B \cap L$ are closed linear subspaces of A. Thus, $B \cap R$ and $B \cap L$ are closed right and left ideals of B. Again, by Part(iv) of Proposition A.15.2, $(B \cap R) + (B \cap L)$ is a closed linear subspace of B, obviously contained in the closed linear subspace $B \cap (R + L)$ of B.

By our pre-assumption that $(B \cap R) + (B \cap L)$ is dense in $B \cap (R + L)$, we get the equality

$$B \cap (R + L) = (B \cap R) + (B \cap L)\,.$$

(Without this "density assumption" the formula would be wrong: $A : \mathbb{C} \oplus \mathbb{C}$, $B := \mathbb{C} \cdot (1,1)$, $L := \mathbb{C} \oplus 0$, $R := 0 \oplus \mathbb{C}$. $A = L + R$, $B \cap L = 0$, $B \cap R = 0$ and $B \cap (L + R) = B$.)

But requires an additional selection process that also shows that

$$B/(B \cap (R + L)) \to A/(R + L)$$

is isometric, respectively, is completely isometric, by tensoring all with M_n ...

Needs to show that $\operatorname{dist}(b, R + L) = \operatorname{dist}(b, B \cap (R + L))$ ...

This seems to work only with a suitable iteration process, until $B \otimes \mathbb{K} \subseteq A \otimes \mathbb{K}$ is relatively weakly injective.

But above procedure has to be suitably changed to an inductive procedure that allows to construct a suitable separable C^*-subalgebra $B \subseteq C \subseteq A$ by Corollary A.15.3.

Moreover it hold then that the natural C^*-monomorphism $\{p\}' \cap B_e \to \{p\}' \cap A$ is surjective by separability of $A/(R + L)$.

???? WHERE it is SHOWN ????

$L \cap B_e = B_e \cdot (J \cap B_e)$.

Recall here that $J \subseteq \{p\}' \cap A$ is the kernel of the surjective C^*-morphism $\{p\}' \cap A \to pA^{**}p$.

This implies that $B_e \cdot (J \cap B_e) = L \cap B_e$, $B_e/(L \cap B_e + L^* \cap B_e) \cong M$. and $B_e/(J \cap B_e) \cong M$, with all isomorphisms canonical.

It follows that the separable C^*-algebra $J \cap B_e$ contains a strictly positive contraction and the C^*-subalgebra $\pi^{-1}(\mathbb{C} \cdot 1_M) \subseteq (\{p\}' \cap B_e)$ contains a positive contraction a_0 with $\pi(a_0) = 1_M$ and $(1 - a_0)a_0$ strictly positive in $J \cap B_e$.

Here the set $\Omega \subseteq A$ is not considered.

Consider e.g. the separable C^*-subalgebra $A_1 := C^*(\Omega, B_0) \subseteq A$.

It it a question if A_1 has to be suitably enlarged (still separable and "suitable").

This bigger separable C^*-algebra B_e should have the property that B_e plays the same role as A.

Say, A itself is separable. $M := A/(L^* + L)$, $D := L^* \cap L$, $\mathcal{N}(A, D)/D \cong M$,

$\{p\}' \cap A? =? \mathcal{N}(A, D)$??

Then $L_1 := A_1 \cap (A^{**}(1 - p))$ is a closed left-ideal of A_1, $D_1 := L_1^* \cap L_1$ is a hereditary C^*-subalgebra and

????

$A_1/(L_1^* + L_1) \cong pA_1p \cong M$.

????

$\mathcal{N}(D_1, A_1) \supseteq A_1 \cap \{p\}'$ contains $\phi(CM)$ and is (in ???) the multiplicative domain of $a \in A_1 \mapsto pap \in pA^{**}p$.

$\mathcal{N}(D_1, A_1)/D_1 \cong pA^{**}p$? $p(a - \phi(f_0 \otimes pap))p = 0$ for all $a \in A$.

Need $g \in A_1 \cap \{p\}'$ with $g \geq 0$, $\|g\| = 1$, $pg = p$, $g(1 - g)$ strictly positive in D_1.

Then $g^n(X - \phi(f_0 \otimes pXp))g^n \to 0$ for $X \in A_1$ and $n \to \infty$.

Needs that kernel of $A_1 \to pA^{**}p$ is equal to sum $L_1^* + L_1$????

Needs to construct a "regular" separable enlargement of A_1 by "reduction to separable case".

???? By Lemma A.21.1 there exists ????

`Next is OLD version?????`

The proof refines the proofs of Lemmas 2.1.7 or **??** and Remark 2.1.16

Let $M := pA^{**}p$ of finite linear dimension, $C := \{p\}' \cap A$ and $\pi\colon f \in C \to fp \in M$ the C^*-morphism from C into M considered in Part (ii). Then π is an epimorphism by (ii). We define $\varphi\colon \mathrm{C}_0((0, 1], M) \to M$ by $\varphi(f) := f(1)$ for $f \in \mathrm{C}_0((0, 1], M)$.

By Proposition A.8.4, $\mathrm{C}_0((0,1],M)$ is projective. It says that there exists a C^*-morphism $\psi_0\colon \mathrm{C}_0((0,1],M)\to C$ with $\pi\circ\psi=\varphi$, i.e., $\psi(f_0\otimes a)p=a$ for all $a\in M$.

The convex set $S_p(A)$ $(=C_p(A^{**}))$ of positive functionals $\rho\in A^*=(A^{**})_*$ with $\rho(p)=\|\rho\|$ is $\sigma(A^*,A)$-closed, because this set is naturally isomorphic to the *compact* quasi-state space of $M=pA^{**}p$. Thus, the projection $p\in A^{**}$ is closed, and there is a closed left-ideal $L\subseteq A$ such that $q:=1-p$ is the open support projection of L in A^{**}, i.e., $A^{**}q$ is the weak*-closure of L in A^{**}.

It follows, that $\|pap\|=\mathrm{dist}(a,qA^{**}+A^{**}q)=\mathrm{dist}(a,L^*+L)$. In particular, L^*+L is the kernel of the c.p. map $a\in A\to pap\in M$. We get $a-\psi_0(f_0\otimes pap)\in L^*+L$, because $a-\psi_0(f_0\otimes pap)\in A$ and $p(a-\psi_0(f_0\otimes pap))p=0$ for all $a\in A$.

More ????????

Since Ω is a compact subset of A, there is a separable C^*-subalgebra E_0 of A with $\Omega\cup\psi_0(\mathrm{C}_0((0,1],M))\subseteq E_0$.

Since E_0 is separable, there exist a separable C^*-subalgebra $E_1\subseteq A$ with $E_0\subseteq E_1$, $a-\psi_0(f_0\otimes pap)\in(E_1\cap L)^*+(E_1\cap L)$ and $\|pap\|=\mathrm{dist}(a,(E_1\cap L)^*+(E_1\cap L))$ for all $a\in E_0$.

Next (commented?) comes also from Lemma B.14.1.

to be filled in

Part(iii) of Lemma 2.1.15, ??

If $\gamma\colon M_{n_k}\cong Mq_k$ is a *-isomorphism from M_{n_k} onto Mq_k, then there is a natural *-isomorphism τ from F_k onto $F_{k,1}\otimes M_{n_k}$ such that $F_{k,1}=\overline{yEy}=\overline{y(F_k)y}$ for $y:=\psi_0(f_0\otimes\gamma(p_{11}))$ and $\tau(\psi_0(f_0\otimes\gamma(\alpha)))=y\otimes\alpha$ for $\alpha\in M_{n_k}$.

Step ?:
(Recall here that $q:=1-p$ is the above defined open support projection of a hereditary C*-subalgebra of A.)

Text/Statement:

For every separable C^*-subalgebra $C\subseteq A$, there exists a positive contraction $g\in\{p\}'\cap A$ with $gp=p$ and $\lim_n\|g^ncg^n\|=\|pcp\|$ for all $c\in C$.

It is important that $p\in A^{**}$ is "closed", – in the sense that $1-p\in A^{**}$ is the "open" support projection of a hereditary C^*-subalgebra $D\subseteq A$ of A.

Step ?:
More general case:

If $J\lhd B$ contains a strictly positive element $e\in J_+$, and if the C^*-morphism

$$\varphi\colon\ \mathrm{C}_0((0,1],M)\to B/J$$

has kernel $\ker(\varphi)=\mathrm{C}_0((0,1),M)$, – i.e., there is a *-monomorphism $\lambda\colon M\to B/J$ with $\varphi(f)=\lambda(f(1))$ for $f\in\mathrm{C}_0((0,1],M)$ –, then the homomorphic lift $\psi\colon\mathrm{C}_0((0,1],M)\to B$ of φ can be chosen such that $\lim\|g^ne\|=0$ for $g:=\psi(f_0\otimes 1_M)$, where $f_0\in\mathrm{C}_0(0,1]$ is given by $f_0(t):=t$. □

REMARK 2.1.16. The Parts (i,ii) of Lemma 2.1.15 show that there exist a non-zero C^*-morphism $\psi\colon\ \mathrm{C}_0((0,1], M_n) \to A$ if a C^*-algebra A has an irreducible representation $\rho\colon A \to \mathcal{L}(\mathcal{H})$ of dimension $\geq n$. If we let $f_k := \psi(f_0^{1/2} \otimes p_{k,1})$, $e := \psi(f_0 \otimes p_{11})$ with $f_0(t) = t$ for $t \in [0,1]$, we get the following Part (i) and its consequences in Parts (ii), (iii) and (iv):

(i) If A has an irreducible representation of dimension $\geq n$ then there exist elements $e \in A_+$ and $f_1, \ldots, f_n \in A$ that satisfy the relations $\|e\| = 1$ and $f_j^* f_k = \delta_{j,k} e$ for $j, k = 1, \ldots, n$.

(ii) In particular, if $a \in A$ and a^*Aa is not a commutative algebra, then there exists $b \in a^*Aa$ with $b^2 = 0$ and $\|b\| = 1$. (Local Glimm halving Lemma, cf. [**616**, lem. 6.7.1]).

(iii) If each non-zero hereditary C^*-subalgebra D of A has no character, then for every $n \in \mathbb{N}$ and non-zero hereditary C^*-subalgebra D of A we find $e, f_1, \ldots, f_n \in D$ with the relations listed in Part (i).

(In particular, if A is antiliminary, then each non-zero hereditary C^*-subalgebra D of A is also antiliminary, and the D contain a copy of $M_n \otimes \mathrm{C}_0(0,1]$ for each $n \in \mathbb{N}$.)

(iv) If, moreover, A is *strictly antiliminary* (also called *residually antiliminary*, cf. Definition 2.7.2) in the sense that each non-zero quotient A/J of A is antiliminary in the sense of [**616**, sec. 6.1.1], then for every non-zero hereditary C^*-subalgebra D of A, every pure state ρ on D and every $n \in \mathbb{N}$ there exists a C^*-morphism $\psi\colon\ \mathrm{C}_0((0,1], M_n) \to D$ with

$$\rho(\psi(f_0 \otimes p_{11})) = 1\,.$$

(And this is equivalent to the property that no hereditary C^*-subalgebra of A has a non-zero character.)

Indeed, the Parts (i,ii) of Lemma 2.1.15 apply to all non-zero hereditary C^*-subalgebras $D \subseteq A$, because the property that *each D has no character* implies that all non-zero hereditary C^*-subalgebras $D \neq \{0\}$ of A can not have any irreducible representation of finite dimension (even can not contain non-zero compact operators in its images).

LEMMA 2.1.17. *Let ρ a pure state on a C^*-algebra A. We denote the corresponding irreducible representation with cyclic vector ξ_ρ by $D_\rho\colon A \to \mathcal{L}(L_2(A,\rho))$.*

Then there exists for each $a \in A_+$ and $\varepsilon > 0$ a contraction $g := g(a,\varepsilon) \in A$ such that

$$\|D_\rho(a)\| \,<\, \varepsilon \,+\, \rho(g^*ag)\,. \tag{1.7}$$

In particular, for each (fixed) pure state ρ on a simple *C^*-algebra A,*

$$\|a\|^2 \;=\; \sup\{\,\rho(g^*(a^*a)g)\,;\; g \in A,\ \|g\| \leq 1\}\,.$$

Here $L_2(A,\rho) := A/L_\rho$ with $L_\rho := \{a \in A\,;\; \rho(a^*a) := 0\,\}$. It satisfies for all $a \in A$ that $\rho(a) \;=\; \langle D_\rho(a)\xi_\rho, \xi_\rho\rangle$.

More precisely there exists $h^* = h \in A$ with $\|h\| \leq \pi$ such that $\exp(ih) \in A + \mathbb{C} \cdot 1 \in A^{**}$ satisfies Inequality (1.7) with $\exp(ih)$ in place of g. One can find a contraction $f \in A_+$ such that $D_\rho(f)\xi_\rho = \xi_\rho$. Then $g := \exp(ih)f \in A$ satisfies Inequality (1.7).

PROOF. The connected component $\mathcal{U}_0(\widehat{A})$ of 1 in the unitary group $\mathcal{U}(\widehat{A})$ of the unitization

$$\widehat{A} := A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A) \subseteq A^{**}$$

operates transitive on the unit sphere of the Hilbert space $\mathcal{H} := L_2(A, \rho) = D_\rho(A)x_\rho$, because the (reformulated) Kadison transitivity in Lemma 2.1.15(ii,iii) implies that for every $y \in \mathcal{H}$ the natural c.p. contraction

$$V : a + \xi 1 \mapsto p(D_\rho(a) + \xi 1)p$$

maps the multiplicative domain of V onto $p\mathcal{L}(\mathcal{H})p$, where $p \in \mathcal{L}(\mathcal{H})$ is the orthogonal projection onto the complex linear span of $\{y, \xi_\rho\}$, i.e., $p\mathcal{H}$ is 1-dimensional or is 2-dimensional. If it is one-dimensional, then $y = \alpha \cdot \xi_\rho$ with $\alpha = \exp(i\gamma)$ for some $\gamma \in [0, 2\pi)$.

If $\|y\| = 1$ and $U \in p\mathcal{L}(\mathcal{H})p \cong M_2(\mathbb{C})$ is (partial) unitary with $U^*U = p = UU^*$ and $U(\xi_\rho) = y$, then U is an exponential in $p\mathcal{L}(\mathcal{H})p$, because all unitaries in $M_2(\mathbb{C})$ are exponentials, i.e., $U = p\exp(ik)p = \exp(ik)p$ for some $k^* = k \in p\mathcal{L}(\mathcal{H})p$ with $\|k\| \leq \pi$.

By Part (ii) of Lemma 2.1.15 there exists $h^* = h \in \{p\}' \cap A$ with $pD_\rho(h) = k$. It follows for $T := D_\rho(h)$ in $\mathcal{L}(\mathcal{H})$ that

$$\exp(iT)p = p\exp(iT) = \exp(ik)p = \exp(ik) - (1 - p).$$

In particular, $\exp(iT)(\xi_\rho) = \exp(ik)(\xi_\rho) = y$.

If we extend D_ρ naturally to a *-representation of $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ then this says that $D_\rho(\exp(ih))\xi_\rho = y$. If we use that $b\exp(ih) \in A$, then we get that $D_\rho(b\exp(ih))(\xi_\rho) = D_\rho(b)(y)$, and – therefore – that

$$\rho(\exp(ih)^* b^* b \exp(ih)) \;=\; \|D_\rho(b)y\|^2 .$$

We apply this precise general observation to get the desired more relaxed estimate:

Given $0 \neq a \in A_+$ and $\varepsilon \in (0, 1/2)$ we let $\delta := \varepsilon/\mu$ with $\mu := 2(1 + \|a\|^{1/2})$. Then $\delta^2 + 2\delta\|a\|^{1/2} < \varepsilon$.

There exists $y \in \mathcal{H}$ with $\|y\| = 1$ and $\|D_\rho(a^{1/2})\| \leq \delta + \|D_\rho(a^{1/2})y\|$. It implies

$$\|D_\rho(a)\| < \varepsilon + \langle D_\rho(a)y, y\rangle .$$

As we have seen above, there exists $h^* = h \in A$ with $\rho(\exp(ih)^* a \exp(ih)) = \|D_\rho(a^{1/2})y\|^2$. Thus,

$$\|D_\rho(a)\| \;<\; \varepsilon \;+\; \rho(\exp(ih)^* a \exp(ih)) .$$

We get the "relaxed" Inequality (1.7) for $g := f_\delta(a)\exp(ih)$, where $f_\delta(t) := \min(1, \delta^{-1}\max(t-\delta, 0))$ with sufficiently small $\delta \in (0, 1/4)$. □

REMARK 2.1.18. There are several other methods to prove the surjectivity of $\pi\colon (\{p\}' \cap A) \ni a \mapsto pa \in pA^{**}p$ for "closed" projections $p \in A^{**}$, e.g. by modifying the original Kadison transitivity theorem [**400**, thm. 5.4.3, thm. 5.4.5], or – more comfortable – its generalization by S. Sakai [**704**, thm. 1.21.16], but one can directly use the far more general results mentioned in below given Remark 2.1.19.

`Give here also reference to Appendix A!!!`

Here is one of the very elementary ways: A closer inspection of there original proofs shows that both cited original results (of Kadison or Sakai) did prove only that the map

$$(\{p\}' \cap (A + \mathbb{C}\cdot 1)) \ni a \mapsto pa \in pA^{**}p$$

is surjective for $A + \mathbb{C}\cdot 1 \subseteq A^{**}$.

`Check this critics on the citations to Kadison or Sakai again! Simply to avoid a mistake!`

But a simple trick shows that one can replace here $A + \mathbb{C}\cdot 1$ by A itself ([10]):

Let $D_k\colon A^{**} \to \mathcal{L}(\mathcal{H}_k)$ normal and surjective with supports $Q_k \in Z(A^{**})$ and let $p_k := D_k(p) = D_k(Q_k p)$, – as considered in Lemma 2.1.15(i).

Further let D_0 denote the restriction of $D_1 \oplus \cdots \oplus D_n$ to $pA^{**}p$. Consider a unitary $w \in pA^{**}p$ of $pA^{**}p$ and define $u_k := D_k(w) \in p_k\mathcal{L}(\mathcal{H}_k)p_k$.

The advanced forms of the Kadison transitivity theorem [**704**, thm. 1.21.16], (that follows also directly from [**400**, thm. 5.4.5] or [**616**, thm. 2.7.5]) applied to the unitization $\widehat{A} = A + \mathbb{C}\cdot 1 \subseteq A^{**}$ of A and to the pairwise inequivalent irreducible representations $D_k\colon \widehat{A} \to \mathcal{L}(\mathcal{H}_k)$ show the following: There is a unitary $U \in \widehat{A}$ with $p_k D_k(U) p_k = u_k$ for $k = 1, \ldots, n$.

Thus $D_0(pUp - w) = 0$ for the above defined D_0. Since $D_0|pA^{**}p$ is faithful, we get $pUp = w$. It implies $pU^*pUp = p = pUpU^*p$, $pU(1-p) = 0 = (1-p)Up$ and $(pU - Up) = 0$. It shows that the C^*-morphism $\psi\colon a \in C_1 \mapsto ap \in pA^{**}p$ is a *-epimorphism from $C_1 := \{p\}' \cap (A + \mathbb{C}1)$ onto $pA^{**}p$.

On the other hand, by $1\cdot p = p = p\cdot 1$, $C_1 = C + \mathbb{C}1$ for $C := \{p\}' \cap A$ and $C = C_1$ or $C_1/C \cong \mathbb{C}$. Obviously, in the second case $\psi(C)$ is a closed ideal of $pA^{**}p$ of co-dimension *at most* one!

If we use that $pA^{**}p$ has finite linear dimension, then the latter implies that $pA^{**}p$ has a character, given by a central projection q of $pA^{**}p$.

Now comes the "dirty trick": We replace A by $A \otimes M_2$ and p by

$$p \otimes 1_2 \in A^{**} \otimes M_2 \cong (A \otimes M_2)^{**}.$$

Then $(p \otimes 1_2)(A \otimes M_2)^{**}(p \otimes 1_2) \cong pA^{**}p \otimes M_2$ has no character anymore.

[10] This trick was *not* observed or precisely mentioned in any of the 3 above cited books.

Above considerations (applied to $A \otimes M_2$ instead on A) show that the image of the natural C^*-morphism

$$\eta\colon \{p \otimes 1_2\}' \cap (A \otimes M_2) \to pM^{**}p \otimes M_2$$

is an ideal of $pA^{**}p \otimes M_2$ of co-dimension ≤ 1. But $pM^{**}p \otimes M_2$ has no character. Thus, η must be surjective.

The surjectivity of η implies the surjectivity of the map π defined by $\pi(a) := ap$ for $a \in \{p\}' \cap A$. Indeed: Let $x \in pA^{**}p$. There exists $b \in \{p \otimes 1_2\}' \cap A \otimes M_2$ with $b \cdot (p \otimes 1_2) = x \otimes e_{11}$. It follows that there is $a \in A$ with $(1 \otimes e_{11})b(1 \otimes e_{11}) = a \otimes e_{11}$ and $ap = x$, where $e_{11} \in M_2$ is the upper-left matrix unit. Moreover,

$$[a, p] \otimes e_{11} = (1 \otimes e_{11})[b, p \otimes 1_2](1 \otimes e_{11}) = 0\,.$$

This proves that the C^*-morphism $\pi\colon \{p\}' \cap A \to pA^{**}p$ is surjective. It will be used in the proof of Lemma 2.1.15(ii), by appying first one of the cited (older) textbook versions of the Kadison transitivity theorem.
HERE ends Remark 2.1.18

REMARK 2.1.19. One can shorten the proof of all Parts of Lemma 2.1.15 considerably by using more general results that we anyway have to use for our embedding results in Chapter 5.

First check that each projection $p \in A^{**}$ of finite rank (in the sense that $pA^{**}p$ has finite dimension) is a "closed" projection in A^{**}, i.e., $(1-p)$ is the supremum of an upward directed net of positive contractions in A, or equivalently:
The closed left ideal $L := A \cap (A^{**}(1-p))$ of A has an open right support projection that is equal to $1-p$. This means that the weakly closed left ideal $A^{**}(1-p)$ of A^{**} is the bi-polar of a closed ideal $L := A \cap (A^{**}(1-p))$ of A (that is, in our here considered special case, the intersection of finitely many left-kernels $L_\rho := \{a \in A\,;\ \rho(a^*a) = 0\}$ of pure states ρ on A).

This idea has been used in the proof of [**704**, thm. 1.21.16] ([11]), but in the there considered special case it was not completely proved that the natural C^*-morphism

$$\mathcal{N}(A, L^* \cap L) \to (A/(L^* + L)) \cap \mathcal{M}(A/(L^* + L)) \subseteq pA^{**}p$$

by using the natural isomorphism $pA^{**}p \cong A^{**}/((1-p)A^{**} + A^{**}(1-p))$, is a *surjective C^*-morphism* with kernel $\operatorname{Ann}(A, L^* \cap L)$ – that is a closed ideal of $\mathcal{N}(A, L^* \cap L)$.

The reason for the surjectivity of the map from normalizers onto the multiplier algebra of the quotient C^*-space is that the closed unit ball of A maps *onto* the closed unit ball of $A/(R+L)$ if R is a closed right-ideal of A and L is closed left-ideal of A.

`It should be shortened by reference to the appendices`

It comes by using the bipolar theorem for

?????

[11] Rewritten here in our notation / terminology – and not as in [**704**]

those convex subsets of Banach spaces that are closures of its *open* (again convex) interiors and there bi-duals from the more universal (and almost trivial) *perturbation property* of maps $a \in M \mapsto paq \in pMq$ for projections p, q in W*-algebras M:
If $a, b \in M$ satisfy $\|a\|, \|b\| < 1$ and $\|paq - pbq\| < \varepsilon$ then there is $c \in M$ with $\|c\| < 1$, $pbq = pcq$, and $\|a - c\| \leq \varepsilon + \sqrt{2\varepsilon}$.

(A study of $\|a\|\, u_1$ and $\|b\|\, u_2$ in place of a and b defined by suitably "iterated" construction of "Halmos" unitaries ([12]) $u_1, u_2 \in M_3(M)$ leads to an idea of proof of this non-obvious fairly sharp inequality.)

In the very special case considered in Lemma 2.1.15(iii) we have the particular situation that $\mathcal{M}(A/(L^* + L)) = pA^{**}p + (1-p)A^{**}(1-p)$ where $L := A \cap (A^{**}(1-p))$ is nothing else the intersection of the "left"-kernels of finitely many pure states on A. The C^*-subalgebra $B := \{p\}' \cap A$ of A maps always onto $\mathcal{M}(A/(L^* + L)) \cap pAp$, which is in this special situation equal to $pA^{**}p$ because p is a closed projection.

We can then simply consider the multiplier algebra $\mathcal{M}(A//E) \cong pA^{**}p$ of the unital operator system $A/(L^* + L) =: A//E$ for $E := L^* \cap L$ to get the desired "precise" variant of an "advanced" Kadison transitivity theorem by the projectivity of cones over C^*-algebras of finite dimension.
HERE ends Remark 2.1.19
More precise and short references to the Appendix Section would be useful.

In the three following lemmata $\mathbb{C}^n$ denotes the complex Hilbert space $\ell_2(n)$ of dimension $n \in \mathbb{N}$. Next one uses an extension that reduce all to the until case:

Lemma 2.1.20. *Let C a unital C^*-algebra, $B \subseteq C$ a C^*-subalgebra and $V \colon B \to M_n = \mathcal{L}(\mathbb{C}^n)$ a c.p. contraction.*

There exists a unital c.p. map $U \colon C \to M_n$ and a positive contraction $d \in (M_n)_+$ with $V(b) = dU(b)d$ for all $b \in B$.

Moreover, $d = V^{**}(1)^{1/2}$ for $1 \in B^{**}$.

Proof. The "Stinespring" dilation of the c.p. map V is given by some Hilbert space $\mathcal{H}_0$, a *-representation $\rho \colon B \to \mathcal{L}(\mathcal{H}_0)$ and a linear map $g \colon \mathbb{C}^n \to \mathcal{H}_0$ such that $V(b) = g^* \rho(b) g$.

We can suppose here that ρ is non-degenerate, i.e., $\mathcal{H}_0 = \overline{\rho(B)\mathcal{H}_0}$, because otherwise we can replace g by $q \cdot g$ and ρ by $\rho(b)|q\mathcal{H}_0$, where q is the orthogonal projection from $\mathcal{H}_0$ onto the closure of the linear span of $\rho(B)\mathcal{H}_0$. Then the normalization $\overline{\rho} \colon B^{**} \to \mathcal{L}(\mathcal{H}_0)$ satisfies $\overline{\rho}(1) = \mathrm{id}_{\mathcal{H}_0}$ and the second conjugate V^{**} of $V \colon B \to M_n$ is given by $V^{**} = g^* \overline{\rho}(\,\cdot\,) g$. It implies $V^{**}(1_{B^{**}}) = g^* g$.

There exist a unital *-representation $D \colon C \to \mathcal{L}(\mathcal{H})$ and an isometry $I \colon \mathcal{H}_0 \to \mathcal{H}$ such that $\rho(b) = I^* D(b) I$ for all $b \in B$. This can be shown e.g. by using extension

[12] See Remark 4.2.4 for the definition of Halmos unitaries.

of positive functionals, cf. [**616**, prop. 3.1.6]. The map $W(c) := I^*D(c)I$ is a unital c.p. map $W\colon C \to \mathcal{L}(\mathcal{H}_0)$ with $W|B = \rho$.

Let $p \in M_n$ denote the support projection of $g^*g \in M_n$. There is partial isometry $z \in \mathcal{L}(\ell_2(n), \mathcal{H}_0)$ with $z(g^*g)^{1/2} = g$ and $z^*z = p$. Let $d := (g^*g)^{1/2}$. We can define a c.p. contraction $U_0\colon C \to M_n$ by $U_0(c) := z^*W(c)z$ with $U_0(1) = p$. Then $V(b) = g^*\rho(b)g = dz^*W(b)zd = dU_0(b)d$. Let $\mu\colon C \to \mathbb{C}$ a state on C and define $U_1(c) := \mu(c)\cdot(1-p)$. The sum $U := U_0 + U_1$ is a c.p. map from C into M_n with $U(1_C) = 1_n$ and $dU(b)d = V(b)$.

If $1_C \in B$ then $V^{**}(1_C) = dU^{**}(1_C)d = dU(1_C)d = d^2$.

If $1_C \notin B$ then the C^*-subalgebra $B + \mathbb{C}\cdot 1_C$ of C has a non-zero character χ with $\chi(B) = 0$. It extends to a state μ on C with $\mu(B) = \{0\}$. If we define U_1 with this state μ, then we get $U(b) = U_0(b)$ for $b \in B$ and $dU(b)d = V(b) = g^*\rho(b)g$.

Now observe that $V^{**}(x) = g^*\overline{\rho}(x)g$ for $x \in B^{**}$. It follows $V^{**}(1_{B^{**}}) = g^*g = d^2$, because $V^{**}(x) = dU_0^{**}(x)d$ for $x \in B^{**}$. □

The following "existence" Lemma 2.1.21 is needed, because we consider e.g. in Proposition 2.2.1 non-separable purely infinite simple C^*-algebras, and then non-separable ultra-powers and corona algebras. It requires to check some cases for non-separable behavior.

Recall that the projections $P = P^* = P^2$ with separable $P\mathcal{H}$ generate (– in a pure algebraic manner! –) a non-unital simple closed ideal J of $\mathcal{L}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ if the Hilbert space $\mathcal{H}$ is not separable. In case $n = 1$ the assumptions of next Lemma allow that λ is a state on $\mathcal{L}(\mathcal{H})$ with $\lambda|\pi_{\mathbb{K}}^{-1}(J) = 0$, then λ is zero on $P\mathcal{L}(\mathcal{H})P$. It is one of the reasons for our "local redefining" of λ on separable C^*-subalgebras B of $\mathcal{L}(\mathcal{H})$ like in next Lemma ...

LEMMA 2.1.21. *Let $\mathcal{H}$ a non-separable Hilbert space, $B \subseteq \mathcal{L}(\mathcal{H})$ a separable C^*-subalgebra, $Q \in \mathcal{L}(\mathcal{H})$ an orthogonal projection onto a separable subspace of $\mathcal{H}$ and $\lambda\colon \mathcal{L}(\mathcal{H}) \to M_n$ a c.p. map with $\lambda(\mathbb{K}(\mathcal{H})) = 0$.*

Then there exists an orthogonal projection $P \in \mathcal{L}(\mathcal{H})$, a (new) unital c.p. map $\rho\colon P\mathcal{L}(\mathcal{H})P \to M_n$ and an element $g \in (M_n)_+$, such that $P\mathcal{H}$ is separable, $PQ = Q$, $Pb = bP$ for all $b \in B$, and with $\lambda(b) = g\rho(Pb)g$, for all $b \in B$, and $\rho(P\mathbb{K}P) = \{0\}$.

PROOF. Still to be checked again:

Consider the separable C^*-subalgebra C of $\mathcal{L}(\mathcal{H})$ generated by $\{1, Q, R\} \cup B$, where R denotes the orthogonal projection onto $(B \cap \mathbb{K})\cdot\mathcal{H}$ and let $1 := \mathrm{id}_{\mathcal{H}}$. ([13])

The subspace $R\mathcal{H}$ is separable because $B \cap \mathbb{K}$ is separable and, therefore, contains an in $B \cap \mathbb{K}$ strictly positive compact operator b_0. It has has above defined R as its support projection in $\mathcal{L}(\mathcal{H})$. The C^*-algebra C is unital and $1 \in C$. Take a sequence $c_1, c_2, \ldots$ in C that is dense in the unit-ball of C, and find finite-dimensional subspaces $X_n \subseteq \mathcal{H}$ with the property that $\|c_k|X_n\| + 2^{-n} \geq \|c_k\|$ for

[13] Here the notation is different from that in the proof of Lemma 2.1.20.

$k \leq n$. Let X_∞ denote the separable linear span of the X_n and of $Q\mathcal{H}$. Now take the smallest C-invariant closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ containing X_∞. It is separable by separability of C and X_∞. Then $Q\mathcal{H} \subseteq \mathcal{H}_0$ and the restriction $c|\mathcal{H}_0$ of the elements $c \in C$ is a faithful unital C^*-algebra morphism D from C into $\mathcal{L}(\mathcal{H}_0)$ given by $D(c) := c|\mathcal{H}_0 = (c \cdot P)|\mathcal{H}_0$, where we denote by P the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$.

Then $\mathbb{K}(\mathcal{H}_0) = P\mathbb{K}(\mathcal{H})P$, and

$$\|\pi_{\mathbb{K}(\mathcal{H}_0)}(D(c))\| = \|\pi_{\mathbb{K}(\mathcal{H})}(c)\| \quad \forall\ c \in C\,,$$

because $D(c) \in \mathbb{K}(\mathcal{H}_0)$, if and only if, $c \in \mathbb{K}(\mathcal{H})$.

Since C is a separable C^*-algebra, the closed ideal $C \cap \mathbb{K}$ of C contains a strictly positive contraction $e \in C_+$ that is a strictly positive element of $C \cap \mathbb{K}$, and the support projection of e is equal to P and commutes with C.

It holds

$$D(C) \cap \mathbb{K}(\mathcal{H}_0) = D(C \cap \mathbb{K})\,,$$

and, for $c \in C$, $D(c) = (c \circ P) \in P\mathbb{K}P \cong \mathbb{K}(\mathcal{H}_0)$ if and only if $c \in \mathbb{K}$. And this implies that $\|\pi_{\mathbb{K}}(c)\| = \|\pi_{\mathbb{K}(\mathcal{H}_0)}(D(c))\|$ for all $c \in C$.

Thus, $C/(C \cap \mathbb{K}) \cong (C + \mathbb{K})/\mathbb{K} \subseteq \mathcal{L}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ and $D(C)/(D(C) \cap \mathbb{K}(\mathcal{H}_0)) \cong (D(C) + \mathbb{K}(\mathcal{H}_0))/\mathbb{K}(\mathcal{H}_0)$ are isomorphic C^*-algebras, and up to natural isomorphisms, $C/(C \cap \mathbb{K}) \subseteq \mathcal{L}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ and $D(C)/(D(C) \cap \mathbb{K}(\mathcal{H}_0))$ contained in $\mathcal{L}(\mathcal{H}_0)/\mathbb{K}(\mathcal{H}_0)$.

Because $\lambda\colon \mathcal{L}(\mathcal{H}) \to M_n$ is a c.p. map with $\lambda(\mathbb{K}(\mathcal{H})) = 0$, the restriction to $C + \mathbb{K}(\mathcal{H})$ defines a c.p. map from $C + \mathbb{K}$ to M_n with kernel $\mathbb{K}$. Let $[\lambda]\colon (C + \mathbb{K})/\mathbb{K} \to M_n$ the c.p. map with $\lambda = [\lambda] \circ \pi_{\mathbb{K}}$ The above discussed isomorphism $[D]$ from $(D(C) + \mathbb{K}(\mathcal{H}_0))/\mathbb{K}(\mathcal{H}_0)$ onto $(C + \mathbb{K})/\mathbb{K}$ allows to define a c.p. map ??????

$C/(C \cap \mathbb{K}) \cong D(C)/(D(C) \cap \mathbb{K})$ to M_n and then from ?????

....The $\rho\colon \mathcal{L}(\mathcal{H}_0) \to M_n$ comes ... later ... as unital c.p. map $\rho(T) :=$????? by use of Lemma **??** extension ... of the

Then P commutes with the elements of $b \in C$ and $\mathrm{dist}(\mathbb{K}, b) = \mathrm{dist}(\mathbb{K}, P \cdot b)$ for $\mathbb{K} = \mathbb{K}(\mathcal{H})$. This distance is also identical with $\mathrm{dist}(P\mathbb{K}P, b)$ and with $\|\pi_{\mathbb{K}}(b)\| = \|\pi_{C \cap \mathbb{K}}(b)\|$ for $b \in C$. Moreover $C \cap \mathbb{K} = C \cap P\mathbb{K}P$, $C/C \cap \mathbb{K}$ is natural isomorphic to $(C + \mathbb{K})/\mathbb{K}$ and is isomorphic to $\pi(D(C) + \mathbb{K}(\mathcal{H}_0))/\mathbb{K}(\mathcal{H}_0)$.

The point for the latter is: 1) $b \to bP$ is a C^*-morphism, that is faithful on B? and $\mathbb{K} \cap (B \cdot P) = (B \cap \mathbb{K}) \cdot P$?

??? We can now the restriction λ to $b \in B$ of λ given by $\rho(Pb) =: \lambda(b)$, extend to $\mathcal{L}(P\mathcal{H})$ with $\rho(\mathbb{K} \cap \mathcal{L}(P\mathcal{H})) = \{0\}$ because $\mathbb{K} \cap \mathcal{L}(P\mathcal{H}) = P\mathbb{K}P \cong \mathbb{K}(P\mathcal{H})$ and $\lambda(B) \cap (P \cdot \mathbb{K} \cdot P) = \lambda(B \cap \mathbb{K})$

$\lambda|(B \cap P \cdot \mathbb{K} \cdot P)$

????

Or extend $\rho(Pb)$ to $P\mathcal{L}(\mathcal{H})P \cong \mathcal{L}(P\mathcal{H})$ with $\rho(P\mathbb{K}(\mathcal{H})P) = \{0\}$???

Here is used that $\rho(\mathbb{K}) = 0$ ... ???

To be filled in !! ?? □

The below given Lemma 2.1.22 and its proof shows that the generalizations of the classical Weyl–von-Neumann Theorem along the lines of D. Voiculescu is in fact a very special case of Proposition 5.4.1 that generalizes the Weyl–von-Neumann Theorem in an m.o.c. cone equivariant manner, – in particular in an ideal-system preserving manner if those m.o.c. cones are suitably chosen.

The space $\mathbb{C}^n$ means the n-dimensional complex Hilbert space $\ell_2(n)$ in next Lemma 2.1.22. Similar results can be shown also in the real case with almost obvious modifications here and in Lemma 2.1.20.

Lemma 2.1.22 (J. Glimm, D. Voiculescu). *Suppose that $A \subseteq \mathcal{L}(\mathcal{H})$ is a C^*-algebra and that $S\colon A \to M_n = \mathcal{L}(\mathbb{C}^n)$ is a completely positive contraction that annihilates $A \cap \mathbb{K}(\mathcal{H})$, i.e., satisfies $S(A \cap \mathbb{K}(\mathcal{H})) = \{0\}$.*

Then for every compact subset $\Omega_1 \subseteq A$, every compact subset $\Omega_2 \subseteq \mathcal{H}$ and every $\varepsilon > 0$, there is a linear isometry ?? perhaps only: contraction ?? $v\colon \mathbb{C}^n \to \mathcal{H}$ such that, for all $a \in \Omega_1$, $y \in \Omega_2$ and $x \in \mathbb{C}^n$ holds:

$$\|S(a) - v^*av\| < \varepsilon \qquad \textit{and} \qquad |\langle v(x), y\rangle| < \varepsilon\|x\|\,. \tag{1.8}$$

If, moreover, $\mathrm{id}_{\mathcal{H}} \in A + \mathbb{K}(\mathcal{H})$ and $S(a) = 1_n$ for all $a \in A$ with $a \in \mathbb{K}(\mathcal{H}) + \mathrm{id}_{\mathcal{H}}$, or if $\mathrm{id}_{\mathcal{H}}$ is not contained in $A + \mathbb{K}(\mathcal{H})$, then there exists an isometry *$v\colon \mathbb{C}^n \cong \ell_2(n) \to \mathcal{H}$ that satisfies the inequalities* (1.8).

What are the differences between this *isometries*? check the proof again!

Notice that $\mathrm{id}_{\mathcal{H}} \notin A + \mathbb{K}(\mathcal{H})$ if $A + \mathbb{K}(\mathcal{H})$ is *not* unital. In particular $A + \mathbb{K}(\mathcal{H})$ is not unital if A is stable or if $A \subseteq \mathbb{K}(\mathcal{H})$.

Here $\mathbb{C}^n$ is the n-dimensional complex Hilbert space $\ell_2(n)$, and $1_n \in M_n \cong \mathcal{L}(\ell_2(n))$ is then the identity map of $\mathbb{C}^n$. It is crucial for applications that the map v can be chosen always as a contraction, respectively as an isometry in the mentioned two special cases.

Proof of Lemma 2.1.22. We use Lemma 2.1.20 for the reductions to the unital case and (later) reduction to the unital separable case for a unital c.p. map $T\colon (A + \mathbb{K} + \mathbb{C} \cdot \mathrm{id}_{\mathcal{H}})/\mathbb{K} \to M_n$. Here $\mathcal{H}$ is not necessarily separable, but restriction to separable C^*-subalgebras of A and then to separable subspaces of $\mathcal{H}$ allows a reduction to separable closed subspaces of $\mathcal{H}$. This reductions to the separable case allows to use (and cite) then several places where this cases are considered (with slightly different proofs).

The original S is then obtained by "compression" as in the Lemma 2.1.20, that leads to the case where $v\colon \mathbb{C}^n \to \mathcal{H}$ is then only a contraction.

In the following, $\mathbb{K}$ means $\mathbb{K}(\mathcal{H})$ and $1 := \mathrm{id}_{\mathcal{H}}$. If 1 is not contained in $A+\mathbb{K}$ then $A+\mathbb{K}$ is a closed ideal of the C^*-algebra $A+\mathbb{K}+\mathbb{C}\cdot 1 \subseteq \mathcal{L}(\mathcal{H})$, and $A\cap\mathbb{K}$ is a closed ideal of A and is also a closed ideal of $A+\mathbb{K}$ with $(A+\mathbb{K})/\mathbb{K} \cong A/(A\cap\mathbb{K})$.

Thus, $S\colon A\to M_n$ with $S(A\cap\mathbb{K}) = \{0\}$ can be considered as a completely positive contraction $T\colon (A+\mathbb{K})\to M_n$ with $T(\mathbb{K}) = 0$.

If $1\in A+\mathbb{K}$, i.e., there exists $a\in A$ with $a\in\mathbb{K}+1$ and $S(a) = 1_n$ for all $a\in A$ with $a\in\mathbb{K}+1$, then T is unital.

If 1 is not contained in $A+\mathbb{K}$, then $A+\mathbb{K}$ is an essential ideal of $A+\mathbb{K}+\mathbb{C}\cdot 1$, and the minimal extension T_e of $T\colon A+\mathbb{K}\to M_n$ to a positive map from $A+\mathbb{K}+\mathbb{C}\cdot 1$ to M_n is given by the restriction to $A+\mathbb{K}+\mathbb{C}\cdot 1\subseteq (A+\mathbb{K})^{**}$ of the bi-adjoint $T^{**}\colon (A+\mathbb{K})^{**}\to M_n$ of T.

??? Compare Lemma 2.1.20 !!!

$1\in A+\mathbb{K}(\mathcal{H})$ and $S(a) = 1_n$ for all $a\in A$ with $a\in\mathbb{K}(\mathcal{H})+\mathbb{C}\cdot 1$

???

that annihilates $A\cap\mathbb{K}(\mathcal{H})$, i.e., satisfies $S(A\cap\mathbb{K}(\mathcal{H})) = \{0\}$.

The Stinespring dilation of $S\colon A\to M_n\cong\mathcal{L}(\ell_2(n))$ with ?????

We use here the notation $\mathbb{C}^n := \ell_2(n)$, i.e., $\mathbb{C}^n$ is equipped with the norm $\|(\alpha_1,\ldots,\alpha_n)\|_2 := (\sum_k |\alpha_k|^2)^{1/2}$. And "c.p." means "completely positive".

The estimates in the inequalities (1.8) have to be shown only for the compact subsets $\Omega_1\subseteq A$ and $\Omega_2\subseteq\mathcal{H}$, and for given $\varepsilon>0$. It implies that it suffices to *consider* in place of A and $\mathcal{H}$ the separable C^*-subalgebra $C^*(\Omega_1)\subseteq A$ and the closed separable subspace $\mathcal{H}_0\subseteq\mathcal{H}$ that contains Ω_2 and is invariant under $C^*(\Omega_1)$. They satisfy $C^*(\Omega_1)\cap\mathbb{K}(\mathcal{H}_0)\subseteq A\cap\mathbb{K}(\mathcal{H})$, i.e., we can suppose that A and $\mathcal{H}$ *are separable from now on.* But there are critical points here: Possibly we have to enlarge the separable closed subspaces $\mathcal{H}_0\subseteq\mathcal{H}$ and the separable C^*-algebra $A_0\subseteq A$ with $A_0\mathcal{H}_0\subseteq\mathcal{H}_0$ in the cases where $\mathrm{id}_{\mathcal{H}}\in A+\mathbb{K}(\mathcal{H})$ to get that $\mathrm{id}_{\mathcal{H}_0}\in A_0+\mathbb{K}(\mathcal{H}_0)$. We find $k\in\mathbb{K}(\mathcal{H})$ and $a_1\in A$ with $k+a_1 = \mathrm{id}_{\mathcal{H}}$ ...

Then we can ...

In case that ...

and to check that we can select the new separable C^*-algebra A_0 (in place of A) and , such that the new Hilbert space $\mathcal{H}_0$ and the C^*-algebra A_0 is chosen big enough ...

that in the cases where $\mathrm{id}_{\mathcal{H}}\in A+\mathbb{K}(\mathcal{H})$ and $S(a) = 1$ for all $a\in A$ with $a\in\mathbb{K}(\mathcal{H})+\mathrm{id}_{\mathcal{H}}$, or where $\mathrm{id}_{\mathcal{H}}$ is not contained in $A+\mathbb{K}(\mathcal{H})$,

the new $A_0|\mathcal{H}_0\subseteq\mathcal{L}(\mathcal{H}_0)$ given now by

again satisfies $\mathrm{id}_{\mathcal{H}_0}\in A_0+\mathbb{K}(\mathcal{H}_0)$ and $S(a) = 1$ for all $a\in A_0$ with $a\in\mathbb{K}(\mathcal{H}_0)+\mathrm{id}_{\mathcal{H}_0}$, respectively again satisfies

????????????

and

Details ?????.

Reduction of the general case to the case where A and $\mathcal{H}$ are separable:
The compact sets $\Omega_1 \subset A$ and $\Omega_2 \subset \mathcal{H}$ generate a separable C^*-subalgebra B of A and a B-invariant separable closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ such that $\Omega_2 \subset \mathcal{H}_1$ and $\psi\colon b \in B \mapsto b|\mathcal{H}_1$ is a faithful C^*-morphism and $\psi(B \cap \mathbb{K}(\mathcal{H})) = \psi(B) \cap \mathbb{K}(\mathcal{H}_1)$.

If $\mathrm{id}_{\mathcal{H}} \in A + \mathbb{K}(\mathcal{H})$ and $S(a) = 1$ for all $a \in A$ with $a \in \mathbb{K}(\mathcal{H}) + \mathrm{id}_{\mathcal{H}}$,

or if $\mathrm{id}_{\mathcal{H}}$ is not contained in $A + \mathbb{K}(\mathcal{H})$, ...

Notice that Ω_1 and Ω_2 contain for each ε finite subsets X_1 and X_2 such that the $\varepsilon/3$ balls around them cover all of Ω_1, respectively Ω_2. Then all predicted conclusions

(still to be proved !)

are reduced to the study of the case where the algebra A and the Hilbert space $\mathcal{H}$ are *separable* and the subsets $\Omega_1 \subset A$ and $\Omega_2 \subset \mathcal{H}$ are finite sets.

Further reductions of the general case with separable A and $\mathcal{H}$:

We start with a reduction of all cases to the case where $\mathrm{id}_{\mathcal{H}} \in A + \mathbb{K}(\mathcal{H}) = A$, $S(\mathrm{id}_{\mathcal{H}}) = 1_n$ and $S(\mathbb{K}(\mathcal{H})) = \{0\}$.

`Next step not done yet !!!`
`but cited 3 places where it was done for separable` C^*`-subalgebras of the Calkin algebra.`

Below ????

Then we can give references to different nice proofs of this very special case, i.e., where A is separable, $\mathrm{id}_{\mathcal{H}} = 1_A$ and $S(\mathrm{id}_{\mathcal{H}}) = 1$, i.e., where $n = 1$ and S is the restriction to A of a state on the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}(\mathcal{H})$.

It says essentially that the restriction of a state S on the Calkin algebra to a separable C^*-subalgebra of $A \subseteq \mathcal{L}(\mathcal{H})$ is a point-wise limit on A of a net of vector states that converges on $\mathbb{K}(\mathcal{H})$ point-wise to zero.

The reduction to the special separable unital case uses Lemma 2.1.20.

`Refer to above Lemma ???`

The following observation is easy to verify:
If D and E are (arbitrary) C^*-algebras, $T\colon D \to E$ a c.p. map and $J \subseteq D$ closed ideal with $T(J) = \{0\}$, the the class-map $[T]_J\colon D/J \to E$ is a c.p. map.

We apply this observation and Lemma 2.1.20 to $C := \mathcal{L}(\mathcal{H})/\mathbb{K}$, $B := (A + \mathbb{K})/\mathbb{K} \cong A/(A \cap \mathbb{K})$ and the quotient map $V := [S]\colon B \to M_n$.

This reduces all cases to the special case where $1 := \mathrm{id}_{\mathcal{H}} \in A + \mathbb{K}$ and $S(a) = 1_n$ for all $a \in 1 + \mathbb{K}$, and then to $\mathcal{H} \cong \ell_2(\mathbb{N})$ and separable A. Now we can cite textbooks and papers for the ???? remaining two steps: ???? Which steps???

The proof for this special case reduces to the well-known more special case where A and $\mathcal{H}$ are separable and $n = 1$:

Each of [**798**](1976), [**43**](1977) or [**207**, lem. II.5.2](1996) give a reduction of the unital separable case with $n > 1$ to the case $n = 1$.

Then all is reduced to the case of unital separable A and $n = 1$, i.e., where $M_n = \mathbb{C}$ and S is a state on A that annihilates $A \cap \mathbb{K}(\mathcal{H})$. This is the basic case, that has been shown in each of the references Glimm [**323**](1960), Dixmier [**217**, lem. 11.2.1](1969), [**123**] (1976), or [**207**, lem. II.5.1](1996) with slightly different proofs requiring different basic knowledge. □

`Next: Still to be checked:`

Remark 2.1.23. The Part(iii) of Lemma 2.1.15 and Lemma 2.1.22 and 2.1.15 imply together in the special case of a non-elementary simple C^*-algebra A the following observation:

Let $V\colon A \to M_n$ a completely positive contraction. Then, for every compact subset $\Omega \subseteq A$, pure state ρ on A and $\varepsilon > 0$, there exist contractions $e, b_1, \ldots, b_n \in A$ with the properties $b_j^* b_k = \delta_{j,k} e$, $\rho(e) = 1$ and

$$\|b_j^* a b_k - V(a)_{j,k} e\| < \varepsilon \quad \text{for} \quad j,k \in \{1,\ldots,n\} \text{ and all } a \in \Omega .$$

`!!! Wanted result:`

There exists contractions $b_1, \ldots, b_n \in A$ and $e \in A_+$ with $\|e\| = \rho(e) = 1$, $b_j^* b_k = \delta_{jk} e$ such that $\|\Delta(a)\| < \varepsilon$ for $a \in \Omega$ where

$$\Delta(a) := [b_1, \ldots, b_n]^* a [b_1, \ldots, b_n] - V(a) \otimes e \in M_n(A) .$$

It is equivalent to:

$$\|V(a)_{j,k} e - b_j^* a b_k\| < \varepsilon/(n+1) \quad \text{for all } j,k \in \{1,\ldots,n\},\ a \in \Omega .$$

(Here one can take partial isometries b_k and a projection $e \in A$ if A is simple and has *real rank zero*.)

Indeed:

Let $d_\rho\colon A \to \mathcal{L}(\mathcal{H})$ the irreducible representation corresponding to some pure state ρ on A.

The simplicity of A allows us to identify the elements $a \in A$ with $d_\rho(a) \in \mathcal{L}(\mathcal{H})$ for $\mathcal{H} := A/L_\rho$. Then $d_\rho(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$, because A is non-elementary by assumption.

We ?? get ?? by Lemma **??** ???

Let V be a completely positive contraction $gW(\cdot)g$ from $\mathcal{L}(\mathcal{H})$ to M_n, where W is unital and $0 \le g \le 1$ in $g \in (M_n)_+$, $\|g\| \le 1$.

The Lemma 2.1.22 implies the existence of an isometry $I\colon \mathbb{C}^n \to \mathcal{H}$ with the property that $\|I^* d_\rho(a) I - V(a)\| < \varepsilon/3$ for all $a \in \Omega$.

Let $x_j := Ie_j \in A/L_\rho$ for the canonical ONB of $\mathbb{C}^n = \ell_2(n)$ with norm $\|(\alpha_1, \ldots, \alpha_n)\|^2 = \sum_j |\alpha_j|^2$. The Lemma 2.1.15 gives for the irreducible representation d_ρ the existence of contractions $b_1, \ldots, b_n \in A$ with $b_j + L_\rho = x_j$, $\|b_1\| = 1$ and $b_j^* b_k = \delta_{j,k} b_1^* b_1$, $j, k \in \{1, \ldots, n\}$, such that $\rho(b_1^* b_1) = 1$.

The quotient map $A \to A/L_\rho$ has the property that the closed unit-ball of A maps onto the closed unit-ball of the Hilbert space A/L_ρ , cf. Remark **??**.

The point is here:
We find a C^*-morphism $\psi\colon \mathrm{C}_0((0,1], M_n) \to A$ such that $d_\rho(\psi(f_0 \otimes 1_n)) = P_n$ for the orthogonal projection onto the linear span of $x_1, \ldots, x_n$, and

$$\rho(f_0 \otimes p_{j,j}) = Ie_j = x_j \ \ ???$$

Take $b_k := f_0^{1/2} \otimes p_{k,1}$. Could replace f_0 by f_0^n with the property that

$$\rho(\psi(f_0^m \otimes 1_n) a (f_0^m \otimes 1_n)) - \rho(a) \otimes 1_n$$

becomes small ?? ??????

2. Characterizations of simple purely infinite algebras

Among the simple C^*-algebras, the purely infinite C^*-algebras can be characterized in terms of fairly different criteria but those turn out to be equivalent in this special case. The most useful and applicable criteria are the more complicate looking once. We provide a selection of those criteria in the Proposition 2.2.1 without claiming completeness or originality. The list of this equivalent properties is the work of many hands. We tried to find the most simple explanations for this equivalences. But notice that we give later several other different criteria that also characterize pure infiniteness of *simple* C^*-algebras A, see e.g. Proposition 2.2.5(iv), Corollary 2.4.6, Corollary 2.2.11, Corollary 3.2.16(ii) in conjunction with Remark 3.2.17, and, last but not least, the most important for our applications:
a simple non-elementary C^*-algebra A has the WvN-property – given in Definition 1.2.3 –, if and only if, A is purely infinite, cf. Chapters 3 and 5 for basic applications of the WvN-property (= *Weyl – von Neumann Property*).

There exist fairly different properties that also characterize separable non-elementary simple purely infinite *nuclear* C^*-algebras A by simplicity of $F(A) := (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$ in [**448**, thm. 2.12], or by the “unrestricted” validity of an analog of the Weyl–von-Neumann–Voiculescu theorem, cf. Corollary 5.7.3.

Most of the criteria in the parts of Proposition 2.2.1 itself do *not* imply simplicity, only the properties in Parts (ii), (iv), (vi), (vii), (xv) and (xvi) of Proposition 2.2.1 imply also that A is simple. Therefore we prefer to add here the requirement that A is simple and non-zero.

What about Part (ix)? (Same number now???)

The properties in Parts (iv), (vii) and (viii) are only equivalent if A is simple, because weakly purely infinite C^*-algebras are not necessarily simple.

But (iv) and (vii) both imply simplicity ... Are (iv) and (vii) “formally” equivalent? Formally (iv) implies (vii), and (vii) implies (viii). ...

They are almost obviously the same in case of simple C^*-algebras, but (vii) (and (iv)?) imply the simplicity of A. ?????

Several of those properties are not equivalent to each other if the C^*-algebra A is *not* simple as e.g. those in Parts (i) and (ii).

Part(ii) of Proposition 2.2.1 expresses only the Definition 1.2.1 of not necessarily simple purely infinite C^*-algebras in the special case of simple C^*-algebras – in a way that this new definition of a “simple purely infinite” C^*-algebras A contains also the simplicity of A and is clearly not equivalent to the property in Part(x) that works also for non-simple C^*-algebras A.

This many equivalent properties allow to say: If A is simple and has one of the properties listed in Proposition 2.2.1, then A has each of the listed properties, e.g. if A is p.i. in sense of J. Cuntz, then it has also all the properties listed e.g. in (vi), (ix), (x) and (xvi). For example, by Parts (ix) *and* (x), a non-zero simple C^*-algebra A is p.i. if and only if every σ-unital hereditary C^*-subalgebra D of A is stable or has a properly infinite unit $1_D := p$, i.e., $D = pAp$ with in A properly infinite projection p.

We say that a simple C^*-algebra A is **non-elementary** if A is not isomorphic to the algebra of compact operators $\mathbb{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ of finite or infinite dimension, in particular $A \ncong \mathbb{C}$.

`Needed for Citation ?:` Which parts show directly that
Property pi(1) [pi(n) $\Leftarrow$ (iv)] is equal to Property *pi*-1 $\Leftarrow$ (ii);
pi-n $\Leftarrow$ (xi), (xiv)?, **??**
What about (x), (xii), (xiii)?

PROPOSITION 2.2.1. *Suppose that A is a (non-zero)* simple *C^*-algebra. Then the following possible properties* (i)–(xvii) *of A are equivalent to each other:*

(i) *A is purely infinite in the sense of J. Cuntz* [**172**, p. 186]*, i.e., every non-zero hereditary C^*-subalgebra D of A contains an infinite projection $p \in D$* ([14]).
(ii) *A is purely infinite in the sense of Definition 1.2.1, i.e., $A \neq \mathbb{C}$, and for each $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon > 0$, there exists $c \in A$ with $\|b - c^*ac\| < \varepsilon$* ([15]).
(iii) *A is* locally *purely infinite in the sense of Definition 2.0.3, i.e., every non-zero hereditary C^*-subalgebra E of A contains a non-zero stable C^*-subalgebra $B \subseteq E$.*

[14] The projection p is then *properly* infinite by simplicity of A, cf. Lemma 2.1.6. In particular, A has the ”small projections” property (SP) that says that every non-zero hereditary C^*-subalgebra D of A contains a non-zero projection.

[15] There is $c \in A$ with $\|c\| = 1$ and $\|b - c^*ac\| < \varepsilon$, cf. Proof of (iv)$\Rightarrow$(ii) or Proof of (iii)$\Rightarrow$(ii).

(iv) *The algebra A is non-elementary and, for every $b \in A_+$ with $\|b\| = 1$ and every $\varepsilon > 0$, there exists a number $m := m(b, \varepsilon) \in \mathbb{N}$ (allowed here to depend on both of b and ε) such that, for every $a \in A_+$ with $\|a\| = 1$, there exist elements $c_1, \ldots, c_m \in A$ (depending on a, b, ε) that satisfy $\|b - \sum_{j=1}^{m} c_j^* a c_j\| < \varepsilon$.* [16]

(v) *A is* strongly *purely infinite in the sense of Definition 1.2.2, i.e., for every $a_1, a_2 \in A_+$ and $\varepsilon > 0$ there exist $d_1, d_2 \in A$ with $\|d_i^* a_i a_j d_j - \delta_{ij} a_i a_j\| < \varepsilon$ for $i, j = 1, 2$.*

(vi) *A is algebraical simple in the sense that $A \neq \mathbb{C}$ and for every $a, b \in A$ with $a \neq 0$ there are $x, y \in A$ with $xay = b$.*

(vii) *There exists (general, – not depending on a, b, ε –) $n \in \mathbb{N}$ such that A is not isomorphic to $M_k(\mathbb{C})$ for $k = 1, \ldots, n$, and, for every non-zero $a, b \in A_+$ and $\varepsilon > 0$, there exists $d_1, \ldots, d_n \in A$ such that $\|b - \sum_{j=1}^{n} d_j^* a d_j\| < \varepsilon$.* [17]

(viii) *A is simple and is* weakly *purely infinite in the sense of Definition 2.0.4.* [18]

(ix) *A is non-elementary and every non-zero non-unital σ-unital hereditary C^*-subalgebra D of A is stable.*

(x) *A has real rank zero and every non-zero projection in A is properly infinite.*

(xi) *Every non-zero element $a \in A_+$ is infinite (and A is non-zero and simple).* [19]

(xii) *There exists $n \in \mathbb{N}$ such that A is not isomorphic to $M_k(\mathbb{C})$ for each $k < n$ and each n-homogenous non-zero element $a \in A_+$ is infinite.*

(xiii) *There exists $n \in \mathbb{N}$ such that $a \otimes 1_n$ is infinite in $A \otimes \mathbb{K}$ for each non-zero $a \in A_+$.*

(xiv) *A has property pi-n of Definition* **??** *for some $n \in \mathbb{N}$.*

(xv) *A is non-elementary and, for every $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and every $\varepsilon > 0$, there exists $\nu := \nu(b, a, \varepsilon) \in \mathbb{N}$ (depending on each of a, b, ε) such that, for* every *$k = 1, 2, \ldots$, there exist* contractions *$c_1, c_2, \ldots, c_\nu \in A$*

---

[16] `check again, and give reference to` $W(A)$: Essentially, the assumptions of Part (iv) say that $[(b - \varepsilon)_+] \leq m(b, \varepsilon)[a]$ in the Cuntz semigroup $W(A)$, and are fulfilled with a fixed constant number $m(b, \varepsilon) := n$ if A is simple and has Property pi(n) of Definition 2.0.4 for some $n \in \mathbb{N}$.

[17] This property follows from Property pi(n) in Definition 2.0.4 in the special case that A is simple. It is *formally NOT* a special case of Part (iv).

[18] This Part is almost the same as in Part (vii). It is not a special case of Part (iv), because it implies no simplicity.

[19] Notice here our overall pre-assumption that A is simple and non-zero. It does not directly imply that A has Property pi(1) of Definition 2.0.4 or Property *pi*-1 of Definition **??**.

Property pi(1) says in case of simple A, that $A \neq \mathbb{C}$, and for every elements $a \in A_+$, $b \in A$ and $\varepsilon > 0$, that there exists $d_1, d_2 \in A$ with $\|d_1 a d_2 - b\| < \varepsilon$.

pi-1 says: Every non-zero element is properly infinite. But since A has no non-zero non-trivial quotients it is equivalent to require only that each $0 \neq a \in A$ is infinite.

But how to formulate the latter understandable?

(depending on k, a, b, ε, but with same number $\nu(b, a, \varepsilon) \in \mathbb{N}$) such that $\|b - \sum_{j=1}^{\nu} c_j^* a^k c_j\| < \varepsilon$. ([20]).

(xvi) *For each $n \in \mathbb{N}$ holds: If $V : A \to M_n$ and $W : M_n \to A$ are completely positive contractions, then for each finite subset $F \subset A$ and $\varepsilon > 0$ there exists contractions $d_1, d_2 \in A$ (depending on $W \circ V, F, \varepsilon$) with $d_1^* d_2 = 0$ and*

$$\|(W \circ V)(a) - d_k^* a d_k\| < \varepsilon \quad \textit{for all} \ \ a \in F, \ k \in \{1, 2\}.$$

(xvii) *For every non-zero $e \in A_+$ there exists in $C^*(\overline{eAe}, 1) \subseteq \mathcal{M}(A)$ an element that is left-invertible in $A + \mathbb{C}1$ but is not right-invertible.*

We shorten the proof of the equivalences of conditions in (ix, x) with the other conditions in Parts (i)–(viii), (xi)–(xiv) by using a stability criterium for σ-unital C^*-algebras of Hjelmborg and Rørdam [**373**] – cf. also our Corollary 5.5.1 with an alternative proof of this criterium –, and we use the characterizations of real rank zero C^*-algebras given by L.G. Brown and G.K. Pedersen in [**113**] as those C^*-algebras that have the property that every σ-unital hereditary C^*-subalgebra contains an approximate unit consisting of projections. See Remark 2.2.2 concerning applications of the (almost ridiculous) Part (xvii) and is applications.

PROOF. All in Part (o) has to be discussed and/or shifted to other places.
But remove or change references from (o) to more relevant!!
Needed/used? is reduction to separable case (Perhaps in Appendix A or B)

(o): We start with this preliminary observation (o) that is related to the conditions in Parts (iv) and (vii) but uses formally weaker assumptions. It has (implicit) applications in proofs of Parts (i)–(xvi) and in the proof of Corollary 2.4.6.

TO BE DONE !!!

Define the numbers $m(a, b, \varepsilon)$, $\mu(a, b, \varepsilon)$, $n(a, b, \varepsilon)$ $\nu(b, a, \varepsilon)$
and others carefully and relate them
to the reductions to separable cases ...
Do we really need for (xv)⇒(iv) the case
where the c_k are required to be contractions?

Those numbers should be also discussed in Appendices !!!! ????

Let A a *simple* and non-elementary C^*-algebra, $b \in A_+$ with $\|b\| = 1$, fixed from now on, and define maps $(0, 1] \ni \varepsilon \mapsto m(b, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ with the following property:

Let $a, b \in A_+$ and $\varepsilon > 0$.

[20] In general, e.g. for non-simple A, the numbers $\nu(b, a, \varepsilon) \in \mathbb{N} \cup \{+\infty\}$, can be bigger than the $m(b, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ in Part (iv): The c_j in (iv) are not necessarily contractions.

We say that b can be ε-accessed from a in $m := m(a,b,\varepsilon)$ steps if there exists $c_1,\ldots,c_m \in A$ with $\|b - \sum_{k=1}^m c_k^* a c_k\| < \varepsilon$. If A is not simple, then we allow here $m := +\infty$ if b is not in the closed ideal $\overline{\mathrm{span}(AaA)}$ generated by $a \in A_+$.

(It should be almost the smallest number m with $(b-\varepsilon)_+ \otimes p_{1,1} \precsim a \otimes 1_m$ in $M_n(A)$???????????)

Compare $(b-\varepsilon)_+ \precsim a \otimes 1_m$, i.e., $[(b-\varepsilon)_+] \leq m[a]$ in $\mathrm{Cu}(A)$, carefully with $m := m(a,b,\varepsilon)$. The point is that we allow that m variates with ε considerably. So it is not at point of start the same as $b \precsim a \otimes 1_m$ with fixed $m \in \mathbb{N}$

What is this/next ? Put it to right place!

For each $a, b \in A_+$ there exists a simple separable C^*-subalgebra $B \subseteq A$ with $a, b \in B$ and $n(c,d,\varepsilon;B) = n(c,d,\varepsilon;A)$ and $\nu(c,d,\varepsilon;B) = \nu(c,d,\varepsilon;A)$ for all $c, d \in B_+$.

In case of non-simple A we allow $m(a,b,\varepsilon) := \infty$ if b is not in the closed ideal of A generated by a, and can define $m(a,b,\varepsilon) := m \in \mathbb{N}$ if $(b-\varepsilon/2)_+$ is in the closed ideal of A generated by a.

?? with no restrictions on the values (except $m(b,\varepsilon) \neq \infty$ in case of simple A).

Let $X \subset A$ any subset.

We say that b is *approximately accessible in $m(b,\varepsilon)$-steps* from X, if for each $a \in X$ with $\|a\| = 1$ and each $\varepsilon > 0$ there exist $c_1,\ldots,c_m \in A$ – with $m := m(b,\varepsilon)$ – such that $\|b - \sum_{k=1}^m c_k^* a c_k\| < \varepsilon$. We require here no bound for the norms $\|c_k\|$ $(k = 1,\ldots,m)$.

Is it then not simply, -- in case $b \geq 0$ --, $[(b-\varepsilon)_+] \leq m(b,\varepsilon)[a]$ in $\mathrm{Cu}(A \otimes \mathbb{K})$????

Equivalent is that there exists $\gamma \in (0,\varepsilon)$ and $d_1,\ldots,d_m$ with $(b-\gamma)_+ = \sum_{k=1}^m d_k^* a d_k$.

Here $m := m(b,\varepsilon)$ depends on both of b and ε. But by our *assumption* this number m is independent from $a \in A_+ \setminus \{0\}$. Clearly, the $c_1,\ldots,c_m$ depend also on a. It implies, by Lemma 2.1.9, in the notation of Definition 2.4.2 that, for all $0 \neq a \in A$, $[(b-\varepsilon)_+] \leq m(b,\varepsilon)[a]$ in the large Cuntz semi-group $\mathrm{Cu}(A)$. Then the below given arguments can be roughly outlined by saying that for $0 \neq a \in A_+$ and $n \in \mathbb{N}$ there exists $0 \neq g \in A_+$ with $n[g] \leq [a]$ in case of non-elementary simple A. The latter is wrong for "elementary" $A \cong \mathbb{K}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. There $m(b,\varepsilon)$ behaves almost like the rank of $(b-\varepsilon)_+ \in \mathbb{K}(\ell_2(\mathbb{N}))$.

Suppose that A is a non-elementary simple C^*-algebra and that a given element $b \in A_+$ with $\|b\| = 1$ is *approximately accessible in $m(b,\varepsilon)$-steps* (from every $a \in A_+$ with $\|a\| = 1$).

Then we find for each $a \in A_+$ with $\|a\| = 1$ and each $\varepsilon \in (0,1)$, a contraction *$d := d(a,\varepsilon) \in A$ with $d^* a d = (b-\varepsilon)_+$.*

In particular, $\|b - d^*ad\| \leq \varepsilon$, and we can take $m = 1$ for the numbers $m(b, \varepsilon)$, i.e., $m(b, \varepsilon) := 1$ for all $\varepsilon > 0$.

`Compare next blue with proof of (iv).`
`What is shorter or more transparent?`

Indeed: Let $a \in A_+$ with $\|a\| = 1$ and $\varepsilon \in (0,1)$. Define $\delta := \varepsilon/2$, $f_0(t) := t$, $f_1(t) := \max(0, t - (1-\delta))$ and $f_2(t) := (1-\delta)^{-1}\min(t, 1-\delta)$, for $t \in [0,1]_+$. Notice $\|f_0 - f_2\| = \delta$ if $1 - \delta$ is in the spectrum of a. Thus if $\|a - f_2(a)\| \leq \delta$.

Then $g := (a - (1-\delta))_+ = f_1(a)$ has norm $\|g\| = \delta$. It implies $D := \overline{gAg} \neq 0$. The element $a_1 := (1-\delta)^{-1}\big(a - (a - (1-\delta))_+\big) = f_2(a)$ satisfies $ga_1 = g = a_1 g$ and $a \leq a_1 \leq (1-\delta)^{-1}a$.

Let $n := m(b, \delta) \in \mathbb{N}$ from now on in this part of proof. (Notice that we require for the given map $\varepsilon \mapsto m(b, \varepsilon)$ no kind of regularity or general bound and require only that $m(b, \varepsilon) < \infty$ for all $\varepsilon \in (0,1)$.)

The hereditary C^*-subalgebra $D = \overline{gAg}$ of A is simple and is non-elementary by assumption on A and because $g \neq 0$. Thus, D admits an irreducible representation $\rho\colon D \to \mathcal{L}(\mathcal{H})$ such that $\rho(D) \cap \mathbb{K}(\mathcal{H}) = \{0\}$. It implies that $\mathcal{H}$ is not finite-dimensional. In particular, there exists a projection $Q \in \mathbb{K}(\mathcal{H})$ of rank $= n$. Let $\lambda\colon M_n \to Q\mathbb{K}(\mathcal{H})Q$ an isomorphism from M_n onto $Q\mathbb{K}(\mathcal{H})Q$, and denote by $\mu\colon \varphi \mapsto \varphi(1)$ the evaluation map from $\mathrm{C}_0((0,1], M_n)$ onto M_n.

By Lemma 2.1.15(ii), there exists a C^*-morphism $\psi\colon \mathrm{C}_0((0,1], M_n) \to D$ with $(\rho \circ \psi)(\varphi) = \lambda(\varphi(1)) = (\lambda \circ \mu)(\varphi)$ for all $\varphi \in \mathrm{C}_0((0,1], M_n) \to D$. Let $h := \psi(f_0 \otimes p_{11}) \in D$. Then $\|h\| = 1$ and $n = m(b, \varepsilon/2)$ imply that there are elements $c_1, \ldots, c_n \in A$ with $\| b - \sum_k c_k^* h c_k \| < \delta$ – by the assumption that b is *approximately accessible in* $n := m(b, \delta)$*-steps*. The Lemma 2.1.9 shows the existence of a contraction $e_0 \in A$ such that $(b - \delta)_+ = \sum_k d_k^* h d_k$ for $d_k := c_k e_0$, $k \in \{1, \ldots, m\}$. Notice $\sum_k d_k^* h d_k = e^*e$ with $e := \sum_k \psi(f_0^{1/2} \otimes p_{k,1}) d_k$, and where $p_{k,\ell}$ denote the matrix units of M_n.

The element e is contained in the closed left-ideal $L := D \cdot A$ by Lemma 2.1.7(o) because $\psi(\mathrm{C}_0(0,1] \otimes M_n) \subseteq D$, and the equation $a_1 g = g$ implies that $a_1 x = x = x a_1$ for all $x \in D = \overline{gAg}$, and that $a_1 y = y$ for all $y \in L$. In particular, $(b - \delta)_+ = e^*e = e^* a_1 e$ and $\|e\|^2 = \|b\| - \delta = 1 - \delta$.

Thus, we obtain that $\|(b-\delta)_+ - e^* a e\| \leq (1-\delta)\|a_1 - a\| \leq (1-\delta)\delta$. By Lemma 2.1.9, there exists a contraction $f \in A$ such that $\big(b - (1 - \varepsilon/8)\varepsilon\big)_+ = d^*ad$ for $d := ef$. In particular, $\|b - d^*ad\| < \varepsilon$.

We consider in Proposition 2.2.1 also non-separable simple C^*-algebras A. Therefore we "relax" the definition of "approximately accessible" elements $b \in A_+$ with $\|b\| = 1$ by considering certain separable simple C^*-subalgebras B of A with $b \in B$ that we define in the following manner:

We define numbers $n(b, a, \varepsilon; B)$ for C^*-algebras B and non-zero $a, b \in B$ with b in the closed ideal $J(a)$ of B generated by a, by letting $n := n(b, a, \varepsilon; B)$ the minimal number $n \in \mathbb{N}$ such that there exist $c_1, \ldots, c_n, d_1, \ldots d_n \in B$ with

$$\| b - \sum_{j=1}^{n} d_j^* a c_j \| < \varepsilon .$$

Here we can take $d_j = c_j$ if $a, b \in B_+$ by Remark B.15.1. We can define $m(b, \varepsilon; B) \in \mathbb{N} \cup \{+\infty\}$ as the least upper bound of the numbers $n(b, a, \varepsilon; B)$ if B is simple.

`Give ref.s to def's of` $n(b, \ldots)$`!` The definitions of $n(b, a, \varepsilon; B) \in \mathbb{N}$ and $m(b, \varepsilon; B) \in \mathbb{N}$ for non-zero $a, b \in B_+$ says that $\sup_{a \in B_+ \setminus \{0\}} n(b, a, \varepsilon; B) = m(b, \varepsilon; B)$ if B is simple, the $m(b, \varepsilon; B)$ is defined as above `Where exactly?` and if the set of numbers $n(b, a, \varepsilon; B)$ is bounded.

Let A a simple C^*-algebra. We say that $b \in A_+$ with $\|b\| = 1$ is *local approximately accessible* if, – for every simple separable C^*-subalgebra $B \subseteq A$ with $b \in B$ and with the property that $n(x, y, \varepsilon; B) = n(x, y, \varepsilon; A)$ for all $x, y \in B_+$ – there exists a number $m := m(b, \varepsilon; B) \in \mathbb{N}$ such that, for each $a \in B_+$ with $\|a\| = 1$ there exist $f_1, \ldots, f_m \in A$ with $\|b - \sum_k f_k^* a f_k\| < \varepsilon$. This implies that $n(b, a, \varepsilon; A) \leq m(b, \varepsilon; B)$.

Now we use that $n(b, a, \varepsilon; B) = n(b, a, \varepsilon; A)$ for the selected B and obtain that b *is approximately accessible in* $m := m(b, \varepsilon; B)$*-steps* from each $a \in B_+$ with $\|a\| = 1$ (with the $c_1, \ldots, c_m$ inside B).

It says in particular that b is approximately accessible in B by $m(b, \varepsilon, B)$ steps. We have seen above that this implies that b is approximately accessible by one step from each $a \in B_+$ with $\|a\| = 1$. By Proposition B.15.2(iii,iv), there exists for each $a, b \in A_+$ a simple separable C^*-subalgebra $B \subseteq A$ with $a, b \in B$ and $n(b, a, \varepsilon; B) = n(b, a, \varepsilon; A)$. Thus, $n(b, a, \varepsilon; A) = 1$ for all non-zero $a \in A_+$. Where we use that $n(b, a, \varepsilon; B) \leq m(b, \varepsilon; B)$ for all non-zero $a \in B_+$.

(i)⇒(iii): Let $D \subseteq A$ be a non-zero hereditary C^*-subalgebra. The definition of J. Cuntz [**172**, p. 186] says that D contains an infinite projection $p \neq 0$. It means that pAp contains a C^*-subalgebra that is isomorphic to the Toeplitz algebra $\mathcal{T} := C^*(s\,;\; s^*s = 1)$. Thus, D contains an isomorphic copy of the compact operators $\mathbb{K} \subseteq \mathcal{T}$.

(iii)⇒(ii): Obviously $A \neq \mathbb{C}$.

Let $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $1 > \varepsilon > 0$ be given, and let $\eta := \varepsilon/3$.

Then $e := (a - (1 - \eta))_+ \in A_+$ and $D := \overline{eAe}$ are non-zero. By assumption, D contains a non-zero stable C^*-subalgebra $E \subseteq D$.

The multiplier algebra $\mathcal{M}(E)$ of the stable C^*-algebra E contains a copy $C^*(s_1, s_2, \ldots\,;\; s_i^* s_j = \delta_{ij} 1) \subseteq \mathcal{M}(E)$ of $\mathcal{O}_\infty$ unitally, because $\mathcal{L}(\ell_2(\mathbb{N}))$ is unitally contained in $\mathcal{M}(E)$, cf. Remark 5.1.1(8) or Lemma 2.1.7(iv).

If $d \in E_+ \subseteq D_+$ with norm $\|d\| = 1$, then the elements $f_j := s_j d^{1/2}$ ($j = 1, 2, \ldots$) are contractions in $E \subseteq D$ that satisfy $f_i^* f_j = \delta_{i,j} d$.

Since A is simple, we find $g_1, \ldots, g_n \in A$ with $\|b - \sum g_k^* d g_k\| < \eta$. Let $h := \sum_{1 \leq k \leq n} f_k g_k \in D \cdot A$, cf. Lemma 2.1.7(ii). Then $\|h^* h - b\| < \eta$ and, thus,

$0 < 1-\eta < \|h\|^2 < 1+\eta$. On the other hand, $(a-e)h = (1-\eta)h$ because $h = \lim e^{1/n}h$ and $(a-e)e^{1/n} = (1-\eta)e^{1/n}$. Let $c := \|h\|^{-1}\cdot h$, then $\|c\| = 1$, $h = \|h\|\cdot c$ and

$$c^*ac - b = c^*\big(a-(1-\eta)^{-1}(a-e)\big)c + (c^*c - h^*h) + (h^*h - b)\,.$$

This gives the estimate

$$\|c^*ac-b\| \le \|a-(1-\eta)^{-1}(a-e)\| + |1-\|h\|^2| + \|h^*h-b\| < 3\eta = \varepsilon\,.$$

(ii)⇒(i): The compact operators on a Hilbert space $\mathcal{H}$ of dimension >1 do not satisfy the criteria listed under (ii), because a rank-one projection is not equivalent to a rank-two projection.

We postpone for a moment the below given proof of the existence of a non-zero projection $p \in \overline{eAe} \subseteq E \subseteq A$ for any given non-zero hereditary C^*-subalgebra E of A and a suitable non-zero $e \in E_+$ with $\|e\| = 1$.

Let $p \in A$ any non-zero projection. The unital C^*-subalgebra $D := pAp$ (with unit p) contains $f \in D_+$ with $\|f\| = 1$ and $0 \in \operatorname{Spec}_D(f)$, because otherwise $pAp \cong \mathbb{C}\cdot p$ by the Gelfand-Mazur theorem, and A must be isomorphic to the algebra $\mathbb{K}(\mathcal{H})$ of the compact operators on some Hilbert space $\mathcal{H}$. But this has been excluded above.

By assumption, we find $c \in A$ with $\|c^*fc - p\| < 1/2$. Then $f^{1/2}cp \in pAp$ is left-invertible in pAp, but is not right-invertible in pAp, because $\|c\|^2 b \ge (b^{1/2}cp)(pcb^{1/2})$. It shows that every non-zero projection $p \in A$ is infinite. (And is then properly infinite inside pAp by Lemma 2.1.6 because A is simple.)

It remains to show that every non-zero hereditary C^*-subalgebra E of A contains a non-zero projection. We find a "scaling element" in E and modify the method of J. Cuntz and B. Blackadar in [**78**] to produce from this element a non-zero projection in E:

Take $e \in E_+$ with $\|e\| = 1$ and define the commuting positive contractions $a := (4e-3)_+$, $b := 4e-(4e-1)_+$ and $g := 2e-(2e-1)_+$ in $C^*(e)_+ \subset E_+$. Notice that $\|a\| = \|b\| = \|g\| = 1$, $ga = a$, $bg = g$ and, therefore, $(b-t)_+g = (1-t)g$ and $(b-t)_+a = (g-t)_+a = a$ for all $t \in [0,1)$.

If g is a projection then this is a non-zero projection in E, and nothing has to be shown anymore. Otherwise, $0 \ge g - g^2 = g(1-g) \ne 0$, and we can proceed further in the following way:

By Part (ii) there exists $c \in A$ with $\|c^*a^2c - b\| < (3/5)^2$. Then Lemma 2.1.9 gives a contraction $d \in A$ that satisfies

$$d^*c^*a^2cd = (b-(3/5)^2)_+\,.$$

It follows, by using $5^2 - 3^2 = 4^2$, that the element $z := (5/4)acd$ satisfies

$$z^*z = (1-(3/5)^2)^{-1}(b-(3/5)^2)_+\,.$$

The element z is in the hereditary C^*-subalgebra E because $a, b \in E$. Moreover $\|z\| = 1$, $gz = z$ (by $ga = a$), and $z^*zg = g = gz^*z$ (by $(b-t)_+g = (1-t)g$ for

$t := 9/25$). It implies $(g-g^2)z = 0$ and $(g-g^2)(1-z^*z)^{1/2} = 0$. Notice that $(z^*z)z = z$ (by $z = gz$ and $z^*zg = g$). Therefore, $(1-z^*z)^{1/2}z = 0$. Let

$$u := z + (1-z^*z)^{1/2} \in \widehat{E} := E + \mathbb{C}\cdot 1 \subseteq \mathcal{M}(E).$$

Then $(g-g^2)uu^* = 0$ and

$$u^*u = 1 + z^*(1-z^*z)^{1/2} + (1-z^*z)^{1/2}z + 1 - z^*z = 1.$$

This shows that u is a non-unitary isometry in $\widehat{E}$, and that $p := 1 - uu^* \in E$ is a projection in E with $(g-g^2) \leq p$. Thus, $p^*p = p \neq 0$.

(iv)⇒(ii): The conditions in Part(iv) on A require that A is non-elementary and they obviously imply that A is simple. Simple non-elementary C^*-algebras are *antiliminary* in the sense that each non-zero hereditary C^*-subalgebra contains a non-zero 2-homogenous element, cf. Remark 2.1.16(ii,iii).

Let $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $1 > \varepsilon > 0$, define $\eta := \varepsilon/3$. Then $e := \big(a - (1-\eta)\big)_+ \in A_+$ is non-zero. Let $m := m(b,\eta)$. By Remark 2.1.16(iii), we find in the non-zero hereditary C^*-subalgebra $D := \overline{eAe}$ an element $d \in D_+$ and $f_1, f_2, \ldots f_m \in D$ with $\|d\| = 1$ and $f_i^*f_j = \delta_{i,j}d$ for $i,j = 1,\ldots,m$, because A is antiliminary.

By assumption, we find $c_1,\ldots,c_m \in A$ with $\|b - \sum c_k^*dc_k\| < \eta$. Let $h := \sum_{1\leq k\leq m} f_kc_k$.

Then $hh^* \in D$, $\|h^*h - b\| < \eta$, $0 < 1-\eta < \|h\|^2 < 1+\eta$ and $(a-e)h = (1-\eta)h$. We get $\|c^*ac - b\| < 3\eta = \varepsilon$ for $c := \|h\|^{-1}\cdot h$, by the same calculation as in the proof of the implication (iii)⇒(ii).

If A is simple and has Property pi(n) of Definition 2.0.4 for some $n \in \mathbb{N}$ then we can take $m(b,\varepsilon) := n$ for each $b \in A_+$ and the Property pi(n) excludes the cases $A \cong M_k$ for $k \leq n$ in its definition. It follows that A with Property pi(n) can not be an elementary C^*-algebra, because if $A \cong \mathbb{K}(\mathcal{H})$ with $\mathrm{Dim}(\mathcal{H}) > n$ and $a, b \in A$ are contractions with ranks $(Rk)(a) = 1$ and $(Rk)(b) > n$ then there can not exist $c_1,\ldots,c_n \in A$ with $\|b - \sum c_k^*ac_k\| < 1/2$.

(ii)⇒(iv): The simple algebra A can not be elementary, because if $p, q \in A$ are non-zero projections with rank $\mathrm{Rk}(q) = 1$ then there exists $c \in A$ with $\|p - c^*qc\| < 1/2$, and this shows that all projections in $A \setminus \{0\}$ have rank one, i.e., $A \cong \mathbb{C}$. The case $A = \mathbb{C}$ has been excluded in Definition 1.2.1.

Thus A satisfies the assumptions of Part (iv) with $m(b,\varepsilon) = 1$.

(i)⇒(v): We show moreover that for each $a, b \in A_+$ and $\varepsilon > 0$ there exist $d_1, d_2 \in A$ with $d_1^*abd_2 = 0$, $\|a^2 - d_1^*a^2d_1\| \leq \varepsilon$, and $\|b^2 - d_2^*b^2d_2\| \leq \varepsilon$.

We may suppose that $\max(\|a\|,\|b\|) \leq 1$ and that $\varepsilon \in (0,1)$, because we can start here with smaller $\varepsilon > 0$ if necessary.

If $ab = 0$, then let $\beta := \varepsilon/2$, $d_1 := a^\beta$ and $d_2 := b^\beta$. We get $\|d_1^*abd_2\| = 0$, $\|a^2 - d_1^*a^2d_1\| \leq \varepsilon$, and $\|b^2 - d_2^*b^2d_2\| \leq \varepsilon$ by functional calculus, because $0 \leq t^2 - t^{2+\varepsilon} \leq \varepsilon$ for $t \in [0,1]$.

If $ab \neq 0$, then there is $\delta \in (0, \|ab\|^2)$ with $c := (ab^2a - \delta)_+ \neq 0$. The non-zero hereditary C^*-subalgebra $D := \overline{cAc}$ contains a non-zero infinite projection p by assumption (i).

The elements pcp and pab^2ap are strictly positive in pAp, because $p \in \overline{cAc}$ and $c \leq ab^2a$. Thus, pab^2ap and $pa^2p \geq pab^2ap$ are invertible inside the C^*-algebra pAp with unit p.

Let $f := (pab^2ap)^{-1/2} \in (pAp)_+$. It satisfies $fpab^2apf = p$. Since $f = pfp$, and $p \in \overline{cAc}$ we get $fab^2af = p \leq \delta^{-1}ab^2a \leq \delta^{-1}a^2$.

The element $v := baf$ is a partial isometry in A and $q := vv^* = baf^2ab$ is a projection in bAb. The element b^2 is strictly positive in $\overline{bAb}$, thus qb^2q must be invertible in qAq, i.e., there is $\eta > 0$ with $\eta q \leq qb^2q$. It follows that $\eta p = \eta v^*qv \leq v^*qb^2qv = v^*b^2v$, and that v^*b^2v is invertible in pAp.

All infinite projections in the simple algebra pAp are properly infinite by Lemma 2.1.6. Thus, partial isometries $s, t \in A$ with $s^*s = p = t^*t$ and $ss^* + tt^* \leq p$ exist.

Let $g_1 := fs$ and $g_2 := baft = vt$. Then $g_2^*bag_1 = t^*fab^2afs = t^*ps = 0$. Since f, pa^2p and v^*b^2v are invertible positive elements of pAp, we get that $g_1^*a^2g_2 = s^*fpa^2pfs$ and $g_2^*b^2g_2 = t^*v^*b^2vt$ are positive elements in pAp that are invertible inside pAp. It implies the existence of $h_1, h_2 \in (pAp)_+$ with $h_1(g_1^*a^2g_1)h_1 = p$ and $h_2(g_2^*b^2g_2)h_2 = p$. The elements $e_k := g_kh_k$ satisfy $e_1^*a^2e_1 = p = e_2^*b^2e_2$ and $e_2^*bae_1 = 0$.

Lemma 2.1.7(i) applies to pAp with properly infinite p. Thus, there are $x_1, x_2 \in A$ with $x_1^*x_1 = a^2$, $x_2^*x_2 = b^2$, and $x_1x_1^*$, $x_2x_2^* \in pAp$. The elements $d_1 := e_1x_1$ and $d_2 := e_2x_2$ in A satisfy $d_2^*bad_1 = x_2^*e_2bae_1x_1 = 0$, $d_1^*a^2d_1 = a^2$ and $d_2^*b^2d_2 = b^2$.

(v)⇒(xi): Let $a \in A_+$. If we take $a_1 := a_2 := a^{1/2}$ and $\varepsilon > 0$ in (v), then we get $d_1, d_2 \in A$ with $\|d_i^*ad_j - \delta_{ij}a\| < \varepsilon$. Thus, a is properly infinite.

(xi)⇒(vii): The algebra A is simple and non-zero – by our overall pre-assumption. If every non-zero element $a \in A_+$ is infinite and if A is simple then each non-zero a is also properly infinite by Lemma 2.1.6. In particular, A must be non-elementary, because rank-one projections are not infinite in the algebra of compact operators on a Hilbert space.

We show that A satisfies the assumptions of Part (vii) with $n := 1$:

A is non-zero and is not one-dimensional because A is non-elementary. Let $a, b \in A_+$ non-zero and $\varepsilon > 0$. Since A is simple there exist a *minimal* $m \in \mathbb{N}$ such that there exist elements $d_1, \ldots, d_m \in A$ with $\|b - \sum_{k=1}^m d_k^*ad_k\| < \varepsilon$. We show that necessarily $m = 1$ if a is properly infinite:
Suppose that $m > 1$. Then there is $\delta > 0$ with $\delta \cdot 2(\|d_{m-1}\| + \|d_m\|)^2 < \varepsilon - \|b - \sum_k d_k^*ad_k\|$. Since $a \in A_+$ is properly infinite, there are $e_1, e_2 \in A$ with $\|a - e_k^*ae_k\| < \delta$ and $\|e_1^*ae_2\| < \delta$. Let $f_{m-1} := e_1d_{m-1} + e_2d_m$ and $f_k := d_k$ for $j \leq m-2$. Then $\|b - \sum_{k=1}^{m-1} f_k^*af_k\| < \varepsilon$, which contradicts the minimality of $m > 1$ with this property.

(vii)⇒(iv): Let $n \in \mathbb{N}$ such that A is not isomorphic to $M_k(\mathbb{C})$ for each $k \leq n$ and that, for every non-zero $a, b \in A_+$ and $\varepsilon > 0$, there exists $d_1, \ldots, d_n \in A$ such that $\|b - \sum_{k=1}^n d_k^* a d_k\| < \varepsilon$.

We show that A satisfies the conditions of Part (iv) with $m(b, \varepsilon) := n$ for all $b \in A_+$ and $\varepsilon > 0$ with $\|b\| = 1$:

Suppose that $A \cong \mathbb{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then each projection $p \in \mathbb{K}(\mathcal{H})$ has rank $\leq n$. Thus, $\mathcal{H}$ has dimension $\leq n$, and A is isomorphic to M_k for some $k \leq n$. Since the latter is excluded by the assumptions, A must be non-elementary.

Let $a, b \in A_+$ with $\|a\| = 1 = \|b\|$ and $\varepsilon > 0$. We let $\delta := \varepsilon/4$.

Consider $f := \big(a - (1 - \delta)\big)_+ \in A_+$. Then $f \neq 0$ and $(1 - a)f \leq \delta f$. By conditions in (vii), there are $d_1, \ldots, d_n \in A$ with $\|b - \sum_{k=1}^n d_k^* f d_k\| \leq \delta$. The elements $e_k := f^{1/2} d_k$ satisfy $\| \sum_k e_k^* e_k\| \leq 1 + \delta$ and $e_k^*(1 - a)e_k \leq \delta e_k^* e_k$. Thus, $\| \sum_k e_k^*(1 - a)e_k\| \leq (1 + \delta)\delta$. We let $c_k := (1 + \delta)^{-1/2} e_k$. Then $\|c_k\| \leq 1$, $\| \sum_k c_k^*(1 - a)c_k\| \leq \delta$, and $\|b - \sum_k c_k^* c_k\| \leq 2\delta$.

It follows $\|b - \sum_{k=1}^n c_k^* a c_k\| \leq 3\delta < \varepsilon$.

(viii)⇒(vii): Our overall assumption is that A is simple. For simple A the Definition 2.0.4 of weak pure infiniteness reformulates obviously as follows:
There exists (general) $n \in \mathbb{N}$ such that for each $a, b \in A_+$ with $a \neq 0$ and every $\varepsilon > 0$, there exists $d_1, \ldots, d_n \in A$ – depending on (a, b, ε) – such that

$$\|b - \sum_{j=1}^{n} d_j^* a d_j\| < \varepsilon$$

and the algebra $\ell_\infty(A)$ has no irreducible representation of dimension $\leq n$.

The latter implies for simple A that the algebra A can not be isomorphic to $M_k(\mathbb{C})$ for some $k \leq n$, because otherwise then $\ell_\infty(A) \cong M_k(\ell_\infty)$ would have irreducible representations of dimension $\leq n$, which is forbidden by (viii).

(i)⇒(vi): We may suppose that $\|a\| = 1$. Then the hereditary C^*-algebra E generated by $(a^*a - 1/2)_+$ contains an infinite projection $p \in E$. It follows that pa^*ap is invertible in pAp, and the positive element $v := (pa^*ap)^{-1/2} \in pAp$ satisfies $(va^*)av = p$. Since p is infinite, there exists a (non-zero) stable hereditary C^*-subalgebra $D \subseteq pAp$.

For every (full) stable hereditary C^*-subalgebra $D \subseteq A$ and every $b \in A$ there exists $d \in A$ with $dd^* \in D$ and $d^*d = (b^*b)^{1/4}$ by Lemma 2.1.7. Let $b = w(b^*b)^{1/2}$ the polar decomposition of b in A^{**}. The element $z := w(b^*b)^{1/4} \in A^{**}$ is in A, because

$$z = \lim_n b(1/n + (b^*b)^{1/4})^{-1}$$

in norm and $(1/n + (b^*b)^{1/4})^{-1}$ is a multiplier of A in A^{**}. Clearly $zd^*d = z(b^*b)^{1/4} = b$.

Let $x := zd^*va^*$ and $y := vd$. Then $xay = zd^*va^*avd = zd^*pd = b$.

(Alternatively one could combine Lemma 2.1.7 and [**616**, prop. 1.4.5] and gets almost the same decomposition.)

(vi)⇒(ii): Let $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon \in (0,1)$, and let $\delta := \varepsilon/3$. There are $x, y \in A$ with $x(a-(1-\delta))_+ y = b^{1/2}$. It follows that $x \neq 0$ and $e^*e \geq b$ for $e := \|x\|(a-1+\delta)_+^{1/2} y$. By Lemma 2.1.9 there exists a contraction $f \in A$ with $(ef)^*(ef) = (b-\delta)_+$. The element $d := ef$ is a contraction in A, and dd^* is contained in the hereditary C^*-subalgebra of A that is generated by $(a-(1-\delta))_+$, because

$$dd^* \leq ee^* \leq \|x\|^2\|y\|^2(a-(1-\delta))_+ .$$

Thus, $d^*h(a)d = d^*d$ and $\|d^*d - d^*ad\| \leq \|h(a) - a\| \leq \delta$ for the function $h(t) := \min((1-\delta)^{-1}t, 1)$ with property $(t-(1-\delta))_+ h(t) = (t-(1-\delta))_+$ for $t \in [0,1]$. Since $\|d^*d - b\| \leq \delta$, we get $\|d^*ad - b\| < \varepsilon$.

(ii)⇒(viii): Clearly, if A satisfies (ii) then A satisfies Condition (i) of Definition 2.0.4 of 1-purely infinite algebras.

Let $a \in A_+$ with $\|a\| = 1$, $b := (b_1, b_2, \ldots)$ a positive contraction in $\ell_\infty(A)$, and $\varepsilon \in (0,1)$. Define $g := \eta^{-1}(a-(1-\eta))_+$ for $\eta := \varepsilon/8$ and let $\Delta(a) := (a, a, \ldots) \in \ell_\infty(A)$ for $a \in A$. Then $\|g\| = 1$ and, by Part (ii), there exists elements h_k with $\|b_k - h_k^* g^2 h_k\| < \eta$ for $k \in \mathbb{N}$. It implies $\|gh_k\| < (1+\eta)^{1/2}$ and with $d_k := (1+\eta)^{-1/2} g h_k$, that $\|d_k\| < 1$ and $\|b_k - d_k^* a d_k\| \leq \varepsilon/2$ for $k \in \mathbb{N}$, because

$$b - d_k^* a d_k = (b - h_k^* g^2 h_k) + (1+\eta) d_k^*(1-a) d_k + \eta d_k^* a d_k .$$

This implies $\|b - d^*\Delta(a)d\| < \varepsilon$ in $\ell_\infty(A)$ for the contraction $d := (d_1, d_2, \ldots) \in \ell_\infty(A)$.

Since A has no non-zero character by assumption in (ii) and since $\Delta(A)$ generates $\ell_\infty(A)$ as a closed ideal of $\ell_\infty(A)$, it follows that $\ell_\infty(A)$ does not have a non-zero character. Thus, condition (ii) of the Definition 2.0.4 of 1-purely infinite algebras is also satisfied.

(ix)⇒(iii): Let D a non-zero hereditary C^*-subalgebra of A and $a \in D_+$ with $\|a\| = 1$. Then the non-zero σ-unital hereditary C^*-subalgebra $E := \overline{aDa} \subseteq D$ is non-unital – or – zero is isolated in the spectrum of a. In the first case E is stable by assumption of (ix). In the second case $E = pDp$ is simple and unital with unit given by the projection $p := \varphi(a)$ using the function

$$\varphi(t) := \min(1, \max(0, \alpha t - \gamma/2)),$$

where γ denotes the minimum of $\mathrm{Spec}(a) \setminus \{0\}$ and $\alpha := (\gamma+2)/2\gamma$. In the second case we can consider a maximal commutative C^*-subalgebra C of the unital C^*-algebra pDp. It is not difficult to check that a unital commutative C^*-algebra C with the property that every $c \in C_+$ with $\|c\| = 1$ has all spectral values $t \in (0,1) \cap \mathrm{Spec}(c)$ isolated from 0 and 1 must be of finite (linear) dimension. But this would cause that E must be isomorphic to M_n for some $n \in \mathbb{N}$. And this would imply that also D and A are elementary, but is excluded by the assumptions on A in Part (ix).

Thus, $C \subseteq E$ contains at least one $a \in C_+$ with $\|a\| = 1$, 0 not isolated from the spectrum of $a \in C$ and $F := \overline{aAa} \subseteq D$. Then F is a non-zero stable hereditary C^*-subalgebra of D by assumption of Part (ix). Hence, the algebra D contains in both cases a non-zero stable C^*-subalgebra.

(x)⇒(i): Our overall pre-condition on A requires that A is simple and is non-zero.

Suppose that A has real rank zero and every non-zero projection $p \in A$ is infinite. Then the simplicity of A implies that each projection p is properly infinite in A by Lemma 2.1.6.

Since every non-zero projection $p \in A$ is properly infinite, it suffices to show that a given non-zero σ-unital hereditary C^*-subalgebra D of A contains a non-zero projection.

By a result of L.G. Brown and G.K. Pedersen [**113**], A has real rank zero ($RR(A) = 0$), if and only if, every non-zero σ-unital hereditary C^*-subalgebra D of A contains an approximate unit consisting of projections. In particular, each nonzero hereditary D contains a projection $p \neq 0$. That is properly infinite by our assumptions.

(Notice here that the definition of the "real rank" in [**113**] – specialized to the case $RR(A) = 0$ – says that A has real rank zero, if and only if, the self-adjoint invertible elements of $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ are dense in $A_{sa} + \mathbb{R} \cdot 1$. It is shown in [**113**] that this implies the property (FS), i.e., that the self-adjoint elements with finite spectrum are dense in A_{sa}. It is proven there that this is equivalent to the property that every nonzero hereditary C^*-subalgebra $D \subseteq A$ has an approximate unit of D consisting of projections.)

(i)⇒(x): Let $p \in A$ a non-zero projection. Then $E := pAp$ is a non-zero hereditary C^*-subalgebra of A. Since A is purely infinite in the sense of J. Cuntz, E contains a non-zero infinite projection $q \leq p$. Thus p is also infinite. Moreover p is properly infinite because A is simple, cf. Lemma 2.1.6.

By [**113**], A has real rank zero, if and only if, each non-zero σ-unital hereditary C^*-subalgebra D of A contains an approximate unit consisting of projections.

Recall that D is σ-unital, if and only if, $D = \overline{eAe}$ for some (inside D) strictly positive contraction $e \in D_+$. D is then unital, if and only if, 0 is isolated in the spectrum of e.

Thus, to conclude, it suffices to show that every non-unital σ-unital D is stable, because then $D \cong D \otimes \mathbb{K} \cong pAp \otimes \mathbb{K}$ for each non-zero projection $p \in D$ by L.G. Brown's stable isomorphism theorem [**107**], and D has an approximate unit consisting of projections $P_n \in D$ given by $P_n := p \otimes (e_{11} + \ldots + e_{nn})$, using the natural isomorphism $D \cong pAp \otimes \mathbb{K}$.

The stability of $D = \overline{eAe}$ follows from the fact that for each $\varepsilon > 0$ there exists $d \in D$ with $\|d^*ed\| \leq \varepsilon$ and $d^*d = (e - \varepsilon)_+$, because the (two-sided) annihilator $F := \mathrm{Ann}((e - \varepsilon)_+, D)$ of $(e - \varepsilon)_+$ in D is a non-zero hereditary C^*-subalgebra of

D, and contains a non-zero infinite projection $q \in F$ by assumptions in Part (i). Thus, there exists $d \in D$ with $d^*(e-\varepsilon)_+ d = 0$ and $d^*d = e$, by Lemma 2.1.7. The stability of D follows now from [**373**] – or alternatively from our Corollary 5.5.1 of our generalized version of the WvN-theorem.

(i)⇒(ix): The proof of the implication (i)⇒(x) applies almost verbatim. It shows

`Can delete next ''red/blue'' to get ''new version of (ix)''`

that every non-zero projection in A is infinite (even properly infinite by Lemma 2.1.6), and

that every non-zero non-unital σ-unital hereditary C^*-subalgebra D of A is stable.

Alternatively, the stability of non-zero non-unital σ-unital hereditary C^*-subalgebras D of purely infinite C^*-algebra follows also directly from [**462**, thm. 4.24]: If D is σ-unital, purely infinite, and has no unital quotient, then D stable. It uses [**373**]. Here we can use instead also Corollary 5.5.1 that does not use [**373**] for its proof and is not circular intertwined with proofs given here in Chapter 2 or on other places.

[Check/decide Alternative:]

A is non-elementary and every non-zero non-unital σ-unital hereditary C^*-subalgebra D of A is stable.

Still applicable for new Part (ix)?

Proof of alternative formulation of (ix) out of given (old?) formulation of Part (ix):

By assumption A is non-elementary, i.e., $A \ncong \mathbb{K}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$.

In particular pAp is not of finite (linear) dimension for each non-zero projection $p^*p = p \in A$. It follows that every maximal commutative C^*-subalgebra C of pAp contains a positive element $b \in C \subseteq pAp$ with the property that 0 is not isolated in the spectrum of b. This implies that the non-zero σ-unital hereditary C^*-subalgebra $D := \overline{bAb}$ of A is not unital. Hence, D is stable by the assumptions in Part (ix). By our "overall assumption": A is simple. Thus D generates A as ideal of A. Then Part (ii) of Lemma 2.1.7 shows that there exists $c, d \in A$ with $d^*d = p = c^*c$, $cc^* = dd^* \in D \subseteq pAp$ and $c^*d = 0$. Thus, p is properly infinite in A, by assumptions.

(xii)⇒(xi): Suppose that A is simple and that there exists $n \in \mathbb{N}$ such that each n-homogenous non-zero element $a \in A_+$ is infinite, and that A is not isomorphic to $M_k(\mathbb{C})$ for $k = 1, \ldots, n-1$.

Since A is simple, it follows that each n-homogenous $a \in A_+ \setminus \{0\}$ is properly infinite in A. The algebra A is non-elementary, because the projections of rank equal to n in the algebra of compact operators on Hilbert spaces H of dimension $\geq n$ are n-homogenous, but – obviously – are not properly infinite.

It follows that the non-zero closed hereditary C^*-subalgebras D of A are also non-elementary. Hence, each non-zero D contains non-zero n-homogenous elements. It implies that every non-zero element $a \in A_+$ majorizes a non-zero properly infinite element. Thus, each non-zero $a \in A_+$ is itself properly infinite by Lemma 2.1.6.

(xi)⇒(xiii): It is obvious with $n := 1$.

(xiii)⇒(xii): Suppose that A is simple and there exists $n \in \mathbb{N}$ with the property that $a \otimes 1_n$ is infinite in $A \otimes \mathbb{K}$ for each non-zero $a \in A_+$. Then A is non-elementary because this can not happen for $a \in A_+$ with non-zero $a = a^2$ and $aAa = \mathbb{C} \cdot a$.

Apply Lemma 2.1.6 to the infinite elements $a \otimes 1_n$ and get that $a \otimes 1_n$ is *properly* infinite in the simple C^*-algebra $A \otimes \mathbb{K}$ for each $a \in A_+$.

Let $a \in A_+$ an n-homogenous contraction and $\varphi \colon \mathrm{C}_0((0,1], M_n) \to A$ a C^*-morphism with $a = \varphi(f_0 \otimes 1_n)$ for $f_0(t) := t$ $(t \in [0,1])$. If p_{jk} denote the matrix units, then $1_n := p_{11} + \ldots + p_{nn}$ is the unit element of M_n. Let $c := \varphi(f_0 \otimes p_{11}) \in A_+$.

The elements $a \otimes p_{11} = \varphi(f_0 \otimes 1_n) \otimes p_{11}$ and $c \otimes 1_n = \varphi(f_0 \otimes p_{11}) \otimes 1_n$ are Murray–von-Neumann equivalent in $A \otimes M_n \cong M_n(A)$, because $f_0 \otimes 1_n \otimes p_{11}$ is unitarily equivalent to $f_0 \otimes p_{11} \otimes 1_n$ in $\mathrm{C}_0(0,1] \otimes M_n \otimes M_n$ by a unitary in $1 \otimes M_n \otimes M_n$.

It is easy to check that proper infiniteness of positive elements is invariant under Murray–von-Neumann equivalence, because the proper infiniteness of an element a means $(a \oplus a) \approx a$ with Cuntz equivalence $\approx$. Therefore proper infiniteness passes to all elements $b \approx a$. Use that $\sim_{MvN}$ implies the (weaker) equivalence $\approx$ to obtain that $a \otimes p_{11}$ is properly infinite in $A \otimes \mathbb{K}$. Thus, each positive n-homogenous element a in $A \cong A \otimes p_{11}$ is properly infinite in $A \otimes \mathbb{K}$. The definition of proper infiniteness shows that the positive element a is also properly infinite in A itself, because the definition of properly infinite elements at the beginning of Section 1 shows that an inside A properly infinite element a is always in the hereditary C^*-subalgebra $D := \overline{aAa} \cong D \otimes p_{11}$ properly infinite if $a \otimes p_{11}$ is properly infinite in $A \otimes \mathbb{K}$, cf. the argument in proof of Proposition 2.2.5(i).

(xiii)⇔(xiv): Suppose that there exists $n \in \mathbb{N}$ such that $a \otimes 1_n$ is infinite in $M_n(A)$ for each non-zero $a \in A_+$. Since $M_n(A)$ is simple, $a \otimes 1_n$ is properly infinite by Lemma 2.1.6, i.e., A has property pi-n of Definition **??**.

The implication (xiv)⇒(xiii) is obvious by the Definition of property pi-n.

(ii)⇒(xv): If A satisfies (ii), then A is non-elementary and the additional condition $\|c\| \leq 1$ can be derived like in the proof of the implication (vi)⇒(ii).

It says that A satisfies the conditions in Part (xv) with $\nu(a, b, \varepsilon) := 1$ for all $a, b \in A_+$ and $\varepsilon > 0$.

(xv)⇒(iv): The conditions on the algebra A in Part (xv) imply that A is simple, because the existence of the numbers $\nu(a, b, \varepsilon) < \infty$ implies that the element b is contained in the ideal generated by a for each $a, b \in A_+$ with $\|a\| = 1$.

Only non-elementary A are allowed in Part (xv) by assumptions. Thus, to verify that the assumptions of Part (xv) imply the assumptions in Part (iv), we are going to show that there exists, for each $b \in A_+$ with $\|b\| = 1$ and each $\varepsilon > 0$, a number $m := m(b, \varepsilon) \in \mathbb{N}$ that satisfies the conditions in Part (iv), i.e., has the property that for any $a \in A_+$ with $\|a\| = 1$ there exists an "m-step inner" c.p. map $V := f_1^*(\cdot)f_1 + \cdots + f_m^*(\cdot)f_m$ with $\|b - V(a)\| < \varepsilon$. (If this is the case, then the above proven equivalence of Parts (iv) and (ii) shows that, moreover, $m(b, \varepsilon) = 1$ for all $b \in A_+$ with $\|b\| = 1$ and $\varepsilon > 0$.)

Notice that the definition of the numbers $m := m(b, \varepsilon) \in \mathbb{N}$ in Part (iv) is given by the upper bound (over all $a \in A_+$) of the numbers $m := m(b, a, \varepsilon) \in \mathbb{N}$ that are defined as follows:
Given non-zero $a \in A_+$ and $\varepsilon > 0$, then there exists a minimal $m \in \mathbb{N}$ such that there exist elements $f_1, \ldots, f_m \in A$ that satisfy the inequality $\|b - \sum_{\ell=1}^{m} f_\ell^* a f_\ell\| < \varepsilon$. Here without any proposed bound for the norms $\|f_\ell\|$.

The reader should observe here that the number $\nu := \nu(b, a, \varepsilon) \in \mathbb{N}$ is differently defined. It is the minimal number ν that satisfies the following condition:
For each power $\ell \in \mathbb{N}$ there exist $\nu(b, a, \varepsilon)$ *contractions* $c_1, \ldots, c_\nu \in A$ – depending on $\varepsilon > 0$, $b, a \in A_+$ and $\ell \in \mathbb{N}$ – such that $\| b - \sum_{k=1}^{\nu} c_k^* a^\ell c_k \| < \varepsilon$. The crucial point is here that we *require* here that this number ν is independent from the $\ell \in \mathbb{N}$.

We consider first the case of separable A, and discuss later the reduction to the study of suitable simple separable C^*-subalgebras B of A that have the same values $\nu(b, a, \varepsilon; B) = \nu(b, a, \varepsilon; A)$ for all $a, b \in B$.

We show for separable simple A that if $\nu(a, b, \varepsilon) < \infty$ for all $a \in A$ and $\varepsilon > 0$ then $m(b, \varepsilon) < \infty$ for all $\varepsilon > 0$. The latter causes that Part (iv) applies to A and shows that $m(b, \varepsilon) = 1$, i.e., that A is purely infinite by the implication (iv)⇒(ii).

We use in the separable case Part (iii) of Lemma 2.1.15, Lemma 2.1.17 and the classical "excision" Proposition A.21.4 for the proof.

Move the Excision Lemma back to Chp.2. !!?

Let $b \in A_+$ with $\|b\| = 1$ and let ρ denote a pure state on A with $\rho(b) = 1$.

Any pure state on A would do the job!
Only the below defined c plays a role.

By the above cited Lemmata

(???? Cite more precise !!!)

there exists a positive contraction $c \in A_+$ with $\|c\| = 1$ such that $\rho(c) = 1$, and for all $a \in A$,

$$\lim_n \|c^{2n}\rho(a) - c^n a c^n\| = 0\,.$$

By assumptions of Part (xv), there are numbers $\nu(b, c^2; \varepsilon) < \infty$, for every $\varepsilon > 0$ with the in Part (xv) quoted properties. But for fixed b and $\varepsilon > 0$ they could depend possibly on the contraction $c \in B_+$.

Let $\varepsilon > 0$ given and let $\delta := \varepsilon/(2\nu(b, c^2; \varepsilon/2))$.

Let $a \in A_+$ non-zero, w.l.o.g., $\|a\| = 1$. By Inequality (1.7) in Lemma 2.1.17 there exists a contraction $g \in A$ with $1 = \|a\| \leq \delta + \rho(g^*ag) \leq \delta + 1$. Thus, there exists $n_0 \in \mathbb{N}$ with $\|\rho(g^*ag)c^{2n} - c^n g^*agc^n\| < \delta$ for all $n \geq n_0$. Hence,

$$\|c^{2n} - c^n g^*agc^n\| < 2\delta \quad \text{for all } n \in \mathbb{N}, n \geq n_0 .$$

By assumptions of Part (xv), we find contractions $d_1, \ldots, d_\nu \in A$ with $\nu := \nu(b, c^2; \varepsilon/2)$ a fixed number, but with the d_k also depending from the power $n \in \mathbb{N}$ of $(c^2)^n$, such that

$$\|b - \sum_{k=1}^{\nu} d_k^* c^{2n} d_k\| < \varepsilon/2 .$$

Then $\|f_k^* a f_k - d_k^* c^{2n} d_k\| < \delta$ for the contractions $f_k := gc^n d_k$ $(k = 1, \ldots, \nu)$, and therefore

$$\|b - \sum_{k=1}^{\nu} f_k^* a f_k\| < \varepsilon/2 + 2\nu \cdot \delta \leq \varepsilon .$$

It shows that for $a, b \in A_+$ and $\varepsilon > 0$ and $\|a\| = \|b\| = 1$ holds $n(a, b, \varepsilon) \leq \nu(b, c^2, \varepsilon/2) < \infty$, with suitable fixed $c \in A_+$ and $\|c\| = 1$. Thus $m(b, \varepsilon) < \infty$ for each $b \in A_+$ with $\|b\| = 1$ and $\varepsilon > 0$.

This means that – in case that A is separable – the conditions in Part (xv) imply the conditions in Part (iv).

Case with separable A is ready here.

Now we discuss how to reduce to the separable case.

By Proposition B.15.2(iii,iv), there exists for each $a, b \in A_+$ a separable C^*-subalgebra $B \subseteq A$ with $a, b \in B$ and the property that $n(e, d, \varepsilon; B) = n(e, d, \varepsilon; A)$ and $m(e, d, \varepsilon; B) = m(e, d, \varepsilon; A)$ for all $\varepsilon > 0$ and $d, e \in B_+$, where for given C^*-subalgebras $B \subseteq A$ the numbers $n := n(e, d, \varepsilon; B)$ are defined for each $d, e \in B_+$ and $\varepsilon > 0$ as the minimal number $n \in \mathbb{N}$ with the property that there exists n contractions $c_1, \ldots, c_n \in B$ with $\|e - \sum_{k=1}^n c_k^* d c_k\| < \varepsilon$. The above defined (possibly infinite) numbers $\nu(b, a, \varepsilon; B) \in \mathbb{N} \cup \{+\infty\}$ is then equal to $\sup_{\ell \in \mathbb{N}} n(b, a^\ell, \varepsilon; B)$.

In particular this causes that B is simple if A is simple, and that $\nu(a, b, \varepsilon; B) = \nu(a, b, \varepsilon; A)$ for $a, b \in B_+$.

Since we can find for each $a, b \in A_+$ a separable C^*-subalgebra B with this property, we get finally that A satisfies the assumptions of Part (iv), i.e., $n(b, a, \varepsilon; A) = 1$ for all non-zero $a \in A_+$. Where we use $n(b, a, \varepsilon; B) \leq m(b, \varepsilon; B)$ for all non-zero $a \in B_+$.

We start now the study of the above proposed separable case:

We reduce the general case to the separable case by Proposition B.15.2(iii,iv).

It says that, for each $a, b \in A_+$, there exists a simple separable C^*-subalgebra $B \subseteq A$ with $a, b \in B$ and $n(c, d, \varepsilon; B) = n(c, d, \varepsilon; A)$ and $\nu(c, d, \varepsilon; B) = \nu(c, d, \varepsilon; A)$ for all $c, d \in B_+$.

In particular, $n(b,a,\varepsilon;A) = n(b,a,\varepsilon,B) = 1$. Thus, $n(b,a,\varepsilon) = 1$ for all $a,b \in A_+$ with $\|a\| = \|b\| = 1$. It implies finally that Part (xv) implies Part (iv) with $n(a,b;\varepsilon) = 1$.

(xvi)⇒(ii): Let $a,b \in A_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon > 0$. There is a character χ on $C^*(a)$ with $\chi(a) = 1$. The character χ extends to a linear functional ρ on A with $\rho(a) = 1 = \|\rho\| = \|\chi\|$ by Hahn-Banach extension. Consider the maps $V\colon A \to \mathbb{C}$ and $W\colon \mathbb{C} \to A$ defined by $V(x) := \rho(x)$ for $x \in A$ and $W(\xi) := \xi \cdot b$. The maps V and W are completely positive, have norms $= 1$ and satisfy $W(V(a)) = b$. By assumptions of Part (xvi), there exist $d_1, d_2 \in A$ with $d_1^* d_2 = 0$ and $\|b - d_k^* a d_k\| = \|W(V(a)) - d_k^* a d_k\| < \varepsilon$ for $k \in \{1,2\}$. Thus, A satisfies the condition in Part (ii).

TEXT-Info (xvi):

For each finite subset $F \subset A$, each $n \in \mathbb{N}$, $\varepsilon > 0$, each completely positive contractions $V\colon A \to M_n$, $W\colon M_n \to A$ there exists contractions $d_1, d_2 \in A$ (depending from F, V, W and ε) with $d_1^* d_2 = 0$ and

$$\|(W \circ V)(x) - d_k^* x d_k\| < \varepsilon \quad \text{for all} \quad x \in F,\ k \in \{1,2\}\,.$$

From Proposition 3.1.9(ii):

Suppose that A has the property, that for every positive contraction $e \in A_+$ and $\varepsilon > 0$ there exist contractions $e_1, e_2 \in A$ with $\|e_j^* e_k - \delta_{jk} e\| < \varepsilon$ for $j,k \in \{1,2\}$.

This property of $e \in A_+$ has been studied also further above in Chp.2...??? Why refer to Chapter 3? Find it and compare the ref.s !!!

Then W can be "orthogonal approximated in norm" – in the sense that for each $\gamma > 0$ there exist c.p. maps $U_0, U_1\colon M_n \to A$ such that $U_k(1_n) \leq V(1_n)$, $\|U_k - W\| < \gamma$ and that U_0 and U_1 are "elementary" and "orthogonal" in the sense that there are column-matrices $d_k \in M_{n,1}(A)$ with $U_k(\beta) = d_k^* \beta d_k$ and $d_0^* \beta d_1 = 0$ for all $\beta \in M_n \cong M_n(\mathbb{C} \cdot 1) \subseteq M_n(\mathcal{M}(A))$ and $k \in \{0,1\}$.

(xi)⇒(xvi): Let $F \subset A$ a finite subset, $\varepsilon > 0$, $V\colon A \to M_n$, and $W\colon M_n \to A$ completely positive contractions. Let $b := W(1_n)$.

Let $0 < \delta := \varepsilon/?????$, ρ a pure states on A and $d_\rho\colon A \to \mathcal{L}(\mathcal{H})$ with $\mathcal{H} := L_2(A,\rho) := A/L_\rho$ the corresponding irreducible representation with cyclic vector x_0 satisfying $\rho(a) = \langle d_\rho(a) x_0, x_0 \rangle$. Here $L_\rho := \{a \in A\,;\ \rho(a^*a) = 0\}$.

By Lemma 2.1.22 there exist an isometry I from $\ell_2(n)$ into $\mathcal{L}(\mathcal{H})$ with $\|I^* d_\rho(a) I - V(a)\| < \delta$ for $a \in F$. By simplicity of A we can take any irreducible representation $\rho\colon A \to \mathcal{L}(\ell_2)$ and find a suitable isometry T from $\mathbb{C}^n$ into ℓ_2 such that $\|V(a) - T^* \rho(a) T\| < \delta$ for all $a \in F$.

By assumption of Part (xi), for each $a \in A_+$ there exists $d, e \in A$ with $d^* e = 0$ and $d^* d = (a - \gamma)_+ = e^* e$. This allows to apply Proposition 3.1.9(ii).

`Is it better to use an elementary tool? Not from Chp.3??`

It says that each c.p. contraction $W\colon M_n \to A$ can be approximated in norm by c.p. maps $U\colon M_n \to A$ with $U(1_n) \leq W(1_n)$, that are "elementary" in the sense that there are column-matrices $d \in M_{n,1}(A)$ with $U(\alpha) = d^*\alpha d$ for $\alpha \in M_n$.

This means that we can find $d \in M_{n,1}(A)$ with $\|W(Y) - d^*Yd\| < \delta\|Y\|$ for all $Y \in M_n$ and $d^*d = \sum_k d_k^* d_k \leq W(1_n)$. In particular $\|d^*d\| \leq 1$, because W is a c.p. contraction.

`Give reference for next ''blue''`

`Seems to be better to factorize over a fixed copy of` M_n `sitting in` A `itself ?`

By Part (i) and Lemma 2.1.6 there exists a properly infinite projection $0 \neq p^*p = p \in A$. Let $s, t \in pAp$ partial isometries with $s^*t = 0$ and $s^*s = t^*t = p$. Consider the partial isometry $v_k := t^k s$

(i)⇒(xvii): Let D a non-zero hereditary C^*-subalgebra of A and $p \in D$ an infinite projection. This means that $pDp = pAp$ contains a non-unitary isometry, i.e., there exists partial isometry $v \in D$ with $v^*v = p$, $vv^* \leq p$ and $vv^* \neq p$. Let 1 denote the unit of $\mathcal{M}(A)$. Then $T := (1-p) + v \in D + 1_{\mathcal{M}(A)}$ is a non-unitary isometry in the C^*-algebra $D + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$.

(xvii)⇒(iii): Let $D \subseteq A$ a non-zero hereditary C^*-subalgebra of A and $0 \neq e \in D_+$. The hereditary C^*-subalgebra $E := \overline{eAe}$ is non-zero. Let 1 denote the unit-element of A if A is unital, otherwise let it denote the unit $1_{\mathcal{M}(A)}$ of the multiplier algebra of A.

By assumptions of Part (xvii), there exists an element $T \in E + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ that is left-invertible in $A + \mathbb{C} \cdot 1$ but is not right-invertible in $A + \mathbb{C} \cdot 1$.

Let $L \in A + \mathbb{C} \cdot 1$ a left inverse for T, i.e., $LT = 1$. Then $\|L\|^2 T^*T \geq T^*L^*LT = 1$. It implies that $(T^*T)^{-1/2} \in E + \mathbb{C} \cdot 1$ exists and that $S := T(T^*T)^{-1/2} \in E + \mathbb{C} \cdot 1$ is a non-unitary isometry in $E + \mathbb{C} \cdot 1$. The S is non-unitary, because otherwise T would be also right-invertible.

It follows that $p := 1 - SS^*$ is a non-zero projection in E and the hereditary C^*-subalgebra F of E generated by p and $S^n p (S^*)^n$ for $n \in \mathbb{N}$ is a non-zero stable C^*-subalgebra of $E \subseteq D$. □

Remark 2.2.2. The Part (xvii) of Proposition 2.2.1 is a "poor-man variant" of a criterium of L.G. Brown, P. Friis and M. Rørdam. See [**109**] and [**305**]:

If A is simple and satisfies the Friis-Rørdam "Property IR*", then there is a strict alternative that excludes each other:*
A is purely infinite or A has "stable rank one".

More precisely, the alternative for simple A with Property *IR* is: A is purely infinite or each element of $A + \mathbb{C} \cdot 1_{\mathcal{M}(A)} \subseteq \mathcal{M}(A)$ is contained in the norm closures of the *invertible* (!) elements of $A + \mathbb{C} \cdot 1_{\mathcal{M}(A)}$.

See Section 9 in Appendix B for the definitions of Friies-Rørdam "Property *IR*" and "stable rank one" and some explanations, in particular why Part (xvii) of

Proposition 2.2.1 is one of the observations for the proof of the Brown-Friis-Rørdam alternative for simple C^*-algebras with Property *IR* between pure infiniteness and "stable rank one".

Next lemma will be used later to prove in some cases pure infiniteness of tensor products of C^*-algebras.

LEMMA 2.2.3 ([**93**], lem. 2.15). *Let D be a non-zero hereditary C^*-subalgebra of the minimal C^*-algebra tensor product $A \otimes B$ of C^*-algebras A and B.*

Then there exists $0 \neq z \in A \otimes B$ with $zz^* \in D$ and $z^*z = e \otimes f$ for some non-zero $e \in A_+$ and $f \in B_+$.

If $d \in D_+$ and pure states $\varphi \in A^*$, $\psi \in B^*$ are given with $(\varphi \otimes \psi)(d) > 0$, then such element $z \in A \otimes B$ can be found such that, moreover, $\varphi(e)\psi(f) > 0$.

PROOF. Let $d \in D_+$ with $\|d\| = 1$, and let $C := A \otimes B$. The minimal C^*-algebra tensor product is the same as spatial tensor product with respect to the direct sum of irreducible representations of A and B (cf. [**704**, prop. 1.22.9], [**766**]). Thus, there are pure states φ on A and ψ on B such that $(\varphi \otimes \psi)(d) > 0$.

We can assume from now on, that fixed *pure* states φ and ψ and a fixed contraction $d \in D_+$ with $(\varphi \otimes \psi)(d) > 0$ are given, i.e., we prove also the second part of this Lemma (2.2.3) at the same time.

Then $a := (\varphi \otimes \mathrm{id}_B)(d) \in B_+$ is a non-zero contraction and $0 < \psi(a) \leq \|a\|$.

Let $\delta := \psi(a)/2$ and $f := (a - \delta)_+^2$. Thus $0 < \delta \leq 1/2$, $f \in B_+$ and $\psi(f) > 0$, because $\psi(f)^{1/2} \geq \psi(f^{1/2}) \geq \psi(a) - \delta > 0$.

There exists a separable C^*-subalgebra G of A such that d is in the closure of the *algebraic* tensor product $G \odot B$, because d is the limit of a sequence in $A \odot B$. By Proposition A.21.4 and Lemma A.21.3 there exists $b \in A_+$ with $\|b\| = 1 = \varphi(b)$ (for all $m \in \mathbb{N}$) such that

$$\lim_n \|b^n g b^n - \varphi(g) b^{2n}\| = 0 \quad \text{for all } g \in G\,.$$

The state φ extends to a state φ_e on the unitization $A + \mathbb{C}1 \subset \mathcal{M}(A)$ of A with $\varphi_e(1 - b) = 0$. It gives $\varphi_e(a(1 - b)) = 0$ by Cauchy inequality. In particular, $\varphi(a) = \varphi(ab)$ for all $a \in A$ and $\varphi(b^n) = 1$ for all $n \in \mathbb{N}$. Thus, the restriction of φ to $C^*(b) \subseteq A$ is a character on $C^*(b)$.

The maps

$$T_n \colon \ y \in A \otimes B \mapsto (b^n \otimes 1)y(b^n \otimes 1) - (b^{2n} \otimes (\varphi \otimes \mathrm{id})(y))$$

converge on $G \otimes B$ point-wise to zero, because T_n is a difference of two completely positive contractions on C and tends on $G \odot B$ point-wise to zero.

Thus, there exists n with $\|T_n(d)\| < \delta^2$, i.e.,

$$(b^{2n} \otimes a) - \delta^2 \ \leq \ (b^n \otimes 1)d(b^n \otimes 1)$$

in the unitization of C. Let $g := b^{2n}$ and $t := d^{1/2}(b^n \otimes 1)((g \otimes a) - \delta^2)_+^{1/2}$. Then we get $((g \otimes a) - \delta^2)_+^2 \leq t^*t$ and $t\, C\, t^* \subseteq d^{1/2}\, C\, d^{1/2} \subseteq D$.

Now let $e := (g - \delta)_+{}^2 \in A_+$. Using that $\varphi(g) = \|g\| = 1$ and $g \geq 0$, i.e., that φ is a character on $C^*(g)$, and that $0 < \delta \leq 1/2$, we get $\varphi(e) = (1 - \delta)^2 > 0$.

Recall that $f := (a - \delta)_+^2$. Functional calculus in $C^*(g) \otimes C^*(a)$ shows that $e \otimes f \leq ((g \otimes a) - \delta^2)_+)^2$.

If $t = (tt^*)^{1/2}v$ is the polar decomposition of t in the second conjugate of C, then $vxv^* \in \overline{t\, C\, t^*} \subseteq D$ and $vx^{1/2} \in C$ for every $x \in C$ with $0 \leq x \leq t^*t$, because $x^{1/2}$ is in the norm closure of $t^*\, C\, t$ and $vt^* = (tt^*)^{1/2}$.

Since $e \otimes f \leq t^*t$ we get that $z := v((g - \delta)_+ \otimes (a - \delta)_+)$ is in $A \otimes B$, and e, f, z satisfy $z^*z = e \otimes f$, $zz^* \in D$ and $\varphi(e)\psi(f) > 0$. □

Next Lemma 2.2.4 allows later applications by using suitable central sequences.

LEMMA 2.2.4. *Let $F \subseteq A \setminus \{0\}$ a finite subset of non-zero elements in a C^*-algebra A.*

Suppose that $f^* \in F$ for each $f \in F$ and that, for every $\mu \in (0, 1)$, there exist contractions $s_1, s_2 \in \mathcal{M}(A)$ *(depending on μ) with the properties that for all $f \in F$ and $j \in \{1, 2\}$ holds:*

$$\|s_j f - f s_j\| \; + \; \|(1 - s_j^* s_j)f\| \; + \; \|(s_1 s_1^* s_2 s_2^*)f\| \; < \; \mu \, \|f\| \,. \tag{2.1}$$

Then, for each finite sequence $d_1, d_2, \ldots, d_n \in \mathcal{M}(A)$ and $\varepsilon > 0$, there exists elements $g_1, g_2 \in A$ with $g_1^* g_2 = 0$, $g_k^* g_k \leq \sum_{\ell=1}^n d_\ell^* d_\ell$ for $k \in \{1, 2\}$ and

$$\|g_j^* f g_k \; - \; \delta_{j,k} \cdot \sum_{\ell=1}^n d_\ell^* f d_\ell\| \; < \; \varepsilon \cdot \| \, f \, \| \,, \quad \forall \; 0 \neq f \in F, \; j, k \in \{1, 2\} \,. \tag{2.2}$$

PROOF. We show below that the existence of contractions $s_1, s_2 \in \mathcal{M}(A)$ satisfying the inequalities (2.1) for each $\mu > 0$ and fixed given F implies that we can find for the given $f \in F$ the following, – formally stronger –, existence property (*):

(*) *For each $\gamma \in (0, 1)$ there exist contractions $t_1, \ldots, t_{2n} \in A$ with the property that $t_j^* t_k = 0$ for $j \neq k$ and $\|t_j^* f t_k - \delta_{j,k} f\| < \gamma$ for all $j, k \in \{1, \ldots, 2n\}$ and $f \in F$.*

If we have shown the existence (*) – for each given $\gamma > 0$ – then we can use the contractions $t_1, \ldots, t_{2n}$ to define $g_1 := \sum_{\ell=1}^n t_\ell d_\ell$ and $g_2 := \sum_{\ell=1}^n t_{\ell+n} d_\ell$. Clearly, $g_1^* g_2 = 0$ and $g_k^* g_k \leq \sum_{\ell=1}^n d_\ell^* d_\ell$ for $k \in \{1, 2\}$. This elements $g_1, g_2 \in A$ satisfy then the Inequalities (2.2):

Indeed, consider the matrices $S[f] := [\delta_{j,k} f] \in M_{2n}(A)$ and $T[f] := [t_j^* f t_k] \in M_{2n}(A)$, and the columns $Z_1 := [d_1, \ldots, d_n, 0, \ldots, 0]^\top \in M_{2n,1}(\mathcal{M}(A))$ and $Z_2 := [0, \ldots, 0, d_1, \ldots, d_n]^\top \in M_{2n,1}(\mathcal{M}(A))$. Then, for $f \in F$, $g_j^* f g_k = Z_j^* T[f] Z_k$ and

$\delta_{j,k}\sum d_\ell^* f d_\ell = Z_j^* S[f] Z_k$. Thus,

$$\|g_j^* f g_k - \delta_{j,k}\sum_\ell d_\ell^* f d_\ell\| \,\le\, \|S[f]-T[f]\|\cdot \|\sum_{\ell=1}^{n} d_\ell^* d_\ell\|\,.$$

The entries of the self-adjoint matrix $S[f]-T[f] \in M_{2n}(A)$ have norms $\le \gamma$ by Property (*). A simple matrix-norm estimate says that an operator $m\times m$ matrix $[x_{jk}] \in M_m(A)$ has operator norm $\le m\max\{\|x_{j,k}\|\,;\ 1\le j,k\le m\}$ in $M_m(A)$, cf. Section 19 of Appendix B and Remark 2.1.10, thus: $\|S[f]-T[f]\| \le 2n\cdot\gamma$.

Hence, we get elements $g_1, g_2 \in A$ that satisfy all desired properties if we can show – for given $\varepsilon \in (0,1)$ – the existence of the contractions t_k with above properties (*) for suitable $\gamma \in (0,1)$ – that satisfies e.g.

$$2n\gamma \cdot \big(1+\sum_\ell \|d_\ell\|^2\big) \,\le\, \varepsilon\,.$$

Now we are going to derive the existence of contractions $t_1,\ldots,t_\ell,t_{2n} \in A$ for given $\gamma \in (0,1)$ with the above listed property (*) for given $\gamma \in (0,1)$ fixed given finite subset $F \subset A$, with $f^* \in F$ for $f \in F$.

We avoid complicate iterated estimates for commutators of polynomials and use instead its translation into precise algebraic relations in corona spaces – together with a sort of approximate semi-projectivity for relations like $x_j^* x_k f = \delta_{j,k} f$ and $\|x_k\| \le 1$ for $j,k \in \{1,\ldots,2n\}$ and f in a finite set.

Indeed, we derive from the existence of s_1, s_2 that satisfy the inequalities (2.1) for $\mu = 2^{-m}$ $(m = 1,2,\ldots)$ the following observation ([21]):

`Above are used other/same letters, give new notation!`

For each $n \in \mathbb{N}$ there exists of $2n$ contractions $T_\ell = (t_{\ell,1}, t_{\ell,2},\ldots) \in \ell_\infty(A)$ $(\ell \in \{1,2,\ldots,2n\})$ that have the following properties:

$T_\ell^* T_k = 0$ for $k \ne \ell$, $T_\ell f - f T_\ell \in c_0(A)$ and $f - T_\ell^* T_\ell f \in c_0(A)$ for all $f \in F$, – where we identify here $f \in F$ with $(f,f,\ldots) \in \ell_\infty(A)$.

We can find finally the desired contractions $t_1,\ldots,t_\ell,\ldots t_{2n} \in A$ – with the above discussed desired properties – by picking up entries $t_\ell := t_{\ell,n_0}$ for suitable (fixed) $n_0 \in \mathbb{N}$ from the elements $T_\ell = (t_{\ell,1},\ldots,t_{\ell,n_0-1},t_{\ell,n_0},\ldots)$.

Thus, the proof is complete if we have shown the *existence* of the contractions $T_\ell \in \ell_\infty(A)$ $(\ell = 1,\ldots,2n)$ with the above proposed property,

To get such $T_\ell \in \ell_\infty(A)$, we modify the elements $s_1, s_2 \in \mathcal{M}(A)$ that satisfy the inequalities (2.2) with help of a quasi-central approximate unit of A in the sense of [**616**, thm. 3.12.14], cf. also Part (3) of Remarks 5.1.1:

If $\tau \in (0,1)$ is given, then we use that $F \subseteq A$ is a finite set and find – in a quasi-central approximate unit of A – a positive contraction $e \in A_+$ such that, for

[21] Here $n \in \mathbb{N}$ can not be replaced by "∞" because the used projectivity property does not work for *infinitely many* generators! Says that the estimates are delicate and need some care.

$j \in \{1,2\}$ and $f \in F$,

$$\|s_j e - e s_j\| < \tau \quad \text{and} \quad \|ef - f\| + \|fe - f\| < \tau \|f\| .$$

It follows for $0 \neq f \in F$ that $\|ef - fe\| < \tau\|f\|$, $\|efe - f\| < \tau\|f\|$, $\|e^2 f - f\| \leq 2\tau\|f\|$ and that $\|s_1 e^2 s_1^* s_2 e^2 s_2^* - e^2 s_1 s_1^* s_2 s_2^* e^2\| \leq 4\tau$. It allows to calculate that the contractions $s_1 e, s_2 e \in A$ satisfy, for $f \in F$ and $j \in \{1,2\}$, that the sum of norms

$$\|(s_j e)f - f(s_j e)\| + \|f - (s_j e)^*(s_j e)f\| + \|(s_1 e)(s_1 e)^*(s_2 e)(s_2 e)^* f\|$$

is less or equal to the sum

$$\|s_j f - f s_j\| + \|f - s_j^* s_j f\| + \|s_1 s_1^* s_2 s_2^* f\| + 7\tau \cdot \|f\| .$$

Thus, if the contraction $e \in A_+$ is chosen as above for sufficiently small $\tau > 0$, then we obtain that the contractions $s_1 e$ and $s_2 e$ satisfy again the Inequality (2.1) in place of s_1 and s_2. It follows that our assumptions imply that for each $\mu \in (0,1)$ there are contractions $s_1, s_2 \in A$ that satisfy the inequalities (2.1). Thus, we can find the s_1, s_2 with this properties in A itself.

Now let $A_\infty := \ell_\infty(A)/c_0(A)$, consider A as C^*-subalgebra of A_∞ via the natural C^*-monomorphism $A \ni a \mapsto (a, a, \ldots) + c_0(A)$, and let $C := C^*(F) \subseteq A \subseteq A_\infty$ the separable C^*-subalgebra of A generated by the finite set F.

Let $s_{1,n}, s_{2,n} \in A$ contractions that satisfy the inequalities (2.1) for $s_{j,n}$ ($j \in \{1,2\}$) in place of s_j and with $\mu := 2^{-n}$ ($n = 1, 2, \ldots$). Define contractions $S_j \in A_\infty$ ($j \in \{1,2\}$) by $S_j := (s_{j,1}, s_{j,2}, \ldots, s_{j,n}, \ldots)$.

Here is ... ???

`Compare above with below def.s and notations !!! ???`

We use now that the universal C^*-algebra

$$C^*(x_1, \ldots, x_n \,;\; \|x_k\| \leq 1\,,\; x_j^* x_k = 0\;,\; j \neq k\,,\, j,k = 1, \ldots, n\,)$$

is projective, cf. Corollary A.8.5. It shows the existence of the above predicted contractions $T_1, \ldots, T_{2n} \in \ell_\infty(A)$ with orthogonal ranges ([22]).

This method allows to translate the question, if contractions $t_1, \ldots, t_{2n}$ with the above quoted properties exist, into the question, if we can find isometries

$$D_1, \ldots, D_{2n} \in (C' \cap A_\infty)/\operatorname{Ann}(C, A_\infty)$$

with mutually orthogonal ranges, i.e., with $D_j^* D_k = \delta_{jk} 1$ for $j,k \in \{1, \ldots, 2n\}$.

It works well because we can lift the isometries D_j by Corollary A.8.5 to contractions

`HERE are the` E_j `defined. Somewhere below called` T_j `???`

[22] Instead using the projectivity of this universal C^*-algebra we could use – only for this part of the proof – that the below considered subalgebras and sub-quotients of A_∞ are all "locally" sub-Stonean to get the desired lifts with pairwise *orthogonal* ranges. But it isn't much simpler all together.

$E_j := (e_{j,1}, e_{j,2}, \ldots) + c_0(A) \in C' \cap A_\infty$ with $E_j^* E_k = 0$ for $j \neq k$ and $E_j f - f E_j = 0$ and $f - E_j^* E_j f = 0$, where we identify here the $f \in F$ with $(f, f, \ldots) + c_0(A)$.

Then we can use the Corollary A.8.5 a second time to lift the contractions $E_j \in A_\infty = \ell_\infty(A)/c_0(A)$ to contractions $T_j \in \ell_\infty(A)$ with mutually orthogonal ranges, i.e., $\|T_j\| = 1$, $T_k^* T_j = 0$, $T_j f - f T_j, f - T_j^* T_j f \in c_0(A)$, where we identify $f \in F$ with $(f, f, \ldots) \in \ell_\infty(A)$.

Recall that $C \subseteq A \subset A_\infty$ denotes the separable C^*-subalgebra of A generated by $F \subseteq A_\infty$. The two-sided annihilator of C in A_∞ is defined by

$$\mathrm{Ann}(C, A_\infty) := \left\{ X \in A_\infty\, ;\ X \cdot C = \{0\} = C \cdot X \right\}.$$

It is easy to see that $\mathrm{Ann}(C, A_\infty)$ is a closed ideal of the relative commutant $C' \cap A_\infty$ of $C \subseteq A_\infty$, cf. [**448**] for similar definitions and considerations in the – perhaps for some readers more delicate – case of ultra-powers $A_\omega := \ell_\infty(A)/c_\omega(A)$, in place of $A_\infty := \ell_\infty(A)/c_0(A)$.

The contractions $t_1, \ldots, t_{2n} \in A$, with the property that

$$t_j^* t_k = 0 \quad \text{and} \quad \|t_j^* f t_k - \delta_{j,k} f\| < \gamma \quad \text{for all } j, k \in \{1, \ldots, 2n\} \text{ and } f \in F,$$

can be obtained by lifting the

`Where` E_k `!!!,` T_k `was defined?`

to contractions T_k with pairwise orthogonal ranges in $\ell_\infty(A)$, where we use that F is a finite subset of A, because then we can select suitable “coordinates” $t_k := T_{k,n_0}$ (n_0 sufficiently large and then fixed) of $T_k = (T_{k,1}, T_{k,2}, \ldots) + c_0(A)$ in an obvious manner such that the considered inequalities are fulfilled.

`Here !!! and a bit above` $\Uparrow$ `starts notation disorder.`
`New attempts below?` E `and` T `and` S `same?:`

The assumptions allow to find for $\mu_n = 2^{-n}$, $n = 2, 3, \ldots$ sequences $(s_{j,1}, s_{j,2}, \ldots)$ of contractions in A such that $s_{j,n}$, $j \in \{1, 2\}$, define an element $S_j := (s_{j,1}, s_{j,2}, \ldots) \in \ell_\infty(A)$ ($j \in \{1, 2\}$) such that, for $f \in F$ and $k \in \{1, 2\}$,

$$S_j f - f S_j\,, \quad f - (S_j^* S_j) f\,, \quad (S_1 S_1^* S_2 S_2^*) f \in c_0(A)\,.$$

Let $A_\infty := \ell_\infty(A)/c_0(A) \supset A$, and consider the bounded sequences in A modulo $c_0(A)$ as elements of A_∞, i.e., $S_k := (s_k^{(1)}, s_k^{(2)}, \ldots) + c_0(A)$ where we embed here A into A_∞ by $a \mapsto (a, a, \ldots) + c_0(A)$.

Then it happens in the algebra A_∞ that $\|S_k\| \leq 1$, $S_1^* S_2 = 0$, $S_k f = f S_k$ and $(1 - S_k^* S_k) f = 0$ for all $f \in F$ and $k \in \{1, 2\}$.

It implies $(1 - S_k^* S_k)(f^* f + f f^*) = 0$ for all $f \in F = F^*$. Since $1 - S_k^* S_k \geq 0$ it follows that $F \cup F \cdot F$ is contained in the two-sided annihilator of the positive element $2 - S_1^* S_1 - S_2^* S_2$ of A_∞. Thus, the C^*-algebra $C := C^*(F) \subseteq A \subseteq A_\infty$ is contained in the two-sided annihilator of $2 - S_1^* S_1 - S_2^* S_2$.

Using that $f^* \in F$ for all $f \in F$ we get that $F \subseteq \{S_1, S_1^*, S_2, S_2^*\}'$.

It follows that the elements of the C^*-subalgebra C of $B := A_\infty$???? generated by F commute with S_1 and S_2. Thus $S_1, S_2 \in C' \cap B$, i.e., the elements S_k commute with all elements in the C^*-subalgebra $C \subseteq B$ generated by F.

Since $F^* = F$, we get that C is the closure of the linear span of products of elements $f^* + f$, f^*f and ff^* with $f \in F$.

This implies that $(1 - S_k^* S_k)C = \{0\}$, $S_k c - cS_k = 0$ for all $c \in C$, ... $(k \in \{1, 2\})$.

???? CHECK !!! ???

$Y := \operatorname{Ann}(1 - S_k^* S_k, B) \subseteq B := \ell_\infty(\mathcal{M}(A))/c_0(A) \supseteq A_\infty$ is a hereditary C^*-subalgebra of B with $F \subseteq Y$. It follows $C := C^*(F) \subseteq Y$. Thus $(1 - S_k^* S_k) \in C' \cap B$.

Thus we can consider first the two-sided annihilator $\operatorname{Ann}(C, B)$ of C in B and the commutator $C' \cap B$ of C in B.

Then $\operatorname{Ann}(C, B)$ is a closed ideal of $C' \cap B$. If we find isometries $T_1, \ldots, T_n \in (C' \cap B)/\operatorname{Ann}(C, B)$ with pairwise orthogonal ranges. Then we can lift them to contractions $t_1, \ldots t_n$ in $C' \cap B$ with pairwise orthogonal ranges such that $t_j^* c t_k = \delta_{j,k} c$ for all $1 \le j, k \le n$ and $t_j^* t_k = 0$ for $j \ne k$.

If $d_1, \ldots, d_n \in \mathcal{M}(A) \subseteq B$ are given, then $g := \sum_k t_k d_k$ satisfies $g^* g \le \sum_k d_k^* d_k$ and $g^* f g = \sum d_k^* f d_k$.

Now one can take take ??or?? find mutually orthogonal representing sequences for the element $g \in B$ and of the $d_1, \ldots, d_n$.

By Corollary A.8.5, we get that the universal C^*-algebra generated by n contractions with orthogonal ranges is projective.

We consider first the case of isometries $s_1, s_2 \in \mathcal{M}(A)$ with the quoted properties, to make the idea of the proof transparent. The additional study for the approximate version will be added at the end of the proof.

Notice that the isometries s_1, s_2 with $s_k^* s_\ell = \delta_{k,\ell}$ $(k, \ell \in \{1, 2\})$ define isometries by $t_1 := s_1$ and $t_{n+1} := s_2^n s_1$ for $n = 1, 2, \ldots$ that satisfy $t_m^* t_n = \delta_{m,n} 1$ for $n, m \in \mathbb{N}$.

If $a \in F$ satisfies $\|s_k a - a s_k\| < \gamma$ $(k = 1, 2)$ for some $\gamma \in (0, \infty)$ and $k \in \{1, 2\}$ then, using that $t_m^*(a t_n - t_n a) = t_m^* a t_n - \delta_{m,n} a$, we get

$$\|t_m^* a t_n - \delta_{m,n} a\| \le \|a t_n - t_n a\| < n\gamma .$$

Follows from

$$\|a(b_1 \cdot b_2) - (b_1 \cdot b_2)a\| \le \|b_2\| \|[a, b_1]\| + \|b_1\| \|[a, b_2]\| .$$

`What about working in` $\mathcal{M}(A)_\infty$
`and replacing then` γ `by zero,` e.g.
$s_1, s_2 \in \mathcal{M}(A)_\infty$ `with` $s_1^* s_2 = 0$, $(1 - s_k^* s_k)a = 0$ `and` $[a, s_k] = 0$
`for` $a \in F$, $k \in \{1, 2\}$ `...`
`It implies` $(s_2^k s_1)^* a (s_2^\ell s_1) = \delta_{k,\ell} a$ `for` $a \in F$, $k, \ell \in \{0, 1, \ldots, n\}$, `with` $s_2^0 := 1$.

`The problem is:`
`Does there exist a ''local unit''` $e \in \mathcal{M}(A)_\infty$ `with` $0 \leq e \leq 1$ `for` F `such that` $ea = a = ae$ `for` $a \in F$ `and` $(s_2^k s_1)^* e (s_2^\ell s_1) = \delta_{k,\ell} e$ `??`

Let $s_1, s_2 \in A$ contractions with $s_1^* s_2 = 0$, $\|s_k a - a s_k\| < \gamma \cdot \|a\|$ and $\|(1 - s_k^* s_k)a\| < \gamma \cdot \|a\|$ $(k \in \{1,2\})$ for all $a \in F$ and some $\gamma \in (0,1)$. We can define contractions $t_n \in \mathcal{M}(A)$ by $t_1 := s_1$ and $t_{n+1} := s_2^n s_1$ for $n = 1, 2, \ldots$, and show that, for each $a \in F$,

$$\|t_n a - a t_n\| \leq n \cdot \gamma \|a\| \quad \text{and} \quad \|t_m^* a t_n - \delta_{m,n} a\| \leq 2 \min(m,n) \cdot \gamma \|a\| \,.$$

It implies that $\|(1 - t_n^* t_n)a\| \leq 3n \cdot \gamma \|a\|$.

We use the convention $s_2^0 := 1 \in \mathcal{M}(A)$. By assumptions, $\|t_1 a - a t_1\| \leq \gamma \|a\|$ for $t_1 := s_1$ and $a \in F$. Then $[a, xy] = x[a,y] + [a,x]y$ shows that $t_{n+1} a - a t_{n+1} = s_2^n (s_1 a - a s_1) + (s_2^n a - a s_2^n) s_1$ and

$$\|s_2^n a - a s_2^n\| \leq n \|s_2\|^{n-1} \|s_2 a - a s_2\| \leq n \cdot \gamma \|a\| \,.$$

Since s_1 and s_2 are contractions we get $\|t_n a - a t_n\| \leq n \cdot \gamma \|a\|$ for all $n = 1, 2, \ldots$.

An estimate for $\|t_m^* a t_n - \delta_{m,n} a\|$:
Let $s_k^0 := 1$ for $k \in \{1,2\}$. It suffices to consider the case where $1 \leq m \leq n$ because $(t_m^* a t_n)^* = t_n^* a^* t_m$ and $a^* \in F$ for $a \in F$. If $1 \leq m \leq n$ then $t_m = s_2^{m-1} s_1$, $t_n = s_2^{m-1} s_2^{n-m} s_1$ and $\delta_{n,m} = 1$ if and only if $n = m$.

$X := (s_2^*)^{m-1} a s_2^{m-1} - a$, $Y := s_1^* X s_2^{n-m} s_1$ and $Z := s_1^* a s_2^{n-m} s_1 - \delta_{m,n} a$, satisfy $Y + Z = t_m^* a t_n - \delta_{m,n} a$. Thus $\|t_m^* a t_n - \delta_{m,n} a\| \leq \|X\| + \|Z\|$.

If $\delta_{m,n} = 1$ then $m = n$ and $Z = s_1^* a s_1 - a$. If $\delta_{m,n} = 0$ then $m < n$, $Z = s_1^* (s_2 a - a s_2)(s_2^{n-m-1} s_1)$. The latter implies $\|Z\| \leq \gamma \|a\|$.

The equations $s_k^* a s_k - a = s_k^* (a s_k - s_k a) - (1 - s_k^* s_k) a$ imply $\|s_k^* a s_k - a\| \leq 2 \cdot \gamma \|a\|$ for $k \in \{1,2\}$. In particular, $\|Z\| \leq 2 \cdot \gamma \|a\|$ if $\delta_{n,m} = 1$. We show that $\|X\| \leq 2(m-1) \cdot \gamma \|a\|$:

Let $p > 1$. Then $(s_2^*)^p a (s_2)^p - a = \sum_{\ell=1}^{p} (s_2^*)^{p-\ell} (s_2^* a s_2 - a)(s_2)^{p-\ell}$.

Hence, $\|(s_2^*)^p a (s_2)^p - a\| \leq 2p \cdot \gamma \|a\|$.

Summing up, our calculations show that

$$\|t_m^* a t_n - \delta_{m,n} a\| \leq \|X\| + \|Z\| \leq 2 \min(m,n) \cdot \gamma \|a\| \,.$$

`Consider here the easy case of isometries` s_1, s_2 `.`

Let $a \in F$, $\varepsilon > 0$ and $d_1, \ldots, d_n \in A$ given, take $\mu := \| \sum_k d_k^* d_k \|$.

We find for each $\gamma > 0$ isometries $s_1, s_2 \in \mathcal{M}(A)$ with $\|s_k a - a s_k\| < \gamma$???? for $k = 1, 2$ and $a \in F$.

Let ?????

$$\gamma := \varepsilon / \big(1 + 2n \, (\|d_1\| + \ldots + \|d_n\|)^2\big) \,.$$

By assumptions, we find isometries $s_1, s_2 \in \mathcal{M}(A)$ with $s_1^* s_2 = 0$ and $\|s_k a - a s_k\| < \gamma$ for $k \in \{1,2\}$. The above from the s_1, s_2 defined isometries $t_1, \ldots t_n$ satisfy

$$\|d_i^*(t_i^* a t_j - \delta_{ij} a) d_j\| < n\gamma \|d_i\| \|d_j\| \,.$$

Let $c := \sum t_i d_i$. Then $c^*c = \sum_i d_i^* d_i$, $c^*ac - \sum_i d_i^* a d_i \; = \; \sum_{i,j=1}^n d_i^*(t_i^* a t_j - \delta_{ij} a) d_j$ and

$$\|c^*ac - \sum_{i=1}^n d_i^* a d_i\| \;\le\; \sum_{i,j=1}^n \|d_i^*(t_i^* a t_j - \delta_{ij} a) d_j\| \;<\; \varepsilon\,.$$

Detailed calculations:

$c^*ac = \sum_{i,j} x_i^* t_i^* a t_j x_j$.

$\|c^*ac - b\| \le \|c^*ac - \sum x_i^* a x_i\| + \|b - \sum x_i^* a x_i\|$.

$\|b - \sum x_i^* a x_i\| < \varepsilon/2$ by property of the chosen x_i .

$\sum_i x_i^* a x_i = \sum_{i,j} x_i^* \delta_{ij} a x_j$.

$$\sum_{i,j=1}^n \|x_i^*(t_i^* a t_j - \delta_{ij} a) x_j\| \le \sum_{i,j=1}^n \|x_i\| \cdot \|x_j\| \cdot \|t_i^* a t_j - \delta_{ij} a\| \,.$$

Using that $\|[a, s_k]\| \le \gamma := \varepsilon/2n(1 + (\sum_i \|x_i\|)^2)$, and $n \max_{k=1,2} \|[a, s_k]\| \ge \|a t_j - t_j a\| \ge \|t_i^* a t_j - \delta_{ij} a)\|$ for $i, j \le n$, we get for $\sum_{i,j=1}^n \|x_i\| \cdot \|x_j\| \cdot \|t_i^* a t_j - \delta_{ij} a)\|$ the estimate $n(\|x_1\| + \ldots + \|x_n\|)^2 \cdot \gamma \le \varepsilon/2$.

Recall here that $\gamma := \varepsilon / \big(2n(1 + (\sum_i \|x_i\|)^2)\big)$. END OF THE EASY CASE is here !!!???

Let $a \in F$, $\varepsilon > 0$ and $d_1, \ldots, d_n \in A$ given, take $\mu := \|\sum_k d_k^* d_k\|$.

By assumptions, we find for each $\delta > 0$ contractions $s_1, s_2 \in \mathcal{M}(A)$ with $s_2^* s_1 = 0$ and $\|s_k a - a s_k\| < \delta$ and $\|(1 - s_k^* s_k) a\| < \delta$ for $k = 1, 2$ and $a \in F$.

Attempt (?? !!!): $g := \sum_{k=1}^n t_k d_k$ for sufficiently small γ.

Calculate $\|g^* a g - \sum_k d_k^* a d_k\|$ for $a \in F$. Uses matrix-norm of $[t_j^* a t_k - \delta_{jk} a]$ for contractions $a \in F$ and $\|\sum d_k^* d_k\|$ with suitable bound. □

We list some *sufficient criteria* for a C^*-algebra A to be simple and purely infinite:

Proposition 2.2.5. *Let A be a non-zero C^*-algebra. Each one of the following permanences or properties* (i)–(v) *implies that A is simple and purely infinite.*

(i) *A is a hereditary C^*-subalgebra of a simple purely infinite C^*-algebra.*
(ii) *$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$ for a simple purely infinite C^*-algebra B and the compact operators $\mathbb{K}$ on some Hilbert space $\mathcal{H}$.*
(iii) *A is isomorphic to the inductive limit* $\operatorname{indlim}_n(h_n \colon A_n \to A_{n+1})$ *of non-zero C^*-algebras $A_n \neq \mathbb{C} \cdot 1$ with non-zero C^*-morphisms h_n that satisfy:*
   (*) *For every positive integer n, every $\varepsilon > 0$ and every $a, b \in (A_n)_+$ with $\|b\| \le \|a\| = 1$, there exists a contraction $c \in A_{n+1}$ such that $\|c^* h_n(a) c - h_n(b)\| < \varepsilon$.*

(In particular, the property (*) is satisfied if all A_n are simple and purely infinite.)

(iv) *The C^*-algebra A is simple and has the following property (Δ):*

(Δ) *For every $a \in A_+$ and $\varepsilon > 0$, the multiplier algebra $\mathcal{M}(A)$ contains isometries s_1, s_2, depending on (a, ε), with*

$$s_1^* s_2 = 0 \quad \text{and} \quad \|s_j a - a s_j\| < \varepsilon \quad \text{for } j \in \{1, 2\}\,.$$

(v) *The C^*-algebra A is isomorphic to the relative commutant $D' \cap B$ of D in B, where D is a separable C^*-subalgebra of a unital C^*-algebra B that satisfies following properties (α) and (β):*

(α) $D' \cap B \neq \mathbb{C}1$.

(β) *For every $a \in (D' \cap B)_+$ with $\|a\| = 1$, there exists a contraction $s \in B$ with $d = s^* a d s$ for every $d \in D$.*

REMARK 2.2.6. It is evident that Proposition 2.2.5(iii) and Proposition 2.2.1(ii) together imply that *inductive limits of simple purely infinite C^*-algebras are again simple purely infinite C^*-algebras* ([23]).

Parts (i) and (ii) of Proposition 2.2.5 together show invariance of the class of simple p.i. C^*-algebras under Morita equivalence.

Notice also that every simple purely infinite C^*-algebra A is isomorphic to each of the types of algebras considered in Parts (i)–(iii):

Indeed, take $B := A \otimes \mathbb{K}$ to fulfill the conditions in Parts (i) and (ii), and take $A_n := A$ and $h_n := \mathrm{id}_A$ for the conditions in Part (iii). The Condition (*) in Part (iii) is satisfied for $h_n = \mathrm{id}_A$, because if $a, b \in A_+$ are contractions with $\|a\| = 1$ and $\|b\| = 1$ then there exists $c \in A$ with $\|c^* a c - b\| < \varepsilon$ by Part (ii) of Proposition 2.2.1. The arguments in the proof of the implication (iv)⇒(ii) or of the implication (iii)⇒(ii) for Proposition 2.2.1(ii,iii,iv) show that we can take here $c \in A$ with $\|c\| \leq 1$.

That the above examples satisfy the assumptions (i–iii) can be seen from

Parts (??? to be listed !!!) of Proposition 2.2.1. (Must show for (i,ii) that $A \otimes \mathbb{K}$ is p.i. if A p.i. and that pi passes to non-zero hereditary C^*-subalgebras. The latter is done in the proof.)

For the general general property pi(1) (which is equal to "purely infinite" in sense of Definition 1.2.1) the passage to $M_2(A)$ is equivalent to the (not obvious) proof that Property pi(1) is equal to Property pi-1.

But for simple A the equivalence of pi(1) and pi-1 is given by of Parts (???) and (???) of Proposition 2.2.1 ??? Condition (*) in Part (iii) implies injectivity

of the h_n:
If the h_n are non-zero (as required !) then Condition (*) causes that each h_n is injective.

[23] If they are "non-zero" in the sense that the all C^*-morphisms $\psi_{m,n} \colon A_m \to A_n$ are non-zero!

Indeed, since h_n is non-zero there exists $b \in (A_n)_+$ with $\|b\| = 1$ and $\|h_n(b)\| = 1$. Let $a \in (A_n)_+$ with $\|a\| = 1$. Then condition (*) on h_n provides $c \in A_{n+1}$ with $\|c^* h_n(a)c - h_n(b)\| < 1/2$. It implies $h_n(a) \neq 0$, and that each h_n is faithful.

The Property (Δ) of Part (iv) holds for all σ-unital purely infinite simple C^*-algebras A.

Indeed, by Proposition 2.2.1(v) all simple purely infinite C^*-algebras are strongly purely infinite, Theorem **??** (A.spi.implies.M(A).spi???.is where?) says that the multiplier algebra $\mathcal{M}(A)$ of every σ-unital strongly purely infinite C^*-algebra A is again strongly purely infinite and Theorem **??** (nuclear.embed.of.exact) says that each nuclear C^*-morphism $h\colon C \to E$ of a separable exact C^*-algebra into a strongly purely infinite "corona" C^*-algebra E – e.g. as it is the case for $E = \mathcal{M}(A)_\infty := \ell_\infty(\mathcal{M}(A))/c_0(\mathcal{M}(A))$ – extends to a C^*-morphism $h_{\text{ext}}\colon C \otimes \mathcal{O}_\infty \to E$ with $h_{\text{ext}}(c \otimes 1) = h(c)$. Then apply this to

$$\iota\colon C^*(a, 1) \to \mathcal{M}(A) \subseteq \mathcal{M}(A)_\infty .$$

It gives that all (not necessary simple) σ-unital strongly purely infinite C^*-algebras A satisfy Property (Δ) of Part (iv) of Proposition 2.2.5.

The approximate decomposition condition (Δ) in Part (iv) of Proposition 2.2.5 is satisfied for every not necessarily simple but *strongly* purely infinite σ-unital C^*-algebra A, because for them $C' \cap \mathcal{M}(A)_\omega$ is purely infinite for every nuclear embedded separable exact C^*-subalgebra C of A_ω. This applies to $C := C^*(\pi_\omega(a, a, \ldots)) \in A_\omega$ and yields the existence of the $s_1, s_2 \in \mathcal{M}(A)$ with the property in Part (iv).

Simple purely infinite C^*-algebras are strongly purely infinite by Proposition 2.2.1(v). Thus, the condition in Proposition 2.2.5(iv) is moreover *equivalent* to pure infiniteness of σ-unital *simple* C^*-algebras A.

Proof of Proposition 2.2.5. The assumptions in Parts (i), (ii) or (iv) assume or imply the simplicity of A. But we prove also the similar statement for non-simple C^*-algebras if possible, i.e., if the condition itself implies the simplicity.

???

It is not difficult to see that each of the properties (i)–(iv) of A in Proposition 2.2.5 implies that A is simple and that A is *not* isomorphic to the compact operators on some Hilbert space. `Why it is easy to see from (iv)?` i.e., where A is simple and has Property (Δ): The $\mathbb{K}(\mathcal{H})$ do not satisfy (Δ).

(i): Part (i) follows straight from the definition of J. Cuntz, cf. [**172**, p. 186] or Part (i) of Proposition 2.2.1, in case of simple C^*-algebras.

But we use here the general Definition 1.2.1, because it is also suitable for non-simple C^*-algebras B. The proof that non-zero hereditary C^*-subalgebras $D \subseteq B$ satisfy Part (ii) of Definition 1.2.1 is easy:
If $a, b \in D_+$ and $\varepsilon > 0$ are given with $a \neq 0$ and b in the closed ideal of B generated by a, then, by Definition 1.2.1, there exists $g \in B$ with $\|g^* a g - b\| < \varepsilon/3$, because

$b \in D_+$ is in the closed ideal of D generated by a, then it is also in the closed ideal of B generated by a.

Let $\gamma := \|g\|$, $\delta := \varepsilon/(3\gamma^2+3)$ and $\psi(t) := \min\big(1, \delta^{-1}\cdot \max(0, t-\delta)\big)$.

Define $g_1 \in D$ by $g_1 := \psi(a)^{1/2} g \psi(b)^{1/2}$. Then $\|g_1^* a g_1 - b\| < \varepsilon$.

But we must prove that quotients B/J of B can not contain non-zero projections $p \in B/J$ with $p(B/J)p = \mathbb{C}\cdot p$. This requires to show first that quotients B/J are also purely infinite in sense of Definition 1.2.1.

It is obvious from the Definition 1.2.1 that non-zero ideals J of B (respectively non-zero quotients B/J of B) can not have a non-zero character, because otherwise such a character *extends* to, respectively *defines*, a non-zero character of B.

If $a, b \in B_+$ and $\pi_J(b)$ is in the closed ideal of B/J that is generated by $\pi_J(a)$ the for $\varepsilon > 0$ there exists $\delta > 0$ and $c_1, \ldots, c_n \in B$ with $(b-\varepsilon/3)_+ - \sum_{k=1}^n c_k^*(a-\delta)_+ c_k \in J$. By Part (ii) of Definition 1.2.1 there exists $d \in B$ with

$$\|d^* a d - \sum_{k=1}^n c_k^*(a-\delta)_+ c_k\| < \varepsilon/3\,.$$

This implies together that

$$\|\pi_J(b) - \pi_J(d)^*\pi_J(a)\pi_J(d)\| < \varepsilon\,.$$

The map $\pi_J\colon B \to B/J$ is surjective. Hence B/J is again purely infinite in the sense of Definition 1.2.1 for each closed ideal $J \neq B$ of B.

Let $D \subseteq B$ is a non-zero hereditary C^*-subalgebra of purely infinite C^*-algebra B. We show that D can not have a non-zero character (and is therefore also purely infinite in sense of Definition 1.2.1):

Suppose that D has a non-zero character $\xi\colon D \to \mathbb{C}$. The kernel $K \subset D$ of ξ is a closed ideal of D. The closed ideal J of B generated by K has the property that $J \cap D = K$, cf. ?????

Cite here the intersection Lemma ????

The argument (for the in Definition 1.2.1 required non-existence of characters on D) is the following – that works also in the non-simple case:

THE NEXT HERE is an application of the study of residually anti-liminariy C^*-algebras. Refere to there!!!

By assumption (i) of Definition 1.2.1, the algebra B has no characters. But the Property (ii) of Definition 1.2.1 implies then moreover that no irreducible representation λ of B contains a non-zero compact operator in its image, because otherwise λ must be necessarily one-dimensional representation, by Property 1.2.1(ii) and the combination of the Kadison transitivity theorem and the semi-projectivity of $C_0((0,1], M_n)$ applied in Lemma 2.1.15, i.e., λ would be a character on B. (Notice here that the Properties 1.2.1(i,ii) pass to quotients B/J of B and hereditary C^*-subalgebras of B also in case where B is non-simple.)

The property of a C^*-algebra B having only irreducible representations that do not contain non-zero compact operators passes as a property to all non-zero quotients of B.

It passes also to all non-zero hereditary C^*-subalgebras of B, because for hereditary $D \subseteq B$ each irreducible representation $\rho\colon D \to \mathcal{L}(\mathcal{H}_1)$ extends, up to unitary equivalence, uniquely to an irreducible representation $\lambda\colon B \to \mathcal{L}(\mathcal{H}_1 \oplus_2 \mathcal{H}_2)$ such that $\lambda(d) = \rho(d) \oplus 0$ for $d \in D$.

(ii): By Part (i), the Definition 1.2.1 of pure infiniteness passes to non-zero hereditary C^*-subalgebras. In case of simple C^*-algebras it is equivalent to all other definitions of pure infiniteness (except being only *stably properly infinite* !).

It remains to observe that the simple C^*-algebra $B \otimes \mathbb{K}$ is purely infinite in the sense of J. Cuntz if B is purely infinite:
Let $D \subseteq B \otimes \mathbb{K}$ be any non-zero hereditary C^*-subalgebra.

By Lemma 2.2.3 there exists non-zero $z \in B \otimes \mathbb{K}$ with $zz^* \in D$ and $z^*z = b \otimes a$ for some (non-zero) $b \in B_+$ and $a \in \mathbb{K}_+$. Then $w = z(1 \otimes v)$ satisfies $w^*w = b \otimes p_{11}$ and $ww^* \in D$ for a suitable element $v \in \mathbb{K}$. Thus, D contains the closure of $w(B \otimes \mathbb{K})w^*$ which is isomorphic to the (non-zero) closure of bBb (cf. Remark 2.3.1). Now use that $b \neq 0$ and B is purely infinite in the sense of J. Cuntz. It says that $\overline{bBb}$ and D contain both (non-zero) infinite projections $p \in \overline{bBb}$ and $q \in D$, that are Murray–von-Neumann equivalent in $B \otimes \mathbb{K}$. Thus, $B \otimes \mathbb{K}$ is purely infinite in the sense of Cuntz.

(iii): We establish here a more general *necessary and sufficient* condition for inductive limits of C^*-algebras to be simple and purely infinite. It is easy to see that the assumptions of Part (iii) satisfy this condition:

Let $h_n\colon A_n \to A_{n+1}$ be a sequence of C^*-morphisms. Then for $m < n$

$$h_{m,n} := h_{n-1} \cdot \ldots \cdot h_{m+1} h_m \colon A_m \to A_n$$

is a C^*-morphism. We embed the inductive limit $B := \operatorname{indlim}(h_n\colon A_n \to A_{n+1})$ naturally in $(\prod_{n=1}^\infty A_n)/(\bigoplus_{n=1}^\infty A_n)$ in the below described way.

We explain sometimes our viewpoint by use of the different notations

$$(\prod_{n=1}^\infty A_n)/(\bigoplus_{n=1}^\infty A_n) = \ell_\infty(A_1, A_2, \ldots)/c_0(A_1, A_2, \ldots) \subseteq \mathcal{L}(\mathcal{H})_\infty$$

if the A_n are considered as C^*-subalgebras of $\mathcal{L}(\mathcal{H})$.

The canonical morphisms $h_n^\infty\colon A_n \to B$ are then given by

$$h_n^\infty(a) := (0, \ldots, 0, a, h_{n,n+1}(a), h_{n,n+2}(a), \ldots) + \oplus_k^\infty A_k \qquad \forall\, a \in A_n.$$

Thus $\ker(h_{n,k}) \subseteq \ker(h_{n,k+1}) \subseteq \ker(h_n^\infty)$ and the kernel of h_n^∞ is the closure of the union of the kernels of $h_{n,k}$ for $k = n+1, n+2, \ldots$. If the h_n are injective, then the $h_{m,n}\colon A_m \to A_n$ and the $h_n^\infty\colon A_n \to B$ are injective. In this case it follows $\dim(B) > 1$ if $\dim(A_{n_0}) > 1$ for some n_0.

In the inductive limit B holds $h_m^\infty(A_m) \subseteq h_n^\infty(A_n)$ for $m < n$, and B is the closure of $\bigcup h_n^\infty(A_n)$. We show that
B is simple and purely infinite, if and only if,

(1) $\mathrm{Dim}(B) > 1$, *and*
(2) *for every $m \in \mathbb{N}$, $\varepsilon > 0$, and $e, f \in (A_m)_+$ with $\|f\| \leq 1 = \|e\| = \|h_{m,m+k}(e)\|$ for $k = 1, 2, \ldots$, there exist $p > n > m$ and a contraction $g \in A_n$ such that the distance of $h_{m,n}(f) - g^* h_{m,n}(e) g$ from* $\ker(h_{n,p})$ *is less than ε.*

Note that the distance from $c := h_{m,n}(f) - g^* h_{m,n}(e) g$ in A_n to $\ker(h_n^\infty)$ is less than ε, if and only if, $\|h_n^\infty(c)\| < \varepsilon$, if and only if, there exists $p > n$ such that c has distance from $\ker(h_{n,p})$ less than ε.

For the proof of Proposition 2.2.1(ii) it suffices to consider the only elements a and b in $\bigcup_n h_n^\infty(A_n)$ with $0 \leq a$, $0 \leq b$ and $\|a\| = \|b\| = 1$. Let such a and b given and let $m \in \mathbb{N}$ such that $a, b \in h_m^\infty(A_m)$. Then there exist contractions $e, f \in (A_m)_+$ with $h_m^\infty(e) = a$ and $h_m^\infty(f) = b$. It implies $1 \leq \|h_{m,m+k}(e)\| \leq \|e\| \leq 1$ and $\|f\| = 1$.

By (2) there exits, for given $\varepsilon > 0$. indices $n > m$ and a contraction $g \in A_m$ such that $\|b - d^* a d\| < \varepsilon$ for the contraction $d := h_m^\infty(g)$. Thus, (1) and (2) imply together that B is simple and purely infinite.

Conversely, suppose that B is simple and purely infinite. Then $\mathrm{Dim}(B) > 1$ by Proposition 2.2.1. Let $\varepsilon > 0$ and $e, f \in (A_m)_+$ with $\|f\| \leq 1 = \|h_{m,m+k}(a)\| = \|a\|$.

Then $a := h_m^\infty(e)$ and $b := h_m^\infty(f)$ are in B_+ and $\gamma := \|b\| \leq 1 = \|a\|$. By Proposition 2.2.1(ii), there exist a contraction $d \in B$ such that $\|b - \gamma d^* a d\| < \varepsilon$. If we replace d by a small perturbation of d it satisfies the same inequality. Therefore we can assume that $d \in h_n^\infty(A_n)$ for some $n > m$. If $g \in A_n$ is a contraction with $h_n^\infty(g) = \gamma^{1/2} d$, then there exists $p > n$ such that (2) is satisfied for $m, e, f, \varepsilon, g, n, p$. Thus (1) and (2) is satisfied if B is simple and purely infinite.

(iv): The algebra A is supposed to be simple by assumption in Part (iv). But we show that all – not necessarily simple – non-zero C^*-algebras A with Property (Δ) are purely infinite in sense of Definition 1.2.1:

If the algebra $A \neq \{0\}$ satisfies Property (Δ), then A can not have a character because $\mathcal{M}(A)$ contains two isometries s_1, s_2 with orthogonal ranges. (In particular A is not commutative, especially $A \neq \mathbb{C}$.)

Let $a, b \in A$ with $\|a\| = 1$ and let $\varepsilon > 0$.

We can consider here also non-simple A with property (Δ). But then we must require in addition that b is in the closed ideal generated by a, i.e., that $\|\pi_J(b)\| = 0$ if $\|\pi_J(a)\| = 0$ for all ideals J of A. It is enough to know this for all primitive ideals J of A.

Since b is in the ideal generated by a (that is automatic if A is simple and a is non-zero), there are $x_1, x_2, \ldots, x_n$ in A such that $\|b - \sum_k x_k^* a x_k\| < \varepsilon/2$.

The set $X := \{a\} \subset A$ satisfies the assumptions of Lemma 2.2.4. Thus, there exists for $x_1, \ldots, x_n$ an element $d \in A$ with

$$\|d\|^2 = \|\sum_k x_k^* x_k\| \quad \text{and} \quad \|d^* a d - \sum_k x_k^* a x_k\| < \varepsilon/2 .$$

In particular, $\|b - d^* a d\| < \varepsilon$.

Thus, Property (Δ) implies that A satisfies all conditions of the Definition 1.2.1 of purely infinite C^*-algebras. If A is simple, then A satisfies the criterium in Proposition 2.2.1(ii) for A being purely infinite. It causes that A is strongly p.i. in sense of Definition 1.2.2 if A is simple, cf. Proposition 2.2.1(v).

(v): We use criterion (ii) of Proposition 2.2.1. Notice first that $A := D' \cap B \neq \mathbb{C}$ by our assumptions. For $a \in (D' \cap B)_+$ and $\|a\| = 1$ there are unital *-epimorphisms $h_1\colon \mathrm{C}(\mathrm{Spec}(a)) \otimes D \to C^*(a, D)$ and $h_2\colon \mathrm{C}(\mathrm{Spec}(a)) \otimes D \to D$ given by $h_1(f \otimes d) = f(a)d$ and $h_2(f \otimes d) = f(1)d$ for $f \in \mathrm{C}(\mathrm{Spec}(a))$ and $d \in D$. If $f \otimes d$ is in the kernel of h_1, and $d \neq 0$, then $f(a) = 0$, because D is simple. Thus $f = 0$ on $\mathrm{Spec}(a)$. Therefore, by Lemma 2.2.3, the kernel of h_1 is zero. By assumption, there exists an isometry s in B with $s^* f(a) d s = h(f(a)d) = h_2((h_1^{-1})(f(a)d)) = f(1)d$ for $f \in C(\mathrm{Spec}(a))$ and $b \in D$. This yields $s \in D' \cap B$ and $s^* a s = 1$. □

Corollary 2.2.7. *The algebra $\mathcal{O}_\infty$ is simple and purely infinite.*

If C is a unital C^*-algebra with properly infinite unit element and B is any C^*-algebra, then $B \otimes^{\max} C \otimes^{\max} C \otimes^{\max} \cdots$ and its quotient $B \otimes^{\min} C \otimes^{\min} C \otimes^{\min} \cdots$ are purely infinite.

In particular, $B \otimes \mathcal{O}_\infty$ (and thus $B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$) are simple and purely infinite if B is simple.

We remind our convention that $\otimes$ always denotes the minimal (= spatial) C^*-algebra tensor product $\otimes^{\min}$.

`Check consistency of notations:`
$\otimes^{\min}$, $\otimes_{\min}$, $\otimes$, $\otimes^{\max}$, $\otimes_{\max}$, $\widehat{\otimes}$
`Where used? For what?`

Other tensor products will be denoted by $\odot$ (= algebraic tensor product) and $\otimes^{\max}$ (= maximal tensor product = universal in both variables short exact tensor product functor). Notice that there exist (at least) continuously may tensor product functors on the category of separable C^*-algebra with very different partial injectivity or (short-) exactness properties, see Ozawa and Pisier [**602**] if combined with considerations in [**431**].

Proof. Check and sort proof again or move it or parts of it to the appendices.

`Give here exact ref's to` $\mathcal{E}_n$ `in Appendices !!! ??`

In proof of property (sq) for $\mathcal{E}_n$ in Chp. 4
is already said something about the ''algebraic'' version of $\mathcal{E}_n$.
Give ref's to all places with $\mathcal{E}_n$ and $\mathcal{O}_n$ discussion.

The C^*-algebra $\mathcal{E}_n$ is the universal C^*-algebra generated by n isometries with orthogonal ranges, i.e.,

$$\mathcal{E}_n := C^*(s_1, \ldots, s_n\,;\; s_k^* s_\ell \,=\, \delta_{k\ell} 1)\,,$$

and the C^*-algebra $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by a (countable) sequence of isometries with mutually orthogonal ranges:

$$\mathcal{O}_\infty := C^*(s_1, s_2, \ldots\,;\; s_k^* s_\ell = \delta_{k\ell} 1\,,\; k, \ell = 1, 2, \ldots)\,.$$

It implies immediately that the C^*-algebra $\mathcal{O}_\infty$ is isomorphic to the inductive limit of the sequence of the natural unital C^*-morphisms $h_n \colon \mathcal{E}_n \to \mathcal{E}_{n+1}$ defined by $h_n(s_k) := s_k$ for $k = 1, \ldots, n$.

We are going to show that the C^*-morphisms h_n are injective and satisfy the condition (*) of Proposition 2.2.5(iii). It implies that $\mathcal{O}_\infty$ is simple and purely infinite.

Let $T_q(w) := s_{k_1} s_{k_2} \cdot \ldots \cdot s_{k_q}$ for the "word" $w = k_1 k_2 \ldots k_q$ of length $q \in \mathbb{N}$ with $k_1, k_2, \ldots, k_q \in \{1, 2, \ldots, n\}$.

Denote by A_n the (algebraic) linear span of elements 1, $T_q(w)$, $T_q(w)^*$ and $T_p(v) T_q(w)^*$ with $p, q \in \mathbb{N}$, v a word of length $= p$ and w a word of length $= q$. Calculation shows that A_n is a *-subalgebra of $\mathcal{E}_n$. Since A_n contains the canonical generators of $\mathcal{E}_n$ the vector space A_n is a *dense* *-subalgebra of $\mathcal{E}_n$.

Let $p_n := 1 - (s_1 s_1^* + \ldots + s_n s_n^*)$. Then p_n is a projection in $\mathcal{E}_n$ that satisfies $p_n T_q(w) = 0$ for "words" w in the "alphabet" $\{1, \ldots, n\}$ if w is not the "empty" word (i.e., if the "lengths" q of w is not zero).

It implies that $p_n A_n p_n = \mathbb{C} \cdot p_n$ and – therefore – that $p_n \cdot \mathcal{E}_n \cdot p_n = \mathbb{C} \cdot p_n$, i.e., p_n is a minimal projection in $\mathcal{E}_n$ that generates a closed ideal $J(p_n)$ of $\mathcal{E}_n$ that is isomorphic to an elementary simple C^*-algebra, i.e., $J(p_n) \cong \mathbb{K}(\ell_2(\mathbb{N}))$.

Clearly, $T_q(w) p_n \neq 0$ because $T_q(w)^* T_q(w) = 1$. But $p_n T_r(v)^* T_q(w) p_n = 0$ for $w \in \{1, \ldots, n\}^q$ and $v \in \{1, \ldots, n\}^r$ if $r \neq q$, or if $r = q$ and $v \neq w$.

This shows that $J(p_n) \cong \mathbb{K}(\ell_2(\mathbb{N}))$, i.e., the closed ideal $J(p_n)$ of the C^*-algebra $\mathcal{E}_n$ generated by $p_n := 1 - (s_1 s_1^* + \ldots + s_n s_n^*)$ is isomorphic to $\mathbb{K}(\ell_2(\mathbb{N}))$, and p_n is a minimal projection of $J(p_n) \cong \mathbb{K}$.

The results of J. Cuntz in [**169, 172**] show that the ideal $J(p_n)$ is an *essential* ideal of $\mathcal{E}_n$ and that $\mathcal{O}_n \cong \mathcal{E}_n / J(p_n)$ is simple.

We explore her a different method to show that $J(p_n)$ is an essential ideal of $\mathcal{E}_n$ by using the "gauge action" action of S^1 on $\mathcal{E}_n$ defined by $\gamma(z)(s_n) := z s_n$ on $\mathcal{E}_n$ (calculated modulo $J(p_n)$ in $\mathcal{O}_n$) for all $z \in S^1 \subseteq \mathbb{C}$

But it shows not that $\mathcal{O}_n := \mathcal{E}_n / J(p_n)$ is simple, instead only the – here sufficient – result that $\mathcal{O}_n$ contains no nontrivial closed ideal that is invariant under the "gauge action" of S^1 defined by $\gamma(z)(s_n) := zs_n$ on $\mathcal{E}_n$ (calculated modulo $J(p_n)$ in $\mathcal{O}_\mathrm{n}$) for all $z \in S^1 \subseteq \mathbb{C}$ and, of course, ([24]).

([25]).

It satisfies $\gamma(z)(e_n) = e_n$ for all $z \in S^1$. Therefore, the closed ideal $J(e_n)$ of $\mathcal{E}_n$ generated by p_n is invariant under the action of the gauge group. To distinguish between the gauge action on $\mathcal{E}_n$ and $\mathcal{O}_\mathrm{n}$ we denote by $[\gamma](z)$ the action on $\mathcal{O}_\mathrm{n}$.

The projection $p_n \in \mathcal{E}_n$ is fixed by the natural/canonical circle action given by $\gamma(z)(s_k) := z \cdot s_k$ on $\mathcal{E}_n$ for $z \in \mathbb{C}$ with $|z| = 1$. This action is called "gauge action" and is well-defined by the universality of the defining relations for $\mathcal{E}_n$. Clearly, the ideal $J(p_n)$ generated by p_n is invariant under the circle action $\gamma(z)$,because $\gamma(z)(p_n) = p_n$ for $z \in S^1 \subseteq \mathbb{C}$. It follows immediately that also the set K of all elements $b \in \mathcal{E}_n$ with $b \cdot J(p_n) = \{0\}$ is a closed ideal of $\mathcal{E}_n$ that is invariant under the gauge action, i.e., that $\gamma(S^1)(K) \subseteq K$. This annihilator K of the ideal $J(p_n)$ is a closed ideal K of $\mathcal{E}_n$ and must be $\gamma(S^1)$-invariant – by $\gamma(S^1)$-invariance of $J(p_n)$. Therefore $\pi_{J(p_n)}(K)$ must be a closed ideal of $\mathcal{O}_\mathrm{n}$ that is invariant under the gauge action $[\gamma]$ (modulo $J(p_n)$) on $\mathcal{O}_\mathrm{n}$. Moreover, $\pi_{J(p_n)}|K$ is faithful on K and is equivariant with respect to the gauge actions γ on $\mathcal{E}_n$ and $[\gamma]$ on $\mathcal{O}_\mathrm{n}$.

The fix-point algebra of $[\gamma]$ on $\mathcal{O}_\mathrm{n}$ is isomorphic to the UHF-algebra M_{n^∞} as a study of the algebraic dense subalgebra $\pi_{J(p_n)}(A_n)$ of $\mathcal{O}_\mathrm{n}$ shows. In particular it contains no non-trivial ideals. It allows to apply the following general elementary observation: Each continuous action of a compact group G on a C^*-algebra B has a fix-point algebra that contains an approximate unit of B, this happens also for every G-invariant closed ideal I of B, and the G-fix-point algebra of I is a closed ideal of the fix-point subalgebra of G in B.

If we apply this to $I := \pi_{J(p_n)}(K)$ and the circle action $[\gamma]\colon S^1 \to \mathrm{Aut}(\mathcal{O}_\mathrm{n})$ then we get that the intersection of I with the fixed point-algebra of $[\gamma]$ is an ideal of the fix-point algebra of $[\gamma]$ that contains a strictly positive element of $\pi_{J(p_n)}(K)$. Straight forward calculation shows that the fix-point algebra of the gauge action on $\mathcal{O}_\mathrm{n}$ is isomorphic to the simple UHF algebra M_{n^∞}. It proves that the ideal $\pi J(p_n)(K) \cong K$ of $\mathcal{O}_\mathrm{n}$ can only be zero or all of $\mathcal{O}_\mathrm{n}$, i.e., $K = \{0\}$ or $1 \in \pi J(p_n)(K)$.

Suppose that $1 \in \pi_{J(p_n)}(K)$. The orthogonality $K \cdot J(p_n) = 0$ implies that the quotient map $\pi_{J(p_n)}$ is faithful on K. Hence, K contains a projection $Q \in K$ with $\pi_{J(p_n)}(Q) = 1$, and it follows that $Q \cdot J(p_n) \subseteq K \cdot J(p_n) = \{0\}$ and that

[24] This arguments are elementary but do not prove the simplicity of $\mathcal{O}_\mathrm{n}$.

[25] It proves the simplicity of $\mathcal{O}_\mathrm{n}$ not completely. A complete proof of the simplicity and nuclearity of $\mathcal{O}_\mathrm{n}$ can be obtained by showing that the canonical conditional expectation onto the fix-point algebra $\cong M_{n^\infty}$ of the gauge action is an approximately inner c.p. map on $\mathcal{O}_\mathrm{n}$. Equivalently, by nuclearity and simplicity of M_{n^∞}, that the natural conditional expectation P_n is "faithful" as c.p. map and must preserve all closed ideals, i.e., for each positive $a \in \mathcal{O}_\mathrm{n}$ the element $P_n(a) \in M_{n^\infty} \subset \mathcal{O}_\mathrm{n}$ is contained in the ideal generated by by a. This can be seen by approximating a by elementary elements b and then show that $P_n(b)$ can be approximated up to $\varepsilon > 0$ by X^*bX for a suitable elementary element X. It shows that also that all positive elements of $\mathcal{O}_\mathrm{n}$ are purely infinite.

$\pi_{J(p_n)}(1-Q)=0$, i.e., $1-Q\in J(p_n)$. Since Q is a projection with $Qx=0=xQ$ for all $x\in J(p_n)$. It follow that $P:=1-Q$ is a unit element for $J(p_n)$. But we have seen above that $J(p_n)\cong \mathbb{K}(\ell_2(\mathbb{N}))$ is not unital. This shows that only the case $K=\{0\}$ remains, and this implies that $J(p_n)$ is an essential ideal of $\mathcal{E}_n$ (as discussed above).

It gives us the here needed injectivity of the natural map from $\mathcal{E}_n$ into $\mathcal{E}_{n+1}$????

The $[\gamma]$-simplicity and nuclearity of $\mathcal{O}_{\rm n}$ follows from the fact that M_{n^∞} is the fixed point algebra of the circle action (and that S^1 is a compact group), and that the by this action define conditional expectation P_n from $\mathcal{O}_{\rm n}$ onto M_{n^∞} defined by the integral over this action is an *approximately inner* u.c.p. map. There exists an explicit approximation by inner c.p. maps. It is a 1-step innerness that proves also the pure infiniteness if the elements of $\mathcal{O}_{\rm n}$. It can be seen by considering "algebraic" elements.

But we do not us it this here. See ... ... give Citation ???? to Appendix

Since, as above shown, $a\in\mathcal{E}_n\mapsto L(a)\in\mathcal{M}(J(p_n))$ with $L(a)b=ab$ for all $b\in J(p_n)$ is a faithful unital C^*-morphism, we obtain for $a\in(\mathcal{E}_n)_+$ with $\|a\|=1$ that there exists a partial isometry $z\in J(p_n)\cong\mathbb{K}$ with $z^*z=p_n$ and $\|p_n-z^*az\|<\varepsilon$.

We define $c:=h_n(z)s_{n+1}h_n(b^{1/2})\in\mathcal{E}_{n+1}$. Notice that $h_n(p_n)=p_{n+1}+s_{n+1}s_{n+1}^*\in\mathcal{E}_{n+1}$ and $p_{n+1}s_{n+1}=0$. This shows that

$$\|c^*h_n(a)c-h_n(b)\|<\varepsilon\,.$$

It means that the canonical morphisms $h_n\colon\mathcal{E}_n\to\mathcal{E}_{n+1}$ satisfies the criterium (*) of Proposition 2.2.5(iii). Thus $\mathcal{O}_\infty=\operatorname{indlim}_{n\to\infty}(h_n\colon\mathcal{E}_n\to\mathcal{E}_{n+1})$ is simple and purely infinite.

Part (iv) of Proposition 2.2.5 implies that $B\otimes C\otimes C\otimes\cdots$ is simple and purely infinite if B is simple and C is a simple C^*-algebra with properly infinite unit element 1_C, because then $1_C\in\mathcal{O}_\infty\subseteq\mathcal{E}_2\subseteq C$ (with *-monomorphism defined by $\mathcal{O}_\infty\ni s_n\mapsto t_2^{n-1}t_1\in\mathcal{E}_2$ for $\mathcal{O}_\infty=C^*(s_1,s_2,\ldots)$ and $\mathcal{E}_2=C^*(t_1,t_2)\subseteq C$.

For each $a\in B\otimes C\otimes C\otimes\cdots$ and $\varepsilon>$ there exists $a'\in B\otimes C\otimes\cdots\otimes C\otimes 1_C\otimes 1_C\otimes\cdots$ with $\|a'-a\|<\varepsilon/2$ and a' commutes with a copy of $\mathcal{O}_\infty\cong 1_B\otimes 1_C\otimes\cdots\otimes 1_c\otimes\mathcal{O}_\infty\otimes 1_C\otimes\cdots$.

The proof shows also for every (not necessarily simple) C^*-algebra B that $B\otimes C\otimes C\otimes\cdots$ (with $\otimes:=\otimes^{\min}$) is strongly purely infinite if C is a unital properly infinite C^*-algebra and the argument works also for finitely many $a_1,\ldots,a_n\in B\otimes C\otimes C\otimes\cdots$.

Can also use dichotomy for tensor prod.s of simple $C*$-alg.s
if B and C are *simple* !!!
Then it is only needed that both are non-elementary and that one of
B or C is stably infinite. ∎

LEMMA 2.2.8. *Let* $\varepsilon \in (0, 1/4)$ *and* $a, b \in A_+$ *contractions that satisfy*

$$\|ab^{1/2} - b^{1/2}a\| < \varepsilon \qquad \text{and} \qquad \|aba\| > 1 - \varepsilon .$$

Then $\|b^{1/2}a^2b^{1/2} - aba\| < 2\varepsilon$ *and the non-zero hereditary* C^**-subalgebra* $E := \overline{abaAaba}$ *of* $\overline{aAa}$ *has the property that*

$$\|(1 - b)^{1/2}d\| \le \sqrt{3\varepsilon} \cdot \|d\| \quad \text{for all } d \in E . \tag{2.3}$$

Notice here for later applications that $(1 - b)^{1/2}$ is a (positive) contraction.

PROOF. Suppose that $e \in A_+$ is a contraction with $3/4 < 1 - \varepsilon < \|e\| \le 1$, and let $f := (e - (1 - \varepsilon))_+$. Then $0 \le f \ne 0$, $\|f\| \le \varepsilon$ and $E := \overline{fAf}$ is a non-zero hereditary C^*-subalgebra of A. We use that

$$(1 + 2\varepsilon - t)(t - 1 + \varepsilon)^{1/n} \le 3 \cdot \varepsilon \quad \text{for all } t \in [1 - \varepsilon, 1] \text{ and } n \in \mathbb{N} ,$$

to obtain that $\|(1+2\varepsilon-e)f^{1/n}\| \le 3\varepsilon$ for all $n \in N$. Since the $f^{1/n}$ commute with the positive element $(1 + 2\varepsilon - e)$ and f is a strictly positive element in E, we get that $\|d^*(1 + 2\varepsilon - e)d\| \le 3\varepsilon \cdot \|d\|^2$ for all $d \in E$. It shows that

$$\|(1 + 2\varepsilon - e)^{1/2}d\| \le \sqrt{3\varepsilon}\|d\| \quad \text{for all } d \in E . \tag{2.4}$$

We apply this observation to our special case:

Let $a, b \in A_+$ elements that satisfy the in Lemma 2.2.8 listed pre-assumptions with $0 < \varepsilon < 1/4$, and let $x := b^{1/2}a$. Then $\|x\| \le 1$ and $\|x^* - x\| = \|ab^{1/2} - b^{1/2}a\| < \varepsilon$. It gives

$$\|b^{1/2}a^2b^{1/2} - aba\| = \|xx^* - x^*x\| < 2\varepsilon .$$

Thus, $b^{1/2}a^2b^{1/2} - aba \le 2\varepsilon \cdot 1$. Together with $0 \le a^2 \le 1$ it implies

$$0 \ge 1 - b \le 1 - b^{1/2}a^2b^{1/2} \le 1 + 2\varepsilon - aba .$$

Now let $e := aba$ and $f := (e - (1 - \varepsilon))_+$ build from a and b with the listed properties. This $e, f \in A_+$ have the previously considered properties $3/4 < 1 - \varepsilon < \|e\| \le 1$, because $a, b \in A_+$ are contractions that satisfy $\|aba\| > 1 - \varepsilon$ by assumptions on a, b and ε .

The monotony of the C^*-norm on positive parts of C^*-algebras implies that for all elements $d \in E := \overline{fAf}$ holds

$$\|d^*(1 - b)d\| \le \|d^*(1 + 2\varepsilon - aba)d\| .$$

The above shown Inequality 2.4 says that $\|d^*(1 + 2\varepsilon - e)d\| \le 3\varepsilon \cdot \|d\|^2$ for all $d \in E := \overline{fAf}$, where we let here $e := aba$ and $f := (aba - (1 - \varepsilon))_+$.

Using again monotony of the C^*-norm on A_+ we get that

$$\|d^*(1 - b)d\| \le \|d^*(1 + 2\varepsilon - aba)d\| \le 3\varepsilon \cdot \|d\|^2 ,$$

for all $d \in E := \overline{fAf}$. This is equal to the Inequality (2.4). □

LEMMA 2.2.9. *Let A a non-zero σ-unital C^*-algebra, and $S \in \mathcal{M}(A)_+$ a positive contraction with norm $\|\pi_A(S)\| = 1$ for the image $\pi_A(S)$ of S in the corona $\mathrm{Q}(A) := \mathcal{M}(A)/A$ of A.*

(i) *There exists a positive contraction $T \in \mathcal{M}(A)_+$ with $\|\pi_A(T)\| = 1$ and $(1 - S)T \in A$.*
(ii) *If A is* antiliminary *in the sense of* [**616**, sec. 6.6.1]*, cf. our Definition 2.7.2, i.e., if a^*Aa is non-commutative for each non-zero $a \in A$, then there exists a contraction $T \in \mathcal{M}(A)$ with $T^2 = 0$, $\|\pi_A(T)\| = 1$ and $(1 - S)T$, $T(1 - S) \in A$.*
(iii) *If A is simple and non-elementary then for each $a \in A_+$ with $\|a\| = 2$ there exists a sequence of contractions $d_1, d_2, \ldots \in A$ such that $(a-(a-1)_+)d_n = d_n$, $\|d_n\| = 1$, $d_m^* d_n = 0$ and $d_n(d_m)^* = 0$ for $m, n \in \mathbb{N}$, and such that the sum $\sum_n d_n^* d_n$ is strictly convergent in $\mathcal{M}(A)$ to a contraction $T := \sum_n d_n^* d_n \in \mathcal{M}(A)_+$ that satisfies $\|\pi_A(T)\| = 1$ and $(1 - S)T \in A$.* ([26]).

PROOF. The corona algebra $\mathrm{Q}(A) := \mathcal{M}(A)/A$ is non-zero (and therefore is unital) by existence of a contraction $S \in \mathcal{M}(A)_+$ with $\|\pi_A(S)\| = 1$.

(i) and preparations for (ii): Let $e \in A_+$ with $\|e\| = 1$ strictly positive in A, i.e., $\overline{eAe} = A$. For every separable C^*-subalgebra $B \subseteq \mathcal{M}(A)$, e.g. for $B := C^*(e, S)$ with the considered contraction $S \in \mathcal{M}(A)_+$, we find a linear filtration $X_n \subseteq B$ with $S, e \in X_1$, $\dim(X_n) < \infty$ and $X_n^* = X_n \subseteq X_{n+1}$ – as considered in Remark 5.1.1(3) of Chapter 5 – and an approximate quasi-central unit $(e_n)_{n\in\mathbb{N}}$ of form $e_n := f_n(e)$ with continuous functions $f_n \in \mathrm{C}_0(0, 1]$ with $f_0(t) := t$, $f_n f_{n+1} = f_n$ for $n \geq 1$, $\|f_0 - f_n f_0\| \to 0$, and, for $b \in X_n$,

$$\| [f_n(e), b] \| \; + \; \|[(f_m(e) - f_n(e))^{1/2}, b] \| \; < \; 4^{-n}\|b\| \, .$$

Then, for every sub-selection $g_k := f_{n_k}(e)$, $n_{k+1} > n_k \geq k > 0$ (and $g_0 := 0$), we can build the completely positive unital map $V \colon \ell_\infty(\mathcal{M}(A)) \to \mathcal{M}(A)$ with help of the strictly convergence of sums

$$V(a_1, a_2, \ldots) := \sum_{k=1}^{\infty} (g_k - g_{k-1})^{1/2} a_k (g_k - g_{k-1})^{1/2}$$

for bounded sequences $a_1, a_2, \ldots$ in $\mathcal{M}(A)$. Notice that $b - V(b, b, \ldots) \in A$ for all $b \in B$ and that $V(ba_1, ba_2, \ldots) - bV(a_1, a_2, \ldots) \in A$ for each sequence $(a_1, a_2, \ldots) \in \ell_\infty(\mathcal{M}(A))$ and every $b \in B$. This is the case because V is a unital completely positive map from $\ell_\infty(\mathcal{M}(A))$ into $\mathcal{M}(A)$ with $V(b, b, \ldots) = b$ for all $b \in B \subseteq \mathcal{M}(A)$, i.e., the images $(b, b, \ldots)$ by the diagonal embedding of $b \in B$ in $\ell_\infty(\mathcal{M}(A))$ are contained in the multiplicative domain of V.

(See Remarks 5.1.1(2,...,5) for more details on this standard constructions.)

If $c_1, c_2, \ldots \in A$ is a bounded sequence that satisfies the condition $c_k = c_k(g_k - g_{k-1}) = (g_k - g_{k-1})c_k$ (e.g. if $c_k g_k = c_k$ and $c_k g_{k-1} = 0$) for $k \in \mathbb{N}$ then

[26] But the sum $d_1 + d_2 + \cdots$ is in general not strictly convergent in $\mathcal{M}(A)$.

$\|\pi_A(V(c_1, c_2, \ldots))\| = \limsup_n \|c_n\|$, $c_k c_{k+1} = 0 = c_k^* c_{k+1}$ and the series $\sum_k c_k$ is strictly convergent in $\mathcal{M}(A)$ and $\sum_k c_k = V(c_1, c_2, \ldots)$.

If moreover $\lim_k \|c_k S - c_k\| = 0$, then $V(c_1, c_2, \ldots)S - V(c_1, c_2, \ldots) \in A$, because of $V(a_1, a_2, \ldots) \in A$ if $\lim \|a_n\| = 0$.

It follows $T - TS \in A$ and $\|\pi_A(T)\| = 1$ for the contraction $T := V(c_1, c_2, \ldots)$ with $c_1, c_2, \ldots \in A$ contractions that satisfy $\limsup_n \|c_n\| = 1$ and $\lim_n \|c_n S - c_n\| = 0$ and $c_n(g_n - g_{n-1}) = c_n = (g_n - g_{n-1})c_n$. Similarly also $T - ST \in A$ if here in addition the contractions c_n satisfy $\lim \|c_n - Sc_n\| = 0$.

If, moreover, those c_n can be fund such that $c_n^2 = 0$ in addition, then $T^2 = 0$, i.e., in the hereditary C^*-subalgebra $\mathrm{Ann}(1 - \pi_A(S))$ of elements $X \in \mathrm{Q}(A)$ with $X(1 - \pi_A(S)) = 0$ and $X^*(1 - \pi_A(S)) = 0$ is not Abelian.

Notice that $\mathrm{Ann}(1 - \pi_A(S))$ is automatically contained in $\pi_A(S)\,\mathrm{Q}(A)\pi_A(S) \subseteq D$ if $S \in \mathcal{M}(A)_+$ is a positive contraction with $\pi_A(S) \in D$.

The above outlined method requires a careful inductive choice of the $g_k := e_{n_k} := f_{n_k}(e)$ for suitable $n_1, \ldots, n_k, \ldots$ and then of non-zero hereditary C^*-subalgebras E_k with the property that for all $d \in E_k$ holds $d(g_{k+1} - g_k) = d$ and $\|dS - d\| < 1/2^k$.

(Those n_k and E_k can be selected suitably with help of Lemma 2.2.8.)

Then all elements $X = \pi_A(V(d_1, d_2, \ldots))$ are in $\mathrm{Ann}(1 - \pi_A(S))$ if $d_k \in E_k$ for $k = 1, 2, \ldots$ and $\limsup_n \|d_k\| = \|X\|$. If $d_k^2 = 0$ for all $k \in \mathbb{N}$ it follows that $X^2 = 0$.

`Compare below text with above. Give precise selection rule.`

Let $S \in \mathcal{M}(A)_+$ with $\|S\| = \|\pi_A(S)\| = 1$ and $B := C^*(S, e)$. Choose $e_n := f_n(e)$ for B as above described for $n = 1, 2, \ldots$.

We select $g_k := e_{n_k} := f_{n_k}(e)$ such that there exists a non-zero hereditary C^*-subalgebra E_k of A with $\|d - Sd\| < 2^{-n}$ and $d(g_k - g_{k-1}) = d$ for all $d \in E_k$.

Then $V(d_1, d_2, \ldots)$ defines a C^*-morphism from $\prod_k E_k := \ell_\infty(E_1, E_2, \ldots)$ into $\mathcal{M}(A)$ with $V(c_0(E_1, E_2, \ldots)) \subseteq A$, and $(1-S)V(d_1, d_2, \ldots), V(d_1, d_2, \ldots)(1-S) \in A$ for all $(d_1, d_2, \ldots) \in \ell_\infty(E_1, E_2, \ldots)$.

We find $m_1 \in \mathbb{N}$ with $\|e_{m_1} S e_{m_1}\| \geq 1 - 2^{-2}$ and $\|[S, e_{m_1}]\| < 2^{-2}$.

by Lemma 2.2.8 ??

Let E_1 the hereditary C^*-algebra generated by $((e_{m_1} S e_{m_1}) - 1/2)_+$. Then $e_{m_1+1} d = d$ for all $d \in E_1$, and $\|d - Sd\| \leq$?????.

Take ????

???? with the above listed properties such that $\|c_k\| = 1$ in addition. Then $T := V(c_1, c_2, \ldots) \in \mathcal{M}(A)$ has the desired property $T - ST \in A$.

Define contractions $c_1, c_2, \ldots \in A_+$ and $n_k \in \mathbb{N}$ step-wise:

Recall that $\|(e_n - e_m)S\| \geq$????. Take $g_1 := e_2$, i.e., $k_1 := 2$, and find in the hereditary C^*-algebra generated by e_1Se_1 let $c_1 := e_1 = e_1e_2$ for Part (i).

Find for Part (ii) in the hereditary C^*-subalgebra E_1 generated by e_1Se_1 an element with $d_1^2 = 0$ and $\|d_1\| = 1$ if $E_1 \neq \{0\}$

and ????.

Then $\|c_1\| = 1$ and $c_1(e_3 - e_0) = c_1$.

Then $\|(1-e_4)S(1-e_4)\| = 1$ in $B \subset \mathcal{M}(A)$ and we find $n_k \in \mathbb{N}$, $n_{k+1} \geq 2+n_k$ with $\|(e_{n_{k+1}-1} - e_{n_k+1})^{1/2}S^{1/2}\| > 1-2^{-n_k}$. It follows that in the closed right ideal L_k generated by $(e_{n_{k+1}-1} - e_{1+n_k})^{1/2}S^{1/2}A$ contains a positive contraction c_k with $\|c_kS\| \geq 1 - 2^{-k_1}$.

The $c_1, c_2, \ldots$ and $g_1, g_2, \ldots$ have the desired properties.

(ii): The pairwise orthogonal non-zero hereditary C^*-subalgebras E_k considered in the proof Part (i) contain elements $c_k \in E_k$ with $c_k^2 = 0$, $\|c_k\| = 1$. Then $T = \sum_k c_k = V(c_1, c_2, \ldots)$ satisfies $T^2 = 0$, $\|T\| = \|\pi_A(T)\| = 1$ and $TS - S, ST - S \in A$.

(iii): With the hereditary C^*-subalgebras E_k considered in Parts (i,ii) we need only to find in the hereditary C^*-subalgebra $F := \overline{(a-1)_+A(a-1)_+}$ a sequence of elements $a_1, a_2, \ldots \in F_+$ with $\|a_n\| = 1$, $a_ma_n = 0$ for $m \neq n$. Then the simplicity of A allows to find $d_1, d_2, \ldots \in A$ with $d_n^*d_n \in E_n$, $d_n(d_n)^* \in \overline{a_nAa_n}$ and $\|d_n\| = 1$. This is possible for simple non-elementary A. □

Lemma 2.2.10. *Let A a σ-unital C^*-algebra and let $D \subseteq \mathrm{Q}(A)$ a non-zero* simple *hereditary C^*-subalgebra of $\mathrm{Q}(A)$. Then D has the following properties:*

(i) *D is purely infinite or is "elementary"* ([27]).

(ii) *If A is antiliminary, in the sense of Definition 2.7.2, then D is non-elementary, i.e., D can not contain a non-zero minimal projection.*

(iii) *The corona $\mathrm{Q}(A)$ of A is purely infinite if $\mathrm{Q}(A)$ is simple (and non-zero).*

If $\mathrm{Q}(A)$ is simple then, for each $S \in \mathcal{M}(A)_+$ with $\|\pi_A(S)\| = 1$, there exists a contraction $d \in \mathcal{M}(A)$ with $1 - d^*Sd \in A$.

Proof. (i): The non-zero simple C^*-subalgebra $D \subseteq \mathrm{Q}(A)$ satisfies the condition (xv) of Proposition 2.2.1 if D is non-elementary:

Let $a, b \in D_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon > 0$. There exist $R, S \in \mathcal{M}(A)_+$ with $\pi_A(R) = b$, $\pi_A(S) = a$, $\|R\| = 1$ and $\|S\| = 1$.

By Lemma 2.2.9(i) there exists $T \in \mathcal{M}(A)_+$ with $\|T\| = \|\pi_A(T)\| = 1$ and $TS - T \in A$.

Let $g := \pi_A(T)$. We get $g = aga \in D$, because $g = ga = ag$ and D is hereditary in $\mathcal{M}(A)$. Since D is simple and $g \in D_+$ has norm $\|g\| = 1$ we find for every $\varepsilon > 0$ a number $\nu \in \mathbb{N}$ such that there exists *contractions* $d_1, \ldots, d_\nu \in D$ with

[27] The simple D is "elementary", if and only if, D contains a non-zero projection $p \in D$ with $p\,\mathrm{Q}(A)p = \mathbb{C} \cdot p$. Then $D \cong \mathbb{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

$\|b - \sum d_j^* g^2 d_j\| < \varepsilon$. Define $\nu(b, a, \varepsilon)$ as this $\nu \in \mathbb{N}$. Then $\|b - \sum_{j=1}^{\nu} c_j a^k c_j\| < \varepsilon$ with contractions $c_j := g d_j$ for each $k \in \mathbb{N}$.

Thus, D is purely infinite by Proposition 2.2.1(xv).

(ii): Let $p \in D \subseteq \mathrm{Q}(A)$ a non-zero projection. There exists a positive contraction $S \in \mathcal{M}(A)_+$ with $\pi_A(S) = p$.

By Lemma 2.2.9(ii) there exists $T \in \mathcal{M}(A)$ such that $b := \pi_A(T) \in \mathrm{Q}(A)$ satisfies $\|b\| = 1$, $b^2 = 0$ and $pb = b = bp$. Thus $b \in p\,\mathrm{Q}(A)p \subseteq D$ and $b \notin \mathbb{C} \cdot p$.

(iii): Suppose that $0 \neq p$ is a projection in $\mathrm{Q}(A)$ with $p\,\mathrm{Q}(A)p = \mathbb{C} \cdot p$ and that $\mathrm{Q}(A)$ is simple. Then there exists $d_1, \ldots, d_n \in \mathrm{Q}(A)$ with $\sum_k d_k^* p d_k = 1$. It follows that $\mathrm{Q}(A) \cong M_m(\mathbb{C})$ for some $m \leq n$.

The following argument shows that σ-unital A is unital (i.e., $\mathrm{Q}(A) = \{0\}$) if every Abelian C^*-subalgebra of $\mathrm{Q}(A)$ is finite dimensional.

If $a \in A_+$ is a strictly positive contraction then $\mathcal{M}(C^*(a))/C^*(a) \subseteq \mathrm{Q}(A)$ by a natural unital *-monomorphism.

Let $X \subseteq (0, 1]$ denote the non-zero values in the spectrum of a in $C^*(a) \cong \mathrm{C}_0(X)$. If 0 is in the closure of X, then $\mathrm{C_b}(X)/\,\mathrm{C}_0(X)$ infinite dimensional (as vector space) because then $\ell_\infty(\mathbb{N})/c_0(\mathbb{N})$ is a quotient of it.

Thus, A is unital if $\mathrm{Q}(A)$ has only finite dimensional abelian C^*-subalgebras (which is equal to $\mathrm{Q}(A)$ finite-dimensional). □

Corollary 2.2.11. *Let A a non-zero C^*-algebra, and $\mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))$.*

(i) *The stable corona $\mathrm{Q}^s(A) := \mathcal{M}(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$ is simple, if and only if,*
 (a) *A is σ-unital and simple,* and
 (b) *A is either purely infinite or is "elementary" (i.e., $A \otimes \mathbb{K} \cong \mathbb{K}$).*
 If $\mathrm{Q}^s(A)$ is simple then $\mathrm{Q}^s(A)$ is purely infinite.

(ii) *The ultrapower A_ω is simple, if and only if, either A is simple and purely infinite, or $A \cong M_n$ for some $n \in \mathbb{N}$.*

(iii) *The hereditary C^*-subalgebra of A_ω generated by $A \subseteq A_\omega$ is simple, if and only if, either A is simple and purely infinite or A is isomorphic to the compact operators $\mathbb{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$.*

Remarks 2.2.12. `To be re-read again and sorted !!!`

(1) There is an interesting observation of the author that is related to Part (iii) of Corollary 2.2.11, but with a technically much more involved proof (cf. [**448**, thm. 2.12]).

`HERE check "cite" (of article and of thm.)  of new version !!!`

It says:

If A is separable, then the algebra $F(A) := (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$ is simple, if and only if, A is nuclear (sic !) and simple *and, either A is purely infinite, or $A \otimes \mathbb{K} \cong \mathbb{K}$. The isomorphism $A \otimes \mathbb{K} \cong \mathbb{K}$ is equivalent to $F(A) \cong \mathbb{C}$, because only $A = M_n$ or $A = \mathbb{K}$ are isomorphic to hereditary C^*-subalgebras of $\mathbb{K}$.*

Here $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, is a "free" ultrafilter on $\mathbb{N}$, $A_\omega := \ell_\infty(A)/J_\omega$ is the (ω-) ultrapower of A with

$$J_\omega := \{(a_1, a_2, \ldots) \in \ell_\infty(A) \,;\, \lim_{n\to\omega} \|a_n\| = 0\,\}\,,$$

and $\mathrm{Ann}(A, A_\omega)$ is the two-sided annihilator of A in A_ω, that is obviously an ideal in $A' \cap A_\omega$.

(Hint: The main reason for this result is that the elements $d := \pi_\omega(d_1, d_2, \ldots) \in A_\omega$ with the property that there exists a bounded sequence of nuclear maps $V_n \colon A \to A$ such that $\omega - \lim \|d_n^* a d_n - V_n(a)\| = 0$ for all $a \in A$, build a non-zero closed right ideal of A_ω. It is a left module for $A' \cap A_\omega$, is not contained in $\mathrm{Ann}(A, A_\omega)$ and it contains an element in $A' \cap A_\omega$ that is not in the two-sided annihilator $\mathrm{Ann}(A, A_\omega)$. Now the reader can start here/his exercise by combining this with (ii).)

No other single property of separable non-elementary C^*-algebras characterizes the pi-sun C^*-algebras by only one condition!

(2) ????

Compare next ''blue'' with further below

Part(i) of Corollary 2.2.11 was proved by M. Rørdam [**674**, thm. 3.2] for the special case of non-elementary unital A.

(2??) The Parts (i,ii) of Corollary 2.2.11 has been proved for *unital* A by M. Rørdam ([**674**, thm. 3.2], [**681**]).

Then H. Lin [**517**, thm. 3.8] studied the case of non-elementary σ-unital A and observed that A must be algebraically simple. It implies that $A \otimes \mathbb{K}$ contains a scaling element.

The latter is equivalent by [**78**, thm. 1.2] to the existence of a projection in $A \otimes \mathbb{K}$. This proves the general result by reduction to the unital case considered in [**674**, thm. 3.2] by M. Rørdam.

Notice that we use a different and more elementary idea in the proof of Part (i).

Part (i) of Corollary 2.2.11 has also been obtained by S. Zhang [**842**] under the additional pre-assumptions that A is σ-unital and has real rank zero. Then H. Lin [**517**, thm. 3.8] extended the result [**674**, thm. 3.2] of M. Rørdam by replacing the requirement that A is unital by the requirement that A σ-unital.

The new argument of

S. Zhang ... or was it Lin? ...

is that stable σ-unital simple A with simple $\mathcal{M}(A)/A$ contains a "scaling" element in the sense of B. Blackadar and J. Cuntz [**78**].

They show in [**78**] that it allows to define a non-zero projection in A, cf. proof of the implication (iii)⇒(ii) for Proposition 2.2.1 for a similar argument. This allows to apply [**674**, thm. 3.2] also for the σ-unital case without the extra assumption that A is unital.

Later H. Lin proved in [**526**] that, if A is simple, σ-unital and non-unital then simplicity of $\mathcal{M}(A)/A$ is equivalent to $\mathcal{M}(A)/A$ being purely infinite and simple.

But he contributed also examples of

separable ??? `or non-separable??` `non-stable`

simple non-unital C^*-algebras A with $\mathcal{M}(A)/A$ simple but where A is not purely infinite (but is then at least "algebraically simple").

It follows that *a non-stable version of Part (i) of Corollary 2.2.11(i) can not exist*!

Our proofs of Propositions 2.2.1 and 2.2.5 and Corollary 2.2.11 are more general and elementary in a sense, because Remark 2.2.6, Lemma 2.2.3 and the used technical Lemmata A.21.3 and 2.1.22 and the passage to the non-separable case in Proposition A.21.4 are elementary basic observations in C^*-algebra theory (that we here often apply).

Our Corollary 5.7.3 characterizes the *nuclear* simple unital p.i. C^*-algebras $A \neq \mathbb{C}$ by the property that for each separable unital C^*-subalgebra $C \subseteq \mathcal{M}(A \otimes \mathbb{K})$ and *each* unital C^*-morphism

$$h\colon C \to \mathcal{M}(A \otimes \mathbb{K}) \qquad \text{with} \quad h(C \cap (A \otimes \mathbb{K})) = \{0\}$$

there exists a unitary $U \in \mathcal{M}(A \otimes \mathbb{K})$ such that

$$c \oplus h(c) - U^* c U \in A \otimes \mathbb{K} \quad \text{for all } c \in C\,.$$

The proof uses Proposition 2.2.5(vi) and a Weyl–von-Neumann type theorem, that is related to an absorption result of G. Elliott and D. Kucerovsky, cf. [**264**] and a correction (of what?) observed by J. Gabe [**309**].

Recall that the Calkin algebra $\mathrm{C}(\ell_2) = \mathrm{Q}(\mathbb{K}) = \mathcal{L}(\ell_2)/\mathbb{K}$ is simple and purely infinite.

Proposition 2.2.5(iii) shows that *any unital simple C^*-algebra A that contains an approximately central sequence of copies of $\mathcal{O}_\infty$ is purely infinite.*

We can replace $\mathcal{O}_\infty$ in the statement of Proposition 2.2.5(iii) by $\mathcal{E}_2 := C^*(s, t\colon s^*s = t^*t = 1, s^*t = 0)$ or any C^*-algebra with properly infinite unit because

$$\mathcal{E}_2 \subseteq \mathcal{O}_\infty \subseteq \mathcal{E}_2 \subseteq \mathcal{O}_\mathrm{n}$$

(via suitable unital monomorphisms) for every $n > 1$. See Chapter 10 on related topics.

(3??) $F(\mathcal{L}(\ell_2)/\mathbb{K}(\ell_2)) \cong \mathbb{C}$. `Mention here also the result of Farah and Phillips` on $F\big(\mathcal{L}(\ell_2(\mathbb{N}))\big)$. It is simple if and only if one uses Axiom of Choice?

Proof of Corollary 2.2.11. (i): Let $\mathrm{Q}^\mathrm{s}(A) := \mathcal{M}(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$, and $a \in A_+$ with $\|a\| = 1$.

If the unital C^*-algebra $\mathrm{Q}^\mathrm{s}(A)$ is simple then there exist $c_1, \ldots, c_n \in \mathcal{M}(A \otimes \mathbb{K})$ with $c_1^*(a \otimes 1)c_1 + \ldots + c_n^*(a \otimes 1)c_n = 1 + d$ for some $d \in A \otimes \mathbb{K}$, – here $1 \in \mathcal{M}(\mathbb{K}) \cong$

$\mathcal{L}(\ell_2(\mathbb{N}))$, by using that $\mathcal{M}(A) \otimes \mathcal{M}(\mathbb{K})$ is naturally contained in $\mathcal{M}(A \otimes \mathbb{K})$ and identifying $\mathcal{M}(\mathbb{K})$ with $1_{\mathcal{M}}(A) \otimes \mathcal{M}(\mathbb{K})$ naturally. We find a projection $p_n \in \mathbb{K}$ with $\|(1 \otimes p_n)d(1 \otimes p_n) - d\| < 1/2$. Thus, there is an isometry $s \in \mathcal{M}(\mathbb{K})$ with $\|(1 \otimes s)^* d(1 \otimes s)\| < 1/2$. If we let $g_k := c_k(1 \otimes s)$, then the positive element $T := g_1^*(a \otimes 1)g_1 + \ldots + g_n^*(a \otimes 1)g_n$ satisfies $\|1 - T\| < 1/2$ in $\mathcal{M}(A \otimes \mathbb{K})$, and the elements $e_1, \ldots, e_n \in \mathcal{M}(A \otimes \mathbb{K})$ given by $e_k := g_k T^{-1/2}$ satisfy with $\sum_k e_k^*(a \otimes 1)e_k = 1$.

It implies that $f_n := \sum_k e_k^*(a \otimes p_n)e_k \in A \otimes \mathbb{K}$ is a countable approximate unit of $A \otimes \mathbb{K}$ if p_n is an approximate unit of $\mathbb{K}$ (consisting of projections). In particular, each non-zero element $a \in A_+$ is not contained in a non-trivial ideal of A. It proves that A *is σ-unital and simple* if $\mathrm{Q^s}(A)$ is simple.

In case where A is the algebra of compact operators on a Hilbert space $\mathcal{H}$, it implies that $\mathcal{H}$ must be separable. Hence $A \otimes \mathbb{K} \cong \mathbb{K}$, if A is isomorphic to the compact operators on a Hilbert space, i.e., $A \cong M_n$ or $A \cong \mathbb{K}(\ell_2(\mathbb{N}))$.

It remains to consider the case where A is not isomorphic to the compact operators on a Hilbert space.

We show that the non-elementary simple stable σ-unital C^*-algebra $B := A \otimes \mathbb{K}$ satisfies in this case the criteria in Part (vx) of Proposition 2.2.1 for the pure infiniteness of B (thus also of A) if A is non-elementary:

Let $a, b \in B_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon \in (0, 1/2)$, and define $e := (a - (1-\varepsilon))_+$ and $c := a - e$.

Since B is again non-elementary, the non-zero hereditary C^*-subalgebra $D := \overline{eBe}$ of B contains a sequence of mutually orthogonal positive contractions $f_1, f_2, \ldots \in D_+$ with $\|f_n\| = 1$. Notice that $dc = cd = (1-\varepsilon)d$ for all $d \in D$. Let $s_1, s_2, \ldots \in \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B)$ a sequence of isometries with the property that $\sum_n s_n s_n^*$ converges strictly to $1_{\mathcal{M}(\mathbb{K})} = 1_{\mathcal{M}(B)}$. Then $\sum_k s_n f_n^2 s_n^*$ converges strictly to an element $S \in \mathcal{M}(B)$ with norms $\|S\| = \|\pi_B(S)\| = 1$.

By Lemma 2.2.10(i) the simplicity of $\mathcal{M}(B)/B$ implies that $\mathcal{M}(B)/B$ is purely infinite. Thus, there exists a contraction $Z \in \mathcal{M}(B)$ with $1 - Z^*SZ \in B$ by using Parts (ii) or(vi) of Proposition 2.2.1.

It follows that there exist $m \in \mathbb{N}$ with $\|1 - s_m^* Z^* S Z s_m\| < 1/2$ and $1 - s_m^* Z^* S Z s_m \in B$. It implies there exists $G \in \mathcal{M}(B)$ with $G^*SG = 1$. Then $b = y^*Sy$ with $y := Gb^{1/2} \in B$. The strict convergence of $\sum_k s_n f_n^2 s_n^*$ to S implies that

$$b = \lim_p \sum_{n=1}^{p} y^* s_n f_n^2 s_n^* y$$

in the norm of B. It means that b is the limit of the increasing sequence $x_1^* x_1, x_2^* x_2, \ldots$ in B, where $x_p := \sum_{n=1}^{p} f_n s_n^* y \in B$, and where we use that $f_n f_m = 0$ for $m \neq n$. (The sequence $x_1, x_2, \ldots$ itself is not necessarily convergent in B.)

Now we use that $f_n \in D$ and get that $cx_p = (1-\varepsilon)x_p$, $x_p^* c x_p = (1-\varepsilon)x_p^* x_p$ and $c \leq a \leq 1$. Thus, $(1-\varepsilon)x_p^* x_p \leq x_p^* a x_p \leq x_p^* x_p$, and $\|b - x_p^* a x_p\| \leq \|b - x_p^* x_p\| + \varepsilon$. Since $b = \lim_p x_p^* x_p$, we get that $\lim_p \|x_p\| = \|b\|^{1/2}$. Thus, B is simple and is purely infinite if B is not elementary.

(ii,iii): If J is a non-zero closed ideal of A, then J_ω is a non-zero closed ideal of A_ω that has non-trivial intersection with the hereditary C^*-subalgebra $D := D(A) := \overline{A \cdot A_\omega \cdot A}$ of A_ω generated by A. Thus $J = A$ and A is simple if D is simple.

If $A = M_n$, then $A_\omega = M_n$.

If $A \cong \mathbb{K}(\mathcal{H})$ and if the Hilbert space $\mathcal{H}$ has infinite dimension, then $D(A)$ is a hereditary C^*-subalgebra of the closed ideal of A_ω that is generated by a rank-one projection in $A \cong \mathbb{K}(\mathcal{H})$ (in fact the closed ideal of A_ω generated by $A = \mathbb{K}(\mathcal{H})$ is naturally isomorphic to the compact operators $\mathbb{K}(\mathcal{H}_\omega)$ on $\mathcal{H}_\omega$). But the whole ultrapower $\mathbb{K}(\mathcal{H})_\omega$ of the compact operators $\mathbb{K}(\mathcal{H})$ on $\mathcal{H}$ (with $\mathrm{Dim}(\mathcal{H}) = \infty$) is *not* simple, because $\mathbb{K}(\mathcal{H}_\omega) \neq \mathbb{K}(\mathcal{H})_\omega$. The latter can be seen directly or by the fact that $\mathbb{K}(\mathcal{H})_\omega$ contains the CAR-algebra as a subalgebra (and, therefore, is not of type I).

If A is not isomorphic to the compact operators on a Hilbert space, and if $D(A) = A(A_\omega)A$ is simple, then A must satisfy the criteria in Proposition 2.2.5(iv): The element that is represented by the constant sequence $(b, b, \ldots) \in A$ is in the closed ideal generated by the element with representing sequence $(a^3, a^4, a^5, \ldots) \in A(A_\omega)A \subseteq A_\omega$ if $a \in A_+$, $\|a\| = 1$ and there is $c \in A_+$ with $c\,a = a$, $\|c\| \leq 1$, compare Remark 2.2.6.

Conversely, if A is simple and purely infinite, then Proposition 2.2.1(ii) implies that for positive contractions $a, b \in A_\omega$ with $\|a\| = \|b\| = 1$ there is a contraction $d \in A_\omega$ with $d^* a d = b$. This can be seen with help of representing sequences $a = \pi_\omega(a_1, a_2, \ldots)$, $b = \pi_\omega(b_1, b_2, \ldots)$ and $d = \pi_\omega(d_1, d_2, \ldots)$, because one can find the representing sequences such that $\|a_n\| = \|b_n\| = 1$ for $n = 1, 2, \ldots$. □

3. Proof of the Dichotomy Theorem E for simple C^*-algebras

The proof of the in the Introduction 1 stated Theorem E uses some elementary observations, and from our general study of p.i. algebras only that for simple C^*-algebras pure infiniteness and local pure infiniteness are the same.

Among the needed observations is that, *for every $n \in \mathbb{N}$, every non-elementary simple C^*-algebra A and every non-zero hereditary C^*-subalgebra D of A, there exists a non-zero n-homogenous element $a \in D_+$*, cf. Remark 2.1.16.

We call a simple C^*-algebra A **non-elementary** if A is simple (and non-zero) and is not isomorphic to the algebra of compact operators on some Hilbert space (of finite or infinite dimension). An element $a \in D_+ \subseteq A$ is n-**homogenous** if there exists an isomorphism $\lambda\colon C^*(a) \otimes M_n \to D$ with $\lambda(a \otimes 1_n) = a$.

Notice that then $\overline{aDa} \cong D_0 \otimes M_n$ for the closure D_0 of $\lambda(a \otimes e_{11}) D \lambda(a \otimes e_{11})$.

The following remark will be used in the proof of Theorem E. It is stated here more general than needed in the proof of Theorem E, because the more general statement will be used in Chapters 4, 5 and 7.

REMARK 2.3.1. Let $A \subseteq E$ C^*-algebras, $x \in A$ and $v \in E$ with $\|v\| \leq 1$, $xv^* = (xx^*)^{1/2}$ and $v^*x = (x^*x)^{1/2}$.

If $D_1, D_2 \subseteq A$ denote the hereditary C^*-subalgebras of A that are generated by x^*x respectively by xx^*, then $\varphi(a) := vav^*$ defines an isomorphism φ from D_1 onto D_2.

For example, if we let $E := A^{**}$ then such an element $v \in A^{**}$ is given by the partial isometry v arising in the polar decomposition $x = v(x^*x)^{1/2}$ of x in A^{**}.

DETAILS FOR REMARK 2.3.1. The mappings $v(\cdot)v^*$ and $v^*(\cdot)v$ are completely positive contractions on E, and map D_1 into D_2 respectively D_2 into D_1. Indeed:

$$\begin{aligned} vD_1v^* = v(\overline{x^*Ax})v^* &\subseteq \overline{vx^*Axv^*} = \overline{(xx^*)^{1/2}A(xx^*)^{1/2}} = D_2, \\ v^*D_2v = v^*(\overline{xAx^*})v &\subseteq \overline{v^*xAx^*v} = \overline{(x^*x)^{1/2}A(x^*x)^{1/2}} = D_1. \end{aligned}$$

The equations $xv^*vx^* = xx^*$ and $x^*vv^*x = x^*x$ imply, that $\varphi(ab) = \varphi(a)\varphi(b)$, for $a, b \in x^*Ax$, and that $xv^*v = x$, $vv^*x = x$, i.e., that φ and $v^*(.)v|\, D_2$ are inverse to each other. □

PROOF OF THEOREM E.. (i): By a result of M. Takesaki [**767**, cor. IV.4.21], $A \otimes B$ is simple if A and B are simple, because the C^*-tensor product $\otimes := \otimes^{\min}$ is the completion of the algebraic tensor product $A \odot B$ with respect to the *minimal* C^*-seminorm $N(\cdot)$ on $A \odot B$ with the property that $N(a \otimes b) \neq 0$ for all non-zero $a \in A_+$ and $b \in B_+$, [**767**, thm. IV.4.19]. It turns out that this norm is the same as the "spatial" tensor product norm on $A \odot B \subseteq \mathcal{L}(\mathcal{H}_1 \otimes_2 \mathcal{H}_2)$ given by any faithful representations $A \subseteq \mathcal{L}(\mathcal{H}_1)$ and $B \subseteq \mathcal{L}(\mathcal{H}_2)$, cf. [**767**, lem. IV.4.11].

It suffices to consider the case where A is not stably finite, because for all C^*-algebras A and B holds $A \otimes B \cong B \otimes A$.

If A is not stably finite then for every non-zero hereditary C^*-subalgebra F of A there is a positive integer $n = n(F)$ such that $M_n(F)$ contains an infinite projection $p \in M_n(F)$. It follows that $p(M_n(F))p$ contains an isomorphic copy of the Toeplitz algebra $C^*(s\,;\ s^*s = 1)$ and, therefore, $M_n(F)$ contains a copy of algebra of compact operators $\mathbb{K} \subset C^*(s\,;\ s^*s = 1)$, cf. [**172**].

Now let $D \subseteq A \otimes B$ be a non-zero hereditary C^*-subalgebra of $A \otimes B$. By Lemma 2.2.3 and Remark 2.3.1, there exist non-zero hereditary C^*-subalgebras $E \subseteq D$, $F \subseteq A$ and $G \subseteq B$ such that $E \cong (F \otimes G)$. Here F, G and E denote the hereditary C^*-subalgebras that are generated by the elements e, f and zz^* of Lemma 2.2.3, and the natural isomorphism is given by the partial isometry v in the second conjugate of $A \otimes B$ which appears in the polar decomposition of z, cf. Remark 2.3.1.

We show that $F \otimes G$ contains a non-zero *stable* C^*-subalgebra:

The algebra $F \otimes M_n \cong M_n(F)$, for the number $n := n(F)$ as above defined, contains a hereditary C^*-subalgebra D_1 that is stable and non-zero.

By assumption, the algebra B is non-elementary, i.e., B is simple and is not isomorphic to the compact operators on some Hilbert space. Hence, B is necessarily antiliminary, i.e., every non-zero hereditary C^*-subalgebra G of B has no finite-dimensional irreducible representations. This applies to our above defined G:

By Remark 2.1.16, the non-zero hereditary C^*-subalgebra G of B contains a non-zero hereditary C^*-subalgebra G_1 which is isomorphic to $M_n \otimes G_2$ for a suitable non-zero hereditary C^*-subalgebra G_2 of G_1.

We get that

$$D_1 \otimes G_2 \subseteq F \otimes M_n \otimes G_2 \cong F \otimes G_1 \subseteq F \otimes G \cong E \subseteq D\,,$$

and the C^*-algebra $D_1 \otimes G_2$ is non-zero and stable. Thus, the given non-zero hereditary C^*-subalgebra $D \supseteq E$ contains a non-zero stable C^*-subalgebra.

In conclusion, every non-zero hereditary C^*-subalgebra $D \subseteq A \otimes B$ contains a non-zero stable C^*-subalgebra. By Proposition 2.2.1(iii), this is equivalent to the pure infiniteness of the simple C^*-algebra $A \otimes B$.

(ii): The algebras A and B are simple and nuclear, because $A \otimes B$ is simple and nuclear by assumption in Part (i): E.g. A is nuclear, because $\theta \circ \mathrm{id}_{A\otimes B} \circ \eta = \mathrm{id}_A$, for the completely positive maps

$$\eta \colon A \ni a \mapsto a \otimes b \in A \otimes B \text{ and } \theta := (\mathrm{id}_A \otimes \psi) \colon\ A \otimes B\ \to A$$

for some $b \in B_+$ with $\|b\| = 1$ and a state ψ with $\psi(b) = 1$, and $\mathrm{id}_{A\otimes B}$ is nuclear. The simplicity of $A \otimes^{\min} B$ implies that A and B must be simple, because the minimal tensor product is a bi-functor with respect to A and B.

Suppose that $A \otimes B$ is *not* purely infinite. Then A and B are both stably finite by Part (i). By [**342**] (see [**441**] for the non-unital case) there are non-zero lower semi-continuous semi-finite traces on A_+ and B_+. Thus, there is a non-zero lower semi-continuous trace on $(A \otimes B)_+$, which contradicts $T^+(A \otimes B) = 0$ (for the simple stably infinite $A \otimes B$). □

4. On absence of infinitesimal sequences

The – not so deep – results of this section are mostly known to experts (since about 1970?) and have been considered as “folklore”, but we could not find always elementary and precise references for them.

Check here, and sort, definitions and blue discussions on Cuntz semi-groups! e.g. ‘‘$\precsim$’’ and ‘‘$\approx$’’ given in Def. 2.5.1.
Go into opposite direction to restrict to minimal necessary pre-info ...
Move then all further details to an overview section,
e.g. to a Section 5 in Appendix A,

or to places where they will used first time.
Check again the def.'s for $\mathrm{CS}(A)$.

The here considered "small" Cuntz semigroup $\mathrm{CS}(A)$ has not been used ever before, despite it reflects the original version of the semigroups introduced and considered in [**170, 171**] by J. Cuntz, but here we generalize it to the cases of non-simple and non-unital C^*-algebras. The different generalization of the definition of J. Cuntz require some explanations at the beginning, because we use here the – not so common – "smallest possible" variant of the original definition of J. Cuntz. It is the "small Cuntz semi-group" $\mathrm{CS}(A)$ – near to the original definition of J. Cuntz,

??? even if in that time there was not the Pedersen ideal invented ???:

We explain it with help of the **Pedersen ideal** $\mathrm{Ped}(B)$ of a C^*-algebra B: It is the smallest *dense* ideal of a C^*-algebra B. It is algebraical generated by the set of all ε-cut-downs $(b-\varepsilon)_+$ of elements $b \in B_+$ with $\varepsilon \in (0, \|b\|)$.

Check next blue again!

It considers only classes of elements in the $\mathrm{Ped}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$, i.e., the minimal dense ideal of $A \otimes \mathbb{K}$. If we identify $M_n(A)$ in the natural way with $(1 \otimes p_n)(A \otimes \mathbb{K})(1 \otimes p_n)$ then $\mathrm{Ped}(A \otimes \mathbb{K})$ is in general not contained in $\bigcup_n M_n(A)$ and $M_n(A) \neq M_n(\mathrm{Ped}(A))$,

$$\bigcup_n M_n(A) \not\supset \mathrm{Ped}(A \otimes \mathbb{K}) \subset A \otimes \mathbb{K}\,.$$

On the other hand $\bigcup_n M_n(\mathrm{Ped}(A))$ is contained in $\mathrm{Ped}(A \otimes \mathbb{K})$ and contains all representatives, but is not identical to $\mathrm{Ped}(A \otimes \mathbb{K})$...

For each element $a \in \mathrm{Ped}(A \otimes \mathbb{K})$ there exists some element $b \in \bigcup_n M_n(\mathrm{Ped}(A))$ with the property that b^*b is MvN-equivalent to a^*a, i.e., $b^*b \sim_{MvN} a^*a$.

There are natural semigroup morphisms

$$\mathrm{V}(A) \to \mathrm{CS}(A) \to \mathrm{W}(A) \to \mathrm{Cu}(A)\,. \tag{4.1}$$

They are given by the map that respects the $\approx$-relation between elements, cf. Definition 2.4.1, for the below defined or "recalled" semigroups $\mathrm{V}(A)$, $\mathrm{CS}(A)$, $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$.

Here $\mathrm{V}(A)$ denote the Murray–von-Neumann equivalence ($\sim_{MvN}$) classes of projections in $A \otimes \mathbb{K}$, cf. Definition 2.0.1, that is in general considerably stronger relation than the ($\approx$) relation.

Beginning with $\mathrm{CS}(A)$, the natural morphisms of pre-ordered semigroups in (4.1) are *injective*, but the natural semi-group morphism $\mathrm{V}(A) \to \mathrm{CS}(A)$ is in general not necessarily injective, e.g. . $\mathrm{CS}(\mathcal{O}_2) = \mathrm{CS}(\mathcal{O}_\infty) = \{[0], [1]\}$ but ??????

E.g. $\mathrm{V}(\mathcal{O}_2) = \{[0], [1]\}$ and $\mathrm{V}(\mathcal{O}_\infty)\backslash\{[0]\}$ is naturally isomorphic to $\mathbb{Z}$, $\mathrm{CS}(A) = \mathrm{W}(A) = \mathrm{Cu}(A) = \{[0], [1]\}$ for all purely infinite simple unital C^*-algebras A.

In particular their equivalence classes have nothing to do with with its K_*-theory!

More generally, for all simple purely infinite C^*-algebras A holds $\mathrm{CS}(A) = \{[0],[p]\} = \mathrm{Cu}(A)$, with any non-zero projection $p \in A \otimes \mathbb{K}$, i.e., for each $a, b \in (A \otimes \mathbb{K}) \setminus \{0\}$ there exist $c, d, e, f \in A$ with $cad = b$ and $ebf = a$.

If non-zero A is simple and stably projection-less then $\mathrm{V}(A) = \{0\}$, but $\mathrm{CS}(A) \neq \{0\}$.

But one can adjoin a unit to A and then consider the kernel of $a + z \cdot 1 \to z \cdot 1$ ...

Next ‘‘filtration’’ has to be explained -- if true

There are natural filtrations (indexed by sub-semigroups of $\mathbb{R}_+ \cup \{+\infty\}$) and certain order topologies that allow to see that the right ones are completions of the left ones (except $\mathrm{V}(A)$, where this happens only in the case where A has real rank zero).

This does not say that the *set of elements* $a \in X_b \subseteq A \otimes \mathbb{K}$ that represent the class of $b \in \bigcup_n M_n(A)$ in $A \otimes \mathbb{K}$ with respect to the below defined equivalence relation $a \approx b$ (given by $a \precsim b$ and $b \precsim a$ in $A \otimes \mathbb{K}$) is the same as the class of $c \in \bigcup_n M_n(A)$ with $c \approx a$

?????

This causes that it is in general not clear under what circumstances $c \in A \otimes \mathbb{K}$ and $c \precsim a \in M_n(A)$ implies that $c \approx b$ for some $b \in \bigcup_n M_n(A)$ ([28]).

The point is that $a \approx b$ in $A \otimes \mathbb{K}$ and $a, b \in M_n(A)_+$ (respectively $a, b \in \mathrm{Ped}(A \otimes \mathbb{K})_+$) implies that $a \approx b$ also inside $M_n(A)$ (respectively inside $\mathrm{Ped}(A \otimes \mathbb{K})$).

The "$\approx$" classes become in $A \otimes \mathbb{K}$ much bigger. For example, in case $A := \mathrm{C}_0(0,1]$, we can take the increasing pice-wise linear functions $h_n \in \mathrm{C}_0(0,1]_+$ given by $h_n(t) := \min(1, \max(2^n t - 1, 0))$ for $t \in [0,1]$, $(n = 0, 1, \ldots)$. (We use them for some constructions, see Section 22 in Appendix A.)

If we define $f_n(t) := \left(2^{-n}(h_{n+1}(t) - h_n(t))\right)^{1/2}$ and let $T := \sum_{n=1}^{\infty} f_n \otimes p_{1,n}$, then $T \in \mathrm{C}_0(0,1] \otimes \mathbb{K}$ and $TT^* = (\sum f_n^2) \otimes p_{11} = e_0 \otimes p_{11}$ for $e_0(t) := t$, but $T^*T \notin \bigcup_n M_n(A)$.

It is not clear if in general the natural *image* of $\mathrm{W}(A)$ in $\mathrm{Cu}(A)$, i.e., the image of the map that sends classes in $\mathrm{W}(A)$ into the corresponding – usually much bigger classes (of $\approx$-equivalent elements in $A \otimes \mathbb{K}$) which are elements of $\mathrm{Cu}(A)$ is a *hereditary* sub-semigroup in $\mathrm{Cu}(A)$ with respect the natural order in Abelian semi-groups. (It has to do with variants of the so-called "radius of comparison".)

I.e. it is unknown if $\mathrm{W}(A)$ "is" a "hereditary" sub-semigroup of $\mathrm{Cu}(A)$, i.e., it is not known if $c \in A \otimes \mathbb{K}$ and $c \precsim a$ for some $a \in M_m(A)$ and $m \in \mathbb{N}$ implies the existence of some $n \in \mathbb{N}$ and $b \in M_n(A)$ such that $c \approx b$. See [**85**] for an additional property on A (that is inspired by the "bounded radius of comparison" for vector bundles) which implies that $\mathrm{W}(A)$ is "hereditary" in $\mathrm{Cu}(A)$.

[28] It has to do with the classical idea of a "radius of comparison" for vector bundles on compact manifolds.

It is not difficult to see that for each element $a \in \mathrm{Ped}(A \otimes \mathbb{K})_+$ there exists $n \in \mathbb{N}$ and $b \in (\mathrm{Ped}(A) \otimes M_n)_+$ such that $a \sim_{MvN} b$. But there are elements in $a \in \mathrm{Ped}(A \otimes \mathbb{K})_+$ that are not itself contained in the algebraic tensor product $A \odot F$, where here $F \subset \mathbb{K}$ means here the *-algebra of finite rank operators on $\ell_2(\mathbb{N})$.

It shows that $\mathrm{CS}(A)$ is isomorphic to a $\precsim$-hereditary sub-semigroup both of $\mathrm{W}(A)$ and of $\mathrm{Cu}(A)$, that $\mathrm{W}(A)$ is in a natural manner a sub-semigroup of $\mathrm{Cu}(A)$ and that $\mathrm{CS}(A)$ and $\mathrm{Cu}(A)$ are stable invariants of C^*-algebras A, i.e., there are natural isomorphisms $\mathrm{CS}(A) \cong \mathrm{CS}(A \otimes \mathbb{K})$ and $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathbb{K})$. Moreover, it is not difficult to check that $\mathrm{W}(A \otimes \mathbb{K}) = \mathrm{Cu}(A)$ in a natural way (using the flip on $\mathbb{K} \otimes \mathbb{K}$).

This natural isomorphisms are induced by the embedding $T \in \mathbb{K} \mapsto T \otimes p_{11} \in \mathbb{K} \otimes \mathbb{K}$ that extends to a *-monomorphism of $A \otimes \mathbb{K}$ onto the full stable corner $A \otimes \mathbb{K}) \otimes p_{11}$ of $A \otimes \mathbb{K} \otimes \mathbb{K}$. The *-endomorphism is unitarily homotopic to an isomorphism from $A \otimes \mathbb{K}$ onto $A \otimes \mathbb{K} \otimes \mathbb{K}$, because the *-monomorphism $T \in \mathbb{K} \mapsto T \otimes p_{11} \in \mathbb{K} \otimes \mathbb{K}$ is unitarily homotopic in sense of Definition 5.0.1 to an isomorphism from $\mathbb{K}$ onto $\mathbb{K} \otimes \mathbb{K}$ by a norm-continuous path $t \in [0, \infty) \to U(t)$ of unitary elements in $\mathcal{M}(\mathbb{K} \otimes \mathbb{K})$. (See the comments following Definition 5.0.1 for more details.)

?? Verification discussion?:

Take an isometry S from $\ell_2(\mathbb{N} \times \mathbb{N})$ onto $\ell_2(\mathbb{N})$, that is defined by a suitable bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. It defines an isomorphism from $\mathbb{K}$ onto $\mathbb{K} \otimes \mathbb{K}$ by $a \mapsto S^* a S$. Let p_{11} the upper-left minimal projection in $\mathbb{K}$. There is an isometry R from ℓ_2 onto $\ell_2 \otimes e_1$ given by $R(x) = x \otimes e_1$. It defines the map $a \to RaR^* = a \otimes p_{11}$. Then $a \to SRaR^*S^* = TaT^*$ is a C^*-morphism from $\mathbb{K}$ into $\mathbb{K}$ given by an isometry $T \in \mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$. If we find a norm-continuous path $t \mapsto U(t)$ in the unitaries of $\mathcal{M}(\mathbb{K})$ with $U(t)^* TaT^* U(t) \to a$ for all $a \in \mathbb{K}$ then

$$S^* a S = \lim_{t \to \infty} (S^* U(t) S)(a \otimes p_{11})(S^* U(t) S)^* .$$

Consider projections in $\mathbb{K}$ with $q_1 \leq q_2 \leq \cdots$ and $\lim_n \|q_n a q_n - a\| = 0$ for all $a \in \mathbb{K}$ and then find $t \mapsto U(t)$ norm-continuous with $U(t)^* T q_n T^* U(t) = q_n$ for $t \geq n = 1$.

Compare next with comments below Def. 5.0.1

The latter holds because $\mathcal{M}(\mathbb{K} \otimes \mathbb{K}) \cong \mathcal{L}(\ell_2)$, $\mathcal{U}(\mathcal{L}(\ell_2)) = \mathcal{U}_0(\mathcal{L}(\ell_2))$ and all isometries $s \in \mathcal{L}(\ell_2)$ with $\mathrm{Dim}((1 - ss^*)\ell_2) = \infty$

Are (???) unitary equivalent. Perhaps only in case that $\bigcup_n (1 - s^n (s^n)^*)\ell_2$ is dense in ℓ_2???

Needs to study the unitary part of s for the unitary equivalence? Does the standard part for Wold decomposition $U \oplus (T \otimes \mathrm{id}_{\mathcal{H}_3})$ – up to unitary equivalence of $\mathcal{H}$ with $\mathcal{H}_1 \oplus (\ell_2 \otimes \mathcal{H}_3)$, where $T \in \mathcal{L}(\ell_2)$ is the Toeplitz shift (unilateral shift) and U is a unitary on $\mathcal{H}_1$ – always "absorb" the unitary part?

Or try to show that there is a unitary V from $\ell_2(\mathbb{N}) \otimes \ell_2(\mathbb{N})$ onto $\ell_2(\mathbb{N})$ such that $V \circ (1 \otimes S) = s \circ V$?

Give Ref for precise unitary equivalence!!!

But there exists also something explicit ?????

Alternatively, one can take the isometry $R := 1 \otimes s \in \mathcal{M}(A \otimes \mathbb{K} \otimes \mathbb{K})$ with $RR^* = 1 \otimes p_{11}$. It gives $RbR^* \sim_{MvN} b$ (in $A \otimes \mathbb{K} \otimes \mathbb{K}$ itself) and $RbR^* = h(b) \otimes p_{11}$ for well-defined $h(b) \in A \otimes \mathbb{K}$ for all $b \in (A \otimes \mathbb{K} \otimes \mathbb{K})$. Then check that h is an isomorphism from $A \otimes \mathbb{K} \otimes \mathbb{K}$ onto $A \otimes \mathbb{K}$ and that $h(\cdot) \otimes p_{11}$ is unitary homotopic to id on $A \otimes \mathbb{K} \otimes \mathbb{K}$.

This monomorphism defines isomorphisms from $\mathrm{CS}(A)$ onto $\mathrm{CS}(A \otimes \mathbb{K})$ and from $\mathrm{Cu}(A)$ onto $\mathrm{Cu}(A \otimes \mathbb{K})$, because each positive element of $\mathrm{Ped}(A \otimes \mathbb{K})$ is Murray–von-Neumann equivalent ($\sim_{MvN}$ equivalent) to an element of $\bigcup_n M_n(\mathrm{Ped}(A))$ by Pedersen's characterization of sets of generators of the minimal dense ideal $\mathrm{Ped}(B)$ of a C^*-algebra B, as the ε-cut-down's $(a - \varepsilon)_+$ for $a \in G$ in any – fixed given – subset $G \subseteq B_+$ that generates a dense two-sided ideal of B.

The semigroup $\mathrm{V}(A)$ consists of the Murray–von-Neumann equivalence classes of projections in $A \otimes \mathbb{K}$ (It is zero if A is stably projection less). Projections are always in the Pedersen ideal $\mathrm{Ped}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$ and $p \precsim q$ implies that there exists a partial isometry $v \in A \otimes \mathbb{K}$ with $v^*v = p$ and $vv^* \leq q$. Thus, $v^*qv = p$.

The map from $\mathrm{V}(A)$ to $\mathrm{CS}(A \otimes \mathbb{K})$ is the natural one and is not necessarily injective, because properly infinite full projections in $A \otimes \mathbb{K}$ (if exist) are all $\approx$-equivalent but need not to be $\sim_{MvN}$ equivalent in $A \otimes \mathbb{K}$:

The existence of partial isometries $v, w \in A \otimes \mathbb{K}$ with $v^*v = p$, $vv^* \leq q$, $w^*w = q$ and $ww^* \leq p$ does not imply that p and q are MvN-equivalent. The existence of such v, w is equivalent to the existence of projections $p_1 \leq q_1 \leq p$ with $q_1 \sim_{MvN} q$ and $p_1 \sim_{MvN} p$. The latter does not imply $q \sim_{MvN} p$ because p might be an infinite projection (e.g. $p_1 \neq p$).

The equation $\mathrm{CS}(A) = \mathrm{CS}(A \otimes \mathbb{K})$ can be seen as follows:

Notice that $\mathrm{Ped}(A \otimes \mathbb{K})$ is the same as the algebraic ideal of $A \otimes \mathbb{K}$ generated by $\mathrm{Ped}(A) \otimes p_{11}$. Here $\mathrm{Ped}(A)$ denotes the *Pedersen ideal* of A. If $a, b \in \mathrm{Ped}(A) \otimes M_n$ with $a \precsim b$ in $A \otimes \mathbb{K}$ are given, then $a \precsim b$ in $\mathrm{Ped}(A) \otimes M_n$, i.e., if $a = \lim_n d_n^* b c_n$ for suitable sequences $c_n, d_n \in \mathcal{M}(A \otimes \mathbb{K})$. We can replace the c_n and d_n by $e_n := (b - 1/n)_+^{1/n} c_n (a - 1/n)_+^{1/n}$ and $f_n := (b - 1/n)_+^{1/n} d_n (a - 1/n)_+^{1/n}$. Then $e_n, f_n \in \mathrm{Ped}(A \otimes M_n) = \mathrm{Ped}(A) \otimes M_n$ and $a = \lim_n f_n^* b e_n$.

The ideal $\mathrm{Ped}(A)$ is algebraically generated by the ε-cut-downs $(a - \varepsilon)_+$ of positive contractions $a \in A_+$, and the ideal $\mathrm{Ped}(A) \otimes p_{11}$ of $A \otimes p_{11}$ generates (algebraically) a minimal dense ideal ideal of $A \otimes \mathbb{K}$ – as one can see from Lemma 2.1.9. This implies that for each element b in the Pedersen ideal $\mathrm{Ped}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$ there exists $n = n(b) \in \mathbb{N}$, contractions $e_1, f_1, \ldots, e_n, f_n \in A_+$ and $d_1, d_2 \in A \otimes \mathbb{K}$ with $f_k e_k = e_k$ for $k = 1, \ldots, n$, $b^*b = d_1^* \operatorname{diag}(e_1, \ldots, e_n) d_2$. It follows that $b \approx b^*b \sim_{WvN} cc^*$ with $2cc^* \operatorname{diag}((f_1 - 1/2)_+, \ldots, (f_n - 1/2)_+) = cc^*$ for suitable $c \in A \otimes \mathbb{K}$. Thus, $cc^* \in \mathrm{Ped}(A \otimes M_n) = M_n(\mathrm{Ped}(A))$, and each $b \in \mathrm{Ped}(A \otimes \mathbb{K})$ is $\approx$-equivalent to some element in $\bigcup_n M_n(\mathrm{Ped}(A)) \subseteq \mathrm{Ped}(A \otimes \mathbb{K})$.

(But notice here that $\mathrm{Ped}(A\otimes\mathbb{K})\neq\bigcup_n \mathrm{Ped}(A)\otimes M_n$ in general, if the M_n are considered here as $P_n\mathbb{K}P_n$ with P_n of rank $= n$, and $P_n\leq P_{n+1}$ and $\bigcup_n P_n\mathbb{K}P_n$ dense in $\mathbb{K}$, e.g. this happens even for $A:=\mathbb{C}$ or $A:=\mathrm{C}_0(0,1]$.)

We use here the (in literature not common) small "local" hereditary sub-semigroup $\mathrm{CS}(A)$ of the "larger" Cuntz-semigroups $\mathrm{W}(A)\subseteq\mathrm{Cu}(A)$. for an "infinitesimal" characterization of simple p.i. C^*-algebras. Therefore we discuss the differences and different applications before we return to the study of p.i. simple C^*-algebras. The definition of J. Cuntz in [**171**] itself did not care about the possibly different definitions of the related semi-group, and that they have different properties – even for simple ASH C^*-algebras or Cuntz-Pimsner algebras.

The below displayed elements and semigroups show that ????? what ???

$\mathbb{Z}_+\cong\mathrm{V}(\mathbb{C})=\mathrm{CS}(\mathbb{C})=\mathrm{W}(\mathbb{C})=\mathrm{CS}(A)\neq\mathrm{W}(A)$ if $A=\mathbb{K}$ because then $\mathrm{W}(\mathbb{K})=\mathrm{Cu}(\mathbb{K})\cong\mathbb{Z}_+\cup\{+\infty\}$. The isomorphism $\mathrm{Cu}(\mathbb{K})\to\mathbb{Z}_+\cup\{+\infty\}$ is given by $[a]\mapsto\mathrm{Rk}(a)$ the rank of a.

Moreover $\mathrm{W}(A)\neq\mathrm{Cu}(A)$ if A is unital and stably finite.

There exists a C^*-algebra A with $\mathrm{CS}(A)\neq\mathrm{W}(A)$

`Really?` $A=c_0$ `?, Cite ???`

`Above examples correct? Below considerations?`

The "small" Cuntz algebras $\mathrm{CS}(A)$ have in particular the following extra property:

The C^*-algebra A is simple if and only if for each non-zero $a,b\in\mathrm{Ped}(A)$ there exists $m,n\in\mathbb{N}$ with $[a]\leq m[b]$ and $[b]\leq n[a]$ in $\mathrm{CS}(A)$.

This is not the case for the semigroup $\mathrm{W}(A)$.

Example of simple A and elements $a\in M_m(A)$ and $b\in M_n(A)$ with the property that $[b]\not\leq k[a]$ for all $k\in\mathbb{N}$: $A=\mathbb{K}$, $b:=\mathrm{diag}(1,1/2,1/3,\ldots)$ $a:=\mathrm{diag}(1,0,0,\ldots)$.

`Compare Definition 2.5.1 !!`

Definition 2.4.1. Let B denote a non-zero (not necessarily simple) C^*-algebra and let $b,c\in B$. We write $b\precsim c$ if there exist sequences $d_n,e_n\in B$ such that $b=\lim_n d_n\,c\,e_n$.

We say that b and c are **Cuntz equivalent** if $b\precsim c$ and $c\precsim b$ and denote this by $b\approx c$.

It is easy to see that $a\precsim b$, and $b\precsim c$ implies $a\precsim c$, and that the relation $b\approx c$ is an equivalence relation on B. It will be also denoted sometimes by $b\sim c$. The corresponding equivalence classes, say of $b\in B$ will be denoted by $[b]$. Thus $b\approx c$ or $[b]=[c]$ are equivalent to $b\precsim c$ and $c\precsim b$.

In some special (notational) situation $b\approx c$ will be also denoted by $b\sim c$ or by $[b]=[c]$. We write later also $[b]\leq[c]$ for $b\precsim c$.

`Decide here for` $\approx$ `or` $\sim$ `!!! ?? ??`

Then $\approx$ becomes an equivalence relation on B. The $\approx$-class of $b \in B$ will be denoted by $[b]$ (and sometimes more precisely by $[b]_B$).

It is easy to see that $\precsim$ is a transitive relation on $B \times B$ with $b \precsim b$ and $0 \precsim b$ for all $b \in B$. And it is not difficult to see that $b \precsim b^*b \precsim b$, and $b \precsim c$ for $0 \leq b \leq c$.

There is an in some cases different equivalence relation $[a] = [b]$ that considers the isomorphism classes of the right Hilbert B-modules $\overline{aB}$ and $\overline{bB}$.

`Find reference for above and next red!?`
`The source is a bit semi-philosophic ... (Elliott and Co.)`

I.e., if there exists a linear isometry T from the closed right ideal $R_a := \overline{aB}$ generated by a onto $R_b := \overline{bB}$ that satisfies with $(Tx)c = T(xc)$ for all $c \in B$??

Gives that $T(aa^*)^{1/m} = \lim_n bx_{m,n}$????? Not really understood!!!

(??? What happens with "open" support projections P and Q in B^{**} of aa^* and bb^*. What gets lost in the exact or amenable case?

Seems to be that T is given by a partial isometry in B^{**} with $T = QTP$ and

We can here always suppose that the sequences (d_n) and (e_n) satisfy $d_n \in (bb^* - 1/n)_+^{1/n} \cdot B \cdot (cc^* - 1/n)_+^{1/n}$ and $e_n \in (c^*c - 1/n)_+^{1/n} \cdot B \cdot (b^*b - 1/n)_+^{1/n}$. It does not change the definition of " $\precsim$ " in Definition 2.4.1.

Sometimes we write more precisely $b \precsim_B c$ (or later $[b]_B \leq [c]_B$ for its equivalence classes) if $b, c \in A \subseteq B \subseteq D$, $a \precsim b$ in B but not $b \precsim c$ in A. Clearly $[b]_D \leq [c]_D$ if $[b]_B \leq [c]_B$ and $B \subseteq D$.

The definition of $\approx$ implies that $b \precsim c$, $d \approx b$ and $c \approx e$ imply $d \precsim e$. Therefore we can define on the classes $[b]$ and $[c]$ a pre-order $[b] \leq [c]$ if $b \precsim c$. ([29])

The MvN-equivalence $b \sim_{MvN} c$ of elements $b, c \in B_+$ – defined by the existence of $d \in B$ with $d^*d = b$ and $dd^* = c$ – implies that $b \approx c$, i.e., $[b] = [c]$. It is also easy to see that $a, b \in B_+$ and $a \leq b$ imply that $[0] \leq [a] \leq [b]$.

If $\mathcal{M}(B)$ is properly infinite – as it is e.g. the case for $B := A \otimes \mathbb{K}$ –, then we can define on the $\approx$-classes $[b]$ a commutative and associative addition by $[b] + [c] := [sbs^* + tct^*]$, where $s, t \in \mathcal{M}(B)$ are isometries with orthogonal ranges: $s^*s = 1$, $t^*t = 1$ and $s^*t = 0$. It is easy to see that it is well-defined on the family of $\approx$-classes $[b]$ and that it is independent from the chosen isometries s and t (with $s^*t = 0$). See Lemma 4.2.6 and its proof for the transformation of those generalized Cuntz-sums.

It studies the invariance for unitary equivalence classes if $ss^* + tt^* = 1$ or in case that $1 - ss^* + tt^*$ is full and infinite. But that can be carried over to $\approx$-classes in case where $1 - ss^* + tt^*$ is full and infinite e.g. by passage to st, t ...

Need Remark (!) to Lemma 4.2.6, that explains that in the sufficiently degenerated case this shows also the $\approx$-invariance, $a_1 \approx a_2$, $b_1 \approx b_2$ implies that $a_1 \oplus_{s,t} b_1 \approx a_1 \oplus_{q,r} b_1$ if $s, t, q, r \in \mathcal{M}(A)$ are isometries with $s^*t = 0$ and $q^*r = 0$.

---

[29] The classes will be sometimes also denoted by $[b]$, $\langle b \rangle$ or $[b]_\approx$, simply because for the later used several equivalence classes similar notations have to be used.

We define now a *large* (or *maximal*) version of the Cuntz semigroup that contains the usual used semigroup $W(A)$ and our very *small* version

DEFINITION 2.4.2. We define here a **large Cuntz-semigroup** $\mathrm{Cu}(A)$ by taking above $B := A \otimes \mathbb{K}$, i.e.,

$$\mathrm{Cu}(A) := \{[a]_{\approx}\,;\ a \in A \otimes \mathbb{K}\}$$

consisting of the $\approx$-classes $[a]$ with $a \in A \otimes \mathbb{K}$ and with addition defined by

$$[a] + [b] := [a \oplus b]$$

the direct sum of representatives of the classes: $[a] + [b] := [a \oplus_{s,t} b]$ with help of any unital copy of $\mathcal{E}_2 = C^*(s,t)$ in $\mathcal{M}(A \otimes \mathbb{K})$.

We define $\mathrm{W}(A)$ as the sub-semigroup of $\mathrm{Cu}(A)$ given by the classes $[a] \in \mathrm{Cu}(A)$ with $a \approx b$ for some $b \in \bigcup_n M_n(A) \subseteq A \otimes \mathbb{K}$.

The usual definition of $W(A)$, e.g. in ?????? is given on the disjoint union of the sets $M_n(A)$ and use of "canonical" unital C^*-morphisms from $M_m \oplus M_n$ into M_{m+n}. This definitions of $\mathrm{W}(A)$ are equivalent on $\approx$-classes, because the cartesian sum

$$M_m(A) \oplus M_n(A) \to M_{m+n}(A)$$

defined by

$$(b_1, b_2) \mapsto b_1 \oplus b_2 \in M_{m+n}(A) \subseteq A \otimes \mathbb{K} \quad \text{for} \quad b_1 \in M_m(A)\,,\ b_2 \in M_n(A)\,,$$

– using the natural C^*-morphisms $M_n(A) \cong A \otimes M_n$ and $M_m \oplus M_n \to M_{m+n}$ –, is unitary equivalent to the map

$$(b_1, b_2) \mapsto b_1 \oplus_{s,t} b_2 \in sM_m(A)s^* + tM_n(A)t^*\,,$$

by a suitable unitaries $U_{n,m} \in \mathcal{M}(A \otimes \mathbb{K})$.

But notice here that $b_1 \oplus_{s,t} b_2$ is not necessarily in $\bigcup_n M_n(A)$ – interpreted as $\bigcup_n A \otimes M_n \subset A \otimes \mathbb{K}$ anymore. But there exists for given $m.n \in \mathbb{N}$ isomorphisms of $A \otimes \mathbb{K}$ that are approximately inner inside $\mathcal{M}(A \otimes \mathbb{K})$ and define a "natural" isomorphism from $sM_m(A)s^* + tM_n(A)t^* \subset A \otimes \mathbb{K}$ onto $(A \otimes M_m) \oplus (A \otimes M_n) \subseteq A \otimes M_{m+n} \subset A \otimes \mathbb{K}$.

The "large" semi-group $\mathrm{Cu}(A)$ is often not useful for our application, because it contains in the case of σ-unital A always an additively absorbing "infinite" idempotent $[e]$, e.g. let $f \in A_+$ a strictly positive contraction and define a strictly positive contraction $e \in A \otimes \mathbb{K}$ by $e := \sum_n n^{-1/4} f^{1/n} \otimes p_{n,n}$. Then $[b] \leq [b] + [e] = [e]$ in $\mathrm{Cu}(A)$ for all $b \in A \otimes \mathbb{K}$. In particular, the Grothendieck group $\mathrm{Gr}(\mathrm{Cu}(A))$ is zero for each σ-unital C^*-algebra A. The reasons for $[b] + [e] = [e]$ are that for all C^*-algebras B with strictly positive element $e \in B$ holds $b \precsim e$ for all $b \in B$, $e \oplus_{s,t} e$ is again strictly positive in $A \otimes \mathbb{K}$, $e \approx 0 \oplus e \precsim b \oplus e$, $b \precsim e$ and $e \oplus e \approx e$.

DEFINITION 2.4.3. We define the **local Cuntz semigroup** as the sub-semigroup $\mathrm{CS}(A)$ of $\mathrm{Cu}(A)$ given by

$$\mathrm{CS}(A) := \{[a] \in \mathrm{Cu}(A)\,;\ a \in \bigcup_n M_n(\mathrm{Ped}(A))\,\}\,.$$

Notice here that for the Pedersen ideals always holds $\bigcup_n M_n(\mathrm{Ped}(A)) \subseteq \mathrm{Ped}(A \otimes \mathbb{K})$, but also that for each positive element b in the Pedersen ideal $\mathrm{Ped}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$ there exists $n \in \mathbb{N}$ and a positive element $c \in M_n(\mathrm{Ped}(A))_+$ that is Murray–von Neumann equivalent to b. Therefore $\mathrm{CS}(A)$ and $\mathrm{Cu}(A)$ are both invariants of the Morita equivalence classes of σ-unital C^*-algebras A.

The most common sub-semigroup of the "large" Cuntz semigroup $\mathrm{Cu}(A)$ is its sub-semigroup $\mathrm{W}(A)$ consisting of the $\approx$-classes of elements in $\bigcup_n M_n(A) \subseteq A \otimes \mathbb{K}$.

It does not matter if we consider in the definition $\bigcup_n M_n(A)$ as "disjoint" union of the sets $M_n(A)$, or if we consider $M_n(A)$ in a natural way as $M_n(A) \oplus 0_p \subset M_{n+p}(A) \subset \mathcal{M}(A \otimes \mathbb{K})$ and "identify" Cuntz-equivalent elements, because (it is easy to see that) inside $A \otimes \mathbb{K}$ holds

$$SaS^* + TbT^* \approx a \oplus b \approx \mathrm{diag}(a, b)$$

for $a \in M_n(A)$, $b \in M_p(A)$ and any isometries $S, T \in \mathcal{M}(A \otimes \mathbb{K})$ with $S^*T = 0$.

Further notice that $\mathrm{W}(A) = \mathrm{CS}(A) = \mathrm{CS}(A \otimes \mathbb{K})$ if A is *unital*, but that $\mathrm{W}(A \otimes \mathbb{K}) = \mathrm{Cu}(A)$ is not isomorphic to $\mathrm{W}(A)$ if A is stably finite.

An alternative way – inspired by [**171**] – to define $\mathrm{W}(A)$ and our version $\mathrm{CS}(A)$ is the following (that is near to the original definition of J. Cuntz in [**171**]):

Take a unital copy of $\mathcal{O}_2$ given by isometries $s, t \in \mathcal{M}(\mathbb{K})$ with $ss^* + tt^* = 1$. Define a copy of $\mathcal{O}_2$ in $\mathcal{M}(A \otimes \mathbb{K})$ by $S, T \in \mathcal{M}(A) \otimes \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(A \otimes \mathbb{K})$ by $S := 1_{\mathcal{M}(A)} \otimes s$ and $T := 1_{\mathcal{M}(A)} \otimes t$ (recall that $\otimes$ denotes here the minimal C^*-algebra tensor product, that is also called *spatial* tensor product).

Let $F \subset \mathbb{K}(\ell_2)$ denote the the set of all operators of finite rank on $\ell_2(\mathbb{N})$. It is an ideal of $\mathbb{K}(\ell_2)$, because it is just the Pedersen ideal of $\mathbb{K}(\ell_2)$ because it is algebraically generated by all elements $(T - \varepsilon)_+$ for compact positive operators $T \in \mathbb{K}(\ell_2)_+$. Notice that the algebraic tensor product $F \odot F$ is a dense *-subalgebra of the C*-algebra tensor product of $\mathbb{K}(\ell_2) \otimes \mathbb{K}(\ell_2) \cong \mathbb{K}(\ell_2 \otimes_2 \ell_2)$. But is not an ideal of $\mathbb{K}(\ell_2) \otimes \mathbb{K}(\ell_2)$, because, e.g., the operator $X := \sum_n 2^{-n} P_n \otimes P_n$ (with $P_n(v) :=< v, e_n > \cdot e_n$ for $v \in \ell_2$ and $\{e_1, e_2, ..., e_n, ...\}$ an orthogonal basis of ℓ_2) is the orthogonal projection of vectors $w \in \ell_2 \otimes_2 \ell_2 \cong \ell_2$ on the vector $v_X := \sum_n 2^{-n} e_n \otimes e_n$. Thus X is a projection in $\mathbb{K}(\ell_2 \otimes_2 \ell_2)$ but is not in the algebraic tensor product $F \otimes F \subseteq \mathbb{K}(\ell_2 \otimes_2 \ell_2)$. But the algebraic ideal (!) of $\mathbb{K}(\ell_2 \otimes_2 \ell_2)$ generated by the algebra $F \otimes F$ is again the minimal dense ideal of $\mathbb{K}(\ell_2 \otimes_2 \ell_2)$.

The set F is identical with the norm-dense algebraic *ideal* of the closed ideal $\mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))$ *generated* by its subset $\bigcup_n M_n$ – where we identify here M_n naturally with $p_n \mathbb{K} p_n$ and p_n is the orthogonal projection onto the linear span of $\{e_1, \ldots, e_n\}$ for the canonical basis of $\ell_2(\mathbb{N})$.

Obviously, F is a *-ideal of $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$, that is dense in $\mathbb{K}$, and is *minimal* with this properties. Thus, F is exactly the Pedersen ideal $\mathrm{Ped}(\mathbb{K})$ of $\mathbb{K}$, i.e., is the minimal dense ideal of $\mathbb{K}$.

But the algebraic tensor product ???? ???

The algebraic tensor products $A \odot F$ and $\mathrm{Ped}(A) \odot F \subseteq A \odot F$ are dense in the algebra $A \otimes \mathbb{K}$ and are algebraic *-ideals of the *algebraic* (!) tensor product $\mathcal{M}(A) \otimes^{alg} \mathcal{M}(\mathbb{K})$ that is in general strictly contained in $\mathcal{M}(A \otimes \mathbb{K})$.

By the definition of the Pedersen ideal, the algebraic ideal of $A \otimes \mathbb{K}$ generated by $\mathrm{Ped}(A) \odot F$ contains the Pedersen ideal $\mathrm{Ped}(A \otimes \mathbb{K})$ (:= minimal dense ideal of $A \otimes \mathbb{K}$).

This follows from the general observation (that is easy to see) that $\mathrm{Ped}(A) \otimes \mathrm{Ped}(B) \subseteq \mathrm{Ped}(A \otimes B)$, because for positive contractions $a \in A_+$ and $b \in B_+$ the products $(a - \varepsilon)_+ \otimes (b - \delta)_+ \leq ((a \otimes b) - \gamma)_+$ for $a \in A_+$, $b \in B_+$, $\varepsilon > 0$, $\delta > 0$, and $0 < \gamma < (\varepsilon \cdot \delta)/2$

$A \otimes \mathbb{K}$???? and generates ????. Above we have seen that sometimes $\mathrm{Ped}(A) \odot F$ is not an (algebraic) ideal of $A \otimes \mathbb{K}$.

Thus are in general not algebraic *-ideals of the C^*-algebra $\mathcal{M}(A) \otimes \mathcal{M}(\mathbb{K})$. The "sum" $X \oplus Y := SXS^* + TYT^*$ for given isometries, e.g. $S := 1 \otimes s$ and $T := 1 \otimes t$ is well defined and preserve $\approx$ (and $\sim_{MvN}$) classes of the elements in $A \odot F$, respectively in $\mathrm{Ped}(A) \odot F$.

The classes of elements in $A \odot F$ build $\mathrm{W}(A)$ and the classes of elements in $\mathrm{Ped}(A) \otimes F$ build $\mathrm{CS}(A)$.

The definition of the Pedersen ideal of a C^*-algebra shows that $\mathrm{Ped}(A \otimes \mathbb{K})$ is identical with the algebraic ideal of $A \otimes \mathbb{K}$ generated by $\mathrm{Ped}(A) \otimes F$, in particular $F = \mathrm{Ped}(\mathbb{K})$.

????

The *set* $F \cdot F$ of products $f_1 \cdot f_2$ is in in general not identical with F, i.e., $F \odot F$ (perhaps) generates F of $A \otimes B$ even in case $A = \mathbb{K}$ $B = \mathbb{K}$.

Here is something wrong!!!

$F = \mathrm{Ped}(\mathbb{K})$ seems to be OK! This are all elements in $\mathbb{K}$ with finite ranks.

But $F \odot F$ is not the Pedersen ideal of $\mathbb{K} \otimes \mathbb{K}$ but is contained in the Pedersen ideal and generates the Pedersen ideal of $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$.

Consider it in $\ell_2 \otimes \ell_2$????

Thus, $\mathrm{CS}(A \otimes \mathbb{K}) \cong \mathrm{CS}(A)$ in a natural way. ???? Checked ???

??? Moreover $(A \otimes \mathbb{K}) \odot F$ is "locally" isomorphic to $A \otimes \mathbb{K} \otimes \mathbb{K}$????

The only stably invariant versions of the original Cuntz semigroups are $\mathrm{CS}(A) \cong \mathrm{CS}(A \otimes \mathbb{K})$ and $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathbb{K}) \cong \mathrm{W}(A \otimes \mathbb{K})$, where we take for $\mathrm{Cu}(A)$ "our" – above given – Definitions.

That $\mathrm{W}(A \otimes \mathbb{K}) \cong \mathrm{Cu}(A \otimes \mathbb{K})$ follows from $A \otimes \mathbb{K} \otimes M_n \cong A \otimes \mathbb{K} \otimes e_{11}$ by *-conjugating with an isometry in $1_{\mathcal{M}(A)} \otimes \mathcal{M}(\mathbb{K} \otimes M_n)$.

The "large" Cuntz semigroup can give some sort of informations about the countably generated Hilbert-modules over A.

But notice that there are examples of C^*-algebras A where $\mathrm{CS}(A)$, $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$ are all different, e.g. for $A := c_0(\mathbb{N}) \otimes \mathrm{C}([0,1]^\infty)$, the zero sequences in $\mathrm{C}([0,1]^\infty)$.

`last example A OK ??  Check it??`

But, since they are all contained in $\mathrm{Cu}(A)$, one can carry out all calculations inside $\mathrm{Cu}(A)$, – except some relations in their corresponding (pre-ordered) Grothendieck groups, i.e. given by additional absorbing elements.

`Check Def. Cuntz semi-group. Compare with that in Sec. ??? in App. A`

The difference between $\mathrm{Cu}(A)$ and its hereditary sub-semigroup $\mathrm{CS}(A)$ becomes visible in case of simple C^*-algebras A, because then $\mathrm{CS}(A)$ is a simple semi-group, i.e., for any two non-zero elements $[a], [b] \in \mathrm{CS}(A)$ there exist $m, n \in \mathbb{N}$ such that $[b] \leq m[a]$ (i.e., $b \precsim a \otimes 1_m$) and $[a] \leq n[b]$. But in general, for

$$a \in \mathrm{Ped}(A)_+ \cong \mathrm{Ped}(A)_+ \otimes p_{11} \subseteq \mathrm{Ped}(A \otimes \mathbb{K})$$

in the Pedersen ideal of A there does not exist a positive integer m such that $[e] \leq m[a]$, i.e., $e \precsim a \otimes 1_m$, for the above defined "infinite" element $[e] \in \mathrm{Cu}(A)$. It would imply that $3m[a] \leq m[a]$ and causes the existence of a scaling element $b \in A \otimes \mathbb{K}$. But the existence of non-zero scaling elements causes the existence of a non-zero projection $p \in A \otimes \mathbb{K}$ and $k \geq 1$ with $2k[p] \leq k[p]$.

`Exactly as ''absorbing'' is defined? Check this:`

Thus, the "large" semi-groups $\mathrm{Cu}(A)$ of σ-unital A contain always a kind of "absorbing" infinite elements by this trivial reason (and is therefore not suitable for our purpose).

If A is simple and σ-unital then $\mathrm{Cu}(A) = \mathrm{CS}(A)$ if and only if A is stably infinite, i.e., if and only if $A \otimes \mathbb{K}$ contains an infinite projection $p \neq 0$ (which is then a properly infinite projection by Lemma 2.1.6) ([30]).

`A better ''defining'' name could be:`
`''compatibly pre-ordered additive semi-group''`

DEFINITION 2.4.4. Let $\mathcal{S} = (S, +, 0, \leq)$ a **compatibly pre-ordered additive semigroup** (shortly named as **c.p.a.** semigroup) with order-minimal zero-element 0, i.e., $(S, +)$ is a commutative semigroup, and for $x, y \in S$ holds: $x + 0 = x$, $y \leq x + y$, and that $x \leq 0$ implies $x = 0$.

We say that $\mathcal{S}$ is **simple** if for every for every $x, y \in S \setminus \{0\}$ there exist $n \in \mathbb{N}$ with $y \leq nx$.

A sequence $(x_1, x_2, \ldots)$ with $x_n \in S$ is **infinitesimal** (with respect to given $y \in S \setminus \{0\}$) if $k_n \to \infty$ (by $n \to \infty$) for each sequence of natural numbers $k_1, k_2, \ldots \in \mathbb{N}$ with the property that $y \leq k_n x_n$ for all $n \in \mathbb{N}$.

[30] We have introduced here the "small" variant $\mathrm{CS}(A)$ of a Cuntz semi-group for a transparent formulation of the proof and statement of Corollary 2.4.6.

Clearly *every infinitesimal sequence* $(x_1, x_2, \ldots)$ *satisfies that the set* $X := \{x_1, x_2, \ldots\} \subseteq S$ *is an infinite subset of* S.
Indeed, consider – more general – any finite subset X of S and let $y \in S \setminus \{0\}$. Then – alternatively – there exists $x_{n_0} \in X$ such that there is no $k \in \mathbb{N}$ with $y \leq kx_{n_0}$ or there exists $k(y) \in \mathbb{N}$ with $y \leq k(y)x_n$ for all $x_n \in X$. Hence, the sequence $x_1, x_2, \ldots \in X$ is not infinitesimal (with respect to y).

Thus, *finite* pre-ordered additive semigroups $\mathcal{S} = (S, +, 0, \leq)$ with order-minimal zero-element can not contain an infinitesimal sequence.

In particular, the semi-group $S = \{0, y\}$ with relations $0 \leq y$ and $2y = y$ has no infinitesimal sequence.

The pre-ordered semi-group $(\mathbb{Z}_+, +)$ is simple and does not contain any infinitesimal sequence $(x_1, x_2, \ldots) \in \mathbb{Z}_+^\infty$, because for $0 < y \in \mathbb{Z}$ holds that the relation $y \leq m \cdot x$ implies that $m, x \in \mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$, i.e., $1 \leq x$ and $y \leq y \cdot x$. Thus, with $k(y) := y$ we can see that $(\mathbb{Z}_+, +)$ *contains no infinitesimal sequence.*

LEMMA 2.4.5. *Let* $\mathcal{S} = (S, +, 0, \leq)$ *a simple pre-ordered additive semi-group.*

Then $S \setminus \{0\}$ *contains no infinitesimal sequence, if and only if, for each* $y \in S \setminus \{0\}$ *there exists* $k(y) \in \mathbb{N}$ *such that* $y \leq k(y)x$ *for all* $x \in S \setminus \{0\}$.

PROOF. Suppose that for each $y \in S \setminus \{0\}$ there exists $k(y) \in \mathbb{N}$ – depending only on y – such that $y \leq k(y)x$ for all those $x \in S \setminus \{0\}$ that have the property that there exists $\ell(x, y) \in \mathbb{N}$ with $y \leq \ell(x, y)x$.

Let $(x_1, x_2, \ldots)$ is a sequence in S and $k_1, k_2, \ldots$ a sequence in $\mathbb{N}$. If $y \in S \setminus \{0\}$ satisfies $y \leq k_n x_n$ then $x_n \neq 0$ and $y \leq k(y)x_n$ for all $n \in \mathbb{N}$, i.e., $(x_1, x_2, \ldots)$ is not an infinitesimal sequence. Thus, S contains no infinitesimal sequences (with respect to any y).

Now suppose that S contains no infinitesimal sequence (with respect to $y \in S \setminus \{0\}$) and that S is simple. We show that this implies the existence of $k(y) \in \mathbb{N}$ with $y \leq k(y)x$ for all $x \in S \setminus \{0\}$.

Notice that if $z \in S \setminus \{0\}$, then there are $m, n \in \mathbb{N}$ with $y \leq mz$, $z \leq ny$ for suitable $m, n \in \mathbb{N}$, and that if $(x_1, x_2, \ldots)$ is a sequence in $S \setminus \{0\}$, then this sequence is infinitesimal with respect to z, if and only if, $(x_1, x_2, \ldots)$ is infinitesimal with respect to y. This is because, e.g. $z \leq nk_n x_n$ if $y \leq k_n x_n$.

The simplicity of S implies that, for each $x \in S \setminus \{0\}$, there exists $n \in \mathbb{N}$ with $y \leq nx$. Let $\ell(y, x) \in \mathbb{N}$ denote the minimal number $n \in \mathbb{N}$ with this property. We show that the subset $M_y := \{\ell(y, x)\,;\ 0 \neq x \in S\}$ is bounded:

Otherwise there exists a sequence $(x_1, x_2, \ldots)$ in $S \setminus \{0\}$ with $\ell(y, x_n) \to \infty$ for $n \to \infty$.

Let $k_1, k_2, \ldots \in \mathbb{N}$ with $y \leq k_n x_n$. Then $k_n \geq \ell(y, x_n)$ by definition of $\ell(y, x_n)$. Thus, $k_n \to \infty$. It means that $(x_1, x_2, \ldots)$ is an infinitesimal sequence in $S \setminus \{0\}$. Its existence contradicts our assumption on infinitesimal sequences.

It implies that the number $k(y) := \max M_y \in \mathbb{N}$ exists and satisfies $y \leq k(y)x$ for all $x \in S \setminus \{0\}$. □

... Where is the definition of "non-elementary" ?

COROLLARY 2.4.6. *A non-elementary simple C^*-algebras A is purely infinite, if and only if, the "small" Cuntz semigroup* $\mathrm{CS}(A)$ *of A does not contain an infinitesimal sequence, in the sense of Definition 2.4.4.*

If a non-elementary simple C^*-algebra A has real rank zero and the natural image of $\mathrm{V}(A)$ *in* $\mathrm{CS}(A)$ *does not contain an infinitesimal sequence for* $\mathrm{CS}(A)$ *then A is purely infinite.*

If A is an elementary simple C^*-algebra then $\mathrm{CS}(A) \cong (\mathbb{Z}_+, 0, +)$, *and* $(\mathbb{Z}_+, 0, +)$ *does not contain an infinitesimal sequence.*

PROOF. If A is elementary, then the pre-ordered semi-group $\mathrm{CS}(A)$ is isomorphic to the ordered additive semi-group of non-negative integers $\mathbb{Z}_+ = \{0, 1, \ldots\}$, given by semi-group isomorphism $[a] \mapsto \mathrm{Rk}(a)$ for $a \in \mathrm{Ped}(\mathbb{K})$ (where $\mathrm{Rk}(a)$ denotes the rank of $a \in \mathbb{K}$, it is on other places also denoted by: $\mathrm{rank}(a)$).

Above we have seen that the semigroup $(\mathbb{Z}_+, 0, +)$ contains no infinitesimal sequence – in sense of our Definition 2.4.4.

Recall that $[a] = [a^*] = [a^*a]$ in $\mathrm{CS}(A)$ for $a \in \mathrm{Ped}(A \otimes \mathbb{K})$, cf. [**175**], – or use our Lemma 2.5.3(xi) and Lemma A.6.1.

If A is purely infinite and simple, then Definition 2.4.1 of Cuntz equivalence classes $[a]$ and Proposition 2.2.1(ii) implies that $[a] = [a^*a] = [b^*b] = [b]$ for any non-zero $a, b \in A \otimes \mathbb{K}$, cf. Lemma 2.5.3(xi), and that A is *non-elementary* (i.e., A is not isomorphic to the compact operators on a Hilbert space). Thus, $\mathrm{CS}(A) = \{[0], [a]\}$ for any non-zero element $a \in A$, $[0]$ is the zero of $\mathrm{CS}(A)$, and $2[a] = [a] > [0]$. By the considerations on finite additive semi-groups (above here and in Lemma 2.4.5) this finite pre-ordered semi-group with zero element does not contain any infinitesimal sequence in the sense of Definition 2.4.4.

To show the opposite direction we suppose that *A is simple and non-elementary, and that* $\mathrm{CS}(A)$ *does not contain an infinitesimal sequence.*

It is easy to see that for all simple $A \neq \{0\}$ the "small" Cuntz semigroup $\mathrm{CS}(A)$ is simple (and vice versa).

By Lemma 2.4.5, the non-existence of infinitesimal sequences in simple semigroups is equivalent to the existence of – "universal" = only from y depending – numbers $\nu(y) \in \mathbb{N}$ with the property that $y \leq \nu(y)x$ for all non-zero elements x.

Thus, for every non-zero $b \in \mathrm{Ped}(A)$ there exist $\nu(b) \in \mathbb{N}$ such that $[b] \leq \nu(b) \cdot [a]$ for every non-zero $a \in \mathrm{Ped}(A)$. But this implies the following:

Let $a, b \in A_+$ with $\|a\| = 1 = \|b\|$ and $\varepsilon \in (0, 1)$. If we let $\delta := \varepsilon/3$ then $(a-\delta)_+$ and $(b-\delta)_+$ are in $\mathrm{Ped}(A)$. Thus, there exists numbers $n := n(b, \varepsilon) := \nu((b-\delta)_+) \in \mathbb{N}$, – depending only from b and $\varepsilon > 0$ – with the property $(b - \delta)_+ \otimes e_{11} \precsim$

$(a-\delta)_+ \otimes 1_n$ in $A \otimes M_n$. In particular, there exists $c_1, \ldots, c_n \in A$ with

$$\|b - (c_1^* a c_1 + \ldots + c_n^* a c_n)\| < \varepsilon\,.$$

It says that A satisfies the criteria in Part (iv) of Proposition 2.2.1. Therefore A is purely infinite.

Now we consider the more special case where A satisfies the additional pre-assumptions that A is a simple, non-elementary C^*-algebra, that has real rank zero, again with the additional non-degeneracy property that there exists $0 \neq y \in \mathrm{Ped}(A \otimes \mathbb{K})_+$

with what ????

Then there exists a non-zero projection $q \in A \otimes \mathbb{K}$ and $m, n \in \mathbb{N}$ with $[q] \leq m[y]$ and $[y] \leq n[q]$.

Suppose that there are $a_1, a_2, \ldots \in \mathrm{Ped}(A \otimes \mathbb{K})_+$ with $\|a_k\| = 1$ that represent an infinitesimal sequence $[a_1], [a_2], \ldots \in \mathrm{CS}(A)$, then nonzero projections

$$p_k \in \overline{(a_k - 1/2)_+ (A \otimes \mathbb{K})_+ (a_k - 1/2)_+}$$

build an infinitesimal sequence in $[p_1], [p_2], \ldots \in \mathrm{CS}(A)$ with respect to $[q]$.

By assumptions, we have excluded the existence of infinitesimal sequences defined by projections in the Cuntz semigroup $\mathrm{CS}(A)$. Thus, $\mathrm{CS}(A)$ does not contain any infinitesimal sequence, and the above considered cases apply here also to A and prove purely infiniteness of A. □

5. Properly infinite elements in non-simple C^*-algebras

The study of non-simple purely infinite algebras requires basics on majorization in the sense of J. Cuntz and properly infinite elements as defined by M. Rørdam. Some needed properties of the majorization relation $b \precsim c$ are considered here. Others are listed and studied in Appendix A, see e.g. the Lemma A.6.1.

Our Definition 1.2.1 becomes more transparent and applicable by [**462**, thm. 4.16]. It says that non-zero C^*-algebras A are purely infinite in the sense of Definition 1.2.1, if and only if, each element $a \in A_+$ is *properly infinite* in the sense of following Definition 2.5.1. We outline a proof of this result, because the behavior of properly infinite elements play a fundamental role in proofs of our main results.

Versions of $\mathrm{Cu}(A)$ have been defined also before!!? Refer to this Defs.!!
Compare also Definition 2.4.1 for the relation $\precsim$, ...
There exist other Defs of infinite or properly infinite elements of $C*$-algebras.
Even on the level elements are sometimes diferent.
Only in case of elements in $s$imple $C*$-algebras all the definitions are equivalent ...

DEFINITION 2.5.1. Let a, b, c elements of a C^*-algebra A.

We say that c **majorizes** b (denoted by $b \precsim c$) if there exist sequences of elements $d_n, e_n \in A$ with $b = \lim_n e_n c\, d_n$ ([31]). It is easy to check that for $b \in A_+$ and $c \in A$ holds $b \precsim c$ if and only if $(b - \varepsilon_n)_+ \precsim c$ for a zero-sequence (ε_n) in $(0, \|b\|)$.

Refer here to Appendix Section for $\precsim$ (continuity) properties.

The relation $\precsim$ is transitive and becomes "reflexive" with respect to the as follows defined equivalence relation " $\approx$ " :

We write $a \approx b$ if $a \precsim b$ *and* $b \precsim a$. The $\approx$-class of $a \in \bigcup_n M_n(A) \subset M_\infty(A)$ will be denoted also by $[a]_\approx$, or simply by $[a]$ if it can not mixed up in some places with the considerably smaller class $[a]_{MvN}$ of elements in $\bigcup_n M_n(A)$ that are Murray–von-Neumann equivalent to a. The definition shows that the $\approx$-classes of elements in $M_\infty(A)$ are (relatively) closed subsets, and that the set of $b \in M_\infty(A)$ with $b \approx a$ are closed sets (relatively to $M_\infty(A)$).

Then a partial order $[b] \leq [a]$ can be defined by $b \precsim a$.

There are natural embeddings $a \in A \mapsto a \oplus 0 \in M_2(A)$ and – more generally – $M_m(A) \oplus 0_n \subset M_{m+n}(A)$, consider them as a common $*$-subalgebra $\bigcup_n M_n(A)$ of $M_\infty(A) \supseteq A \otimes \mathbb{K}(\ell_2)$. In this way the $\approx$-classes become an pre-ordered abelian semi-group with addition $[a] + [b] := [a \oplus b]$ and the relation $[a] \leq [b]$ is compatible with this addition.

We denote the corresponding pre-ordered semi-group by $\mathrm{W}(A)$, and call it the **Cuntz semigroup** of A. In case of non-unital A the semigroup $\mathrm{W}(A)$ can be bigger than the "small" local Cuntz semigroup $\mathrm{CS}(A)$ of Definition 2.4.3 and is for all stably finite σ-unital C^*-algebras A smaller than the in Definition 2.4.2 defined (large) general Cuntz semi-group $\mathrm{Cu}(A)$, because $\mathrm{Cu}(A)$ contains always an "infinite" absorbing idempotent, cf. the remarks below Definition 2.4.3. If A is unital, then obviously $\mathrm{W}(A) = \mathrm{CS}(A)$.

The element $a \in A$ **absorbs** the element $b \in A$, if a majorizes $a \oplus b$, i.e., if $a \oplus b \precsim a \oplus 0$ in $M_2(A)$ (The matrix $a \oplus b := \mathrm{diag}(a, b) \in M_2(A)_+$ denotes the diagonal 2×2-matrix with diagonal entries a and b).

Let $I(a) \subseteq A$ denote the set of $b \in A$ that are absorbed by a, i.e.,

$$I(a) := I_A(a) := \{ b \in A\, ;\ [b] + [a] \leq [a] \} .$$

We write $I(a)$ if it should be clear which C^*-algebra A is considered in that moment. Notice that the natural semigroup morphism $I_B(a) \to I_A(a)$ for $a \in B \subset A$ is not necessarily injective. It was denoted by $J(a)$ in [**462**, def. 3.11].

Check if notation is consistent to other use of $I(a)$, $J(a)$, $I_A(a)$, ... etc.

Give an example where $I_B(a) \to I_A(a)$ for some $a \in B \subset A$ is not injective.

[31] The sequences (d_n) and (e_n) are here not necessarily required to be bounded.

Then $a \in A$ is called **finite** if $I(a) = \{0\}$, and **infinite** if $I(a) \neq \{0\}$.

We say that $a \in A$ is **properly infinite**, if $a \in I(a)$, i.e., if $a \oplus a \precsim a \oplus 0$ in $M_2(A)$, here we allow only $a \neq 0$ ([32]), i.e., $a \in A$ is properly infinite if and only if $2[a] \leq [a]$ in the pre-ordered Abelian semi-group $\mathrm{Cu}(A)$ of Cuntz equivalence classes in $A \otimes \mathbb{K}$ with $[a]$ corresponding naturally to the class of $a \otimes p_{11}$ in $\mathrm{Cu}(A)$, cf. Definition 2.4.2.

There exist several other formulations of infiniteness of elements or of Next Remark 2.5.2 is used in the proofs of Part(viii) of Lemma 2.5.3 and of Lemma 2.5.4.

Remark 2.5.2. If $d \in A_+$ and $b \in D := \overline{dAd}$ then $b \precsim d$ in D and A. But from $b \precsim d$ in A one gets only that $b \in J(d)$, where $J(d)$ denotes the closed ideal of A generated by d.

Indeed, the existence of elements $e_n, f_n \in D$ with $b = \lim_n e_n\, d\, f_n$ follows from the equation $D = \overline{D \cdot d \cdot D}$, that is implied by the equations $D = \overline{d \cdot D}$ and $D = D \cdot D$ ([33]).

Moreover, this shows that $b \precsim c$ if $c \in A$ is properly infinite in A and b is contained in the closed ideal of A generated by $c \in A$.

Obviously, the Murray–von-Neumann equivalence $a \sim_{MvN} b$ of Definition 2.0.1 implies $a \approx b$.

Recall that a positive element $a \in B_+$ is **stable** by Definition 2.1.1 if the hereditary C^*-subalgebra $\overline{aBa}$ is stable (or zero). By Part (viii) of the following Lemma 2.5.3, every non-zero *stable* element is properly infinite. But properly infinite elements, e.g. projections, that are not stable.

```
Concerning several changes of references:
Observe changes in order of items of Lemma 2.5.3:
(ix) to (vii), (x) to (viii), (xi) to (ix), (xii) to (x), (xiii) to
(xi), ...  ,
and then in (vii) to (xii) to (i), (viii) to (xiii) to (ii),
(xiv) to (iii), and (xv) to (iv), ...

Go to all places where Lemma 2.5.3 is cited, and check if ref's are
corrected !!!
Give ref.  to Def.  of properly infinite "element"s
and infinite C*-algebras.
Ref.s and Cite bring ORDER !!!  Is it 2.5.1 for elements ??
```

[32] We allow 0 to be a properly infinite element on some places to simplify notations!

[33] Here $D \cdot D$ it the *set* of products $\{ab;\ a, b \in D\}$, but is equal to the closed linear span of this products, cf. Lemma 2.1.7(o).

LEMMA 2.5.3. *Let* $I(a) := I_A(a)$ *denote the set of elements* $b \in A$ *that are "absorbed" by* $a \in A$*, i.e.,* $b \in I_A(a)$ *if and only if* $b \oplus a \precsim a$ *inside* $M_2(A)$*, i.e.,* $[b] + [a] \leq [a]$ *in* $\mathrm{W}(A)$ *.*

Have we here only to consider $\mathrm{W}(A)$ *or also the bigger* $\mathrm{Cu}(A)$*?*

(i) $I_A(a)$ *is a* closed ideal *of* A *that satisfies* $I_A(a) = I_A(c)$ *for all* $c \in A$ *with* $a \approx c$ *.*

In particular, if $a \in A_+$ *and* $b^* = b \in M_2(I_A(a))$ *then* $|b| \in M_2(I_A(a))$ *and, for all* $d, e \in A$*,* $|b| + [d,e]^* a [d,e] \precsim a$ *in* $M_2(A)$ ([34]).

(ii) *If* $D \subseteq A$ *is a hereditary* C^**-subalgebra of* A *and if* $a \in D$*, then* $I_D(a) = I_A(a) \cap D$*.*

(iii) *The element* $\pi_{I_A(a)}(a) = a + I_A(a)$ *is always* finite *in* $A/I_A(a)$ *.*

(iv) *Every* C^**-morphism* $\varphi\colon A \to B$ *maps* $I_A(a)$ *into* $I_B(\varphi(a))$*.*

In particular, if $J \triangleleft A$ *is a closed ideal and if* $\pi_J(a)$ *is finite in* A/J*, then* $I_A(a) \subseteq J$ *,* ([35])*.*

(v) *If* J *is a closed ideal of* A *and* $J \subseteq I_A(a)$*, then* $\pi_J(I_A(a)) = I_{A/J}(\pi_J(a))$*.*

In particular, $a \in A$ *is* properly infinite *in* A*, if and only if,* $\pi_J(a)$ *is infinite in* A/J *for* every *closed ideal* $J \triangleleft A$ *with* $a \notin J$*.*

(vi) *If* $a, b \in A_+$ *are orthogonal, then* $I(a) + I(b) \subseteq I(a+b)$*.*

(vii) *If* $a \in A_+$*, then* $a \approx a + b$ *for all* $b \in I(a)_+$*.*

In particular, if $a \in A_+$ *is* properly *infinite then the elements* $a + x$ *are properly infinite for each positive element* $x \in \overline{\mathrm{span}(AaA)}$ *.*

(viii) *If* $d \in A_+$ *is non-zero and the hereditary* C^**-subalgebra* $D := \overline{dAd}$ *of* A *has the property that* $\mathcal{M}(D)$ *contains isometries* $s_1, s_2 \in \mathcal{M}(D)$ *with orthogonal ranges, i.e.,* $s_j^* s_k = \delta_{j,k} 1$*, then* d *is properly infinite in* A*.*

In particular, if $d \in A_+ \setminus \{0\}$ *is* stable *in the sense of Definition 2.1.1, i.e.,* $D := \overline{dAd}$ *is stable, then* d *is properly infinite in* A*.*

(ix) *An element* $a \in A_+$ *is properly infinite, if and only if, for every* $\varepsilon > 0$*, there exists* $x = x(\varepsilon) \in M_{1,2}(A)$ *such that* $x^* a x = (\mathrm{diag}(a,a) - \varepsilon)_+ = (a - \varepsilon)_+ \otimes 1_2$ *in* $M_2(A)$ *.*

(x) *If* $0 \neq a \in A_+$ *and there exists* $\delta > 0$ *such that* $(a - \nu)_+$ *is properly infinite for every* $\nu \in (0, \delta)$*, then* a *is properly infinite in* A*.*

(xi) *If* $a \approx b$ *and* a *is properly infinite, then* b *is properly infinite.*

In particular, the elements a*,* a^*a*,* aa^* *and* $(a^*a)^{1/2}$ *are all properly infinite if one of them is properly infinite.*

PROOF. We use in the proof of Part (i) the precise notation $X \oplus 0_m \in M_{n+m}(A)$ if $X \in M_n(A)$, but in the proofs of other parts we write simply X for $X \oplus 0_m$.

(i): More generally we define $I_n := I_n(a)$ for $a \in A$ as the *set* of $x \in M_n(A)$ with $a \oplus x \precsim a \oplus 0_n$ in $M_{n+1}(A)$. In particular $I_1(a) = I_A(a)$. Below given arguments show that the I_n are closed ideals of $M_n(A)$ and $I_n = M_n(I_A(a))$ for $n \in \mathbb{N}$. Clearly

[34] Identify here a with $a \otimes p_{11}$ in $A \otimes M_2$, or with the matrix $[a_{j,k}]$ given by $a_{1,1} := a$ and $a_{j,k} = 0$ if $j \neq 1$ or $k \neq 1$.

[35] Case $I_A(a) = \{0\}$ and $I_{A/J}(\pi_J(a)) = A/J$ can appear, cf. Examples 2.5.12 and 2.5.13.

we get the same subset of $M_n(A)$ if we consider the elements $x \in M_n(A)$ with the property $x \oplus a \precsim 0_n \oplus a$ in $M_{n+1}(A)$.

In the case where $x \in M_n(A)_+$ and $a \in A_+$ this is equivalent to $(x-\gamma-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a$ for all $\varepsilon \in (0,\gamma)$ and $\gamma \in (0,\|x\|)$, because it implies $(x-\gamma)_+ \oplus a \precsim a$ for every $\gamma \in (0,\|x\|)$ and then the relation $x \oplus a \precsim a$. Both implications follow from the continuity properties of the relation $\precsim$ on $M_2(A)$, respective from the fact that $I_n(a)$ is closed in $M_n(A)$, as shown now in more detail:

Temporarily and only here, we denote by $\xi(a \oplus 0_n)$ the set of elements $z \in M_{n+1}(A)$ with $z \precsim (a \oplus 0_n)$. We write this as $z \precsim a$ via identifying A with $A \otimes e_{11} \cong A \oplus 0_n$. Likewise, we can use here instead the identification of A with $A \otimes e_{n+1,n+1} \cong 0_n \oplus A$ in $M_{n+1}(A)$.

By Lemma A.6.1(iv,viii), the set $\xi(a \oplus 0_n)$ is a *closed* subset of $M_{n+1}(A)$ that satisfies $X\xi(a \oplus 0_n)Y \subset \xi(a \oplus 0_n)$ for all $X, Y \in \mathcal{M}(M_{n+1}(A)) \cong M_{n+1}(\mathcal{M}(A))$. More precisely, $\xi(a \oplus 0_n)$ is the closure in $M_{n+1}(A)$ of the set of elements $X(a \oplus 0_n)Y$ with $X, Y \in \mathcal{M}(M_{n+1}(A)) \cong M_{n+1}(\mathcal{M}(A))$.

The sets $I_n(a)$ are closed subsets of $M_n(A)$ because

$$a \oplus I_n(a) \;=\; (a \oplus M_n(A)) \cap \xi(a \oplus 0_n)\,.$$

This equation shows also that $I_n(a)$ invariant under right and left multiplications by operators in $\mathcal{M}(M_n(A))$.

In particular, $I_A(a) = I_1(a) \subseteq A$ and the sets $I_n(a) \subseteq M_n(A)$ are *closed* in $M_n(A)$ and satisfy $cI_n(a)d \subseteq I_n(a)$ for all $c, d \in \mathcal{M}(M_n(A))$.

We derive that if $x \in I_m(a)$ and $y \in I_n(a)$, then $x \oplus y \in I_{m+n}(a)$:

Notice that in $M_{m+n+1}(A)$ for all $v \in M_m(A)$ and $z \in M_n(A)$ holds

$$(a \oplus v) \oplus z \approx a \oplus (v \oplus z) \approx a \oplus (z \oplus v) \approx (a \oplus z) \oplus v\,,$$

where the middle equivalence is given by $1_{\mathcal{M}}(A) \oplus S$ for some suitable inner automorphism S of M_{n+n}. The elements $x \in I_m(a) \subseteq M_m(A)$ and $y \in I_n(a) \subseteq M_n(A)$ satisfy $a \oplus x \precsim a \oplus 0_m$ in $M_{m+1}(A)$ and $a \oplus y \precsim a \oplus 0_n$ in $M_{n+1}(A)$.

It follows that

$$a \oplus (x \oplus y) \approx (a \oplus x) \oplus y \precsim (a \oplus 0_m) \oplus y$$

in $M_{m+n+1}(A)$. Then

$$(a \oplus 0_m) \oplus y \approx (a \oplus y) \oplus 0_m \precsim (a \oplus 0_n) \oplus 0_m \approx a \oplus 0_{m+n}\,.$$

Together it gives that

$$a \oplus (x \oplus y) \precsim a \oplus 0_{m+n}$$

in $M_{m+n+1}(A)$, i.e., $x \oplus y \in I_{m+n}(A)$. We can also use the relations

$$a \oplus x \precsim a \oplus 0_m\,,\;\; a \oplus (0_m \oplus y) \precsim a \oplus 0_{m+n}$$

and Lemma A.6.1(ii) to get the relation:

$$a \oplus (x \oplus y) \approx (a \oplus x) \oplus y \precsim (a \oplus 0_m) \oplus y \precsim a \oplus 0_{m+n}\,.$$

Since $M_{m+1}(A)$ is naturally isomorphic to the hereditary C^*-subalgebra D of $b \in M_{m+n+1}(A)$ with $b = c \oplus 0_n$ for some $c \in M_{m+1}(A)$, and since $a \oplus 0_m \oplus 0_n$ and $a \oplus x \oplus 0_n$ are both in D, we get that $x \in M_m(A)$ is in $I_m(a)$, if and only if, $x \oplus 0_n \in I_{m+n}(a)$, cf. Lemma A.6.1(viii).

If $x, y \in I_n$, then $a \oplus (x+y) \oplus 0_n \precsim a \oplus (x \oplus y) \precsim a \oplus 0_{2n}$ in $M_{2n+1}(A)$, because $(x+y) \oplus 0_n \precsim x \oplus y$ in $M_{2n}(A)$, cf. Lemma A.6.1(x).

Thus, $I_n(a)$ is also additively closed, i.e., is a closed *ideal* of $M_n(A)$. In particular there is a unique ideal J of A such that $I_n(a) = J \otimes M_n \cong M_n(J)$ and this ideal is determined by the equation $J \otimes e_{11} = (A \otimes e_{11}) \cap I_n(a)$.

Then $I(a) \oplus 0_{n-1} = I_n(a) \cap (A \oplus 0_{n-1})$ by the above gives definitions of $I_n(a) \subseteq M_n(A)$ and $I(a) \subseteq A$. Combined with the natural isomorphism $A \oplus 0_{n-1} \cong A \otimes e_{11}$ it gives that $I_n(a) = M_n(I(a))$.

Alternatively one could use here that $I(a) \oplus 0_{n-1}$ is a full hereditary C^*-subalgebra of the ideal $I_n(a) \subseteq M_n(A)$, to obtain that $I_n(a) = M_n(I(a))$.

For all elements $X \in I_n(a) = M_n(I(a))$ holds $X \oplus a \approx a \oplus X \precsim a \oplus 0_n$ inside $M_{n+1}(A)$.

If $a \approx c$ and $x \in I(a)$, then $x \in I(c)$, because

$$c \oplus x \precsim a \oplus x \precsim a \oplus 0 \precsim c \oplus 0 \,.$$

It implies then $I_n(a) = I_n(c)$ for all $n \in \mathbb{N}$.

It holds $X + Y \precsim X \oplus Y$ for all $X, Y \in M_n(A)$ by Lemma A.6.1(x), and $|Z|^2 = Z^2 \approx Z$ for all selfadjoint $Z \in I_n(a)$. It implies that $|Z| \oplus a \precsim a$ if $Z \oplus a \precsim a$. Thus, $|b| \oplus [d,e]^* a [d,e] \precsim |b| \oplus a \precsim a$ for each $a \in A_+$ and $b^* = b \in M_2(I(a))$.

(ii): Clearly, $I_D(a) \subseteq I_A(a) \cap D$ by definitions of $I_A(a)$ and $I_D(a)$ in case of $a \in D$. If $x \in I_A(a) \cap D$, then $a \oplus x \precsim a \oplus 0$ in $M_2(A)$. Since $M_2(D)$ is hereditary in $M_2(A)$ and $a \oplus x, a \oplus 0 \in M_2(D)$, it follows $x \in I_D(a)$ by Lemma A.6.1(viii).

(iii): The observation in Part (iii) is equivalent to [**462**, lem. 3.13]. We give an alternative proof:

Let $a \in A_+$ and denote by $\pi := \pi_{I(a)} \colon A \to A/I(a)$ the quotient map, where here $I(a) := I_A(a)$.

Let $x \in A_+$ with $\pi(x) \oplus \pi(a) \precsim \pi(a)$, i.e., with $\pi(x) \in I(\pi(a))_+$. Here the short notation $I(\pi(a))$ means – more precisely – the ideal $I_{A/I(a)}(\pi(a))$ of $A/I(a)$, and $\pi(x)$ denotes the element $\pi_{I(a)}(x)$ of $A/I(a)$.

We show below that this implies $\pi(x) = 0$, i.e., show that $x \in I_A(a)$. It follows then that $I(\pi(a)) = 0$, which means that the element $\pi(a)$ is finite in $A/I(a)$.

Since $I(a)$ is a closed ideal of A, it suffices to show that $(x - \gamma)_+ \in I(a)$ for each $\gamma \in (0, \|x\|)$, i.e., that $(x - \gamma)_+ \oplus a \precsim a$.

By definition of $\precsim$ in $A/I(a)$, for each $\varepsilon \in (0,\gamma)$ there exist $\delta := \delta(\varepsilon) > 0$ and $d, e \in A$ with

$$[\pi(d),\pi(e)]^*\pi((a-\delta)_+)[\pi(d),\pi(e)] = (\pi(x)-\gamma-\varepsilon)_+ \oplus (\pi(a)-\varepsilon)_+ \,.$$

We can rewrite this equivalently as "$Y \in M_2(I(a))$" for the selfadjoint element $Y \in M_2(A)$ defined by

$$Y := \big((x-\gamma-\varepsilon)_+ \oplus (a-\varepsilon)_+\big) - [d,e]^*(a-\delta)_+[d,e] \;\in M_2(I(a))\,.$$

It follows that

$$(x-\gamma-\varepsilon)_+ \oplus (a-\varepsilon)_+ = Y + [d,e]^*(a-\delta)_+[d,e] \le |Y| + [d,e]^*(a-\delta)_+[d,e] \in M_2(A)\,.$$

Since $Y \in M_2(I(a))$ we get $|Y| \in M_2(I(a))$ and by Part (i), this implies that $|Y| + [d,e]^*(a-\delta)_+[d,e] \precsim a\,$.

Thus $(x-\gamma-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a$ in $M_2(A)$. This happens for every $\varepsilon \in (0,\gamma)$, and implies $(x-\gamma)_+ \oplus a \precsim a$, i.e., $(x-\gamma)_+ \in I(a)$ for each $\gamma \in (0,\|x\|)$. Thus $x \in I(a)$, because $I_A(a)$ is closed.

(iv): By Lemma A.6.1(iii), if $a \oplus x \precsim a \oplus 0$ and

$$\psi := \mathrm{id} \otimes \varphi \colon M_2 \otimes A \,\to\, M_2 \otimes B\,,$$

then $\psi(a \oplus x) \precsim \psi(a \oplus 0)$ in $M_2(B)$. Thus, $\varphi(I_A(a)) \subseteq I_B(\varphi(a)) \subseteq B$.

(v): It is – up to suitable notation – an almost trivial consequence of parts (iii) and (iv). Let $J \subseteq I(a)$ a closed ideal. Define $K := \pi_J(I_A(a)) = I_A(a)/J \lhd A/J$, $L := I_{A/J}(a+J) \lhd A/J$, $B := A/I_A(a)$ and $b := \pi_{I_A(a)}(a) \in B$ (To inflate not iterated indices).

Notice that $I_B(b) = 0$ by Part (iii), i.e., $b := \pi_{I_A(a)}(a)$ is finite in B. By Part (iv) we get $K \subseteq L$ in A/J.

Now, under natural identification of $(A/J)/K$ with $B := A/I_A(a)$, we get $\pi_K(\pi_J(x)) = \pi_{I_A(a)}(x)$ for all $x \in A$. Thus, Part (iv) says also that

$$\pi_K \colon A/J \to (A/J)/K \cong B$$

maps $L := I_{A/J}(\pi_J(a))$ into

$$I_B\big(\pi_K(\pi_J(a))\big) \;=\; I_B\big(\pi_{I_A(a)}(a)\big) \;=\; I_B(b) = \{0\}\,.$$

It implies that also $L \subseteq K$. Now $L = K$ rewrites as $\pi_J(I_A(a)) = I_{A/J}(\pi_J(a))$.

(vi): If $a, b \in A_+$ are orthogonal, and $x \in I(a)$, $y \in I(b)$, then $(a+b) \oplus 0 \sim a \oplus b$, and $a \oplus b \oplus x \oplus y \precsim a \oplus b$, i.e., $x \oplus y \in I(a \oplus b) = I_2(a+b)$ – with notations in proof of Part(i). Then $x + y \in I(a+b)$ follows finally from $(x+y) \precsim x \oplus y$.

(vii): $a \oplus a \precsim a$ implies $I_A(a) \supseteq \overline{\mathrm{span}\, AaA}$. Thus, $(a+x) \oplus 0 \precsim a \oplus x \precsim a \oplus 0$ for positive $x \in \overline{\mathrm{span}\, AaA}$. This and $a \le a + x$ imply $a \sim a + x$.

(viii): Let $0 \ne d \in A_+$ a *stable element* in A. Then $D \cong D \otimes \mathbb{K}$ for $D := \overline{dAd}$ by Definition 2.1.1. Then $\mathcal{M}(D)$ contains isometries $s_1, s_2 \in \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(D)$ with orthogonal ranges, i.e., $s_1^* s_2 = 0$.

Let $d \in A_+$ a non-zero positive element, $D := \overline{dAd}$ and suppose that $s_1, s_2 \in \mathcal{M}(D)$ are isometries with $s_1^* s_2 = 0$. Thus, $d \sim_{MvN} d_k := s_k d s_k^* = g_k^* g_k$ for $g_k := d^{1/2} s_k$ and $k \in \{1,2\}$, because $g_k g_k^* = d$.

The element $d_1 + d_2 = d \oplus_{s_1,s_2} d \in D$ is Cuntz-equivalent in $M_2(A)$ to $\operatorname{diag}(d,d) = d \oplus d$. Hence $\operatorname{diag}(d,d) \precsim d_1 + d_2 \in D$ in $M_2(D)$. Since d is a strictly positive element of D we get that $d \oplus_{s_1,s_2} d \precsim d$ by Remark 2.5.2.

Thus, $d \oplus d \precsim d \oplus 0$ in $M_2(D)$. Hence d is properly infinite in D and in A in the sense of Definition 2.5.1.

(ix): If $a \in A_+$, then $a \oplus a \precsim a$, if and only if, for every $\varepsilon > 0$, there exists $x \in M_{1,2}(A)$ such that $x^* a x = ((a \oplus a) - \varepsilon)_+$, cf. Lemma A.6.1(vi).

(x): $((a \oplus a) - \nu)_+ = (a - \nu)_+ \oplus (a - \nu)_+ \precsim (a - \nu)_+ \leq a$ for all $\nu \in (0, \delta)$ implies $a \oplus a \precsim a$ by Lemma A.6.1(vii).

(xi): $b \oplus b \precsim a \oplus a \precsim a \precsim b$ if $a \approx b$ and a is properly infinite. Thus $b \in I(b)$ by Lemma A.6.1(ii,i). It applies to $|a| = (a^* a)^{1/2} \approx a \approx a^* a \sim a a^*$ by Lemma A.6.1(v). □

MORE of ABOVE? OR LESS?

A rather practical elementary *sufficient* criteria of – not necessarily proper – infiniteness is contained in the following lemma:

LEMMA 2.5.4. *An element $0 \neq a \in A_+$ is infinite in A*

Where is "infinite" defined? Give Reference !!

if there exists non-zero positive $c \in D := \overline{aAa}$ with the property that, for each $\varepsilon > 0$, there exists $b \in D_+$ and $d \in A$ with

More BETTER And Clear DETAILS !!!

$c \oplus a \precsim a$ (depending on ε) with $bc = 0$ and $d^* b d = (a - \varepsilon)_+$.

In particular, $a \geq 0$ is infinite in A if there exists a non-zero projection $p \in \overline{aAa}$ that is infinite in A .

Notice that the element $c \geq 0$ is untouched here – in the sense that the d can be chosen that $cd = 0$ –, because we can replace b by $b^{1/3}$ and d by $b^{1/3} d$. Recall that the notation $x \oplus y$ denotes the the 2×2 matrix a_{ij} with entries $a_{1,2} = 0 = a_{2,1}$, $a_{1,1} = x$ and $a_{2,2} = y$.

MORE details !!!!! ???

PROOF. Let $D := \overline{aAa}$ and non-zero $c \in D_+$ with proposed property: For each $\varepsilon \in (0, \|a\|)$ there exist $b \in D_+$ with $bc = 0$ and $d \in A$ with $d^* b d = (a - \varepsilon)_+$.

Then $c + b \in D_+$ implies that $c + b \precsim a$, cf. Remark 2.5.2, and $c \oplus b = g^* g \sim_{MvN} g g^* = c + b \oplus 0$ in $M_2(D) \subseteq M_2(A)$, by the row matrix $g \in M_2(D)$ with entries $g_{11} := c^{1/2}$, $g_{12} := b^{1/2}$ and $g_{2,1} = 0 = g_{22}$.

The equation $d^*bd = (a-\varepsilon)_+$ implies then, that $c \oplus (a-\varepsilon)_+ \precsim c \oplus b$. Using that $\sim_{MvN}$ implies $\approx$, we get that $c \oplus (a-\varepsilon)_+ \precsim a \oplus 0$ in $M_2(A)$ for each $\varepsilon > 0$. But this says that $c \oplus a \precsim a \oplus 0$ in $M_2(A)$ by lower semi-continuity of $\precsim$. Thus, $c \in I_A(a)_+$ and a is infinite in A by Definition 2.5.1 and Lemma 2.5.3(i).

Now we consider the case of $a \in A_+$ with the property that there exists an inside A infinite projection $0 \neq p \in D := \overline{aAa}$. We show below that this implies the existence a non-unitary isometry $T \in \mathcal{M}(D)$, with $1 - TT^* \in D$. It follows that $c := (1-TT^*)a(1-TT^*) \neq 0$ and $b := Ta^{1/3}T^*$, $d := Ta^{1/3}$ satisfy $bc = 0$, $d^*c = 0 = cd$ and $d^*bd = a$ ([36]).

By Definition 2.5.1 and Lemma 2.5.3 the projection $p \in D$ is infinite if $I_D(p) \neq 0$, i.e., there exists nonzero $b \in I_D(p)_+$ with $b \oplus p \precsim p$. We can suppose here that $\|b\| = 1$ and $b \in pAp \subseteq D$. Then $b \oplus p \precsim p$ implies that there exists $G \in M_2(D)$ with

$$\|G^* \operatorname{diag}(p,0)G - \operatorname{diag}(p,b)\| < 1/2\,.$$

We apply Lemma 2.1.9 to this inequality and get a contraction $d \in M_2(D)$ with $(Gd)^* \operatorname{diag}(p,0)(Gd) = \operatorname{diag}((p-1/2)_+, (c-1/2)_+)$. This rewrites as a row $f := [f_1, f_2] := \operatorname{diag}(p,0)Gd \in M_{1,2}(D)$, and the equations $f_1^*pf_1 = (1/2)p$, $f_1^*pf_2 = 0$ and $f_2^*pf_2 = (b-1/2)_+$. In particular, $v := \sqrt{2}pf_1 \in D$ is a partial isometry with $v^*v = p$ and $g := pf_2f_2^*p \neq 0$. Then $gv = 0 = v*g$, $vv^* \leq p$ and $c \leq q := p - vv^* \leq p$.

It gives that $q \neq 0$ and that $p - q = vv^*$, $v^*v = p$. The operator $T := (1_{\mathcal{M}(D)} - p) + v \in 1 + D$ is an isometry with $T^*T = 1$ and $0 \neq 1 - TT^* = q \in D$. □

`Next: General criteria of infiniteness. Not urgent!`

Question 2.5.5. What about the necessary direction of Lemma 2.5.4 for infiniteness?

Let $a \in A_+$ with $\|a\| = 1$ and $D := \overline{aAa}$. Consider D as a C^*-subalgebra of $D_\infty := \ell_\infty(D)/c_0(D)$ by the map $b \in D \mapsto (b, b, \ldots) + c_0(D)$.

Is a infinite in A if a is infinite in D_∞?

Is a infinite in A if a is infinite in the ultrapower A_ω?

Suppose that $a \in A_+$ is infinite in A. Does there exist in $D_\infty := \ell_\infty(D)/c_0(D)$ a (fixed) positive $c \in D_\infty$ with $\|c\| = 1$ such that for each $\varepsilon > 0$ there exists $d \in D_\infty$ with $d^*c = 0$ and $d^*d = (a-\varepsilon)_+$?

Can we find in $(D_\infty)_+$ positive contractions $b, c \in A_\infty := \ell_\infty(A)/c_0(A)$ with $bc = 0$, and for every $\varepsilon > 0$ some (bounded) $d_\varepsilon \in A_\infty$ with $d^*bd = \psi((a-\varepsilon)_+)$ for $\psi \colon A \to A_\infty$ defined by $\psi(a) := (a, a, \ldots) + c_0(A)$?

An attempt is the following: Suppose that non-zero $a \in A_+$ is infinite in A, i.e., $I_A(a) \neq 0$ by Definition 2.5.1 and Lemma 2.5.3(i). Take $g \in I_A(a)_+$ with $\|g\| = 1$.

[36] It shows also that there exists $q = (1-TT^*) \in pDp$ with $a \precsim (1-q)a(1-q)$ in A. So q can play the role of c – instead the inside qAq invertible qaq –, but needs an approximation argument ... an exercise for the reader.

Then $g \oplus a \precsim a$ by definition of $\precsim$ in Definition 2.5.1, i.e., for every sufficiently small $\varepsilon > 0$ there exist $\delta \in (0, \varepsilon)$ and $d_1, d_2 \in A$ with $d_1^*(a-\delta)_+ d_1 = (a-\varepsilon)_+$, $d_2^*(a-\delta)_+ d_2 = (g-1/2)_+$ and $d_2^*(a-\delta)_+ d_1 = 0$. We could consider $c_\varepsilon := (a-\delta)_+^{1/2} d_2 d_2^* (a-\delta)_+^{1/2}$, and $b_\varepsilon := ((a-\delta)_+ d_1 d_1^* (a-\delta)_+)^{1/4}$

?? Perhaps power $^{1/3}$ better?? Check it!

The polar decomposition of $(a-\delta)_+^{1/2} d_1$ is equal to BZ with a partial isometry $Z \in A^{**}$ and $B := b_\varepsilon^2$. We can take $d_\varepsilon := b_\varepsilon \cdot Z$. Get $b_\varepsilon \cdot c_\varepsilon = 0$, $d_\varepsilon^* b_\varepsilon d_\varepsilon = (a-\varepsilon)_+$, $c_\varepsilon \leq \|d_2\|^2 \cdot (a-\delta)_+$ and

$\|b_\varepsilon\|^2 = \|a\| - \varepsilon$???,

$\|c_\varepsilon\| = \|(g-1/2)_+\|$.

We only know that $\delta \in (0, \varepsilon)$ ... no reasonable functional dependence. ?????

The cases of pi(n) are similar to pi(1)?

Notice that for all positive elements X, Y in a C^*-algebra A with $\|X\| = \|Y\| = 1$ and $X \in \overline{(Y-1/2)_+ A (Y-1/2)_+}$ holds that $X(Y-(Y-1/2)_+) = (1/2)X$ and for $Z := 2(Y-(Y-1/2)_+) - 2(X-(X-1/2)_+)$ that $\|Z\| = 1$ and $Z(X-1/2)_+ = 0$. If $(X-1/2)_+$ is in the ideal generated by Z, and if each element of this ideal is in the closure of the *set* of products $A \cdot Z \cdot A$ (not its linear span), then Z is infinite.

COROLLARY 2.5.6. *For (non-zero) C^*-algebras A following properties* (1)-(4) *are equivalent:*

(1) *A is purely infinite in sense of Definition 1.2.1.*
(2) *A satisfies property pi*(1) *of Definition 2.0.4.*
(3) *A satisfies property pi*-1 *of Definition* **??**.
(4) *Every non-zero element of A_+ is properly infinite in A in sense of Definition 2.1.1.*

PROOF. (3)⇔(4): The definitions of property pi-n in Definition **??** imply in case $n = 1$ that A satisfies pi-1, if and only if, each element of A is properly infinite in the sense of Definition 2.1.1.

(4)⇒(2): (Compare Lemma 2.1.2 for more details.) Suppose that every non-zero element $a \in A_+$ is properly infinite in A in sense of Definition 2.1.1.

Then it is easy to see that each non-zero element of $\ell_\infty(A)_+$ is properly infinite, and that this property pass to all non-zero quotients of $\ell_\infty(A)$. In particular, $\ell_\infty(A)$ can not have a non-zero character, because $1 \in \mathbb{C}$ is not infinite in $\mathbb{C}$.

If a is properly infinite and b is in the closed ideal generated by a then there exist for $\varepsilon > 0$ elements $c_1, d_1, \ldots, c_m, d_m \in A$ with $\|b - \sum_{\ell=1}^m c_k a d_k\| < \varepsilon/2$. The proper infiniteness of a causes that there exists $g, h \in M_{1,m}(A)$ with

$$\|1_m \otimes a - g^*(a \oplus 0_{m-1})h\| < \varepsilon/(2m^2(\max_k(\|c_k\| + \|d_k\|)))\,.$$

Thus, there exist $e, f \in A$ with $\|b - eaf\| < \varepsilon$. It shows that A satisfies property pi(1) of Definition 2.0.4.

(2)⇒(1): The property pi(1) in Definition 2.0.4 requires from A that $\ell_\infty(A)$ has no non-zero character, and that, for each element $a \in A_+$ and $c \in A_+$ in the closed ideal generated by a and for each $\varepsilon > 0$, there exists $d \in A$ with $d^*ad = (c-\varepsilon)_+$.

It implies immediately that A is purely infinite in sense of Definition 1.2.1, because this requires only that A has no non-zero character and that for every $a \in A_+$ and every positive c in the closed ideal generated by a, and every $\varepsilon > 0$, there exists an element $d \in A$ such that $\|c - d^*ad\| < \varepsilon$.

(1)⇒(4): Suppose that A has no character and that for every non-zero $a \in A$, every $b \in A$ in the closed ideal $J(a) \subseteq A$ generated by a, and every $\varepsilon > 0$ there exists $d, e \in A$ (depending from a, b, ε) with $\|dae - b\| < \varepsilon$. It it easy to reformulate this property with help of polar decomposition and Lemma 2.1.9 as follows:
The C^*-algebra A has no character and, for every positive contractions in $a, b \in A_+$, with b in the closed ideal $J(a) \subseteq A$ generated by a and $\varepsilon > 0$, there exists $\delta > 0$ and $d \in A$ with $d^*(a-\delta)_+d = (b-\varepsilon)_+$.

The properties of A carry over to each non-zero quotient A/K of A by a closed ideal K of A, because if A has no non-zero character then A/K can not have a non-zero character, and if $e, f \in (A/K)_+$, $f \in J(e)$ in A/K and $\varepsilon \in (0, \|f\|)$ given, then there exist $a \in A_+$ with $e = \pi_K(a)$ and elements $g_1, \ldots, g_n \in A$ such that $\pi_K(b) = (f - \varepsilon/2)_+$ for $b := g_1^*ag_1 + \ldots + g_n^*ag_n \in J(a)$. There exists $d \in A$ and $\delta > 0$ with $d^*(a-\delta)_+d = (b-\varepsilon/2)_+$. It says that $(f-\varepsilon)_+ = \pi_K(d)^*(e-\delta)_+\pi_K(d)$ for given $\varepsilon \in (0, \|f\|)$ and suitable $\delta \in (0, \|e\|)$.

By Part (v) of Lemma 2.5.3, non-zero elements $a \in A$ are *properly* infinite, if and only if, $\pi_J(a)$ is infinite for each closed ideal J of A with $a \notin J$.

Above we have seen that non-zero quotients A/K satisfy the same properties as A. Therefore it suffices to show in general that each non-zero element of A_+ is infinite if A is purely infinite in sense of Definition 1.2.1.

If $D \subseteq A$ is a non-zero hereditary C^*-subalgebra of A then D can not have a character $\chi\colon D \to \mathbb{C}$:

Indeed: Suppose that a non-zero character χ on D exists and let $I \subseteq D$ the kernel ideal of χ.

Let $J \subseteq A$ the closed ideal J of A generated by the kernel I of χ. Then the ideal J has the property that $\mathbb{C} \cong D/I \subseteq A/J$ is a 1-dimensional hereditary C^*-subalgebra of A/J, because $J \cap D = I$ by Lemma ??.

Let $p \in D/I \subseteq A/J$ the unique projection with $\mathbb{C}\cdot p = D/I$, then every element in the closed ideal K of A/J generated by p is itself in $\mathbb{C} \cdot p$. Thus, J must be the kernel of a character on A. But this is forbidden for A in Definition 1.2.1.

It follows that each non-zero hereditary C^*-subalgebra D of A has no character. It implies by Lemma 2.1.15(ii) that the D contains a 2-homogenous element $g := \psi(f_0 \otimes 1_2)$ for some non-zero *-morphism $\psi\colon \mathrm{C}_0((0,1], M_2) \to D$ with $\|g\| = 1$.

Now let $a \in A_+$ non-zero and $\|a\| = 1$. The hereditary C^*-subalgebra $D := \overline{(a-1/2)_+A(a-1/2)_+}$ contains a 2-homogenous element $g := \psi(f_0 \otimes 1_2)$ with $\|g\| = 1$.

Let $c := \psi((f_0 - 1/2)_+ \otimes p_{22}) \in D$, $d := 2\psi((f_0 - (f_0 - 1/2)_+) \otimes p_{22}$, and $e := 2(a - (a-1/2)_+) - d$. Then $dc = c = cd$, $da = d = ad$, a is in the ideal of $\overline{aAa}$ generated by e, and $ec = 0$.

There exists a pure state $\rho\colon A \to \mathbb{C}$ with $\rho(a) = 1$. Let $d_\rho\colon A \to \mathcal{L}(\mathcal{H})$ the corresponding irreducible representation and $x \in \mathcal{L}(\mathcal{H})$ with $\|x\| = 1$ and

$\rho(a) = \langle d_\rho x, x\rangle$

Perhaps it is easier to show first that $(a-1/2)_+A(a-1/2)_+$ can not have a character

By Lemma **??**

NO-NO! need here a new criteria for infiniteness! Which one? ???

Part (v) of Lemma 2.5.3

to show that a (respectively $\pi_J(a)$) is infinite.

Something like:

?????????????? PROOF? □

There are places, e.g. Lemma 2.6.8(i) - or ?? -, with results partly similar to the in Lemma 2.5.7 considered!

LEMMA 2.5.7. *Let $p, q \in B$ projections and $b \in B_+$. If $(b \oplus q) \precsim (p \oplus 0)$ in $M_2(B)$, then for each $0 < \varepsilon < \min(\|b\|, 1)$ there exists an element $d = d(\varepsilon) \in B$ and a partial isometry $z = z(\varepsilon) \in B$ – depending both on ε –, such that for $r := z(z^*)$ holds $r \leq p$, $z^*z = q$, $(p-r)d = d$ and $d^*d = (b-\varepsilon)_+$.*

Special cases are:

(i) *If, moreover, b is a projection then this implies that $b \oplus q$ is in $M_2(B)$ MvN-equivalent to a sub-projection of $p \oplus 0$, i.e., there are projections $p_1, p_2 \in B$ such that $p_1p_2 = 0$, $p_1 + p_2 \leq p$, $p_1 \sim_{MvN} b$ and $p_2 \sim_{MvN} q$.*

(ii) *If $p, q \in B$ are projections that satisfy $p \precsim q$ then there exists a projection $r \leq q$ that is Murray–von-Neumann equivalent to p, i.e., there exists a partial isometry $v \in B$ with $v^*v = p$ and $vv^* =: r \leq q$.*

In particular, $p \approx q$, if and only if, there exist partial isometries $v, w \in B$ such that $v^*v = q$, $vv^* \leq p$, $w^*w = p$ and $ww^* \leq q$ ([37]).

(iii) *If B is unital and $1 := 1_B$, then there exists $0 \neq c \in B$ with $c \oplus 1 \precsim 1$, if and only if, B contains a non-unitary isometry.*

And this is equivalent to $I(1) \neq \{0\}$ by the definition of the ideal $I(1)$ given in Lemma 2.5.3.

[37] Compare Remark 2.5.8.

(iv) *The unit* $1 := 1_B$ *is properly infinite in* B, *– in the sense that there exists isometries* $S, T \in B$ *with* $S^*T = 0$ *–, if and only if,* $1 \oplus 1 \precsim 1 \oplus 0$ *in* $M_2(B)$*, if and only if,* $1 \in I(1)$*, if and only if,* $I(1) = B$ *.*

PROOF. Recall that $b \approx b^*b \approx (b^*b)^{1/2}$. It implies that the relation $b \oplus q \precsim p$ is equivalent to $(b^*b)^{1/2} \oplus q \precsim p$. Thus, we can suppose that b is positive in all later considered cases.

The relation $b \oplus q \precsim p \oplus 0$ implies by Lemma A.6.1(iv) that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and a matrix $T \in M_2(B)$ such that

$$(b - \varepsilon)_+ \oplus (q - \varepsilon)_+ \;=\; ((b \oplus q) - \varepsilon)_+ \;=\; T^*((p \oplus 0) - \delta)_+ T\,.$$

Since $\varepsilon < \min(\|b\|, 1)$, it follows that the left side is non-zero, and therefore $\delta < 1$. Thus,

$$(b - \varepsilon)_+ \oplus (1 - \varepsilon)p \;=\; (1 - \delta)S^*(p \oplus 0)S\,,$$

where $S := [x, y]$ is the first row of T (and we can suppose that $px = x$ and $py = y$). We get that $x^*py = 0$ and $S^*(p \oplus 0)S = \mathrm{diag}(x^*px, y^*py)$. It implies that $(b - \varepsilon)_+ = (1 - \delta)x^*px$ and $(1 - \varepsilon)p = (1 - \delta)y^*py$ We define a partial isometry by

$$z := z(\varepsilon) \;:=\; (1 - \varepsilon)^{-1/2}(1 - \delta)^{1/2}py$$

and an element $d := d(\varepsilon) \in B$ by

$$d \;:=\; (1 - \delta)^{1/2}px\,.$$

Straight calculation shows that z and d have the proposed properties.

Proof of the special cases:

(i): If b is a projection and $\varepsilon < 1/2$ then let $p_1 := r = z(\varepsilon) \cdot z(\varepsilon)^*$ and $p_2 := (1 - \varepsilon)^{-1} d(\varepsilon) \cdot d(\varepsilon)^*$. This works because $b = (1 - \varepsilon)^{-1} d(\varepsilon)^* d(\varepsilon)$.

(ii): It if moreover $b = 0$, then (i) says $p_1 = 0$ and that $r := p_2 \subseteq p$ and $v := (1 - \varepsilon)^{-1/2} d(\varepsilon)$ satisfy $v^*v = q \leq p$ and $vv^* = r$.

(iii): The definition of the ideal $I(1) \subseteq B$ with $1 := 1_B$, given in Lemma 2.5.3, says that $c \in I(1)$ if and only if $c \oplus 1 \precsim 1$. Thus, $I(1) \neq \{0\}$, if and only if, there exists non-zero $c \in B$ with $c \oplus 1 \precsim 1$.

If there exists $0 \neq c \in B$ with $c \oplus 1 \precsim 1$, then $b \oplus 1 \precsim 1$ for $b := c^*c$, because $b \approx c$.

Let $q := 1$, $p := 1$, $b := c^*c$ and $\varepsilon := \|b\|/2$ in the general case. Then there exists elements $d \in B$ and $z \in B$ with $z^*z = 1$, $d^*d = (b - \|b\|/2)_+$ and $(1 - zz^*)d = d$. It follows that z is a non-unitary isometry in B.

If B contains a non-unitary isometry $z \in B$, then $c \oplus 1 \precsim 1$ for $c := 1 - zz^* \neq 0$, because $c \oplus 1 \approx c \oplus zz^*$ and $1 - zz^* \oplus zz^* = R^*(1 \oplus 0)R$ for the row $R := [1 - zz^*, zz^*]$,. Thus, $(1 - zz^*) \oplus 1 \precsim 1$.

(iv): The subset $I(1) \subseteq B$ is a closed ideal of B by Lemma 2.5.3(i). Thus, $1 \in I(1)$, if and only if, $B = I(1)$. By Definition of the ideal $I(1)$ of B, $1 \in I(1)$, if and only if, $1 \oplus 1 \precsim 1 \oplus 0$ in $M_2(B)$.

The latter $1 \oplus 1 \precsim 1$ is the case of Part (i) with $b := 1$, $q := 1$ and $p := 1$. By Part (i), there are projections $p_1, p_2 \in B$ such that $p_1 p_2 = 0$, $p_1 + p_2 \leq 1$, $p_1 \sim_{MvN} 1$ and $p_2 \sim_{MvN} 1$. It says that there exists elements $S, T \in B$ with $S^*S = 1 = T^*T$ with $S(S^*) = p_1$ and $T(T^*) = p_2$. This are isometries with $S^*T = S^*(SS^*)(TT^*)T = S^*p_1p_2T = 0$. □

Remark 2.5.8. The relations $v^*v = q$, $vv^* \leq p$, $w^*w = p$ and $ww^* \leq q$ in Part(ii) of Lemma 2.5.7 do not imply W-vN equivalence: This can be seen in stable simple purely infinite C^*-algebras: The W-vN equivalence classes of projections in $B \otimes \mathbb{K}$ correspond bijective to the elements of $\mathrm{K}_0(B)$ but all non-zero projections in $p, q \in B \otimes \mathbb{K}$ satisfy $p \approx q$.

Corollary 2.5.9. *Let A a unital C^*-algebra. Then 1_A is properly infinite in A, if and only if, A/J contains a non-unitary isometry for each closed ideal $J \neq A$ of A.*

Proof. Part (v) of Lemma 2.5.3 says that 1 is properly infinite in A, if and only if, $\pi_J(1) = 1_{A/J} \in A/J$ is infinite in A/J for each closed ideal $J \neq \{0\}$ of A. By Part (iii) of Lemma 2.5.7, $1_{A/J}$ is infinite in A/J, if and only if, there exists a non-unitary isometry in A/J. □

Remark 2.5.10. The closed ideal $I_A(a)$ is always contained in the closed ideal $J(a)$ of A generated by a, and $J(a) = I_A(a)$ if and only if a is properly infinite. Then $J(a)$ is equal to the closure of $\bigcup_n A \cdot (a^*a - 1/n)_+ \cdot A$, i.e., of the *set* of products $b \cdot (a^*a - 1/n) \cdot c$, with $b, c \in A$ ([38]).

The Definition 2.0.4 of property pi(n) immediately allows to see that it implies that each non-zero $(n+1)$-homogenous element is properly infinite in A if A has property pi(n). But the Corollary 2.7.18 of Proposition 2.7.16 gives the better result that all non-zero n-homogenous elements are properly infinite in A with property pi(n). It implies that $\ell_\infty(A)$ has no irreducible representation on a Hilbert space of dimension $\geq n$, and that property pi(n) passes to $\ell_\infty(A)$ if A has property pi(n).

Property pi-n in Definition **??** implies property pi(m) for some $m \leq n$, but no explicit *lower* bound for m is known. The non-trivial direction is, that each C^*-algebra with property pi(m) has property pi-n for some $n \geq m$, but it seems that some property of the ideal system of the multiplier algebra plays a certain role.

For real-rank zero C^*-algebras they are all the same and say that each non-zero projection is infinite.

Some sensible property of the definitions of infiniteness of elements can be seen in the C^*-subalgebra $A \subseteq \mathrm{C}([0,1], \mathcal{O}_2)$ of operator-valued functions

$$f\colon [0,1] \to \mathcal{O}_2 := C^*(s_k\,;\; s_k^*s_k = 1 = s_1s_1^* + s_2s_2^*)$$

with $f(1) \in M_{2^\infty} \subseteq \mathcal{O}_2$. There $f \in A_+$ is infinite if and only if $f(1) = 0$. The – inside A – properly infinite element f given by the function $f(t) := (1-t)s_1s_1^* \in$

[38] But it seems that the latter property does not imply always that a is properly infinite. Examples for that?

$C_0([0,1),\mathcal{O}_2)$ has the property that no element $g\in A_+$ with $\|g\|\leq 1$ and $gf=f$ can be infinite in A.

LEMMA 2.5.11. *Let* $b:=\sum_{k=1}^m a_k$ *with* $a_1,\ldots,a_m\in A_+$.

If the elements $a_k\otimes 1_n$ *are all properly infinite in* $M_n(A)$, *i.e., if* $a_k\otimes 1_{n+1}\precsim a_k\otimes 1_n$ *for* $k=1,\ldots,n$, *then* $b\otimes 1_{m\cdot n}$ *is properly infinite in* $M_{m\cdot n}(A)$.

PROOF. Notice that $b\precsim (a_1\oplus\ldots\oplus a_m)\leq b\otimes 1_m$ in $M_m(A)$, and that $(a_1\oplus\ldots\oplus a_m)\otimes 1_{n+1}\precsim (a_1\oplus\ldots\oplus a_m)\otimes 1_n$ in $M_{m\cdot n}(A)$, because the $a_k\otimes 1_n$ are properly infinite in $M_n(A)$.

Induction over $k\in\mathbb{N}$ gives that $(a_1\oplus\ldots\oplus a_m)\otimes 1_{k+n}\precsim (a_1\oplus\ldots\oplus a_m)\otimes 1_n$.

Gives $b\otimes 1_{k+n}\precsim b\otimes 1_{mn}$. for all $k\in\mathbb{N}$. In particular, $b\otimes 1_{1+(mn)}\precsim b\otimes 1_{mn}$, i.e., $b\otimes 1_{mn}$ is properly infinite. □

EXAMPLE 2.5.12. There exist a closed ideal $J\lhd A$ and $a\in A_+$ with $J\subseteq D:=\overline{aAa}$, such that D is stably finite and $I(d)=\{0\}$ for all $d\in D$ (hence A is stably finite), but $\pi_J(a)=a+J$ is properly infinite in A/J.

Indeed: By an "opposite version" of a theorem of Glimm in [**438**, cor.1.4(v)] there exists a hereditary C^*-subalgebra D of the CAR-algebra $B:=M_{2^\infty}$ such that the unital two-sided normalizer algebra $A:=\mathcal{N}(D,B)\subseteq B$ of D in B satisfies $A/D=\mathcal{O}_2$. Then $a:=1\in A$ satisfies that the C^*-subalgebra $\overline{aAa}=A\subseteq B$ is stably finite, i.e., $I(a\otimes 1_n)=\{0\}$ for all $n\in\mathbb{N}$. But $\pi_J(a)$ is the properly infinite unit of $\mathcal{O}_2$ for $J:=D\lhd A$.

EXAMPLE 2.5.13. The unit element 1 of the unitization $\widetilde{A}$ of $C_0((0,1],\mathcal{O}_2)$ is a *stably finite* element of $\widetilde{A}$.

More generally, the unit element 1_A of the "join" algebras

$$A:=\mathcal{E}_{\max}(B,C)\subseteq C([0,1],B\otimes^{\max}C)$$

of unital algebras B and C, or its quotient

$$A:=\mathcal{E}(B,C)\subseteq C([0,1],B\otimes^{\min}C)$$

is stably finite if $1_n\otimes 1_B$ finite in $M_n\otimes B$ for all $n\in\mathbb{N}$.

Even more generally, 1_A is infinite in A if and only if $1_A=T^*T$ for some non-unitary isometry $T\in A$, cf. Lemma 2.6.8(i).

In the next lemma we can replace the pair $(\varepsilon,2\varepsilon)$ with $0<\varepsilon<\|a\|/2$ obviously by any pair of positive reals $0<\delta<\gamma<\|a\|$, simply by passing from a to $f(a)$ with a suitable strictly increasing continuous real function $f\in C_0(0,\|a\|]_+$ on $[0,\|a\|]$ with $f(0)=0$.

LEMMA 2.5.14. *Let* B *a* C^**-algebra,* $a\in B_+$ *non-zero,* $\varepsilon\in(0,\|a\|/2)$ *and denote by* J *the closed ideal that is generated by the two-sided annihilator*

$$\operatorname{Ann}((a-2\varepsilon)_+,B)\;:=\;\{\,b\in B\,;\;b\,(a-2\varepsilon)_+\;=\;0\;=\;(a-2\varepsilon)_+\,b\,\}$$

of $(a-2\varepsilon)_+$ *in* B.

(i) *If* $(a-\varepsilon)_+$ *is in* J *then* $J = B$*, i.e., then* $\mathrm{Ann}((a-2\varepsilon)_+, B)$ *is a* full *hereditary C^*-subalgebra of* B*.*

(ii) *The hereditary C^*-subalgebra* $\mathrm{Ann}((a-2\varepsilon)_+, B)$ *is full in* B *if there exists* $b \in B$ *that satisfies the equations:*

$$b^*(a-2\varepsilon)_+ = 0 \quad and \quad b^*b = (a-\varepsilon)_+ \,. \tag{5.1}$$

If, moreover, the elements of $\mathrm{Ann}((a-2\varepsilon)_+, B)$ *are properly infinite, then conversely the fullness of* $\mathrm{Ann}((a-2\varepsilon)_+, B)$ *implies the existence of* $b \in B$ *that satisfies Equations* (5.1)*.*

More generally, $\mathrm{Ann}((a-2\varepsilon)_+, B)$ *is full in* B *if there exist elements* $u, v \in B$ *and* $\delta > 0$ *such that* $(vv^* - \delta)_+ = (a-2\varepsilon)_+$*,* $u^*(vv^*-\delta)_+ = 0$*, and* $[u] \geq [v]$ *in* $\mathrm{Cu}(B)$*.*

(iii) *If* $J \neq B$ *then* B/J *is unital, and* $\pi_J((a-\varepsilon)_+)$ *is invertible in* B/J *with spectrum in* $[\,\varepsilon,\, \|a\| - \varepsilon\,]$*.*

In particular, $\pi_J(a)$ *is invertible in* B/J*, if and only if,* $\pi_J(a) \neq 0$*.*

Proof. Let $\varepsilon, \mu > 0$. Define $f_\mu(t) := \min\{\,1,\, (t/\mu - 1)_+\,\}$ for $t \in [0, \infty)$, i.e., $f_\mu(t) := 0$ for $t \leq \mu$, $f_\mu(t) := t/\mu - 1$ for $t \in [\mu, 2\mu]$ and $f_\mu(t) := 1$ for $t \in [2\mu, \infty)$. Notice that $\mu f_\mu(t) \leq \max(t-\mu, 0)$ and $(1 - f_\mu(t))\max(t - 2\mu, 0) = 0$.

(i,iii): Let $a \neq 0$, $0 < \varepsilon < \|a\|/2$ and let J denote the closed ideal J of B, generated by $\mathrm{Ann}((a-2\varepsilon)_+, B)$.

The ideal $C^*(a) \cap J$ of the C^*-subalgebra $C^*(a) \subseteq B$ contains the closed ideal K of $C^*(a)$ that is generated by $(1 - f_\varepsilon(a)) \cdot C^*(a)$ because $f_\varepsilon(t) = 1$ for $t \geq 2\varepsilon$.

It follows that the non-zero values of the spectrum of $\pi_J((a-\varepsilon)_+)$ are contained in the spectrum of $\pi_K((a-\varepsilon)_+)$ in $C^*(a)/K$. The latter is contained in $[\varepsilon, \|a\| - \varepsilon]$ if $2\varepsilon < \|a\|$.

The elements $(1 - f_\varepsilon(a))a$ and $f_\varepsilon(a) - f_\varepsilon(a)^2$ are contained in $K \subseteq J$. It follows that $P := \pi_J(f_\varepsilon(a))$ is a projection in B/J.

The equation $(1 - f_\varepsilon(a))(a - 2\varepsilon)_+ = 0$ implies that $b - f_\varepsilon(a))b \in J$ for all $b \in B$, because

$$(1 - f_\varepsilon(a))B(1 - f_\varepsilon(a)) \subseteq \mathrm{Ann}((a-2\varepsilon)_+, B) \subseteq J\,.$$

Thus, P is the unit element of B/J if $J \neq B$, and then $\varepsilon P \leq \pi_J((a-\varepsilon)_+) \leq (\|a\| - \varepsilon)P$. We get that $\pi_J((a-\varepsilon)_+)$ and $\pi_J(a)$ are invertible in B/J with inverses of norms $\leq \varepsilon^{-1}$, respectively $\leq (2\varepsilon)^{-1}$.

The equation $J = B$ is equivalent to $(a-\varepsilon)_+ \in J$ because that latter causes $P = 0$.

(ii): If $a \in J = B$ then there exist $b_1, \ldots, b_n \in B$ with $b_k b_k^*(a - 2\varepsilon)_+ = 0$ and $\sum_{k=1}^n b_k^* b_k = (a - \varepsilon/2)_+\,$.

If the elements of $\mathrm{Ann}((a-2\varepsilon)_+, B)$ are properly infinite, then it follows that there exists $\gamma > 0$ and $d_1, d_2 \in B$ such that

$$\sum_{k=1}^{n} (d_1 b_k d_2)^* (f-\gamma)_+ (d_1 b_k d_2) \;=\; (a-\varepsilon)_+$$

for the element $f := \sum_{k=1}^{n} b_k b_k^*$ of $\mathrm{Ann}((a-2\varepsilon)_+, B)$.

By assumption on $\mathrm{Ann}((a-2\varepsilon)_+, B)$, the element f is properly infinite. It implies that there exist $g_1, \ldots, g_n \in B$ such that $g_j^* f g_k = \delta_{jk}(f-\gamma)_+$. The element $b := \sum_{k=1}^{n} f^{1/2} g_k d_1 b_k d_2$ satisfies $bb^*(a-2\varepsilon)_+ = 0$ and $b^*b = (a-\varepsilon)_+$.

If there exists $b \in B$ with $bb^*(a-2\varepsilon)_+ = 0$ and $b^*b = (a-\varepsilon)_+$, then $(a-\varepsilon)_+ \in J$. It implies $J = B$ by Part (i). (This particular observation do not use an assumption on infiniteness of elements in $\mathrm{Ann}((a-2\varepsilon)_+, B)$.)

Suppose, more generally, that there exist elements $u, v \in B$ and $\delta > 0$ such that $(vv^* - \delta)_+ = (a-2\varepsilon)_+$, $u^*(vv^*-\delta)_+ = 0$, and $[u] \geq [v]$ in $\mathrm{Cu}(B)$.

Then $vv^* \precsim u^*u$, and there exist $d \in B$ with $d^*u^*ud = (vv^*-\delta/2)_+ \sim_{MvN} (v^*v - \delta/2)$. The element $b := ud$ satisfies $b^*(vv^*-\delta)_+ = 0$ and $b^*b = (vv^*-\delta/2)_+$.

Thus $\mathrm{Ann}((vv^*-\delta)_+, B) = \mathrm{Ann}((a-2\varepsilon)_+, B)$ is full in B, because we can take $(vv^*, \delta/2)$ in place of (a, ε) in our former considerations.

In case where $\mathrm{Ann}((a-2\varepsilon)_+, B)$ is full and purely infinite, we can find the above considered element b and can define $u := b$, $v := b^*$, $\delta := \varepsilon$. They satisfy the requirements on v, w and δ, because $b^*(b^*b-\varepsilon)_+ = b^*(a-2\varepsilon) = 0$ and $[b] \geq [b]$. □

`Does there exists a ''relative'' variant for some spectral p.i. elements in` $e \in J$??

LEMMA 2.5.15 ([**462**]). *Let $a \in A_+$ with $A = \overline{aAa}$, and let $J \lhd A$ a closed ideal, such that $\pi_J(a) = a + J$ is properly infinite in A/J. Then:*

(i) *For $b, c_1, c_2, c_3 \in A_+$ holds $c_1 \oplus c_2 \oplus c_3 \precsim b \oplus 0 \oplus 0$ in $M_3(A)$, if and only if, for every $\varepsilon > 0$, there are $f_1, f_2, f_3 \in bA$ and $\gamma > 0$ such that*

$$f_j^*(b-\gamma)_+ f_k \;=\; \delta_{j,k}(c_k-\varepsilon)_+ \quad \text{for } j,k = 1,2,3\,.$$

(ii) *For every $\varepsilon > 0$,there exist elements $a_1, a_2 \in A_+$, $e \in J_+$ and $n \in \mathbb{N}$, such that $a_1 a_2 = 0$, $(a_1 + a_2)e = 0$, and $(a-\varepsilon)_+ \precsim a_k \oplus (e \otimes 1_n)$ for $k = 1, 2$.*

(iii) *If each $0 \neq e \in J_+$ is properly infinite, then a is properly infinite.*

PROOF. We give here a proof that contains more details than in [**462**].

(i): Let $c := c_1 \oplus c_2 \oplus c_3$ and $\varepsilon > 0$. By Lemma A.6.1(vi), there are $F \in M_3(A)$ and $\delta > 0$ with $(c-\varepsilon)_+ \;=\; F^*((b-\delta)_+ \oplus 0 \oplus 0)F$.

We define $f_k := \xi(b)F_{1,k}$ for $k \in \{1,2,3\}$, where $\xi \in \mathrm{C}_0(0, 1+\|b\|]$ is given by $\xi(t) := \min(1, (2/\delta)\max(0, t-\delta/2))$.
Then $f_k \in bA$ and the row $G := [f_1, f_2, f_3] \in M_{1,3}(A) \subset M_3(A)$ satisfies

$$(c-\varepsilon)_+ = F^*((b-\delta)_+ \oplus 0_2)F \;=\; G^*((b-\delta)_+ \oplus 0_2)G\,.$$

(ii): Let $\varepsilon \in (0, \|\pi_J(a)\|)$ and $\mu := \varepsilon/3$. Since $\pi_J(a) \otimes 1_3 \precsim \pi_J(a)$ by proper infiniteness of $\pi_J(a)$, and since always $a \approx a^2$, there exist $g_1, g_2, g_3 \in A/J$ and $\delta > 0$ with $g_i^* \pi_J((a-\delta)_+) g_j = \delta_{i,j} \pi_J((a-\mu)_+)$. If we let $x_k := \pi_J((a-\delta)_+)^{1/2} g_k$, then

$$(x_k x_k^* - \mu)_+ \sim (x_k^* x_k - \mu)_+ = \pi_J((a-2\mu)_+)\,,$$

in A/J, and there is a (unique) C^*-morphism $\varphi\colon \mathrm{C}_0(0,1] \otimes M_3 \to A/J$ with $\varphi(f_0 \otimes p_{ij}) = x_i(x_j^*)$. The C^*-algebras $\mathrm{C}_0(0,1] \otimes M_n$ are projective by Proposition A.8.4 in the sense of Definition A.8.1 in Appendix A . It implies that there exists a C^*-morphism $\psi\colon \mathrm{C}_0(0,1] \otimes M_3 \to A$ with $\pi_J \circ \psi = \varphi$, cf. also Lemma 2.1.15 and Remark 2.1.16 concerning the projectivity of $\mathrm{C}_0((0,1], M)$, for C^*-algebras M of finite dimension.

Let $b_k := \psi(f_0 \otimes p_{kk})$, for $k = 1, 2, 3$ and take $y_k \in A$ with $\pi_J(y_k) = x_k$. Then $b_1 \sim b_2 \sim b_3$, $b_j b_k = 0$ for $j \neq k$, and define

$$z_k := \big(y_k^*(b_k - \mu)_+ y_k\big) - \big((a-2\mu)_+ \cdot (a-\mu)_+\big) \in J\,.$$

Thus, $(a-2\mu)_+ \precsim (b_k - \mu)_+ \oplus v$ for $k = 1, 2$ and $v := |z_1| + |z_2|$. It follows, that there exists $\gamma > 0$ such that, for $k = 1$ and $k = 2$, holds

$$(a-\varepsilon)_+ \precsim (b_k - \mu)_+ \oplus (v-\gamma)_+\,.$$

The annihilator $D := \mathrm{Ann}((b_1 + b_2 - \mu)_+, A)$ is a hereditary C^*-subalgebra of A that contains b_3 . It implies that $b_1 + b_2$ is in the closed ideal generated by D, and that D is full by Lemma 2.5.14. Since D is full, also $J \cap D$ is full in J , i.e., $\mathrm{span}(J(D \cap J)_+ J)$ is dense in J .

It follows that there exist $d_1, \ldots, d_n \in J$ and $e \in (D \cap J)_+$ with $\sum_j d_j^* e d_j = (v-\gamma)_+$. In particular, $(v-\gamma)_+ \precsim e \otimes 1_n$.

(iii): Let $\varepsilon > 0$, $a_1, a_2, e \in A_+$ and $n \in \mathbb{N}$ as in Part (ii). Then

$$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a_1 \oplus a_2 \oplus (e \otimes 1_{2n})\,.$$

Since e is properly infinite, and since a_1, a_2, e are pairwise orthogonal, we get:

$$((a \otimes 1_2) - \varepsilon)_+ \precsim a_1 \oplus a_2 \oplus e \sim a_1 + a_2 + e \precsim a\,,$$

because a is a strictly positive element of A. Since $\varepsilon > 0$ is arbitrary, we obtain $a \otimes 1_2 \precsim a$. □

DEFINITION 2.5.16. We say that $0 \neq b \in A_+$ is **spectral properly infinite** if $(b-t)_+$ is properly infinite in A for each $t \in (0, \|b\|)$. (See Section 1 or Definition 2.5.1 of properly infinite elements.)

LEMMA 2.5.17. *Let A a C^*-algebra and $\varepsilon > 0$.*

(o) *Each $b \in A_+ \setminus \{0\}$ is spectral properly infinite, if and only if, A is a purely infinite C^*-algebra.*

(i) *If $b \in A_+$ is spectral properly infinite in sense of Definition 2.5.16, then elements $d_1, d_2 \in A$ with $d_j^* b d_k = \delta_{j,k}(b-\varepsilon)_+$ can be found with norm-bound $\|d_j\|^2 \leq 2\|b\|/\varepsilon$.*

(ii) *If the positive part A_+ of a C^*-algebra A contains a dense subset $X \subseteq A_+$ such that each $b \in X$ is spectral properly infinite, then A is purely infinite, i.e., each non-zero $a \in A_+$ is properly infinite.*

PROOF. (o): By definition, A is purely infinite if each non-zero element $a \in A_+$ is properly infinite. In particular $(b-t)_+$ is properly infinite in A for each $b \in A_+$ and $t \in (0, \|b\|)$.

(i): Let $b \in A_+$ spectral properly infinite, and $\varepsilon = 2\delta \in (0, \|b\|)$. There exist $g_1, g_2 \in A$ with $g_j^*(b-\delta)_+ g_k = \delta_{jk} \cdot (b-2\delta)_+$ because $(b-\delta)_+$ is properly infinite. Then $d_k := b^{-1/2}(b-\delta)_+^{1/2} g_k$ is in A and $\|d_k\|^2 \leq \delta^{-1}\|(b-2\delta)_+\|$, because $\delta(t-\delta)_+ \leq t(t-\delta)_+$ for $t \in (0, \infty)$, and $A_+ \ni a \mapsto \|g_k^* a\, g_k\| \in \mathbb{R}_+$ is order monotone.

(ii): Let $a \in A_+$ with $\|a\| = 1$ and $\varepsilon > 0$. There exists $b \in X$ with $\|b - a\| < \gamma := \varepsilon/3$. It implies $\|(b-2\gamma)_+ - a\| < 3\gamma$, because $\|(b-2\gamma)_+ - b\| \leq 2\gamma$.

By Lemma 2.1.9 there exists *contractions* $d_1, d_2 \in A$ with $d_1^* a d_1 = (b-\gamma)_+$ and $d_2^*(b-2\gamma)d_2 = (a-3\gamma)_+$.

By assumption, each $b \in X$ is spectral properly infinite, thus $(b-\gamma)_+$ is properly infinite for each $\gamma \in (0, \|b\|)$, i.e., $(b-\gamma)_+ \oplus (b-\gamma)_+ \precsim (b-\gamma)_+$, and we find $e_1, e_2 \in A$ with $e_j^*(b-\gamma)_+ e_k = \delta_{jk}(b-2\gamma)_+$. It follows that $c_k := d_1 e_k d_2$ satisfies $c_j^* a c_k = \delta_{jk}(a-3\gamma)_+$. Thus, each $a \in A_+$ is properly infinite and A is purely infinite. □

EXAMPLE 2.5.18. Let $a_1, a_2, \ldots \in \mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))_+$ defined by $a_n := \mathrm{diag}(1, (n+1)^{-1}, (n+1)^{-2}, \ldots)$ then each a_n is properly infinite, and the sequence $(a_1, a_2, \ldots)$ is decreasing with norm-limit $(1, 0, 0, \ldots)$ in $\mathbb{K}$. Thus, convergent sequence of of properly infinite contractions have in general no properly infinite limit.

Are they really properly infinite?

Does there exist $d_k \in \mathbb{K}$ with

$$\mathrm{diag}(1, 1, 1/n, 1, n, 1/n^2, 1/n^2, \ldots) = \lim_k d_k^* \,\mathrm{diag}(1, 1/n, 1/n^2, \ldots) d_k$$

in norm?

$(1, 1/n, 1/n^2, \ldots) \precsim (1, 1/n^2, 1/n^4, 1/n^6, \ldots) \oplus (1/n, 1/n^3, 1/n^5, 1/n^7 \ldots) \approx (1, 1/n, 1/n^2, 1/n^3, \ldots) \oplus (1, 1/n, 1/n^2, 1/n^3, \ldots)$ via $d_k := a_k \oplus b_k$ $a_k := (1, n^{1/2}, n, 1/n^6, \ldots)$

The above technical lemmata on the properties of the Cuntz-relation $\precsim$ and $\approx$ yield almost immediately the following permanence properties of the class of purely infinite C^*-algebras.

Notice that $A \otimes \mathrm{C}_0(0,1]$ is not in this list.

PROPOSITION 2.5.19 (Permanence of pure infiniteness). *The class of (not necessarily simple) purely infinite C^*-algebras A is invariant under following operations:*

(i) *passage to quotients* A/J,
(ii) *inductive limits,*
(iii) *infinite direct products* $\ell_\infty(A_1, A_2, \ldots)$ *and ultrapowers* A_ω,
(iv) *passage to hereditary* C^**-subalgebras* $D = DAD$,
(v) *Morita equivalence, and*
(vi) *extensions.*

PROOF. Recall that A is purely infinite, if and only if, for each $a \in A$,

$$a \in I(a) := \{b \in A;\ b \oplus a \precsim a\},$$

and recall that $I(a)$ is a closed ideal of A by Lemma 2.5.3(i).

(i): If $a \in A \setminus J$ is properly infinite in A, i.e., $a \in I(a)$, then $\pi_J(a) = a + J$ is properly infinite in A/J, i.e., $\pi_J(a) \in I(\pi_J(a))$ because $\pi_J(I(a)) \subseteq I(\pi_J(a))$ by Lemma 2.5.3(v). Thus, each quotient A/J by a closed ideal $J \neq A$ of A is purely infinite if the C^*-algebra A is purely infinite.

(ii): If $\{ h_{\sigma,\tau} \colon A_\sigma \to A_\tau\}_{\tau,\sigma \in T}$ is a directed net of C^*-morphisms,

$$A := \operatorname{indlim}_{\tau \to T}(h_{\sigma,\tau} \colon A_\sigma \to A_\tau)$$

is the (canonical) inductive limit of C^*-algebras, then A is generated by the upward directed net of the images of $h_{\sigma,\infty} \colon A_\sigma \to A$. Thus, A contains an upward directed net of C^*-subalgebras $h_{\sigma,\infty}(A_\sigma)$ that are quotients of purely infinite C^*-algebras. By Part (i), this C^*-subalgebras are purely infinite. The elements of the images build together a dense *-subalgebra of A. The elements $(a^*a - t)_+$ in the positive part of this *-subalgebra are properly infinite. By Lemma 2.5.17(ii) this implies that A is purely infinite.

(iii): Let $A_1, A_2, \ldots$ a sequence of purely infinite C^*-algebras, $a_n \in A_n$ positive contractions (possibly $a_n = 0$), and $\varepsilon > 0$.

For $n \in \mathbb{N}$ with $a_n \neq 0$ there exist by Lemma 2.5.17(i) elements $d_1^{(n)}, d_2^{(n)} \in A_n$ with

$$\|d_k^{(n)}\|^2 \leq 2\|a_n\|/\varepsilon$$

and

$$(d_j^{(n)})^* a_n d_k^{(n)} = \delta_{j,k}(a_n - \varepsilon)_+ .$$

Let $a := (a_1, a_2, \ldots)$ and $d_k := (d_k^{(1)}, d_k^{(2)}, \ldots)$ $(k = 1, 2)$, where we put $d_k^{(n)} = 0$ if $a_n = 0$. Then $d_k \in \ell_\infty(A_1, A_2, \ldots)$ and $d_j^* a d_k = \delta_{j,k}(a - \varepsilon)_+$. Hence, each positive contraction in $\ell_\infty(A_1, A_2, \ldots)$ is properly infinite.

The ultrapower A_ω is purely infinite by Part (i), because it is a quotient of $\ell_\infty(A)$.

(iv): If D is a hereditary C^*-subalgebra of A, $a \in D$ and a is properly infinite in A, i.e., $a \in I_A(a)$, then a is properly infinite in D, i.e., $a \in I_D(a)$, because $I_D(a) = I_A(a) \cap D$ by Lemma 2.5.3(ii). Hence, D is purely infinite if A is purely infinite.

(v): Clearly, a C^*-algebra B is purely infinite if there exists a purely infinite C^*-algebra D and a *-isomorphism from D onto B,

Two C^*-algebras B_1 and B_2 are Morita equivalent (in the category of C^*-algebras), if there exists a C^*-algebra A and *-monomorphisms $\varphi_k \colon B_k \to A$ with $k \in \{1, 2\}$ such that $D_k := \varphi_k(B_k)$ are *full* hereditary C^*-subalgebras of A. Here "*full*" means that A is the closure of the ideal in A generated by D_k.

We know from Part (iv) that a hereditary C^*-subalgebra D of A is purely infinite if A is purely infinite.

It follows that the invariance of pure infiniteness under Morita equivalence is equivalent to the following statement:
C^*-algebras A are purely infinite if they contain a full hereditary C^*-subalgebra D that is purely infinite.

We show more generally that a purely infinite hereditary C^*-subalgebra D of A generates a purely infinite closed ideal J of A.

Let D a purely infinite hereditary C^*-subalgebra of A and let $J := \overline{\mathrm{span}(ADA)}$ the ideal J of A generated by D.

Let $a \in J_+ \subseteq A_+$ and $\varepsilon > 0$. There exist $b_1, \ldots, b_n \in D_+$ and $f_1, \ldots, f_n \in A$ with

$$\| \sum_{k=1}^{n} f_k^* b_k f_k - a \| < \varepsilon/2\,.$$

It follows for $b := b_1 + \cdots + b_n \in D_+$ that $a \leq \varepsilon/2 + \sum_{k=1}^n f_k^* b f_k$, calculated in $A + \mathbb{C}1 \subseteq \mathcal{M}(A)$. Let $\gamma := \| \sum_{k=1}^n f_k^* f_k \|$ and $\nu := \varepsilon/(2 + 3\gamma)$.

By assumption on D, $b \in D_+$ is properly infinite in D. Thus, $b \otimes 1_n \precsim b$ inside D, and there exist $e_1, \ldots, e_n \in A$ with $e_j^* b e_k = \delta_{jk}(b - \nu)_+$. The element $g := \sum_k e_k d_k$ satisfies $g^* b g = \sum_{k=1}^n d_k^* (b - \nu)_+ d_k$ and $a \leq \varepsilon/2 + g^* b g$.

By Lemma 2.1.9, there exists a contraction $h \in A$ such that $(a - \varepsilon)_+ = z^* z$ for $z := (\sum_k (b - \nu)_+^{1/2} d_k) h$.

Since $zz^* \in D$ and $z^* z = (a - \varepsilon)_+$ we get $[zz^*] = [z^* z] = [(a - \varepsilon)_+] \leq [a]$ and $[zz^*] + [zz^*] \leq [zz^*] \leq [a]$. It follows that $[(a - \varepsilon)_+] + [(a - \varepsilon)_+] \leq [a]$ for all $\varepsilon > 0$. But this is equivalent to $[a] + [a] \leq [a]$, i.e., to $a \oplus a \precsim a$ in J, and a is properly infinite in J. This shows that for each $a \in J_+ \setminus \{0\}$, and shows that J is purely infinite.

Alternatively expressed: Take $a \in J_+$ and verify that for each $\varepsilon \in (0, \|a\|)$ the element $(a - \varepsilon)_+$ is M-vN-equivalent to some element in D_+. The use that proper infiniteness is invariant under M-vN equivalence.

(vi): Let $J \triangleleft A$ a purely infinite closed ideal with purely infinite quotient A/J, and let $a \in A_+$.

We can replace A by its hereditary C^*-subalgebra $D := \overline{aAa}$. Then still $D \cap J$ and $D/(D \cap J) \cong \overline{\pi_J(a)(B/J)\pi_J(a)}$ are purely infinite by Parts (i) and (iv). Then

aDa is dense in D and $\pi_J(a)$ is properly infinite in $D/(D\cap J)$. Thus, Lemma 2.5.15(iii) applies and a is properly infinite in A.

This applies to each $a \in A_+$ and shows that A is purely infinite. □

We underline that some proofs of similar permanence properties for the classes of weakly purely infinite and of strongly purely infinite C^*-algebras are much more involved. The latter class of C^*-algebras is relevant for our classification. Up to now (2019) it is unknown if this three infiniteness properties (locally, weakly, pi(1)) are different or not, even in the case of separable nuclear C^*-algebras. It is also unknown if $\mathrm{C}([0,1], A)$ p.i. if A is p.i.

QUESTION 2.5.20. Suppose that A is a C^*-algebra with properly infinite unit element and let $a \in A$ a contraction.

Does there exist isometries $s, t \in A$ with $3\|s^*at\| < 2$?

(The latter is the "squeezing property" of Definition 4.2.14.)

`Compare next with Proof of Prop. 4.2.15(b) and with Prop. ??!!`
`New references give different results??`

The property that for each contraction $a \in A$ there exist isometries $s, t \in A$ with $3\|s^*at\| < 2$ is equivalent to the "*squeezing*" Property (sq) for A, cf. Definition 4.2.14.

It is easy to see that it holds for all unital *strongly* purely infinite C^*-algebras, because it is true for all contractions $a \in A$, if and only if, the positive 2×2-matrix $M := [b_{jk}]$ with entries $b_{11} := b_{22} := 1$ and $b_{21}^* = b_{12} := a$ can be approximately diagonalized with help of a diagonal matrix $D := \mathrm{diag}(d_1, d_2)$ up to $\varepsilon \in (0, 1/2)$, i.e.,

$$\| 1_2 - D^*MD \| < \varepsilon .$$

The Lemma 4.2.13 allows to show that all unital purely infinite (not necessarily simple) C^*-algebras A have the squeezing property, cf. Part (b) of Proposition 4.2.15.

The Property (sq) alone does not imply that a properly infinite unital C^*-algebra A is purely infinite, e.g. take $A := \mathcal{M}(B)$ for $B := \mathrm{C}([0,1], \mathbb{K})$ or $B := M_{2^\infty} \otimes \mathbb{K}$. In the case $B := M_{2^\infty} \otimes \mathbb{K}$ the corona $\mathrm{Q}(B)$ has Property (sq) and is residually antiliminary, but is *not weakly p.i.*.

Question concerning Property (sq):
What happens in case of "properly" infinite non- p.i. *simple* unital A ? Examples? Relations to (Q.1) in real rank zero case?

`Optional one could attempt to use`
`--e.g. in case of linearly ordered ideal lattice --`
`following Definition 2.5.21.`
`But it could be that it is equivalent to ''residually antiliminary''`

Definition 2.5.21. We call a C^*-algebra B "**locally infinite**" if for every pure state ρ on B and each hereditary C^*-subalgebra D of B with $\|\rho|D\| = 1$ there exists a contraction $d := d(D, \rho) \in D$ with $\rho(d^*d) = 1$, $\rho(dd^*) = 0$ and $(dd^* - 1/3)_+ \leq d^*d$.

Where is here really "infiniteness" ??? What is it in case of simple C*-algebras with RR=0?

Definition 2.5.21 implies:

A "locally infinite" C^*-algebra B is **residual antiliminary** in the sense of Definition 2.7.2, because the property of "local infiniteness"

(i) passes to each non-zero quotient and every non-zero hereditary C^*-subalgebra, and
(ii) implies that a pure state ρ on a non-zero hereditary C^*-algebra D of B can not be a non-zero character on D.

It is not difficult to see that for pure states ρ on $D \subseteq B$ ...

Does "l.p.i." or "pi(n)" imply this? (Both pass to hereditary C^*-subalgebras and quotients!)

If $E \subseteq D$ is hereditary, stable and σ-unital with $\rho(E) \neq \{0\}$ then there exists an element $d := d(\rho, D) \in E$ with the properties in Definition 2.5.21...

6. Cases: many projections, linear ordered ideal-lattice ...

`Still to be worked out in detail!!!`

Desired Main Result:

If A has the property that for each closed ideal $J \neq A$ and any non-zero hereditary C^*-subalgebra D of A/J contains a (non-zero) projection $p \in D$, then A is strongly purely infinite, if and only if, A is locally purely infinite, ...

Definition 2.6.1. We say that a C^*-algebra A is *rich of projections* if, for each closed ideal $J \neq A$ of A and non-zero hereditary C^*-subalgebra D of A/J, there exists a non-zero projection $p \in D$.

In particular, A has then the "small projections" property (SP) that requires that every non-zero hereditary C^*-subalgebra D of A contains a non-zero projection.

Among the C^*-algebras that are "rich of projections" are the C^*-algebras A with *real rank zero*, a property that is equivalent to the property that each hereditary C^*-subalgebra D of A contains an approximate unit of D consisting of projections in D, cf. [**73**, thm. 6.5.6].

(We do not know if C^*-algebras that are "rich of projections" have real rank zero, because projections usually do not lift to projections.

It is also not clear if every non-zero hereditary C^*-subalgebra $D \subseteq A$ contains an approximate unit for D consisting of convex combinations of projections ...)

LEMMA 2.6.2. *Let A a unital C^*-algebra. If for each pure state ρ of A there exits a stable C^*-subalgebra $D \subseteq A$ with $\rho(D) \neq \{0\}$ then there exists $n \in \mathbb{N}$ such that $1_A \otimes 1_n \in A \otimes M_n$ is properly infinite in $A \otimes M_n$.*

If there exists a stable C^*-subalgebra $D \subseteq A$ that generates A (in the sense that the linear span of $A \cdot D \cdot A$ is dense in A), then 1_A itself is properly infinite.

PROOF. The algebraic union V of the upward directed family of finite sums of stably generated ideals of A is an ideal of A that is dense in A by its definition, because V can not be in the kernel of any pure state. It implies that $1 \in V$. Thus, 1 is contained in an ideal that is a finite sum of stably generated ideals, i.e., there exists – by assumptions and Part (ii) of Lemma 2.1.7 – stable hereditary C^*-subalgebras $D_1, \ldots, D_n \subset A$ and elements $d_j \in A$ $(j = 1, \ldots, n)$ with $d_1^* d_1 + \ldots + d_n^* d_n = 1$ and $d_j d_j^* \in D_j$.

The $n \in \mathbb{N}$ with this property can be chosen minimal with the property that the sum of ideals generated by the D_j contains 1, because if $d_j d_j^*$ and $d_k d_k^*$ both are in a closed ideal J that is generated by some stable C^*-subalgebra D, then one can find $d \in A$ with $dd^* \in D$ and $d^* d = d_j^* d_j + d_k^* d_k$, because $\mathcal{M}(D)$ contains isometries $S_1, S_2 \in \mathcal{M}(D)$ with $S_1^* S_2 = 0$.

The stability of the D_j causes that its two-sided multiplier algebra $\mathcal{M}(D_j)$ contains a copy B_j of $\mathcal{E}_{2n}$ generated by $2n$ isometries $S_{jk} \in B_j$, $k \in \{1, \ldots, 2n\}$, with mutually orthogonal ranges:
$S_{jk}^* S_{j,\ell} = \delta_{k,\ell} 1_j$, where 1_j denoted the unit element of $\mathcal{M}(D_j)$.

Consider the two $n \times n$ matrices $T_0 := [T_{j,k,0}]$ and $T_1 := [T_{j,k,1}]$ in $M_n(A)$ with entries $T_{j,k,0} := S_{j,k} d_k$ and $T_{j,k,1} := S_{j,(k+n)} d_k$ for $j, k \in \{1, \ldots, n\}$ then $T_x^* T_y = \delta_{x,y} \cdot 1_n$ for $x, y \in \{0, 1\}$.

Thus, $1_n \in M_n(A)$ is properly infinite.

In case that a single hereditary C^*-subalgebra D generates A then there exist $d_1, d_2 \in A$ with $d_1 d_1^* + d_2 d_2^* \in D$, $d_1^* d_2 = 0$ and $d_1^* d_1 = d_2^* d_2 = 1$. □

LEMMA 2.6.3. *The class of C^*-algebras $A, B, \ldots$ that are rich of projections is invariant under following operations:*

(i) *passage to non-zero hereditary C^*-subalgebras $D \subseteq A$,*
(ii) *passage to non-zero quotients A/J,*
(iii) *extensions $0 \to A \to E \to B \to 0$,*
(v) *building inductive limits,*
(iv) *passage to a Morita equivalent algebras. In particular A is rich of projections, if and only if, $A \otimes \mathbb{K}$ is rich of projections.*

Each C^*-algebra A that is rich of projections is the inductive limit of the net of separable C^*-subalgebras that are rich of projections and are relatively weakly injective in A.

PROOF. `to be filled in` ?? □

PROPOSITION 2.6.4. *Suppose that a C^*-algebra A is rich of projections in the sense of Definition 2.6.1. Then the following properties (i)–(iii) are equivalent:*

(i) *For each closed ideal $J \neq A$ and nonzero hereditary C^*-subalgebra $D \subseteq A/J$ the algebra D contains a (non-zero) infinite projection $q \in D$.*
(ii) *A is locally purely infinite in sense of Definition 2.0.3.*
(iii) *A is purely infinite, cf. Definition 1.2.1.*

The C^*-algebra A is strongly purely infinite in sense of Definition 1.2.2, if A is purely infinite and has the stronger property that, for each closed ideal J of A and each hereditary C^*-subalgebra $D \subseteq A/J$ there exists a projection $p \in A \setminus J$ with $\pi_J(p) \in D$.

We do not know if for separable C^*-algebras that are "rich of projection" in the fairly weak sense of ?????

Compare with Proposition 2.6.5!

Notice that Part (i) of Proposition 2.6.4 is just the the definition of J. Cuntz for pure infiniteness.

PROOF. Obviously (iv) implies (iii). Part (iii) is equivalent to properties pi(1) and pi-1 on A by Corollary 2.5.6, and property pi(n) implies that A is locally purely infinite, i.e., Part (ii).

⇐ References?

(ii)⇒(i):

(i)⇒(iii):

(iii)⇒(ii):

Still not understood... (iii)⇒(iv): Perhaps wrong ?? **??** □

From (ii) to (i):

First step:

Suppose that A is is locally purely infinite, i.e., for each non-zero $b \in A_+$ and pure state λ on $E := \overline{bAb}$ there exists a C^*-morphism $\psi \colon \mathrm{C}_0(0,1] \otimes \mathbb{K} \to E$ with $\lambda \circ \psi \neq 0$.

We show that it implies that, for each closed ideal $J \neq A$ of A, every nonzero hereditary C^*-subalgebra $D \subseteq A/J$ and every non-zero projection $p \in D$, there exists $n \in \mathbb{N}$ such that $p \otimes 1_n$ is properly infinite in $D \otimes M_n$. (The multiplicity n could depend here from p, D and J.)

Let J, $D \subseteq A/J$ given and $p \in D \subseteq A/J$ a non-zero projection. We are going to show that for each pure state ρ on the unital C^*-algebra $pDp = p(A/J)p$, – i.e., with $\rho(p) = 1$ –, there exist a stable C^*-subalgebra $G \subseteq pDp$ with $\rho(G) \neq \{0\}$.

By Lemma 2.6.2 it follows that the upward directed net of finite sums of stably generated ideals of pDp must contain p and this implies that $p \otimes 1_n$ is properly infinite in $pDp \otimes M_n$ for suitable $n \in \mathbb{N}$.

There is a nonzero contraction $b \in A_+$ with $\pi_J(b) = p$ and a pure state and $\rho\colon A/J \to \mathbb{C}$ with $\rho(p) = 1$. The state $\xi := \rho \circ \pi_J$ satisfies $\xi(b) = 1$ and is a pure state on $F := \overline{bAb}$. Since A is w.p.i., there exists a C^*-morphism $\psi\colon \mathrm{C}_0(0,1] \otimes \mathbb{K} \to F$ with $\xi \circ \psi \neq 0$. Thus $\rho \circ (\pi_J \circ \psi) \neq 0$. Now the Lemma 2.6.2 applies, and shows that there exists $n \in \mathbb{N}$ such that $p \otimes 1_n$ is properly infinite in $pDp \otimes M_n$.

We combine this with the assumption that A is "rich of projections" in sense of Definition 2.6.1, ...

Let $0 \neq a \in A_+$ and $J \subseteq A$ a closed ideal of A with $a \notin J$, $\delta \in (0, \|\pi_J(a)\|)$, then let $E := \overline{(a-\delta)_+ A (a-\delta)_+}$ and $D := \pi_J(E) \cong E/(E \cap J)$. Then D is a non-zero hereditary C^*-subalgebra of A/J. By the general pre-assumption that A is "rich of projections", there exists a non-zero projection $q \in D \subseteq A/J$.

The above given arguments show that, for a pure state ρ on A/J with $\rho(q) = 1$ there exists a nonzero stable C^*-subalgebra of qDq given by the image of a non-zero morphism $\psi\colon \mathrm{C}_0(0,1] \otimes \mathbb{K} \to qDq = q(A/J)q$ with $\rho \circ \psi \neq 0$. Now we can consider the non-zero hereditary C^*-subalgebra $F := \overline{cAc}$ for some positive contraction $c \in A_+$ with $\pi_J(c) = \psi(f_0 \otimes p_{11})$. Then $G := \pi_J(F)$ is a hereditary C^*-subalgebra of A/J.

By assumption that ... 12.11.2018, not clear which one...

Let $b \in E_+$ a contraction with $\pi_J(b) = p$ and ρ a pure state on A/J with $\rho(p) = 1$ and $\rho(b) = \|b\| = 1$, i.e., there is a pure state ρ_1 on A/J with $\rho := \rho_1 \circ \pi_J$ and $\rho_1(p) = 1$.

`sort best from below:`

By Part (ii), ... there exists C^*-morphism $\psi\colon \mathrm{C}_0(0,1] \otimes \mathbb{K} \to F := \overline{bAb} \subseteq E$ with $\rho \circ \psi \neq 0$.

Let $F := \overline{bAb} \subseteq A$, then $\pi_J(F) = pDp$. By the assumption that A is locally purely infinite there exist a non-zero C^*-morphism $\psi\colon \mathrm{C}_0((0,1], \mathbb{K}) \to F$ with $\rho \circ \psi \neq 0$.

Thus, $\pi_J \circ \psi$ is non-zero, and its image is a non-zero stable hereditary C^*-subalgebra of $\pi_J(F) \subseteq pDp$.

Define $f_0 \in \mathrm{C}_0(0,1]$ by $f_0(t) := t$, and let $H := \psi(f_0 \otimes p_{11})$. Then $G := \overline{H \cdot A \cdot H)} \subset F$ is a hereditary C^*-subalgebra of A with $\pi_J(G) = \overline{HAH}$ ...

`check again`

Then the hereditary C^*-subalgebras of $pDp = \pi_J(F)$ defined by $\overline{\pi_J(G) D \pi_J(G)}$ is a stable hereditary C^*-subalgebra of pDp.

It contains the nonzero hereditary C^*-subalgebra generates a non-zero stable C^*-subalgebra of pDp.

Has then A then the additional property that for each closed ideal J of A and hereditary C^*-subalgebra D of A with $\pi_J(D) \neq \{0\}$ there exists a projection $p \in D$ with $\pi_J(p)$ infinite or zero?

Answer unknown! The assumptions say only that that there is a non-zero projection $q \in \pi_J(D)$. But it is not clear under which circumstances it can be lifted to a projection in $p \in D$ ...

Then A has the property that for each pure state ρ on $\pi_J(D) \subset A/J$ there exists a properly infinite projection $p \in D$ with $\rho(p) \neq 0$?

For separable C^*-algebras A the prime and primitive ideals are the same, cf. [**616**, Prop. 4.3.6]. Thus, if one can reduce a property to (suitably selected) separable C^*-subalgebras and pure states, then separating pure states (from zero) is the same as separating factorial states ...

It seems that one must require also that $\rho(p) \neq 0$ for a given nonzero pure state ρ on D ??

Or that $p = \pi_J(q)$ comes from a projection q

No! It is inclusive in case of pure state!:

One get this, if ρ is any nonzero state on A with $\rho(J) = \{0\}$ and with kernel $I \geq J$ of the cyclic representation defined by ρ. If $\rho|D \neq 0$, then $\pi_I(D) \neq 0$, and $\pi_I(D)$ is a non-zero hereditary C^*-subalgebra of A/I. If $q \in A$ is a nonzero projection with

with by passing to $E := \pi_I(D)$

PROPOSITION 2.6.5. *If a C^*-algebra A has real rank zero, then following properties of A are equivalent:*

(i) *A is p.i.*
(ii) *For every non-zero projection $p \in A$ there exist partial isometries $u, v \in A$ with $u^*u = v^*v = p$ and $uu^* + vv^* \leq p$.*
(iii) *Every quotient of A is purely infinite in the sense of Cuntz, i.e., for every hereditary C^*-subalgebra D and every closed ideal J of A which does not contain D, there is a (non-zero) infinite projection in $D/(D \cap J)$.*
(iv) *A is locally purely infinite.*
(v) *A_∞ has no non-trivial l.s.c. quasi-trace.*

`Compare with Proposition 2.6.4!`

See [**93**, thm. 4.17] for the equivalence of (iii) and (iv). The equivalence of (i), (ii) and (iii) follows from Corollaries 2.15.7(ii) and 2.15.8. Recall here that an infinite projection p is not necessarily properly infinite. It can be shown to be properly infinite only if all $\pi_J(p)$ are infinite or zero for all closed ideals J of A.

PROPOSITION 2.6.6. *If A is a C^*-algebra with linear ordered lattice $\mathcal{I}(A)$ of closed ideals and A is locally purely infinite, then A strongly purely infinite.*

There exist at most one non-zero simple quotient A/J and at most one non-zero simple closed ideal $I \subseteq A$.

If, in addition, A is separable (or σ-unital = A has a strictly positive element) and has no unital quotient, then A is stable.

There are/can exist non-simple unital or stable simple quotients, and there can be non-simple unital quotients:

Consider $\mathcal{L}(\ell_2(N))$ or any extension of a simple separable C^*-algebra by the compact operators.

DEFINITION 2.6.7. A C^*-algebra B has the **small projection property** if every non-zero hereditary C^*-subalgebra D of B contains a non-zero projection.

(If this happens also for all quotients, i.e., for non-zero hereditary $D \subseteq B/J$, then we say that B is *rich of projections*, cf. Definition 2.6.1.)

Suppose that a C^*-algebra B has the small projection property. Then we say that B "*has sufficiently many* **locally bounded dimension functions**" if for each non-zero contraction $a \in B_+$ and each non-zero projection $p \in \overline{aBa}$ there exists a *dimension function* $D\colon B \to [0, \infty]$ with

$$0 < D(p) < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} D((a - 1/n)_+) < \infty .$$

We do not require that the dimension function D is lower semi-continuous on $\overline{a\,B_+\,a}$ or bounded. In particular it may happen that $D(a) = \infty$.

Are above Lemma and next two Lemmas 2.6.8 and 2.6.9 used? cited? Otherwise put it into Appendix A/B. Eg. with title "Pure infiniteness for algebras that are rich of projections"?.

There are in next? section some study of projections in locally p.i. algebras. Compare!!!

Recall that $I(a)$ is the closed ideal of A defined for $a \in A$ in Definition 2.5.1 and Lemma 2.5.3(i).

LEMMA 2.6.8. *Let A a C^*-algebra.*

(i) *If $p \in A$ is a projection, then $I(p) \neq 0$, if and only if, pAp contains a non-unitary isometry, i.e., if and only if p is Murray–von-Neumann equivalent to a proper sub-projection $q \leq p$, $q \neq p$ in A.*

   In particular, a projection p is infinite with respect to the family of projections of A, i.e., there exists a non-zero projection $r \in A$ such that $p \oplus r$ is MvN-equivalent in $M_2(A)$ to $p \oplus 0$, if and only if, p is infinite as an element of A_+, i.e., if there exists non-zero $a \in A$ with $[p] + [a] \leq [p]$ in $\mathrm{Cu}(A)$.

(ii) `Exists also in other lemmas:` *If $q \in \overline{cAc}$ is a properly infinite projection, then $c \oplus q \precsim c$, i.e., $q \in I(c)$.*

(iii) `Exists also in other lemmas:` *If p is a properly infinite projection in A and $p \precsim c \in A_+$, then $p \in I(c)$, i.e., $c \oplus p \precsim c$. In particular c is infinite in A.*

(iv) `Where is it used?:` *If A has the "small projection property" of Definition 2.6.7 and has "sufficiently many locally bounded dimension functions" in the sense of Definition 2.6.7, then each non-zero element $a \in A_+$ is finite, i.e., $I(a) = \{0\}$, which says for $b \in A_+$ that $b \oplus a \precsim a$ implies $b = 0$.*

Proof. (i): Let $p \oplus x \precsim p$ with $0 \neq x \in A_+$ and $0 < \varepsilon < \min(\|x\|, 1)$, $\delta \in (0, \varepsilon)$ and $f = [f_1, f_2] \in M_{1,2}(A) \subseteq M_2(A)$ with $f^*(p-\delta)_+ f = (p-\varepsilon)_+ \oplus (x-\varepsilon)_+$. Notice that $(p-t)_+ = (1-t)p$ for $t \in [0,1]$. Then $(1-\delta) f_1^* p f_1 = (1-\varepsilon)p$ and $f_1^* p f_2 = 0$, but $(1-\delta) f_2^* p f_2 = (x-\varepsilon)_+ \neq 0$. Thus, $V := (1-\varepsilon)^{-1/2}(1-\delta)^{1/2} p f_1 p$ is a non-unitary isometry in pAp.

Conversely, if $V \in pAp$ is a non-unitary isometry, then $p \oplus (p - VV^*) = f^* p f$ for $f := [V, (p - VV^*)]$.

(ii): Let $c \in A_+$ and $q \in \overline{cAc}$ a properly infinite projection. Then the projection $q \in A$ is also properly infinite in the hereditary C^*-subalgebra $qAq \subseteq \overline{cAc}$ by Part (ii) of Lemma 2.5.3. Thus, we may suppose that c is a strictly positive element of A. Let $d := (1-q)c(1-q)$. Then $d + q$ is also a strictly positive element of $A = \overline{cAc}$, and therefore, $(d+q) \approx c$.

Take $u, v \in qAq$ that satisfy $u^*v = 0$, $u^*u = v^*v = q$ and define

$$f := [(d^{1/2} + u), v] \in M_{1,2}(A)\,,$$

then in $M_2(A)$ holds

$$(d+q) \oplus q \approx (d+q) \oplus q = f^*f \approx ff^* = d + uu^* + vv^* \leq d + q\,.$$

Thus, $c \oplus q \approx (d+q) \oplus q \precsim d + q \approx c$.

(iii): If $p^* = p^2 = p \precsim c$ for $c \in A_+$, then there is $d \in A$ with $d^*cd = (1/2)p$, and $q = 2c^{1/2}dd^*c^{1/2}$ is a projection in $\overline{cAc}$ with $q \sim p$. If p is properly infinite (in pAp), then q is properly infinite in $\overline{cAc}$. Thus $p \oplus c \approx q \oplus c \precsim c$ by Part (ii), i.e., $p \in I(c)$.

(iv): Let $a \in A_+$ and *suppose* that $I(a) \neq \{0\}$, i.e., that there exists non-zero $b \in A_+$ with $b \oplus a \precsim a$.

Then $a \oplus b \precsim a$ implies $b \precsim a$. The latter says that there exists, for each $\varepsilon \in (0, \|b\|)$, some $\gamma \in (0, \varepsilon)$ and $d \in A$ with $d^*(a-\gamma)_+ d = (b-\varepsilon)_+$.

Since A has the "small projection" property by assumptions in (iv), there exists a projection $q \in A$ with

$$0 \neq q \in \overline{(b-\varepsilon)_+ A (b-\varepsilon)_+}\,.$$

In particular

$$q \precsim (a \oplus q) \precsim (a \oplus b) \precsim a\,.$$

Since $q \in A$ is a projection with $q \precsim a$, there are $\delta \in (0, \|a\|)$, a partial isometry $v \in A$ such that $vv^* = q$ and $p := v^*v \in \overline{(a-\delta)_+ A (a-\delta)_+}$. It follows that $a \oplus p \precsim a$ inside $\overline{aAa}$, because

$$(a \oplus p) \sim (a \oplus q) \precsim (a \oplus (b-\varepsilon)_+) \precsim a\,.$$

It implies that, for each $\varepsilon \in (0, \|a\|)$, there exists $\gamma \in (0, \|a\|)$ and $d = [d_1, d_2] \in \mathcal{M}_{1,2}(A)$ with

$$d^*((a-\gamma)_+ \oplus 0)d = (a-\varepsilon)_+ \oplus p\,.$$

Now we apply the assumption on the (local) existence of locally bounded dimension functions: There is a dimension function $D\colon \bigcup_n M_n(\overline{aAa}) \to [0,\infty)$ with $D(p) > 0$ and

$$\theta := \sup_{\gamma>0} D\big((a-\gamma)_+\big) = \sup_k D\big((a-1/k)_+\big) < \infty\,.$$

We get, for each $\varepsilon \in (0, \|a\|)$, that

$$\theta \geq D\big((a-\gamma)_+\big) \geq D(p) + D\big((a-\varepsilon)_+\big)\,.$$

Since $\theta = \sup\{D((a-\varepsilon)_+)\,;\ \varepsilon \in (0, \|a\|)\,\}$ this contradicts $D(p) > 0$, and – by assumptions on A – that $a \oplus b \precsim a$. $\square$

LEMMA 2.6.9. *Let A denote an AW*-algebra, a Rickart algebra , or let A the C^*-subalgebra of B^{**} generated by the elements in the σ-up-hull of a C^*-algebra B in B^{**}.*

We need "finiteness" of the projections and that each element $a \in B_+$ has a "support projection" $p_a := 1 - P$, where $PB = R_a$ for $R_a = \{b \in B\,;\ ab = 0\}$. Moreover we need that $p_a = \bigvee\{p_{(a-t)_+}\,;\ t \in (0, \|a\|]\}$.

(i) *There is no (non-zero) properly infinite element in a finite Rickart C^*-algebra.*

(ii) *If A is an AW*-algebra and if $b \in A_+$ is properly infinite, then the support projection p_b of b is a properly infinite projection.*

(iii) *If, for every $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ and $a \in A_+$ with properly infinite support projection p_a such that ??????? and $(b-\varepsilon)_+ \leq a \leq (b-\delta)_+$, then b itself is properly infinite (in the AW*-algebra A).*

PROOF. (i): We use only that the Rickart C^*-algebras and AW*-algebras A have the "small projection" property because they have real rank zero as C^*-algebras.

And we claim that A has sufficiently many "locally bounded" dimension functions if A is a finite Rickart algebra or finite AW*-algebras A. Then Part (iv) of Lemma 2.6.8 applies.

`Are the` *`finite`* `2-q-traces on` $M_2(A)_+$ `separating from` 0
`for the non-zero projections` p `in`
`finite Rickart` *`C*`*`-algebra or AW*-algebras` A`??`

If A is a finite AW*-algebra, then there is a unique center-valued 2-quasi-trace $T\colon A_+ \to \mathcal{Z}(A)$ on a A_+ (cf. [**79**, p. 320]).

It extends to $T_2\colon M_2(A)_+ \to \mathcal{Z}(M_2(A)) \cong \mathcal{Z}(A)$.

The restrictions to (the positive parts of maximal) commutative C^*-subalgebras C of $M_2(A)$ are faithful, because $T_2(p) \neq 0$ for non-zero projections $p \in C$, see [**64**, Chap. 6]. Thus, $0 \leq T_2(a) \neq 0$ for $0 \neq a \in A_+$.

Let $\mathrm{diag}(a,p) \precsim \mathrm{diag}(a,0)$?????

Let $\chi\colon \mathcal{Z}(A) \to \mathbb{C}$ a character with $\chi(T(a)) = \|T(a)\|$. The dimension function $D\colon M_\infty(A) \to [0,\infty)$ corresponding to the 2-quasi-trace $\chi \circ T\colon A_+ \to [0,\infty)$ must satisfy $D(a) \neq 0$. Hence, a can not be properly infinite.

Check above and below arguments for Rickart algebras again.
The $\mathcal{Z}$-bimodule A is not a $\mathcal{Z}$-bundle ??!!!
The fibers are not simple in general.
And $[\chi \circ T]$ need not to be faithful on the fibers.

(ii): The element b is properly infinite inside the Rickart C^*-algebra or AW*-algebra $B := p_b A p_b$.

Now let $q \in \mathcal{Z}(B)$ a central projection that is finite in B.

Then every element of qB is finite, because otherwise there exists

– by the consideration in proof of Part (i) ??? –

an infinite projection r in qB and $q = r + (q-r)$ is infinite in $rBr + \mathbb{C}\cdot(q-r)$.

The element qb is zero or properly infinite in qB if b is properly infinite, by Lemma A.6.1(iii), ?????

Since qB is finite, it follows that $qb = 0$. Only 0 is orthogonal to b in $p_b B p_b$. This implies that the support projection p_b is properly infinite in the AW*-algebra A.

Same argument applies for Rickart algebras etc.

(iii): Suppose that, for every $\varepsilon > 0$, there exists $\delta > 0$ and $a \in A_+$ with properly infinite support projection p_a such that

?????? and $(b-\varepsilon)_+ \leq a \leq (b-\delta)_+$. Then

$$(b-\varepsilon)_+ \otimes 1_2 \leq \|b\|(p_a \otimes 1_2) \precsim p_a \leq p_{(b-\delta)_+} \leq \delta^{-1} b\,.$$

Thus, b is properly infinite, by Part (ix) of Lemma 2.5.3. □

End of Rickart-algebra considerations!!!
Cited below? ??

HERE ENDS PART 1 of Chp. 2

7. Residually antiliminary C^*-algebras

We define here the class of "residual antiliminary" C^*-algebras A and obtain for this particular class of (not necessarily simple) C^*-algebras A some relations between the there formally different definitions (and classes) of pure infiniteness.

In a not obvious sense the class of residually antiliminary C^*-algebras is the "strict opposite" of the class of C^*-algebras of type I, – that are also called "post-liminal" C^*-algebras ([39]).

Lemma 2.7.1. *Let $\mathbb{K}(\mathcal{H})$ denote the algebra of compact operators on a Hilbert space $\mathcal{H}$, and let $A \subseteq \mathbb{K}(\mathcal{H})$ a C^*-subalgebra of of $\mathbb{K}(\mathcal{H})$.*

If A is irreducible on $\mathcal{H}$, – in the sense that Ax is dense in $\mathcal{H}$ for every non-zero $x \in \mathcal{H}$ –, then it follows that $A = \mathbb{K}(\mathcal{H})$.

Proof. The C^*-subalgebras A of $\mathbb{K}(\mathcal{H})$ are generated by projections $p \in A$ of finite rank, because each $T^* = T \in A \subseteq \mathbb{K}(\mathcal{H})$ has discrete spectrum that related to operators $P \in A$ of finite rank. The transitivity of A on $\mathcal{H}$ of PAP in $P\mathbb{K}(\mathcal{H})P$ shows that $PAP = P\mathbb{K}(\mathcal{H})P$. In particular, this shows that A contains projections of rank $= 1$. The transitivity of A in $\mathcal{H}$ implies that A contains all rank-one projections in $\mathbb{K}(\mathcal{H})$. Since $\mathbb{K}(\mathcal{H})$ is generated as C*-algebra from its rank-one projection, this shows that $A = \mathbb{K}(\mathcal{H})$. □

Definition 2.7.2. A (non-zero) C^*-algebra A is **residual antiliminary** if, for each closed ideal $I \neq A$ of A and each non-zero $b \in A/I$ the hereditary C^*-subalgebra $E := \overline{b^*(A/I)b}$ of A/I is *not* commutative.

Is it not identical to the property that each non-zero hereditary C^*-subalgebra $D \subseteq A$ of A has not a character?
(= D has not a 1-dimensional quotient-algebra)
What is perhaps different?

Here there is also a good place for my separate notes on the ultra-power property: If $c_0(A_1, A_2, ...)$ has no quotient of dimension $= n$, then $\ell_\infty(A_1, A_2, \ldots)$ has no quotient of dimension $= n$. (Same with ultra -powers.)

A C^*-algebra B (as e.g. $B := A/I$) is **antiliminary** if B contains no non-zero hereditary Abelian C^*-subalgebra, i.e., $\overline{b^*Bb}$ is not Abelian for every non-zero $b \in B$, compare [**616**, sec. 6.1.1].

Notice that Definition 2.7.2 equivalently says that a C^*-algebra B is (only) *antiliminary*, if and only if, the hereditary C^*-subalgebra $\overline{b^*Bb}$ of B is not Abelian for every non-zero $b \in B$, and the Definition says that a C^*-algebra A is *residual antiliminary*, if and only if, for every non-zero $a \in A$ the hereditary C^*-subalgebra $D := \overline{a^*Aa}$ of A has no non-zero character.

List of topics to be considered:

(0.1) Each irreducible representation of A comes fro a pure state of ρ of A. If A is residually anti-liminary then the image $d_\rho(A)$ of every irreducible representation $d_\rho \colon A \to \mathcal{L}(\mathcal{H}_\rho)$ does not contain a non-zero compact operator. Here d_ρ is defined by (some suitable) pure state ρ on A.

[39] Could be called also "residual liminal" by some intuition based on old history of math ...

(Proof: A non-zero intersection of $d_\rho(A)$ with the algebra $\mathbb{K}(\mathcal{H}_\rho)$ of compact operators on $\mathcal{H}_\rho$ must necessarily be the image $d_\rho(I)$ of a non-zero closed ideal I of A, e.g. $I := (d_\rho)^{-1}(d_\rho(A) \cap \mathbb{K}(\mathcal{H}_\rho)$.

If the C^*-subalgebra $J := \mathbb{K} \cap d_\rho(A)$ of $\mathbb{K}$ acts irreducible on $\mathcal{H}_\rho$ then necessarily $J = \mathbb{K}$. Every

again an irreducible C*-subalgebra of the compact operators, because otherwise ?????????

This can only happen if $d_\rho(A)$ contains all compact operators.)

(0.2) If A has the property that $d_\rho(A)$ for every irreducible representation d_ρ of A does not contain a non-zero compact operator, then this property passes to all non-zero hereditary C^*-subalgebras $D \subseteq A$ of A.

In particular, then all non-zero hereditary C^*-subalgebras D of A have no characters.

And this implies that no non-zero hereditary C^*-subalgebra D of A has a non-zero quotient C^*-algebra D/J with finite dimension.

(????? Fist attempt of proof ... : ????)

Suppose that $d_0 \colon D \mapsto \mathcal{L}(H)$ is an irreducible *-representation such that $d_0(D)$ contains a non-zero compact operator.

Let $x_0 \in H$ with $\|x_0\| = 1$, and let ρ_0 denote the pure state of D given by $\rho_0(a) :=< ax_0, x_0 >$, and *suppose* that the corresponding irreducible representation $d_0 \colon D \mapsto \mathcal{L}(H)$ contains nonzero compact operators in its image.

The inverse image $I := (d_0)^{-1}(d_0(D) \cap \mathbb{K}(H))$ of the ideal of compact operators defines a closed ideal I of D.

The closed ideal J of A generated by I and I itself have the following properties:

(0.2. i) The set $I \subseteq A$ is also a hereditary closed C^*-subalgebra of A, because I is hereditary in D as every closed ideal of D. (And the hereditary C*-subalgebras have a lattice order given by its open support projections in A^{**}.)

(0.2. ii) We show that $I = J \cap D$, that the support projections of the hereditary C^*-subalgebras J, D and I of A are *open* projections in A^{**}, and that the support projection of J is in the centre of A^{**} and the support projections of I in the centre of D^{**} – here considered as hereditary W*-subalgebra of A^{**}.

(We obtain implicit that $p_J \cdot p_D = p_I$ for the corresponding open questions in A^{**}.)

Proof of $I = J \cap D$:

The ideal J is the closed *linear span* of AIA. The intersection $J \cap D$ is an ideal of D, obviously with $I \subseteq J \cap D$. The latter because $I = I \cdot I \cdot I$ (as for every C^*-algebra) and $I^3 \subseteq AIA \subseteq J$.

The closed linear span L of $A \cdot I$ is a left ideal of A with bi-polar (here = second conjugate) $L^{**} = A^{**}p_L$, where p_L is an open projection in A^{**} (with respect to $\sigma(A^{**}, A^*)$ - topology on A^{**}).

and more ???????? what is needed below ????

The element $d \in D$ is in $J \cap D$ if there exists for each $\varepsilon > 0$ elements $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in A$ and $c_1, \ldots, c_n \in I$ with the property

$$\|d - \sum_k a_k c_k b_k\| < \varepsilon .$$

If we use an approximate quasi-central unit (e_τ) of D and an approximate quasi-central unit (f_τ) of I, then the sum $\sum_k a_k c_k b_k$ can be replaced by suitable summands

$$\sum_k (e_\tau a_k f_\tau c_k f_\tau b_k e_\tau)$$

with suitable index τ such that also $e_\tau d e_\tau - d$ is sufficiently small. The elements $e_\tau a_k f_\tau$ and $c_k f_\tau b_k e_\tau$ are then in DAI respectively in IAD.

(0.3) Question:
Does for every separable C^*-subalgebra $B \subseteq A$ of A exists a separable C^*-subalgebra $C \subseteq A$, such that C has the property in (0.1) – for irreducible $d(C)$ –, if A has the property in (0.1)?

(0.4) Suppose that A is separable and satisfies the conditions (0.1). Can then $\mathcal{M}(A)$ have an irreducible representation of finite dimension? e.g. a character?

(0.5a) ?????????

(0.5b) If $\mathcal{M}(A)$ has no irreducible representation of dimension $\leq n$ then, A, $\ell_\infty(A)$ and $\ell_\infty(\mathcal{M}(A))$ (and all non-zero quotient C*-algebras and ideals of them) have no irreducible representation of dimension $\leq n$.

Suppose that one of A, $\ell_\infty(A)$ or $\ell_\infty(\mathcal{M}(A))$ have an irreducible representation ρ of dimension $k \leq n$ (here k = dimension of the corresponding Hilbert space).

Thus, suppose that one of the cases $\rho(A) = M_k$, $\rho(\ell_\infty(\mathcal{M}(A))) = M_k$, or $\rho(\ell_\infty(A)) = M_k$ with some $k \leq n$ appears.

In the first case ρ extend a normal unital morphism from A^{**} onto M_k with $k \leq n$. Since $A \subseteq \mathcal{M}(A) \subseteq A^{**}$, this shows that, in the case $\rho(A) = M_k$ with $k \leq n$, there must exists an irreducible representation of $\mathcal{M}(A)$ of some dimension $k \leq n$.

In the case of an irreducible representation $\rho\colon \ell_\infty(\mathcal{M}(A)) \to M_k$ it is always $\rho(1, 1, \ldots) = 1_k$ in M_k. The natural diagonal embedding Δ given by $y \in \mathcal{M}(A) \mapsto (y, y, \ldots) \in \ell_\infty(\mathcal{M}(A))$ is unital and $\rho \circ \Delta\colon \mathcal{M}(A) \to M_k$ is a unital C*-algebra morphism from $\mathcal{M}(A)$ into M_k.

(0.6) Question:
Suppose that A is separable and $\mathcal{M}(A)$ has an irreducible representation of finite dimension. Has then A_∞ (or A_ω) an irreducible representation of finite dimension?

Hope? Question: Are elements of $\mathcal{M}(A)/A$ expressible as sum of two commuting elements of A_∞? (perhaps then with a shift of one those 2 elements?)

Are separable parts of $\mathcal{M}(A)/A$ contained in sums of suitable separable ?????

Using a quasi-diagonal approximate unit of A, we can find 2 hereditary C^*-subalgebras of $\ell_\infty(A)$ that have irreducible ????

If $\mathcal{M}(A)$ has no irreducible representation of finite dimension, then $\mathcal{M}(A)/A$ (??? and ???? $\ell_\infty(A)$????) have no irreducible representation of finite dimension.

Suppose that $\ell_\infty(A)$ has no irreducible representation of finite dimension. Does it imply that $\mathcal{M}(A)$ has no irreducible representation of finite dimension (at least in case that A is σ-unital, or separable)?

(0.7) If any irreducible representation of A does not contain non-zero compact operators in its image, then each hereditary C^*-subalgebra of A has this property.

(1) l.p.i, weakly p.i., p.i., and strongly p.i. C^*-algebras are all residually antiliminary.

(1.0) It suffices to show that locally p.i. C^*-algebras are residual antiliminary. This can be seen as follows:

Foe each non-zero quotients $D/(D \cap J)$, where J is a closed ideal of A and $D \subseteq A$ is a hereditary C^*-subalgebra of A, there exist a pure state ρ on D with $\rho(D \cap J) = \{0\}$ and a positive contraction $d \in D_+$ with $\rho(d) = 1$ (by using Kadison transitivity theorem, or Lemma **??**). By Definition 2.0.3 of locally purely infinite C^*-algebras, there exists a C^*-morphism $h \colon \mathrm{C}_0((0,1], \mathbb{K}) \to \overline{dAd} \subseteq D$ with $\rho \cdot h \neq 0$. It shows that $D/(D \cap J)$ can not be one-dimensional (i.e., is $\ncong \mathbb{C}$). This shows that A must be residual antiliminary.)

The implications "strongly p.i. $\Rightarrow$ p.i. $\Rightarrow$ weakly p.i. $\Rightarrow$ locally p.i. " show then that the all sorts of "purely infinite" C^*-algebras are residually antiliminary.

(2) ??? The free product A of countably many algebras A_n, $A_n \cong \mathrm{C}_0((0,1], \mathbb{K}(\ell_2(\mathbb{N}))$ has no finite-dimensional irreducible representation, but $\ell_\infty(A)$ has a character (and moreover irreducible representations of every dimension n for each $n \in \mathbb{N}$).

This is not proven completely, because we have only an example where representations $\rho_n \colon A \to \mathbb{K}$ "converge" to a character on some *-subalgebra of $\ell_\infty(A)/c_0(A)$ that contains in its kernel an ideal J of A such that A/J is commutative.

But A itself has no character. Thus, ??? $\rho_\infty(A) \subseteq c_0(A)$???...

It is not clear if ρ_ω itself has abelian image on all elements of $\ell_\infty(A)/c_0(A)$.

It is not clear if this A is residually antiliminary, or can be used to construct a residually antiliminary C^*-algebra B with the property that $\ell_\infty(B)$ has a character.

(3) M_n has the $n \times n$-matrix $[a_{jk}]$ with $1 = a_{12} = a_{23} = \cdots = a_{n-1,n}$ (and zero at all other places) as generator.

(4) Let I_n denote the ideal of A that is generated by all homomorphisms of $\mathrm{C}_0((0,1], M_n)$ into A,

Then A/I_n has only irreducible representations of dimension $\leq n-1$:

Suppose that A/I_n has an irreducible representation $d\colon A/I_n \to \mathcal{L}(\mathcal{H})$ with dimension of $\mathcal{H}$ being $\geq n$. Let $P \in \mathbb{K}(\mathcal{H})$ a projection of rank $= n$, then, by the generalized Kadison transitivity Lemma **??** and semi-projectivity of $\mathrm{C}_0((0,1], M_n)$, cf. by Proposition **??**, there exist a C^*-morphism $h\colon \mathrm{C}_0((0,1], M_n) \to A$ with $d \circ \pi_{I_n}(f_0 \otimes 1_n) = P$. But $h(\mathrm{C}_0((0,1], M_n)) \subseteq I_n$ by definition of I_n and therefore $d \circ \pi_{I_n}(f_0 \otimes 1_n) = 0$ (with $f_0(t) = t$ on $(0,1]$). This contradicts the existence of $P \in \mathbb{K}(\mathcal{H})$ of rank $= n$ and shows that the dimension of $\mathcal{H}$ is $\leq n-1$.

Moreover the ideal I_n has no irreducible representations of dimension $< n$: Indeed, there would be an irreducible representation $d\colon A \to \mathcal{L}(\mathcal{H})$ with $k := \dim(\mathcal{H}) < n$ and $d|I_n \neq 0$. But, by definition of I_n, there exists a C^*-morphism $h\colon \mathrm{C}_0((0,1], M_n) \to I_n$ such that $d \circ h\colon \mathrm{C}_0((0,1], M_n) \to \mathcal{L}(\mathcal{H})$ is not $= 0$. But there does not exist a C^*-morphism from $\mathrm{C}_0((0,1], M_n)$ onto onto M_k for $k < n$.

What is with the (perhaps bigger) ideal J_n generated by contractions a in A with $\|a^{n-1}\| = 1$ and $a^n = 0$? (Notice that J_n contains I_n!)

The universal C^*-algebra $A_n := C^*(a^*, a\,;\ \|a^{n-1}\| = 1 \geq \|a\|, a^n = 0)$ has no irreducible representations of dimension $\leq n-1$.

$A_2 := C^*(b^*, b\,;\ \|b\| = 1, b^2 = 0)$ has a C^*-morphism φ into A_n give by $\varphi(b) := a^{n-1}$, or by $\varphi(b) := a^\ell$ with $\ell \geq n/2$.

There must be an irreducible representation on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ with $\|x\| = 1$, $\|a^{n-1}x\| = 1$, because $\|a^{n-1}\| = 1$. Consider the vectors $\{x, ax, a^2x, \ldots, a^{n-1}x\}$. They have all the norm $= 1$. It generates an a-invariant (i.e., with $aL \subseteq L$) linear subspace $L \subseteq \mathcal{H}$ of dimension $\leq n$. It contains the a-invariant subspaces a^kL generated by $\{a^kx, a^{k+1}x, \ldots, a^{n-1}x\}$ of dimension $\leq n-k$: $\{\mathbb{C} \cdot a^{n-1}x\} = a^{n-1}L \subseteq a^{n-2}L \subseteq \ldots \subseteq \ldots \subseteq a^2L \subseteq aL$.

If $k \in \{1, \ldots, n-2\}$ exists with $\mathrm{Dim}(a^kL) = \mathrm{Dim}(a^{k-1}L)$ then $a|a^{k-1}L$ is bijective on $a^{k-1}L$. Hence, $a^{k-1}L = a^kL$ and $a|a^kL$ is again bijective. This contradicts $a^nL = \{0\}$, and shows that linear span of $\{x, ax, a^2x, \ldots, a^{n-1}x\}$ is vector space of dimension $= n$.

In particular, $M_{n-1}(\mathbb{C})$ can not contain a contraction a with $\|a^{n-1}\| = 1$ and $a^n = 0$.

`Beginning from here, above considerations are not (!) integrated so far.`

Let $\mathcal{H}$ of dimension $m < n$, $a \in \mathcal{L}(\mathcal{H})$, with $\|a\| \leq 1$, $\|a^{n-1}\| = 1$ and $a^n = 0$. Can suppose that $C^*(a^*, a\,;\ \|a^{n-1}\| = 1 \geq \|a\|, a^n = 0)$ is there irreducible represented. The above considerations show that $\mathcal{H}$ must have dimension $\geq n$.

Find $x \in \mathcal{H}$ with $\|x\| = 1$, $\|a^{n-1}x\| = 1$. It must exists because $\|a^{n-1}\| = 1$. It follows that the vectors $x, ax, a^2x, \ldots, a^{n-1}x$ necessarily have all the norm $= 1$.

They can not be linearly independent, because the dimension of $\mathcal{H}$ is $= m < n$. Let $\mathcal{H}_0$ denote the linear span of $\{x, ax, a^2x, \ldots, a^{n-1}x\}$. It has some dimension $=: m < n$.

There is a linear map $T\colon \mathbb{C}^n \to \mathcal{H}$ defined by $T(e_1) := x$ and $T(e_k) := a^{k-1}x$ for $k = 2, \ldots, n$. The linear map T satisfies $TS_n = aT$ for the linear map $a \in \mathcal{L}(\mathcal{H})$ and the map $S_n \in \mathcal{L}(\mathbb{C}^n)$ defined by $S_n(e_k) := e_{k+1}$ $(k = 1, \ldots, n-1)$ and $S_n(e_n) := 0$. The map S_n satisfies $S_n^\ell(e_k) = e_{k+\ell}$ for $k + \ell \leq n$ and $S_n^\ell(e_k) = 0$ for $k + \ell > n$. In particular, $S_n^{n-1}(e_1) = e_n$, $S_n^{n-1}(e_k) = 0$ for $1 < k \leq n$ and $S_n^n = 0$. It implies $T(S_n(e_k)) = T(e_{k+1}) = a^k x = aT(e_k)$, for $k < n$ and $T(S_n(e_n)) = 0 = a^n x = aT(e_n)$, because $T(e_n) = a^{n-1}x$ and $a^n = 0$. Thus $TS_n = aT$ and $T\colon \mathbb{C}^n \to \mathcal{H}_0$ is surjective.

More?

Move it to some Appendix ??.

(5) The ideal I_n is identical with the hereditary C^*-subalgebra D_n of A that is build by convex combinations of all positive n-homogenous elements of A. (Because the D_n is invariant under inner automorphisms of A induced by unitaries in $\mathcal{U}_0(A + \mathbb{C}1)$.)

(6) Conjecture: If A is σ-unital and its asymptotic central sequence algebra $F(A) := (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$ has no characters, then A_ω is residual anti-liminary.

Attempt:
Let $h \in A_\omega$ a positive contraction. $D := \overline{hA_\omega h}$. Then there is a unital C^*-morphism from $F(A)$ into $\mathcal{M}(D)$ (? To be checked !!!?). The D can not have a character.

Thus A_ω is residual antiliminary.

(7) If A has only l.s.c. 2-quasi-traces that take the values 0 and $+\infty$ then A is residually antiliminary.

(8) All simple non-elementary C^*-algebras are residually antiliminary.

Equivalently expressed, cf. [**616**, sec. 6.1.1], B is "antiliminary" if B does not contain a non-zero "abelian element" $b \in B_+$. The element $b \in B_+$ is called "abelian" if bBb is a commutative subalgebra of B, cf. [**616**, sec. 5.5.1].

The following Proposition 2.7.7 and the Lemmata 2.7.3 and 2.7.13 will be used for the recognition and applications of residual antiliminary C^*-algebras.

Compare Remark A.5.8 and Lemma 2.1.7(o,v) with next Lemma and its proof.

LEMMA 2.7.3. *Let D a hereditary C^*-subalgebra of a C^*-algebra A. Then the map $J \mapsto J \cap D$ is a* surjective *map from the lattice $\mathcal{I}(A)$ of closed ideals of A onto $\mathcal{I}(D)$. In particular, $I = D \cap \overline{\mathrm{Span}(AIA)}$ for each closed ideal of D.*

PROOF. The latter identity can be seen easily with help of an approximate unit $\{e_\tau\}$ of D, by using that $DAD = D$, cf. Lemma 2.1.7(o). It shows the surjectivity of the map $J \mapsto J \cap D$, because $D \cap J = I$ for $J := \mathrm{Span}(AIA)$ if I is a closed ideal I of $D \subseteq A$. $\square$

Why should this non-zero character exist???

LEMMA 2.7.4. *Let D a C^*-algebra and $c \in D_+$ with $\|c\| = 1$ and suppose that*

$$\chi\colon E := \overline{cDc} \to \mathbb{C}$$

is a non-zero character.

Then $0 < \chi(c) \leq 1$, and, for each non-zero positive contraction $b \in D_+$ with $\|c - b\| < \chi(c)/3$ and $\delta \in (\|c - b\|, \chi(c)/3)$, the hereditary C^*-subalgebra $\overline{(b-\delta)_+ D(b-\delta)_+}$ of D has a non-zero character.

PROOF. By assumptions, the hereditary C^*-subalgebra $E := \overline{cDc}$ of D has non-zero character $\chi\colon E \to \mathbb{C}$ with kernel ideal $I \subset \overline{cDc}$, and $\|c\| = 1$. Since c is a strictly positive element of E it follows that $0 < \chi(c) \leq 1$.

Let $I \subseteq E$ the kernel ideal of the character χ, and let $J \subseteq D$ the closed ideal of D generated by I, i.e., $J := \overline{\mathrm{Span}(D \cdot I \cdot D)}$. It can be shown that $J \cap E = I$ by using an approximate unit of I, cf. Lemma 2.7.3.

It implies that $\pi_J(I) = \mathbb{C} \cdot p$ for some non-zero projection $p \in D/J$ that satisfies $p(D/J)p = \mathbb{C} \cdot p$ and $\pi_J(c) = \chi(c) \cdot p$. In particular, $\|\pi_J(c)\| = \chi(c)$.

The ideal K of D/J generated by the 1-dimensional hereditary C^*-subalgebra

$$\pi_J(E) = E/(J \cap E) = E/I = \mathbb{C} \cdot p = p(D/J)p$$

is equal to $\pi_J(\overline{\mathrm{Span}(DcD)})$. The ideal K is also equal to the closure of the linear span $\mathrm{Span}((D/J)p(D/J))$. Thus, K is isomorphic to the algebra of compact operators $\mathbb{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. The projection $p \in K$ has rank equal to one if considered as element in $\mathbb{K}(\mathcal{H}) \cong K$ because $pKp = \mathbb{C} \cdot p$.

Let $b \in D_+$ a with $\|b\| \leq 1$, $\|b - c\| < \chi(c)/3$ and $\delta \in (\|b - c\|, \chi(c)/3)$.
HERE XXX check again !!!

There exists a contraction $d \in D$ with $d^* c d = (b - \delta)_+$ by Lemma 2.1.9. It implies that in the ideal K of D/J the element $\pi_J((b-\delta)_+) = \chi(c)\pi_J(d)^* p \pi_J(d)$ has at most the rank one. It means that, if $\pi_J((b-\delta)_+) \neq 0$, then there exists a projection $q \in K \subseteq D/J$ with $q(D/J)q = \mathbb{C} \cdot q$ and

$$\pi_J((b-\delta)_+) = \|\pi_J((b-\delta)_+)\| q\,.$$

Thus, if $\pi_J((b-\delta)_+) \neq 0$ then the π_J defines a C^*-algebra epimorphism from the hereditary C^*-subalgebra $\overline{(b-\delta)_+ D(b-\delta)_+}$ of D onto the algebra $\mathbb{C} \cdot q$ in D/J. This shows that $\overline{(b-\delta)_+ D(b-\delta)_+}$ has a non-zero character if $\mathcal{I}(\pi_J((b-\delta))\| > 0$.

We finish the proof by showing that $\mathcal{I}(\pi_J((b-\delta))\| > 0$:
We have $\|(\pi_J((b-\delta)_+)\| = \|(\pi_J(b) - \delta)_+\| \geq \|\pi_J(b)\| - \delta$ and $\|\pi_J(b)\| \geq \|\pi_J(c)\| - \|\pi_J(c-b)\| \geq \chi(c) - \|c - b\|$. Thus,

$$\|(\pi_J((b-\delta)_+)\| \geq \chi(c) - \|c - b\| - \delta > \chi(c)/3\,.$$

In particular, $\|(\pi_J((b-\delta)_+)\| > 0$. □

The next Lemma 2.7.5 could be improved with respect to the possible estimates, but it is good enough for our applications.

LEMMA 2.7.5. *Let A a C^*-algebra and $B \subseteq A$ a C^*-subalgebra. If A contains a non-zero projection p with the properties $pAp = \mathbb{C} \cdot p$ and that the distance of p to B is less than $1/9$, i.e., $\operatorname{dist}(p, B) < 1/9$, then there exists a projection $q \in B$ with $\|p - q\| < 1$ and $qBq = \mathbb{C}q$.*

Moreover, there is a unitary $U \in \mathcal{U}_0(\mathcal{M}(A)) \cap (1 + A)$ with $q := U^*pU \in B$.

PROOF. Recall here that $\mathcal{U}_0(\mathcal{M}(A))$ denotes the in operator-norm connected component of 1 in the unitaries $\mathcal{U}(\mathcal{M}(A))$. There is a positive linear functional ψ on A with $\psi(a)p = pap$ for all $a \in A$, because $pAp = \mathbb{C} \cdot p$. It shows that ψ must be a pure state on A. Notice that there exists only one state on A with this property, and the property that the closed ideal $J \subseteq A$ of A generated by p is isomorphic to the compact operators $\mathbb{K}(\mathcal{H})$ of some Hilbert space $\mathcal{H}$, i.e., $J \cong \mathbb{K}(\mathcal{H})$.

By assumption, there exist $b \in B$ with $\|b - p\| < 1/9$. The selfadjoint $f := (1/2)(b^* + b) \in B$ is the real part of b and satisfies again $\|f - p\| < 1/9$. It says, for $\delta \in (0, 1/9 - \|s - p\|)$ and the unit $1 \in B^{**}$, that

$$-(1/9 - \delta)1 \;\leq\; p + f_- - f_+ \;\leq\; (1/9 - \delta)1\,.$$

Let χ a character on $C^*(f) \subseteq B \subseteq A$ with $\chi(f_-) = \|f_-\|$ and $\chi(f_+) = 0$. It extends to a pure state on B and then, – further on –, to a pure state on $\rho \colon A \to \mathbb{C}$ with the properties that $\rho(f_-) = \|f_-\|$ and $\rho(f_+) = 0$. If we apply it to above inequality, then we get $\rho(p) + \|f_-\| \leq (1/9 - \delta)_+$ and $0 \leq \rho(p) \leq 1$. Thus, $\|f_-\| < 1/9$ and positive part $c := f_+$ satisfies $\|c - p\| < 2/9$, and $c \geq 0$.

Let $\xi \in (\|c - p\|, 2/9)$. It satisfies the inequalities $\|c - p\| < \xi$ and, therefore, $1 - \xi < \|c\| < 1 + \xi$. In particular, $0 < \xi < \|c\|$ because $9\,\xi < 2$.

Lemma 2.1.9 gives a contraction $d \in A$ with $d^*pd = (c - \xi)_+ \neq 0$, because $\|c - p\| < \xi$ and $\|c\| > \xi$.

It implies that $q := (\|c\| - \xi)^{-1}(c - \xi)_+ \in B$ is a rank-one projection in the ideal $J \cong \mathbb{K}$ of A generated by p.

They imply following identities and estimates: $(\|c\| - \xi) \cdot q = (c - \xi)_+$, $1 - \xi < \|c\| < 1 + \xi$,

$$\|q - (c - \xi)_+\| = |1 - (\|c\| - \xi)| \;=\; 1 + \xi - \|c\| \;<\; 2\xi\,,$$

and

$$\|(c - \xi)_+ - p\| \;\leq\; \|c - p\| + \|c - (c - \xi)_+\| \;<\; 2\xi\,.$$

It gives the desired estimate:

$$\|q - p\| \;\leq\; \|q - (c - \xi)_+\| + \|(c - \xi)_+ - p\| \;<\; 4\xi \;< 1\,.$$

The inequality $\|p - q\| < 1$ implies for the two rank-one projections $p, q \in J \cong \mathbb{K}(\mathcal{H})$ that they are identical if they are linear dependent. But if they are not linear dependent then the support project Q of $p + q$ in J has rank two, i.e., $p, q \in QJQ \cong M_2$. Then there exists a unitary $V \in \mathcal{U}_0(QJQ)$ with $V^*pV = q$. The unitary $U := V + (1 - Q) \in 1_{\mathcal{M}(A)} + A$ has the proposed properties. □

Corollary 2.7.6. *Let A a C^*-algebra and $b_1, b_2, \ldots \in A_+$ a sequence contractions that converge to a contraction $c \in A_+$ with $\|c\| = 1$, i.e., $\lim_n \|c - b_n\| = 0$.*

If the hereditary C^*-subalgebras $\overline{(b_n - \rho)_+ A (b_n - \rho)_+}$ have no non-zero characters for every $n \in \mathbb{N}$ and every rational number $\rho \in (0, \|b_n\|)$, then the hereditary C^*-subalgebra $\overline{(cAc)_+}$ can not have a non-zero character.

Proof. If $\overline{cAc}$ has a non-zero character then infinitely many of the hereditary C^*-subalgebras $\overline{(b_n - \rho)_+ A (b_n - \rho)_+}$ must have a character, by Lemma 2.7.4. It would contradict the assumptions. □

Proposition 2.7.7. *Following properties of a C^*-algebras A are equivalent:*

(i) *A is residual antiliminary in the sense of Definition 2.7.2.*
(ii) *Each hereditary C^*-subalgebra D of A does not have a non-zero character.*
(iii) *Each non-zero hereditary C^*-subalgebra D of A does not have irreducible representations that contain non-zero compact operators in its image.*
(iv) *No irreducible representation $\rho\colon A \to \mathcal{L}(\mathcal{H})$ of A contains a non-zero compact operator in its image.*
(v) *There exists a norm-dense subset $S \subseteq \{a \in A_+ \,;\, \|a\| = 1\}$ with the property that for each $b \in S$ and each rational numbers $\rho \in (0, 1)$ the hereditary C^*-subalgebra $\overline{(b - \rho)_+ A (b - \rho)_+}$ of A has no non-zero character.*

Proof. We give indirect proofs for all implications, except for the trivial implications (ii)⇒(v) and (iii)⇒(iv).

(i)⇒(ii): Suppose that D is a hereditary C^*-subalgebra of A that has a non-zero character character $\chi\colon D \to \mathbb{C}$ and let I denote the kernel of χ. By Lemma 2.7.3 there exists a closed ideal J of A that satisfies $J \cap D = I$. Then $E := \pi_J(D) \cong \mathbb{C}$ is a hereditary C^*-subalgebra of A/J and there is a unique non-zero projection $p \in E$ with $E = \mathbb{C} \cdot p$. Then $E = \overline{p^*(A/J)p}$ is a non-zero commutative and hereditary C^*-subalgebra of the quotient A/J of A. It contradicts that A is residual antiliminary.

(ii)⇒(iii): Suppose that there exists a hereditary C^*-subalgebra D of A that admits an irreducible $\rho\colon D \to \mathcal{L}(\mathcal{H})$ with the property that $\rho(D)$ contains a non-zero compact operator $T \in \rho(D)$. Let $I \subset D$ denote the kernel of ρ.

Since irreducible representations of C^*-algebras are cyclic with respect to each non-zero vector, one can show that the non-zero C^*-subalgebra $\rho(D) \cap \mathbb{K}(\mathcal{H})$ of $\mathbb{K}(\mathcal{H})$ is an irreducible C^*-subalgebra of $\mathbb{K}(\mathcal{H})$. It implies that $\mathbb{K}(\mathcal{H}) \subseteq \rho(D)$. Thus, there exist a projection $p \in \mathbb{K}(\mathcal{H})$ of rank one with the property $p \in \rho(D)$. It implies $p\rho(D)p = \mathbb{C} \cdot p$. The C^*-subalgebra $E := \rho^{-1}(\mathbb{C} \cdot p) = \rho^{-1}(p\rho(D)p)$ of D is a hereditary C^*-subalgebra of D, – hence is also hereditary in A –, and $E/(I \cap E)$

is one-dimensional, i.e., E is a hereditary C^*-subalgebra of A that has a character. It contradicts that E can not have a character by (ii).

(iii)⇒(i): Suppose that A is not residual antiliminary in in the sense of Definition 2.7.2.

Then there exists a closed ideal $I \neq A$ of A and a non-zero element $b \in A/I$ such that the hereditary C^*-subalgebra $E := \overline{b^*(A/I)b}$ of A/I is commutative. Let $D := \pi_I^{-1}(E)$ and $\xi \colon E \to \mathbb{C}$ a non-zero character of E. Then D is a non-zero hereditary C^*-subalgebra of A, and $\chi := \xi \circ (\pi_I | D)$ is a non-zero character on D, contradicting Property (iii) on A.

(iii)⇒(iv): Consider $D := A$.

(iv)⇒(iii): Suppose that D is a hereditary C^*-subalgebra of A, $\rho_0 \colon D \to \mathcal{L}(\mathcal{H}_0)$ an irreducible representation and that $T \in \rho_0(D)$ is a non-zero compact operator on $\mathcal{H}_0$. Take $e \in D$ with $\rho_0(e) = T$. We can suppose that $e \geq 0$ and $T \geq 0$ by replacing them otherwise by e^*e and T^*T.

The irreducible representation ρ_0 of D "extends" to an irreducible representation $\rho_1 \colon A \to \mathcal{L}(\mathcal{H}_1)$ in the sense that there exists an isometry $I \colon \mathcal{H}_0 \to \mathcal{H}_1$ such that $\rho_1(d) \circ I = I \circ \rho_0(d)$ and $\rho_1(d)x = 0$ for all $x \in \mathcal{H}_1 \top I(\mathcal{H}_0)$ and for all $d \in D$. Thus, $e \in D \subseteq A$ satisfies that $\rho_1(e)$ is a non-zero compact operator on $\mathcal{H}_1$. This contradicts (iv).

(v)⇒(ii): The assumptions of Part (v) imply by Corollary 2.7.6 that $\overline{bAb}$ has no non-zero character for all $b \in A_+$ with $\|b\| = 1$.

It follows that each hereditary C^*-subalgebra D of A does not have a non-zero character:

Suppose that there exists a non-zero character ξ on D. Then there exists some $a \in D$ with $\xi(a) = 1$. Let $b := a^*a - (a^*a - 1)_+$, then $b \in D_+$, $\|b\| = 1$ and $\xi(b) = 1$. Thus, the restriction of ξ to $\overline{bAb}$ is a non-zero character. But this impossible. □

LEMMA 2.7.8. *A C^*-algebra B has no irreducible representation of dimension $\leq n$, if and only if, the closed ideal J_n of B generated by all elements $b \in B$ with the property $b^{m-1} \neq 0$ and $b^m = 0$ and $m \geq n$, is equal to B.*

If B has an irreducible representation of dimension $\geq n$ then B contains an element with $b^{n-1} \neq 0$ and $b^n = 0$.

If B contains an element with $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$, then B has an irreducible representation with dimension $\geq n$.

Thus, if B has only irreducible representations of dimension $< n$, then B can not contain b with $b^{n-1} \neq 0$ and $b^n = 0$.

If B does not contain elements $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$ then every irreducible representation of B has dimension $< n$.

Let J_n denote the closed ideal of B generated by all elements $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$.

(1) B/J_n has only irreducible representation on Hilbert spaces of dimension $\leq n$.

It is equivalent to the property that for every irreducible representation $\rho\colon B \to \mathcal{L}(\mathcal{H})$ of dimension $\dim(\mathcal{H}) > n$ there exists a contraction $b \in B$ with $b^{n+1} = 0$, $b^n \neq 0$ and $\rho(b) \neq 0$. (Because this implies that $\rho|J_n$ is an irreducible representation of the J_n.)

Proof of (1):
Let $\rho\colon B \to \mathcal{L}(\mathcal{H})$ an irreducible representation and $\dim(\mathcal{H}) > n$.

By (sharpened form of) the generalized Kadison transitivity theorem there exists C^*-algebra homomorphism $\varphi\colon \mathrm{C}_0((0,1], M_{n+1}) \to B$ such that $f \to \rho \circ \varphi(f)$ has kernel $\mathrm{C}_0((0,1), M_{n+1})$, and defines a (non-zero) C^*-morphism $\eta\colon M_{n+1} \to \mathbb{K}(\mathcal{H})$ of M_{n+1} into $\mathbb{K}(\mathcal{H})$. Let $d := e_{1,2}+e_{2,3}+\cdots+e_{n-1,n}$ Then $b := \varphi(f_0 \otimes d) \in B$ satisfies $b^{n-1} \neq 0$ and $b^n = 0$.

(2) Every irreducible representation $\rho\colon B \to \mathcal{L}(\mathcal{H})$ of dimension $\dim(\mathcal{H}) \leq n$ contains J_n in its kernel ideal.

This is equivalent to the property of J_n that says that J_n is contained in the intersection of the kernels of all irreducible representations $\rho\colon B \to \mathcal{L}(\mathcal{H})$ with $\mathrm{Dim}(\mathcal{H}) \leq n$.

To show this, we have to prove that there does not exist an irreducible representation $\rho\colon B \to \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ such that $\rho(b) \neq 0$ for some $b \in B$ in the given set of generators for J_n.

The generators of J_n are the elements $b \in B$ with the property $b^{n-1} \neq 0$ and $b^n = 0$.

Thus we have to determine: What is the dimension of a Hilbert space $\mathcal{H}$ if $\mathcal{L}(\mathcal{H})$ contains a contraction T with $T^n = 0$ and $T^{n-1} \neq 0$.

We show that for a Hilbert space $\mathcal{H}$ holds:

$\mathrm{Dim}(\mathcal{H}) \geq n$, if and only if, $\mathcal{L}(\mathcal{H})$ contains an operator of finite rank $T \in \mathcal{L}(\mathcal{H})$ with the property $T^n = 0$ and $T^{n-1} \neq 0$.

Equivalently this says that if $\mathrm{Dim}(\mathcal{H}) \leq n$ then there does not exists an operator $T \in \mathcal{L}(\mathcal{H})$ with $T^{m-1} \neq 0$ and $T^m = 0$ if $m > n$.

(There exists T with $T^{n-1} \neq 0$ and $T^n = 0$ if $\mathrm{Dim}(\mathcal{H}) = n$, but no T with $T^{m-1} \neq 0$ and $T^m = 0$ if $\mathrm{Dim}(\mathcal{H}) < m < \infty$)

The images $\mathcal{H}_k := T^k\mathcal{H}$ are linear subspaces for $k \in \{0, \ldots m\}$, where we let $T^0 := \mathrm{id}_{\mathcal{H}}$. They are closed subspaces of $\mathcal{H}$, because $\mathrm{Dim}(\mathcal{H}) \leq n$. Obviously $\mathcal{H}_{k+1} \subseteq \mathcal{H}_k$, and the $\mathcal{H}_k$ are invariant subspaces of T, i.e., $T\mathcal{H}_k \subseteq \mathcal{H}_k$.

If there is $k \in \{0, \ldots, m-1\}$ with $\mathcal{H}_k = \mathcal{H}_{k+1}$ and $\mathcal{H}_k$ then is $T|\mathcal{H}_k$ is a bijective linear map on $\mathcal{H}_k$. It implies that $T^\ell \neq 0$ for all $\ell \in \mathbb{N}$, if $\mathcal{H}_k = \mathcal{H}_{k+1}$ for some $k \in \{0, 1, \ldots, m-1\}$. It would contradict the assumption that $T^m = 0$.

Thus, $\mathrm{Dim}(\mathcal{H}_{k+1}) < \mathrm{Dim}(\mathcal{H}_k) \neq \{0\}$ for $k \in \{0, 1, \ldots, m-1\}$ and $\mathcal{H}_m = T^m\mathcal{H} = \{0\}$, because $T^{m-1} \neq 0$ and $T^m = 0$.

It follows that ????????????

Since $T^m = 0$, by assumption, this can not happen. Thus, $\mathrm{Dim}(T^{k+1}\mathcal{H}) < \mathrm{Dim}(T^k\mathcal{H})$ for all $k \in \{0, \ldots, m\}$. $\{0\} = \mathcal{H}_m$ and $\mathrm{Dim}(H_{k+1}) < \mathrm{Dim}(H_k)$ for $k \in \{0, 1, \ldots, m\}$.

It follows that $\mathrm{Dim}(Hi) \geq m - 1$???

(2) If B has an irreducible representation ρ of dimension $m > n$ then there exists an element

????????

If B/J_n has an irreducible representation $\rho \colon B/J_n \to M_n$

of dimension m or n here ?????

then the semi-projectivity of $\mathrm{C}_0((0,1], M_m)$ shows that there is a C^*-morphism $\psi \colon \mathrm{C}_0((0,1], M_m) \to B$ with $(\rho \circ \psi)(f) = f(1)$ for $f \in \mathrm{C}_0((0,1], M_m)$. Then $b := \psi(f_0 \otimes (p_{1,2} + p_{2,3} + \ldots + p_{n_1,n})$ satisfies $b^{m-1} \neq 0$ and $b^m = 0$. Thus, $m < n$ if $b \notin J_n$.

Need also to show that J_n has no irreducible representation of dimension $< n$ B/J_n has only irreducible representation of dimension $< n$. It is NOT clear if it is so!

We know only that if B has an irreducible representation $D \colon B \to M_k$ of dimension $k < n$ then B contains a contraction $b \in B$ with $\|b^\ell\| = 1$ for $\ell < k$ and $b^k = 0$, because B contains then an image $\psi(M_k(\mathrm{C}_0(0,1]))$ of $M_k(\mathrm{C}_0(0,1])$ by some C^*-morphism ψ with $D \circ \psi(f_0 \otimes 1_n) = 1_k$. In this (very special) situation one has moreover that $D(b)^\ell \neq 0$ for $1 \leq \ell < k$. The b is given by $\psi(f_0 \otimes Q)$ for $Q := \sum_{\ell=1}^{k-1} p_{\ell, 1+\ell}$.

But $b \in B$ with this properties can be found if B has any irreducible representation D of dimension $\geq k$. Simply by considering the the hereditary C^*-subalgebra that maps onto $M_k \oplus 0_{n-k} \subseteq M_k \oplus M_{n-k} \subseteq M_n$.

If B does not contain an element $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$ then B can not have any irreducible representation of dimension $\geq n$.

The problem is, if the relations are "liftable", i.e., if B/J does not contain an element $c \in B/J$ with $c^{n-1} \neq 0$ and $c^n = 0$

(in particular B/J can not have an irreducible representation of dimension $\geq n$),

is then every element $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$ contained in J?

The problem is, that it can be that $\pi_J(b) \neq 0$, but also $\pi_J(b^\ell) = 0$ for some $1 < \ell < n - 1$. This can happen in case $B = B_1 \oplus B_2$ if $b = b_1 \oplus b_2$ and

????. $b^n = 0$ for $n := \max(n_1, n_2)$ if

?????

REMARK 2.7.9. The property that the closed ideal $J_n(B)$, generated by all elements $b \in B$ with $b^{n-1} \neq 0$ and $b^n = 0$, is equal to B, like considered in Lemma 2.7.8 does not pass in general to $\ell_\infty(B)$ or its quotients, as e.g. the ultrapowers B_ω.

It is still not clear if this is a true counter examples!
Can a sequence of surjective (!) $C*$-morphisms $h_n : B \to \mathbb{K}$
produce a morphism from B_ω into $\mathbb{K}_\omega$ that is not surjective?

The ultrapower B_ω of the free product B of countably many copies of $\mathrm{C}_0((0,1], \mathbb{K}(\ell_2))$ could have a character, even if B has no irreducible representation of finite dimension?

It is only known that the restriction to $B \subset B_\omega$ is a character.

Is the ideal J_n different from the ideal generated by all n-homogenous positive elements?

Case $n = 2$, i.e., general $C^*(b^*, b)$ with $b^2 = 0$?

Then $C^*(b^*, b) \cong M_2 \otimes C^*(b^*b)$ by letting b correspond to $p_{1,2} \otimes (b^*b)^{1/2}$:

The polar decomposition $b = v(b^*b)^{1/2}$ in $C^*(b^*, b)^{**}$ has the property that $b = (bb^*)^{1/2}v$. In particular, $b^* = v^*(bb^*)^{1/2}$ is the polar decomposition of b^*, $v(b^*b)v^* = bb^*$ and $v^*(bb^*)v = b^*b$.

Let $f := (b^*b + bb^*)^{1/2} = (b^*b)^{1/2} + (bb^*)^{1/2}$ in $C^*(b^*, b)$. The partial isometry v commutes with f: $vf = v(v^*v)(b^*b)^{1/2} = b$ and $fv = (bb^*)^{1/2}v = b$. Notice that the *non-zero* values in the spectra of f and $(b^*b)^{1/2}$ are the same.

Thus, there is a C^*-morphisms ρ from $C^*(v^*, v) \otimes C^*(f)$ onto $C^*(b^*, b)$ with the property that $\rho(v \otimes f) = b$ and $\rho((vv^* + v^*v) \otimes f) = f$. Since $C^*(v^*, v) \cong M_2$, we get an isomorphism from $M_2(C^*(f))$ onto $C^*(b^*, b)$.

PROOF. Let $D_n \in M_n$ denote the matrix with entries $d_{k,k+1} = 1$ for $k = 1, \ldots, n-1$ and $d_{k,\ell} = 0$ on all other places. Then $D_n^{n-1} \neq 0$ but $D_n^n = 0$.

Let $J_n \subseteq B$ the closed ideal of B that is generated by all elements $b \in B$ with $b^n = 0$ and $b^{n-1} \neq 0$.

We show that B/J_n is sub-homogenous with irreducible *-representations of dimension $< n$, and that J_n has only irreducible representation of dimension $\geq n$.

Let $\rho \colon B \to M_m$ a *-morphism, and $b \in B$ with $b^{n-1} \neq 0$ but $b^n = 0$ with $\rho(b^{n-1}) \neq 0$...

Need: $C^*(b^*, b)$ is n-homogenous (or n-sub-homogenous C^*-subalgebra) if $b^n = 0$ and $b^{n-1} \neq 0$???

Suppose that $\rho \colon J_n \to \mathcal{L}(\mathcal{H})$ has an irreducible representation of dimension $m < n$, then exists an element $b \in B$ with $b^n = 0$ and $b^{n-1} \neq 0$, and $\rho(b) \neq 0$ ...

HERE IS SOMETHING wrong??

We need also to show that each irreducible representation with $\rho|J_n \neq 0$ has dimension $\geq n$.

This could be shown by proving directly that J_n is also generated by all n-homogenous elements of B.

At least, it is easy to see that the image of all non-zero C^*-morphisms $\psi\colon \mathrm{C}_0((0,1], M_n) \to B$ is contained in J_n, because $\psi(f_0 \otimes D_n)$ is contained in J_n, where D_n means the matrix with entries $d^{(n)}_{j,k} = 1$ for $k := j+1$ and $j = 1, \ldots, n-1$ and let $d^{(n)}_{j,\ell} := 0$ if $\ell - j \neq 1$. The element $f_0 \otimes D_n$ generates $\mathrm{C}_0((0,1], M_n)$ as a C^*-algebra, because $t \cdot D_n$ generates M_n as C^*-algebra for each $t \in (0,1]$ (Use here the variant of the classical Stone-Weierstrass theorem for type-I C^*-algebras, e.g. [**704**, cor. 4.7.8]).

The question is, if $b \in B$ with $b^n = 0$ and $b^{n-1} \neq 0$ generates an n-homogenous C^*-subalgebra?

What about $b - b^{n-1}$? Has it a good polar decomposition?

$(1-b)(b + b^2 + \ldots + b^{n-1}) = b - b^n$ and $b^n = 0$ if b is as above.

How about explicit construction?

$b(b^{n-1}\mathcal{H}) = \{0\}$.

Let $\rho\colon B \to \mathcal{L}(\mathcal{H})$ and irreducible with $\rho(J_n) = \{0\}$. Then the dimension of $\mathcal{H}$ is $< n$:

Suppose that the dimension of $\mathcal{H}$ is $\geq n$. Then there exists a subspace $L \subseteq \mathcal{H}$ of dimension n, a C^*-morphism λ from M_n into $\mathcal{L}(\mathcal{H})$ with $\lambda(1_n) = P_L$ the orthogonal projection onto L, and a C^*-morphism ψ from $\mathrm{C}_0((0,1], M_n)$ into B with the property $\rho \circ \psi = \lambda$ (by Kadison transitivity theorem and projectivity of cones over finite-dimensional C^*-algebras).

We take in B the element $C := \psi(f_0 \otimes D_n)$. It satisfies $C^{n-1} \neq 0$ but $C^n = 0$. Thus, $C \in J_n$ and $\rho(C) \neq 0$, in contradiction to the requirement $\rho(J_n) = \{0\}$.

The family of elements $b \in B$ with $b^n = 0$ and $b^{n-1} \neq 0$ is invariant under all automorphisms of B ... Thus, the hereditary C^*-algebra generated by them is an ideal. The same happens with n-homogenous elements.

Use Lemma 2.1.15(iii)

The case $n = 1$, i.e., $b^2 = 0$ is the interesting case.

Is the C^*-subalgebra generated by them an ideal of B? □

LEMMA 2.7.10. *The following properties of C^*-algebras A are equivalent:*

(i) *For every irreducible representation $d\colon A \to \mathcal{L}(\mathcal{H})$ the image $d(A)$ does not contain a nonzero compact operator on H, i.e., $d(A) \cap (C)(\mathcal{H}) = \{0\}$.*
(ii) *Every non-zero hereditary C^*-subalgebra D of A has no character.*
(iii) *For every non-zero hereditary C^*-subalgebra D of A, $k \in N$ and state ρ on D there exists a C^*-algebra morphism $h\colon C_0((0,1], M_k) \to D$ with $\rho(h((0,1] \otimes 1_k)) = 1$.*

LEMMA 2.7.11. *Let A a C^*-algebra. Following observations are nearly obvious:*

(i) *If $B \subseteq C \subseteq A$ are C^*-subalgebras of A and if an element $b \in B_+$ has the property that $\overline{bBb}$ has no non-zero character, then $\overline{bCb}$ and $\overline{bAb}$ have no non-zero characters.*

(ii) *If $c \in A_+$ and $\overline{cAc}$ has no non-zero character, $C \subseteq A$ is a separable C^*-subalgebra of A with $c \in C$, then there exists a separable C^*-subalgebra $B \subseteq A$ such that $C \subseteq B$ and $\overline{cBc}$ has no non-zero character.*

(iii) *Let $X \subseteq A_+$ be a countable subset of elements with the property that $\overline{xAx}$ has no non-zero character for each $x \in X$. Then there exists a separable C^*-subalgebra $C \subseteq A$ with $X \subseteq C$ and the property that $\overline{xCx}$ has no non-zero characters for each $x \in X$.*

(iv) *Let C a C^*-algebra and let $X \subseteq C_+$ a countable subset of contractions that is norm-dense in the set of positive contractions of C. If all elements $x \in X$ have the property that $\overline{(x-\rho)_+C(x-\rho)_+}$ has no nonzero character for all $\rho \in (0, \|x\|)$, then C^*-algebra C is residually antiliminary (and separable).*

PROOF. Part (i) is obvious, because the restriction to $\overline{bBb}$ of a character on $\overline{bCb}$ or $\overline{bAb}$ would be a (non-zero) character on $\overline{bBb}$.

(ii): TEXT(ii): If $c \in A_+$ and $\overline{cAc}$ has no non-zero character, $C \subseteq A$ is a separable C^*-subalgebra of A with $c \in C$, then there exists a separable C^*-subalgebra $B \subseteq A$ such that $C \subseteq B$ and $\overline{cBc}$ has no non-zero character.

If $c \in A_+$ and $D := \overline{cAc}$ has no non-zero character, then $\overline{cAc}$ is – as a C^*-algebra – generated by its 2-homogenous elements. They are determined by the elements $a \in \overline{cAc}$ with $a^2 = 0$ and $\|a\| = 1$. This is, because $C^*(a^*, a) \subseteq A$ is then a 2-homogenous C^*-algebra. And for every pure state ρ on $\overline{cAc}$ there is contraction $a \in \overline{cAc}$ with $\rho(a^*a) = 1$ and $a^2 = 0$. Indeed, let $\rho \in A^*$ a pure state with $\rho(c) \neq 0$ then the irreducible representation $d_\rho \colon A \to \mathcal{L}(\mathcal{H})$ with cyclic vector $x \in \mathcal{L}(\mathcal{H})$, i.e., $d_\rho(A)x = \mathcal{H}$, $\|x\| = 1$ and $\langle d_\rho(a)x, x\rangle = \rho(a)$ is not a character if an only if $\dim(\mathcal{H}) > 1$. Thus there exists $y \in \mathcal{H}$ with $\|y\| = 1$ and $\langle x, y\rangle = 0$.

By ???? some of above lemmata ???

Let $E \subseteq A$ a separable C^*-subalgebra of A that contains c, let $C := \overline{cEc} \subseteq D$.

If ????

C^*-subalgebra with $c \in C_+$???

(iii): remember TEXT(iii): Let $X \subseteq A_+$ be a countable subset of elements with the property that $\overline{xAx}$ has no non-zero character for each $x \in X$. Then there exists a separable C^*-subalgebra $C \subseteq A$ with $X \subseteq C$ and the property that $\overline{xCx}$ has no non-zero characters for each $x \in X$.

(iv): Let $c \in C_+$ with $\|c\| = 1$. Then there exists a sequence $x_1, x_2, \ldots \in X$ such that $\lim_n \|x_n - c\| = 0$. The assumption that $\overline{(x_n-\rho)_+C(x_n-\rho)_+}$ has no non-zero character for all rational numbers $\rho \in (0, \|x_n\|)$ implies that $\overline{cCc}$ has no non-zero character, cf. Corollary 2.7.6.

`Complete proof be filled in ... ??`

□

PROPOSITION 2.7.12. *The class* $\mathcal{RAL}$ *of the* 'residual anti-liminary' *C^*-algebras, defined in Definition 2.7.2, is invariant under each of following operations:*

(i) *Passage to non-zero quotients,*
(ii) *Passage to non-zero hereditary C^*-subalgebras (including closed ideals),*
(iii) *Forming of finite direct sums,*
(iv) *Inductive limits,*
(v) *Maximal tensor products $A \otimes_{\max} B$, ????,*
(vi) *Infinite direct c_0-sums $A_1 \oplus A_2 \oplus \cdots$,*
(Follows from (iii) and (iv))
(v) *Passage from A to $\ell_\infty(A)$.*

If A is residual anti-liminary, then for each countable subset S of A there exists a separable C^*-subalgebra $B \subseteq A$ such that $S \subseteq B$ and B is residual anti-liminary.

In particular, each residual anti-liminary C^*-algebra A is the inductive limit of its separable residual anti-liminary C^*-subalgebras (in its natural containment order as indices of the order).

WHAT ABOUT: tensor products?,

PROOF. The proposed *permanence properties* of the class of residual anti-liminary C^*-algebras A can be seen by following indirect arguments:

If A/J is a non-zero quotient of A and $E \subseteq A/J$ a closed hereditary C^*-subalgebra of A/J and has a non-zero character $\chi\colon E \to \mathbb{C}$ (i.e., if A/J does not satisfy (ii) in place of A), then $D := \pi_J^{-1}(E)$ is a non-zero hereditary C^*-subalgebra of A, and D has the non-zero character $\chi \circ \pi_J$. But this is impossible by property (ii) of A.

Thus, E must satisfies Part(ii) if A satisfies Part(ii), i.e., E must be residual anti-liminary if A is residually anti-liminary. □

Very important open Question:
Is $\ell_\infty(A)$ residual anti-liminary if A is residually anti-liminary?

(It seems to be very difficult to decide, despite it can be reduced to the case of separable A.)

Ques: Is every separable subset of a residual anti-liminary C^*-algebra A contained in a residual anti-liminary separable C^*-subalgebra of A?

Is a C^*-subalgebra of a residual anti-liminary separable C^*-algebra again residual anti-liminary?

Is the inductive limit of a sequence of residual antiliminary separable C^*-algebras again residual anti-liminary?

Has positive answer, using following obvious fact:

(The C can be generated by a countable family of – inside C – 2-homogenous contractions ????)

Combine this with following observation:

If $b \in A_+$ is an element such that $\overline{bAb}$ has no non-zero character, then there exists a separable C^*-subalgebra $B \subseteq A$ such that $b \in B$ and $\overline{bBb}$ has no character.

It shows then by countability of a dense subset of the positive contractions in a separable C^*-algebra C:

If C is a separable C^*-subalgebra of A and $X \subset C_+$ is a countable subset of the positive contractions that is dense in $\{c \in C_+ ; \|c\| = 1\}$ then there exists a separable C^*-subalgebra D of A such that $C \subseteq D$ and for each $x \in X$ and each rational number $\rho \in (0, \|x\|)$ the hereditary C^*-subalgebra $\overline{(x-\rho)_+ D (x-\rho)_+}$ has no non-zero character.

??? To be shown !!! ???

One can show that this implies that $\overline{cDc}$ has no non-zero character for every $c \in C_+$.

In this way we find a sequence $C_1 := C \subseteq C_2 \subseteq C_3 \ldots$ of separable C^*-subalgebras C_n of A, and a countable subset S of the positive contractions $D := \overline{\bigcap_n C_n} \subseteq A$ that is dense in $\{d \in D_+ ; \|d\| = 1\}$ and has the property that $\overline{cDc}$ has no non-zero character for each $c \in S$.

Ques: Is the inductive limit of residual antiliminary C^*-algebras again residually anti-liminary?

(Seems to be: $D \subset E$ hereditary, $E = \overline{\bigcup_\tau E_\tau}$ with E_τ residual anti-liminary, τ in upward directed net, $E_\tau \leq E_\sigma$ if $\tau < \sigma$... D with character ξ.

Then by Remark/Lemma **??** ???: If $D \subseteq E$ is a hereditary C^*-subalgebra of E and J is a closed ideal of D, the the ideal I of E generated by J satisfies $D \cap I = J$.

We consider ideal I of E generated by kernel of ξ (on D), and then the ideal J generated by the image of D in E/I. It would show that $\|\pi_I(f)\| = \|\pi_J(f)\|$ for every $f \in E_\tau$. This leads to a contradiction. Thus, D can not have a character.

Thus, we can consider the quotient $E/I \supseteq D/J$. But D/J is one-dimensional: there is a projection $p \in D/J \subseteq E/I$ with $\mathbb{C} \cdot p = D/J$. and $p(E/I)p = D/J$

Thus E/I contains a minimal projection p with $p(E/I)p = \mathbb{C} \cdot p$.

The ideal K of E/I generated by p is necessarily isomorphic to $\mathbb{K}(\mathcal{H})$ of some Hilbert space.

The C^*-algebras $\pi_I(E_\tau) \cong E_\tau/(E_\tau \cap I)$ are again residual anti-liminary by Prop. ????(??).

Therefore there intersections with K must be $= \{0\}$. It follows that $\|\pi_K(\pi_I(e))\| = \|\pi_I(e)\|$ for all $e \in E_\tau$ for each τ.

We use now that $\bigcup_\tau \pi_I(E_\tau)$ is dense in $\pi_I(E)$:

There exists τ and $e \in \pi_I(E_\tau)_+$ with $\|e\| \leq 1$ and $\|p-e\| < 1/4$. This implies that $\|e\| = \|\pi_K(e)\| < 1/4$ but $\|e\| \geq \|p\| - \|p-e\| \geq 1 - 1/4 = 3/4$ (because $1/2 = \|e\| + \|p-e\| \geq \|p\| = 1$) a contradiction.

Compare Remark 2.1.16(iv):
Item(iv):
If, moreover, A is *strictly antiliminary* (also called *residually antiliminary*, cf. Definition 2.7.2) in the sense that each non-zero quotient A/J of A is antiliminary in the sense of [**616**, sec. 6.1.1]

(Here footnote:

It is equivalent to the formally stronger assumption that no hereditary C^*-subalgebra of A has a non-zero character.),

then for every non-zero hereditary C^*-subalgebra D of A, every pure state ρ on D and every $n \in \mathbb{N}$ there exists a C^*-morphism $\psi\colon \mathrm{C}_0((0,1], M_n) \to D$ with $\rho(\psi(f_0 \otimes p_{11})) = 1$.

Big big QUESTION:
Does here exists a non-zero element $e \in A_+$ such that the *hereditary* C^*-subalgebra $\mathrm{Ann}(e) := \{a \in A\,;\; ae = 0\}$ of A generates A as an ideal of A if A is residually antiliminary ????

In the case of separable A one could look to $e := (\varphi(f_0 \otimes p_{11}) - \varepsilon)$ with φ and ε suitable...?

LEMMA 2.7.13. *Let $\rho\colon D \to \mathcal{L}(\mathcal{H})$ an irreducible representation that does not contain a non-zero compact operator in its image $\rho(D)$, and $(p_1, p_2, \ldots, p_m)$ a sequence of orthogonal projections $p_n \in \mathcal{L}(\mathcal{H})$ of finite rank $k_n \in \mathbb{N}$, $1 \leq n \leq m$.*

Then D contains k_n-homogenous positive contraction $e_n \in D_+$ with $p_n\rho(e_n) = \rho(e_n)p_n = p_n$.

If the $p_1, \ldots, p_m$ are mutually orthogonal, then the e_n can be found mutually orthogonal.

PROOF. Use the "advanced version" of Kadison transitivity in Lemma 2.1.15(ii).
The case of mutually orthogonal $p_1, \ldots, p_m$ can be studied by considering the projection $p_1 + \cdots + p_m$. □

The following Lemma 2.7.14 lists some later used reformulations of the Definition 2.7.2.

It could be that some items are only one-sided implications... Check it!

LEMMA 2.7.14. *Following properties of a C^*-algebra $A \neq \{0\}$ are equivalent (???):*

(i) *Any irreducible representation of A does not contain a non-zero compact operator in its image, i.e., for every irreducible representation $\rho\colon A \to \mathcal{L}(\mathcal{H})$ holds that $\rho(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$.*

(ii) *A is residual antiliminary in sense of Definition 2.7.2.*

(iii) *Each hereditary C^*-subalgebra D of A has no (non-zero) character.*

(iv*) *[Definition 2.7.2 of "residual antiliminary" !!!] Every non-zero quotient A/I of A is antiliminary for each closed ideal $I \neq A$ of A.*

[??? Follows immediately from [**616**, prop. 6.2.8]. *????*

???? Take $J := \pi^{-1}(\mathbb{K}(\mathcal{H}))$. If $x \in J$ then how to find an other representation ???] .

(v) *For each closed ideal $I \neq A$ and $x \in A \setminus I$ there exists an irreducible representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$ with $\pi(I) = \{0\}$ and $\pi(x) \notin \mathbb{K}(\mathcal{H})$.*

(vi) *All non-zero quotients A/I of A do not contain a "minimal" projection $p \in A/I$ (i.e., with the property $p(A/I)p = \mathbb{C}p \neq \{0\}$).*

(viii) *Each non-zero hereditary C^*-subalgebra D of A has no non-zero quotient of Type I.*

What means Type I here?

(It implies (iii): D has no character.)

(Implies: Each non-zero quotient of D contains a non-zero n-homogenous positive contraction.)

(ix) *For each non-zero hereditary C^*-subalgebra D of A and each pure state λ on A with $\lambda(D) \neq 0$ and each $n \in \mathbb{N}$ there exists an n-homogenous element $e \in D_+$ with $\lambda(e) > 0$.*

(x) *For every $n \in \mathbb{N}$, every closed ideal I of A, and every hereditary C^*-subalgebra D of A with $\pi_I(D) \neq \{0\}$ there exists an n-homogenous positive contraction $a \in D$ with $\|\pi_I(a)\| = 1$.*

(xi ???) *More needed ???*

PROOF. To be filled in ??

The problems are ?: Given a non-zero hereditary Abelian C^*-subalgebra D of A. Does there exists an irreducible representation ρ of A such that $\rho(D)$ is 1-dimensional ????

(extend a character ξ of D to a pure state on A/I and use that $D \cap I$ the the ideal I of A generated by the kernel of the character ???????

Let D any hereditary C^*-subalgebra of A and ξ a character of D, $K \subseteq D$ the kernel of ξ.

Citation of Lemma for next is where ???:

The closed ideal of ideal J of A generated by K has the property that $J \cap D = K$.

Thus, $\pi_J(D) \cong D/K \cong \mathbb{C}$, and therefore, $\pi_J(D) = \mathbb{C} \cdot p$ for some projection $p \in A/K$ with $p(A/K)p = \mathbb{C} \cdot p$.

?(x)⇒(ix): trivial.

?(ix)⇒(x): indirect.

The equivalence of (?v) and (?vi) (= old ???) follows immediately from [**616**, prop. 6.2.8].

The other implications follow from the fact that for each closed ideal J of a hereditary C^*-subalgebra D of A the closed ideal $I := \overline{\mathrm{Span}(AJA)}$ of A satisfies $I \cap D \supseteq J$.

latter OK ????? □

LEMMA 2.7.15. *Let A a C^*-algebra, $C_n := \mathrm{C}_0((0,1], M_n) = M_n(\mathrm{C}_0((0,1])$ for $n > 1$, $C_\infty := \mathrm{C}_0((0,1], \mathbb{K}(\ell_2(\mathbb{N}))$ and D a hereditary C^*-subalgebra of A.*

Let $\kappa \in \{2, 3, \ldots\}$ or $\kappa = \infty$. Then the following properties (i) *and* (ii) *of A and D are equivalent (for the same fixed κ).*

(i) *For every* pure *state ρ on A with $\rho|D \neq 0$ there exists a C^*-morphism $\varphi\colon C_\kappa \to D$ with $\rho(\varphi(C_\kappa)) \neq \{0\}$.*
(ii) *For* every *non-zero positive functional ρ on A (or on D) with $\rho|D \neq 0$ there exists a C^*-morphism $\varphi\colon C_\kappa \to D$ with $\rho(\varphi(C_\kappa)) \neq \{0\}$.*

Moreover, if Part (i) *holds then D has no irreducible representation $\pi\colon D \to \mathcal{L}(\mathcal{H})$ with dimension $\mathrm{Dim}(\mathcal{H}) < \kappa$.*

In case of $\kappa \in \mathbb{N}$, i.e., with $\kappa < \infty$, this property of D is moreover equivalent to Part (i)*.*

PROOF. For the following we use that for each closed ideal J of D the ideal I of A generated by J satisfies $J = I \cap D$, cf. Lemma **??**.

Consider the closed ideal J_κ of D generated by the images $\varphi(C_\kappa)$ of all possible C^*-algebra morphisms $\varphi\colon C_\kappa \to D$.

If $J = D$ then clearly Part(ii) holds, – and implies Part(i).

Conversely, if Part (i) holds then $J_\kappa = D$ because, otherwise, there would exists a pure state λ on D with $\lambda(J_\kappa) = \{0\}$. The pure state λ has a unique extension to a pure state ρ on A.

This shows the equivalence of Parts (i) and (ii).

If Part (i) holds, then D has no irreducible representation $\pi\colon D \to \mathcal{L}(\mathcal{H})$ with dimension $\mathrm{Dim}(\mathcal{H}) < \kappa$:

Indeed, if $\pi\colon D \to \mathcal{L}(\mathcal{H})$ is an irreducible representation, then the vector state $\lambda(d) := \langle \pi(d)x, x\rangle$, for given $x \in \mathcal{H}$ with $\|x\| = 1$, is a pure state on D that uniquely extends to a pure state ρ on A. By Part (i) there exists a C^*-morphism $\varphi\colon C_\kappa \to D$ with $\rho(\varphi(C_\kappa)) \neq \{0\}$.

The positive functionals ρ and λ coincide on $\varphi(C_\kappa) \subseteq D$. Thus $\pi(\varphi(C_\kappa)) \subseteq \mathcal{L}(\mathcal{H})$ is isomorphic to a non-zero quotient C^*-algebra of $\mathrm{C}_0((0,1], M_\kappa)$ and has a dimension $\geq \kappa^2$, considered as vector space. Thus, $\mathcal{L}(\mathcal{H})$ has vector-space dimension $\geq \kappa^2$. But this me means that $\mathrm{Dim}(\mathcal{H}) \geq \kappa$.

In case $\kappa = \infty$ this should be read as: $\mathcal{H}$ must be of infinite dimension, because then $\rho(\varphi(C_\kappa))x$ is as a vector space infinite dimensional for each $x \in \mathcal{H}$ with $\rho(\varphi(C_\kappa))x \neq \{0\}$ ([40]).

Above we have seen that the property of $D \subseteq A$ and $2 \leq \kappa \leq \infty$, as stated in Part (i), implies that D has no quotient that is isomorphic to M_κ if $2 \leq \kappa < \infty$, i.e., D has no irreducible representation on a Hilbert space of dimension $< \kappa$.

The opposite direction of this implication holds in case of $1 < \kappa \in \mathbb{N}$, i.e., if D has no irreducible representation $\pi \colon D \to \mathcal{L}(\mathcal{H})$ with dimension $\mathrm{Dim}(\mathcal{H}) < \kappa$, then the hereditary subalgebra $D \subseteq A$ satisfies Part (i):

Let ρ a pure state on A with $\rho|D \neq 0$. Then there is an irreducible representation $\pi \colon A \to \mathcal{L}(\mathcal{H})$ with cyclic vector x such that $\mathcal{H} = \pi(A)x$ and $\rho(a) = \langle \pi(a)x, x\rangle$ for $a \in A$. The closed linear subspace $\mathcal{H}_0 := \pi(D)x$ is invariant under $\pi(D)$ and the restrictions $\psi(d) := \pi(d)|\mathcal{H}_0$ of $\pi(d)$ to $\mathcal{H}_0$ for $d \in D$ define an irreducible *-representation ot D on $\mathcal{H}_0$. We denote by P the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}_0$ and let $y := \|Px\|^{-1/2} \cdot Px$. Then $\|Px\| = \|\rho|D\| > 0$ and $\lambda(d) := \langle \psi(d)y, y\rangle$ is pure state on D of norm $= 1$.

Now we use the assumption that D has no irreducible representation on a Hilbert space of dimension $< \kappa$, i.e., that D has no quotient D/I that is isomorphic to M_m for some $m < \kappa$. It follows that $\mathrm{Dim}(\mathcal{H}_0) \geq \kappa$. Let $P \in \mathcal{L}(\mathcal{H}_0) \subseteq \mathcal{L}(\mathcal{H})$ a projection with $\mathrm{Dim}(P\mathcal{H}) = \kappa$. By Lemma **??**

there exists a C^*-morphism $\varphi \colon C_\kappa \to D$ with $\varphi(C_\kappa) = P\mathcal{L}(\mathcal{H}_0)P = P\mathcal{L}(\mathcal{H})P$ and $\psi \circ \varphi(f_0 \otimes P) = P$. Thus, $\rho(\varphi(C_\kappa)) \neq \{0\}$ and $D \subseteq A$ fulfill Part (i).

Thus, in case where $2 \leq \kappa < \infty$, the non-existence of irreducible representations $\rho \colon D \to \mathcal{L}(\mathcal{H})$ with $\mathrm{Dim}(\mathcal{H}) < \kappa$ is equivalent to Part (i) with this κ. □

The following equivalent formulations l.p.i.(1) and l.p.i(2) of local pure infiniteness imply that l.p.i. C^*-algebras A are residually antiliminary in the sense of Definition 2.7.2 ????? where ????? :

l.p.i.(1):
The C^*-algebra A is locally purely infinite if, for every non-zero hereditary C^*-subalgebra B of A and every pure state ρ of B, there exists a σ-unital stable C^*-subalgebra $D \subseteq B$ with $\rho|D \neq 0$.

Indeed, if $B \subseteq A$ is a hereditary C^*-subalgebra of A and ρ is a pure state of B, then ρ can be uniquely extended to a pure state ρ_{ext} on A with $\rho_{ext}|B = \rho$ and there exists a positive contraction $e \in B_+$ with $\rho(e) = 1$.

Let $D := \overline{eAe}$. Then $D \subseteq B$ and $\rho_{ext}(a) = \rho(eae)$ for all $a \in A$.

By assumption there exists a a stable C^*-subalgebra $D \subseteq E$ with $\rho(D) \neq \{0\}$.

[40] Notice here that $\rho \circ \varphi$ is non-zero, but is not necessarily cyclic or even irreducible, because $\varphi(C_\kappa)$ is in general not necessarily hereditary in D.

It follows that ????

for every non-zero $a \in A_+$ and pure state ρ on A with $\rho(a) > 0$, there exists a stable C^*-subalgebra $D \subseteq \overline{aAa}$ with $\rho(D) \neq \{0\}$.

Thus, ?????

Conclusion:

This property implies in particular that every nonzero hereditary C^*-subalgebra B of A can not have a character ρ, because the non-zero restriction $\rho|D$ to any stable non-zero $D \subseteq B$ can not be character.

Thus, locally purely infinite C^*-algebras are residual antiliminary in sense of Definition 2.7.2.

l.p.i.(2): The C^*-algebra A is locally purely infinite if, for every $d \in A_+$ with $\|d\| = 1$ and every state ρ of A with $\rho(d) \neq 0$, there exists $b \in D := \overline{dAd}$ with the properties that $0 \leq b\,,\ \rho(b) > \|\rho|D\|/2\,,\ \|b\| = 1$ and $\overline{bAb}$ is stable.

`Here: One of the (old! last) estimate is not clear ?!`

Question: Carries the latter property over to $A_\infty := \ell_\infty(A)/c_0(A)$?

Can one at least show that A_∞ is residually antiliminary if A is residual antiliminary?

It can be reduced to the separable case, because one can consider finite free products of $\mathrm{C}_0((0,1], M_n)$, $n = 2, 3, \ldots$. And build suitable inductive limits ...? Look if ultrapowers of this inductive limits have ultra-powers with non-zero characters.

One could try to construct an inductive limit of free products of $\mathrm{C}_0((0,1], M_n)$ that is "universal" for residual antiliminary separable C^*-algebras.

Proof of l.p.i. $\Rightarrow$ l.p.i.(1):

Definition 2.0.3 implies l.p.i.(1): If ρ is a pure state on E then there exists, e.g. by cf. Lemma 2.1.15(ii), a contraction $a \in E_+$ with $\rho(a) = 1$. We can extend ρ to a pure state on A. Thus, there exists a stable C^*-subalgebra $D \subseteq \overline{aAa} \subseteq E$ with $\rho|D \neq 0$ by Definition 2.0.3.

`Next ques should be somewhere discussed`

Question:

Let A a residually antiliminary C^*-algebra and $b \in A_+$ with $\|b\| = 1$ and ρ a state on A with $\rho(b) > 2/3$. Does there exist a general continuous function $f(t) > 0$ with $\lim_{t\to 0} f(t) > 0$ such that there exist always an n-homogenous element $a \in A_+$ with $a \min((1 + 1/n)b, 1) = a$ and $\rho(a) > f(1/n)$?????

`Sort above Lemmata and its Proofs!`
`Until here some are on wrong place ??`

The following Proposition 2.7.16 shows that possible differences between the definitions of Properties pi-n and pi(n) disappear if the multiplier algebra $\mathcal{M}(A)$ of A has a properly infinite unit.

PROPOSITION 2.7.16.
Suppose that A satisfies the following properties (i) *and* (ii) *:*

(i) *For each n-homogenous element $a \in A_+$ and each $b \geq 0$ in the closed ideal $J(a)$ of A generated by a there exists a sequence of elements $d_1, d_2, \ldots \in A$ with $a + b = \lim_n d_n^* a d_n$.*

(ii) *A has no irreducible representations $\rho\colon A \to \mathcal{L}(\mathcal{H})$ of dimension $\leq n$ (of the Hilbert space $\mathcal{H}$).*

Then every non-zero n-homogenous positive element is properly infinite inside the C^*-algebra A .

We do *not* require in Part (i) that $ab = 0$ for the in Part (i) considered sums $a + b$.

The property in Part (i) holds for A if each n-homogenous element of A is properly infinite, but Part (ii) does not follow from Part (i), e.g. for the C^*-algebra $A := \mathcal{O}_2 \oplus M_{n-1}$. (Each non-zero n-homogenous element $a = x \oplus y$ satisfies $a = x \oplus 0$ with $x \neq 0$ and $J(a) = \mathcal{O}_2 \oplus 0$. Then there exists d with $d^* a d = 1 \oplus 0$.)

A trivial example is $A := M_{n-1}$, it satisfies condition (i) trivially, because only 0_{n-1} is n-homogenous in M_{n-1}, but A satisfies not (ii).

PROOF. The idea of the proof are the following:

(α) The assumptions (i) and (ii) pass to all non-zero quotients of all non-zero hereditary C^*-subalgebras of A.

(In particular, A is residually antiliminary in sense of Definition 2.7.2, because this permanence implies that non-zero hereditary C^*-subalgebras can not have contractions.)

(β) It follows from this permanence properties that it suffices to show that each non-zero n-homogenous positive contraction $X \in A_+$ is not finite in A (i.e., is infinite in A).

(γ) To show (β) we shall find for the given X non-zero n-homogenous positive contractions $T \leq S \leq X$ and $R \leq X$ such that $R + S = X$, $TS = T \neq 0$, S in the ideal generated by R and $TR = 0$.

Then it follows that $X \precsim R$, because $X = R + S$ is in the ideal generated by the n-homogenous R. This implies together with $R + T \leq R + S = X$, that $X \approx R$ and then that $R \approx T + R$.

The relations $R \approx T + R$, $T \neq 0$ and $TR = 0$ imply that R is infinite in A. Thus X is infinite in A.

Now we prove this claims and carry out this constructions explicitly:

The Part(i) says that if $\varphi\colon \mathrm{C}_0(0,1] \otimes M_n \to A$ is a C^*-morphism, then for $c := \varphi(f_0 \otimes 1_n)$ holds: If b is in the closed ideal generated by $a := \varphi(f_0 \otimes p_{11})$ then $b \precsim c$. Notice that c is MvN-equivalent in $M_n(A) \subset M_\infty(A)$ to the n-fold sum $\mathrm{diag}(a, a, \ldots, a) = a \oplus a \oplus \cdots \oplus a$.

If expressed in $W(A)$ it says:

If $b \in A$ satisfies $[b] \leq m[a]$ then this implies $[b] \leq \min(m, n)[a]$.

It implies that for every nonzero $n+1$-homogenous element $b := \psi(f_0 \otimes 1_{n+1})$ the n-homogenous element $c := \psi(f_0 \otimes 1_n)$ is properly infinite in A, because it says in $W(A)$ that $[b] = n[a] + [a] = [c] + [a] \leq n[a] = [c]$ and then inductively (over k) that $[c] + k[a] \leq [c]$ and finally that $[c] + [c] \leq [c]$, i.e., that c is properly infinite in A.

The arguments show also that A has the property that all n-homogenous elements $c = \psi(f_0 \otimes 1_n)$ of A "absorb" all elements $b \in A_+$ – in the sense that $b \oplus c \precsim c$ - with $bc = 0$ and and b is in the closed ideal of A that is generated by c ([41]).

It is easy to see that this property passes to all non-zero hereditary C^*-subalgebras $D \subseteq A$ of A.

The passage to the quotient goes as follows: Every selfadjoint contraction $b - c \in A/J$ (with $bc = 0$, and $0 \leq b \leq 1$ and $0 \leq c \leq 1$) can be lifted to a selfadjoint contraction in A, and every n-homogenous positive contraction in A/J can be lifted to an n-homogenous contraction in A, cf. Proposition A.8.4 (or [**540**, cor. 3.8]).

Now we show by an indirect argument that this implies that the C^*-algebra A is residually antiliminary in the sense of Definition 2.7.2 using the characterization in Part (iv) of Proposition 2.7.7:

Let $\rho \colon A \to \mathcal{L}(\mathcal{H})$ an irreducible representation of A. Suppose that there exists a positive contraction in $a \in A_+$ with $0 \neq \rho(a) \in \mathbb{K}(\mathcal{H})$. Then the dimension of $\mathcal{H}$ is $\geq n+1$ by assumption in Part (ii) of Proposition 2.7.16. Moreover then $\rho(A) \cap \mathbb{K}(\mathcal{H})$ is a non-zero C^*-subalgebra of $\mathbb{K}(\mathcal{H})$ that acts irreducibly on $\mathcal{H}$. Thus, $\rho(A) \cap \mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$, and we find an projection $P = P^*P \in \mathbb{K}(\mathcal{H})$ of rank $n+1$.

It follows that there exists a C^*-subalgebra B of A such that $\rho|B$ is a epimorphism from B onto $P\mathbb{K}(\mathcal{H})P \cong M_{n+1}$. The projectivity of $\mathrm{C}_0((0,1], M_{n+1})$, cf. Proposition A.8.4 (or [**540**, cor. 3.8]), shows that there exists an $(n+1)$-homogenous element $b := \psi(f_0 \otimes 1_{n+1}) \in B \subseteq A$ with $\rho(b) = P$. But above we have seen that $[b] = (n+1)[\psi(f_0 \otimes p_{11})]$ satisfies $2[b] \leq [b]$, i.e., the b is properly infinite in A. It is easy to see that non-zero image P of the properly infinite element $b \in A$ are properly infinite. Since P is not properly infinite this contradicts the existence of P and that of non-zero elements in $\rho(A) \cap \mathbb{K}(\mathcal{H})$.

Hence, $\rho(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$ for every irreducible representation $\rho \colon A \to \mathcal{L}(\mathcal{H})$ of A, i.e., A is residually antiliminary.

Change notation, use only two functions f_1, f_2.!?
Remove blue repeat of assumptions?

Suppose that each n-homogenous element $a \in A_+$ has the following property:

[41] It seems that this property is not equivalent to the stated assumptions on A.

For every positive element $b \in \overline{\mathrm{span}(AaA)} =: J(a)$ *there exist* $d_1, d_2, \ldots \in A$ *with* $\lim_k d_k^* a d_k = a + b$*, and that* A *has no irreducible representation of dimension* $\leq n$*.*

We show the following observations (1.0)-(1.3), (2.1) and (2.2), and get finally in (3.0) the desired conclusion.

(1.0) The C^*-algebra A is residual antiliminary in sense of Definition 2.7.2.

(1.1) The properties (i) and (ii) of A pass to each non-zero hereditary C^*-subalgebra D of A, and to all quotients A/I for closed ideals $I \neq A$.

(It is done at the beginning ?)

(1.2) All non-zero $(n+1)$-homogenous elements of $(A/I)_+$ are infinite in A/I.

(It is done at the beginning ?)

(1.3) Non-zero quotients A/I of A can not contain a "minimal" projection $p \in A/I$ with $p(A/I)p = \mathbb{C}p$. (Should be equivalent to "A is residual antiliminary"???)

In particular, any irreducible representation of A does not contain a non-zero compact operator in its image, and each non-zero hereditary C^*-subalgebra D of A has no character.

`(1.3) has been discussed further above or below. Synchronize!`

(2.1) Every non-zero n-homogenous $a \in A_+$ is infinite in A.

(2.2) Every non-zero n-homogenous $a \in (A/I)_+$ is infinite in A/I for every closed ideal $I \neq A$ of A. [Follows immediately from (2.1) and (1.1) by replacing A by A/I.]

(3) Final conclusion from (2.1) and (2.2): Every non-zero n-homogenous element $a \in A_+$ is properly infinite in A, because it says equivalently – by Lemma 2.5.3(v) – that $\pi_I(a)$ is infinite in A/I or $a \in I$ for every closed ideal I of A.

Ad(1.1): Let $\{0\} \neq D \subseteq A$ a hereditary C^*-subalgebra of A and $0 \neq a \in D_+$, $b \in D_+$ in the closed ideal $\overline{\mathrm{span}(DaD)}$ of D generated by a. Then $b \in \overline{\mathrm{span}(AaA)}$ and there exist $d_1, d_2, \ldots \in A$ with $\lim_k d_k^* a d_k = a + b$. The elements $e_k := a^{1/m_k} d_k (a+b)^{1/k}$ are in D and satisfy $\lim_k e_k^* a e_k = a + b$ if the $m_k > k$ are chosen such that $\|a^{1/2} a^{1/m_k} - a^{1/2}\| < 2^{-n} \cdot (\|d_k\|^2 + 1)^{-1}$.

If $\rho \colon D \to \mathcal{L}(\mathcal{H})$ is an irreducible representation ??????????

properties (i) and (ii) of A pass to each non-zero

?? hereditary ??

C^*-subalgebras D of A:

Let $I \neq A$ a closed ideal of A. Then A/I can not have an irreducible representation ρ of dimension $\leq n$, because $\rho \circ \pi_I$ would be an irreducible representation of A on a Hilbert space of dimension $\leq n$, that we have excluded by assumption (ii).

Now let $0 \neq c \in (A/I)_+$ an n-homogenous element and $g \in (A/I)_+$ in the ideal generated by c. We can suppose that $\|c\| = 1$, otherwise replace c by $\|c\|^{-1}c$. Then there exists a C^*-morphism $\psi\colon \mathrm{C}_0((0,1], M_n) \to A/I$ with $c := \psi(f_0 \otimes 1_n)$.

The C^*-algebra $\mathrm{C}_0((0,1], M_n)$ is projective, cf. Proposition A.8.4. Thus there exists a C^*-morphism $\varphi\colon \mathrm{C}_0((0,1], M_n) \to A$ with $\pi_I \circ \varphi = \psi$. The element $a := \varphi(f_0 \otimes 1_n) \in A_+$ satisfies $\pi_I(a) = c$.

Let $\varepsilon \in (0,1)$. There are $h_1, h_2, \ldots, h_m \in A$ with $\|g - \sum_k \pi_I(h_k)^* c \pi_I(h_k)\| < \varepsilon/2$, because the element g is contained in the ideal generated by c.

Property (i) of n-homogenous $a \in A_+$ and that $b := \sum_k h_k^* a h_k$ give the existence of $d \in A$ with $\|(a+b) - d^*ad\| < \varepsilon/2$. Thus $\|(c+g) - \pi_I(d)^* c \pi_I(d)\| < \varepsilon$.

This shows Part (1.1), i.e., that A/I satisfies the assumptions (i) and (ii) – with c and A/I in place of a and A.

Ad(1.2): Let $c \in A_+$ an $(n+1)$-homogenous element. By Lemma 2.5.3(v) it suffices to show that $\pi_I(c)$ is infinite in A/I for all closed ideals I of A with $c \notin I$. By (1.1) the non-zero quotients A/I of A satisfy again conditions (i) and (ii) with $0 \neq \pi_I(a) \in (A/I)_+$ in place of $0 \neq a \in A_+$. Therefore, it suffices to show that each non-zero $(n+1)$-homogenous element in A_+ is infinite in A:

If $\psi\colon \mathrm{C}_0(0,1] \otimes M_{n+1} \to A$ is a non-zero C^*-morphism and $c := \psi(f_0 \otimes 1)$, then $a := \psi(f_0 \otimes (1 - p_{11}))$ is an n-homogenous element and $b := \psi(f_0 \otimes p_{11})$ is in the ideal generated by a, because $b = zz^*$ and $z^*z = \psi(f_0 \otimes p_{22}) \leq a$ for $z := \psi(f_0^{1/2} \otimes p_{12})$. Thus, $b + a = \lim_n d_n^* a d_n$ for a suitable sequence $d_1, d_2, \ldots \in A$ by assumptions on A and a. In particular $c = b + a \precsim a$, $ab = ba = 0$ and $b \neq 0$, i.e., a is infinite in sense of Definition 2.5.1. Then $a \leq c \precsim a$ shows that $c \approx a$, and c is infinite in A.

Ad(1.3): Suppose that $p \in A/I$ is a projection with $p(A/I)p = \mathbb{C} \cdot p$. Then the closed ideal K of A/I generated by p is isomorphic to $\mathbb{K}(\mathcal{H}) \cong K$ for some Hilbert space $\mathcal{H}$.

Suppose that $\mathrm{Dim}(\mathcal{H}) > n$. Then there exists an $n+1$-homogenous element in K (given by any projection $p \in K$ of rank $= n+1$). But this is not possible, because all $n+1$-homogenous elements in A/I are properly infinite by Step (1.2).

It follows that $\mathrm{Dim}(\mathcal{H}) \leq n$. Then $\mathbb{K}(\mathcal{H}) \cong K$ is unital and its unit is in the center of A/I. It would define an irreducible representation of A of dimension $\leq n$. But this has been excluded by the assumption (ii) on A. Hence, a minimal projection p can not exist in any quotient of A.

If D is a non-zero hereditary C^*-subalgebra of A ?????

Ad(1.4): Let $\{0\} \neq D \subseteq A$ a hereditary C^*-subalgebra of A and $0 \neq Y \in D_+$ an n-homogenous element and $b \in D_+$ in the ideal $J(Y)$ generated by Y.

(1.4.a) An element $b \in D$ is in the ideal of D generated by Y inside D, if and only if, b is in the ideal $J(Y)$ of A generated by Y, i.e., $D \cap J(Y) = I(Y)$ for the ideal of $I(Y)$ of D generated by Y.

Compare here also Remark A.5.8 or Lemma 2.7.3!

This is because if $f_1, \ldots, f_m \in A$ satisfy $\|b - \sum_k f_k^* Y f_k\| < \varepsilon$ then for suitable $\beta \in (0, 1/2)$ and $g_k := Y^\beta f_k b^\beta$ holds $\|b - \sum_k g_k^* Y g_k\| < \varepsilon$.

Moreover then there exists a sequence of elements $d_n \in A$ with $\lim_n d_n^* Y d_n = Y + b$. Again we can suppose that the d_n are in D, by replacing the d_n by $Y^{\beta_n} d_n (Y + b)^{\beta_n}$ for a suitable zero sequence $\beta_n \in (0, 1/2)$.

(1.4.b)

D has no irreducible representation of dimension $\leq n$:

An irreducible representation of D of finite dimension extends to an irreducible representation ρ of A that contains the compact operators in its image. Let I denote the kernel of ρ. It would imply that a quotient A/I of A contains a projection p with $p(A/I)p = \mathbb{C} \cdot p$. We have seen in

Ad(1.2)

that this is impossible.

Thus, D has no irreducible representation of finite dimension. In particular, D has no character.

Ad(2): Every nonzero n-homogenous $Y \in A_+$ is infinite in A:

Let $Y \in A_+$ non-zero and n-homogenous. Since $Y \approx \|Y\|^{-1} Y$, we can suppose that $\|Y\| = 1$. Then there exists $\psi\colon \mathrm{C}_0(0, 1] \otimes M_n \to A$ with $\psi(f_0 \otimes 1_n) = Y$, where $f_0(t) = t$ on $[0, 1]$.

Consider the functions $f_1(t) := \min(3t, 1)$, $f_2(t) := \min(\max(3t - 1, 0), 1)$, and $f_3(t) := \max(3t - 2, 0)$ in $\mathrm{C}_0(0, 1]_+$. They are increasing with $f_k(1) = 1$ and $f_k(0) = 0$ $(k = 1, 2, 3)$ and satisfy $f_0 \leq f_1 \leq 3f_0$, $f_2 f_1 = f_2$, $f_3 f_2 = f_3$, $f_1 + f_2 + f_3 = 3f_0$, $f_3 = (3f_0 - 2)_+$, $f_2 = (3f_0 - 1)_+ - (3f_0 - 2)_+$ and $f_1 = 3f_0 - (3f_0 - 1)_+$.

Let $X_1 := 3Y - (3Y - 1)_+ = \psi(f_1 \otimes 1_n)$, $X_2 := (3Y - 1)_+ - (3X - 2)_+ = \psi(f_2 \otimes 1_n)$ and $X_3 := (3Y - 2)_+ = \psi(f_3 \otimes 1_n)$. Then the X_k are all n-homogenous, have norms $\|X_k\| = 1$, and $X_{k+1} = X_{k+1} X_k$ (for $k = 1, 2$). Moreover $Y \leq X_1 \leq 3Y$, in particular $Y \approx X_1$.

The element $Z := \psi(f_3 \otimes p_{11})$ has norm $\|Z\| = \|X_3\| = 1$, because the elements $\psi(f_3 \otimes p_{jj})$ $(j = 1, \ldots, n)$ are pairwise orthogonal, are MvN-equivalent and their sum is $X_3 = \psi(f_3 \otimes 1_n)$. It implies $\|\psi(f_3 \otimes p_{jj})\| = \|X_3\| = 1$ for $1 \leq j \leq n$.

Moreover we get that $Z\psi(f_2 \otimes p_{11}) = Z = Z\psi(f_1 \otimes p_{11})$ because $f_3 f_2 = f_3 = f_3 f_1$. Let $D := \overline{ZAZ}$ the hereditary C^*-subalgebra of A generated by Z.

In Ad(2) we have seen that D has no character. In particular D is not Abelian.

A non-abelian C^*-algebra D contains a nonzero element $x \in D$ with $\|x\| = 1$ and $x^2 = 0$. I.e., D contains a 2-homogenous element $\varphi\colon \mathrm{C}_0(0, 1] \otimes M_2 \to D$ with $\varphi(f_0 \otimes p_{12}) = x$. In particular $\|\varphi(f_0 \otimes 1_2)\| = 1$.

(It can be seen from the existence of an irreducible representation of dimension ≥ 2 and a strong variant of the Kadison transitivity theorem, - Lemma 2.1.15(ii)-, and projectivity of $\mathrm{C}_0(0, 1] \otimes M_2$.

Give ref's or cite!?)

Now we use the natural isomorphism $\overline{X_3AX_3} \cong D \otimes M_n$ for $X_3 := \psi(f_3 \otimes 1_n)$ in a way that $\psi(f_3 \otimes p_{jk})$ corresponds to to $Z \otimes p_{jk} \in D \otimes M_n$. Notice that $X_2(Z \otimes p_{jk}) = (Z \otimes p_{jk}) = (Z \otimes p_{jk})X_2$ under this natural identification.

We build the elements $g_1, g_2 \in \mathrm{C}_0(0,1]$ by $g_1 := 2f_0 - (2f_0 - 1)_+ = \min(2f_0, 1)$ and $g_2 := 2f_0 - \min(2f_0, 1) = \max(2f_0 - 1, 0)$. Notice that $g_1 + g_2 = 2f_0$, $g_2 = (2f_0 - 1)_+$, $g_1 g_2 = g_2$ and $g_1(1) = g_2(1) = 1$. In particular $\|\varphi(g_1 \otimes e_{jk})\| = 1 = \|\varphi(g_1 \otimes 1_2)\|$, $\|\varphi(g_2 \otimes 1_2)\| = 1$ and $\varphi(g_1 \otimes 1_2)\varphi(g_2 \otimes 1_2) = \varphi(g_1 \otimes 1_2)$.

Let $E \subseteq A$ the hereditary C^*-subalgebra of A defined by

$$E := \overline{\psi(f_1 \otimes 1_n)A\psi(f_1 \otimes 1_n)}\,.$$

There is a natural isomorphism γ from E onto the C^*-algebra $F \otimes M_n$, where F is the hereditary C^*-subalgebra of A given by

$$F := \overline{\psi(f_1 \otimes p_{11})A\psi(f_1 \otimes p_{11})}\,.$$

Notice that $\psi(f_1 \otimes p_{11})\varphi(h) = \varphi(h)$ for all $h \in \mathrm{C}_0(0,1] \otimes M_2$, because $\psi(f_1 \otimes p_{11})d = d$ for all $d \in D$.

We define $S := \varphi(g_2 \otimes e_{11}) \otimes 1_n$, $T_1 := \varphi(g_1 \otimes e_{11}) \otimes 1_n$, $T_2 := \varphi(g_1 \otimes e_{22}) \otimes 1_n$, $R := X_1 - T_1 = [\psi(f_1 \otimes p_{11}) - \varphi(g_1 \otimes e_{11})] \otimes 1_n$. Notice that $T_1 S = S$ implies $RS = 0$, that $S \leq T_1 \sim_{MvN} T_2$ and $T_2 \leq R$. It follows that T_1 and S are in closed ideal generated by R. Since $\psi(f_1 \otimes p_{11})\varphi(g_1 \otimes e_{jj}) = \varphi(g_1 \otimes e_{jj})$ it follows that R is a positive n-homogenous element in A and that $RT_2 = T_2 = T_2 R$ implies that $T_2 \leq R$ and $\|R\| = 1$.

Clearly $R \leq X_1$ and $X_1 = R + T_1$. Thus $R \precsim X_1$. Since R is n-homogenous and $T_1 \approx T_2 \leq R$ it follows by assumption on n-homogenous elements in A that $X_1 = R + T_1 \precsim R$.

Thus, $X_1 \approx R$. It implies that the X_1 is infinite in A if and only if R is infinite in A. Since $Y \approx X_1$ (by $Y \leq X_1 = 3Y - (3Y - 1)_+ \leq 3Y$) this would cause also that Y is infinite in A.

Since $RS = 0$, $\|S\| = 1$ and $S \leq T_1 \approx T_2 \leq R$ we have by assumption on n-homogenous elements $R \in A_+$ that there exist $d_1, d_2, \ldots \in A$ with $S + R = \lim_n d_n^* R d_n$. This implies $S + R \precsim R$. On the other hand $(S + R) \oplus 0 = [S^{1/2}, R^{1/2}][S^{1/2}, R^{1/2}]^*$ and $S \oplus R = \mathrm{diag}(S, R) = [S^{1/2}, R^{1/2}]^*[S^{1/2}, R^{1/2}]$ in $M_2(A)$, i.e., $(S + R) \oplus 0 \sim_{MvN} S \oplus R$ in $M_2(A)$. It implies that $S \oplus R \precsim R \oplus 0$ in $M_2(A)$. Thus R, X_1 and Y are infinite elements of A_+.

Since the considered property of Y pass to each quotient A/I of A with $Y \notin I$, it follows finally that Y is properly infinite in A by Lemma 2.5.3(v). □

The following Corollaries 2.7.18 and 2.7.19 are consequences of Proposition 2.7.16.

But we have here inserted also other permanence properties.
Move this to other places.

PROPOSITION 2.7.17. *The Property pi-n of a C^*-algebra A implies that A has Property pi(n).*

Both properties imply that all non-zero n-homogenous elements in A are properly infinite.

Both properties pass to non-zero hereditary C^*-subalgebras and to non-zero quotients.

If, for each $a \in A_+$ and $\varepsilon \in (0, \|a\|)$, there exists elements $d_1, \ldots, d_n \in A$ with $d_j^* d_k = \delta_{jk}(a - \varepsilon)_+$, then Property pi($n$) implies also Property pi-n.

`Not verified, needs special topology of` prime(A)`!:`

??? And then both are equivalent to the property that A has no irreducible representation of dimension $\leq n$ and all n-homogenous elements are properly infinite.???

The Property pi(n) (and therefore also the Property pi-n) implies that A is residually antiliminary in sense of Definition **??**.

The algebra $\mathbb{K}(\mathcal{H})$ of compact operators in any Hilbert space $\mathcal{H}$ can not have one of the properties pi-n or pi(n). In particular, C^*-algebras with this properties have no irreducible representations of finite dimension.

PROOF. The Property pi-n implies Property pi(n):

Let $a \in A_+$, $b \in A_+$ in the closed ideal generated by a, and $\varepsilon > 0$, then there exists $\delta > 0$ and $d_1, \ldots, d_m \in A$ with $\sum_{k=1}^{m} d_k(a - 2\delta)_+(d_k)^* = (b - \varepsilon)_+$. Let $\ell \in \mathbb{N}$ with $n \leq (\ell - 1) \cdot n \leq m \leq \ell \cdot n$. Then the row-matrix $D \in M_{1,m}$ with entries $[d_1, \ldots, d_m]$ in $M_{1,m}$ satisfies $D((a - 2\delta)_+ \otimes 1_m)D^* = (b - \varepsilon)_+$.

By assumption of Property pi-n, the diagonal matrix

$$\operatorname{diag}((a - \gamma)_+, \ldots, (a - \gamma)_+) = (a - \gamma)_+ \otimes 1_n$$

is properly infinite inside $M_n(A)$ for each $\gamma \in (0, \|a\|)$. Thus, there exists an operator matrix $R \in M_{m,n}(A)$ with

$$R((a - \delta)_+ \otimes 1_n)R^* = (a - 2\delta)_+ \otimes 1_m\,.$$

The row matrix $D \cdot R \in M_{1,n}(A)$ satisfies $(D \cdot R)((a - \delta)_+ \otimes 1_n)(D \cdot R)^* = (b - \varepsilon)_+$. This shows that an element b in the in the in closed ideal generated by a non-zero element $a \in A$ can be approximated by sums of n products $cad \in A \cdot a \cdot A$.

The Properties pi-n and pi(n) pass to non-zero hereditary C^*-subalgebras D, in particular to non-zero closed ideals J, and they pass to all non-zero quotients.

`What happens with stabilizations?` $A \otimes M_n$?

The algebra $\mathbb{K}(\mathcal{H})$ of compact operators in any Hilbert space $\mathcal{H}$ can not have one of the properties pi-n or pi(n). In particular, C^*-algebras with this properties have no irreducible representations of finite dimension. □

COROLLARY 2.7.18. *If the C^*-algebra A has Property pi-n of Definition **??** or Property pi(n) of Definition 2.0.4, then every non-zero n-homogenous element of A_+ is properly infinite.*

PROOF. The Property pi-n implies Property pi(n) by Lemma 2.1.2

Let $a \in A_+$, $b \in A_+$ in the closed ideal generated by a, and $\varepsilon > 0$, then there exists $\delta > 0$ and $d_1, \ldots, d_m \in A$ with $\sum_{k=1}^{m} d_k(a-2\delta)_+(d_k)^* = (b-\varepsilon)_+$. Let $\ell \in \mathbb{N}$ with $n \leq (\ell - 1) \cdot n \leq m \leq \ell \cdot n$. Then the row-matrix $D \in M_{1,m}$ with entries $[d_1, \ldots, d_m]$ in $M_{1,m}$ satisfies $D((a-2\delta)_+ \otimes 1_m)D^* = (b-\varepsilon)_+$.

By assumption of Property pi-n, the diagonal matrix

$$\mathrm{diag}((a-\gamma)_+, \ldots, (a-\gamma)_+) = (a-\gamma)_+ \otimes 1_n$$

is properly infinite inside $M_n(A)$ for each $\gamma \in (0, \|a\|)$. Thus, there exists an operator matrix $R \in M_{m,n}(A)$ with

$$R((a-\delta)_+ \otimes 1_n)R^* = (a-2\delta)_+ \otimes 1_m .$$

The row matrix $D \cdot R \in M_{1,n}(A)$ satisfies $(D\cdot R)((a-\delta)_+ \otimes 1_n)(D\cdot R)^* = (b-\varepsilon)_+$. This shows that an element b in the closed ideal generated by a non-zero element $a \in A$ can be approximated by sums of n elements *cad* in $A \cdot a \cdot A$.

The Properties pi-n and pi(n) pass to non-zero hereditary C^*-subalgebras D, in particular to non-zero closed ideals J, and they pass to all non-zero quotients.

`What happens with stabilizations?` $A \otimes M_n$?

The algebra $\mathbb{K}(\mathcal{H})$ of compact operators in any Hilbert space $\mathcal{H}$ can not have one of the properties pi-n or pi(n). In particular, C^*-algebras with this properties have no irreducible representations of finite dimension.

Let ρ a pure state on A, $L := \{a \in A\,;\ \rho(a^*a) = 0\}$, $\mathcal{H} := A/L$, and $D\colon A \to \mathcal{L}(\mathcal{H})$ the corresponding irreducible representation. Then there exists a contraction $e \in A_+$ with $\rho(e) = 1 = \|\rho\|$. By Property pi-n, the diagonal matrix $\mathrm{diag}(e, \ldots, e)$ is infinite in $M_n(A)$.

In case of Property pi-n, the Definition **??** rather directly implies that all n-homogenous non-zero elements in A with Property pi-n are properly infinite in A:

If $a \in A_+$ is n-homogenous and $\|a\| = 1$, then there exists a C^*-morphism $\psi\colon \mathrm{C}_0(0,1] \otimes M_n \to A$ with $\psi(f_0 \otimes 1_n) = a$, where $f_0(t) = t$ for $t \in [0,1]$.

Let $F := f_0 \otimes 1_n$ $G := f_0 \otimes e_{11}$ in $\mathrm{C}_0(0,1] \otimes M_n$, the equivalence $\psi(F) \otimes e_{11} \sim_{MvN} \psi(G) \otimes 1_n$ in $A \otimes M_n$ and an isomorphism

$$A \supseteq \overline{\psi(F)A\psi(F)} \cong M_n(\,\overline{\psi(G)A\psi(G)}\,) \subseteq M_n(A)$$

that maps the n-homogenous element $\psi(F) \in \overline{\psi(F)A\psi(F)} \subseteq A$ to the element $\psi(G) \otimes 1_n \in A \otimes M_n$. The element $\psi(G) \otimes 1_n$ is properly infinite in $A \otimes M_n$ by Definition **??**. It is then automatic also properly infinite inside the hereditary C^*-subalgebra $\overline{\psi(G)A\psi(G)} \otimes M_n$ of $A \otimes M_n$ generated by $\psi(G) \otimes 1_n$. But there is an isomorphism from the C^*-algebra $M_n(\overline{\psi(G)A\psi(G)}$ onto the hereditary C^*-subalgebra $\overline{aAa} = \overline{\psi(F)A\psi(F)}$ with the n-homogenous element $\psi(F) = a$ corresponding to the in $\overline{\psi(G)A\psi(G)} \otimes M_n$ properly infinite $\psi(G) \otimes 1_n$. Thus, the n-homogenous $a \in A_+$ is properly infinite in A, and this shows that the property pi-n of A implies that each non-zero n-homogenous element $\psi(F) \in A$ is properly infinite in A.

In the case of Property pi(n) the proper infiniteness of non-zero n-homogenous elements can be derived with help of the (not so trivial) Proposition 2.7.16:
The Definition 2.0.4 requires (!) that A has no irreducible representation on a Hilbert space of dimension $\leq n$ and that A has the property that each element c in the closed ideal $J(a)$ generated by $a \in A$ satisfies $[c] \leq n[a]$ in the (large) Cuntz algebra $\mathrm{Cu}(A)$.

If moreover $a, c \in A_+$ are non-zero then the latter means that, for each $\varepsilon > 0$ there exists $\delta \in (0, \|a\|)$ and $d_1, \ldots, d_n$ such that

$$(c - \varepsilon)_+ \,=\, d_1^*(a - \delta)_+ d_1 + \ldots + d_n^*(a - \delta)_+ d_n \,.$$

In special case where $a \in A_+$ is n-homogenous and $\|a\| = 1$, there exists $\psi \colon \mathrm{C}_0(0,1] \otimes M_n \to A$ such that $a = \psi(F)$ for $F := f_0 \otimes 1_n$. Let $b := \psi(f_0 \otimes p_{11}) \in A_+$. Then the element $c \in J(a) = J(b)$ satisfies also that $c \in J(b)$ and now $[c] \leq n[b] = [a]$.

Thus, we can in this special case the above defined n-homogenous element $a := \psi(F)$, the element $b := \psi(G)$ and get for every $c \in J(a)_+$, that $c + a \in J(a) = J(b)$ and $[c + a] \leq n[b] = [a]$. It gives that $c + a \precsim a$ in A for all n-homogenous $a \in A_+$ and $c \in J(a)_+$. The additional requirement that A has no irreducible representation of dimension $\leq n$ allows to apply Proposition 2.7.16 to get that all non-zero n-homogenous elements of A are properly infinite in A. □

The Corollary 2.7.18 implies immediately the following Corollary:

COROLLARY 2.7.19. *If the unit element $1_{\mathcal{M}(A)}$ of the multiplier algebra is properly infinite, i.e., if there exists isometries $S, T \in \mathcal{M}(A)$ with $S^*T = 0$, then A has Property pi-n of Definition* **??**, *if and only if, A has Property pi(n) of Definition 2.0.4.*

`Check next proof again. Something not clear?`

PROOF. Since $\mathcal{M}(A)$ is unitally contained in $\mathcal{M}(\ell_\infty(A))$, $\mathcal{M}(A_\infty)$ and $\mathcal{M}(A_\omega)$, it follow that this algebras have no finite-dimensional quotient C^*-algebras.

In fact, the properties pi-n and pi(n) are equivalent if for every element $a \in A_+$ and $\varepsilon \in (0, \|a\|)$ there exist elements $b_1, \ldots, b_n \in A$ with $b_j^* b_k = \delta_{j,k}(a - \varepsilon)_+$. In case of Corollary 2.7.19 one can use $b_k := S^k T \cdot (a - \varepsilon)_+^{1/2}$, for $k = 1, \ldots, n$.

The $b := \sum_{k=1}^n b_k b_k^* \in A_+$ is a properly infinite n-homogenous element and $b \otimes p_{11}$ is approximately equivalent in $M_n(A)$ to the diagonal $n \times n$-matrix $C := \mathrm{diag}((a - \varepsilon)_+, \ldots, (a - \varepsilon)_+) \in M_n(A)$.

But the question is if C is properly infinite in $M_n(A)$, i.e., if $C\oplus 0 \approx C\oplus (a-\varepsilon)_+$ in $M_{n+1}(A)$.

If $c=\sum_{k=1}^m d_k^* a d_k$, then for every $\varepsilon>0$ there exists $\delta>0$ and $e_j\in A$ such that $(c-\varepsilon)_+ = \sum_{j=1}^n e_j^*(a-\delta)_+ e_j$.

Thus ??????

Then $\sum_k (b_k)^* b_k = n\cdot (a-\delta)_+$ and $\sum_k b_k(b_k)^*$ is an n-homogenous element that is equivalent to $(a-\varepsilon)_+\otimes 1_n$ in $M_n(A)$.

???? □

Next BLUE is not proven yet. Perhaps it is wrong ...

The free product $A:=C_1 * C_2 * \cdots$ of a sequence of $C_1, C_2, \ldots$, where the C_n are copies $C_n\cong \mathrm{C}_0((0,1],\mathbb{K}(\ell_2(\mathbb{N})))$ of the cones over the compact operators, has no irreducible representation of finite dimension but the ultrapower A_ω of A has a character (and has irreducible representation of each finite dimension). But this example A is not residually antiliminary:
A has e.g. the compact operators $\mathbb{K}$ on $\ell_2(\mathbb{N})$ as its quotient, because $\mathbb{K}$ is the quotient of one of the freely generating copies C_n of $\mathrm{C}_0((0,1],\mathbb{K})$.

If there is an example of a separable C^*-algebra A that has no characters but A_∞ has (non-zero) characters then the free product F of countably many copies of $M_2(\mathrm{C}_0(0,1])$ must have this property: The algebra A must be a quotient of F, because the C^*-subalgebra B of A generated by a dense sequence in the set of all 2-homogenous contractions contains all 2-homogenous elements in A_+. It is then not difficult to see that for each $a\in A_+$ the closed hereditary C^*-subalgebra $\overline{aAa}$ contains a positive contraction $b\in B_+$ such that $B\cap \overline{aAa}=\overline{bBb}$ and that alternatively $\overline{bAb}=\overline{aAa}$ or $\overline{bAb}$ is an ideal of $\overline{aAa}$ with the property that $\overline{aAa}/\overline{bAb}$ is commutative.

The convex cone K_2 in A_+ generated by all 2-homogenous positive contractions in A should be identical with A_+ , and F has no characters.

It follows from the invariance of K_2 under automorphisms of A and that the (norm-) closure of K_2 is hereditary and K_2 separates the pure states of A (by Kadison transitivity theorem, cf. Lemma **??**).

It needs a Lemma that says if $\overline{aAa}$ has no characters and $0\leq a\leq b_1+\ldots+b_n$, with b_k all 2-homogenous and positive, than a can be approximated by convex combinations of positive 2-homogenous elements in $\overline{aAa}$ (In case that A is residually antiliminary this happens for every non-zero $a\in A_+$).

8. Basics on quasi-traces

Begin of eksec2-Part2.tex !!!

The main destinations are:

1) If $A \otimes C^*_{red}(F)$ has no non-trivial l.s.c. (additive) trace, then $A \otimes C^*_{red}(F)$ is s.p.i., if F is one of the free groups F_n, $n \in \{2, 3, \ldots\} \cup \{\infty\}$.

(The exactness of the reduced group C^*-algebra $C^*_{red}(F)$ ensures that the questions are reduced to the observation that all quasi-ideals of $M \otimes C^*_{red}(F)$ are ideals and to the non-existence of central states on M if $M \otimes C^*_{red}(F)$ has no non-zero central state.)

2) If A_ω has no non-trivial l.s.c. 2-quasi-trace then A_ω is weakly purely infinite (it implies that A is weakly p.i.).

3) If $A \otimes \mathcal{Z}$ has no non-trivial l.s.c. quasi-trace, then $A \otimes \mathcal{Z}$ is s.p.i.

Work here with Rørdam condition on the Cuntz semi-group $\mathrm{Cu}(A)$ and using central sequences of the join algebra $\mathcal{E}(M_p, M_q)$ in $\mathcal{Z}$.

4) Permanence properties for s.p.i. algebras (including extensions, coronas etc.) tensor with exact C^*-algebras ...

5) Special cases of continuous fields with s.p.i. fibres ...

6) $A \otimes \mathcal{O}_\infty \cong A$ if A is separable, nuclear and s.p.i.

More ?????

Before we give in the next section some results concerning the relations between 2-quasi-traceless C^*-algebras and purely infinite C^*-algebras, we report some basic knowledge on 2-quasi-traces, give necessary definitions and explain results that we apply here.

`We have all the Definitions in the first section`
`of Chapter 2 or in the Introduction`

DEFINITION 2.8.1. A map $\tau\colon A_+ \to [0, \infty]$ is a **quasi-trace** if $\tau(0) = 0$ and $\tau(a + b) = \tau(a) + \tau(b)$ for all *commuting* $a, b \in A_+$, and $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$. It is **lower semi-continuous** (l.s.c.) if $\tau(a) = \sup_{\delta>0} \tau((a - \delta)_+)$ for each $a \in A_+$ ([42]). If $\mu\colon A_+ \to [0, \infty]$ is any quasi-trace then $\mu_*(a) := \sup_{\delta>0} \mu\big((a-\delta)_+\big)$ is an l.s.c. quasi-trace. It satisfies $\mu_* \le \mu$

`Check this:` and $\mu(a) = \mu_*(a)$ for all $a \in A_+$ with $\mu(a) < \infty$.

The quasi-trace τ is a **2-quasi-trace** if there is a quasi-trace

$$\tau_2\colon M_2(A)_+ \to [0, \infty]$$

such that $\tau(a) = \tau_2(a \otimes p_{11})$ for all $a \in A$.

We say that A is **(2-quasi-) traceless** (respectively A is **traceless**) if all *lower semi-continuous* 2-quasi-traces (respectively all l.s.c. and additive τ) on A_+ take only the values 0 and $+\infty$.

[42] Then τ is an l.s.c. function from A_+ into $[0, +\infty]$ in the ordinary sense with respect to the norm-topology on A_+ and the Hausdorff topology on $[0, +\infty] \cong [0, 1]$.

```
Hold next ready for later needed reference!
But shift all further explanations to Appendix B.
Including all hints for proofs.
```

PROPOSITION 2.8.2. *Let* $\tau\colon A_+ \to [0,\infty]$ *a lower semi-continuous* 2*-quasi-trace. Then*

(1) $\tau\colon A_+ \to [0,\infty]$ *is an order-monotone map on* A_+*,*
(2) $\tau(a+b) \le 2(\tau(a)+\tau(b))$ *for all* $a, b \in A_+$*,*
(3) $\{a \in A\,;\ \tau(a^*a) = 0\}$ *is a closed ideal of* A*,*
(4) $\{a \in A\,;\ \tau(a^*a) < \infty\}$ *is an algebraic* **-ideal of* A*.*
(5) *If* $\mathcal{U} := M_2 \otimes M_3 \otimes \cdots$ *denotes the universal UHF algebra then there exists a unique l.s.c. "local" quasi-trace* $\mu\colon (\mathcal{U}\otimes A)_+ \to [0,\infty]$ *with* $\mu(1\otimes a) = \tau(a)$ *for all* $a \in A_+$ *.*
(6) *For each* $n \in \mathbb{N}$ *there exists a unique lower s.c. quasi-trace*
$$\tau_n\colon (A\otimes M_n)_+ \to [0,\infty]$$
with $\tau(a) = \tau_n(a\otimes e_{11})$ *for all* $a \in A$*.*

Every l.s.c. "local" quasi-trace on a C^*-algebra A of real rank zero, e.g. on an AW*-algebra or a Rickard algebra, is a 2-quasi-trace. (And it can be interpreted as some sort of evaluations on the J. von Neumann "perspectives" for the related affine geometry.)

It is (2019?) a still open question if unital 2-quasi-traces on AW*-algebras are always additive, i.e., are unital traces.

The minimal requirements on a C^*-algebra A for having the property that all quasi-traces on A are 2-quasi-traces has to do with the question if all quasi-ideals of $\ell_\infty(A)/c_0(A)$ are ideals ... (Consider here also the quasi-traces that take only the values 0 and ∞ ...

But there exists a unital l.s.c. quasi-traces on $C([0,1]) * C([0,1])$ and on $C^*(F_2) = C(S_1) * C(S_1)$ that are not a 2-quasi-traces, – do not mix it up here with the reduced group C^*-algebra $C^*_{red}(F_2)$, which is an exact C^*-algebra.

Notice here that e.g. $t \in [0,1] \to e^{it} \in S_1$ defines a unital epimorphism from $C(S_1)$ onto $C([0,1])$. Thus we have only to give a ???

The τ_2-existence condition in Definition 2.8.1 of 2-quasi-trace τ is equivalent to each of the two following inequalities for all $a, b \in A_+$:

$$\tau(a+b)^{1/2} \le \tau(a)^{1/2} + \tau(b)^{1/2}\,,$$

(which was 1991 observed by U. Haagerup [**342**]), and the 2-sub-additivity:

$$\tau(a+b) \le 2(\tau(a)+\tau(b))\,.$$

The 2-sub-additivity follows from Haagerup's inequality, because for $x, y, z \in [0,+\infty]$ the inequality $x \le y+z$ implies $x^2 \le 2(y^2+z^2)$. But it can be seen also directly from the τ_2-existence by considering the matrix $C := a^{1/2}\otimes p_{11} + b^{1/2}\otimes p_{21}$

by using that $C^*C = (a+b)\otimes p_{11}$, $CC^* \leq 2(a\otimes p_{11} + b\otimes p_{22})$ and that $\tau_2(a\otimes p_{11} + a\otimes p_{22}) = \tau(a)+\tau(b)$ for all $a,b\in A_+$.

A now classical result of U. Haagerup [**342**], says that all 2-sub-additive states on exact C^*-algebra are additive.

The question if every l.s.c. 2-quasi-trace on a C^*-algebra A is additive is an open question. A possible positive answer is equivalent to the likely more than 65 (?) years old conjecture of I. Kaplansky that every finite AW*-factor is a W*-factor.

`Next requires details ...`

J.F. Aarnes [**3**] has described a rather natural quasi-state τ on $\mathrm{C}([0,1]^2)_+$ that has the property that $\tau|C^*(1,f)_+$ is additive if all winding numbers $w(\sigma,f,z)$ of $f\in \mathrm{C}([0,1]^2)$ are zero for all Jordan curves $\sigma\colon S^1\to[0,1]^2$ and all $z\in\mathbb{C}\setminus f(S^1)$.

There exists unital type-I C^*-algebras A, given by a suitable extension $0\to c_0(\mathbb{K})\to A\to \mathrm{C}([0,1]^2)\to 0$, and a unital map $\mu\colon A_+\to[0,\infty)$, defined by the Aarnes quasi-state on $\mathrm{C}([0,1]^2)$, that has the property that the restriction $\mu|C_+$ for each commutative C^*-subalgebra $C\subset A$ is additive on C_+, $\mu(a^*a)=\mu(aa^*)$ for all $a\in A$, but there exists $a,b\in A_+$ with $\mu(a)=0$, $\mu(b)=0$ and $\mu(a+b)>2$. In particular this implies that μ is a quasi-trace state that is not a 2-quasi-trace.

An other example is the free unital product $B:=\mathrm{C}([0,1])\star\mathrm{C}([0,1])$ and the unital quasi-trace given by $\tau_A\circ\pi$, π denotes here the canonical unital epimorphism $\pi\colon B\to\mathrm{C}([0,1]^2)$ and τ_A is the Aarnes quasi-state on $\mathrm{C}([0,1]^2)$, cf. [**3**].

(Compare Section **??** for more information on the existence of non-2-sub-additive quasi-traces!)

In general an l.s.c. quasi-trace $\tau\colon A_+\to[0,\infty]$ is sub-additive (= 1-sub-additive), if and only if, τ is additive, if and only if, τ "extends" to $A\otimes C^*_{red}(F_2)$ in the sense that there exists an l.s.c. quasi-trace ρ on $(A\otimes C^*_{red}(F_2))_+$ with $\rho(a\otimes b)=\tau(a)\mathrm{tr}(b)$, where tr is the unique trace state on the reduced group C^*-algebra $C^*_{red}(F_2)_+$ of the free group F_2 on two generators.

(Here it is important to remind that $C^*_{red}(F_n)$, for $n=2,3,\ldots$ and $C^*_{red}(F_\infty)$ contain the Jiang-Su algebra $\mathcal{Z}$ unitally, cf. [**667**, section 6.3].)

`Give reference for above`

Every l.s.c. 2-quasi-trace $\tau\colon A_+\to[0,\infty]$ on an *exact* C^*-algebras A is additive, because it "extends" to an l.s.c. quasi-trace ρ on $(A\otimes C^*_{red}(F_2))_+$ with $\rho(a\otimes b)=\tau(a)\mathrm{tr}(b)$. (Here we use only that $C^*_{red}(F_2)$ is simple, has a unique trace state and is unitally contained in M_ω.)

`Where is` M_ω `defined?`

`Give example for next blue text??`

Notice that on *traceless* C^*-algebras A there can still exist non-zero 2-quasi-traces τ on the positive part D_+ of some hereditary C^*-subalgebras D of A that have also non-zero finite values on D_+, but those 2-quasi-traces can not have lower semi-continuous extensions to A_+ as 2-quasi-traces.

`Examples for above green text?  Case of ideal?`

IS IT REALLY TRUE, e.g. for simple C^*-algebras and degenerate quasi-traces?

If D is a hereditary C^*-subalgebra of A with l.s.c. 2-quasi-trace τ on $D_+ \cap A$, then τ should extend to A.

First extend to $E := I(D)_+$ for the ideal $E := I(D)$ generated by D by extending to $D \otimes \mathbb{K}$ and then using that $D \otimes \mathbb{K} \cong E \otimes \mathbb{K}$ for σ-unital hereditary C^*-subalgebras D and E that generate the same closed ideal of A.

Is it really on $D \cong D \otimes p_{11}$ the same 2-q-trace?

We recall here that not necessarily bounded lower semi-continuous 2-quasi-traces are in one-to-one correspondence to lower semi-continuous dimension functions D (= sub additive rank functions), and, therefore, are always order monotone maps.

(what we can see directly from the definition???) See Section 7 in Appendix A, ????? e.g. [**342**] and [**93**, sec. 2.9] more precise in BlanKir2 ???? or Appendix A??? for more details.

REMARK 2.8.3. The C^*-algebra A is (2-quasi-) *traceless*, if and only if, A has the property that

(*) for every $a \in A_+$ and $\varepsilon > 0$ there exists $n := n(a, \varepsilon) \in \mathbb{N}$ with

$$(a - \varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n \quad \text{for all} \quad n \geq n(a, \varepsilon)\,.$$

.

Compare Corollaries A.13.10 and A.7.3, or [**462**, prop. 5.7] for a proof.

Notice that this can be expressed equivalently in the Cuntz semigroup $W(A)$ as $2n \cdot [(a - \varepsilon)_+] \leq n \cdot [a]$ for all $n \geq n(a, \varepsilon)$.

Variation of ε and using that $((a - \varepsilon)_+ - \delta)_+ = (a - (\varepsilon + \delta))_+$ shows that

(**) for each $a \in A_+$, $\varepsilon > 0$ and $k > 1$ there exist a number $n(a, \varepsilon, k)$ such that $kn \cdot [(a - \varepsilon)_+] \leq n[a]$ for all $n \geq n(a, \varepsilon, k)$.

In particular there exists, for $n \geq n(a, \varepsilon, k)$, an element $d \in M_{1,k}(A \otimes M_n)$ with

$$d^*(a \otimes 1_n)d = ((a - 2\varepsilon)_+ \otimes 1_n) \otimes 1_k\,.$$

9. Pure infiniteness versus (2-quasi-) traceless.

LEMMA 2.9.1. *Let A and B C^*-algebras, and denote by $A \otimes B$ the minimal C^*-algebra tensor product, and by $P^\times$ the natural continuous map from the cartesian product of T_0-spaces* $\mathrm{prime}(A) \times \mathrm{prime}(B)$ *into the T_0-space* $\mathrm{prime}(A \otimes B)$*, that is defined by*

$$P^\times(I, J) := (I \otimes A) + (B \otimes J)\,.$$

(i) *If A or B is exact, then $P^\times$ is a (surjective) homeomorphism.*

(ii) *Suppose that $P^{\times}$ is surjective, D is a hereditary C^*-subalgebra of $A \otimes B$ and $I \in \mathcal{I}(A \otimes B)$ is a prime ideal*

`Or I must it be primitive ?? Check original!`

that does not contain D.

Then there exist $a \in A_+$, $b \in B_+$, $c \in A \otimes B$, pure states λ on A, and ρ on B such that

(α) $\lambda \otimes \rho(I) = \{0\}$,

(β) $c c^* \in D$ *and* $c^* c = a \otimes b$,

(γ) $\lambda(a) = \|a\| = 1$ *and* $\rho(b) = \|b\| = 1$.

(iii) *Suppose that the C^*-algebras A and B have the property that $P^{\times}$ is surjective.*

If A or B is (2-quasi-)-traceless then $A \otimes B$ is (2-quasi-)-traceless.

Notice that Part (ii) implies that – in case where $P^{\times}$ is surjective – the set of elements $\mathcal{F} := \{a \otimes b\, ;\; a \in A_+\, ,\; b \in B_+\}$ is a *filling family* for $A \otimes B$ in sense of [**469**, def. 4.2].

`Not.published? No!: It is now published, see [469]. But one has to pay 30 British pound.`
`But arXiv:1503.08519v2 is fairly readable... Check typos again?`
`Check new references !!!`

PROOF. (i): See [**93**, prop. 2.16 and prop.2.17(2)].

(ii): See Lemma 2.2.3 (cf. also [**93**, lem. 2.18]),

`Lemma 2.2:`

Let D be a non-zero hereditary C^*-subalgebra of the minimal C^*-algebra tensor product $A \otimes B$ of C^*-algebras A and B.

Then there exists $0 \neq z \in A \otimes B$ with $zz^* \in D$ and $z^*z = e \otimes f$ for some non-zero $e \in A_+$ and $f \in B_+$.

If $d \in D_+$ and pure states $\varphi \in A^*$, $\psi \in B^*$ are given with $(\varphi \otimes \psi)(d) > 0$, then the element $z \in A \otimes B$ can be found such that, moreover, $\varphi(e)\psi(f) > 0$.

(iii): By symmetry it suffices to consider the case where B is (2-quasi-) traceless. For each fixed $a \in A_+$, the map

$$\tau_a \colon b \in B_+ \mapsto \tau(a \otimes b) \in [0, \infty]$$

is an l.s.c. 2-quasi-trace on B_+. The τ_a can only take the values 0 and $+\infty$, because B is (2-quasi-) traceless. We show that this implies that τ can take only the values 0 and $+\infty$ by lower semi-continuity of τ:

The set of $g \in (A \otimes B)_+$ with $\tau(g) = 0$ is the positive part J_+ of a closed ideal J of $A \otimes B$ by Proposition 2.8.2.

Let $e \in (A \otimes B)_+$ with $\tau(e) < \infty$. To prove that $\tau(e) = 0$ it suffices by lower semi-continuity of τ to show that, for each $\varepsilon > 0$, the hereditary C^*-subalgebra D generated by $(e - \varepsilon)_+$ is contained in J.

Suppose that D is *not* contained in J. Then there exists a primitive ideal of I $A \otimes B$ such that $J \subseteq I$ and $D \not\subseteq I$.

Let $a \in A_+$, $b \in B_+$, $c \in A \otimes B$, $\lambda \in A^*$, and $\rho \in B^*$ as in Part (ii). Then $a \otimes b = c^*c \notin I$ but $cc^* \in D$, in particular $a \otimes b \notin J$, i.e., $0 < \tau(a \otimes b) \leq \infty$ and $\tau_a \colon x \in B_+ \mapsto \tau(a \otimes x)$ is an l.s.c. 2-quasi-trace on B_+.

The latter contradicts the assumption that B is traceless. Thus, $D \subseteq J$ and $\tau(e) = 0$ for each $e \in (A \otimes B)_+$ with $\tau(e) < \infty$. □

The Remark 2.8.3 has the following consequences for tensor products of A with UHF algebras, and for algebras A with (2-quasi-) traceless ultra-powers of A:

Corollary 2.9.2. *Let A a (2-quasi-) traceless C^*-algebra, E an exact C^*-algebra and B an UHF algebra (of infinite dimension) then $E \otimes A \otimes B$ is a purely infinite C^*-algebra.*

In fact $E \otimes A \otimes B$ is also *strongly* purely infinite, because one can decompose B into a tensor product $B = B_1 \otimes B_2$ with B_1 and B_2 UHF-algebras of infinite dimensions, cf. Proposition **??**.

Proof. By Lemma 2.9.1(iii) $E \otimes A$ is (2-quasi-) traceless. Thus we can rename $E \otimes A$ by A for the following.

We express B as a tensor product $B = C_1 \otimes C_2 \otimes \cdots$ with $C_k = M_{n_k}$ such that $n_{k+1} > 2 \cdot n_1 \cdot n_2 \cdot \ldots \cdot n_k$. The union of the increasing sequence of C^*-subalgebras

$$A_m := A \otimes C_1 \otimes C_2 \otimes \cdots \otimes C_m \otimes 1_{n_{m+1}} \otimes 1_{n_{m+2}} \otimes \cdots$$

of $A \otimes B$ is dense in $A \otimes B$. To simplify notation we write simply C_m for

$$1_{\mathcal{M}(A)} \otimes 1_{n_1} \otimes \cdots 1_{n_{m-1}} \otimes C_m \otimes 1_{n_{m+1}} \otimes \cdots .$$

The algebras A_m are matrix algebras over A and are q-trace-less because A is q-trace-less. Thus $A \otimes B$ must be q-trace-less. (We can use here also exactness of B and Part (iii) of Lemma 2.9.1.)

By Remark 2.8.3 we get that for each positive $a \in A_m$ and $\varepsilon > 0$ there exists $n(a, \varepsilon) \in \mathbb{N}$ with $(a - \varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ for all $n \geq n(a, \varepsilon)$.

It implies $2n[(a - \varepsilon)_+] \leq n[a]$ for all $n \geq n(a, \varepsilon)$.

One needs here that the numbers $n(a, \varepsilon)$ for given $\varepsilon > 0$ can be chosen in the list of finite products of "remaining" numbers $n_k = \mathrm{Dim}(C_k)^{1/2}$ starting from $k := m + 1$.

```
???? Next not correct !!! ???
```

For each $a \in A_m$ there exists $n \geq m$ such that $(n + 1)[a] \leq n[a]$ in $\mathrm{Cu}(A_m)$. Then for all $k \geq n$ and $\ell > 1$ holds $k[a] \geq k\ell[a]$.

Indeed: We get by induction $(n + r)[a] \leq n[a]$ for all $r \in \mathbb{N}$. Let $r := (k \cdot \ell) - n$. Then $k\ell[a] \leq n[a] \leq k[a]$.

In particular, it follows that for each $\varepsilon > 0$ there exist sufficiently big $k := k(a, \varepsilon) \in \mathbb{N}$, $\delta > 0$ and $d_1, d_2 \in A_k \subset A \otimes B$ such that $d_i^* a d_j = \delta_{ij}(a - \varepsilon)_+$.

This is the case because the relative commutant

$$(A_m)' \cap \mathcal{M}(A) \otimes B \subseteq (A_m)' \cap \mathcal{M}(A \otimes B)$$

of $A_m \subseteq A \otimes B$ in $\mathcal{M}(A) \otimes B$ contains sufficiently big unital matrix algebras C_m, and there is $n \in \mathbb{N}$ with $a \otimes 1_{2k} \precsim a \otimes 1_k$ for all $k \geq n$.

It follows that $A \otimes B$ is purely infinite by Lemma 2.5.17(ii), because the positive elements in $\bigcup_{m \in \mathbb{N}} A_m$ are dense in $(A \otimes B)_+$ and are spectral properly infinite in $A \otimes B$. □

M. Rørdam [**690**, prop. 2.2] gets the following observation using results in the Jiang-Su paper [**391**]: *There is a unital embedding of $E(m, n) := \mathcal{E}(M_m, M_n)$ into $\mathcal{Z}$ for every pair of natural numbers $m < n$ that are relatively prime.*

Recall that a C^*-algebra B is called "**(2-quasi)-traceless**" if every l.s.c. 2-quasitrace $\tau \colon B_+ \to [0, \infty]$ takes only the values 0 or $+\infty$. The lower semi-continuity of the 2-quasitrace and the 2-sub-additivity are important ingredients of this terminology, because the l.s.c. 2-sub-additive maps τ on A_+ with values in $\{0, \infty\}$ and $\tau(a^*a) = \tau(aa^*)$ correspond one-to-one to the lattice of closed ideals $J \triangleleft A$. The bijection is given by $\tau_J(a) := \|\pi_J(a)\| \cdot \infty$ and $J_\tau := \{a \in A\,;\ \tau(a^*a) = 0\}$.

Proposition 2.9.3 ([**463**]). *The ultra-power A_ω of a C^*-algebra A is (2-quasi)-traceless, if and only if, there exists general $n \in \mathbb{N}$ such that $a \otimes 1_n$ is properly infinite in $A \otimes M_n$ for every non-zero $a \in A_+$, i.e., A has property pi-n in sense of Definition* **??**.

In this case A is also pi(m) in sense of Definition 2.0.4, with $m \leq n$ (cf. Proposition **??** *NO other !).*

??? Strange formulation:

Recall that A has property pi-n if and only if A is m-purely infinite for some $m \leq n$

and that pi-n implies pi(n),

but nothing is known over the converse ...

or $m \leq n^2$?

by Proposition **??**, i.e., A is pi(m) in sense of Definitions 2.0.4 with $m \leq n$.

Proof. Suppose that A_ω has no non-zero 2-quasitrace. By Remark 2.8.3 we get that for each positive $b \in A_\omega$ and $\varepsilon > 0$ there exists $n(b, \varepsilon) \in \mathbb{N}$ with $(b - \varepsilon)_+ \otimes 1_{2n} \precsim b \otimes 1_n$ for all $n \geq n(b, \varepsilon)$.

Give reference

Suppose that there does not exist an $n \in \mathbb{N}$ such that $a \otimes 1_n$ is properly infinite in $A \otimes M_n$ for every $a \in A_+$.

Then there exists a sequence of positive contractions $a_k \in A_+$ and and $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$, and $a_k \otimes 1_{n_k}$ is not properly infinite in $A \otimes M_{n_k}$ for each $k \in \mathbb{N}$.

If we use observation (x) of Lemma 2.5.3 then we find $\varepsilon_k \in (0,1)$ with $\varepsilon_{k+1} < \varepsilon_k$ such that

$$(a_k \otimes 1_{2n_k} - \varepsilon_k)_+ \not\precsim a_k \otimes 1_{n_k} \,.$$

There are $m_k \in \mathbb{N}$ such that $(\varepsilon_k)^{1/m_k} > 1/2$ for each $k \in \mathbb{N}$. Let $b_k := a_k^{1/m_k} \in A$ and $b := \pi_\omega(b_1, b_2, \ldots)$ in A_ω. It satisfies

$$(b - 1/4)_+ \otimes 1_{2n} \not\precsim b \otimes 1_n \,.$$

This contradicts that A_ω admits no l.s.c. 2-quasi-trace. □

It is not clear if the following Definition 2.9.4 is useful for general study:

Definition 2.9.4. Let C a unital C^*-algebra and suppose that there exist contractions $e_1, e_2 \in C_+$ and $d_{1,1}, \ldots, d_{1,m}, d_{2,1}, \ldots, d_{2,m} \in C$ such that $e_1 e_2 = 0$ and $\sum_{k=1}^m d_{j,k}^* e_j d_{j,k} = 1$ for $j = 1, 2$.

The smallest number $m \in \mathbb{N}$ with the property that such $(e_j, d_{j,k})$ exist will be denoted by $m(C) \in \mathbb{N} \cup \{+\infty\}$. We call $m(C)$ the "**minimal orthogonality**" of C.

And we define a *universal* unital C^*-algebra

$$G_m := C^*(e_j, d_{j,1}, \ldots, d_{j,m};\, j = 1, 2, \langle R \rangle\,)$$

with above considered relations (R) on the e_j and $d_{j,k}$.

Next Lemma 2.9.5 and its proof are wrong because we can take $V := \mathrm{id}_{M_n}$, but there does not exist a unitary $U \in M_n$ with $U^* p_{11} U = (1/n) 1_n$, because the of norms $(1, 1/n)$ and ranks $(1, n)$ are different.

Lemma 2.9.5. *Let $V \colon M_n \to A$ a completely positive map.*

Then there exists a unitary $U \in M_n$ such that $V(U^* p_{11} U) = 1/n V(1_n)$, $V(U^* p_{22} U) = 1/(n-1) V(1_n - p_{11})$ for $1 \le k \le n$.

Lemma 2.9.6. *Let A a unital C^*-algebra. Then A has no character, if and only if, there exists $m \in \mathbb{N}$ and contractions $e_{j,k} \in A$, $j = 1, \ldots, m$, $k \in \{1, 2\}$, that satisfy the relations: $\sum_j e_{j,k}^* e_{j,k} = 1$, $e_{j,1}^* e_{j,2} = 0 = e_{j,2}^* e_{j,1}$ and*

????? Look also to centalseq. paper with M.Rordam!// Perhaps this can not be realized?:

$e_{j,1}^* e_{j,1} = e_{j,2}^* e_{j,2}$????? because it is likely that

$T^{-1/2} p_{1,1} T^{-1/2} \neq T^{-1/2} p_{2,2} T^{-1/2}$

It is even not an estimate available !!!

([43]).

[43] The relations do not imply – or require – that $e_{i,k}^* e_{j,k} = 0$ for $i \neq j$.

PROOF. To be filled in ! ??

Let S denote the set of all finite sums $n^{-1}\sum_{k=1}^{n} a_k \in A_+$ of $a_k := \psi_k(f_0 \otimes 1_2)$ where ψ_k denote any non-zero *-homomorphism from $\mathrm{C}_0((0,1], M_2) \cong M_2(\mathrm{C}_0(0,1])$ into A. The set $S \subseteq A_+$ is a convex subset of the contractions in A that is invariant under automorphisms of A. In particular the hereditary convex sub-cone of A_+, i.e., the set of $a \in A_+$ with the property that there exists $b \in A_+$ and $\gamma \in [0,\infty)$ with $a+b \in \gamma \cdot S$, i.e., is generated by S, has as its norm-closure a hereditary closed convex cone that invariant under all automorphisms of A and is generated by all 2-homogenous elements of A_+.

By Lemma ??

Says/should say:

(1) the closures C of a hereditary sub-cone of A_+, defined by a subset $X \subset A_+$ of contractions in A_+, is a hereditary (!!) sub-cone of A_+. It is the positive part $D_+ = C$ of a hereditary C^*-subalgebra D of A.

The D is invariant under conjugation by all unitaries of A (= inner automorphisms) if C it is invariant under conjugation by all $\exp(ih)$ with $h^* = h \in A$ and $\|h\| < 1$.

(2) If a hereditary C^*-subalgebra D is invariant under conjugation by all $\exp(ih)$ with $h^* = h \in A$ and $\|h\| < 1$ then D is a closed ideal of A.

(3) For every element $d \in D_+$ and $\varepsilon > 0$ there exists f in the convex span of X and $\lambda \in (0,\infty)$ with $(d-\varepsilon)_+ \leq \lambda x$. ???? (at least if X is invariant under inner automorphisms of A.)

It is the positive part of the closed ideal of A generated by all images $\psi(f_0 \otimes 1_2)$.

The quotient A/J can not contain a non-zero 2-homogenous element, because $\mathrm{C}_0((0,1], M_k)$ is a projective C^*-algebra for all $k \in \mathbb{N}$, cf. Proposition A.8.4 or [**540**, cor. 3.8].

Thus, J is also the closed ideal of A generated by all commutators $ab - ba$, i.e., is the intersection of the kernels of all characters on A.

By assumption, A has no character. It shows that $A = J$ and $1 \in J$.

By the above study of J_+ there exists $r \in \mathbb{N}$ and a finite sum $\sum_{q=1}^{r} a_q \in A_+$ of 2-homogenous positive elements $a_q := \psi_q(f_0 \otimes 1_2) = \psi_q(f_0 \otimes p_{11}) + \psi_q(f_0 \otimes p_{22})$

such that

$1_A \leq \sum_{q=1}^{r} a_q =: T$. Then $\sum_{q=1}^{r} T^{-1/2} a_q T^{-1/2} = 1$.

But this T is imbalanced with respect to the entries $\psi_q(f_0 \otimes p_{j,k})$ if $a_q = \psi_q(f_0 \otimes p_{1,1})$.

$\psi_q(f_0^{1/2} \otimes p_{11}) + \psi_q(f_0^{1/2} \otimes p_{12}) = \psi_q(f_0^{1/2}, p_{11} + p_{12})$

Let $x := (1/\sqrt{2})(p_{11} + p_{12})$ gives $x^* \neq x$ $x^*x = 1/2(p_{11} + p_{12} + p_{2,1} + p_{2,2})$, and for $y := (1/\sqrt{2})(p_{22} - p_{21})$ gives $y^*y = 1/2(p_{11} - p_{12} - p_{2,1} + p_{2,2})$ and $y^*x = 0$.

Can we find some estimate for x^*x ? $x^*x \leq p_{11} + p_{22}$ because $x^*x + y^*y = p_{11} + p_{22}$

The elements $T^{-1/2}\psi_q(f_0 \otimes 1_2)T^{-1/2} \in A$ with properties $\psi_q \colon \mathrm{C}_0((0,1], M_2) \to A$, $1 \leq q \leq r$ imply the statement ... ???

(Likely ???: $m := 2r$, ???? $e_{j,1} := \psi_j(f_0^{1/2} \otimes p_{11})T^{-1/2}$ for $1 \leq j \leq n$ and $e_{n+j,1} := \psi_{j-n}(f_0 \otimes p_{22})T^{-1/2}$ for $n+1 \leq j \leq 2r$????)

But, what can we show ??? It is:

If unital A has no non-zero character, then there exist m contractions $z_1, \ldots, z_m \in A$ with the properties $z_k^2 = 0$ and

$$\sum_{k=1}^{m} (z_k (z_k)^* + z_k^* z_k) \geq 1 \,.$$

In case of only σ-unital A, with strictly positive element $e \in A_+$ that satisfies $\|e\| = 1$ and has no non-zero character, this should have similar properties:

Likewise this:

For every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$, and $m \in \mathbb{N}$ such that there are exist m contractions $z_1, \ldots, z_m \in \overline{(e-\delta)_+ A (e-\delta)_+}$ with the properties $z_k^2 = 0$, and

$$\sum_{k=1}^{m} (z_k (z_k)^* + z_k^* z_k) \geq (e-\varepsilon)_+ \,.$$

There exists $m \in \mathbb{N}$ and contractions $e_{j,k} \in A$, $j = 1, \ldots, m$, $k \in \{1, 2\}$, that satisfy the relations: $\sum_j e_{j,k}^* e_{j,k} \geq 1$, $e_{j,k}^* e_{j,\ell} = 0$ and $e_{j,k}^* e_{j,k} = e_{j,\ell}^* e_{j,\ell}$ for $k \neq \ell$, (The relations do not imply or require that $e_{i,k}^* e_{j,k} = 0$ for $i \neq j$). □

`Remarks:` The Lemma 2.9.6 implies immediately the following ??????

If B is a σ-unital C^*-algebra, then the unital C^*-algebra

$$F(B) := (B' \cap B_\omega) / \operatorname{Ann}(B, B_\omega)$$

has no character, if and only if there exist $d_{j,k} \in B' \cap B_\omega$ $j = 1, \ldots, m$, $k \in \{1, 2\}$, $d_{j,k}^* d_{j,\ell} = 0$ and $d_{j,k}^* d_{j,k} = d_{j,\ell}^* d_{j,\ell}$ for $k \neq \ell$ and $\sum_j d_{j,k}^* d_{j,k} \in 1 + \operatorname{Ann}(B, B_\omega)$.

Notice that for $\mathcal{E}(M_2, M_3) \subseteq \mathcal{Z} \subseteq \mathcal{O}_\infty$ we get $m(\mathcal{E}(M_2, M_3)) \leq 3$.

`Or` $m(\ldots) \leq 2$ `???`

One can see that the infinite unital free product $C *_1 C *_1 C *_1 \cdots$ of the unital cones $C := \{ f \in \mathrm{C}([0,1], M_2) \, ; \; f(0) = 1_2 \}$ (with unit element $1_C(t) := 1_2$ for all $t \in [0,1]$) has only one character χ given by $f \mapsto f(0) \in \mathbb{C} \cdot 1_2$.

It has to be considered in a natural way as the C^*-algebra inductive limit by the injective maps

$$C \mapsto C *_1 1 \subset C * C \mapsto C *_1 C *_1 1 \subset C *_1 C *_1 C \ldots$$

The unital character χ then extends step by step to a character of

$$C *_1 C *_1 C * \cdots .$$

This should be the only character on it ???

Let D denote the kernel ideal of the character χ. Then $D *_1 1_C$ and $1_C *_1 D$ together generate $C *_1 C$ .....????

PROPOSITION 2.9.7. *Let A a purely infinite C^*-algebra. If there exists $m \in \mathbb{N}$ such that for every finite subset $F \subseteq A$ and $\delta > 0$ there exists a unital C^*-morphism $\phi\colon G_m \to \mathcal{M}(A)$ with $\|[x, \phi(e_j)]\| < \delta$ and $\|[x, \phi(d_{j,k})]\| < \delta$ for $x \in F$, $j = 1, 2$ and $k = 1, \ldots, m$. Then A is strongly p.i.*

PROOF. In the general case, e.g. where A is only pi-n, we get at least that all matrices with n-homogenous diagonal entries are approximately diagonalizable by n-homogenous entries in the diagonal.

The assumptions imply that, for every separable C^*-subalgebra B of A, there exist a unital C^*-morphism $\psi\colon G_m \to \mathcal{M}(A)_\omega$ with zero commutators $[b, \psi(e_j)] = 0$ and $[b, \psi(d_{j,k})] = 0$ for all $b \in B$.

Let $a, b \in A_+$ properly infinite elements and consider the C^*-subalgebra $C^*(a, b)$ of A generated by a and b. Find in A a separable C^*-subalgebra B with the following properties:

(1) $a, b \in B_+$ and a, b are inside B properly infinite.

(2) B contains sequences that represent the generating elements of $\psi(G_m)$ such that $\psi(G_m)$ commutes with B and is in B_ω.

Now we consider the 2×2-matrix $C := [c_{p,q}] = [a, b]^\top [a, b]$ with entries $c_{11} = a^2$, $c_{22} = b^2$, $c_{1,2} = ab$ and $c_{2,1} = ba$. If a and b are properly infinite in A, then they are also properly infinite in A_ω.

This happens also for elements be_1 and ae_2 in A_ω if $be_1 = e_1 b$ and $ae_2 = e_2 a$ if ?????.

?????? IS THE LATTER TRUE ??

More generally let $d_{j,k} \in B' \cap B_\omega$ $j = 1, \ldots, m$, $k \in \{1, 2\}$, $d_{j,k}^* d_{j,\ell} = 0$ and $d_{j,k}^* d_{j,k} = d_{j,\ell}^* d_{j,\ell}$ for $k \neq \ell$ and $\sum_j d_{j,k}^* d_{j,k} \in 1 + \mathrm{Ann}(B, B_\omega)$.

Define properly infinite $a, b \in A_+$ by elements $f_1, \ldots, f_m, g_1, \ldots g_m \in A$ with $f_j^* a^2 f_k = \delta_{j,k}(a^2 - \varepsilon)_+$ and $g_j^* b^2 g_k = \delta_{j,k}(b^2 - \varepsilon)_+$.

Then define $h_1 := \sum_{j=1}^m f_j d_{j,1} \in B_\omega$ and $h_2 := \sum_{j=1}^m g_j d_{j,2} \in B_\omega$

????

Notice that $h_1^* a^2 h_1 = (a^2 - \varepsilon)_+$ and $h_2^* b^2 h_2 = (b^2 - \varepsilon)_+$, but to get $h_1^* ab h_2 = 0$, one needs in addition that $d_{j,k}^* d_{i,\ell} = 0$ for all $j \neq i$ and all $k, \ell \in \{1, 2\}$.

`To be filled in` ?? □

COROLLARY 2.9.8. *Let $\mathcal{Z}$ denote the Jiang-Su algebra. If A is a (2-quasi-) traceless C^*-algebra, then $A \otimes \mathcal{Z}$ is strongly purely infinite.*

PROOF. By Proposition n?? $A \otimes \mathcal{Z}$ is purely infinite.

By [**391**], the algebra $\mathcal{Z}$ contains $\mathcal{E}(M_2, M_3)$ and $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$. Since $m(\mathcal{E}(M_2, M_3)) \leq 3$ it follows that there is a unital C^*-morphism from G_3 into $\mathcal{Z}$.

The isomorphism $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$ implies that there exists a "central" sequence of unital C^*-morphisms from G_3 into $\mathcal{M}(A) \otimes \mathcal{Z} \subset \mathcal{M}(A \otimes \mathcal{Z})$. Now Proposition 2.9.7 applies and shows that $A \otimes \mathcal{Z}$ is strongly purely infinite.

Alternatively one can $\mathcal{Z}$ represent as inductive limit of $\mathcal{E}(M_{2^\infty}, M_{3^\infty})$ given by a suitable endomorphism of $\mathcal{E}(M_{2^\infty}, M_{3^\infty})$. The latter can be described as inductive limit of $\mathcal{E}(M_{2^n}, M_{3^n})$. But $\mathcal{E}(M_{2^n}, M_{3^n})$ contains obviously a central sequence of copies of $\mathcal{E}(M_2, M_3)$.

□

DEFINITION 2.9.9. Let A, B unital C^*-algebras. The **tensorial joint algebra** (also called **joining algebra** , or **winding around algebra**) of A and B is defined in [**448**, following Prop. 1.17] as the C^*-subalgebra of $\mathrm{C}([0,1], A \otimes^{\max} B)$ defined by:

$$\mathcal{E}(A, B) := \{ f \in \mathrm{C}([0,1], A \otimes^{\max} B)\, ;\ f(0) \in A \otimes 1\, ,\ f(1) \in 1 \otimes B \} \tag{9.1}$$

In cases where $A := M_m$ and $B := M_n$, this algebras also are called *dimension-drop* C^*-algebras.

COROLLARY 2.9.10. *If A is a (2-quasi-) traceless C^*-algebra, and if $B := M_p$ and $C := M_q$ are UHF-algebras with infinite supernatural numbers p and q, then the tensor product*

$$A \otimes \mathcal{E}(B, C) \subseteq A \otimes \mathrm{C}([0,1], B \otimes C)$$

of A with the "tensorial joint" algebra $\mathcal{E}(B, C)$ of Definition 2.9.9 is strongly purely infinite.

PROOF. Consider unital C^*-algebras B and C. Let H_1 denote the sub-algebra of $\mathrm{C}[0,1/2] \otimes C$ defined by the maps $f \in \mathrm{C}([0,1/2], C)$ with $f(0) = 1$, and H_2 the sub-algebra of maps $g \in \mathrm{C}[1/2,1] \otimes B$ with $g(1) = 1$. Then there are natural C^*-epimorphisms ϕ_1 from $B \otimes H_1$ onto $B \otimes C$ with $\phi_1(b \otimes f) = b \otimes f(1/2)$ and $\phi_2 \colon H_2 \otimes C \to B \otimes C$ with $\phi_2(g \otimes c) = g(1/2) \otimes c$.

It is easy to see that $\mathcal{E}(B, C) \subseteq \mathrm{C}([0,1], B \otimes C)$ is naturally isomorphic to the pull-back of ϕ_1 and ϕ_2.

It follows that $A \otimes \mathcal{E}(B, C)$ is the pull-back of the epimorphisms $\mathrm{id}_A \otimes \phi_1$ and $\mathrm{id}_A \otimes \phi_2$ onto $A \otimes B \otimes C$. In particular, it is an extension of $A \otimes B \otimes C$ by the ideal $(A \otimes \phi_1^{-1}(0)) \oplus (A \otimes \phi_2^{-1}(0))$ of $A \otimes \mathcal{E}(B, C)$.

Notice that $\mathcal{E}(\mathbb{C}, B)$ is the C^*-subalgebra of $\mathrm{C}([0,1], B)$ of continuos functions $f \colon [0,1] \to B$ wit $f(0) = 1_B$. Notice that $\phi_1^{-1}(0) \cong B \otimes \mathcal{E}(\mathbb{C}, C)$ and $\phi_2^{-1}(0) \cong \mathcal{E}(\mathbb{C}, B) \otimes C$.

For this ideals and for the $A \otimes B \otimes C$ the Lemma **??** applies if $B := M_{2^\infty}$ and $C := M_{3^\infty}$, because all tensorial absorb UHF algebras. Thus they are purely infinite infinite if A is (2-quasi-) traceless.

By Proposition 2.9.12 they are moreover strongly purely infinite because they have central sequences of matrix algebras.

Thus, $A \otimes \mathcal{E}(B, C)$ is an extension of strongly p.i. C^*-algebras if A is (2-quasi-) traceless. By Proposition **??** it follows that $A \otimes \mathcal{E}(B, C)$ is p.i., because the class of purely infinite C^*-algebras is invariant under extensions.

`next Ext.part of Thm. is not present here. Only cited.`

Since extensions of s.p.i. algebras are s.p.i. by Theorem 2.9.13, we get moreover that $A \otimes \mathcal{E}(B, C)$ is strongly p.i.

□

Corollary 2.9.11. *Let A a (2-quasi-) traceless C^*-algebra and $\mathcal{Z}$ the Jiang-Su algebra, then $A \otimes \mathcal{Z}$ is strongly purely infinite.*

Proof. If B is a unital C^*-algebra and $\psi \colon B \to B$ a unital C^*-morphism. The map ψ is called "trace-collapsing" if $\tau_1 \circ \psi = \tau_2 \circ \psi$ for any trace state on B, i.e., $\psi^*(T(A))$ consists of a fix-point of $T(A)$.

By [**693**, thm. 3.4], $\mathcal{Z}$ is the inductive limit

$$\mathrm{indlim}_n \left(\phi_n \colon \mathcal{E}(M_{2^\infty}, M_{3^\infty}) \to \mathcal{E}(M_{2^\infty}, M_{3^\infty}) \right),$$

where $\phi_n := \phi$ is a suitable fixed "trace-collapsing" unital endomorphism ϕ of $\mathcal{E}(M_{2^\infty}, M_{3^\infty})$. "Trace-collapsing" means here that for each trace state ρ on $\mathcal{E}(M_{2^\infty}, M_{3^\infty})$ the state $\rho \circ \phi$ is identical with the state induced on $\mathcal{E}(M_{2^\infty}, M_{3^\infty}) \subseteq \mathrm{C}([0,1], M_{2^\infty} \otimes M_{3^\infty})$ by $\mu(f) = \int_0^1 \tau(f(t))\, dt$, where τ is the unique trace state on $M_{2^\infty} \otimes M_{3^\infty}$.

We have seen above (in a more general case) that $A \otimes \mathcal{E}(M_{2^\infty}, M_{3^\infty})$ is s.p.i. because there exists a "central" sequence of unital C^*-morphisms $\psi_k \colon \mathcal{E}(M_p, M_q) \to \mathcal{E}(M_{2^\infty}, M_{3^\infty})$, where $p := 2^k$ and $q := 3^\ell$ for suitable $k, \ell \in \mathbb{N}$.

?????????

The classes of the purely infinite C^*-algebras and of the strongly purely infinite C^*-algebras are both invariant under inductive limits.

By Theorem **??**(????) the class of s.p.i. algebras is invariant under inductive limits. □

Proposition 2.9.12. *If A is (2-quasi-) traceless and B is a UHF algebra (of infinite dimension), then $A \otimes B$ is s.p.i.*

Proof. We can split B into a tensor-product $B = B_1 \otimes B_2$ where B_1 and B_2 are UHF-algebras of infinite dimension. Then $A \otimes B_1$ is purely infinite by Lemma **??**. Thus we can suppose that A is purely infinite. Let $a, b \in (A \otimes B)_+$

contractions and $\varepsilon > 0$ There exist contractions $c, d \in (A \otimes B)_+$ with $\|a - c\| < \varepsilon/2$ and $\|b - d\| < \varepsilon/2$

???? ???? □

THEOREM 2.9.13. *The class of s.p.i. C^*-algebras is permanent under extensions.*

(But only citation. The known proof is complicate.)

Other operations.?

PROOF. ??? □

COROLLARY 2.9.14. *If A is (2-quasi)-trace-less then $A \otimes \mathcal{Z}$ is strongly p.i.*

PROOF. Use that $\mathcal{Z}$ has central sequences of copies of $\mathcal{E}(M_2, M_3)$. □

Below we give later some results on tensor products that are farer going than those in Corollaries 2.9.2 and 2.9.10, e.g. we show that the tensor products $A \otimes B$ and $A \otimes \mathcal{E}(M_{2^\infty}, M_{3^\infty})$ considered in this corollaries are moreover strongly p.i.

PROOF. to be filled in ?? □

Synchronize below given Def's and statements with further above!!
Some from Paper with Blanchard.
Give extra Definitions (!) for all sorts of pure infiniteness!!!

REMARKS 2.9.15.

(i) If $a \in A_+$, $b \in M_n(A)_+$ and suppose that there is a matrix $e \in M_{m,n}(A)$ with $\|b - e^*(a \otimes 1_m)e\| < \varepsilon$ for some $\varepsilon > 0$, then

$$(b - \varepsilon)_+ = f^*((a - 2\eta)_+ \otimes 1_m)f$$

for some matrix $f \in M_{m,n}(A)$ with $\|f\| \leq \|e\|$ and some $\eta \in (0, \varepsilon/2)$.

The def's (ii) of p.i. etc. are given earlier in Section 1, later again !!! where???

(ii) An element $a \in A_+ \setminus \{0\}$ in a (not necessarily purely infinite) C^*-algebra A is **properly infinite** if, for every $\varepsilon > 0$, there exists a row $d = [d_1, d_2] \in M_{1,2}(A)$ (depending on a and ε) such that $\|d^*ad - a \otimes 1_2\| < \varepsilon$, cf. [**462**, def. 3.2].

This property of a is equivalent to $[a] + [a] \leq [a]$ in $\mathrm{Cu}(A)$, i.e., $a \oplus a \precsim a$, or to $a \in I(a)$ by definition of $I(a)$ in Definition 2.5.1, cf. also Lemma 2.5.3(i).

An element $a \in A_+$ is **properly infinite** if for every closed ideal J of A – that does not contain a – there is an element $h \neq 0$ in $(A/J)_+$ such that for every $\delta > 0$ there exists a row matrix $d = [d_1, d_2] \in M_{1,2}(A/J)$ with $\|d^*\pi_J(a)d - (\pi_J(a) \oplus h)\| < \delta$, cf. Part (v) of Lemma 2.5.3 or [**462**, prop. 3.14].

The latter says that $[h] + [\pi_J(a)] \leq [\pi_J(a)]$ in $\mathrm{Cu}(A/J)$, i.e., $I(\pi_J(a)) \neq 0$ for all $J \in \mathcal{I}(A)$ with $a \notin J$, where $I(b)$ for $b \in \pi_J(A)$ is defined by Definition 2.5.1.

This definition of properly infinite infinite (positive) elements coincides with Definition 2.5.1 of a properly element $a \in A$ by $a \oplus a \precsim a$.

(iii) A C^*-algebra A is purely infinite in the sense of Definition 1.2.1, if and only if, every element $a \in A_+ \setminus \{0\}$ is properly infinite, [**462**, thm. 4.16].

`there is also Ref. in ???? to place here!!`

(iv) All purely infinite C^*-algebras A have the *global* Glimm halving property of Definition *2.15.9*.

Proof. (i): We find $\eta \in (0, \varepsilon/2)$ such that we have $\|b - e^*((a-\eta)_+ \otimes 1_m)e\| < \varepsilon$. By Lemma 2.1.9 there is a contraction $d \in M_n(A)$ such that $f := ed$ is as desired.

(ii?): ??

The element $\pi_{I(a)}(a) = a + I(a)$ is always finite in $A/I(a)$ by Part (iii) of Lemma 2.5.3. Thus above property of $a \in A_+$ implies that $a \in I(a)$, i.e., $a \oplus a \precsim a$.

(ii?): ?? If one applies (i) with $m = 1, n = 2$ then one finds $u, v \in aAa$ with $u^*u = v^*v = (a-\varepsilon)_+$ and $u^*v = 0$ ([**462**, prop. 3.3(v)]), i.e., there exists a row $w = [u, v] \in M_{1,2}(aAa)$ satisfying

$$w^*w = (a-\varepsilon)_+ \otimes 1_2 \quad \text{in } \; A \otimes M_2 \,,$$

cf. also Lemma 2.5.3(ix,xi)

(iv): ??

For each $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$, then $b = vu^* \in aAa$ with u, v from (ii??) verifies $b^2 = 0$ and $\big((a-\varepsilon)_+\big)^2 = v^*bu$, so that $(a-\varepsilon)_+ \in AbA$. □

`How often is` f_ε `defined and mentioned??`

In the following, recall that $f_\varepsilon(t) := \min(1, \max(0, 2t/\varepsilon - 1))$ for $t \geq 0$ and $\varepsilon > 0$. Those special pice-wise linear functions have been used fist in Glimm's paper [**324**] for his pioneering study of C^*-algebras of Type I.

Notice that the Part (i) of the next Lemma 2.9.16 applies in particular to all non-zero elements $a \in A_+$ that are *spectral properly infinite* in sense of Definition 2.5.16.

`Later appears a similar remark!!!`

Lemma 2.9.16. *Let $a \in A_+$ with $\|a\| = 1$, $0 < \delta < \varepsilon < 1$, and let $B := \overline{aAa}$.*

Suppose that for every $\nu \in (\delta, \varepsilon)$, the element $(a-\nu)_+$ is properly infinite *in the sense of Definition 2.5.1.*

(i) *There exists an infinite sequence $w_1, w_2, \ldots \in (a-\delta)_+ A (a-\varepsilon)_+^{1/3}$ such that*

$$w_n^* w_m = \delta_{n,m}\,(a-\varepsilon)_+ \quad \textit{for all } \; m, n \in \mathbb{N}\,.$$

The element $d := \sum_{n \in \mathbb{N}} 2^{-n} w_n w_n^* \in B$ generates a stable *hereditary C^*-subalgebra $E_\varepsilon := \overline{dAd}$ of $(a-\delta)_+ A (a-\delta)_+$ that is naturally isomorphic to $D \otimes \mathbb{K}$ for $D := \overline{(a-\varepsilon)_+ A (a-\varepsilon)_+}$.*

In particular the algebra E_ε is a full hereditary C^*-subalgebra of the closed ideal of A that is generated by $(a-\varepsilon)_+$.

(ii) *If, moreover,* every *non-zero element of B is properly infinite, then there are elements $w_n \in B$ such that $w_n^* w_m = \delta_{n,m} \cdot (a - 2^{-n})_+$ for all $n \geq m \in \mathbb{N}$.*

(iii) *Let $d := \sum_{n\in\mathbb{N}} 2^{-n} w_n w_n^* \in B := \overline{a\,A\,a}$ where the elements $w_1, w_2, \ldots \in B$ are the elements from Part* (ii).

The hereditary C^*-subalgebra $D := \overline{d\,A\,d}$ is contained in B, and there exist elements $z_1, z_2, \ldots \in B$ with $z_j^* z_k = \delta_{j,k} a$ and $z_k z_j^* \in D$.

In particular, the contraction $e := \sum_n 2^{-n} z_n z_n^*$ generates a hereditary C^*-subalgebra $E := \overline{e\,D\,e} = \overline{e\,A\,e}$ of D that is full in B.

(iv) *The hereditary C^*-subalgebra $B := \overline{a\,A\,a}$ is itself stable if each element of B is properly infinite and the annihilators $\mathrm{Ann}((a-1/n)_+, B)$ of $(a-1/n)_+$ are full in B for each $n \in \mathbb{N}$.*

(v) *The annihilators $\mathrm{Ann}((a-1/n)_+, B)$ are full in B for each $n \in \mathbb{N}$ if B has no unital quotient.*

(vi) *The hereditary C^*-subalgebra $B := \overline{a\,A\,a}$ is stable if B has no unital quotient and a is "spectral" properly infinite, i.e., $\psi(a)$ is properly infinite or zero for every function $\psi \in \mathrm{C}_0(0, \|a\|]_+$.*

PROOF. Recall that an element a itself is properly infinite if $(a-\nu)_+$ is properly infinite for every $\nu \in (0, \delta)$ for some $\delta > 0$, and that the elements b^*b, bb^* and $(b^*b)^{1/2}$ are all properly infinite if one of its is properly infinite, cf. Lemma 2.5.3(xi,xii).

by Remark 2.9.15(ii) ?

Find precise Ref.s for next.

(i): We can take $\delta < \delta' < \varepsilon' < \varepsilon$, replace a by $(a-\delta')_+$ and then show moreover that there are elements $c_1, c_2, \ldots$ in the closure of $(a-\delta')_+ A (a-\varepsilon')_+$ such that $c_n^* c_m = \delta_{n,m} (a-\varepsilon')_+$ for $m, n \in \mathbb{N}$. We get the desired w_n as $w_n := c_n \cdot \varphi(c_n^* c_n)^{1/2}$ for the function $\varphi(t) := (t-\varepsilon)/(t-\varepsilon')$ for $t \geq \varepsilon$ and $\varphi(t) = 0$ for $t < \varepsilon$.

Hence, we can rename $(a-\delta')_+$ by a and $\varepsilon' - \delta'$ by ε and show only the existence of $w_k \in \overline{aA(a-\varepsilon)_+}$ with $w_k^* w_j = \delta_{j,k} (a-\varepsilon)_+$.

For $n \in \mathbb{N}$, let $\varepsilon_n := 2^{-n-1}\varepsilon$ and $\mu_n := \sum_{0\leq k\leq n} \varepsilon_k = (1-2^{-n-1}) \cdot \varepsilon < \varepsilon$.

Recall that, if $(a-\delta)_+$ is properly infinite, then, for each $\mu > \delta$, there exist $\tau \in (\delta, \mu)$ and $u, v \in A$ with

$$u^*v = 0\,, \quad u^*u = v^*v = (a-\mu)_+ \quad \text{and} \quad uu^* + vv^* \in (a-\tau)_+ A (a-\tau)_+\,. \tag{9.2}$$

Compare Remark 2.9.15(ii) or Lemma 2.5.3(ix,xi). `or other reference ?`

If we let $v_{-1} := a^{1/2}$, then we can find for each $n \in \mathbb{N}$ inductively – by repeated use of Equation (9.2) – elements $u_n, v_n \in aAa$ such that

(1) u_n, v_n in $v_{n-1} A v_{n-1}^* \subseteq \overline{aAa}$ $(n \in \mathbb{N})$

(2) for every $\nu \in (0, \varepsilon - \nu_n)$, the element $(v_n^* v_n - \nu)_+$ is either zero or properly infinite,
(3) $u_n^* u_n = v_n^* v_n = (v_{n-1}^* v_{n-1} - \varepsilon_n)_+ = (a - \mu_n)_+ \geq (a - \varepsilon)_+$ and
(4) $u_n^* v_n = 0$,

Induction over m shows that condition (1) implies that $v_{n-1} A v_{n-1}^* \subseteq v_{m+1} A v_{m+1}^* \subseteq v_m A v_m^*$ for $m = 1, \ldots, n-2$.

Thus, $u_m^* v_{n-1} A v_{n-1}^* = 0$ and $u_m^* u_n = 0$ for $m < n$ by (1) and (4). Hence, $u_m^* u_n = \delta_{m,n}(a - \mu_n)_+$ for $m, n \in \mathbb{N}$ by (3).

Notice that (1) and (3) imply $u_n, v_n \in v_{n-1} A (a - \mu_{n-1})_+ \subseteq aAa$.

For $n \in \mathbb{N}$, let $\varphi_n \colon \mathbb{R}_+ \to [0,1]$ be the continuous function with $\varphi_n(t) := 0$ for $t < \varepsilon$ and $\varphi_n(t) := (t - \varepsilon)/(t - \mu_n)$ for $t \geq \varepsilon$.

The elements $w_n = u_n f_n(a)^{1/2} \in \overline{aA(a - \varepsilon)_+}$ satisfy the requested relations, because $(t - \mu_n)_+ \varphi_n(t) = (t - \varepsilon)_+$.

Let $B := \overline{(a - \varepsilon)_+ A (a - \varepsilon)_+}$. The relations $w_k^* w_\ell = \delta_{k,\ell}(a - \varepsilon)_+$ imply that the polar decomposition of the w_k define a natural isomorphism $B \cong \overline{w_1 A w_1^*}$ and that the C^*-subalgebra $D := C^*(w_m b w_n^* \, ; \; n, m \in \mathbb{N}, \; b \in A)$ of $\overline{aAa}$ is naturally isomorphic to $B \otimes \mathbb{K}$. In particular D is stable.

The algebra D contains $d := \sum_{n \in \mathbb{N}} 2^{-n} w_n w_n^*$ and is identical with the hereditary C^*-subalgebra $\overline{dAd} \subseteq B$, with d as strictly positive element.

The hereditary subalgebra D generates the same closed ideal of A as the element $(a - \varepsilon)_+$ because $2 w_1^* d w_1 = (w_1^* w_1)^2 = (a - \varepsilon)_+^2$.

(ii): Let $a \in A_+$ with $\|a\| = 1$, and suppose that *every* non-zero element of $B := \overline{aAa}$ is properly infinite and let $D \subseteq B$ (any) *full* hereditary C^*-subalgebra of B.

(α) Suppose that $D \subseteq B := \overline{aAa}$ (with $a \in A_+$, $\|a\| = 1$) is a full hereditary C^*-subalgebra of B and that each non-zero $e \in D_+$ is properly infinite.

Then for each $\gamma \in (0,1)$ there exists elements $u, v \in B$ with $u^* u = v^* v = (a - \gamma/2)_+$, $v^* u = 0$ and $vv^*, uu^* \in D$.

(β) If D is any full hereditary C^*-subalgebra of $B := \overline{aAa}$, where $a \in A_+$ with $\|a\| = 1$, and if

`check here if a weaker property also works!`

every element of D_+ is properly infinite, then for each $\gamma \in (0,1)$ there exists an element $w \in B$ that satisfies $w^* w = (a - \gamma)_+$, $ww^* \in D$, and that the annihilator $\mathrm{Ann}(ww^*, D)$ of ww^* in D is again a *full* hereditary C^*-subalgebra of B.

We postpone the proof of (α) and (β) and show that (β) allows to construct the in Part (ii) requested elements $w_1, w_2, \ldots \in B := \overline{aAa}$ with the relations $w_n^* w_m = \delta_{n,m} \cdot (a - 2^n)_+$ for $n \geq m$.

We proceed by induction using (β):

We find by (β) an element $w_1 \in D_0 := B$ with $w_1^* w_1 = (a - 1/2)_+$ such that $D_1 := \mathrm{Ann}(w_1 w_1^*, B)$ is a full hereditary C^*-subalgebra of B.

Suppose we have selected $w_1, \ldots, w_n \in B$ that satisfy

$$w_k^* w_\ell = \delta_{k,\ell} \cdot (a - 2^{-\ell})_+$$

for $k \leq \ell \leq n$, and that

$$D_n := \mathrm{Ann}(w_1 w_1^* + \cdots + w_n w_n^*, B) = \mathrm{Ann}(w_n w_n^*, D_{n-1})$$

is full in B.

Recall here that $\mathrm{Ann}(e, D) = \mathrm{Ann}(e, B) \cap D$ for hereditary $D \subseteq B$ and $e \in D_+$.

Thus, we can apply (β) to $D := D_n$ and $\gamma := 1/2^{n+1}$ and get $w_{n+1} \in D_n$ with $w_{n+1}^* w_{n+1} = (a - 2^{-(n+1)})_+$ and

$$D_{n+1} := \mathrm{Ann}(w_{n+1} w_{n+1}, D_n) = \mathrm{Ann}(w_1 w_1^* + \cdots + w_{n+1} w_{n+1}^*, B)$$

is full in B. It implies that

$$w_{n+1}^* w_k = \delta_{n+1,k} (a - 2^{-(n+1)})_+ \quad \text{for} \quad k = 1, \ldots, n+1 \ .$$

Proof of observation (α): Let $\gamma \in (0,1)$. Since D is full in B and $a \in B_+$ there exists $d_1, \cdots, d_n \in B$ with $d_k d_k^* \in D$ for $k = 1, \ldots n$ and

$$d_1^* d_1 + \cdots + d_n^* d_n = (a - \gamma/4)_+ \, .$$

It follows $n[e] \geq [(a - \gamma/4)_+]$ in $\mathrm{Cu}(B)$, for $e := d_1 d_1^* + \cdots + d_n d^* \in D$.

Since, by assumptions, $e \in D_+$ is a properly infinite element of B, we get $[e] \geq 2n[e] \geq 2[(a - \gamma/4)_+]$, i.e.,

$$(a - \gamma/4)_+ \oplus (a - \gamma/4)_+ \precsim e \oplus 0 \, .$$

In particular there exists $f_1, f_2 \in B$ and $\mu > 0$ with

$$[f_1^*, f_2^*](e - \mu)_+ [f_1, f_2]^\top = \mathrm{diag}(a - \gamma/2)_+ \oplus (a - \gamma/2)_+ \, .$$

The elements $u := (e - \mu)_+^{1/2} f_1$ and $v := (e - \mu)_+^{1/2} f_2$ satisfy $uu^*, vv^* \in D$, $u^* v = 0$ and $u^* u = v^* v = (a - \gamma/2)_+$.

Proof of observation (β): Each hereditary C^*-subalgebra D of $B := \overline{aAa}$ that is full in B and is purely infinite fulfills the assumptions for Part (α).

Let $\gamma \in (0,1)$ and let $u, v \in B$ the elements from Part (α), i.e., $u^* u = v^* v = (a - \gamma/2)_+$, $u^* v = 0$ and $uu^*, vv^* \in D$.

The function ψ_γ defined by $\psi_\gamma(t) := (t - \gamma)/(t - \gamma/2)$ for $t \geq \gamma$ and $\psi_\gamma(t) := 0$ for $t \leq \gamma$ is in $\mathrm{C}_0(0,1]_+$ and satisfies $\psi_\gamma(t) \leq 1$. Clearly

$$\psi_\gamma(a)(a - \gamma/2)_+ = (a - \gamma)_+ = ((a - \gamma/2)_+ - \gamma/2)_+ \, .$$

Define $w := v \psi_\gamma(a)^{1/2} \in B$. It satisfies $w^* w = (a - \gamma)_+ = (v^* v - \gamma/2)_+$ and $ww^* = (vv^* - \gamma/2)_+$. The latter equation can be seen with help of the polar decomposition $v = z_v (v^* v)^{1/2} = z_v (a - \gamma/2)_+$ using that $f(vv^*) z_v = z_v f(v^* v)$ for $f \in \mathrm{C}_0(0, \|v\|^2]$.

The elements ww^*, vv^* and $d := uv^*$ are in D and satisfy $ww^* = (vv^* - \gamma/2)_+$, $d^*(vv^* - \gamma/2)_+ = vu^*ww^* = 0$ and $[d] = [vv^*]$ in $\mathrm{Cu}(D)$. The latter because $d^*d = vu^*uv^* = v(a - \gamma/2)_+v^* = (vv^*)^2$. Thus, the hereditary C^*-subalgebra $\mathrm{Ann}(ww^*, D)$ is full in D by the last observation in Part (ii) of Lemma 2.5.14. Since D is full in B it implies that $\mathrm{Ann}(ww^*, D)$ is a hereditary C^*-subalgebra of B that is full in B.

(iii): TEXT:

Let $d := \sum_{n\in\mathbb{N}} 2^{-n} w_n w_n^* \in B := \overline{a\,A\,a}$ where the elements $w_1, w_2, \ldots \in B$ are the elements found in Part (ii).

The hereditary C^*-subalgebra $D := \overline{d\,A\,d}$ is contained in B, and there exist elements $z_1, z_2, \ldots \in B$ with $z_j^* z_k = \delta_{j,k} a$ and $z_k z_j^* \in D$.

In particular, the contraction $e := \sum_n 2^{-n}\, z_n\, z_n^* \in B$ generates a hereditary C^*-subalgebra $E := \overline{e\,D\,e} = \overline{e\,A\,e}$ of D that is full in B.

We define for $t \geq 0$ and $m, n \in \mathbb{N}$, $m < n$, the increasing function $\varphi(m, n; \cdot) \in \mathrm{C}_0(0, 1]$ with $\varphi(m, n; t) := 0$ on $[0, 2^{-m}]$ and $\varphi(m, n; t) := \max(t - 2^{-m}, 0)/\max(t - 2^{-n}, 0)$ for $t \geq 2^{-m}$.

Let $\lambda \colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ a bijective map from $\mathbb{N}$ onto $\mathbb{N} \to \mathbb{N}$ and let $M_k := \lambda^{-1}(\{k\} \times \mathbb{N})$.

Notation M_k is not good,
NEXT property HERE necessary?

We can re-index the infinite subsets M_k of $\mathbb{N}$ such that they have the property that $\min(M_k) < \min(M_{k+1})$ for $k \in \mathbb{N}$.

The map λ defines a unique increasing injective map $\mu_k \colon \mathbb{N} \to \mathbb{N}$ with $\mu_k(\mathbb{N}) = M_k$.

We let $a[k, 1] := 1 \in C^*(a, 1) \subseteq \mathcal{M}(B)$,

$$a[k, n+1] := 1 - \varphi(\mu_k(n), \mu_k(n+1); a) \ \in C^*(a, 1)_+\,,$$

and define elements $v_k \in B$ for $k \in \mathbb{N}$ by

$$v_k := \sum_{n\in\mathbb{N}} w_{\mu_k(n)} \cdot a[k, n]^{1/2}\,.$$

The sum converges in B, because $w^*_{\mu_k(n)} w_{\mu_k(m)} = \delta_{m,n}(a - 2^{-n})_+$ and, for $n > 1$,

$$a[k, n]^{1/2} \cdot w^*_{\mu_k(n)} w_{\mu_k(n)} \cdot a[k, n]^{1/2} = (a - 2^{-\mu_k(n)})_+ - (a - 2^{-\mu_k(n-1)})_+\,.$$

It has norm $\leq 2^{-\mu_k(n-1)} - 2^{-\mu_k(n)} < 2^{-\mu_k(n-1)}$. The series $\sum_{n>1}(2^{-\mu_k(n-1)} - 2^{-\mu_k(n)})$ is absolutely convergent with sum $= 2^{-\mu_k(1)}$.

One gets from $w_\ell^* w_m = 0$ for $\ell \neq m$ that

$$v_k[n]^* v_k[n] = (a - 2^{-\mu_k(n)})_+$$

for the partial sums

$$v_k[n] := \sum_{m=1}^{n} w_{\mu_k(m)} \cdot a[k, m]^{1/2}\,.$$

It follows that

$$v_k^* v_k = \sum_n (a - 2^{-\mu_k(n)})_+ a[k,n] = a\,.$$

Since $w_m^* w_n = 0$ and $w_n w_m^* \in D$ for $m \neq n$ and $\mu_k(\mathbb{N}) \cap \mu_\ell(\mathbb{N}) = \emptyset$ for $k \neq \ell$, it follows that $v_k^* v_\ell = \delta_{k,\ell} a$. and $v_\ell v_k^* \in D$.

(iv):

TEXT:

The hereditary C^*-subalgebra $B := \overline{a\,A\,a}$ is itself stable if each element of B is properly infinite and the annihilators $\mathrm{Ann}((a-1/n)_+, B)$ of $(a-1/n)_+$ are full in B for each $n \in \mathbb{N}$.

to be filled in ??

We show that it implies that there exists a zero sequence $\|a\| > \gamma_1 > \gamma_2 > \cdots > 0$ and elements $u_n \in B$ with

(1) $u_n^* u_n = (a - \gamma_n)_+$,
(2) $u_{n+1}^*(a - \gamma_n)_+ = 0$, and
(3) $u_n = \lim_{n\to\infty} u_n (a - \gamma_{n+1})_+^{1/n}$.

What can be done with it ?

Since a is a strictly positive element of B it follows that the criteria for stability of B is satisfied for B. Alone from (1) and (2)?

Indeed we can define for the exhausting family of order preserving injective maps $???_k \colon \mathbb{N} \to M_k := \lambda^{-1}(\{k\} \times \mathbb{N})$ from a partition $\lambda\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

(v): Let J denote the closed ideal of B generated by $\mathrm{Ann}((a-1/n)_+, B)$. By Lemma 2.5.14, B/J is unital and $\pi_J(a)$ is invertible in B/J if $J \neq B$. Thus, $J = B$ if B has no unital quotients. □

REMARK 2.9.17. Let $a \in A_+$ with $\|a\| = 1$. The hereditary C^*-subalgebra $\overline{aAa}$ contains a full stable hereditary C^*-subalgebra D, if and only if, *there exists a sequence of elements* $b_n \in \overline{aAa}$ *with* $b_m^* b_n = \delta_{m,n}(a - 2^{-n})_+$ *for* $m, n \in \mathbb{N}$.

Here "full" – equivalently – means that a is in the closed ideal generated by D, i.e., that $a \in A \cdot D \cdot A = \overline{\mathrm{span}(ADA)}$ by stability of D.

It implies that there are $g_1, g_2, \ldots \in A$ with $g_m^* g_n = \delta_{m,n} a$ and $g_m g_m^* \in D$.

It is not clear when $C^*(b_n b_m^*\,;\ m, n \in \mathbb{N})$ itself is stable.

PROOF. The existence of the quoted sequence $d_1, d_2, \ldots$ is necessary:
If D is a stable and full hereditary C^*-subalgebra of $B := \overline{aAa}$ then there exists $d \in B$ with $d^* d = a$ and $dd^* \in D$, cf. Lemma ??(ii).

Since D is stable, its multiplier algebra $\mathcal{M}(D)$ contains a unital copy of $\mathcal{L}(\ell_2) \supset \mathcal{O}_\infty = C^*(s_1, s_2, \ldots\,;\ s_m^* s_n = \delta_{m,n} 1)$, cf. Remark 5.1.1(8). We can take $b_n := s_n d g_n(a)$, where $g_n(0) := 0$ and $g_n(t) := ((t - 2^{-n})_+/t)^{1/2}$ with $(t-\mu)_+ := \max(0, t-\mu)$ for $t > 0$.

Conversely, the existence of elements $b_n \in \overline{aAa}$ with $b_m^* b_n = \delta_{m,n}(a - 2^{-n})$ (for $m \leq n$) is sufficient for the existence of a full stable hereditary C^*-subalgebra $D \subseteq \overline{aAa}$ that contains a in the ideal ADA of A, because by the below given calculations $C^*(b_m b_n^*\,;\ m,n \in \mathbb{N}\,)$ is isomorphic to the hereditary C^*-subalgebra $E := \overline{d(C^*(a) \otimes \mathbb{K})d}$ of $C^*(a) \otimes \mathbb{K}$ where

$$d := \operatorname{diag}((a - 1/2)_+, (a - 1/4)_+, \ldots)^{1/2} \in \ell_\infty(C^*(a)) \subset \mathcal{M}(C^*(a) \otimes \mathbb{K})\,,$$

i.e., $e := \operatorname{diag}(1/2(a - 1/2)_+, \ldots, 2^{-n}(a - 2^{-n})_+, \ldots)$ is a strictly positive element of $C^*(a) \otimes \mathbb{K}$.

An isomorphism from E onto $C^*(b_m b_n^*\,;\ m,n \in \mathbb{N}\,)$ can be defined as follows:

First define $\lambda\colon C^*(a) \otimes \mathbb{K} \to B \otimes \mathbb{K}$ as the natural embedding. The λ is non-degenerate because $B = \overline{a\,B\,a}$. Therefore the strictly continuous $\mathcal{M}(\lambda)\colon \mathcal{M}(C^*(a) \otimes \mathbb{K}) \to \mathcal{M}(B \otimes \mathbb{K})$ exists and is injective natural inclusion of $\mathcal{M}(C^*(a) \otimes \mathbb{K})$ into $\mathcal{M}(B \otimes \mathbb{K})$. Let $g := \mathcal{M}(\lambda)(d)$ and $h := \lambda(e) = 1/2(a - 1/2)_+ \otimes p_{1,1} + 1/4(a - 1/4)_+ \otimes p_{2,2} + \cdots$. Then $\lambda(E)$ is a non-degenerate C^*-subalgebra of the hereditary C^*-subalgebra $\overline{g(B \otimes \mathbb{K})g} = \overline{h(B \otimes \mathbb{K})h} =: F$. It is easy to see that $\lambda(E) := F \cap \lambda(C^*(a) \otimes \mathbb{K})$, because it is generated by the elements $(a - 2^{-m})^{1/2}(a - 2^{-n})_+^{1/2} \otimes p_{m,n}$, and, even by definition of h, $h = \lambda(e) \in F_+$ is a strictly positive contraction of F and $\lambda(E)$.

The element $P := \sum_n 2^{-n} b_n b_n^* \in B$ is a strictly positive contraction of $C^*(b_m b_n^*\,;\ n,m \in \mathbb{N})$.

Let $T := \sum_n 2^{-n/2} b_n \otimes p_{1,n} \in B \otimes \mathbb{K}$. Then $TT^* = P \otimes p_{1,1}$ and

$$T^*T = \sum_n 2^{-n}(a - 2^{-n})_+ \otimes p_{n,n} = h\,.$$

The polar decomposition $T = U(T^*T)^{1/2}$ of T is given by the partial isometry

$$U := \sum_n v_n \otimes p_{1,n} \;\in\; B^{**} \overline{\otimes} \mathcal{L}(\ell_2) \cong (B \otimes \mathbb{K})^{**}\,,$$

where $(T^*T)^{1/2} = h^{1/2}$ and $(b_n b_n^*)^{1/2} v_n = v_n (b_n^* b_n)^{1/2} = b_n$. Notice that $U^*U = \sum_n v_n^* v_n \otimes p_{n,n}$, and $UU^* = \sum_n v_n v_n^* \otimes p_{1,1}$, the element $v_n^* v_n \in B^{**}$ is equal to the open support projection P_n of $(a - 2^{-n})_+$ in B^{**}, $v_n(a - 2^{-n})_+^{1/2} = b_n$ and $v_m^* v_n = \delta_{m,n} P_n$. In particular $U^*(b_n b_m^* \otimes p_{1,1})U = (a - 2^{-n})_+^{1/2}(a - 2^{-m})_+^{1/2} \otimes p_{n,m}$. Thus

$$U^*(C^*(b_n b_m^*\,;\ m,n \in \mathbb{N}) \otimes p_{1,1})U \;=\; \lambda(E)\,.$$

The C^*-algebra E is isomorphic to a C^*-subalgebra of $C^*(a) \otimes \mathbb{K}$ of the sort considered in Lemma 2.9.18 and contains therefore a full stable C^*-subalgebra by Lemma 2.9.18. □

LEMMA 2.9.18. *Let $\mu_1 > \mu_2 > \cdots$ a strictly decreasing zero sequence in $(0,1)$ and define a positive contraction $d \in \mathrm{C}_0(0,1] \otimes \mathbb{K}$ by*

$$d := \operatorname{diag}(2^{-1}(f_0 - \mu_1)_+\,,\ 2^{-2}(f_0 - \mu_2)_+\,,\ \ldots)^{1/2} \in \mathrm{C}_0(0,1] \otimes c_0 \subset \mathrm{C}_0(0,1] \otimes \mathbb{K}\,,$$

where $f_0 \in \mathrm{C}_0(0,1]$ is given by $f_0(t) := t$.

The hereditary C^*-subalgebra $E := \overline{d\,(\mathrm{C}_0(0,1]\otimes\mathbb{K})\,d}$ of $\mathrm{C}_0(0,1]\otimes\mathbb{K}$ contains a full stable C^*-subalgebra D of E.

PROOF. Let $\lambda\colon \mathbb{N}\to\mathbb{N}\times\mathbb{N}$ a bijective map from $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$.

Then λ defines a decomposition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets $M_n := \lambda^{-1}(\{n\}\times\mathbb{N})$. The natural order in M_n leads to bijective order-preserving surjective maps $\psi_n\colon \mathbb{N}\to M_n$ from $\mathbb{N}$ onto M_n. Let $n_0, n_1\in\mathbb{N}$ with $1\in M_{n_0}$ and with $\min(\mathbb{N}\setminus M_{n_0})\in M_{n_1}$. Then $\psi_{n_0}(k)=k$ for $k=1,\ldots,\psi_{n_1}(1)-1$ and $\psi_n(k)>k$ for all other pairs (n,k).

We define isometries $T_n\in\mathcal{L}(\ell_2)\cong\mathcal{M}(\mathbb{K})$ by $T_n(\eta_k):=\eta_{\psi_n(k)}$ for the canonical ONB $\{\eta_1,\eta_2,\ldots\}$ of $\ell_2(\mathbb{N})$. The T_n satisfy the relations $T_m^*T_n=\delta_{m,n}1$ and $\sum_n T_nT_n^*$ converges strictly to $1\in\mathcal{M}(\mathbb{K})$.

We define $d_k := (1\otimes T_k)d$, then $d_j^*d_k = \delta_{j,k}d^2\in E$ and `Check next`

$$d_kd_k^* = \sum_n 2^{-n}(f_0-\mu_n)_+\otimes e_{\psi_k(n),\psi_k(n)}\,.$$

Since $2^{-n}(f_0-\mu_n)_+ \le 2^{-\psi_k(n)}(f_0-\mu_{\psi_k(n)})$???

$d_kd_k^* =$?????

We get $d_kd_k^*\in E$ and $d_k\in E$, because always $n\le\psi_k(n)$, E is hereditary and $d_k^*d_k = d^2\in E$.

Thus $D := C^*(d_jd_k^*\,;\; j,k\in\mathbb{N})$ is a full and stable C^*-subalgebra of E. □

REMARK 2.9.19. The algebra $E := \overline{d(\mathrm{C}_0(0,1]\otimes\mathbb{K})d}$ of Lemma 2.9.18 is the inductive limit of the hereditary C^*-subalgebras
$E_n := \overline{g_n(\mathrm{C}_0(0,1]\otimes\mathbb{K})g_n}$ with positive contractions $g_n\in\ell_\infty(\mathrm{C}_0(0,1]\otimes\mathbb{K})\subset\mathcal{M}(\mathrm{C}_0(0,1]\otimes\mathbb{K})$ given by the sequences

$$g_n := \mathrm{diag}(2^{-1}(f_0-\mu_1)_+\,,\,\ldots\,,\;2^{-n}(f_0-\mu_n)_+\,,\;2^{-n}(f_0-\mu_n)_+\,,\ldots)^{1/2}\,,$$

that are stationary beginning from n-th entry.

It is likely that the E_n are stable because each $\mathcal{M}(E_n)$ is properly infinite and E_{n+1} is a extensions of $\mathbb{K}$ by

$$E_n\oplus(0_n\oplus\mathrm{C}_0(\mu_{n+1},\mu_n)\otimes\mathbb{K})\,.$$

REMARK 2.9.20. The algebra $D := \overline{d\,A\,d}$ of Part (iii) of Lemma 2.9.16

`give reason for`
`''HAS NO UNITAL QUOTIENT''`

has no unital quotient and, for each element $b\in D_+$ and $\varepsilon>0$, there exist $c := c(b,\varepsilon)\in D$ such that $\|c^*bc\|<\varepsilon$ and $\|c^*c-b\|<\varepsilon$. By a criterium of J. Hjelmborg and M. Rørdam in [**373**], this properties of D imply that D itself is a stable σ-unital full C^*-subalgebra of $\overline{a\,A\,a}$. (Compare also Corollary 5.5.2 for an alternative proof of this stability criterium.)

It follows that *every σ-unital purely infinite C^*-algebra A contains a stable σ-unital hereditary C^*-subalgebra D that is full in A.* (cf. J. Hjelmborg, M. Rørdam, [**373**], [**688**, prop. 5.4]).

10. Local stability and properly infinite full projections

Recall that a non-zero element $a \in A$ is **stable** if $\overline{a^*Aa}$ is stable, cf. Definition 2.1.1 of stable elements and Definition 2.0.3 of locally purely infinite C^*-algebras.

(Perhaps one can here moreover take 1-stable elements or even n-stable elements in sense of Definition 2.1.3?)

Lemma 2.10.1. *A C^*-algebra B is* locally purely infinite *in the sense of Definition 2.0.3, if and only if, for every non-zero $b \in B_+$ and $\varepsilon \in (0, \|b\|)$, there exist $n = n(b, \varepsilon) \in \mathbb{N}$ and* stable *positive elements $a_1, \ldots, a_n$ in the hereditary C^*-subalgebra $\overline{bBb}$ such that $(b-\varepsilon)_+$ is in the ideal of B generated by $\{a_1, \ldots, a_n\}$.*

Proof. Consider the algebraic ideal J_0 that is generated by all stable positive element of $\overline{bBb}$. The closure J of the $*$-ideal J_0 is an ideal of B. We show that $b \in J$:

Suppose $\gamma := \|\pi_J(b)\| > 0$. Then there is a pure state ρ on B/J with $\rho(\pi_J(b)) = \gamma$, simply by extending a character χ on $C^*(\pi_J(b))$ with $\chi(\pi_J(b)) = \|\pi_J(b)\|$ to a pure state ρ on B/J.

Then $\lambda := \rho \circ \pi_J$ is a pure state on B that satisfies $\lambda(b) = \gamma > 0$ and $\lambda(J) = \{0\}$.

Let K the kernel of the irreducible representation $D_\lambda \colon B \to \mathcal{L}(L_2(B, \lambda))$ corresponding to λ. Clearly $J \subseteq K$ follows from $\lambda(J) = 0$.

By Definition 2.0.3 there is a non-zero stable hereditary C^*-subalgebra $D \subseteq \overline{bBb}$ which is not contained in K. Therefore, D is not contained in J. The isomorphism $D \cong F \otimes \mathbb{K}$ (for some C^*-algebra F) shows that there is a separable stable C^*-subalgebra $D_1 \subseteq D$ with $D_1 \not\subseteq J$. Then a strictly positive contraction $a \in (D_1)_+$ is not in J, but a is stable, because $\overline{aBa} = D_1$. This contradicts the assumption $b \notin J$.

The algebraic ideal J_0 generated by the positive stable elements $a \in \overline{bBb}$ is dense in $J \supset \overline{bBb}$ and (therefore) contains the Pedersen ideal of $\overline{bBb}$. In particular, $(b-\varepsilon)_+ \in J_0$ for all $\varepsilon > 0$. □

Lemma 2.10.2. *Each non-zero locally purely infinite C^*-algebra A (in the sense of Definition 2.0.3) is (2-quasi-) traceless.*

Proof. Let $\tau \colon A_+ \to [0, \infty]$ a lower s.c. 2-quasi-trace. If $a \in A_+$ is a non-zero contraction with $\tau(a) \neq \infty$ and if $\delta \in (0, \|a\|)$ then $D := \overline{(a-\delta)_+ A (a-\delta)_+}$ is a hereditary C^*-subalgebra of A with $(a-\delta)_+ \in D_+$, $d\varphi_\delta(a) = \varphi_\delta(a)d = d$ for all $d \in D$, where

$$\varphi_\delta(t) := \min(1, \max(0, 2t/\delta - 1)) \leq 2t/\delta\,,$$

and

$$\tau(d) \leq \|d\| \cdot 2\tau(a)/\delta \quad \text{for all } d \in D_+ . \tag{10.1}$$

We show below that the Inequalities (10.1) yield that $\tau(d) = 0$ for all $d \in D_+$, in particular that $\tau((a-\delta)_+) = 0$. The lower semi-continuity of τ shows then in general that $\tau(a) = 0$ if $\tau(a) < \infty$, i.e., that τ takes only the values 0 and $+\infty$ on A_+. This holds for all l.s.c. 2-quasi-traces τ on A_+ and means that A is traceless.

The Inequalities (10.1) show that τ is bounded on the set of contractions in $D_+ \neq \{0\}$.

`Next similar to above said!!?`

The Inequalities 10.1 hold because $d \leq \|d\|\varphi_\delta(a) \leq \|d\|(2/\delta)a$ for $\varphi_\delta(t) := \min((2/\delta)(t-\delta/2)_+, 1)$, and $C^*(d, \varphi_\delta(a))$ and $C^*(a)$ are commutative C^*-subalgebras of A.

`Compare next with above`
`and Lemma on stably generated C*-alg's.`

If $b \in D_+$ is a non-zero stable contraction, then the multiplier algebra of $\overline{bDb}$ contains a copy of $\mathcal{O}_\infty$ unitally. Thus, there are $d_1, d_2, \ldots \in \overline{bDb}$ with $d_j^* d_i = \delta_{i,j} b$. It implies that $n\tau(d) = \tau(\sum_{j=1}^n (d_j)(d_j)^*) \leq (2/\delta)\tau(a)$ for each $n \in \mathbb{N}$. Thus $\tau(d) = 0$.

Since the set of $d \in D_+$ with $\tau(d) = 0$ is the positive part of some closed ideal of D, and since the ideal generated by the stable elements in D_+ is dense in D by Lemma 2.10.1, we get that $\tau(D_+) = \{0\}$. It shows that, $\tau((a-\delta)_+) = 0$ for every $\delta > 0$. Since τ is lower semi-continuous, we get $\tau(a) = 0$ for all $a \in A_+$ with $\tau(a) \neq \infty$. □

LEMMA 2.10.3. *The primitive ideal space* Prim(A) *can be covered by a countable family of quasi-compact subsets – i.e.,* Prim(A) *is σ-compact –, if and only if, there exists $a \in A_+$ with $J(a) := \overline{\text{span}(AaA)} = A$ (i.e., if and only if, A contains a full σ-unital hereditary C^*-subalgebra $D := \overline{aAa}$).*

The space Prim(A) *is quasi-compact, if and only if, there exists $a \in A_+$ with $\|a + J\| = 1$ for all primitive ideals $J \in$* Prim(A)

PROOF. For $b \in A_+$ with $\|b\| \leq 1$, the **generalized Gelfand transformation** $\widehat{b}\colon \text{Prim}(A) \to \mathbb{R}_+$ is defined by $\widehat{b}(J) := \|b + J\|$ for all $J \in \text{Prim}(A)$. (It is called $\check{b}$ and $\check{A}$ in [**616**].)

Notice here that the function $\widehat{b}$ is lower semi-continuous on the topological space Prim(A), i.e., the $J \in \text{Prim}(A)$ with $\widehat{b}(J) > s \geq 0$ build an open subset of Prim(A).

Let $b \in A$ and $t \in (0, \infty)$. We denote by $C_t(b) \subseteq \text{Prim}(A)$ the set of $J \in \text{Prim}(A)$ with $(\|b + J\| =)\|\pi_J(b)\| \geq t$.

`Relate next together:`

It is known that $C_t(b)$ is a (quasi-) compact subset of Prim(A), cf. [**217**, prop. 3.3.7], [**616**, prop. 4.4.4].

Then $\widehat{b}^{-1}[t, \|b\|] = C_t(b)$, and, therefore, $C_t(b)$ is a quasi-compact subset of $\mathrm{Prim}(A)$.

Let $a \in A$ with $I(a) = A$, then $\| a + J \| := \| \pi_J(a) \| > 0$ for any closed ideal J of A with $J \neq A$.

It follows $\mathrm{Prim}(A) = \bigcup_n C_{1/n}(a)$. Now use that the sets $C_{1/n}(a)$ are quasi-compact. Thus, $\mathrm{Prim}(A)$ is σ-quasi-compact.

Conversely: Let K_n quasi-compact subsets of $\mathrm{Prim}(A)$ such that $\mathrm{Prim}(A) = \bigcup_n K_n$. The supports in $\mathrm{Prim}(A)$ of the functions $\widehat{b}$ with $b \in A_+$ and $\|b\| < 1$ build an upward directed system of open subsets of $\mathrm{Prim}(A)$, because for $b_1, b_2 \in A_+$ with $\|b_i\| < 1$ there is $b_3 \in A_+$ with $\|b_3\| < 1$ and $b_i \leq b_3$ $(i = 1, 2)$, cf. proof of [**616**, thm. 1.4.2], and the generalized Gelfand transformation $b \mapsto \widehat{b}$ is monotone on A_+. For each primitive ideal J of A there is $b \in A_+$ with $\|b\| < 1$ and $\widehat{b}(J) = \|b + J\| > 0$. Thus, by quasi-compactness of K_n, there are $b_n \in A_+$ with $\|b_n\| < 1$ such that K_n is contained in the support of $\widehat{b_n}$. It follows that $a := \sum_n 2^{-n} b_n \in A_+$ satisfies $\|a + J\| > 0$ for all $J \in \mathrm{Prim}(A)$, i.e., $J(a)$ is not contained in any primitive ideal of A. Thus $J(a) = A$.

`Notation overlap:`
`former notation` $I(a)$ `had two different meanings!!!`
`Here we have changed to` $J(a) := \overline{\mathrm{span}(A\,a\,A)}$.

If, moreover, $\mathrm{Prim}(A)$ is quasi-compact, then the open subsets $U_n := \widehat{a}^{-1}(1/n, \infty)$ of $\mathrm{Prim}(A)$ cover $\mathrm{Prim}(A)$. It follows that $U_p = \mathrm{Prim}(A)$ for some $p \in \mathbb{N}$. We can replace a by $f(a)$ for the function $f(t) := \min(p \cdot t, 1)$. Then $\widehat{a} = 1$. □

`Is here the right place to show semi-projectivity of` $\mathcal{E}_n$`?`
`Better put it in the appendices. Near to other` $\mathcal{E}_n$ `stuff?`

For later applications of the next result, recall that locally p.i. algebra are traceless, cf. Lemma 2.10.2.

PROPOSITION 2.10.4. *Suppose that A is traceless (respectively that A is purely infinite) and that* $\mathrm{Prim}(A)$ *is quasi-compact.*

`Give ref. to Def. of "quasi-compact"` (T_0 `spaces)`

Then there exists $n \in \mathbb{N}$ and a full properly infinite projection $p \in M_n(A)$ (respectively a full properly infinite projection $p \in A$).

PROOF. At first, we consider the case where A is purely infinite, i.e., we consider case where the element a is properly infinite for all non-zero $a \in A$. Later we reduce the case of traceless A to the case of p.i. A.

Since $\mathrm{Prim}(A)$ is quasi-compact (by assumption), there exists a positive contraction $a \in A_+$ with $\|a + I\| = 1$ for all primitive ideals $I \in \mathrm{Prim}(A)$ (cf. Lemma 2.10.3).

It follows that $(a-t)_+$ is a full positive contraction in A for all $t \in [0,1)$, i.e., $J((a-t)_+) := \overline{\mathrm{span}(A(a-t)_+A)} = A$.

Since every $(a-t)_+$ is properly infinite and full, there are $d_1, d_2 \in A$ with $d_k^*(a-2/3)_+ d_k = (a-1/4)_+$ $(k=0,1)$ and $d_1^*(a-2/3)_+ d_2 = 0$.

Let $e := (a-2/3)_+^{1/2} d_2 d_2^* (a-2/3)_+^{1/2}$ and $z := (a-2/3)_+^{1/2} d_1 h(a)^{1/2}$ for the continuous function $h(t)$ with $h(t) := 8t$ for $0 \le t \le 1/2$, and $h(t) =: (t-1/4)^{-1}$ for $1/2 \le t \le 1$. It follows, $(z^*z)(zz^*) = zz^*$, $e \ge 0$, $zz^* + e \le 1$, and $J(e) = J((a-1/4)_+) = A$. Let $V := z + (1-z^*z)^{1/2}$. Then V is an isometry in the unitization $\widetilde{A}$ of A with $V^*eV = 0$, i.e., $e \le p := 1 - VV^* \in A$. We get that p is a full projection in A.

Since p is properly infinite, there exists a C^*-morphism $\psi\colon \mathcal{E}_2 \to A$ with $\psi(1) = p$.

We consider now the general case where A itself is only traceless:
The algebra $A \otimes M_{2^\infty}$ is purely infinite if A is traceless (cf. [**462**, thm. 5.9], or Theorem ??).

Thm. on p.i. tensor products ?!

Above we have seen that there exists a C^*-morphism $\psi\colon \mathcal{E}_2 \to A \otimes M_{2^\infty}$ such that $\psi(1) =: p$ is a full projection of $A \otimes M_{2^\infty}$.

Since the union of the C^*-subalgebras $A \otimes M_{2^n}$ is dense in $A \otimes M_{2^\infty}$, we get from Lemma A.2.1 that there are $n \in \mathbb{N}$, a C^*-morphism $\psi_n\colon \mathcal{E}_2 \to A \otimes M_{2^n}$ and a partial isometry $v_n \in A \otimes M_{2^\infty}$ with $v_n v_n^* = \psi(1)$ and $v_n^* v_n = \psi_n(1)$.

Let I denote the closed ideal of $A \otimes M_{2^n}$ generated by $v_n^* v_n$. Then $I = J \otimes M_{2^n}$ for some closed ideal J of A, and $\psi(1) = v_n v_n^*$ is in the closed ideal $J \otimes M_{2^\infty}$ of $A \otimes M_{2^\infty}$. Since $\psi(1)$ is a full projection of $A \otimes M_{2^\infty}$, it follows $J = A$ and that $\psi_n(1) = v_n^* v_n$ must be a full projection of $A \otimes M_{2^n}$. □

COROLLARY 2.10.5. *If A is locally purely infinite and unital, then there is $n \in \mathbb{N}$ such that the unit element $1_A \otimes 1_n$ of $M_n(A)$ is properly infinite.*

PROOF. Locally p.i. algebras are traceless by Lemma 2.10.2. If A is unital, then $\mathrm{Prim}(A)$ is quasi-compact. Thus, by Proposition 2.10.4, there exists $n \in \mathbb{N}$ such that $M_n(A)$ contains a properly infinite full projection p. It follows $p \le 1_A \otimes 1_n \precsim p$, i.e., $1_A \otimes 1_n \approx p$ is properly infinite in $M_n(A)$. □

LEMMA 2.10.6. *If A is locally purely infinite, then for $a \in A_+$ and $\varepsilon > 0$ there exist $n := n(\varepsilon) \in \mathbb{N}$ and a σ-unital stable hereditary C^*-subalgebra $D \subseteq A \otimes M_n$ with $ed = d$ for all $d \in D$ and $e := f_\varepsilon(a) \otimes 1_n$, such that $(a-2\varepsilon)_+ \in (A \otimes M_n) D (A \otimes M_n)$.*

If A has property pi-m, then one can find the $n(\varepsilon)$ such that $n(\varepsilon) \le m$.

In particular, if A is purely infinite, then there is a stable σ-unital hereditary C^*-subalgebra $D \subseteq A$ with $(a-2\varepsilon)_+ \in ADA$ and $f_\varepsilon(a) d = d$ for $d \in D$.

PROOF. Let $B := B_\varepsilon := \overline{(a-\varepsilon)_+ A (a-\varepsilon)_+}$. By definition of l.p.i. algebras, there are $n \in \mathbb{N}$ and stable hereditary C^*-subalgebras $D_k \subseteq B$, $k = 1, \ldots, n$, such that $(a - (3/2)\varepsilon)_+$ is in the closed ideal J generated by $D_1 \cup D_2 \cup \cdots \cup D_n$.

The direct sum $F := D_1 \oplus D_2 \oplus \cdots \oplus D_n$ is stable and is naturally contained in $M_n(B) \cong B \otimes M_n \subseteq A \otimes M_n$.

The algebra $D := F(A \otimes M_n)F \subseteq B \otimes M_n$ is a stable hereditary C^*-subalgebra of $A \otimes M_n$, and D generates the closed ideal $J \otimes M_n$ of $A \otimes M_n$. The element $(a-(3/2)\varepsilon)_+ \otimes 1_n$ is contained in $J \otimes M_n$, and $(f_\varepsilon(a) \otimes 1_n)x = x$ for all $x \in B \otimes M_n$.

If A has property pi-m, then $g(a) \otimes 1_m = g(a \otimes 1_m)$ is properly infinite for all non-zero $g \in \mathrm{C}_0(0, \|a\|]$. Then Lemma **??** says that there is a stable hereditary C^*-subalgebra $D \subseteq B \otimes M_m$ such that the ideal $J \otimes M_m$ generated by D contains $(a - (3/2)\varepsilon)_+ \otimes 1_n$.

`to be filled in` ?? □

LEMMA 2.10.7. *If A is p.i. then for each $a \in A_+$, $\varepsilon > 0$, $n \in \mathbb{N}$ there exists a row $d \in M_{1,n}(A)$ with $dd^* f_\varepsilon(a) = dd^*$ and $d^*d = (a-\varepsilon)_+ \otimes 1_n$.*

Here $f_\varepsilon(t)$ is the continuous function on $[0,\infty)$ with $f_\varepsilon(t) = 1$ on $[\varepsilon, \infty)$, $f_\varepsilon(t) = 0$ on $[0, \varepsilon/2]$ and $f_\varepsilon(t) = 2t/\varepsilon - 1$ on $[\varepsilon/2, \varepsilon]$.

PROOF. Recall $f_\varepsilon(t) := \min\big(1, (2/\varepsilon)\max(0, t-\varepsilon/2)\big)$ for $t \geq 0$ and $\varepsilon > 0$.

Then $(a-\varepsilon)_+$ is contained in the annihilator $\mathrm{Ann}(D)$ of the hereditary C^*-subalgebra $^D := \overline{(1-f_\varepsilon(a))A(1-f_\varepsilon(a))}$ of A.

?? By the observation of M. Rørdam [**688**, prop. 5.4], see also Proposition ??, there is a σ-unital stable hereditary C^*-subalgebra E in $\overline{(a-\varepsilon)_+ A (a-\varepsilon)_+} \subseteq \mathrm{Ann}(D)$, such that $(a-\varepsilon)_+$ is contained in the closed ideal of $\mathrm{Ann}(D)$ generated by E.

It follows from Lemma **??** that there exists $d_1, d_2 \in \mathrm{Ann}(D)$ with $d_1 d_1^* + d_2 d_2^* \in E$, $d_1^* d_2 = 0$ and $d_1^* d_1 = (a-\varepsilon)_+ = d_2^* d_2$. □

`Move to suitable places !!!`

11. Weak pure infiniteness for non-simple C^*-algebras

We have seen in Part (?????) ???? The properties pi(n) and pi-n are the same for C^*-algebras A with the additional property that $1_{\mathcal{M}(A)}$ is properly infinite in $\mathcal{M}(A)$, e.g. as it isthe case if A is stable. Moreover, then both properties coincide with the property that all non-zero n-homogenous elements $a \in A_+$ are properly infinite. This is because,

a) the algebra $\ell_\infty(A)$ is an ideal of $\ell_\infty(\mathcal{M}(A))$ and – therefore – has no quotient of finite dimension if $\mathcal{O}_\infty$ is unitally contained in $\mathcal{M}(A)$ (– more general if $\mathcal{M}(A)$ has no quotient of finite dimension –) and

b) the property pi(n) always implies that an element $a \in A_+$ is *properly infinite* (as defined in Section **??**) if a is n-homogenous.

`Give ref's to ''new'' places where it (= which ??) is proved !`

There could be a considerable shift between the numbers n and the m in general. Notice here that the numbers m and n are only upper estimates. Even in case of separable amenable (non-simple) C^*-algebras we do not know the possible minimal values, except in special cases, e.g. if the algebras absorb the Jiang-Su algebra tensorial.

An estimate for the $n \in \mathbb{N}$ of property pi-n for $A \otimes \mathbb{K}$ for A with property pi-m was given by $n \leq m^2$ in [**463**, prop. 4.5(vi)]. But it is unknown for which $n \in \mathbb{N}$ the stabilization $A \otimes \mathbb{K}$ is pi(n) (and is then there equal to pi-n) if A has property pi(m). It seems to be related to topological properties of the space prime(A) of prime ideals of A.

If A is pi(k) or pi-ℓ for some $k, \ell \in \mathbb{N}$ then there exist a number $n \in \mathbb{N}$ with $n \leq \min(k, \ell)$ such that every quasi-compact subset X of the space of prime ideals prime(A) is covered by (finitely many) open subsets that are dense in prime(A) and correspond to (stable) closed ideals $J \triangleleft A \otimes \mathbb{K}$ that are generated by an n^2-homogenous element of A such that J has property pi(n) and pi-n at the same time as stated in Proposition 2.11.1. It indicates that the relation between between pi(n) and pi-n has something to do with the topology of prime(A) (= prim-ideal space = point-wise completion of Prim(A)).

A partial converse and the passage to stabilization is given in Proposition 2.11.1. Unfortunately we get only "local" equality, despite of the different focus of the definitions.

`Next is in parts only a conjecture!!`

PROPOSITION 2.11.1. *Let A a non-zero C^*-algebra.*

Consider the following properties

(i,a) *A is a pi(n) algebra in sense of Definition 2.0.4.*

(i,b) *$A \otimes \mathbb{K}$ is "locally" a pi(n) algebra, i.e., for each n-homogenous element $b \in A_+$ the ideal $J(b) := \overline{\mathrm{span}(AbA)}$ of A generated by b has the property that $J(b) \otimes \mathbb{K}$ has property pi-n.*

It is known that it is true if b is instead n^2-homogenous in A.

Suppose that A has property pi(n). And that $b \in A_+$ is n-homogenous. Then $(b-\delta)_+$ is n-homogenous for every $\delta \in [0, \|c\|)$ and is properly infinite. Thus, for every $\varepsilon \in (0, \|c\|)$ there exists in $D := \overline{(b-\varepsilon)_+ A (b-\varepsilon)_+}$ an n^2-homogenous element $c \in D_+$ such that $(b-2\varepsilon)_+$ is contained in the ideal $J(c)\overline{AbA}$ of A. Thus $J((b-2\varepsilon)_+) \subseteq J(c) \subseteq J(b)$ and $J((b-2\varepsilon)_+) \otimes \mathbb{K}$ has property pi-n. (The latter because property pi-n passes to hereditary C^*-subalgebras.) Property pi-n is preserved under inductive limits and $J(b) \otimes \mathbb{K}$ is the inductive limit of $J((b-1/n)_+) \otimes \mathbb{K}$, $n \in \mathbb{N}$.

Thus:

If A has property $\mathrm{pi}(n)$ *and* $b \in A_+$ *is* n*-homogenous then than* $J(b) \otimes \mathbb{K}$ *has property* $\mathrm{pi}\text{-}n$*.*

(ii) $A \otimes \mathbb{K}$ *has property* $\mathrm{pi}(n)$*.*

(iii) $A \otimes \mathbb{K}$ *has property* $\mathrm{pi}\text{-}n$ *in sense of Definition* **??**.

(iv) A *has property* $\mathrm{pi}\text{-}n$*.*

(v) $\ell_\infty(A)$ *has property* $\mathrm{pi}\text{-}n$*.*

(vi) $\ell_\infty(A)$ *is locally p.i. in sense of Definition 2.0.3.*

(vii) *If* $J \triangleleft A$ *is a closed ideal of* A*, then* A *has* $\mathrm{pi}\text{-}n$ *(respectively* $\mathrm{pi}(n)$*) with* $n \leq k + \ell$ *if* J *and* A/J *have* $\mathrm{pi}\text{-}k$ *(respectively* $\mathrm{pi}\text{-}k$*) and* $\mathrm{pi}\text{-}\ell$ *(respectively* $\mathrm{pi}\text{-}\ell$*).*

Then (ii) and (iii) are equivalent and imply both of (i,a) and (iv).

If A has property $\mathrm{pi}\text{-}n$ *then* $\ell_\infty(A)$*, all non-zero quotients of A and all non-zero hereditary C^*-subalgebra of A have Property* $\mathrm{pi}\text{-}n$*.*

In particular, A is purely infinite in the sense of Definition 1.2.1, if and only if, each non-zero element $0 \neq a \in A$ *is properly infinite, i.e.,* $a \oplus a \precsim a$ *for each* $a \in A$ *(and means that A has property* $\mathrm{pi}\text{-}1$*).*

Ideas of a proof of Proposition 2.11.1 will be given farer below after we have stated and proven the needed lemmata.

Clearly ???? the trivial implications are ??? (ii)⇔(iii), ??? ??? (ii)⇒(i), and ??? (iii)⇒(iv), the implication (iv)⇒(i) requires only some simple and obvious matrix reformulation of the property $\mathrm{pi}(n)$ in Definition 2.0.4.

The nontrivial implication ??? (i)⇒(ii)? need some more work and preparation.

LEMMA 2.11.2. *Suppose that A has no irreducible representation of dimension* $\leq n$*.*

If A satisfies Part (i) of property $\mathrm{pi}(n)$*, then each non-zero element* $a \in A_+$ *in the ideal J generated by some* $(n+1)$*-homogenous element* $b \in A_+$ *satisfies that* $a \otimes 1_n$ *is properly infinite in* $A \otimes M_n$*.*

(Alternatively, we could require here that $b \in A_+$ *is n-homogenous and that for each* $\varepsilon > 0$ *the ideal generated by* $\mathrm{Ann}((b-\varepsilon)_+, A)_+ := \{c \in A_+ \,;\, c(b-\varepsilon)_+ = 0\}$ *contains* $(b - 2\varepsilon)_+$*.)*

There is (somewhere farer below) a new result that says every non-zero n-homogenous element $b \in A_+$ *is properly infinite in A if A is* $\mathrm{pi}(n)$*. I.e., the latter annihilator condition on* $(b-\varepsilon)_+$ *is not needed.*

If A is pi-n, then for each element $a \in (A \otimes \mathbb{K})_+$ *and* $\varepsilon > 0$ *there exists* $b \in A_+$*,* $d \in A \otimes \mathbb{K}$ *such that* $d^*(b \otimes 1_n)d = (a - \varepsilon)_+$*. It follows that* $c := a \otimes 1_{n^2}$ *is "spectrally properly infinite" for each* $a \in A \otimes \mathbb{K}$*, i.e.,* $\varphi(c^*c)$ *is properly infinite for each* $\varphi \in \mathrm{C}_0((0, \|c\|^2])$*.*

Properties pi(n) and pi-n coincide on stable C^*-algebras.

Holds in general for all C-algebras A with properly infinite* 1 *in* $\mathcal{M}(A)$.

If A has Property pi(n) $\varphi\colon B := \mathrm{C}_0((0,1], M_{n^2+1}) \to A$ *is a* C^**-morphism, then the closed ideal* J *of* A *generated by* $\varphi(B)$ *has the property that* $J \otimes \mathbb{K}$ *has property pi-n and this property coincide there with property pi*(n)*, as for all stable* C^**-algebras.*

In Proposition **??** we list some permanence properties of the class of weakly purely infinite C^*-algebras. We consider weak versions of pure infiniteness because it is usually easier to recognize if a C^*-algebra A is locally or weakly purely infinite in the sense of Definitions 2.0.3 and 2.0.4 e.g. if one considers tensor products, crossed products or continuous fields. Only strongly purely infinite algebras have properties that in good cases allow to classify them by functors into Abelian types of categories. Usually one gets the needed stronger assumptions by combination – say of weak pure infiniteness with corona factorization properties or with the existence of reasonable central sequences.

A basic open question is if property pi(n) or even property l.p.i. for A implies that A has property pi(1) or moreover that A has the in applications needed property that A is *strongly* purely infinite in sense of Definition 1.2.2.

We do not know any example where A is weakly purely infinite but is not pi(1) or where A is l.p.i. but is not weakly purely infinite, and we do not know an example where A is purely infinite in sense of Definition 1.2.1, i.e., is pi(1) in sense of Definition 2.0.4 but is not strongly purely infinite.

It is not difficult to see that $\mathrm{C}([0,1], A)$ is pi(2) if A is purely infinite. More generally, $\mathrm{C}_0(X, A)$ is pi(n) with $n \leq m + \dim(X)$ if A is pi(m) and X is a finite-dimensional and locally compact.

Reference for above? Formula OK?

For the following we need a definition and two lemmata:

DEFINITION 2.11.3. An element $a \in A$ is *spectral properly infinite* inside A if for each $\varepsilon > 0$ the cut down $(a^*a - \varepsilon)_+$ of a^*a is properly infinite in A.

$E_{\mathrm{Spec}}(A) \subseteq A$ (or simply E_{Sp}) will denote the set of all spectral properly infinite elements in A.

$E_{\mathrm{Funct}}(A) \subseteq A$ (or simply E_{Funct}) denotes the set of all elements $a \in A$ with the property $\psi(a^*a) \in E_{\mathrm{Spec}}(A)$ for all functions $\psi \in \mathrm{C}_0(0, \|a\|^2)_+$ with $\psi(a^*a) \neq 0$.

LEMMA 2.11.4. *Let* $E_{\mathrm{spec}}(A) \subseteq A$*, (respectively* $E_{\mathrm{funct}}(A) \subseteq A$*) the sets of all "spectral" (respectively "functional") properly infinite elements in sense of Definition 2.11.3.*

Then $E_{\mathrm{Spec}}(A)$ *and* $E_{\mathrm{Funct}}(A)$ *are closed subsets of* $A \setminus \{0\}$.

PROOF. To be filled in ?? ∎

`Next is still a conjecture ... !!!`

`Good for what? ... replace by other idea !!!`

LEMMA 2.11.5. *Let $a, b \in A$ elements such that $\psi(a^*a)$ is properly infinite in A for every $\psi \in \mathrm{C}_0(0, \|a\|^2]_+$ with $\psi(a^*a) \neq 0$. Let $a = v(a^*a)^{1/2}$ the polar decomposition of a in A^{**}. Then there exists for each $\delta \in (0, \|a\|^2)$ a continuous path of elements $t \in [0,1] \mapsto b(t) \in A$ with $b(t)^*b(t) = ((a^*a) - \delta)_+$ for all $t \in [0,1]$, $b(0) = v((a^*a) - \delta)_+^{1/2}$ and $b(1) = ((a^*a) - \delta)_+^{1/2}$.*

It is *wrong* in this formulation:

Let $A := \mathcal{O}_2 = C^*(s,t\,;\ s^*t = 0, ss^* + tt^* = 1, s^*s = 1 = t^*t)$

and $a := s$, $\delta = 1/2$. Then $v((a^*a - 1/2)_+^{1/2} = 2^{-1/2}s$. There is no continuous path $t \to b(t)$ with $b(t)^*b(t) = 1/2$, $b(0) = 2^{-1/2}s$ and $b(1) = 2^{-1/2}$, because there is no continuous path from s to 1 inside the the set of isometries of $\mathcal{O}_2$.

But if there enough "stable room" orthogonal to $(a^*a - \delta)_+$ and $(aa^* - \delta)_+$ (perhaps moving with the parameter) then there is some hope.

PROOF. (The definition of g_δ has to made precise.) The idea is: For each $\mu, \gamma \in (0, \|a\|^2)$ there exist contractions $s_1, s_2 \in A$ with $s_1^*s_2 = 0$, $s_1^*s_1 = g_{2\delta}(a^*a) = s_2^*s_2$ and $s_1s_1^* + s_2s_2^* \leq g_\delta(a^*a)$ (a "soft" version of $\mathcal{E}_2$ with $g_\delta(t) = 1$ for $t \geq \delta$ and $g_\delta(t) = 0$ for $t \leq \delta/2$ linear on $[\delta/2, \delta]$.) and

$$(a^*a - \gamma)_+^\mu s_k^* s_k (a^*a - \gamma)_+^\mu = (a^*a - \gamma)_+^{2\mu} g_{2\delta}(a^*a)\,.$$

Now consider (with suitable δ), $X := v\cdot((a^*a)^{1/2} - \delta)_+^{1/2} s_1^*$, $Y := ((a^*a)^{1/2} - \delta)_+^{1/2} s_2^*$ and $Z_1 := s_2s_1^* - s_1s_2^*$ (or + ?) $XZ_1Y^* = v((a^*a)^{1/2} - \delta)_+^{1/2} s_1^*s_1s_2^*s_2((a^*a)^{1/2} - \delta)_+^{1/2} =$

$= v((a^*a)^{1/2} - \delta)_+ g_{\delta/2}(a^*a)^2$ ("nearly" XY^*)

$Z_2 := s_1s_1^* + s_2s_2^*$

Has to study $(X + Y)Z_kX^* =$??????

`Idea does not work??`

`To be filled in ??` □

We do not know (2019) if A is purely infinite for separable A if A has the property that there exists a unital C^*-morphism $\psi\colon \mathrm{C}[0,1] \to \mathcal{M}(A)$ into the center of $\mathcal{M}(A)$ such that the "fibers" $A_t := A/I_t$ are all purely infinite. Here the the ideal I_t of A is the closed linear span of $\psi(\mathrm{C}_0([0,1] \setminus \{t\})) \cdot A)$.

(Perhaps, one gets an answer if A has the "corona factorization" property (CFP) in addition? But what have they to do with (CFP)?)

But we know that $\mathrm{C}_0(X, A)$ ($\cong A \otimes \mathrm{C}_0(X)$) is pi-$n$ if A is pi-n and X is any locally compact Hausdorff space:

It reduces to the case of $\mathrm{C}([0,1], A)$, because then one can step up by induction over $m \in \mathbb{N}$ to $\mathrm{C}([0,1]^{2m+1}, A)$. $\mathrm{C}_0(X, A)$ with $\dim(X) \leq m$ is an ideal of a quotient of $\mathrm{C}([0,1]^{2m+1}, A)$, and $\mathrm{C}_0(X, A)$ of an arbitrary (not necessarily metrizable) l.c. Hausdorff spaces X is an inductive limit of an upward directed net of C^*-subalgebras isomorphic to $\mathrm{C}_0(X_\lambda, A)$ with X_λ of finite dimension.

Thus, it reduces to the case of $\mathrm{C}([0,1], A)$, and it is easy to see that is suffices to consider only those elements of $\mathrm{C}([0,1], A)$ that are given by pice-wise linear map ϕ from $[0,1]$ into A_+ with $\|\phi(t)\| \leq 1$, i.e., have to check pice-wise linear maps ϕ from $[0,1]$ into positive contractions in A. In fact it suffices to consider those maps $\phi\colon [0,1] \to A_+$ for given $\delta > 0$ with break points $t_\ell \in [0,1]$ with $t_\ell < t_{\ell+1}$, $t_1 = 0$, $t_m = 1$ and $\|\phi(t_\ell) - \phi(t_{\ell+1})\| < \delta$, but it suffices to consider those paths that satisfy moreover $\phi(t_{2k}) = \phi(t_{2k+1})$ for $n = 2k$, i.e., with $\phi(t) = \phi(t_{2k})$ for $t \in [t_{2k}, t_{2k+1}]$ and only $\phi(t_{2k-1}) \neq \phi(t_{2k})$. This is because any continuous map ψ from $[0,1]$ into the contractions in A_+ can be approximated by such continuous maps ϕ arbitrarily well.

We can go with $\phi(t)$ to the next break point with a constant pice of "path" between.

Let $I_k := [t_{2k-1}, t_{2k}] \subseteq [0,1]$. Then we can "compress" the path $\phi(I_k)$ by a continuous map $c\colon I_k \to A$, given by a path $\{c(t)\,;\ t \in I_k\} \subseteq A$ for I_k of contractions in A to

$$X_k := \big(2^{-1}(\phi(t_{2k-1}) + \phi(t_{2k})) - \delta/2\big)_+\,,$$

i.e., find a contraction $c_k \in \mathrm{C}(I_k, A)$ such that $c_k(t)^*\phi(t)c_k(t) = X_k$ for $t \in I_k$. Here we can apply Lemma 2.1.9 to the elements $a := \phi|I_k$ and $b := 2^{-1}(\phi(t_{2k-1}) + \phi(t_{2k}))$ in the C^*-algebra $\mathrm{C}(I_k, A)$ because

$$\|\phi(t) - 2^{-1}(\phi(t_{2k-1}) + \phi(t_{2k}))\| < \delta/2 \quad \text{for all} \quad t \in I_k\ .$$

Then $\|X_k - \phi(t)\| < \delta$ for $t \in I_k = [t_{2k-1}, t_{2k}]$.

Then we can go back "up to δ" from the "constant" path $(X_k - \delta)_+$ on I_k to $\phi(I_k) \in \mathrm{C}(I_k, A)$, i.e., find by a contraction in $e = \{e(t)\} \in \mathrm{C}(I_k, A)$ with $e(t)^*(X_k - \delta)_+ e(t) = (\phi(t) - 2\delta)_+$ for $t \in [t_{2k-1}, t_{2k}]$. This is possible by Lemma 2.1.9 because $\|(X_k - \delta)_+ - \phi(t)\| < 2\delta$.

We find for the properly infinite element $1_n \otimes X_k \in M_n(A)$ elements $d_k \in M_{2n}(A)$ with $d_k^*((1_n \otimes X_k) \oplus 0_n)d_k = 1_{2n} \otimes (X_k - \delta)_+$, because A has property pi-n.

We define $D_k(t) := ((1_n \otimes c(t)) \oplus 0)d_k(1_{2n} \otimes e_k(t))$.

Then $D_k \in \mathrm{C}([t_{2k-1}, t_{2k}], M_{2n}(A))$ and

$$D_k(t)^*(1_n \otimes \phi(t) \oplus 0_n)D_k(t) = 1_{2n} \otimes (\phi(t) - 2\delta)_+\,.$$

But we do not know if $\mathrm{C}([0,1], A)$ is pi(1) if A is pi(1). By Proposition **??** for every compact space X the algebra $\mathrm{C}(X, A)$ is pi-n (respectively is s.p.i.) if and only if A is pi-n (respectively is s.p.i.). For simple A *all* up to now given definitions of pure infiniteness coincide with strong pure infiniteness.

But for A with property pi(n) we get only that $\mathrm{C}(X, A)$ has pi(m) with estimate $m \leq n \cdot (\mathrm{Dim}(X) + 1)$. ???? (If $X = [0, 1]$ then $m \leq 2n$.)

It would be sufficient to prove that $\mathrm{C}([0, 1], A)$ is pi-n if A is pi-n, cf. proof of Proposition **??**.

OLD or NEW or REMAINING?: The question, if $\mathrm{C}([0, 1], A)$ is pi(1) for purely infinite A, is equivalent to the following:

Let A a purely infinite C^*-algebra, $a, b \in A_+$ positive contraction, $\varepsilon > 0$, $\tau > 0$ and $d_1, d_2 \in A$ with $d_j^* a d_k = \delta_{jk}(a - \varepsilon)_+$ ($j, k \in \{1, 2\}$).

Try to find $e_1, e_2 \in \mathrm{C}([0, 1], A)$ with $\|d_j - e_j(0)\| < \tau$ for $j \in \{1, 2\}$ and

$$e_j(t)^*((1-t)a + tb)e_k(t) = \delta_{jk}\big(((1-t)a + tb) - \varepsilon\big)_+ \qquad \text{for} \quad t \in [0, 1]\,.$$

One can separate the question into the intervals $[0, 1/2]$ and $[1/2, 1]$. $a \leq a + b$, there exist contractions $T, S \in \mathrm{C}([0, 1/2], A)$ with $T(t)^*((1-t)a + tb)T(t) = (1-t)a$ and $S(t)^*((1-t)a + tb)S(t) = tb$, as e.g. the

`Not ready?`

strict ??? limit in ??? $\mathrm{C_b}([0, 1/2], A)$

$$T(t) := \lim_n ((1-t)a + tb + 1/n)^{-1/2}(1-t)^{1/2}a^{1/2}$$

More likely is that one can find a contraction $T \in \mathrm{C}([0, 1/2], A)$ with $T(t)^*(a + t/(1-t)b)T(t) = (a - \varepsilon)_+$ and use this locally ???

It suffices to consider the case of small $\|a - b\|$.

Given $f \in \mathrm{C}([0, 1], A)_+$ with $\|f\| < 1$, we find for each $\varepsilon > 0$ and $t \in [0, 1]$, some $\delta \in (0, 1)$ such that $\|f(t) - f(s)\| < \varepsilon$ for all $s \in [t - \delta, t + \delta] \cap [0, 1]$. There we take $\gamma \in (0, \min(\delta, \varepsilon))$ and a contractions $d_1, d_2 \in \mathrm{C}([0, 1], A)$ with

$$d_1(s)^*(f(s) - \gamma)_+ d_1(s) = (f(t) - \varepsilon)_+$$

and

$$d_2(s)^*(f(t) - 2\varepsilon)_+ d_2(s) \;=\; (f(s) - 3\varepsilon)_+\,.$$

One gets at the end-points $f(t - \delta)$ and $f(t + \delta)$ different rows D, e.g. with $D^* f(t - \delta) D \;=\; \mathrm{diag}((f(t - \delta) - 3\varepsilon)_+, (f(t - \delta) - 3\varepsilon)_+)\,.$????

For *simple* locally purely infinite A the algebra $\mathrm{C}([0, 1], A)$ is strongly purely infinite because simple l.p.i. algebras are *strongly* p.i. by Proposition 2.2.1(v). The later considered applications requires strong pure infiniteness in sense of Definition 1.2.2 for the algebras in question.

We apply permanence properties of the considered class of strongly purely infinite algebras, and sometimes additional properties of A ([44]), to obtain that a particular class of C^*-algebras A has the property that the locally or weakly purely infinite algebras in this particular class are automatically strongly purely infinite. This *unsolved verification problem* restricts the applications of the notion

[44] E.g., that $A \cong A \otimes D$ for some tensorial self-absorbing unital algebra $D \neq \mathbb{C}$.

of purely infinite algebras considerably. It is still unknown if all the definitions of pure infiniteness coincide at least in case of (non-simple) *nuclear* C^*-algebras A. In case of *non-elementary* simple C^*-algebras A, i.e., $A \ncong \mathbb{K}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$, the algebra A is locally p.i. (i.e., is l.p.i. in sense of Definition 2.0.3) if and only if A is strongly p.i. in sense of Definition 1.2.2.

It could be that topological properties of the primitive ideal space $\operatorname{Prim}(A)$ of A plays a role in a final answer. It could also be that non-existence or existence of characters for the C^*-algebras $F(B, A) := (B' \cap A_\omega)/\operatorname{Ann}(B, A_\omega)$ for B any character-less sub-homogenous C^*-subalgebra of A_ω plays a role in a final answer. Unfortunately only rather partial results exist and are not obvious. Possibly related properties could be ideal system equivariant versions of the corona factorization property (CFP), but that is not clear.

In principle, all what we must require for the classification works only for "target" algebras B that are *strongly* purely infinite, e.g. the construction of suitable nuclear *-monomorphisms $A \hookrightarrow B$ of separable exact A into a C^*-algebra B needs that B is *strongly* purely infinite. One can do really nothing with the algebra B if one can only prove pure infiniteness, weak pure infiniteness or even only local pure infiniteness. But sometimes additional properties of B allow to conclude later that B or "sufficiently big" C^*-subalgebras of B are strongly purely infinite. This is the true reason for our study of the "non-strong" versions of pure infiniteness, and that this "weak" properties are easier to check.

12. Non-simple purely infinite algebras

There are several definitions of "pure infiniteness" for C^*-algebras between "locally purely infinite" and "strongly purely infinite". Their interrelations and permanence properties have been studied in [**462**], [**463**], [**92**], [**93**] and [**443**]. The properties are equivalent in the case of algebras of real rank zero or in the case of algebras with finite-dimensional Hausdorff primitive ideal spaces. It is still an open question, whether or not all this properties are equivalent in general. We get some partial results in this direction (cf. also [**443**]). Open problems will be listed in Section **??**. We generalize some methods of Section 2 for the study of pure infiniteness of non-simple algebras.

`HERE comes imported text from other places!`

`Move next blue discussion to later place!`

`Replace citations by similarities ??? here!`

Property pi-n implies property pi(n) on A. Conversely if A has property pi(m) for some m then there exists $n \geq m$ such that A has property pi-n, but no general function $f : \mathbb{N} \to \mathbb{N}$ is known yet with the property that $n \leq f(m)$, such that A has property pi-$f(m)$ if A has property pi(m). Moreover, there exists for every factorial state ρ on A a closed ideal J of A with the properties

(i) $\rho|J \neq 0$,

(ii) J is "essential" in A, i.e., $a \in A$ and $a \cdot J = \{0\}$ implies $a = 0$, and
(iii) the ideal J has property pi-n (if A has property pi(n)).

If A has property pi(m) then there exists $n \geq m$ such that A has property pi-n. But until now no estimating function function $n(t)$ is known with the property that $n(m) \geq m$ if A has property pi(m). It seems to depend from topological properties of Prim(A).

We give an overview about the known results and open questions concerning non-simple (locally, weakly or strongly) purely infinite algebras in Section **??** – with some parts of the needed proofs postponed to later sections and chapters.

By [**463**, lem. 4.4]):

Let a be a non-zero positive element in a C^*-algebra A and let n be a natural number. The following conditions are equivalent:
(i) $a \otimes 1_n$ is properly infinite,
(ii) $a \otimes 1_m \precsim a \otimes 1_n$ for all (natural numbers) $m \in \mathbb{N}$,
(iii) $a \otimes 1_m \precsim a \otimes 1_r$ for all natural numbers r, m with $r \geq n$,
(iv) $a \otimes 1_{n+1} \precsim a \otimes 1_n$,
(v) for each $\varepsilon > 0$ and for each m in $\mathbb{N}$ there is x in $M_{m,n}(A)$ such that x^*x belongs to $M_n(\overline{aAa})$ and $xx^* = (a - \varepsilon)_+ \otimes 1_m$.

Permanence properties of property pi-n:

The property pi-n on A passes to non-zero hereditary C^*-subalgebras $D \subseteq A$, non-zero quotients A/J, and to $A_\infty := \ell_\infty(A)/c_0(A)$ and the ultrapowers A_ω and (that are special quotients of $\ell_\infty(A)$).

If A and B have property pi-n, then $A \oplus B$ has property pi-n, $\ell_\infty(A_1, A_2, \ldots)$ is pi-n for sequences $A_1, A_2, \ldots$ of C^*-algebras.

Both of pi-n and pi(n) pass from A to ultra-powers A_ω.

All n-homogenous elements of A are properly infinite in A itself, if A has property pi-n or pi(n).

The property pi(n) is equal to property pi-n on each closed ideal J of A that is generated by a hereditary C^*-subalgebra D of A with the property that D is n-homogenous in the sense that there exists a hereditary C^*-subalgebra $E \subseteq D$ such that $M_n(E) \cong D$. The maximal closed ideals J with this property have in Prim(A) and prime(A) open supports that are there dense, i.e. there is no non-zero closed ideal orthogonal to J.

Thus, all is reduced to an extension problem for the sum of two ideals ...

The Property pi-n will be preserved by inductive limits,

passes to $M_2(A)$, – and then to stabilizations of A, – ...???

If $A_+ \ni b = \sum_{k=1}^{m} d_k^* a d_k$ for $a \in A_+$, then for each $\varepsilon > 0$, there exists $e_1, \ldots, e_n \in A$ with the property $(b - \varepsilon)_+ = \sum_{j=1}^{n} e_j^* a e_j$. The algebra A has no non-zero quotient A/J of finite dimension, because property pi-n carries over to non-zero quotients, and all non-zero hereditary C^*-subalgebras of A/J. Certainly $\mathbb{C}$ is not pi-n.

More generally, all C^*-algebras with property pi(n) or pi-n are residually antiliminary.

??? Why???

Part (iv) shows only that an n-homogenous element in A is infinite in A itself if there exist something orthogonal to this element...

Needs to to show that the infiniteness in A itself is the same as infiniteness in $M_n(A)$.

$A \in a \approx a \otimes e_{11}$ in $A \otimes M_n$ and $a \otimes e_{11} \approx b \otimes 1_n$ and $(b \otimes 1_n) \oplus 0 \approx (b \otimes 1_n) \oplus c$ in $A \otimes M_{n+1}$, then there must be shown that

$$a \otimes e_{11} \approx a \oplus c = a \otimes e_{11} + c \otimes e_{22}$$

in $A \otimes M_2 \cong M_2(A)$. Here $A \oplus A$ is identified with the diagonal matrices in $M_2(A)$.

It is not difficult to see that $\ell_\infty(A)$ is pi-n if A is pi-n. Therefore it implies that $\ell_\infty(A)$ can not have a finite dimensional quotient, i.e., that property pi-n implies property pi(m) for some $m \leq n$. This unknown $m \in \mathbb{N}$ could depend from the topology of Prim(A). This could be because for T_0 (non-Hausdorff) spaces the covering dimension and the decomposition dimension can be very different. Perhaps also the lattice structure of the family of hereditary C^*-subalgebras (or closed left ideals) could play some role ...

It is easy to see that property pi(n) implies that each non-zero $(n+1)$-homogenous element is properly infinite in A. tion **??** provides the non-trivial fact that in general all n-homogenous elements in A are properly infinite if A has property pi(n).

It implies properties pi(n) and pi-n are the same for C^*-algebras A with the property that $1_{\mathcal{M}(A)}$ is properly infinite in $\mathcal{M}(A)$, e.g. in case where A is stable. Moreover, then both properties coincide with the property that all non-zero n-homogenous elements $a \in A_+$ are properly infinite, cf. Proposition 2.7.16.

The above given Definitions 2.0.4, **??** of local, weak pure infiniteness coincide for simple C^*-algebras, cf. Proposition 2.2.1.

(Was mentioned earlier)

Definition 2.12.1. We call a C^*-algebra A **hyper-antiliminary** if each non-zero quotient C^*-algebra $\ell_\infty(A)/J$ of $\ell_\infty(A)$ is antiliminary (= NGCR).

Is it equivalent to the property that no hereditary C*-subalgebra D of $\ell_\infty(A)$ has a character?

It should be equivalent to the property that $\ell_\infty(A)$ is "residual antiliminary" in the sense of Definition 2.7.2

`FIND or Give general Definition`

The in next Lemma used term "antiliminary" should be recalled "residual antiliminary" ??

LEMMA 2.12.2. *The C^*-algebra A is* hyper-antiliminary, *if and only if, there exist an universal constant $n := n(A,2) \in \mathbb{N}$ such that for each contractions $a_1, a_2, a_3 \in A_+$ with $a_1a_2 = a_1$ and $a_2a_3 = a_2$ there are n contractions $d_1, \ldots, d_n \in a_3Aa_3$ and 2-homogenous positive contractions $e_1, \ldots, e_n \in a_2Aa_2$ such that $a_1 = \sum_{j=1}^{n} d_j^* e_j d_j$.*

If A is hyper-antiliminary then there exist numbers $n(A,k) \in \mathbb{N}$, $k = 2, 3, \ldots$, $n(A, k+1) \geq n(A,k)$, such that the same holds with $n := n(A,k)$ and k-homogenous contractions $e_j \in A_+$.

The C^*-algebra A is hyper-antiliminary *if no non-zero hereditary C^*-subalgebra of $\ell_\infty(A)$ has a character.*

The latter is the case if and only if $D(\ell_\infty(A)) \cap \mathbb{K}(\mathcal{H}) = \{0\}$ for every irreducible representation $D: \ell_\infty(A) \to \mathcal{L}(\mathcal{H})$ of $\ell_\infty(A)$.

PROOF. It is not difficult to see that a C^*-algebra B is antiliminary if and only if $\overline{b\,B\,b}$ has no non-zero character for each positive contraction $b \in B$.

Indeed: If $\overline{b\,B\,b}$ has a character, then this character extends to a pure state ρ on B that defines an irreducible representation D of B that contains the compact operators in the image of B.

Conversely, if D is an irreducible representation and $D(B)$ contains the compact operators, then there exists a contraction $b \in B_+$ such that $D(b)$ is a projection with rank one. Then $\overline{b\,B\,b}$ has a non-zero character.

Moreover, if $J \lhd B$ is a closed ideal such that B/J contains a non-zero Abelian hereditary C^*-subalgebra E, then $\pi_J^{-1}(E)$ is a hereditary C^*-subalgebra of B that has a non-zero character.

Take now $B := \ell_\infty(A)$ and let $b := (b_1, b_2, \ldots)$ with contractions $b_n \in A_+$.

Consider the Abelian C^*-subalgebras $C^*(c_n) \subseteq A$ generated by $c_n := \sum_k 2^{-k} b_{n+k}$.

`to be filled in...` ??

lem:Char.A.hyper.antilim □

PROPOSITION 2.12.3. *Suppose that A locally purely infinite (l.p.i.). Then A has following properties:*

1. *Every hereditary C^*-subalgebra $D \subseteq A$ is l.p.i.*
2. *Every quotient A/J is l.p.i.*
3. *$A \otimes B$ is l.p.i. for every* exact *B.*
4. *A is (2-quasi-) traceless, cf. Lemma 2.10.2.*
5. *If A is l.p.i. then A is residually antiliminary.*

Moreover:

All inductive limits of l.p.i. algebras are l.p.i.

It is not known if $\ell_\infty(A)$ is l.p.i. if A is l.p.i. It would show that "l.p.i." implies "w.p.i." (in some sense).

PROOF. **??** Where is the Def. of "residually anti-liminary"

(Is it the same as: Every hereditary □

LEMMA 2.12.4. *Suppose that A satisfies condition (i) of Definition 2.0.4 of the property* pi(n)*. Then:*

(i) *Property (i) of Definition 2.0.4 for pi(n), i.e., that each element $b \in J(a)$ can be approximated by n-term sums $\sum_{1\leq k\leq n} e_k a d_k$, passes to quotients A/J of A, hereditary C^*-subalgebras of A and to $\ell_\infty(A)$.*

(ii) *This has been improved in between (time?):*

Each non-zero n-homogenous $a \in A_+$ is properly infinite inside $D := \overline{aAa}$ if A satisfies Property (i) of Definition 2.0.4 for pi(n) and A has no irreducible representation of dimension $\leq n$. –

It carries over to n-homogenous elements of $\ell_\infty(A)$ but excludes not irreducible representations of $\ell_\infty(A)$ with dimension $\leq (n-1)$... ????

One point here is that we do not know if characters χ on $\overline{h\ell_\infty(A)h}$ with $\chi(h) = 1$ for $h = (h_1, h_2, \ldots) \in \ell_\infty(A)_+$ with $\|h\| = 1$ (can moreover take here h with $\|h_n\| = 1$ for all $n \in \mathbb{N}$). can be represented on all separable C^*-subalgebras $C \subseteq \ell_\infty(A)$ with $h \in C$ by a sequence ρ_n of pure (!) states on A such that $V : \ell_\infty(A) \to \ell_\infty(\mathbb{C})$ given by

$$V((a_1, a_2, \ldots)) := (\rho_1(a_1), \rho_2(a_2), \ldots)$$

and a character ψ on $\ell_\infty(\mathbb{C})$ have the property that for all $c \in C$ holds $\chi(c) = \psi(V(c))$.

The point is that all bigger separable C^*-sub-algebras in $\overline{h\ell_\infty(A)h}$ must have the same representation of χ.

Then still all is rather complicate: Have to show that there exists 2-homogenous elements $g_n \in A_+$ with $\|g_n\| = 1$ and $\rho_n(h_n^{1/2} g_n h_n^{1/2}) \geq 2/3\rho_n(h_n)$.

(Perhaps even a stable element g_n if in question for other topics?)

Then χ can not be a character on $\overline{h\ell_\infty(A)h}$, provided that $\chi = \psi \circ V$ holds also on the hereditary C^*-algebra generated by the element $(h_1, h_2, \ldots)$ ...

`A lot to check!`

Suppose that non-zero $a \in A_+$ is n-homogenous ([45]).

Let $J := J(a) := \overline{\mathrm{Span}(AaA)}$ denote the closed ideal of A that is generated by $\{a\}$, and let $\mathrm{Ann}(a, J) := \{b \in J\,;\, ab = 0 = ba\}$, i.e., $\mathrm{Ann}(a, J) = J \cap \mathrm{Ann}(a, A)$.

[45] i.e., there exists a C^*-morphism $\psi\colon \mathrm{C}_0(0, \|a\|] \otimes M_n \to A$ with $a = \psi(f_0 \otimes 1_n)$ for $f_0(t) := t$

If the closed two-sided ideal of A generated by $\mathrm{Ann}(a, J)$ *contains* a, *then* a *is properly infinite.*

In particular, if $c \in A_+$ *with* $\|c\| = 1$ *is an* $(n+1)$*-homogenous element of* A, *then* $(c - \delta)_+$ *is properly infinite for each* $\delta \in [0, \|c\|)$.

The dimension of $\mathcal{H}$ *is* $\leq n$, *if there exists an irreducible representation* $D \colon A \to \mathcal{L}(\mathcal{H})$ *with* $D(A) \cap \mathbb{K}(\mathcal{H}) \neq \{0\}$.

(iii) *If* A *has no non-zero quotient* A/J *of dimension* $\leq n^2$, *then* $\overline{a^*Aa}$ *is pi*(n) *for each* $a \in \mathrm{Ped}(A)$ *(*$:=$ *Pedersen ideal, the minimal dense ideal of* A*).*

(iv) *If* $\mathcal{M}(A)$ *has no non-zero quotient of dimension* $\leq n^2$, *then* A *is pi*(n).

(v) *If* A *is* σ*-unital, then, for every* $S, T \in \mathrm{Q}(A) := \mathcal{M}(A)/A$ *(respectively* $S, T \in \mathcal{M}(A)$*) with* S *in the closed ideal generated by* T, *and every* $\varepsilon > 0$, *there exist* $X_1, \ldots, X_{2n} \in \mathrm{Q}(A)$ *(respectively* $X_1, \ldots, X_{3n} \in \mathcal{M}(A)$*) with* $\|S - \sum_j X_j^* T X_j\| < \varepsilon$.

(vi) *If* A *is* σ*-unital and is pi*(n), *then* $\mathcal{M}(A)$ *has no finite-dimensional quotient and is pi*(k) *(for some* $k \leq 3n$*).*

(vii) *If* A *(itself) has no irreducible representations of dimension* $\leq n$, *then* A *is* locally purely infinite *(cf. Def. 2.0.3).*

In particular, then A *is* traceless *(in sense of Def. 2.8.1).*

Proof. (i): The passage to hereditary C^*-subalgebras and to quotients A/J is almost obvious from the definition.

The passage of property pi(n) to $\ell_\infty(A)$ requires an uniform estimate for property (i) of Definition 2.0.4:
If $b_k := S_k a_k S_k^*$, for $a_k, b_k \in A_+$, $S_k \in M_{1,m}(A)$ (for some fixed $m \in \mathbb{N}$) with $\|a_k\| \leq 1$ and $\gamma := \sup_k \|S_k\| < \infty$, then $\|b_k - S_k(a_k - \delta)_+ S_k^*\| \leq \gamma^2 \delta$. This allows to find $T_k \in M_{1,n}(A)$ with $\sup_k \|T_k\| \leq 2/\sqrt{\varepsilon}$, such that $\|b_k - T_k a_k T_k^*\| < \varepsilon$ for all $k \in \mathbb{N}$. Now use that $M_{1,n}(\ell_\infty(A)) = \ell_\infty(M_{1,n}(A))$.

(ii): Let $b := \psi(f_0 \otimes p_{1,1})$. The elements a and b generate the same closed ideal J, and $b \otimes 1_n \sim a \oplus 0_{n-1}$ in $M_n(A)$. For $x \in \mathrm{Ann}(a, J)$ holds $a + x \in J$. Thus $(a + x) \oplus 0_{n-1} \precsim b \otimes 1_n,$. Since $ax = xa = 0$, also $a \oplus x \sim (a + x) \oplus 0$. Together: $a \oplus x \oplus 0_{n-2} \precsim a \oplus 0_{n-1}$, i.e., x is in $I(a) := \{x \in A\,;\ a \oplus x \precsim a\}$ for all $x \in \mathrm{Ann}(a, J)$. By Lemma 2.5.3(i), $I(a)$ is a closed ideal of A. By assumptions, a is in the closed ideal of A that is generated by $\mathrm{Ann}(a, J)_+$. Thus, $a \oplus a \precsim a \oplus 0$ follows from $\mathrm{Ann}(a, J) \subseteq I(a)$.

If $\psi \colon \mathrm{C}_0(0, \gamma] \otimes M_m \to A$ is given with $\gamma := \|c\|$, $m > n$ and $\psi(f_0 \otimes 1_m) = c$ for $f_0(t) := t$, then, for each $0 \leq \delta < \gamma$, the element $(a - \delta)_+ = \psi((f_0 - \delta)_+ \otimes e_n)$ (where $e_n := p_{11} + p_{22} + \cdots + p_{nn}$) satisfies the above considered assumptions on a. Thus, $(a - \delta)_+$ is properly infinite. The element $(c - \delta)_+$ is in the ideal generated by $\{(a - \delta)_+\}$. We get $(c - \delta)_+ \precsim (a - \delta)_+ \leq (c - \delta)_+$.

Thus $(c - \delta)_+$ is properly infinite for all $\delta \in [0, \|c\|)$ ...????

If $D \colon A \to \mathcal{L}(\mathcal{H})$ is an irreducible representation, then the compact operators $\mathbb{K}(\mathcal{H})$ are contained in the image of D, and $\mathbb{K}(\mathcal{H})$ becomes isomorphic to a quotient of an ideal of A. $\mathbb{K}(\mathcal{H})$ satisfies property (i) of Definition 2.0.4, if and only if $\mathcal{H}$ has dimension $\leq n$, because $\mathbb{K}(\mathcal{H})$ contains no properly infinite element, but contains $n+1$-homogenous elements if the dimension of $\mathcal{H}$ is $> n$.

(iii): All finite-dimensional irreducible representations of A are of dimension $\leq n$, by our assumptions on A. But this sort of irreducible representations can not exist by our assumption that A has no non-zero quotient of dimension $\leq n^2$. It implies that $\overline{fM_n(A)f}$ has no finite-dimensional quotient if $0 \neq f \in M_n(A)_+$.

Let $a \in \mathrm{Ped}(A)$. Then $b := a^*a \in \mathrm{Ped}(A)$ and $B := \overline{a^*Aa} = \overline{bAb}$. The (minimal dense) Pedersen ideal of A is algebraically generated by the elements $\{(e-\varepsilon)_+ \,;\; e \in A_+, \varepsilon > 0\}$. Hence, there are $e \in A_+$, $\varepsilon > \delta > 0$ and $d_1, \ldots, d_n$ with $b = d_1^*(e-\delta)_+d_1 + \cdots + d_n^*(e-\delta)_+d_n$ (the n comes from property (i) of Definition 2.0.4).

There are contractions $f, x \in M_n(A)$ and $x \in M_n(A)$ such that $fxx^* = xx^*$ and $f \geq 0$ and $B \otimes p_{11} = \overline{xM_n(A)x^*}$.

It follows that $\ell_\infty(B)$ is isomorphic to a hereditary C^*-subalgebra of $M_n(\ell_\infty(A))$ that is contained in $gM_n(A_\omega)g$ for $g := (f, f, \ldots)$, where we use $\ell_\infty(M_n(A)) \cong M_n(\ell_\infty(A))$.

Since, by Part (i), B, $\ell_\infty(B)$ and $\ell_\infty(A)$ satisfy the condition (i) of Definition 2.0.4, all irreducible representation of $\ell_\infty(B)$ that contain non-zero compact operators in its image are of dimension $\leq n$, and all irreducible representation of $M_n(\ell_\infty(B))$ that contain non-zero compact operators in its image are of dimension $\leq n^2$. Therefore, each irreducible representation of $\ell_\infty(B)$ of finite dimension extends to an irreducible representation of $M_n(\ell_\infty(A))$, that contains compact operators in its image, thus is of dimension $\leq n^2$.

But $gM_n(A_\omega)g$ can not have an irreducible representation of finite dimension, because otherwise, $\overline{fM_n(A)f}$ has a non-zero quotient of finite dimension.

Thus, $\ell_\infty(B)$ has no quotient of dimension $\leq n^2$, i.e., satisfies also condition (ii) of Definition 2.0.4, i.e., B is $\mathrm{pi}(n)$.

(iv): If $\mathcal{M}(A)$ has no non-zero quotient of dimension $\leq n^2$, then $\ell_\infty(\mathcal{M}(A))$ can not have a non-zero quotient C^*-algebra of dimension $\leq n^2$, because $\mathcal{M}(A)$ is unitally contained in $\ell_\infty(\mathcal{M}(A))$.

The same happens with the (essential) ideal $\ell_\infty(A)$ of $\ell_\infty(\mathcal{M}(A))$.

(v): `like in [463] ??`

Next considers relations between separable C^*-subalgebras in $\mathrm{Q}(A)$ and in A_∞ ?

LEMMA 2.12.5. *Suppose that A is σ-unital. Then, for every separable C^*-subalgebra $D \subseteq \mathcal{M}(A)$, there exist a sequence $a_0, a_1, \ldots \in A_+$ of commuting contractions such that $\sum a_n = 1$ (strictly in $\mathcal{M}(A)$) $a_n a_m = 0$ for $|n-m| > 1$, $T - \sum_n a_n^{1/2} T a_n^{1/2} \in A$, and $MT - TM \in A$ for $M := \sum_{n \in X} a_n$, for all $T \in D$ and all $X \subseteq \mathbb{N}$.*

??????????????

to be filled in

In particular, for each separable C^*-subalgebra $C \in \mathrm{Q}(A)$, there exist hereditary C^*-subalgebras $E_1, E_2 \subseteq \ell_\infty(A)$ and mono-morphisms $\psi_j \colon E_j \to \mathcal{M}(A)$ such that $C \subseteq \pi_A(\psi_1(E_1)) + \pi_A(\psi_2(E_2))$. ??????

PROOF. **??** □

(???): If A is σ-unital, then, for each separable C^*-subalgebra $C \in \mathrm{Q}(A)$, there exist hereditary C^*-subalgebras $E_1, E_2 \subseteq \ell_\infty(A)$ and monomorphisms $\psi_j \colon E_j \to \mathcal{M}(A)$ such that $C \subseteq \pi_A(\psi_1(E_1)) + \pi_A(\psi_2(E_2))$. See 2.12.5.

If A is pi(n), then every hereditary C^*-subalgebra E of $\ell_\infty(A)$ does not have non-zero quotient algebras of finite dimension.

Suppose now that $\mathcal{M}(A)$ has an irreducible representation $D \colon \mathcal{M}(A) \to \mathcal{L}(\mathcal{H})$ of finite dimension, then necessarily $D(A) = \{0\}$, i.e., $D = d \circ \pi_A$ for an irreducible representation d of $\mathrm{Q}(A)$. Let $C \subseteq \mathrm{Q}(A)$ a separable C^*-subalgebra with $d(C) = \mathcal{L}(\mathcal{H})$, and E_1, E_2 as above.

?????????????

Then $D \circ \psi_1$ or $D \circ \psi_2$ is non-zero. Thus, there exists a hereditary C^*-subalgebra E of $\ell_\infty(A)$ that has a non-zero quotient of finite dimension. This contradicts the existence

of

???????????????????????????

a non-zero hereditary C^*-algebra E of $\ell_\infty(A)$ and a C^*-morphism $\psi \colon E \to \mathcal{M}(A)$ with $D(\psi(E)) \neq 0$.

(???): `to be filled in`

?? □

Recall that a C^*-algebra A `Check definitions!!` has *property pi-n* (or A is pi-n) if, for each $a \in A$, the n-fold sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$, cf. Definition **??**.

There are other places where this result is discussed and partly proven. Compare and unite them!

LEMMA 2.12.6. *The C^*-algebra A is pi-n, if and only if, $\ell_\infty(A)$ is pi-n, if and only if, A_ω is pi-n.*

If A is pi-n, then all non-zero quotients and non-zero hereditary C^*-subalgebra are pi-n.

If A is pi-n then A is traceless. (In particular, A contains no non-zero hereditary C^*-subalgebra D that has a non-zero character. In particular, A admits no irreducible representation of finite dimension.)

If A is pi-n, then A has property pi(m) for some $m \leq n$.

If A is stable *then A has property pi(n), if and only if, A is pi-n.* More general, this holds if the unit element of $\mathcal{M}(A)$ is properly infinite.

More general:
A has property pi-n if A has property pi(n) and for each element $a \in A_+$ and $\varepsilon > 0$ there exists an n-homogenous element $b \in A_+$ with the property that b is the closed ideal generated by a and $(a-\varepsilon)_+$ is contained in the closed ideal of A generated by b.

In between, question for further studies:
If a, $b \in A_+$ are *residually* properly infinite in A, is then $a+b$ infinite in A?

Give Ref. to *residually* p.i. elements in A_+ !!!

(What about counterexamples?

A can be supposed separable?

Where is the Definition of *residually properly infinite* elements in A_+ ?)

If this is the case, then all pi(n) and pi-n are are equivalent to pi-1.!!! Why that ???

If $J \subseteq A$ is a closed ideal with property pi-n and suppose that A/J has pi-n. It is then known that A has property pi-m for some $m \leq 2n$. can it be improved to $m = n$?

PROOF. `to be filled in` ??

If A is stable, then its multiplier algebra $\mathcal{M}(A)$ contains isometries S, T with $S^*T = 0$. Suppose from now on that A has the property that $\mathcal{M}(A)$ contains isometries $S, T \in \mathcal{M}(A)$ with $S^*T = 0$. The isometries $T_k := T^k S$ have mutually orthogonal ranges: $S^*(T^*)^\ell T^k S = \delta_{k,\ell} 1$.

Let $a, b \in M_n(A)$. Clearly $a \precsim b$ in $M_n(A)$ if and only if $a \precsim b$ in $M_n(\mathcal{M}(A))$.

????????? □

It is sometimes useful and easier, to consider local properties of the ultrapower A_ω of A, instead of local properties of the algebra A itself.

Let $a \in A_+$, $t \in (0, \infty)$ and $V : A \to A$ an *inner* completely positive contraction ([46]). We define $\mu(a, V, t)$ as the smallest $m \in \mathbb{N}$ of *contractions* $d_1, \ldots, d_m \in A$ such that $\sum_{j=1}^m d_j^* a d_j = (V(a) - t)_+$. Here we do not require that $\| \sum_{j=1}^m d_j^* d_j \| \leq 1$.

`Add remark/def concerning control of` $\mathrm{CP}_{\mathrm{in}}(A)$ `from notes (which one ????).` ??

LEMMA 2.12.7. *A C^*-algebra A satisfies condition* (i) *of the Definition 2.0.4 of property pi(n) for some $n \in \mathbb{N}$, if and only if, $\mu(t) := \sup_{a,V} \mu(a, V, t) < \infty$ for*

[46] A c.p. map V is *inner*, if there exist $d_1, \ldots, d_m \in \mathcal{M}(A)$ with $V(a) := \sum d_j^* a d_j$.

each $t \in (0,1)$, where (a, V) runs over all inner c.p. contractions $V \in \mathrm{CP}(A, A)$ and over all positive contractions $a \in A_+$.

PROOF. Suppose that $n := \mu(1/4) < \infty$. Let $0 \neq b \in A_+$ a contraction and $\varepsilon \in (0, 1/2)$ such that $b \in \overline{\mathrm{span}(AaA)} =: J_a$. We show that there are contractions $d_1, \ldots, d_n \in A$ and $\delta > 0$ such that

$$(2/\delta) \sum_{j=1}^{n} d_j^* a d_j \geq (b - \varepsilon)_+ \, .$$

It implies then condition (i) of Definition 2.0.4 by Lemma 2.1.9.

Let $c := b^\alpha$ for $0 < \alpha := -\log(2)/\log(\varepsilon) < 1$, i.e., for $\alpha \in (0,1)$ with $\varepsilon^\alpha = 1/2$. Then $(c - 1/2)_+ = (b^\alpha - 1/2)_+ \geq (1/2)(b - \varepsilon)_+$ and $c \in J_a$. Thus, there exists $e_1, \ldots, e_m \in A$ and $\delta > 0$ with

$$\sum_{k=1}^{m} e_k^* (a - \delta)_+ e_k = (c - 1/4)_+ \, .$$

It holds $a \geq \delta f_\delta(a)$ for the $f_\delta(t) := \min(2/\delta(t - \delta/2)_+, 1)$ for $t \in [0, \infty)$. Define $V \colon A \to A$ for $x \in A$ by

$$V(x) := \sum_{k=1}^{m} e_k^* (a - \delta)_+^{1/2} x (a - \delta)_+^{1/2} e_k \, .$$

Then $\|V\| \leq 3/4$ and $V(f_\delta(a)) = (c - 1/4)_+$. Thus $(V(f_\delta(a)) - 1/4)_+ = (c - 1/2)_+$. It follows, that there are contractions $d_1, \ldots, d_n \in A$ with $n \leq \mu(1/4)$ and

$$\sum_{j=1}^{n} d_j^* f_\delta(a) d_j = (c - 1/2)_+ \geq (1/2)(b - \varepsilon)_+ \, .$$

□

PROPOSITION 2.12.8. *Let A denote a non-zero C^*-algebra and A_ω its ultrapower with respect to some free ultrafilter $\omega \in \gamma(\mathbb{N}) := \beta(\mathbb{N}) \setminus \mathbb{N}$. Then the following properties of A are equivalent:*

(i) *A is weakly purely infinite (i.e., A has property pi(n) for some $n \in \mathbb{N}$).*
(ii) *A_ω is locally p.i.*
(iii) *Every lower semi-continuous 2-quasi-trace of $(A_\omega)_+$ is trivial, i.e., takes only the values 0 and ∞.*
(iv) *There exists $m \in \mathbb{N}$, such that $a \otimes 1_m$ is a properly infinite element of $M_m(A)$ for every non-zero $a \in A_+$, i.e., A is pi-m.*
(v) *The ultrapower A_ω has no finite-dimensional irreducible representation, and, for every sequence of approximately inner completely positive contractions $V_n \colon A \to A$, the ultrapower $V_\omega := \prod_\omega (V_n)$ of $(V_1, V_2, \ldots)$ is an ideal-system preserving c.p. map on A_ω, i.e., V_ω satisfies $V_\omega(J) \subseteq J$ for every closed ideal J of A_ω.*
(vi) *A_ω is pi-m for some $m \in \mathbb{N}$.*

Question: What happens if no closed hereditary C^*-subalgebra of A_ω has a character?

The part (v) says equivalently that ?????

(v): The ultrapower A_ω has no finite-dimensional irreducible representation, and, for every sequence of approximately inner completely positive contractions $V_n\colon A \to A$, the ultrapower $V_\omega := \prod_\omega (V_n)$ of $(V_1, V_2, \ldots)$ is an ideal-system equivariant c.p. map on A_ω, i.e., V_ω satisfies $V_\omega(J) \subseteq J$ for every closed ideal J of A_ω.

(vi): A_ω is pi-m for some $m \in \mathbb{N}$.

The new result says: If A is stable then pi(n) = pi-n and pi(n) requires only
that $\overline{\operatorname{span}(AaA)}$ can be approximated in n steps and that
A has no irreducible representations of dimension $\leq n$.

PROOF. (iv)⇒(i): By Lemma **??**, A is pi(n) for some $n \leq m$.

(i)⇒(ii): By Lemma **??**, $\ell_\infty(A)$ and its quotient A_ω are pi(n). By Lemma **??**, pi(n) implies local p.i.

(ii)⇒(iii): local p.i. implies traceless

(iii)⇒(iv): By Remark 2.8.3, A_ω is traceless, if and only if,

(*) for every $a \in (A_\omega)_+$ and $\varepsilon > 0$ there is $n(a, \varepsilon) \in \mathbb{N}$ such that $(a - \varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ for all $n \geq n(a, \varepsilon)$.

For $b, c \in M_k(A)_+$ holds $b \precsim c$ in $M_k(A)$, if and only if, $b \precsim c$ in $M_k(A_\omega) = (M_k(A))_\omega$. Thus, for every $a \in A_+$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $(a - \varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ in $M_{2n}(A)$. We denote from now on by $n(a, \varepsilon)$ the minimal $n \in \mathbb{N}$ with this property.

Suppose that there exists $\varepsilon > 0$ and a sequence of contractions $a_1, a_2, \ldots \in A_+$ such that $n(a_k, \varepsilon/2) \to \infty$ if $k \to \infty$. Then the positive contraction $a := \pi_\omega(a_1, a_2, \ldots) \in A_\omega$ satisfies

check: ????????????

$(a - \varepsilon/4)_+ \otimes 1_{2n} \not\precsim a \otimes 1_n$ for all $n \in \mathbb{N}$. The latter contradicts property (*) of A_ω.

Thus, there is "universal" $m \in \mathbb{N}$ with the property that $(a-1/2)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ in $M_{2n}(A)$ for each contractions $a \in A_+$ for some $n = n(a, 1/2) \leq m$.

????????????????

(iv)⇒(v): Property (ii) passes to $\ell_\infty(A)$, to its quotient A_ω, and to any non-zero quotients of A_ω. In particular, A_ω cannot have any finite dimensional irreducible representation.

If A satisfies (ii) and if $a = (a_1, a_2, \ldots) \in \ell_\infty(A)_+$ is a representative of $b \in (A_\omega)_+$, then $(V_\omega(b) - 2\varepsilon)_+ = \pi_\omega(c_1, c_2, \ldots)$ with $c_n = (V_n(a_n) - 2\varepsilon)_+$.

Since $a_n \otimes 1_m$ is properly infinite for some fixed $m \in \mathbb{N}$, and since $V_n(a_n)$ is in the ideal generated by a_n, we find, as in the last step of the proof of Proposition

2.15.11, $d_{n,1}, \ldots, d_{n,m} \in A$ with $\|(d_{n,1})^* d_{n,1} + \cdots + (d_{n,m})^* d_{n,m}\| \leq \frac{2\|a_n\|}{\varepsilon}$, such that $c_n = (d_{n,1})^* a_n d_{n,1} + \cdots + (d_{n,m})^* a_n d_{n,m}$. Thus $(V_\omega(b) - 2\varepsilon)_+$ is in the ideal generated by b.

(v)⇒(iv): An indirect argument shows that (v) implies the existence of (universal) $m \in \mathbb{N}$, such that, for contractions $a, b \in A_+$ with b in the ideal generated by a, there are $d_1, \ldots, d_m \in A$ with $\sum d_j^* a d_j = (b - 1/2)_+$. Then $1/2$ can be replaced by any $\varepsilon > 0$ if we replace b by b^α with $\alpha := -2\log 2/\log \varepsilon$.

Moreover, this implies that for every $\varepsilon > 0$ and $b \in A_+$ in the closed ideal generated by $a \in A_+$ there is $\delta > 0$ such that there are

??????????

If such $m \in \mathbb{N}$ exists, then

??????????

passes to A_ω with the same number m.

It implies that a lower semi-continuous 2-quasi-trace τ on A_ω annihilates the closed ideal $J(B)$ of A_ω, which is generated by the image $B := h(\mathrm{C}_0((0,1], M_{2m}))$ of a C^*-morphism h from $\mathrm{C}_0((0,1], M_{2m})$ into A_ω, so far as $\tau|B_+$ is semi-finite. Since A_ω has no irreducible representation of finite dimension, the Pedersen ideal of every closed ideal of A_ω is contained in the union of the direct sums of ideals $J(B)$. Now we use again, that τ is lower semi-continuous, and get that τ can only take the values 0 and ∞. □

It is obvious, that (iii), equivalently, means, that all singly generated ideals of A_ω are closed unions of ultrapowers of sequences of singly generated closed ideals of A and that those ultrapowers are singly generated, and that A_ω has no finite-dimensional quotient.

Corollary 2.12.9. *Let $n \in \mathbb{N}$. Suppose that A satisfies condition (i) in the Definition 2.0.4 of the property pi(n), i.e., for each $a \in A_+$, $b \in J(a) := \overline{\mathrm{span}(AaA)}$ and $\varepsilon > 0$ there exist $d_1, \ldots, d_n \in A$ with $\|b - \sum_k d_k^* a d_k\| < \varepsilon$.*

This property passes to quotients A/J of A and to $\ell_\infty(A)$.

The following properties (i)–(v) *of A are equivalent under condition (i) of Definition 2.0.4:*

(i) *A is pi(n).*
(ii) *There exists some free ultrafilter ω such that A_ω has no irreducible representation of dimension $\leq n$.*
(iii) *$\ell_\infty(A)$ is pi(n).*
(iv) *A_ω is pi(n) for* any *free ultrafilter ω.*
(v) *$\ell_\infty(A)$ has no irreducible representation of dimension $\leq n$.*

A sufficient condition for Part (v) *is that $\mathcal{M}(A)$ has no irreducible representation of dimension $\leq n$.*

And this condition is equivalent to Part (v) *if A is σ-unital.*

PROOF. (i) $\Leftrightarrow$ (v): By Definition 2.0.4 A is pi(n) if and only if it satisfies Condition (i) of Definition 2.0.4 and its Condition (ii), which says that $\ell_\infty(A)$ has no irreducible representation of dimension $\leq n$.

(v)$\Rightarrow$(iv): Each A_ω is a quotient of $\ell_\infty(A)$.

(v)$\Rightarrow$(i): By Lemma 2.12.4(i), the quotient A_ω of $\ell_\infty(A)$ has also property (i) of Definition 2.0.4.

If $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ is given with the property that A_ω has no non-zero irreducible representation of dimension $\leq n$, then A_ω is pi(n) by Part (iii) of

????????????, to be filled in ??

because, for each positive $a \in A_\omega$, there is a positive contraction $e \in A_\omega$ with $ea = a$, in particular, $\mathrm{Ped}(A_\omega) = A_\omega$.

By Proposition 2.12.8, there exists $m \in \mathbb{N}$ such that A is pi-m (for some $m \geq n$).

Then $\ell_\infty(A)$ is pi-m by Lemma 2.12.6.

In particular, $\ell_\infty(A)$ can not have a finite-dimensional quotient.

□

REMARK 2.12.10. If σ-unital A satisfies condition (i) of Definition 2.0.4, then condition 2.0.4(ii) of $\ell_\infty(A)/c_0(A)$ is equivalent to:
$\mathcal{M}(A)$ has no irreducible representation of dimension $\leq n$.

This happens, because – otherwise – a suitable hereditary C^*-subalgebra D of $\ell_\infty(A)/c_0(A)$ has a C^*-morphism into $\mathrm{Q}(A) := \mathcal{M}(A)/A$ with image that is not in the kernel of that irreducible representation.

One can see:
If A_ω has no irreducible representation of dimension $\leq n$, then, for each $\delta > 0$ there is $m = m(\delta) \in \mathbb{N}$ such that, for every contraction $a \in A_+$, there are $n+1$-homogenous contractions $e_1, \ldots, e_m \in A_+$ and contractions $d_1, \ldots, d_m \in A$ with $\sum_k d_k^* e_k f_k = (a - \delta)_+$.

Question: Is each element $b \in \mathrm{Q}(A)_+$ a sum of 4 commuting positive elements that are in the closed ideal generated by the $n+1$-homogenous elements?

The closed ideal J_{n+1} of $\mathrm{Q}(A)$ generated by the $n+1$-homogenous elements of $\mathrm{Q}(A)$ is the intersection of all kernels of the C*-morphisms from $\mathrm{Q}(A)$ into M_k with $k < n+2$. The minimal requirement is therefore that $\mathrm{Q}(A)$ has no irreducible representation into M_k for $k \leq n+1$. ...

The possible differences between properties pi(n) and pi-n can be seen by the following Proposition 2.12.11 (see also Question ??). Here $\mathrm{Ped}(A)$ denotes the *minimal* dense algebraic ideal of A, the "Pedersen-ideal".

PROPOSITION 2.12.11. *Let A an inductive limit of algebras $A_1, A_2, \ldots$.*

(i) *If each A_k is pi(n), then, for every non-zero $a \in \mathrm{Ped}(A)$, the algebra $\overline{a^* A a}$ is pi(n).*

(ii) *If each A_k has property pi(n) and if $\mathcal{M}(A)$ has no non-zero finite-dimensional quotient C*-algebra, then A is pi(n).*
(iii) *If each A_k has property pi-n, then A has property pi-n.*
(iv) *If each A_k is locally p.i., then A is locally p.i.*

The next proposition lists some elementary implications of the properties Xpi of C*-algebras.

PROPOSITION 2.12.12. *If A has Property pi-n, then A is pi(n) (thus is weakly purely infinite).*

If A is pi(n) (respectively has Property pi-n), then $\ell_\infty(A)$, A_ω, every hereditary C-subalgebra D of A and every quotient of A are pi(n) (respectively have Property pi-n).*

This is somewhere below also shown?!

The C-algebra A is purely infinite (in sense of Definition 1.2.1, i.e., A is pi(1)), if and only if, A has property pi-1, i.e., each non-zero $a \in A_+$ is properly infinite.*

If A is weakly purely infinite, then A is locally purely infinite.

Locally purely infinite algebras are traceless.

PROOF. To be filled in ?? □

PROPOSITION 2.12.13. *We have the implications: $A \cong A \otimes \mathcal{O}_\infty$ implies A s.p.i. implies $C' \cap A_\omega$ p.i. for all* separable commutative *subalgebras $C \subseteq A_\omega$*

implies A pi(1) $=$ A pi-1 $=$ A p.i. $=$ A_ω p.i. implies A pi-n (" has property p.i.-n ") $=$ A_ω is p.i.-n implies $A \otimes \mathbb{K}$ p.i.-n^2 and A is pi(n) (including absence of quotients of A_ω of dimension $\leq n^2$). $=$ A_ω pi(n) implies A_ω has property p.i.-m for some $m \geq n$ implies A_ω "traceless" $=$ A_ω w.p.i. (:= pi(n) for some $n \in \mathbb{N}$)

$=$ A_ω locally p.i.

$=$ A_ω satisfies: No non-trivial hereditary C-subalgebra of A_ω has a non-zero character, and for each sequence of inner c.p. contractions $V_n \colon A \to A$ holds $V_\omega(I) \subseteq I$ for each closed ideal I of A_ω.*

$=$ A_ω satisfies:

No non-trivial hereditary C-subalgebra of A_ω has a non-zero character, and, for each separable C*-subalgebra B of A_ω and each commutative C*-subalgebra $C \subseteq B$, there is a separable commutative C*-algebra D with $C \subseteq D \subseteq A_\omega$ and an (in A_ω) approximately inner c.p. contraction $T \colon B \to D$ with $T|C = \mathrm{id}_C$.*

$=$

$C' \cap A_\omega$ is w.p.i. for every finitely generated Abelian C-subalgebra C of A_ω*

$=$

$\{a\}' \cap A_\omega$ is "traceless" for every positive $a \in A_\omega$.

implies

A l.p.i.

implies

A is "traces-less"

implies

$A \otimes B$ is locally p.i. for every non-elementary simple exact C^*-algebra B.

PROPOSITION 2.12.14. *Every Xpi-algebra A is the inductive limit of its separable relative weakly injective C^*-subalgebras $B \subseteq A$ with property Xpi, and the property that for each closed ideal I of B there exists a closed ideal J of A with $I = B \cap J$. Moreover, for every non-zero $b \in A_+$ and separable C^*-subalgebra $C \subseteq \overline{bAb}$ of A with $b \in C$ there exists a separable C^*-subalgebra $B \subseteq \overline{bAb}$ with the properties $C \subseteq B$, $b \in B$ and $c \precsim_B d$ for $c, d \in B$, if and only if, $c \precsim_A d$.*

Here "Xpi" stands for l.p.i., pi(n), pi-n, p.i., semi-s.p.i.(concerning commutants of special??) or s.p.i.

and the W-vN-property?

PROOF. We have to verify that all the definitions of pure infiniteness are "local" in nature. It requires to generalize the properties as relative properties given by certain countable families of semi-metrics on pairs of elements in C^*-subalgebras.

The metrics are: `to be filled in. See Appendix A or B? give Ref's ??`

Use uniform local-global m-almost-halving for n-p.i. permanences.

Use description of l.p.i. by morphisms $h\colon \mathrm{C}_0((0,1], \mathbb{K}) \to A$. □

Give here references of the xpi-definitions !!!

PROPOSITION 2.12.15. *Permanences:*

If A is weakly purely infinite and σ-unital then $\mathcal{M}(A)$ is weakly purely infinite.

(More precisely, $\mathcal{M}(A)$ is pi$(2n)$ or pi$(3n)$???? if A is pi(n) and is σ-unital.)

Passage to hereditary subalgebras.

Stabilization (by $\mathbb{K}$).

Quotients.

Extensions.

But do here not forget that
extensions of stable C-algebras are*
in general not (!) stable again.

$\ell_\infty(A)$, A_ω, $C' \cap A_\omega$ for $C \subseteq A_\omega$ commutative with character space of finite dimension.

continuous fields

C(X)-*algebras*

C$_0(X, A)$ *w.p.i. for* A *w.p.i. and* $\mathrm{Dim}(X) < \infty$

tensor products (min*?*)

PROOF. `to be filled in` ?? □

13. Singly generated ideals of coronas

`Collect here all about ideals of` $\mathcal{M}(B)$ `and` Q(B).
`Also from Chps. 5, 6, 12!!!  ?? The notation` K `for closed ideals of` $M(B)$ `is irritating ...`

`Important!  Check ideal characterization carefully again!!!`

LEMMA 2.13.1. *Let* B *a* σ*-unital stable* C^**-algebra, and* $b \in \mathcal{M}(B)_+$. *Then there exists a sequence* $(a_1, a_2, \ldots) \in \ell_\infty(B)_+$ *such that*

$$(1/2)\| b + B + K \| \leq \| (\sum_n s_n a_n s_n^*) + B + K \| \leq \| b + B + K \| ,$$

for every norm-closed ideal K *of* $\mathcal{M}(B)$ ([47]). *In particular,*

$$I(b) + B = I(\sum_n s_n a_n s_n^*) + B$$

for the norm-closed ideals $I(b)$ *and* $I(\sum_n s_n a_n s_n^*)$ *of* $\mathcal{M}(B)$ *generated by* b *respectively by* $\sum_n s_n a_n s_n^*$.

PROOF. Recall that $\|b + B + K\| = \|\pi_{B+K}(b)\|$ in $\mathcal{M}(B)/(B + K) \cong$ Q$(B)/\pi_B(K)$.

If we use a suitable, approximately with b commuting, commutative approximate unit $e_1, e_2, \ldots \in B_+$ with $e_n e_{n+1} = e_n$, then we can manage that for $g_0 := 0$, $g_n := (e_n - e_{n-1})^{1/2}$ and $b_0 := \sum_n g_{2n} b g_{2n}$, $b_1 = \sum_n g_{2n-1} b g_{2n-1}$, holds $\sum g_n^2 = 1$, $b_0 - (\sum_n g_{2n}^2)b \in B$, $b_1 - (\sum_n g_{2n-1}^2)b \in B$, $b_0 + b_1 - b \in B$ and $\|b_j + K + B\| \leq \|b + K + B\|$ for every closed ideal K of $\mathcal{M}(B)$. Now let $X := \sum g_{2n} b^{1/2} s_{2n}$, $Y := \sum g_{2n-1} b^{1/2} s_{2n-1}$. Then $XX^* = b_0$, $YY^* = b_1$ and $YX^* = 0$. Let $A := X^*X + Y^*Y$. It follows,

$$\|A + K\| = \max(\|X^*X + K\|, \|Y^*Y + K\|) = \max(\|b_0 + K\|, \|b_1 + K\|)$$

and $\|b_0 + b_1 + K\| \leq \|b_0 + K\| + \|b_1 + K\| \leq 2\|A + K\|$ for every closed ideal K of $\mathcal{M}(B)$. In particular, $(1/2)\|b + K + B\| = \|A + K + B\| \leq \|b + K + B\|$, because $b + B = b_0 + b_1 + B$. Thus, the sequence $a_n := b^{1/2} g_n^2 b^{1/2}$ has the desired properties. □

The following Proposition 2.13.2 has applications to ideal-equivariant extension and lifting problems. The stable corona Q(B) of (general) σ-unital and stable B has usually many ideals that are very different from the special ideals $\pi_B(\mathcal{M}(B, J))$, –

[47] Here, the $s_1, s_2, \ldots \in \mathcal{M}(B)$ are isometries with the property that $\sum_n s_n (s_n)^*$ converges strictly to 1, cf. Remark 5.1.1(8).

even if B is stable and separable. Notice that the corona algebra $\mathrm{Q}(B)$ of the stable (not w.p.i.) C^*-algebra $B := \mathrm{C}_0((0,1] \times [0,1]^\infty, \mathbb{K})$ does not have the property derived in Proposition 2.13.2 for s.p.i. algebras.

PROPOSITION 2.13.2. *Suppose that B is a σ-unital and weakly purely infinite C^*-algebra. Let $a, b \in \mathrm{Q}(B) := \mathcal{M}(B)/B$. Then*

$$\|a + \pi_B(\mathcal{M}(B,J))\| \leq \|b + \pi_B(\mathcal{M}(B,J))\|, \qquad \textit{for every } J \in \mathcal{I}(B),$$

implies that the element a is in the closed ideal $I(b)$ of $\mathrm{Q}(B)$ generated by b.

General ideals should be J, K, ..., because $I(b)$ is reserved for the by b absorbed ideal...

PROOF. We may suppose that B is stable, because $B \otimes \mathbb{K}$ is again weakly purely infinite and σ-unital, and there is an isomorphism φ from $\mathrm{Q}(B)$ onto the corner $\pi_{Q\otimes\mathbb{K}}(\mathcal{M}(B) \otimes p_{11})$ of $\mathrm{Q}(B \otimes \mathbb{K})$, that satisfies

$$\varphi(\pi_B(\mathcal{M}(B,J))) = \varphi(\mathrm{Q}(B)) \cap \pi_{Q\otimes\mathbb{K}}(\mathcal{M}(B \otimes \mathbb{K}, J \otimes \mathbb{K}))\,.$$

Now suppose that B is stable, σ-unital and is weakly p.i.

Why $B \otimes \mathbb{K}$ is again weakly purely infinite? Answer: pi-n of B implies pi-n^2 of $B \otimes \mathbb{K}$, see paper with Rørdam .
With pi(n) could be difficulties !!!.
Needs: If $X \in M_2(B)_+$ is given. Then all entries of $\mathrm{diag}(X, \ldots, X) \in M_{2n}(B)$ can be considered as properly infinite elements in $M_2(M_n(B))$, i.e., it needs some sort of “almost” strong pure infiniteness!

`Give Ref for next!:` By Definition **??** and Proposition **??** there exists $n \in \mathbb{N}$ such that B is pi-n in the sense of Definition **??**, i.e., $b \otimes 1_n$ is properly infinite (in $M_n(B)$) for each $b \in B$, cf. Proposition 2.12.8(iv).

Since B is stable, $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_\mathrm{n}$, i.e., there are isometries $T_1, \ldots, T_n \in \mathcal{M}(B)$ such that $T_1T_1^* + \cdots + T_nT_n^* = 1$. Let $\delta_n(b) := T_1bT_1^* + \cdots + T_nbT_n^*$ for $b \in \mathcal{M}(B)$. Then $\delta_n(J) = J \cap \delta_n(\mathcal{M}(B))$ and $\|\delta_n(b) + J\| = \|b + J\|$ for *every* closed ideal J of $\mathcal{M}(B)$ and every $b \in \mathcal{M}(B)$. Moreover, $\delta_n(b) \otimes p_{11} = Z(b \otimes 1_n)Z^* \sim_{\mathrm{MvN}} b \otimes 1_n$ in $M_n(\mathcal{M}(B))$, by the isometry $Z = T_1 \otimes p_{11} + \cdots + T_n \otimes p_{1n} \in \mathcal{M}(B) \otimes M_n$. Thus, $\delta_n(b)$ is properly infinite for every $0 \neq b \in B$.

Let $s_1, s_2, \ldots \in \mathcal{M}(B)$ a sequence of isometries such that $\sum_n s_n(s_n)^*$ converges strictly and unconditionally to $1 \in \mathcal{M}(B)$, cf. Remark 5.1.1(8). This applies also to the sequences of isometries $t_n := T_{j(n)}s_{k(n)}$ and $t_n := s_{k(n)}T_{j(n)}$. Therefore $U := \sum_{j,k} T_js_kT_j^*s_k^*$ converges strictly and unconditionally, cf. Remark 5.1.1(2). The sum U is a unitary in $\mathcal{M}(B)$ by Lemma 5.1.2(i) and satisfies $U\sum_k s_k(\delta_n(b_k))s_k^*U^* = \delta_n(\sum_k s_kb_ks_k^*)$ for every sequence $(b_1, b_2, \ldots) \in \mathcal{M}(B)$. In summary, we get $\|J + \sum_k s_kb_ks_k^*\| = \|J + \sum_k s_k\delta_n(b_k)s_k^*\|$ for every (norm-)closed ideal J of B and every sequence $(b_1, b_2, \ldots) \in \ell_\infty(B)$.

By Lemma 2.13.1, for each $Y \in \mathcal{M}(B)_+$ there is a sequence $(y_1, y_2, \ldots) \in \ell_\infty(B)_+$ such that and

$$(1/2)\operatorname{dist}(Y, B+J) \leq \operatorname{dist}(\sum_n s_n y_n s_n^*, B+J) \leq \operatorname{dist}(Y, B+J)$$

for each closed ideal J of $\mathcal{M}(B)$.

It implies, that $I(\sum_n s_n y_n s_n^*) + B = I(Y) + B$. The new sequence, $(\delta_n(y_1), \delta_n(y_2), \ldots) \in \ell_\infty(B)$ has the same properties as $(y_1, y_2, \ldots)$, but satisfies in addition that $f(\delta_n(y_n)) = \delta_n(f(y_n))$ is properly infinite or zero for every continuous function $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$.

We have seen above:
There are sequences $(c_1, c_2, \ldots), (d_1, d_2, \ldots) \in \ell_\infty(B)_+$ such that $f(c_n)$ and $f(d_n)$ are properly infinite or zero for each $n \in \mathbb{N}$ and every $f \in \mathrm{C}_0(0,1]$, and such that, for $c := \pi_B(\sum_n s_n c_n s_n^*)$ and $d := \pi_B(\sum_n s_n d_n s_n^*)$, holds $I(c) = I(a)$, $I(d) = I(b)$, $\|c + K\| \leq \|a + K\|$ and $\|b + K\| \leq 2\|d + K\|$ for all closed ideals K of $\mathrm{Q}(B)$.

Let $e := 2d$, $e_n := 2d_n$ and $E := \sum_n s_n e_n s_n^*$. Notice that $e = \pi(E)$ and $\pi_B(\sum_n s_n a_n s_n^*)$ generate the same ideal of $\mathrm{Q}(B)$ if there is a surjective map $\lambda \colon \mathbb{N} \to \mathbb{N}$ such that $a_n = e_{\lambda(n)}$ and $\lambda^{-1}(k)$ is finite for all $k \in \mathbb{N}$. Indeed: If σ is a permutation $\mathbb{N}$, then $\sum_n s_n a_{\sigma^{-1}(n)} s_n^* = \sum_n s_{\sigma(n)} a_n (s_{\sigma(n)})^*$ is unitarily equivalent to $\sum_n s_n a_n s_n^*$ in $\mathcal{M}(B)$ by the unitary $\sum_n s_{\sigma(n)} s_n^*$, cf. Lemma 5.1.2(i). After a suitable permutation (of indices) of the sequence $(a_1, a_2, \ldots)$ we may suppose that there is a strictly increasing sequence $1 = m_1 < m_2 < \cdots \in \mathbb{N}$ such that $e_k = a_{m_k} = a_{m_k+1} = \cdots = a_{m_{k+1}-1}$ for $k = 1, 2, \ldots$, i.e., that the sets $\lambda^{-1}(k)$ are now disjoint intervals with $j < i$ for all $j \in \lambda^{-1}(k)$ and $i \in \lambda^{-1}(\ell)$ if $k < \ell$. Let $A_k := \sum_{j=m_k}^{m_{k+1}-1} s_j e_k s_j^*$, $R := \sum_k s_{m_k} e_k (s_{m_k})^*$ and $P_k := \sum_{j=m_k}^{m_{k+1}-1} s_j s_j^*$. Then $\sum_k P_k = 1$, $A_k \in B$, $A := \sum_n s_n a_n s_n^* = \sum_k A_k \geq R$, and there is an isometry $T \in \mathcal{M}(B)$ with $TT^* = \sum_k s_{m_k} s_{m_k}^*$ and $T^*AT = T^*RT = E := \sum_n s_n e_n s_n^*$. Let $\delta > 0$. Since $(e_k - \delta)_+$ is zero or properly infinite, there is $g_k \in BP_k$ such that $\|g_k\|^2 \leq 2/\delta$ and $g_k^* e_k g_k = (A_k - \delta)_+$. The sequence $\sum_k s_k g_k$ is strictly convergent to an element $G \in \mathcal{M}(B)$ with $G^*EG = (A - \delta)_+$, cf. Remark 5.1.1(2).

Further notice, that $\|f(e) + \pi_B(\mathcal{M}(B,J))\| = f(\|e + \pi_B(\mathcal{M}(B,J))\|)$ for increasing continuous functions $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$, because $\|f(e) + \pi_B(\mathcal{M}(B,J))\|$ is the norm of the natural image of $e \in \mathrm{Q}(B)$ in $\mathrm{Q}(B/J)$.

Now we show that, if $\|c + \pi_B(\mathcal{M}(B,J))\| \leq \|e + \pi_B(\mathcal{M}(B,J))\|$ for all $J \in \mathcal{I}(B)$, then $C := \sum_n s_n c_n s_n^*$ is contained in $I(E) + B$:
We have, for $\delta > 0$,

$$\|(c-\delta)_+ + \pi_B(\mathcal{M}(B,J))\| = (\|c + \pi_B(\mathcal{M}(B,J))\| - \delta)_+ \leq \|(e-\delta)_+ + \pi_B(\mathcal{M}(B,J))\|\,.$$

It follows, that $(C - \delta)_+ = \sum_n s_n (c_n - \delta)_+ s_n^* \in B + \mathcal{M}(B, J_k)$ for each $k \in \mathbb{N}$, where J_k denotes the closed ideal of B generated by $\{(e_n - \delta)_+\,;\ n = k, k+1, \ldots\}$.

Let $\sum_n s_n (c_n - \delta)_+ s_n^* = x + y$ with $x \in B$ and $y \in \mathcal{M}(B, J_k)$, then $\lim s_n^* x s_n = 0$ implies that there exists $m \in \mathbb{N}$ (depending on k) such that $\|(c_n - \delta)_+ - s_n^* y s_n\| <$

$\delta/2$ for all $n \geq m(k)$. Thus, there exist contractions $z_n \in B$ with $z_n^* s_n^* y s_n z_n = (c_n - 2\delta)_+$. It follows that $(c_n - 2\delta)_+ \in J_k$ for all $n \geq m$.

We define $m(k) \in \mathbb{N}$ inductively: Let $m(0) = 1$, and let $m(k)$ denote the smallest $m \geq \max(k+2, m(k-1)+2)$ with the property $(c_n - 2\delta)_+ \in J_k$ for all $n \geq m$.

For each $n \geq m(k)$, there are $\ell(n,k) \geq k$ and $h_{n,k} \in B$ such that

$$(c_n - 3\delta)_+ \,=\, h_{n,k}^* \big(\sum_{j=k}^{\ell(n,k)} s_n s_j (e_j - \delta)_+ s_j^* s_n^* \big) h_{n,k} \,.$$

This $\ell(n,k)$ and $h_{n,k}$ exists, because the sums $\sum_{j=k}^{\ell} s_j (e_j - \delta)_+ s_j^*$ are properly infinite, generate J_k, and $(c_n - 3\delta)_+$ is in the Pedersen ideal $\mathrm{Ped}(J_k)$ of J_k. Now let $Y_n := s_n(\sum_{j=k}^{\ell(n,k)} s_j e_j s_j^*) s_n^*$ for $n \geq m_1$, $\psi(t) := (\max(t-\delta, 0)/t)^{1/2}$, and $H_n := \psi(Y_n) h_{n,k} s_n^*$. The operators H_n satisfy $H_n^* Y_n H_n = s_n(c_n - 3\delta)_+ s_n^*$ and $\|H_n\|^2 \leq 2/\delta$. The sums $\sum_n Y_n = \sum_n s_n(s_n^* Y_n s_n) s_n^*$ and $\sum H_n = \sum s_n(s_n^* \psi(Y_n) h_{n,k}) s_n^*$ converge strictly to elements $Y \in \mathcal{M}(B)_+$, $H \in \mathcal{M}(B)$ and satisfy $H^* Y H = \sum_{n \geq m_1} s_n(c_n - 3\delta)_+ s_n^*$, cf. Remark 5.1.1(2). Thus, $(C - 3\delta)_+ - H^* Y H \in B_+$. The construction of Y and H depends on $\delta > 0$.

We are going to show that $Y \precsim A$ in $\mathcal{M}(B)$ for some suitable $A = \sum_m s_m a_m s_m^*$ (of the above considered type). It implies $Y \in I(A) = I(E)$ and, finally, $C \in I(E) + B$.

Consider the subsets

$$X_k := \{(n,j)\,;\; m_k \leq n < m_{k+1}, k \leq j \leq \ell(n,k)\,\} \subseteq \mathbb{N} \times \mathbb{N}\,.$$

The sets X_k are pairwise disjoint and $X := \bigcup_k X_k \subseteq \mathbb{N} \times \mathbb{N}$ has the property, that the map $p_2 \colon X \ni (n,j) \mapsto j \in \mathbb{N}$ is surjective with finite $(p_2)^{-1}(n)$ for each $n \in \mathbb{N}$. The map $p_1 \colon X \ni (n,j) \mapsto n \in \mathbb{N}$ maps X onto $\mathbb{N} \setminus \{1, \ldots, m_1 - 1\}$, and $\lambda_1^{-1}(n)$ is finite for each $n \geq m_1$. We find a bijective map $\xi \colon \mathbb{N} \to X$ from $\mathbb{N}$ onto X. Now let $\lambda := p_1 \circ \xi$ and consider the sequence $(a_1, a_2, \ldots)$ with $a_m = e_{\lambda(m)}$, and let $A := \sum_m s_m a_m s_m^*$ Furthermore, consider the map $(n,j) \mapsto T_{(n,j)} := s_n s_j$. Then

$$Y = \sum_{n \geq m_1} Y_n = \sum_{n \geq m_1} \sum_{j \in p_1^{-1}(n)} T_{(n,j)} c_j T_{(n,j)} = \sum_{m \geq 1} T_{\xi(m)} a_m T_{\xi(m)}^*\,.$$

Thus $Y = Z^* A Z$ for the isometry $Z := \sum s_n T_{\xi(m)}^*$. The latter sum converges strictly by Remark 5.1.1(2), because $T_{\xi(m)}^* T_{\xi(n)} = \delta_{m,n} 1$ and $\sum_m T_{\xi(m)} T_{\xi(m)}^* = \sum_{n,j=1}^{\infty} s_n s_j \chi_{n,j} (s_n s_j)^*$ for $(\chi_{n,j}) \in \ell_\infty(\mathbb{N} \times \mathbb{N}) \subseteq \ell_\infty(\mathcal{M}(N))$ the characteristic function $\chi := \chi(X)$ of X. □

LEMMA 2.13.3. *Suppose that $\mathcal{S}$ is a family of closed ideals of a C^*-algebra A, and that $a, b \in A_+$ are positive contractions. The following properties (i) and (ii) are equivalent.*

(i) *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \mathcal{S}) > 0$ such that $(a - \varepsilon)_+ \in J$ for every $J \in \mathcal{S}$ with $(b - \delta)_+ \in J$.*

(ii) *There exists a continuous strictly increasing function* $f\colon [0,1] \to [0,2]$ *with* $f(0) = 0$ *such that* $\|f(b) + J\| \geq \|a + J\|$ *for all* $J \in \mathcal{S}$.

The property (i) is satisfied for all closed ideals J *of* A *if the element* $a \in A_+$ *is contained in the closed ideal* $I(b)$ *of* A *that is generated by* b.

Proof. If $a \in \overline{\mathrm{span}(AbA)}$, then, for every $\varepsilon > 0$, there are $d_1, \ldots, d_n \in A$ and $\delta := \delta(\varepsilon) > 0$, depending on a, b, ε, such that $\sum_j d_j^*(b-\delta)_+ d_j = (a-\varepsilon)_+$. It implies that $(a-\varepsilon)_+ \in J$ if $(b-\delta)_+ \in J$.
Now we can proof the equivalence of (i) and (ii):

(ii)⇒(i): Let $\varepsilon > 0$ and

$$\delta := \delta(\varepsilon, \mathcal{S}) := f^{-1}(\min(\varepsilon, f(1)))\,,$$

then $(f(b)-\varepsilon)_+ \approx (b-\delta)_+$. In particular, $(b-\delta)_+ \in J$, if and only if, $(f(b)-\varepsilon)_+ \in J$, if and only if, $\varepsilon \geq \|\pi_J(f(b)\| = \|f(b) + J\|$.

Since $\|f(b) + J\| \geq \|a + J\|$, we get $\varepsilon \geq \|\pi_J(a)\|$, i.e., $(a-\varepsilon)_+ \in J$.

(i)⇒(ii): Let $\varepsilon \in (0,1]$, consider the non-empty set $X(\varepsilon) \subseteq (0,1]$ of $\delta \in (0,1]$ with the property that $J \in \mathcal{S}$ and $(b-\delta)_+ \in J$ together imply $(a-\varepsilon)_+ \in J$. Clearly, $X(\varepsilon)$ is an interval $(0, \xi(\varepsilon))$ or $(0, \xi(\varepsilon)]$, and $X(\varepsilon) \subseteq X(\varepsilon_1)$ if $\varepsilon \leq \varepsilon_1$. Define $g(\varepsilon) := \sup X(\varepsilon) = \xi(\varepsilon)$. Then $g\colon (0,1] \to (0,1]$ is a positive and increasing function with $g(1) = 1$ and with following property:
If $g(\varepsilon) > \|b + J\|$ and $J \in \mathcal{S}$, then $\varepsilon \geq \|a + J\|$. (Because $t \geq \|b + J\|$ is equivalent to $(b-t)_+ \in J$.)

Since g is increasing and positive, there exists a strictly increasing continuous function $h\colon [0,1] \to [0,1]$ with $h(0) = 0$ and $h(\varepsilon) < g(\varepsilon)$ for all $\varepsilon \in (0,1]$. If $h(1) < 1$, then we extend h to $[0, 2-h(1)]$ by letting $h(t) := h(1) + (t-1)$ for $t \in (1, 2-h(1)]$, and get $h([0, 2-h(1)]) = [0,1]$. Now define $f\colon [0,1] \to [0,2]$ by $f := h^{-1}$.

Then, $\varepsilon \geq \|f(b) + J\| = f(\|b + J\|)$, if and only if, $h(\varepsilon) \geq \|b + J\|$, which implies $g(\varepsilon) > \|b + J\|$. It follows that $\varepsilon \geq \|f(b) + J\|$ and $J \in \mathcal{S}$ together imply $\varepsilon \geq \|a + J\|$. In particular, there is no ideal $J \in \mathcal{S}$ with $\|a + J\| > \|f(b) + J\|$. □

Corollary 2.13.4. *Suppose that* B *is a* σ*-unital, stable and weakly purely infinite* C^**-algebra, and that* $b \in \mathrm{Q}(B)_+$.

Let $\delta > 0$, *and let* $\mathcal{J}(b,\delta)$ *denote the family of ideals* $\pi_B(\mathcal{M}(B,J))$ *with* $(b-\delta)_+ \in \pi_B(\mathcal{M}(B,J))$. *Define* $I(b,\delta)$ *as the intersection of all ideals in* $\mathcal{J}(b,\delta)$.

Then the closed ideal of $\mathrm{Q}(B)$ *generated by* b *is the same as the closure of the union* $\bigcup_{\delta>0} I(b,\delta)$.

Especially all countably generated ideals of $\mathrm{Q}(B)$ *are of this type.*

If, in addition, B *is simple, then* $\mathrm{Q}(B)$ *is simple and purely infinite.*

Notice that the particular final conclusion says:
$\mathrm{Q}(B)$ is simple and p.i. if B is simple stable and p.i.

The more difficult opposite direction, namely that simplicity of $\mathrm{Q}(B)$ for stable σ-unital B implies that B is simple and p.i., is shown in

reference or cite ??????

PROOF. We combine Proposition 2.13.2 and Lemma 2.13.3, where the system of all ideals $\pi_B(\mathcal{M}(B,J))$ plays the role of $\mathcal{S}$.

All countably generated ideals are singly generated.

If, in addition, B is simple (and is weakly p.i. by assumptions !), then $\mathrm{Q}(B)$ is simple, because every singly generated ideal coincides with $\mathrm{Q}(B)$, cf. ?????

By Lemma 2.12.4(vi), $\mathrm{Q}(B)$ is again weakly purely infinite. The simplicity and the weak pure infiniteness of $\mathrm{Q}(B)$ implies the strong pure infiniteness of $\mathrm{Q}(B)$ by Proposition 2.2.1. □

14. Point-wise approximately inner maps

The technicalities of our proves have to do with the different infiniteness properties of the prime quotients of the considered C^*-algebras, or the properties of their multiplier algebras and corona algebras. Only in very comfortable situations exists a sort of local triviality with the respect to prime quotients that have reasonable infiniteness, factorial or prime properties. We try to describe such problems.

DEFINITION 2.14.1. Let X a locally compact space. We call a c.p. map $V\colon \mathrm{C_b}(X,B) \to \mathrm{C_b}(X,B)$ **point-wise approximately inner**, if there is point-norm continuous map $x \in X \mapsto V_x$ from X into the approximately inner c.p. maps on C^*-algebras B such that $V(f)(x) = V_x(f(x))$ for all $x \in X$ and $f \in \mathrm{C_b}(X,B)$.

Notice here that the positive map V is defined *on all* $f \in \mathrm{C_b}(X,B)$, if and only if, there is a general bound given by $\sup_{x\in X} \|V_x\| < \infty$ on the "fibres" V_x. (We start here with some very elementary stuff!)

DEFINITION 2.14.2. A c.p. map $V\colon A \to A$ is called **ideal system preserving** (or $\mathcal{I}(A)$**-equivariant**) if $V(J) \subseteq J$ for all closed ideals J of A.

One of the critical points is: Let $a, b \in A_+$ contractions with $ab = a$ and suppose that a is infinite (in some suitable sense). Is b infinite? Is it the case if $(a-\delta)_+$ is infinite for each $\delta \in [0, \|a\|)$?

(Notice that $ab = a$ for $a, b \in A_+$ implies that $a = (c-1)_+$ for $c := (b-(b-1)_+) + a$. And then $a(c-a) = a$. But b can not been reconstructed from c and a, because $(b-1)_+$ is then lost be this construction of c.)

What means "infinite" here? One definition is: There exists non-zero $0 \leq e_\delta \in (a-\delta)_+ A_+ (a-\delta)_+$ with

$$e_\delta \oplus (a-\delta/2)_+ \precsim (a-\delta/2)_+ \,.$$

It depends here from the answer to the question if $(a-\delta/2)_+ \oplus (b-(a-\delta/2)_+) \precsim b$. This don't happen if A is commutative!

But something like $(a-\delta/2)_+ \oplus (b-(a-\delta)_+) \precsim b$ in $M_2(C_0(a,b))$??? and $(a-\delta/2)_+ + (b-(a-\delta)_+) \geq$?????

THEOREM 2.14.3. *Let X is a σ-compact l.c. space, and suppose that the C^*-algebra $\mathrm{C}_0(X,B)$ is weakly p.i.*

Then each point-wise approximately inner *c.p. map*

$$V\colon\ \mathrm{C}_\mathrm{b}(X,B) \to \mathrm{C}_\mathrm{b}(X,B)$$

in sense of Definition 2.14.1 is ideal system preserving *(in sense of Definition 2.14.2) completely positive map.*

PROOF. It is not difficult to see that positivity preserving linear maps $T\colon C \to D$ from C^*-algebras C into C^*-algebras D that are defined on all elements of C^*-algebras C are always norm-continuous.

Thus, our map V is bounded and $\sup_{x\in X} \|V_x\| \leq \|V\| < \infty$. The assumption of point-norm continuity of $x \mapsto V_x$ and the obvious estimate $\|V_x\| \leq \|V\|$ implies that the map $x \mapsto V_x(b(x)) \in B$ is continuous on X for each continuous map $b\colon X \to B$.

Since X is σ-compact, there is a function $\gamma \in \mathrm{C}_0(X)$ with $0 < \gamma(x) \leq 1$ for all $x \in X$ and $\max_{x\in X} \gamma(x) = 1$.

We define compact sets $Y_n \subseteq X$ with the properties $Y_n \subseteq Y^\circ_{n+1}$ and $X = \bigcup_n Y_n$ and functions $\alpha_n \in \mathrm{C}_0(X)$ by

$$\alpha_n := 2^{n+1}(\gamma - 2^{-(n+1)})_+ - 2^n(\gamma - 2^{-n})_+$$

and $Y_n := \alpha_n^{-1}(1)$. Notice $\alpha_{n+1}\alpha_n = \alpha_n$ and that the support of α_n is contained in Y_n.

Then $\beta_0 := 0$ and $\beta_n := \alpha_n - \alpha_{n-1}$ for $n \in \mathbb{N}$ satisfy $\beta_{n+k}\beta_n = 0$ for $k > 1$ and $\sum_n \beta_n = 1$ on X. The support of β_n is contained Y_{n+1}.

By assumption, there is $k \in \mathbb{N}$ such that $\mathrm{C}_0(X,B)$ is pi(k). This passes to the quotients $\mathrm{C}(Y,B)$ of $\mathrm{C}_0(X,B)$ for compact subsets $Y \subseteq X$. Let $a \in \mathrm{C}_\mathrm{b}(X,B)_+$, $\varepsilon > 0$ and $y \in Y \subseteq X$. There exists $n(y) \in \mathbb{N}$, $b_1,\ldots,b_n \in B$ such that

$$\|V_y(a(y)) - \sum b_j^* a(y) b_j\| < \varepsilon\,,$$

because $V_y\colon B \to B$ is approximately inner. The continuity of the function

$$x \mapsto \|V_x(a(x)) - \sum b_j^* a(x) b_j\|$$

implies that there is an open neighborhood $U(y)$ of y such that

$$\|V_x(a(x)) - \sum b_j^* a(x) b_j\| < \varepsilon \quad \forall\ x \in U(y)\,.$$

By compactness of Y, thus implies the existence of finitely many $U_1,\ldots,U_m$ and $n(i) \in \mathbb{N}$, $b_j^{(i)} \in B$, $i = 1,\ldots,m$, $j = 1,\ldots,n(m)$ such that the U_i cover Y and $\|V_x(a(x)) - \sum_j (b_j^{(i)})^* a(x) b_j^{(i)}\| < \varepsilon$ for $x \in U_i$. If we take a partition of unit

$f_1, \ldots, f_m \in \mathrm{C}(Y)_+$ with supports of f_i in U_i and $\sum_i f_i = 1$, and let $c_\ell \in \mathrm{C}(Y, B)$ for $\ell = 1, \ldots, n(1) + \cdots + n(m)$ denote one and only one of the $f_i^{1/2} \cdot b_j^{(i)}$, then

$$\|V_x(a(x)) - \sum_\ell c_\ell(x)^* a(x) c_\ell(x)\| < \varepsilon \quad \forall \ x \in Y\,.$$

This shows that the restriction of $x \mapsto V_x(a(x))$ to Y is in the closed ideal generated by $a|Y$, for each $a \in \mathrm{C_b}(X)$. It follows that for $\delta > 0$ and compact $Y \subseteq X$ there exists $d_1, \ldots, d_k$ with

$$\|V_x((a(x) - \delta)_+) - \sum_{j=1}^k d_j(x)^*(a(x) - \delta)_+ d_j(x)\| < \delta \quad \text{for} \ \ x \in Y\,.$$

We get for each of the above constructed Y_n elements $d_1^{(n)}, \ldots, d_k^{(n)} \in \mathrm{C}(Y_n, B)$ such that the last inequality holds for $x \in Y_n$ with d_j replaced by $d_j^{(n)}$. We define elements $e_j, f_j \in \mathrm{C_b}(X, B)$ for $j = 1, \ldots, k$ by

$$e_j(x) := \sum_{n=1}^\infty \beta_{2n-1}(x)^{1/2}(a(x) - \delta)_+^{1/2} d_j^{(2n)}\,,$$
$$f_j(x) := \sum_{n=1}^\infty \beta_{2n}(x)^{1/2}(a(x) - \delta)_+^{1/2} d_j^{(2n+1)}\,.$$

Then $\| \sum_{j=1}^k e_j^* e_j\| \leq \|a\|\|V\| + \delta$, $\| \sum_{j=1}^k f_j^* f_j\| \leq \|a\|\|V\| + \delta$, $g_\delta(a) e_j = e_j$, and $g_\delta(a) f_j = f_j$ for $j = 1, \ldots, k$, where $g_\delta(t) := \min(\delta^{-1} t, 1)$. Since the functions β_{2n} (respectively the β_{2n-1}) have mutually orthogonal supports and $\sum \beta_n = 1$, one can see that

$$\|V((a - \delta)_+) - \sum_{j=1}^k e_j^* g_\delta(a) e_j + f_j^* g_\delta(a) f_j\| \leq \delta\,.$$

Since $g_\delta(a) \leq \delta^{-1} a$, the element $(V((a - \delta)_+) - 2\delta)_+$ is in the closed ideal $I(a)$ of B that is generated by a. The $\delta > 0$ can be taken arbitrarily small with e_j, f_j $(j = 1, \ldots, k)$ depending on δ. Hence, $V(a) \in I(a)$. □

The Theorem 2.14.3 is needed, because even the answer to following questions are still open.

QUESTIONS 2.14.4. Is $\mathrm{C}([0,1]^\infty, A)$ weakly purely infinite if A is purely infinite?

A positive answer would follow from a positive answer of the question: *Is* $\mathrm{C}([0,1], A)$ *purely infinite if* A *is purely infinite?* (In fact a positive answer would imply that $\mathrm{C}_0(X) \otimes A$ is purely infinite if A is purely infinite and X is a locally compact Hausdorff space.)

To understand the non-triviality of this question recall here:
A is purely infinite $\Leftrightarrow$ A is pi(1) $\Leftrightarrow$ A is 1-purely infinite (i.e., A has property pi-1).

We know only that $\mathrm{C}([0,1], A)$ has property pi-2 if A is p.i. So the zoological garden of really different pi-n or pi(n) could be bigger than we hope for. What is the secret behind such possible differences?

COROLLARY 2.14.5. *Suppose that X is a σ-compact, non-compact, locally compact Hausdorff space, and B is a C^*-algebra. Let $Y \subseteq \beta(X) \setminus X$ a closed subset of the corona of X, $A \subseteq \mathrm{C_b}(X,B)|Y$ a separable C^*-subalgebra of $\mathrm{C_b}(X,B)|Y$ and let $C \subseteq A$ a commutative C^*-subalgebra of A.*

If the C^*-algebra $\mathrm{C_0}(X,B)$ is weakly p.i., then there exists a continuous map

$$V\colon X \ni x \to V_x \in \mathrm{CP}(B)$$

from X into the approximately inner c.p. contractions $\mathrm{CP_{in}}(B)$, such that the corresponding c.p. map $V_Y := V|Y\colon \mathrm{C_b}(X,B)|Y \to \mathrm{C_b}(X,B)|Y$ maps A into a commutative C^*-subalgebra $C^*((V_Y(A))$ of $\mathrm{C_b}(X,B)|Y$, fixes the elements of C, and has the property, that $V_Y(b)$ is in the closed ideal of $\mathrm{C_b}(X,B)|Y$ that is generated by b for each positive $b \in \mathrm{C_b}(X,B)\,|Y$.

PROOF. `to be filled in, use w.p.i. of ????` ?? □

Recall that $\mathcal{S}_{\mathrm{in}}(B)$ is the set of all inner c.p. maps V defined by $V(b) := \sum_{k=1}^n d_k^* b d_k$ for some $n := n(V) \in \mathbb{N}$ and some finite sequence $d_1, \ldots, d_n \in B$. The elements of $\mathcal{S}_{\mathrm{in}}(B)$ is an *algebraic* matrix operator-convex cone. Its point-norm closure will be denoted by $\mathrm{CP_{in}}(B)$.

COROLLARY 2.14.6. *Let X be a non-compact locally compact σ-compact Hausdorff space, and let $\emptyset \neq Y \subseteq (\beta X) \setminus X$ denote a (non-empty) closed subset of the corona of X.*

Suppose that, for every point-norm continuous family $x \in X \mapsto V_x \in \mathrm{CP_{in}}(B)$ with $\|\, V_x \,\| \leq 1$, the completely positive contractions

$$V\,|Y\colon\; \mathrm{C_b}(X,B)\,|Y \to \mathrm{C_b}(X,B)\,|Y$$

are ideal-system equivariant, i.e., $V\,|Y(J) \subseteq J$, for each closed ideal of J of $\mathrm{C_b}(X,B)\,|Y$.

Then B has the following property:

There exists $\omega \in \beta(\mathbb{R}_+) \setminus \mathbb{R}_+$ such that, for every continuous path $t \in \mathbb{R}_+ \mapsto a(t) \in B_+$ with $\|a(t)\| \leq 1$ and for every $\varepsilon > 0$, there exists an open subset $U = U(\varepsilon) \subseteq \mathbb{R}_+$, in the ultrafilter of $\mathbb{O}(\mathbb{R}_+)$ defined by ω, with the property that the values $\mu(t) := \mu(a(t), \mathcal{S}_{\mathrm{in}}(B), \varepsilon)$ remain bounded on U (i.e., there is $n(U) \in \mathbb{N}$ such that $\mu(t) \leq n(U)$ for all $t \in U$).

In particular, there exists a sequence $n < t_n < t_{n+1}$ such that $\sup_n \mu(t_n) < \infty$.

If, in addition, X is "disconnected at ∞", i.e., $X = \bigcup U_n$ for compact open subsets $U_1 \subseteq U_2 \subseteq \cdots \subseteq X$, then B has the stronger property (i) of Definition 2.0.4 for some $n \in \mathbb{N}$.

Is the existence of the sequence $(t_n)_n$ equivalent to Property (i) of Definition 2.0.4 for $\mathrm{pi}(n)$ in ???? for some $n \in \mathbb{N}$?

PROOF. Recall that the notation $a\,|Y$ for $a \in \mathrm{C_b}(X,B)$ refers to the fact that $\mathrm{C_b}(X,B)$ is a $\mathrm{C_b}(X)$-algebra, i.e., $\mathrm{C_b}(X)$ is unitally contained in $\mathrm{C_b}(X,\mathcal{M}(B)) \subseteq \mathcal{M}(\mathrm{C_b}(X,B))$, and that $\mathrm{C}(\beta X) \cong \mathrm{C_b}(X)$ (both in a natural way).

By Lemma **??**, it sufficed to show that the convex set $\mathcal{S}_{\mathrm{in}}(B) \subseteq \mathrm{CP_{in}}(B)$ of inner c.p. contraction is controlled in the sense of Definition **??**, because this implies the existence of $n \in \mathbb{N}$ such that B satisfies the property (i) of Definition 2.0.4.

Suppose that $\mathcal{S}_{\mathrm{in}}$ is not controlled.

Then there is $\varepsilon > 0$ and a sequence of contractions $a_1, a_2, \dots \in B_+$ such that $\mu(a_n, \mathcal{S}_{\mathrm{in}}(B), 3\varepsilon)\ > 2^n$, i.e., there are $V_n \in \mathcal{S}_{\mathrm{in}}$ such that the equation $\sum_{j=1}^m d_j^* a d_j = (V_n(a_n) - 3\varepsilon)_+$ with contraction $d_1, \dots, d_m \in B$ implies $m \geq 2^n$.

??

`Change text and proof of cor. 2.14.6 !!!`

`Reduction to case:` B `separable,` $X = \mathbb{R}_+$, $Y = \{\omega\}$, $\omega \in \gamma(\mathbb{R}_+)$.
???

?? `proof has to be revised: Consider the case` $X = [1,\infty)$.

`Need following stronger result:`

There exist $\varepsilon \in (0, 1/4)$, a continuous path $t \in [1,\infty) \mapsto a(t) \in B_+$ with $\|a(t)\| \leq 1$, and a point-norm continuous path $t \in [1,\infty) \mapsto S_t \in \mathcal{S}_{\mathrm{in}}(B)$, such that

$$S_t(a(t)) - 3\varepsilon \leq \sum_{j=1}^m d_j^* a d_j\,,$$

with *contractions* $d_1, \dots, d_m \in B$, implies $m \geq t$ (for each fixed $t \in [1,\infty)$).

Remark: If B is separable and all non-zero σ-unital hereditary ????

`Following seems NOT to work:`

We define continuous $g_n(t)$ on $[1,\infty)$ by $g_n(t) := 0$ for $t \in [1,n]$ and $t \geq n+2$, $g_n(n+1) := 1$, and g_n linear on $[n, n+1]$ and on $[n+1, n+2]$. Then we define $a(t) := \sum_{n=1}^\infty g_n(t) a_n \in B_+$ and $S_t(a) := \sum_{n=1}^\infty g_n(t) S_n(a)$ for $t \in [1,\infty)$. The function $a(t)$ defines a positive contraction $a \in \mathrm{C_b}([1,\infty), B)_+$, the map $t \in [1,\infty) \mapsto S_t(a)$ is continuous for each $a \in B$ and $\|S_t\| \leq 1$. Therefore, $S(f)(t) := S_t(f(t))$ for $f \in \mathrm{C_b}([1,\infty), B)$ defines a positive contraction $S \in \mathrm{CP}(\mathrm{C_b}([1,\infty), B))$.

`We proceed with better definition of` $a(x)$ `and` V_x,
`given further above. (or below?)`

`End: suggestion for revision.`

Since X is σ-compact and locally compact, there is a continuous function $h\colon X \to (0,1]$ such that $h^{-1}[\delta, 1]$ is compact for each $\delta \in (0,1]$. We let $V_x(a) := S_t(a)$ and $b(x) := a(t)$ for $t := h(x)^{-1}$.

By definition of $V\,|Y$, the equation $(V\,|Y)(b\,|Y) = c\,|Y$ holds for the map $x \in X \mapsto c(x) := V_x(b(x))$ in $\mathrm{C_b}(X,B)_+$ with $\|c\| \leq 1$. Notice that $((c\,|Y) - \varepsilon)_+ = (c - \varepsilon)_+ |Y$, because $a \in \mathrm{C_b}(X,B) \mapsto a\,|Y \in \mathrm{C_b}(X,B)\,|Y$ is a *-epimorphism.

By our *assumption*, $(c\,|Y)$ is in the ideal generated by $b\,|Y$.

It follows, that there are $m \in \mathbb{N}$ and contractions $e_1, \ldots, e_m \in \mathrm{C_b}(X, B)\,|Y$, such that $\sum_j e_j^*(b\,|Y)e_j = (c-\varepsilon)_+|Y$. Let $f_j \in \mathrm{C_b}(X, B)$ contractions with $f_j\ |Y = e_j$, then

$$r(x) := \left(\sum_{j=1}^{m} f_j(x)^* b(x) f_j(x)\right) - (V_x(b(x)) - \varepsilon)_+$$

defines a selfadjoint element $r \in \mathrm{C_b}(X, B)$ with $r\,|Y\ = 0$.This means for the bounded continuous non-negative function $\lambda \in \mathrm{C_b}(X) \cong \mathrm{C}(\beta X)$ with $\lambda(x) := \|r(x)\|$ for $x \in X$ that $\lambda\,|Y\ = 0$.

The subset $Z := \{x \in \beta X\,;\ \lambda(x) < \varepsilon\}$ is an open subset of βX with $Z \supset Y$. In particular, $Z \neq \emptyset$. Since X is a dense open subset of βX, it follows that $U := X \cap Z$ is a non-empty open subset of X. The set U can not be contained in a compact subset $K \subseteq X$, because, otherwise, $\lambda(x) \geq \varepsilon$ for all $x \in (\beta X) \setminus X$, which is impossible by $\emptyset \neq Y \subseteq (\beta X) \setminus X$ and $\lambda\,|Y = 0$.

Thus, $\inf\{\,h(x)\,;\ x \in U\,\} = 0$, and there exists a sequence $x_1, x_2, \ldots \in U$ with $\lim h(x_n) = 0$, such that $\lambda(x_n) = \|r(x_n)\|\ < \varepsilon$.

Let $x \in U$ with $t := h(x)^{-1} > 2m$. Then $a(t) = b(x)$ and, with $d_j := f_j(x)$,

$$S_t(a(t)) - 2\varepsilon\ =\ V_x(b(x)) - 2\varepsilon\ \leq\ \sum_{j=1}^{m} d_j^* a(t) d_j\,.$$

The latter inequality contradicts the property $m \geq t$ for $a(t)$, S_t and contractions $d_1, \ldots, d_n \in B$. □

15. More on pi(n)-algebras

Recall here for the following Lemma 2.15.1, that an element ... ????

LEMMA 2.15.1. *Let A a C^*-algebra, $D \subseteq A$ a hereditary C^*-subalgebra of A, $\lambda\colon A \to \mathcal{L}(\mathcal{H})$ an irreducible representation such that $\lambda(D) \neq \{0\}$ and such that $\lambda(A)$ does not contain a non-zero operator of finite rank.*

Let $x \in \mathcal{H}$ with $\|x\| = 1$ and $\lambda(D)x \neq 0$, and denote by Q the orthogonal projection from $\mathcal{H}$ onto $\overline{\lambda(D)x}$.

Then, for each $n \in \mathbb{N}$, there exists an n-homogenous positive contraction $a_n \in D_+$ with $2\|\lambda(a_n)x\| \geq \|Qx\| \geq \|\lambda(a_n)x\|$.

PROOF. The restriction of λ to D and to $\overline{\lambda(D)x} \subseteq \mathcal{H}$ is an irreducible representation of D that does not contain any non-zero compact operators in its image, because otherwise it would contain also non-zero operators of finite rank in its image $\lambda(D) \subseteq \mathcal{L}(\mathcal{H})$.

Since $\lambda(D)x$ is a dense vector space in $Q\mathcal{H}$, we find for each given $n \in N$ a linear subspace $L_n \subseteq L_{n+1}$ of $\lambda(D)x$ of dimension $= n$ such that $\mathrm{dist}(Qx, L_n) \leq \|Qx\|/2$.

Let P_n denote the orthogonal projection onto L_n. The variant of the Kadison transitivity theorem given in Lemma 2.1.15 gives that there is a closed projection $q_n \in D^{**}$ of rank $= n$ such that the normal surjective extension $\overline{\lambda|D}\colon D^{**} \to \mathcal{L}(Q\mathcal{H})$ of the restriction $\lambda|D\colon D \to \mathcal{L}(Q\mathcal{H})$ defines an isomorphism Ψ_n from $q_n D^{**} q_n$ onto $\mathcal{L}(P_n\mathcal{H})$ with $\Psi_n(P_n d P_n) = q_n D(d) q_n$ for $d \in D$. In particular $M_n \cong q_n D^{**} q_n$ for some isomorphism γ of M_n with $q_n D^{**} q_n$. By Lemma 2.1.15 there exists a C^*-morphism h from $M_n \otimes \mathrm{C}_0(0,1]$ into D with $q_n h(M_n \otimes \mathrm{C}_0(0,1)) q_n = 0$ and $q_n h(T \otimes f_0) q_n = \gamma(T)$ for $T \in M_n$. In particular $a := h(1_n \otimes f_0)$ is an n-homogenous element in D_+ with $\lambda(a) = \Psi_n(\gamma(1_n)) = P_n$.

`Check it again!   :`

We get $\lambda(a_n)x = \Psi_n(q_n)x = P_n x$, $L_n = P_n\mathcal{H} = P_n Q\mathcal{H}$, and $Qx = P_n x + (Q - P_n)x$ is an orthogonal decomposition for Qx. It follows $\mathrm{dist}(Qx, L_n) = \|Qx - P_n x\| \leq \|Qx\|/2$ by construction of $L_n = P_n\mathcal{H}$.

The orthogonal decomposition $Qx = P_n x + (Q - P_n)x$ leads to

$$\|Qx\|^2 \;=\; \|P_n x\|^2 + \|Qx - P_n x\|^2 \;\leq\; \|P_n x\|^2 + (\|Qx\|/2)^2$$

and gives that $\|Qx\|^2(3/4) \leq \|P_n x\|^2$. It implies that $\|Qx\| \leq 2\|P_n x\|$. □

PROPOSITION 2.15.2. *If A is a pi(n) C^*-algebra then, for each non-zero hereditary C^*-subalgebra $D \subseteq A$ and each pure state λ on A with $\lambda(D) \neq \{0\}$, there exists a hereditary C^*-subalgebra $E \subseteq D$ with $\lambda(E) \neq \{0\}$ such that all non-zero positive elements $b \in J_E$ in the closed ideal $J_E := \overline{\mathrm{span}(AEA)}$ of A generated by E have the property that $b \otimes 1_n$ is properly infinite in $J_E \otimes M_n$.*

Moreover there exist hereditary C^*-subalgebras $E \subseteq D$ with the above properties that satisfy moreover that the ideal $J_E \otimes \mathbb{K}$ of $A \otimes \mathbb{K}$ generated by E has property pi-n.

The primitive ideal space $\mathrm{Prim}(A) \cong \mathrm{Prim}(A \otimes \mathbb{K})$ has a base of its topology that is generated the open subsets that correspond to closed ideals J of A such that $J \otimes \mathbb{K}$ has property pi-n.

Maximal open subsets with this property have the property that ... ????

Idea of proof: Take a maximal family of pairwise orthogonal n^2-homogenous contraction in A_+. The ideal generated by them has this property.

Recall here that the positive integer n for pi(n)-algebras or for the C^*-algebras with property pi-n is only an upper bound. We do not know if there exists for each $n \in \mathbb{N}$ a pi(n)-algebra that is not a pi($n-1$)-algebra. And it is only for pi(1)-algebras (i.e., for the ordinary "purely infinite" C^*-algebras) known that this class of algebras remains invariant under extensions.

Where it is shown that extensions of pi(1) algebras gives pi(1) algebras. Give cite or ref !!!

But we know only that $\mathrm{C}([0,1], A)$ is pi-2 if A is p.i. (= pi-1). This, and the above Proposition 2.15.2 indicates that some sort of "decomposition dimension" of the saturated quasi-compact subsets of Prim(A) plays a role (it means not the covering dimension which is only for Hausdorff spaces the same as the decomposition dimension).

COROLLARY 2.15.3. *The following properties (i), (ii), (iii) and (iv) are equivalent for a (non-zero) C^*-algebra A.*

(i) *A is purely infinite in sense of Definition 1.2.1 – which is equal to the property pi(1) in Definition 2.0.4 –, i.e., A has no character, and, for each $a, b \in A_+$, with b in the closed ideal of A generated by a, and given $\varepsilon > 0$ there exist $d \in A$ with $d^*ad = (b-\varepsilon)_+$.*

(ii) *A has property pi-1 as defined in Definition **??**, i.e., each non-zero element of A_+ is properly infinite, i.e., for each nonzero $a \in A_+$ and $\delta > 0$ there exist $b, c \in A$ with $\|b^*ac\| < \delta$, $\|b^*ab - a\| < \varepsilon$ and $\|c^*ac - a\| < \varepsilon$.*

(iii) *For each $a \in A_+$ and $\varepsilon > 0$ there exist $d, e \in \overline{aAa}$ with $e^*d = 0$ and $d^*d = e^*e = (a-\varepsilon)_+$.*

(iv) *For each $a \in A_+$ and each closed ideal J of A with $a \notin J$ the element $\pi_J(a) \in A/J$ is infinite in A/J, i.e., there exists non-zero $b \in (A/J)_+$ with $\pi_J(a) \oplus b \precsim \pi_J(a)$.*

Refer below to general pi(n) and $\precsim$ rules, where it is possible?! The part (iv) seems to be the suitable definition of for all non-simple C*-algebras.

PROOF. Notice that the definition of $\precsim_A$ for C^*-algebras A is a *local property* in the sense that that the relation $a \precsim_A b$ holds for $a, b \in A_+$, if and only if, $a \precsim_D b$ in the hereditary C^*-subalgebra $D := \overline{(a+b)A(a+b)}$ of A.

(i)⇒(ii): If $a \in A_+$ is a non-zero 2-homogenous element then each element in the hereditary C^*-subalgebra $D := \overline{aAa}$ is properly infinite in D.

Since a is 2-homogenous there exits ????

Then $D \cong M_2(E)$, where E is a hereditary C^*-subalgebra of A that has the property that all elements ???????

Indeed, by definition of 2-homogenous elements there is C^*-morphism $\psi \colon M_2 \otimes \mathrm{C}_0(0,1] \to D$ with $\psi(1 \otimes f_0) = a$. The element $b := \psi(p_{11} \otimes f_0^{1/3})$ has the property that $b \leq a^{1/3}$ and and $d_1^*bd_1 + d_2^*bd_2 = a$ for $d_1 := \psi(p_{11} \otimes f_0^{1/3}) \in A$ and $d_2 := \psi(p_{12} \otimes f_0^{1/3}) \in A$. By Definition 1.2.1 of purely infinite algebras we get that there exists a sequence $c_n \in A$ with $a = \lim c_n^* b c_n$.

We define $E := \overline{bAb}$. If $e \in E_+$ then $e \oplus e \precsim e$, because $\|g_n e g_n - 1_2 \otimes e\| \to 0$ in $M_2 \otimes E$ for $n \to \infty$, where $g_n := \psi(p_{11} \otimes f_0^{1/n})h_{1,n} + \psi(p_{12} \otimes f_0^{1/n})h_{2,n}$ with ??????

It follows that E is a non-zero hereditary C^*-subalgebra of $D \subseteq A$ with the property that each element of E is properly infinite in A (an thus also in D).

Let $D \subseteq A$ a hereditary C^*-subalgebra such that every non-zero elements $d \in D_+$ is properly infinite in A. Then each $d \in D_+$ is also properly infinite in D by Remark **??**.

We show that each $c \in J_+ := \overline{\operatorname{span} ADA_+}$ is properly infinite. Indeed, for each $\varepsilon > 0$ there exist $d_1, \ldots, d_n \in D_+$ and $f_1, \ldots, f_n \in A$ with $(c - \varepsilon/2)_+ = \sum_{k=1}^n f_k^* d_k f_k$. It follows that there is $g \in A$ with $g^* d g = (c - \varepsilon)_+$ for $d := d_1 + \cdots + d_n$. Thus, the element $(c-\varepsilon)_+$ satisfies $d^{1/2} g g^* d^{1/2} \sim_{MvN} (c-\varepsilon)_+^2$, and, hence, is Murray–von-Neumann equivalent to a properly infinite element of D_+. It follows that c is properly infinite in A by Lemma **??**.

If $a \in A_+$ is the limit of elements $a_n \in A_+$, and there exists $\gamma > 0$ such that for each $\delta \in (0, \gamma)$ and $n \in \mathbb{N}$ the elements $(a_n - \delta)_+$ are properly infinite in A_+, then a is properly infinite in A: Given $\varepsilon \in (0, \gamma)$ then there exist $n_0 \in \mathbb{N}$ with $\|a_n - a\| < \varepsilon/3$ for all $n \geq n_0$. By Lemma 2.1.9 there exist contractions $d_n, e_n \in A$ with $d_n^* a d_n = (a_n - \varepsilon/3)_+$ and $e_n^*(a_n - 2\varepsilon/3)_+ e_n = (a - \varepsilon)_+$. Since $(a_n - 2\varepsilon/3)_+ \oplus (a_n - 2\varepsilon/3)_+ \precsim (a_n - \varepsilon/3)$ it follows that $(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a$. It implies that a is properly infinite in $\overline{aAa}$.

Thus, if J_τ is an upward directed family of closed ideals of A with the property that all non-zero positive elements of $\bigcup_\tau J_\tau$ are properly infinite, then all non-zero positive elements in the closed ideal $J := \overline{\bigcup_\tau J_\tau}$ are properly infinite.

It follows that A contains a maximal ideal J with the property that all nonzero positive elements $a \in J_+$ are properly infinite.

The maximality implies that J is closed in A.

Suppose that $J \neq A$. Since A has no characters, the quotient A/J can not have a character. Thus, there exists a 2-homogenous element $b = \varphi(1_2 \otimes f_0)$ for some non-zero C^*-morphism $\varphi \colon M_2 \otimes \mathrm{C}_0(0,1] \to A/J$. Since $M_2 \otimes \mathrm{C}_0(0,1]$ is a projective C^*-algebra, there exists a C^*-morphism $\psi \colon M_2 \otimes \mathrm{C}_0(0,1] \to A$ with $\pi_J \psi(1_2 \otimes f_0) = b$. The element $a := \psi(1_2 \otimes f_0)$ is a 2-homogenous element in A that is not in J.

Let $h \colon A \to \mathcal{L}(\mathcal{H})$ an irreducible representation of A, ?????

(ii)⇒(iii):

(iii)⇒(i):

(ii)⇒(iv): The property (ii) says $a \oplus a \precsim a$ for each $a \in A$. If J is a closed ideal of A with $a \notin J$, then this implies that $\pi_J(a) \oplus \pi_J(a) \precsim \pi_J(a)$

??????

(iv)⇒(???): The part (iv) says that for each $a \in A_+$ and closed ideal J of A with $\pi_J(a) \neq 0$ there exists non-zero $b \in (A/J)_+$ with $\pi_J(a) \oplus b \precsim \pi_J(a)$.

This implies that a is properly infinite in A, i.e., $a \oplus a \precsim a$ in A by Lemma **??** ???

Let $J := \{b \in A; a \oplus b \precsim a\}$. The set J is by an observation of J. Cuntz [**?**] a closed ideal of A with the property that $\pi_J(a)$ "finite" in A/J, i.e. the only element $c \in A/J$ with with $\pi_J(a) \oplus c \precsim \pi_J(a)$ is

□

Compare next Rem. with Def. 1.2.1 or Def. of pi-1 or pi in 1st Sect. in Chp. 2!!

REMARK 2.15.4. We recall that a (not necessarily simple) C^*-algebra A is **purely infinite** in the sense of Definition 1.2.1, that is equal to property pi(1) in Definition 2.0.4, if

(i) A is non-zero and has no character, i.e., every irreducible representation has dimension > 1, and
(ii) for every positive a in A, every positive b in the ideal generated by a, and every $\varepsilon > 0$ there exists $d \in A$ such that $\|b - d^*ad\| < \varepsilon$.

We write sometimes 'p.i.' for 'purely infinite'.

From the definition it is easy to see that *non-zero quotients of p.i. algebras are p.i.*

A hereditary C^*-subalgebra $D \subseteq A$ can not have a (non-zero) character χ. Indeed, otherwise a character χ would extend to a pure state ρ on A such that the GNS construction defines an irreducible representation $d_\rho \colon A \to \mathcal{L}(L_2(\rho))$ with the property that the algebra of compact operators $\mathbb{K}(L_2(\rho))$ is contained in $d_\rho(A)$. But this is impossible, because the quotient $A/d_\rho^{-1}(0) \cong d_\rho(A)$ is again purely infinite. But a rank-one projection can not be properly infinite.

Since every positive element in the ideal generated by a is in the norm-closure of elements $\sum_n c_n^* a c_n$, we can express (ii) equivalently:
For $a \in A_+$, $c_1, c_2 \in A$ and $\varepsilon > 0$, there is $d \in A$ with $\|c_1^*ac_1 + c_2^*ac_2 - d^*ad\| < \varepsilon$.

One can see that it suffices to consider here c_1, c_2 in a dense subset of the unit-ball of A and a in a dense subset S of A_+ with the property that $(a - 1/n)_+ \in S$ for all $a \in S$ and $n \in \mathbb{N}$.

There is no essentially better reduction to dense subsets of A_+, because e.g. for any unital A the set $A_+ + (0,1) \cdot 1_A$ is dense in A_+ and always $\sum_j c_j^* a c_j = d^*ad$ for given $c_1, \dots c_n \in A$ if $a \in A_+ + (0,1) \cdot 1_A$ and $d := a^{-1/2}(\sum c_j^* a c_j)^{1/2}$. And this is certainly not equivalent to pure infiniteness.

The reader can see from Definition 1.2.1, e.g. that the algebras of compact operators on a Hilbert space can not be purely infinite, and, more generally, that any quotient A/J of a p.i. algebra A can not contain a (non-zero) Abelian hereditary C^*-subalgebra, i.e., A is *strictly antiliminary* (equivalently expressed: no irreducible representation of A contains a non-zero compact operator in its image).

We list in Proposition 2.15.5 some elementary facts on purely infinite non-simple C^*-algebras. The proofs follow in the given order step by step almost trivially from the foregoing result. The reader can find the proofs of the propositions and corollaries 2.15.5–2.6.5 also in [**462**], [**463**] and [**93**]. The results on extensions and passage to stabilizations are not trivial.

Is it good to use ''stably p.i.'' ?

PROPOSITION 2.15.5. *Let $A \neq \{0\}$ a C^*-algebra.*

(o) *A is p.i., if and only if, $M_n(A)$ is p.i., if and only if, $A \otimes \mathbb{K}$ is p.i.*
 `Compare with other places e.g. Part (iv)`
(i) *Let J is a closed ideal of A, with $\{0\} \neq J \neq A$. Then A is p.i. if and only if, J and A/J are p.i.*
(ii) *Every p.i. algebra A has no irreducible representation that contains a (no-zero) compact operator in its image.*
 In particular, A is strictly antiliminary *(i.e., each non-zero quotient of A is NGCR).*
(iii) *Non-zero hereditary C^*-subalgebras of a p.i. algebra are p.i.*
(iv) *The inductive limit of p.i. algebras is p.i.*
(v) *The C^*-algebra $\prod_n B_n$ of bounded sequences $(a_1, a_2, \ldots)$ with $a_n \in B_n$ is p.i. if B_n is p.i. for every $n = 1, 2, \ldots$.*
 In particular, the ultrapowers B_ω of B are p.i. if $M_2(B)$ is p.i.
(vi) *$M_2(A)$ is p.i. if and only if A is p.i. and, for every $a \in A_+$ and $\varepsilon > 0$, there exist $b, c \in A$ with $b^*c = 0$ and $b^*b = c^*c = (a - \varepsilon)_+$.*
(vii) *If A is p.i. and σ-unital, and has no unital quotient, then A is stable.*
(viii) *Zero is the only semi-finite lower semi-continuous 2-quasi-trace on* $\mathrm{Ped}(A)_+$ *if A is p.i.*

Part (vii) generalizes Zhang's dichotomy for simple p.i. algebras. Its proof reduces to a special case of [**373**, prop. 5.1], cf. our Corollaries 5.5.1 and 5.5.3.

The next result shows that the class of p.i. algebras is closed under extensions.

Temporary we call A **stably p.i.** if $M_2(A)$ is p.i. This is equivalent to the property that $A \otimes \mathbb{K}$ is p.i. by Proposition 2.15.5(vi).

PROPOSITION 2.15.6. *Suppose that A is a C^*-algebra, and J is a closed ideal of A, such that J and A/J are both stably p.i. Then A is stably p.i.*

COROLLARY 2.15.7. *Every p.i. C^*-algebra is stably p.i. This means:*

(i) *$A \otimes \mathbb{K}$ is p.i. , if and only if, A is p.i.*
(ii) *A is p.i. if and only if for every $b \in A_+$ and $\varepsilon > 0$ there exists $c, d \in bAb$ with $c^*d = 0$ and $c^*c = d^*d = (b - \varepsilon)_+$.*

COROLLARY 2.15.8. *Let A a unital C^*-algebra. Every non-zero quotient A/J of A contains a non-unitary isometry, if and only if, 1_A is properly infinite, i.e., if and only if, there exists a unital C^*-morphism $\varphi \colon \mathcal{O}_\infty \to A$.*

PROOF. Easy direction: Suppose that 1_A is properly infinite, – i.e., that there is a unital C^*-morphism $\psi \colon \mathcal{E}_2 = C^*(s, t;\ s^*t = 0,\ s^*s = 1 = t^*t) \to A$ –, then the compositions $\pi_J \circ \psi$ with the C^*-epimorphisms π_J deliver non-unitary isometries in A/J for all closed ideals $J \neq A$ of A.

The less trivial opposite direction is an evident consequence of Part (iii) of Lemma 2.5.3: Let $a := 1$ and $J := I(a)$. Then $\pi_J(a)$ is finite in A/J by 2.5.3(iii).

By assumption on the quotients, it means that $A/J = \{0\}$ and $J = I(a) = A$, i.e., there exist $d \in M_2(A)$ with $d^*(1 \oplus 0)d = 1 \oplus 1$. This gives that 1 is properly infinite in A: $s^*s = 1$, $s^*t = 0$ and $t^*t = 1$ for $s := d_{1,1}$, $t := d_{1,2}$.

There is also a different looking "amazing" elementary argument for a proof of the non-trivial direction. We carry it out in any detail (!) – otherwise it would take only three lines. It goes as follows:

Since A itself contains at least one non-unitary isometry t by our assumptions, the set of the projections $p_t := 1 - tt^*$ for all isometries $t \in A$ is not empty. Consider now the well-defined non-zero closed ideal J of A that is generated by all (co-range) projections p_t, where t runs through *all* non-unitary isometries in A. All positive elements of the ideal J can be approximated by elements of the form d^*p_td, because the family of elements of this type is hereditary and $d^*p_sd + e^*p_te = f^*p_{s\cdot t}f$ for $f := p_sd + se$. (Use here that $p_{s\cdot t} = p_s + sp_ts^*$.)

If $J = A$ then we are ready, because then there exist $t, d \in A$ with $t^*t = 1$ and $\|d^*p_td - 1\| < 1/2$. The isometries $s := p_td(d^*p_td)^{-1/2}$ and t have orthogonal ranges $s^*t = 0$.

We show now that A/J does not contain a non-unitary isometry (in every generality for all unital C^*-algebras A that contain a non-unitary isometry). It implies that $J = A$, i.e., $A/J = \{0\}$ if each non-zero quotient of A contains a non-unitary isometry.

Notice for the following, that a unital C^*-algebra B does not contain any non-unitary isometry, if and only if, every left-invertible element of B is also right-invertible (and vice versa).

By definition of J, the image $\pi_J(t) \in A/J$ of each isometry $t \in A$ is a unitary element in A/J because $p_t := 1 - tt^* \in J$. It implies that also the image $\pi_J(b)$ of each right-invertible or left-invertible element $b \in A$ is invertible in A/J, because $t := b(b^*b)^{-1/2}$ (respectively $t := b^*(bb^*)^{-1/2}$) is an isometry.

We show now that – moreover – for every contraction $a \in A$ with $1 - a^*a \in J$ there exists an isometry $t \in A$ and $d \in A$ such that $b := ta - p_td$ satisfies $\|1 - b^*b\| < 1/2$. Then $p_t \in J$ and $\pi_J(t)^*\pi_J(b) = \pi_J(a)$ is invertible in A/J by above remarks.

Indeed, as shown above, there exists $d \in A$ and an isometry $t \in A$ with

$$\|(1 - a^*a) - d^*(1 - tt^*)d\| < 1/2$$

but this means for $b := ta - p_td$ that $\|1 - b^*b\| < 1/2$.

Thus, for every unital C^*-algebra A, the quotient A/J has the property that each isometry in A/J is unitary, if J denotes the closed ideal of A that is generated by $\{1 - tt^* \,;\, t \in A,\, t^*t = 1\}$. Moreover, above considerations show that this ideal J is equal to the ideal $I(1)$ for $1 \in A$ defined in Lemma 2.5.3(iii). □

DEFINITION 2.15.9. We say that A has the **Global Glimm halving property** if, for each $b \in A_+$ and $\delta > 0$, there exist $e \in A$ with $e^2 = 0$ and $e^*e + ee^* \in bAb$ such that $(b - \delta)_+$ is in the ideal generated by e.

Equivalently expressed:
For each $b \geq 0$ and $\delta > 0$, there is a C^*-morphism

$$\varphi\colon M_2(\mathrm{C}_0(0,1]) \to bAb$$

such that $(b-\delta)_+$ is contained in the *closed* ideal of A generated by the image of φ.

`Here seems to be a wrong definition?:` The original "halving property" says that, for given $X \in A_+$ and $\varepsilon > 0$, there exist $d_1, d_2 \in A$ such that $\|d_1^* X d_2\| < \varepsilon$ and $(X-\varepsilon)_+$ is in the closed ideal $J_1 \cap J_2$ of A, where $J_k := J(d_k^* X d_k)$ is the closed ideal generated by $d_k^* X d_k$.

Is it the same? CHECK!

REMARK 2.15.10. Every purely infinite C^*-algebra has the global Glimm halving property by Proposition 2.15.11.

The other sorts of infinity of C^*-algebras are in a sense of local nature that does not shows any understandable relation to "Glimm having" ...

PROPOSITION 2.15.11. *For every non-zero C^*-algebra A the following properties* (i) *and* (ii) *are equivalent:*

(i) *The algebra A has the Global Glimm halving property (Def. 2.15.9), and, for every $b \in A_+$ and every $\varepsilon > 0$, there is an $m = m(b, \varepsilon) \in \mathbb{N}$, – depending only on b and ε (and not on a) –, such that:*
If b is in the closed ideal generated by $a \in A_+$, then there are $d_1, \ldots, d_m \in A$ with

$$\|b - \sum_{1 \leq j \leq m} d_j^* a d_j\| < \varepsilon\,.$$

(ii) *The algebra A is purely infinite.*

PROOF. (ii)⇒(i): If A is purely infinite, then we can take $m(b, \varepsilon) := 1$ for all $b \in A_+$ and $\varepsilon > 0$. We use Lemma 2.5.15

Corollary 2.15.7 to check the global Glimm halving property:
There are $\tau \in (0, \delta)$ and $d_1, d_2 \in A$ with $d_1^*(a-\tau)_+ d_2 = 0$ and $(a-\delta)_+ = d_j^*(a-\tau)_+ d_j$ $(j = 1, 2)$. Let $e := (a-\tau)_+^{1/2} d_2 d_1^* (a-\tau)_+^{1/2}$, then $e^2 = 0$, $ee^* + e^*e \in aAa$, and $d_2^*(a-\tau)_+^{1/2} e (a-\tau)^{1/2} d_1 = (a-\delta)_+^2$.

(i)⇒(ii): The proof is an adaptation of the proof of Proposition 2.2.1(iv):
By induction, we see from condition (ii), that, for every $a \in A_+$, $\delta > 0$, and $m \in \mathbb{N}$, there exist $f_1, \ldots, f_m$ in the closure of $(a-2\delta)_+ A (a-2\delta)_+$, such that $f_j^* f_i = \delta_{ij} f_1^* f_1$, and $(a-3\delta)_+$ is in the closed ideal of A which is generated by $f_0 := f_1^* f_1$. In particular, A can not have a non-zero character.

Now let $a, b \in A_+$, such that b is in the closed ideal generated by a. Let $\varepsilon > 0$ be given. We define $\rho := \varepsilon/3$, and let $m := m((b-\rho)_+, \rho)$.

There exist $\delta > 0$ and $c_1, \ldots, c_n \in A$, such that

$$\sum_{1\leq k\leq n} c_k^*(a-3\delta)_+ c_k = (b-\rho)_+ \,.$$

Let $f_1, \ldots, f_m$ as above and $f_0 := (f_1)^* f_1$. Then $(b-\rho)_+$ is in the ideal generated by f_0. By definition of m, there exist $d_1, \ldots, d_m \in A$, such that

$$\| \sum_{1\leq j\leq m} d_j^* f_0 d_j - (b-\rho)_+ \| < \rho \,.$$

We define $g \in \mathrm{C}_0((0, \|a\|])$ by $g(t) := 0$ for $t \in [0, \delta]$, $g(t) := t/\delta - 1$ for $t \in [\delta, 2\delta]$, and $g(t) = 1$ for $t \in [2\delta, \|a\|]$. Let $h(t) := (g(t)/t)^{1/2}$.

Then $d := h(a) \cdot \sum_{1\leq j\leq m} f_j d_j$ satisfies $\|d^* a d - b\| < \varepsilon \,.$ □

REMARK 2.15.12. In joint works with Étienne Blanchard and Mikael Rørdam the above observation was used to show the following:

Suppose that A is a separable C^*-algebra and that the primitive ideal space Prim(A) *is Hausdorff, i.e., A is the algebra of continuous sections vanishing at ∞ of a continuous field of simple C^*-algebras in the sense of* [**471**] ([48]).

If, in addition, $\dim(\mathrm{Prim}(A)) < \infty$, *then A is p.i. if and only if every* simple *quotient of A is p.i. (i.e., if every fiber A_x of the continuous field on* Prim(A) *associated to A is p.i.)*

The criteria (i) and (ii) for p.i. algebras applies, because there is a "global" version of the "local " Glimm halving lemma, cf. Remark 2.1.16(ii).

The condition (i) comes from the finite dimension of Prim(A) if the fibers are simple and p.i. A "global" version of the "halving" in Remark 2.1.16(ii) probably does not extend to C^*-bundles with non-simple fibers.

It could be conjectured that also continuous fields of C^*-algebras over a Hausdorff space, and with strongly purely infinite fibers are strongly purely infinite, in particular if the algebra of continuous sections again has a Hausdorff space as primitive ideal space.

16. Characterizations of non-simple strongly p.i. algebras

The strongly purely infinite C^*-algebras A, as defined in Definition 1.2.2, are the C^*-algebras that are later used for our variant of the classification program. We need several equivalent characterizations that allow in special cases to verify the strong pure infiniteness of given C^*-algebras. So far it seems that the study of the zoological garden of possibly different versions of pure infiniteness becomes not easier to study in case of separable nuclear C^*-algebras. And this is obvious, because we have to work also with its ultra-powers and corona algebras that are in general not nuclear.

[48] I.e., A is the algebra of continuous sections vanishing at infinity of a continuous field of simple C^*-algebras over a Polish l.c. space.

We do not use here the later proven results of Chapters 9 or 10 right now (just to underline logical consistency), instead we work formally with the "decoy" version $\mathcal{D}_\infty := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$ of $\mathcal{O}_\infty$, to emphasis that we use here only above proved material and results from Appendices A and B.

```
!!!  compare notations in the following with new names
coming from notation in extra papers (started at 2002, now 2018--2021)
```

We recall here the inequalities (2.2) for the definition of *strongly pure infinite* C^*-algebras A:

For any given $a, b \in A_+$ and $\varepsilon > 0$ there exist $d, e \in A$ that satisfy following inequalities 16.1.

$$\| a^2 - d^* a^2 d \| < \varepsilon , \quad \|b^2 - e^* b^2 e\| < \varepsilon \quad \text{and} \quad \|d^* abe\| < \varepsilon . \tag{16.1}$$

If we find for given non-zero $a, b \in A_+$, $\delta \in (0, \max(\|a\|, \|b\|))$ and $\varepsilon \in (0, \delta^2)$, some elements $d, e \in A$ that satisfy the inequality (16.1) with $(a-\delta)_+$ and $(b-\delta)_+$ in place of a and b, then we can find "solutions" d, e of the original inequality (16.1) for a, b itself – but now with the additional upper estimate of the norms $\max\{\|d\| , \|e\|\} \leq (2/\varepsilon) \cdot \max(\|a\|, \|b\|)$ for d and e.

Improvement ??:

Take in $M_2(A)$ the diagonal matrix G with diagonal entries d and e and then the matrix $F := [f_{jk}]$ with entries $f_{11} = a$, $f_{12} = b$ and $f_{21} = 0 = f_{22}$ and diagonal H in $M_2(A)$ with $h_{11} = a^2$, $h_{22} = b^2$ and $h_{12} = 0 = h_{21}$ then $\|G^*(F^*F)G - H\| < 2\varepsilon$.

One has here to find an estimate of the norm $\|G\|$ of G !!!

We need a better estimate to get the formaly stronger matrix diagonalisation property:

For each positive matrix $T = [t_{jk}] \in M_n(A)_+$ and $\varepsilon > 0$ there exists a diagonal matrix G with $\|G * TG - D\| < \varepsilon$ where D is the diagonal of T, i.e. D is the diagonal matrix with entries $d_{jk} := \delta_{jk} \cdot t_{jk}$.

To find an estimate of $\|G\|$ for G, one could use here that, for example,

$$\|d^*(a-\varepsilon)_+^2 d - (a-\varepsilon)_+^2\| < \gamma^2$$

with $\varepsilon \leq \|a\|$ and $\gamma \in (0, \varepsilon)$ implies

$$\|(a-\varepsilon)_+ d\| \leq (\gamma^2 + (\|a\| - \varepsilon)^2)^{1/2} \leq \|a\| .$$

Then notice that $\|a^{-1}(a-\varepsilon)_+ d\| \leq \varepsilon^{-1} \|(a-\varepsilon)_+ d\|$.

REMARK 2.16.1. If A is a strongly purely infinite, J a closed ideal of A and D a non-zero hereditary C^*-subalgebra of A, then A/J and D are again strongly p.i.

To see this, take for $a_1, b_1 \in (A/J)_+$ elements $a, b \in A_+$ with $\pi_J(a) = a_1$ and $\pi_J(b) = b_1$. Find $d, e \in A$ that satisfy inequality (2.2) for a and b then $d_1 := \pi_J(d)$ and $e_1 := \pi_J(e)$ fulfill the Inequality (2.2) for a_1 and b_1 (in place of a, b and d, e).

Let $a, b \in D_+$ and $d, e \in A$ that satisfy the Inequality (2.2). Let $c := a^2 + b^2$ and $\gamma := \|a\|^2 + \|b\|^2$, then, for sufficiently large $n \in \mathbb{N}$, $e_n := c^{1/n} d c^{1/n}$ and $f_n := c^{1/n} e c^{1/n}$ satisfy the Inequality (2.2) with $e_n, f_n \in D$ in place of e, f. It is clear that $\|c\| \leq \gamma$ and it is easy to see that

$$\max(\|a - c^{1/n}a\|\,,\ \|b - c^{1/n}b\|) \ \leq\ \|(1-c^{1/n})c\| \ \leq\ \max\{\,1/(n+1)\,, \gamma\cdot|1-\gamma^{1/n}|\,\}\,.$$

It follows that D is again strongly purely infinite.

`Uses:` $\lim c^{1/n}a = a$ for each $0 \leq a \leq c$: Indeed, it implies $0 \leq a^{1/n} \leq c$ $a^2 \leq \|a\| a \leq \|a\| c$, and therefore

$$\|(1-c^{1/n})a\|^2 = \|(1-c^{1/n})a^2(1-c^{1/n})\| \leq \|a\|\cdot\|(1-c^{1/n})^2 c\| \leq \sup\{(t-t^{1+1/n})^2\,,$$

and

$$\sup\{|t - t^{1+1/n}|\,;\ t \in [0,\gamma]\,\} = \max\{\,1/(n+1)\,, \gamma\cdot|1-\gamma^{1/n}|\,\}\,.$$

LEMMA 2.16.2. *Suppose that a C^*-algebra A contains a dense *-subalgebra B with the properties that $(b^*b - \varepsilon)_+ \in B$ for each $b \in B$ and $\varepsilon > 0$.*

Then A is strongly purely infinite, if and only if, for each $f, g \in B$ and $\varepsilon > 0$, there exist $d, e \in A$ with

$$\max\{\,\|\,(f^*f)^2 \ -\ d^*(f^*f)^2 d\,\|\,,\ \|(g^*g)^2 - e^*(g^*g)^2 e\|\,,\ \|d^*(g^*g)(f^*f)e\|\,\} \ <\ \varepsilon\,. \tag{16.2}$$

The elements $d, e \in A$ in the inequalities (16.2) can be chosen with norms $\leq \varepsilon^{-1}\cdot 2\max\{\,\|f\|^4, \|g\|^4\,\}$.

In particular, inductive limits of strongly purely infinite algebras are strongly purely infinite.

PROOF. If we take in Definition 1.2.2 of s.p.i. C^*-algebras A the elements $a := f^*f$ and $b := g^*g$ then we find $d, e \in A$ that satisfy inequalities (16.2).

Conversely, if $a, b \in A_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon \in (0,1)$ are given, then let $\delta := ???\cdot\varepsilon???$. We find $f = f^*, g = g^* \in B$ with $\|f - a^{1/2}\| < \delta$ and $\|g - b^{1/2}\| < \delta$.

There exist $d, e \in A$ with $\|d\| \leq (2/\varepsilon)????$, ... $\|d^*f^4d - f^4\| < \varepsilon/2$, $\|e^*g^4e - g^4\| < \varepsilon/2$, ... and $\|d^*f^4g^4e - f^4\| < \varepsilon/2$, ... It follows existence of contractions $x, y \in A$ with $x^*f^2x = (a-\gamma)_+$ and $y^*g^2y = (b-\gamma)_+$ ...

By Lemma 2.1.9, there exists contractions $x, y \in A$ with $x^*f_+x = (a^{1/2} - \varepsilon^2/4)_+$

By assumption, there exist exist $d, e \in A$ that satisfy Inequalities (16.2). □

DEFINITION 2.16.3. We say that a C^*-algebra A has the (m.d.) **matrix diagonalization property** if for every matrix $M = [b_{ij}] \in M_2(A)_+$ and $\varepsilon > 0$ there exists a diagonal matrix $D := \mathrm{diag}(d_1, d_2) \in M_2(A)$ with

$$\|D^*MD - \mathrm{diag}(b_{11}, b_{22})\| < \varepsilon\,.$$

PROPOSITION 2.16.4. *The C^*-algebra A is strongly purely infinite, if and only if, A has the matrix diagonalization property (m.d.), cf. Definition 2.16.3.*

PROOF. ??

Suppose that A has the matrix diagonalization property of Definition 2.16.3.

We show that it implies that A is strongly purely infinite in sense of Definition 1.2.2, i.e., that, for every $a, b \in A_+$ and $\varepsilon > 0$, there exist $d, e \in A$ such that

$$\| a^2 - d^* a^2 d \| < \varepsilon , \quad \|b^2 - e^* b^2 e\| < \varepsilon \text{ and } \|d^* abe\| < \varepsilon . \tag{16.3}$$

Here we can apply this formula with non-zero $a, b \in A_+$ replaced by $(a-\gamma)_+$ and $(b-\gamma)_+$ for some $\gamma \in (0, \min(\|a\|, \|b\|))$ we can find a solution (d, e) of the inequalities (16.3) for $\varepsilon := \gamma^2$. Then $\|(a-\gamma)_+ d\|^2 \leq \gamma^2 + (\|a\| - \gamma)^2 \leq \|a\|^2$, and similarly $\|(b-\gamma)_+ e\| \leq \|b\|$.

If $a, b \in A_+$ satisfy $\|a\| = 1 = \|b\|$ and if $\varepsilon \in (0, 1)$, then this gives the general estimate for a solution (d, e) of the inequalities (16.3) with $\max(\|e\|, \|d\|) \leq 1/\varepsilon$, where one uses that e.g. $\|a^{-1}(a-\gamma)_+ d\| \leq \|(a-\gamma)_+ d\| / \gamma$.

Let $a, b \in A_+$ and consider the 2×2-matrix $[a_{ik}] = [a, b]^\top [a, b]$ in $M_2(A)_+$ with entries $a_{11} := a^2$, $a_{22} := b^2$, $a_{21}^* = a_{12} = ab$, and let $\varepsilon > 0$.

By assumption we find $d_1, d_2 \in A$ with

$$\| \operatorname{diag}(d_1, d_2)^* [a_{ik}] \operatorname{diag}(d_1, d_2) - \operatorname{diag}(a, b)\| < \varepsilon .$$

But this implies the Inequalities (16.3) with $d := d_1$ and $e := d_2$.

Conversely suppose that A is strongly purely infinite. Let $c \in A_+$ and take $a := b := c^{1/2}$ in (16.3) for given $\varepsilon > 0$. We observe that the there proposed $d, e \in A$ satisfy $\|d^* cd - c\| < \varepsilon$, $\|e^* ce - c\| < \varepsilon$ and $\|d^* ce\| < \varepsilon$. Which says $\|[d, e]^* \operatorname{diag}(c, 0)[d, e] - \operatorname{diag}(c, c)\| < \varepsilon$ in $M_2(A)$ with $[d, e] := d \otimes p_{11} + e \otimes p_{12}$.

This can be rewritten also as

$$\| \operatorname{diag}(d, e)^* (c \otimes E) \operatorname{diag}(d, e) - \operatorname{diag}(c, c)\| < \varepsilon ,$$

where E denotes the 2×2-matrix with all entries $= 1$.

Now let $M = [b_{ij}] \in M_2(A)_+$ a positive 2×2-matrix and $f := b_{11} + b_{22}$. Then

$$M \leq 2 \operatorname{diag}(b_{11}, b_{22}) \leq 2 \operatorname{diag}(f, f)$$

By operator monotony of the function $t^{1/2}$ this implies that

$$M^{1/2} \leq 2^{1/2} \operatorname{diag}(f^{1/2}, f^{1/2})$$

It follow that

$$M = \lim_{n \to \infty} M^{1/2} \operatorname{diag}(f^{1/n}, f^{1/n}) M^{1/2} ,$$

thus, for $\delta > 0$ there exists $n \in \mathbb{N}$ and $d, e \in A$ such that for $c := f^{1/n}$ (with suitable $n \in \mathbb{N}$) the row matrix

$$[g, h] := [c^{1/2} d, c^{1/2} e] M^{1/2}$$

satisfies

$$\|M - [g, h]^* [g, h]\| < \delta$$

That means, that a given positive matrix $M \in M_2(A)_+$ can be arbitrary well approximated by matrices that are products $[g, h]^* [g, h]$.

Let $u(g^*g)^{1/2} = g$ and $v(h^*h)^{1/2} = h$ the polar decompositions and $\alpha \in (0, 1/2)$ sufficiently small. Define $S := (g^*g)^\alpha u^* = u^*(gg^*)^\alpha$, $T := (h^*h)^\alpha v^*$, $a := (gg^*)^{1/2+\alpha}$ and $b := (hh^*)^{1/2+\alpha}$. Then

$$\operatorname{diag}(S,T)^*[g,h]^*[g,h]\operatorname{diag}(S,T) = [a,b]^*[a,b]$$

and there exist $d, e \in A$ that satisfy the Inequalities (16.3) for a, b with $\max(\|d\|, \|e\|) < 4/\varepsilon$.

Thus, if we have for fixed ε the given 2×2-matrix $M = [b_{ij}] \in M_2(A)_+$ approximated well enough by a matrix $[g,h]^*[g,h]$ and take the α in the above formula small enough then we get

$$\|\operatorname{diag}(d,e)^* M \operatorname{diag}(d,e) - \operatorname{diag}(b_{11}, b_{22})\| < \varepsilon .$$

□

REMARK 2.16.5. Property (m.d.) of a C^*-algebra A implies the (formally stronger) property of A that, for every $m, n \in \mathbb{N}$ with $n > 1$, every positive elements $a_1, \ldots, a_m \in A_+$ and arbitrary elements $x_\ell \in A$ ($\ell = 1, \ldots, n$) there exist *contractions* $d_1, \ldots, d_m \in A$ such that $\|d_j^* a_j d_j - a_j\| < \varepsilon$ for $j = 1, \ldots, m$ and $\|d_j^* x_\ell d_k\| < \varepsilon$ for $\ell = 1, \ldots, n$ and $j \neq k$, $1 \leq j, k \leq m$.

?? [the cite is KirRor2] Compare [**463**, ????], but there are misleading typos that should be read: ?????

DEFINITION 2.16.6. The following properties of C^*-algebras A are between pure infiniteness and strong pure infiniteness.

(lsp) The algebra A is purely infinite and for every non-zero contraction $a \in A$ there exist isometries s, t in the multiplier algebra $\mathcal{M}(D)$ of $D := \overline{(a\,a^* + a^*a)\,A\,(a\,a^* + a^*a)}$ with $s^*t = 0$ and $3 \cdot \|s^*at\| < 2$. (Then we call A **sq-pi**.)

Why not "(lsp)" ? Anyway, it seems to be a bit difficult to use or to compare with others ...

(labs) For every positive contraction $a \in (A_\omega)_+$, there exists a *-monomorphism $\psi\colon C^*(a) \otimes \mathcal{O}_\infty \to A_\omega$ with $\psi(a \otimes 1) = a$. (Then we say that A is **locally $\mathcal{O}_\infty$-absorbing**.)

(lcpi) $\{a\}' \cap A_\omega$ is p.i. for every positive element $a \in (A_\omega)_+$. (A is **locally commutant-p.i.**.)

(cpi) For every separable commutative C^*-subalgebra C of A_ω, there exists a *-monomorphism $\psi\colon C \otimes \mathcal{O}_\infty \to A_\omega$ with $\psi(a \otimes 1) = a$ for $a \in C$. Then A is **commutant purely infinite** or **c.p.i.**.

REMARK 2.16.7. We have following two rows of implications:

s.p.i. ⇔ m.d.p. ⇒ (lsp) ⇒ p.i.

s.p.i. ⇒ (cpi) ⇒ (lcpi) ⇒ (labs) ⇒ p.i.

Here is an equivalent formulations of (c.p.i):

There exist increasing continuous functions $f_n\colon [0,2] \to [0,3]$ with $f_n(0) = 0$ with the property that for n contractions $a_1, \ldots, a_n \in A_+$, there are contractions $d_1, d_2 \in A$ such that

$$\|d_i a_k - a_k d_i\| + \|d_i^* a_k d_j - \delta_{i,j} a_k\| \leq f_n(\gamma(a_1, \ldots, a_n))$$

for $1 \leq k \leq n$, $i, j \in \{1, 2\}$, where we let $\gamma(a_1, \ldots, a_n) := \sup_{1 \leq p,q \leq} \| a_p a_q - a_q a_p \|$.

For every separable and commutative C^*-subalgebra $C \subseteq A_\omega$, the relative commutant $C' \cap A_\omega$ is purely infinite.
(Equivalently: $F(C, A) := (C' \cap A_\omega)/\operatorname{Ann}(C, A_\omega)$ is p.i. for every separable commutative C^*-subalgebra C of A_ω.)

PROPOSITION 2.16.8. *We have the implications:*

$A \cong A \otimes \mathcal{D}_\infty$

implies that

A is s.p.i., if and only if, A_ω is s.p.i.,

and this implies that

$C' \cap A_\omega$ is p.i. for all separable commutative *subalgebras $C \subseteq A_\omega$.*

The latter implies that A is pi(1), iff A pi-1, iff A is p.i. A_ω is p.i.

REMARK 2.16.9. "Strongly" p.i. algebras in the sense of Definition 1.2.2 are p.i.

But it needs some work to see that strongly p.i. algebras in the sense of Definition 1.2.2 have the WvN-property of Definition 1.2.3.

The WvN-property is needed for our classification results. We give in Chapter 3 a proof (based on [**443**] and the joint paper with M. Rørdam [**463**]) that strongly p.i. algebras have the WvN-property.

Compare the remarks at the end of Chapter 3.

$A \otimes F$ is strongly p.i. for $F := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$ and for every C^*-algebra A. Notice that $F \cong \mathcal{O}_\infty$ because $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ and

Give precise citations (!!!) of the work with E. Blanchard and M. Rørdam:

In joint works with E. Blanchard and M. Rørdam about p.i. algebras there are some sufficient criteria for p.i. algebras to be strongly p.i. and the study of some cases where p.i. implies "strongly" p.i. in some cases, e.g.:

If A is p.i. and $\operatorname{Prim}(A)$ is Hausdorff, then A is strongly p.i.

If A is locally p.i. and $\operatorname{Prim}(A)$ is a Hausdorff space of finite dimension, then A is p.i. (and therefore is s.p.i.).

If A is exact and "approximately divisible", ...

Where is the definition of "approximately divisible"? Are they criteria concerning $F(A)$???

... then A is strongly purely infinite if every (additive) lower semi-continuous traces $\tau\colon A_+ \to [0,\infty]$ takes only the values $\{0,\infty\}$. The same happens if $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra.

The algebras $A \otimes \mathcal{O}_\infty$ are always s.p.i. Separable *nuclear* s.p.i. algebras A are isomorphic to $A \otimes \mathcal{O}_\infty$ ([49])

Since $\mathcal{O}_\infty$ is approximately divisible, cf. Remark B.5.1, this implies that a separable nuclear C^*-algebra A is s.p.i., if and only if, A is approximately divisible and has only trivial lower semi-continuous traces.

`give Refs or Cites:`

An other equivalent to s.p.i. for separable nuclear A is that A tensorial absorbs the Jiang-Su algebra $\mathcal{Z}$ and that A has only trivial l.s.c. traces. (The proof is more engaged then the case of "approximate divisibility".)

If A has real rank zero and is *locally* p.i., then A is *strongly* p.i. (More generally, for each projection p in a locally p.i. algebra A there is $n = n(p, A) \in \mathbb{N}$ such that $p \otimes 1_n$ is infinite in $M_n(A)$. This property holds also for each projection r in each quotient A/J of A – with $n(\pi_J(a), A/J)$ possibly different from $n(p, A)$ –, because A/J is locally p.i. too. Since $p \in J$ for projection $p \in A$, or there is stable hereditary C^*-subalgebra D of pAp that is not contained in J. Then $\pi_J(D)$ is non-zero and stable and contains a non-zero projection $r \in \pi_J(D) \subseteq A/J$. Now let $n := n(r, A/J)$. The stability of $\pi(D)$ allows to define a monomorphism from $r(A/J)r \otimes M_n$ into $\pi(D)$. Thus $\pi_J(D)$ contains an infinite projection. The argument works also if each non-zero hereditary C^*-subalgebra E of each quotient A/J of A contains a non-zero projection r. This projection must be properly infinite by the above arguments (applied to r and the quotients A/I for $J \leq I \leq A$). It again follows that every non-zero $a \in A_+$ is properly infinite, i.e., that A is purely infinite, One gets that A is strongly p.i. if A has real rank zero case, because algebras of real rank zero contain sufficiently many locally central elements: namely just the projections.)

If A is weakly p.i. or if $\mathcal{M}(A)$ is (itself) *locally* p.i., then there is $n \in \mathbb{N}$ such that $M_n(\mathcal{M}(A))$ has a properly infinite unit. (The first case has an estimate $n \leq 1 + 2k$ if A is $pi(k)$.)

Remark 2.16.10. If X is any (non-empty) locally compact Hausdorff space and $\omega \in \beta(X)$.

Where is $\beta(X)$ defined?

[49] The stable and the unital case follow from Theorem M or from [**463**, thm. 8.6], this passes to the general case by the extension property of s.p.i. algebras, which implies that the unit of $\mathcal{M}(A)$ is properly infinite for s.p.i. A and that $\mathcal{E}(\mathcal{O}_\infty, \mathcal{O}_\infty)$ is s.p.i. Alternatively one can use the arguments in Chapter 10.

Let I_ω the kernel of the character χ_ω associated to ω. It is given by

$$f \in \mathrm{C_b}(X) \cong \mathrm{C}(\beta(X)) \mapsto f(\omega) := \chi_\omega(f)\,.$$

We define (similar to the case $X = \mathbb{N}$) the C^*-algebra $A_\omega := \mathrm{C_b}(X, A)/J_\omega$, where $J_\omega := I_\omega \cdot \mathrm{C_b}(X, B)$

It is not immediate but similar to the arguments used in case $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ that we have that A is strongly p.i. if A_ω is strongly p.i.

Clearly the case $\omega \in X$ is trivial, because s.p.i. passes to quotients, cf. Proposition **??** or Proposition 2.17.1.

The proof that strong pure infiniteness of A implies strong pure infiniteness of $\mathrm{C_b}(X, A)$ (at least for σ-compact X) and of its quotient A_ω follows from the the strong pure infiniteness of A and factorizes over the strong pure infiniteness of $\mathrm{C_0}(X, A)$ and its multiplier algebra $\mathcal{M}(\mathrm{C_0}(X, A)) = \mathrm{C_{b,st}}(X, \mathcal{M}(A))$ with σ-unital A and σ-compact X.

(Here, $\mathrm{C_{b,st}}(X, \mathcal{M}(A))$ denotes the C^*-algebra of bounded maps $f\colon X \mapsto \mathcal{M}(A)$ with the property that $x \mapsto f(x)a$ and $x \mapsto af(x)$ are in $\mathrm{C_b}(X, A)$ for all $a \in A$.)

Proof. `to be filled in` **??**

We use the norm-function $N\colon \mathrm{C_b}(X, A) \to \mathrm{C_b}(X)_+$, that is defined as $N(f)(x) := \|f(x)\|$. The norm of $\pi_\omega(f) \in A_\omega$ is then given by $\chi_\omega(N(f)) = \|\pi_\omega(f)\|$. If $\chi_\omega(N(f)) = 0$ then $N(f) \in (I_\omega)_+$. We get $\inf_{x\in X} G(x) = 0$ for all $G \in (I_\omega)_+$ if we use now that $I_\omega \cong \mathrm{C_0}(\beta(X) \setminus \{\omega\})$, and that X is dense in $\beta(X)$.

Let $a_1, a_2 \in A$ positive contractions and $\varepsilon > 0$. Since A_ω is s.p.i., there exist contractions $f_1, f_2 \in \mathrm{C_b}(X, A)$ such that the functions

$$g_{j,k} := N(f_j^* a_j^* a_k f_k - \delta_{j,k} a_j^* a_k) \in \mathrm{C_b}(X)_+$$

satisfy $\chi_\omega(g_{j,k}) < \varepsilon/2$ for $j, k \in \{1, 2\}$.

Let $G(x) := \sum_{j,k} (g_{j,k}(x) - \varepsilon/2)_+$. Then $\chi_\omega(G) = 0$, i.e., $G \in I_\omega$. Thus, $\inf_{x\in X} G(x) = 0$. This implies the existence of $x_0 \in X$ with $G(x_0) < \varepsilon/2$. But this yields $g_{j,k}(x_0) < \varepsilon$ for $j, k \in \{1, 2\}$, and $d_1 := f_1(x_0)$ and $d_2 := f_2(x_0)$ satisfy for $j, k \in \{1, 2\}$ the inequalities

$$\|d_j^* a_j^* a_k d_k - \delta_{j,k} a_j^* a_k\| < \varepsilon\,.$$

□

Question 2.16.11. Let $A := \mathrm{C_0}((1/2, 1], \mathbb{K})$ and $F \subseteq C^*((0, 1], \mathcal{M}(A))$ the C^*-subalgebra that is generated by A and $f\colon t \in (0, 1] \mapsto \psi(t) \cdot 1_{\mathcal{M}(A)}$ with $\psi(t) := \min(2t, 1)$.

Is f an infinite element in F? (certainly not properly infinite).

(It seems not to be such ???, even if one tensors again...???)

17. Permanence properties of strongly p.i. algebras

We list some permanence properties of the class of *strongly p.i.* C^*-algebras A (i.e., where A is "s.p.i.", as defined in Definition ?????):

Passage to non-zero hereditary C^*-subalgebras D of A

A is s.p.i. if A contains a full hereditary C^*-subalgebra D such that D is s.p.i.

Morita-equivalence,

Extensions,

passage to quotients

inductive limits

Each s.p.i. A is the inductive limit of the net of separable C^*-subalgebras that are s.p.i. and are relatively weakly injective in A.

Infinite direct products, $\Pi_\infty(A_1, A_2, \ldots)$, (and passage to its quotients, e.g. A_ω if $A_n := A$)

Commutator algebras of separable amenable C^*-subalgebras of A_ω if A is s.p.i.

spatial tensor product with arbitrary C^*-algebras in case that prime-ideals are cartesian. (Is exactness necessary?)

Some crossed products by quantum groups??

PROPOSITION 2.17.1. *If A is strongly p.i., then every hereditary C^*-subalgebra and every non-zero quotient A/J is strongly p.i.*

PROOF. Let $D \subseteq A$ a hereditary C^*-subalgebra, $a_1, a_2 \in D_+$ positive contractions and $\varepsilon > 0$.

Since $\|a^{4\gamma}a^2 - a^2\| \leq 2\gamma$ for positive contractions $a \in A_+$, we get, for $\gamma := \varepsilon/4$ and $k, j \in \{1, 2\}$, that

$$\delta_{j,k}\|a_j a_k - a_j^{2\gamma} a_j a_k a_k^{2\gamma}\| \leq \varepsilon/2\,.$$

There exist $e_1, e_2 \in A$ such that, for $j, k \in \{1, 2\}$,

$$\|e_j^* a_j^\gamma a_j a_k a_k^\gamma e_k - \delta_{jk} a_j^\gamma a_j a_k a_k^\gamma\| \; < \; \varepsilon/2\,.$$

Because a_1^γ and a_2^γ are positive contractions in D, we obtain that the elements $d_1 := a_1^\gamma e_1 a_1^\gamma$ and $d_2 := a_2^\gamma e_2 a_2^\gamma$ are in D and satisfy, for $j, k \in \{1, 2\}$, the inequalities

$$\|d_j^* a_j a_k d_k - \delta_{jk} a_j a_k\| \; < \; \varepsilon/2 + \varepsilon/2\,.$$

If $J \lhd A$ is a closed ideal, $b_1, b_2 \in (A/J)_+$ and $\varepsilon > 0$, then there are $a_1, a_2 \in A_+$ and $e_1, e_2 \in A$ such that $\pi_J(a_k) = b_k$ and $\|e_j^* a_j a_k e_k - \delta_{jk} a_j a_k\| < \varepsilon$ for $j, k \in \{1, 2\}$. Thus, $d_j := \pi_J(e_j)$, $j = 1, 2$ satisfy $\|d_j^* b_j b_k d_k - \delta_{jk} b_j b_k\| \; < \varepsilon$ for $j, k \in \{1, 2\}$. □

PROPOSITION 2.17.2. *If A is a purely infinite C^*-algebra and B is a simple non-elementary C^*-algebra and at least one of A or B is exact, then $A \otimes B$ is* strongly *purely infinite.*

PROOF. Where is the general definition of " non-elementary C^*-algebra " ???

Does it work well if B is UHF?

What is the bound if for every finite sequence $y_1, \ldots, y_n \in B$, $b_1, b_2 \in B_+$ and $\delta > 0$ there exist contractions $z_1, z_2, g_1, g_2, h_1, h_2 \in B$ such that $\|z_1^* z_2\| \leq \delta$, $\|[z_k, y_1]\| \leq \delta$ for $k = 1, \ldots, n$ $g_k^*(z_k^* b_k z_k) g_k + h_k^*(z_k^* b_k z_k) h_k = (b_k - \delta)_+$.

Since one of A or B is exact, we get from Lemma **??** that

$$\mathcal{F} := \{a \otimes b\,;\; 0 \neq a \in A_+, 0 \neq b \in B_+\}$$

is a "filling family" for $A \otimes B$.

Definition of "filling family" for a C^*-algebra ???

The non-zero elements $a \otimes b \in \mathcal{F}$ are properly infinite in $A \otimes B$ and $a \otimes b \sim a \otimes c$ in $A \otimes B$ for $b, c \in B_+ \setminus \{0\}$.

Moreover we find for $\delta > 0$ elements $g_1, \ldots, g_n \in B$ with $\sum g_\ell^* b g_\ell = (c - \delta)_+$. $h_1, \ldots, h_n$ with

$$\mathrm{diag}((a-\delta)_+, \ldots, (a-\delta)_+) = [h_1, \ldots, h_n]^* a [h_1, \ldots, h_n]\,.$$

Then $D := \sum_\ell h_\ell \otimes g_\ell \in$ satisfies $D^*(a \otimes b) D = (a - \delta)_+ \otimes (c - \delta)_+$.

`Give controlling bound !!`

If follows that it is enough to prove that for each $\delta > 0$, elements $a_1, a_2, c_1, \ldots, c_n \in A_+$, $b_1, b_2, d_1, \ldots, d_n \in B_+$, $y := \sum_{k=1}^n c_k \otimes d_k$ there exist non-zero contractions $x_1, x_2 \in B_+$ with $\|x_j b_j x_j\| \geq \|b_j\| - \delta$ and $x_1 b_1 d_k b_2 x_2 = 0$ for $k = 1, \ldots, n$.

Provided one has a controlling bound for the x_j and/or D depending on $\delta > 0$.

... □

THEOREM 2.17.3. *The minimal (= spatial) C^*-algebra tensor product $A \otimes B$ is strongly p.i. if A and B satisfy at least one of the following conditions:*

(1) *A is strongly p.i. and the natural map $(I, J) \mapsto A \otimes J + I \otimes B$ from $\mathrm{prime}(A) \times \mathrm{prime}(B)$ into $\mathrm{prime}(A \otimes B)$ is surjective.*
(2) *A is weakly p.i, and B is exact, simple and non-elementary.*
(??) *?????????????? If A is (quasi-)traceless, B is exact and tensorial absorbs a tensorial self-absorbing separable unital C^*-algebra $\mathcal{D} \ncong \mathbb{C}$.*

PROOF. `to be filled in` **??** □

COROLLARY 2.17.4. *Let A and B C^*-algebras. The algebra $A \otimes B$ is strongly p.i. if A is strongly p.i. and at least one of the C^*-algebras A or B is exact.*

PROOF. Recall that, by our convention, $A \otimes B$ denotes the completion of the algebraic tensor product $A \odot B$ with respect to the spatial C^*-norm. This is the minimal C^*-norm on the algebraic tensor product $A \odot B$ of C^*-algebras A and B, cf. [**766**], [**767**, thm. IV.4.19]. In particular each non-zero closed ideal of $A \otimes B$ has non-zero intersection with $A \odot B$.

More precisely, it is shown in [**766**], but is not explicitly stated there, that if a C^*-semi-norm N on $A \odot B$ has the property that $N(a \otimes b) > 0$ for each non-zero $a \in A_+$ and non-zero $b \in B_+$, then $\|x\|_{\max} \geq N(x) \geq \|x\|_*$ for each $x \in A \odot B$, where we here (temporary and only here) denote by $\|x\|_*$ the spatial norm. This shows that $\|x\|_*$ is the minimal possible C^*-norm on $A \odot B$, from now on denoted by $\|x\|_{\min} := \|x\|_*$ or simply by $\|x\|$.

But this shows even more: If K is a non-zero closed ideal of $A \otimes B$ then there exist non-zero positive contractions $a \in A_+$ and $b \in B_+$ such that $a \otimes b \in K$, because otherwise for the quotient map $A \otimes B \to (A \otimes B)/K$ would hold $\|x\|_{\min} \geq \|\pi_K(x)\| \geq \|x\|_{\min}$ for all $x \in A \odot B$, which implies $K = \{0\}$. Compare also the stronger result of Lemma A.24.1, in Section 24 of Appendix A.

Thus, every non-zero closed ideal K of the spatial tensor product $A \otimes B$ contains a tensor product $I \otimes J$ of non-zero ideals $I \triangleleft A$ and $J \triangleleft B$.

Exactness of a C^*-algebra A means that for every C^*-algebra B and each closed ideal $J \triangleleft B$ the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is short exact. The exactness property of A induces the exactness of I and A/I for each closed ideal $I \triangleleft A$ of A. Moreover the exactness of A yields the local lifting property for its quotients A/I. The latter yields also the short exactness of the sequences

$$0 \to I \otimes B \to A \otimes B \to (A/I) \otimes B \to 0\,,$$

and all together imply – now by the 3×3-lemma (also called “5 of 6 lemma”) of category theory – the short exactness of

$$0 \to A \otimes J + B \otimes I \to A \otimes B \to (A/I) \otimes (B/J) \to 0\,,$$

i.e., if A or B is exact then for each $I \triangleleft A$ and $J \triangleleft B$ there are natural isomorphisms

$$(A \otimes B)/(A \otimes J + B \otimes I) \cong (A/I) \otimes (B/J)\,.$$

Recall that a closed ideal $K \triangleleft C$ of a C^*-algebra is prime, $K \in \operatorname{prime}(C)$, if $K_1 \cap K_2 \subseteq K$ implies always that at least one of K_1 and K_2 is contained in K for ideals $K_1 \triangleleft C$ and $K_2 \triangleleft C$. The original definition says only that $K_1 \cap K_2 = K$ implies $K_1 = K$ or $K_1 = K$, but one can change this into the above more flexible formulation if we replace K_j by $K + K_j$ for $j = 1, 2$. Then it means that $\pi_K(K_1) \cap \pi_K(K_2) = 0$ implies that $\pi_K(K_1) = 0$ or $\pi_K(K_2) = 0$.

Note that all primitive ideals are prime [**616**, prop. 3.13.10], and in case of separable C^*-algebras all prime ideals are primitive, cf. [**616**, prop. 4.3.6.]. For non-separable A the space of prime ideals $\operatorname{prime}(A)$ is the “sobrification” or “point-wise completion” of the space $\operatorname{Prim}(A)$ of primitive ideals. It means that the lattice of open subsets of $\operatorname{Prim}(A)$ is the same as the lattice of open subsets of $\operatorname{prime}(A)$.

If one of A or B is exact, then the map $(I, J) \mapsto A \otimes J + I \otimes B$ from $\mathrm{prime}(A) \times \mathrm{prime}(B)$ into $\mathrm{prime}(A \otimes B)$ is surjective. To see this, observe that, for a prime ideal K of $A \otimes B$, the maximal possible ideals $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$ with $A \otimes J + I \otimes B \subseteq K$ are prime, and that the exactness implies then that $K = A \otimes J + I \otimes B$. The maximal I and J with the properties $I \otimes B \subseteq K$ and $A \otimes J \subseteq K$ are well-defined, because, e.g. the set of all algebraic *-ideals $I \in A$ with $I \otimes B \subseteq K$ is closed under finite and infinite *algebraic* sums and under taking the closure.

Indeed, the epimorphism $A \otimes B \to A/I \otimes B/J$ has, – by exactness of A or of B –, precisely the kernel $A \otimes J + I \otimes B$, as we have seen above. If $\pi \colon A \otimes B \to A/I \otimes B/J$ denotes the quotient map, then $\pi(K)$ becomes an ideal of $A/I \otimes B/J$ that must have trivial intersection $\pi(K) \cap (A/I \odot B/J) = \{0\}$ with the algebraic tensor product $A/I \odot B/J$, because otherwise our above considerations on the minimal (= spatial) C^*-norm (here on $A/I \odot B/J$) show that there would be elements $c \in A_+$ and $d \in B_+$ with $c \otimes d \notin A \otimes J + I \otimes B$ and $c \otimes d \in K$. But the latter would imply, by primeness of K, that also the one of the ideals $A \otimes J_1 + I \otimes B$ or $A \otimes J + I_1 \otimes B$ is contained in K, where $I_1 := I + \overline{\mathrm{span}(AcA)}$ and $J_1 := J + \overline{\mathrm{span}(BdB)}$. This contradicts the maximality of the above defined ideals I and J with the property $A \otimes J + I \otimes B \subseteq K$.

In fact, the map

$$(I, J) \in \mathrm{prime}(A) \times \mathrm{prime}(B) \mapsto A \otimes J + I \otimes B \in \mathrm{prime}(A \otimes B)$$

is moreover a homeomorphism of (sober) locally quasi-compact T_0 spaces if one of A or B is exact.

Thus, Theorem 2.17.3(i) applies. □

Since $\mathrm{C}_0(X, A) \cong \mathrm{C}_0(X) \otimes A$ and $\mathrm{C}_0(X)$ is exact, we get from Theorem 2.17.3, Corollary 2.17.4 and Proposition ??:

Corollary 2.17.5. *For every locally compact Hausdorff space X, $\mathrm{C}_0(X, A)$ is strongly purely infinite, if and only if, A is strongly purely infinite.*

Theorem 2.17.6. *Suppose that A is σ-unital. Then $\mathcal{M}(A)$ is strongly p.i., if and only if, A is strongly p.i.*

Proof. By Proposition 2.17.1 the ideal A of $\mathcal{M}(A)$ is strongly p.i., if $\mathcal{M}(A)$ is strongly p.i.

`to be filled in, see extra SPI-paper` ?? □

Corollary 2.17.7. *Let X a non-empty σ-compact locally compact Hausdorff space. The algebra $\mathrm{C}_\mathrm{b}(X, A)$ is strongly p.i., if and only if, A is strongly p.i., if and only if, $\mathrm{Q}(X, A) := \mathrm{C}_\mathrm{b}(X, A)/\,\mathrm{C}_0(X, A)$ is strongly p.i., if and only if, $\mathrm{Q}(X, A)|F$ is strongly p.i. for a non-empty closed subset F of $\beta(X) \setminus X$.*

Proof. The algebra A is s.p.i. if and only if $A \otimes \mathrm{C}_0(X)$ is s.p.i., by Corollary 2.17.5.

If X is σ-compact, A is s.p.i. and $D \subseteq A$ is a σ-unital hereditary C^*-subalgebra, the algebra $\mathrm{C}_0(X, D) \cong D \otimes \mathrm{C}_0(X)$ is σ-unital and s.p.i. By Theorem 2.17.6, it follows that $\mathcal{M}(D \otimes \mathrm{C}_0(X))$ is s.p.i. if and only if $D \otimes \mathrm{C}_0(X)$ is s.p.i.

The algebra $\mathrm{C}_\mathrm{b}(X, D)$ is an ideal of the algebra $\mathcal{M}(\mathrm{C}_0(X, D))$. It implies that $\mathrm{C}_\mathrm{b}(X, D)$ is s.p.i. if $\mathrm{C}_0(X, D)$ is s.p.i. (and σ-unital).

Since X is σ-compact, the C^*-algebra $\mathrm{C}_\mathrm{b}(X, A)$ is the inductive limit of its C^*-subalgebras $\mathrm{C}_\mathrm{b}(X, D)$ with $D \subseteq A$ σ-unital hereditary C^*-subalgebras of A, we obtain that $\mathrm{C}_\mathrm{b}(X, A)$ is s.p.i. if A is s.p.i.,

Now use that $\mathrm{Q}(X, A)|F$ is a quotient of $\mathrm{C}_\mathrm{b}(X, A)$.

Thus, $\mathrm{Q}(X, A)$ (case $F = \beta(X) \setminus X$), $\mathrm{Q}(X, A)|F$ and, in particular, $\mathrm{Q}(X, A)|\{\omega\} =: A_\omega$ for a point $\omega \in F \subseteq \beta(X) \setminus X$ are strongly p.i. by Proposition 2.17.1.

If A_ω is s.p.i. for some $\omega \in \beta(X)$ then A is s.p.i. by Remark 2.16.10. □

We do not know if it is necessary to suppose that X is σ-compact in Corollary 2.17.7.

If X is not σ-compact then in general $\mathrm{C}_\mathrm{b}(X, A)$ is not anymore the inductive limit of its C^*-subalgebras $\mathrm{C}_\mathrm{b}(X, D)$ with hereditary σ-unital $D \subseteq E$, and $\mathcal{M}(D \otimes \mathrm{C}_0(X))$ is possibly not s.p.i.

The class of strongly purely infinite C^*-algebras is closed under extension by the following:

THEOREM 2.17.8. *Let $J \triangleleft A$ a closed ideal. The C^*-algebra A is strongly purely infinite if and only if J and A/J are strongly purely infinite.*

PROOF. We know that A is purely infinite by Theorem **??**.

The proof can easily reduced to the case, where A is separable and unital (thus contains a copy of $\mathcal{O}_\infty$ unitally), and where J is stable. The proof uses the K_1-injectivity of A as an important tool.

See [**443**] for a proof of Theorem 2.17.8. □

18. Strongly p.i. crossed products and generalized Toeplitz algebras

change whole section.

PROPOSITION 2.18.1. *Suppose that A is a C^*-algebra and that $\alpha \in \mathrm{Aut}(A)$ has the property that $\{0\}$ and A are the only α-invariant ideals.*

simple and p.i. crossed products appear in following cases:

If, for every $a \in A_+ \setminus \{0\}$ and $n \in \mathbb{N}$ there exist $b \in A_+ \setminus \{0\}$ such that $b \leq a$ and $\alpha^k(b)$ are pairwise orthogonal, then $A \rtimes_\alpha \mathbb{Z}$ is simple and purely infinite.

above this is wrongly stated

Full-corner endomorphisms, ...????

There is a projection $p \in Z(A^{**})$ such that $\bigvee_{n\in\mathbb{Z}} \alpha^n(p)$ is faithful for A

(or: is not contained in any α-invariant open central projection of A^{**} ????)

and $\alpha^m(p) \perp \alpha^n(p)$ for $m \neq n$.

For every non-zero α-invariant closed central projection p and every $n \in \mathbb{N}$, $a \in A_+$, $\varepsilon > 0$ there is a non-zero central projection $q \leq p$ with $\alpha^j(q)\alpha^k(q) = 0$ for $-n \leq j, k \leq n$ and $\|ra\| + \varepsilon \geq \|pa\|$. Then $A \rtimes_\alpha \mathbb{Z}$ is simple.

A simple and p.i., for every $n \in \mathbb{N}$ there is a unitary $U_n \in F(A)$ such that $\alpha(U_n) = e^{2\pi i/n}U_n$. Then $A \rtimes_\alpha \mathbb{Z}$ is simple.

The action $F(\alpha)$ on $F(A)$ has full spectrum.

PROOF. ??

(Reduction to separable case?) □

COROLLARY 2.18.2. *Suppose that A is purely infinite and simple, and $\alpha \in \mathrm{Aut}(A)$ is such that α_* has infinite order in $\mathrm{Aut}(K_*(A))$. Then $A \rtimes_\alpha \mathbb{Z}$ is purely infinite and simple.*

PROOF. `to be filled in` ?? □

COROLLARY 2.18.3. *Suppose that B is a simple C^*-algebra and $\psi \colon B \to B$ is an endomorphism from B into B, such that for each $a, b \in B_+$ with $\|b\| = 1 \geq \|a\|$ and for each $\varepsilon > 0$ there are $k \in \mathbb{N}$ and $d \in B$ with $\|\psi^k(a) - d^*bd\| < \varepsilon$ (respectively $\|a - d^*\psi^k(b)d\| < \varepsilon$).*

Then the semi-crossed product $B \rtimes_\psi \mathbb{N}$ is purely infinite.

PROOF. Consider the natural circle action on $B \rtimes_\psi \mathbb{N}$ ...

`to be filled in` ??

□

COROLLARY 2.18.4. *Suppose that B is stable and σ-unital and that*

$$h \colon B \hookrightarrow \mathcal{M}(B)$$

*is a non-degenerate *-monomorphism.*

Let $s, t \in \mathcal{M}(B)$ isometries that are canonical generators of a copy of $\mathcal{O}_2$.

Suppose that the following conditions (i)–(iii) are satisfied:

(i) $h(B) \cap B = \{0\}$.
(ii) *For each $0 \neq b \in B$ the closed ideal of B generated by $h(b)B$ is equal to B.*
(iii) *The C^*-morphism h is approximately unitarily equivalent to the C^*-morphism $\delta_2 \circ h := sh(\cdot)s^* + th(\cdot)t^*$.*

Then the Toeplitz algebra $\mathcal{T}_E$ of the Hilbert B-B bi-module E defined by the left-action of B on B – given by $h \colon B \to \mathcal{M}(B)$ – is KK*-equivalent to B.*

?????????????

PROOF. Consider the endomorphism $\beta\colon C \to C$ on $C := \overline{S}$ for $S := \mathrm{span}(\bigcup_n \mathcal{M}(h)^n(B))$, given by $\beta := \mathcal{M}(h)|C$. Then $(\mathrm{indlim}(\beta\colon S \to S)) \rtimes \mathbb{Z}$ is simple and purely infinite, because Corollary 2.18.3 applies to (C, β). □

`Next Def.s should be quoted...  give them precise !...`

DEFINITION 2.18.5. Let $\mathcal{H}$ denote a Hilbert A-B bi-module.

Let A a separable C^*-algebra and $\mathcal{H}$ a Hilbert A-A bi-module.

$\mathcal{T}(\mathcal{H})$ (generalized) Toeplitz-Fock C^*-algebra corresponding to $\mathcal{H}$.

Let $\mathcal{O}_{\mathcal{H}}$ denote the Cuntz-Pimsner C^*-algebra corresponding to $\mathcal{T}(\mathcal{H})$.

The gauge group is available ...

(where is Def. of "gauge group" ???)

The natural action of bi-module automorphisms and its generalizations. ...

REMARK 2.18.6. Automorphisms of $\mathcal{H}$ act on $\mathcal{T}(\mathcal{H})$.

Special case: circle action. (Def. of "Circle action" ?)

Moreover: epimorphisms and other constructions.

$\mathcal{O}_{\mathrm{n}}$, $A = \mathbb{C}$, $\mathcal{H} \cong \mathbb{C}^n$

$\mathcal{O}_\infty$, $A = \mathbb{C}$, $\mathcal{H} =$ separable Hilbert space of countable dimension.

`Compare Appendix A.`

COROLLARY 2.18.7. *Let A a separable C^*-algebra and $\mathcal{H}$ a*

countably generated

Hilbert A-A bi-module. Then $\mathcal{T}(\mathcal{H})$ is separable.

Quotients of $\mathcal{T}(\mathcal{H})$ can be found by invariant ideals of $\mathcal{H}$.

If A is nuclear (respectively is exact, has WEP = is weakly injective, etc. ...) then $\mathcal{T}(\mathcal{H})$ is nuclear (respectively exact, has WEP = is weakly injective ???, etc....).

If $d(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$ and d is injective then $\mathcal{T}(\mathcal{H}) = \mathcal{O}_{\mathcal{H}}$

$\mathcal{O}_{\mathcal{H}}$ is simple if ??????.

$\mathcal{T}(\mathcal{H})$ is simple, separable, stable and nuclear if A is separable, stable and nuclear, $d(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$ and there are no-invariant ideals (modular quotients).

PROOF. Use Lemma **??**

(ref. is:: " lem:App.A.fix-algebra.of.compact.group")

and the gauge circle action on $\mathcal{T}(\mathcal{H})$ for nuclearity, exactness and WEP (= weak injectivity).

(here WEP means "weak expectation property" ??) □

DEFINITION 2.18.8. An automorphism α of a C^*-algebra A is **properly outer**, if for every $a \in \widetilde{A} = A + \mathbb{C}1$ and every non-zero hereditary C^*-subalgebra B of A holds that

$$\inf\{\, \|xa\alpha(x)\|\,;\ 0 \leq x\ \in B,\ \|x\| = 1\,\} = 0\,.$$

REMARK 2.18.9. Suppose that A is separable.

(1) By [**578**, thm. 6.6], the automorphism α is properly outer, if and only if, for every non-zero α-invariant closed ideal $I \triangleleft A$ and every unitary u in the multiplier algebra $\mathcal{M}(I)$ holds $\|\alpha|I - \mathrm{Ad}_u\| = 2$.

This is also equivalent to:
There is no non-zero α-invariant $I \triangleleft A$ such that there is a *-derivation δ of I and a unitary $u \in \mathcal{M}(I)$, with $\alpha|I = \mathrm{Ad}_u \circ \exp\delta$.

`Check next again`

(2) Proper outer-ness of α is implied by the property that for each primitive ideal J of $A \rtimes_\alpha \mathbb{Z}$ the group $\widetilde{\mathbb{T}}(\alpha, J)$ of $\lambda \in \mathbb{T}$ $(= S^1)$ with $\widehat{\alpha}_\lambda(J) = J$ is non-trivial, where $\widehat{\alpha}\colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_\alpha \mathbb{Z})$ means the dual action of the $\mathbb{Z}$-action $n \in \mathbb{Z} \mapsto \alpha^n$. (See *proof* of [**478**, lem.1.1].)

(3) If A is separable and simple, then

$$\{1\} \neq \widetilde{\mathbb{T}}(\alpha) := \bigcap_J \widetilde{\mathbb{T}}(\alpha, J)$$

implies that α is inner, cf. [**577**].

(4) The subgroup $\widetilde{\mathbb{T}}(\alpha) \subseteq \mathbb{T}$ is called the *strong Connes spectrum* of α, cf. [**477**].

THEOREM 2.18.10. *Suppose that $G \neq \{e\}$ is a countable discrete group, that A is a separable C^*-algebra A of infinite dimension, and that $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A such that the automorphism $\alpha(g)$ of A is a properly outer for each $g \in G \setminus \{e\}$ (cf. Definition 2.18.8).*

If there is (fixed) $n \in \mathbb{N}$ such that, for each $a, b \in A_+$ with $\|a\| \leq 1 = \|b\|$ and $\varepsilon > 0$, there are $g_1, \ldots, g_n \in G$ and $d_1, \ldots, d_n \in A$ with $\|a - \sum_k d_k^* \alpha(g_k)(b) d_k\| < \varepsilon$, then the reduced crossed product $A \rtimes_{\alpha,r} G$ is simple and purely infinite.

PROOF. Let $\{0\} \neq I \triangleleft A$ invariant under $\alpha(G)$, and let $b \in I_+$ with $\|b\| = 1$. Furthermore, let $a \in A_+$ with $\|a\| = 1$ and $\varepsilon > 0$. By assumption, we find $g_1, \ldots, g_n \in G$ and $d_1, \ldots, d_n \in A$ with $\|a + I\| \leq \|a - \sum_k d_k^* \alpha(g_n)(b) d_k\| < \varepsilon$. Thus $a \in I$ and $I = A$. It follows now from the proof of [**478**, thm. 3.1] or by [**578**, thm. 7.2] that $C := A \rtimes_{\alpha,r} G$ is simple. It can not be (linearly) one-dimensional, because A is (isomorphic to) a C^*-subalgebra of C. We can consider A as a subalgebra of C. Suppose that C is of finite dimension, then A is unital and the reduced group C^*-algebra of G is contained in C

...??? further?

We show that the simple C^*-algebra $A \rtimes_{\alpha,r} G$ is locally purely infinite.

`To be filled in` ?? □

DEFINITION 2.18.11. Let $\alpha\colon G \to \mathrm{Aut}(A)$ an action of a discrete group G on a C^*-algebra A. We write also $\alpha(g)$ for the natural extension $\alpha(g)^{**}$ to the second conjugate A^{**} of A.

The action α has the **weak Rokhlin property** if the center of A^{**} contains a projection p with the properties

(i) $p\alpha(g)(p) = 0$ for all $g \in G \setminus \{e\}$, and
(ii) $\sup_{g\in G} \|p \cdot \alpha(g)(a)\| > 0$ for all nonzero $a \in A$.

The action α has the **residual weak Rokhlin property** if the projection $p \in \mathcal{Z}(A^{**})$ satisfies (i) and the following stronger property (ii*) instead of (ii):

(ii*) For each $a \in A$ and every $\alpha(G)$-invariant closed ideal $J \lhd A$ (with central support q_J in A^{**}) $\|a + J\| = \sup_{g\in G} \|(1 - q_J)p \cdot \alpha(g)(a)\|$.

The action α of a locally compact group G on A is **exact** if $J \rtimes_{\alpha|,r} G$ is the kernel of the natural epimorphism $A \rtimes_{\alpha,r} G \to (A/J) \rtimes_{[\alpha],r} G$ for every $\alpha(G)$-invariant ideal $J \lhd A$.

An l.c. group G is **exact** if every action of G on C^*-algebras is exact.

LEMMA 2.18.12. *The weak Rokhlin property passes to subgroups H of G.*

The weak Rokhlin property for $G = \mathbb{Z}$ or $G = \mathbb{Z}_n$ implies that $\alpha(g)$ is properly outer for each $g \in G \setminus \{e\}$.

The residual Rokhlin property passes to quotients.

PROOF. `to be filled in` ?? □

THEOREM 2.18.13. *Suppose that A is a separable C^*-algebra, G a countable discrete* exact *group, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action that satisfies (I) and (II):*

(I) *α satisfies the residual Rokhlin* property, i.e., the center $\mathcal{Z}(A^{**})$ of the second conjugate of A contains a projection P with the following properties*
  (a) *$P\alpha(g)(P) = 0$ for all $g \in G \setminus \{e_G\}$*
  (b) *If $R \leq S$ are $\alpha(G)$-invariant A-open central projections with $R \neq S$, then there is $g \in G$ with $(S - R)\alpha(g)(P) \neq 0$.*

(II) *For any $a, b \in A_+$, $c \in A$ and $\varepsilon > 0$ there are$d_1, d_2 \in A$ and $g_1, g_2 \in G$ with $\|d_1^* a d_1 - \alpha(g_1)(a)\| < \varepsilon$, $\|d_2^* b d_2 - \alpha(g_2)(b)\| < \varepsilon$, and $\|d_1^* c d_2\| < \varepsilon$.*

Then $A \rtimes_{\alpha,r} G$ is strongly purely infinite.

`compare the following with the new observations on HarKir and KirSira`

COROLLARY 2.18.14. *Suppose that A is stable and separable, that $h\colon A \to \mathcal{M}(A)$ is non-degenerate, faithful, and satisfies*

(i) *h and its infinite repeat $\delta_\infty \circ h$ are approximately unitarily equivalent in $\mathcal{M}(A)$. (In particular, $h(A) \cap A = \{0\}$.)*
(iii) *$a \in \mathrm{span}(Ah(a)A)$ for all $a \in A_+$.*

(ii) *if* $J \in \mathcal{I}(A)$ *and* $h(J)A \subseteq J$*, then* $h(J) = h(A) \cap \mathcal{M}(A, J)$.

(iv) $\mathcal{M}(h) \circ h$ *is approximately unitarily equivalent to* h *in* $\mathcal{M}(A)$. Check!!!

Then $\mathcal{T}(\mathcal{H}) = \mathcal{O}_{\mathcal{H}}$ *for the Hilbert* A–A *bi-module* $\mathcal{H} := A$ *defined by* $h\colon A \to \mathcal{M}(A)$*, and* $\mathcal{T}(\mathcal{H})$ *is strongly purely infinite.*

check: *Moreover,* $h\colon A \to \mathcal{M}(A)$ *defines a* $\mathrm{KK}(\mathcal{C}\,;\; A, \mathcal{M}(A))$ *equivalence of* A *and* $\mathcal{T}(\mathcal{H}) = \mathcal{O}_{\mathcal{H}}$ *with respect to the matrix operator convex cone* $\mathcal{C}$ *generated by* h*.* This seems strange? Because $\mathcal{M}(A)$ is often KK-trivial?

CHAPTER 3

Nuclear c.p. maps and operator-convex cones

We have seen in Chapter 2, among others, that simple purely infinite algebras in the sense of J. Cuntz are purely infinite in the sense of Definition 1.2.1. Moreover we have shown that *locally* purely infinite (i.e., l..p.i.) simple C^*-algebras are *strongly* purely infinite in the sense of Definition 1.2.2. Now we study the effect of the property of strong pure infiniteness for C^*-algebras on residual nuclear c.p. maps between them. (Notice that strong pure infiniteness of C^*-algebras and residual nuclearity of C^*-morphisms are in general completely unrelated properties.)

We prove in this Chapter 3, among others, that the ultrapower D_ω and asymptotic corona $\mathrm{Q}(\mathbb{R}_+, D)$ of strongly purely infinite algebras D (cf. Definition 1.2.2) satisfy a local version of *Weyl–von-Neumann property* (WvN-property of Definition 1.2.3), cf. Proposition 3.2.15. It implies that the algebras $\mathrm{Q}(\mathbb{R}_+, D)$ are *strongly purely infinite* in the sense of Definition 1.2.2, because they are rich of approximately inner completely positive maps from separable C^*-subalgebras A into commutative C^*-subalgebras C that fix given elements of $A \cap C$. It is a basic observation for the later applications with help of our cone-related KK-theory and to the proof of its equivalence with cone-related un-suspended $\mathcal{E}$-theory.

The WvN-property allows to prove in Chapter 5 variants of generalized Weyl-von Neumann theorems. We use the results of Chapters 3 and 5 in Chapter 6 for the proof of Theorem A and for the proof of Theorem 6.3.1 that is a special case of the, in the last Chapter 12 finally proven, Theorem K.

Remark 3.0.1. We get later results of the following form, that one can consider as an approximate factorization over the cones of sums of matrix algebras.

Let A and B C^*-algebras, $U\colon A \to B$ a nuclear map, $X \subset A_+$ a finite subset of the positive contractions in A and $\varepsilon > 0$.

Then there exist following

(a) a finite dimensional C^*-algebra F,
(b) a c.p. map $V\colon F \to B$ with $\|V\| = \|U\|$,
(c) a C^*-morphism $\phi\colon \mathrm{C}_0((0,1], F) \to A$
(d) a contraction $d \in A_+$ (almost commuting with $\phi(\mathrm{C}_0((0,1], F))$?)

such that $\|U(dad) - U(a)\| < \varepsilon$ for all $a \in X$, and $\mathrm{dist}(dad, \phi(\mathrm{C}_0((0,1], F))) < \varepsilon$ for all $a \in X$, and $\|V(g(1)) - U(d^n\phi(g)d^n)\| < \varepsilon$ for all $g \in \mathrm{C}_0((0,1], F)$.

This means: We can locally "compress" $X \subset A$ – by some approximately inner c.p. map – into a suitable C^*-subalgebra $\phi(\mathrm{C}_0((0,1], F))$ of A and then apply the given nuclear map U to this compressed subset of $\phi(\mathrm{C}_0((0,1], F))$.

This is then in a sense "almost" the restriction U to X.

HERE IS NEEDED a list of "corona" type algebras:

$\mathrm{Q}(\mathbb{R}_+, B)$ (!!!), $\mathrm{Q}(X, B)$ (X with finite dimension??), $\mathrm{Q}^{\mathrm{s}}(B)$ (!!!), some quotients of those, e.g. ultrapowers ...

FIND list of defining properties! Certainly more than sub-Stonean!?

(At present state, in the terminology of Ilijas Farah ...).

B_∞ and B_ω among them?,

Desire: Should be σ-sub-Stonean.

Moreover, should satisfy Kasparov's Technical Lemma (KTL).

It should also require that they all have a property that there are sufficiently many "good" Abelian C^*-subalgebras that allows to get approximate factorization of separable C^*-subalgebras through algebras with regular abelian C^*-algebras.

Would be interesting to see how is the interplay... σ-sub-Stonean versus (KTL).

Where is the Definition "def:res.nuc.cp.map" **??** ???: of residual nuclear (or "residually nuclear") c.p. maps $V : A \to B$ that are "ideal system preserving", i.e., $V(A \cap J) \subseteq J$ for all $J \in \mathcal{I}(B)$ (in case $A \subseteq B$).

Or that (more generally) an map $\lambda : J \in \mathcal{I}(B) \to \lambda(J) \in \mathcal{I}(A)$ from the system of ideals $J \in \mathcal{I}(B)$ of B to the system of ideals $\mathcal{I}(A)$ is given and we require that $V(\lambda(J)) \subseteq J$. ????

The proof of next Theorem needs:

LEMMA 3.0.2. *Let $A \subseteq B$ C^*-algebras, and suppose that A or B strongly p.i.*

Then each residually nuclear *c.p. contraction $V : A \to B$* ([1]) *is approximately 1-step inner, i.e., for each finite subset $F \subset A$ and $\varepsilon > 0$ there exists a contraction $d \in B$ with $\|V(a) - d^*ad\| < \varepsilon$ for all $a \in F$.*

PROOF. WHERE is the Def. of `to be filled in` **??** □

```
Need for later applications the following 1-step inner-ness
(in blue, to be moved to suitable place).
But perhaps we must suppose that φ is itself nuclear?
Define the notion of ``corona type C*-algebra''
```

THEOREM 3.0.3. *Let A an exact C^*-algebra and $\varphi : A \otimes \mathcal{O}_\infty \to B$ a C^*-morphism, and suppose that $V : A \to B$ is a nuclear c.p. map that has the following Property (*):*

[1] The Definition **??** of residual nuclear c.p. maps includes that V is "ideal system preserving", i.e., $V(A \cap J) \subseteq J$ for all $J \in \mathcal{I}(B)$.

(*) *For every* $a \in A_+$*, its image* $V(a)$ *is contained in the closure of the algebraic ideal* $\mathrm{span}(B\varphi(a \otimes 1)B)$ *of* B *that is generated by* $\varphi(a \otimes 1)$.

Then there exist for every finite subset $X \subseteq A$ *and* $\varepsilon > 0$ *an element* $b \in B$ *with* $\|b\|^2 \leq \|V\|$ *and*

$$\|b^*\varphi(a \otimes 1)b - V(a)\| < \varepsilon \quad \textit{for all } a \in X\,.$$

If, in addition, A *is separable and* B *is a "corona" type algebra, then* V *is one-step "inner", i.e.,* $V(a) = b^*\varphi(a \otimes 1)b$ *for all* $a \in A$ *for some* $b \in B$ *with* $\|b\|^2 \leq \|V\|$.

Proposition 3.0.4. *If* A *is separable and exact,* B *is strongly purely infinite (or if* B *is not necessarily s.p.i., but* A *is strongly p.i.) and* $\varphi\colon A \to B$ *is a nuclear* C^**-morphism, then* φ *"extends" to a* C^**-morphism*

$$\varphi_e\colon\, A \otimes \mathcal{O}_\infty \,\to B_\omega \,:=\, \ell_\infty(B)/c_\omega(B)$$

with $\varphi_e(a \otimes 1) = \iota(\varphi(a))$ *for* $a \in A$ *and* $\iota(b) = (b, b, \ldots) + J_\omega$ *for* $b \in B$.

If B *is a strongly purely infinite "corona" algebra*

? Precise Def. of ''corona C*-algebra'' C?
Perhaps in Chp. 5, or extra section in Appendices?
There should be σ-unital D such that C is a non-zero hereditary subalgebra of $\mathcal{M}(D)/D$,
Characterize coronas by ''better than sub-Stonean''? ,

then φ *itself extends to a* C^**-morphism* $\psi\colon A \otimes \mathcal{O}_\infty \to B$ *with* $\psi(a \otimes 1) = \varphi(a)$ *for* $a \in A$.

`Transfer Remark 3.0.5 to suitable place?!`

Remark 3.0.5. Operator-matrix calculations appear in most chapters. We identify often the ternary rings $M_{p,q}(A) \subseteq M_n(A)$ with $\max(p, q) \leq n$ with the tensor products $M_{p,q} \otimes A$. The results of calculations in both of them look sometimes different if we do not care about different interpretations of this calculations.

This difference between this two conventions appears because the correct multiplication $r \cdot c \in e_{11} \otimes A$ of $r \in M_{1,n} \otimes A$ and $c \in M_{n,1} \otimes A$ comes from the identification of columns in $M_{n,1}(A)$ with tensors in $M_{n,1} \otimes A$ and of rows in $M_{1,n}(A)$ with tensors in $M_{1,n} \otimes A$.

Then $M_{1,n}(A) \cdot M_{n,1}(A) = M_{1,1}(A) = A$ but $(M_{1,n} \otimes A) \cdot (M_{n,1} \otimes A) = e_{1,1} \otimes A$ considered in $M_n \otimes A$. If we use in applications both of them, then we have in some cases to observe that we must identify $a \in A$ with $e_{11} \otimes a \in e_{11} \otimes A$.

In other calculation we have to identify $a \in A$ with $1_n \otimes a$ in $M_n \otimes A$. and M_n with $M_n(\mathbb{C} \cdot 1_{\mathcal{M}(A)}) \subseteq \mathcal{M}(M_n(A))$.

If the multiplier algebra $\mathcal{M}(A)$ contains a copy of $\mathcal{O}_2$ with $1_{\mathcal{M}(A)} \in \mathcal{O}_2$, and all elements have to be considered only up to MvN-equivalence, then such differences are usually not important.

1. Exact C^*-algebras and nuclear maps

We recall some definitions and basic facts concerning nuclear maps, exact C^*-algebras and nuclear C^*-algebras. A C^*-algebra B is called **nuclear** if its algebraic tensor $B \odot C$ product with any other C^*-algebra C admits only one C^*-algebra norm ([2]).

U. Haagerup showed in [**340**] with help of results of A. Connes in [**159**] that a C^*-algebra is nuclear if and only if it is *amenable* ([3]). We don't use the (co-homological) *amenability* here, because we work here with special classes of *nuclear maps*, and therefore, we use characterizations of nuclear maps and nuclear C^*-algebras that we recall in Remark 3.1.2(i).

A C^*-algebra B is called **exact** if, for every short exact sequence of C^*-algebras,

$$0 \to J \to A \to A/J \to 0\,,$$

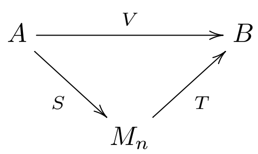
the spatial (or "minimal") tensor products with B of this sequence,

$$0 \to J \otimes B \to A \otimes B \to (A/J) \otimes B \to 0$$

is again exact, i.e., if the closure $J \otimes B$ of the algebraic tensor product $J \odot B$ in $A \otimes B$ is the same as the kernel of the natural C^*-algebra epimorphism $A \otimes B \to (A/J) \otimes B$. With other words: *The algebra B is exact if the functor $A \mapsto A \otimes B$ is a* **short exact functor** *in the category of C^*-algebras.*

Every nuclear C^*-algebra B is exact, because the functor $A \mapsto A \otimes^{\max} B$ is *short-exact* for *every* C^*-algebra B by the "universality" of the maximal C^*-tensor product $A \otimes^{\max} B$ ([4]).

Definition 3.1.1. Let A and B be C^*-algebras and let $V : A \to B$ be a completely positive map. The map V will be called **factorable** if it factorizes through M_n for some n, i.e., if there exist completely positive maps $S : A \to M_n$ and $T : M_n \to B$ such that the diagram



commutes.

By definition, a map $V : A \to B$ is **nuclear** if V can be approximated in point-norm topology by factorable maps (i.e., is the point-wise limit of a net of factorable maps).

---

[2] It was called property (T) by Takesaki [**766**].

[3] The C^*-algebra B is **amenable** if every derivation of an dual Banach B-bi-module X^* is inner, i.e., if X is a Banach B-bi-module and if $\partial : B \to X^*$ satisfies $\partial(ab)(x) = \partial(b)(xa) + \partial(a)(bx)$ for all $a, b \in B$ and $x \in X$, then there is $\rho \in X^*$ with $\partial(a)(x) = \rho(ax - xa)$ for all $a \in B$, $x \in X$.

[4] The algebraic sequence $0 \to J \odot B \to A \odot B \to (A/J) \odot B \to 0$ of algebraic tensor products is always exact, and the same holds then automatic for the maximal C^*-algebra tensor product $\otimes^{\max}$ – in place of $\odot$ – by universality and uniqueness of bi-functor $\odot$.

We say that a completely positive map $V\colon A \to M$ from a C^*-algebra into a von-Neumann algebra M is **weakly nuclear** if it can be approximated in point-$\sigma(M, M_*)$ topology by factorable maps.

The next remark collects a few key observations concerning nuclearity and exactness.

REMARK 3.1.2. In the following denotes $\mathrm{CP}(A, B)$ the (point-norm) closed convex cone of completely positive maps from A into B. The algebraic tensor product and the minimal (respectively maximal) C^*-algebra tensor product of C^*-algebras will be denoted respectively by $A \odot B$ and $A \otimes B$ (respectively by $A \otimes^{\max} B$), corresponding to the minimal C^*-norm $\|\cdot\|_{\min}$ (respectively to the maximal C^*-norm $\|\cdot\|_{\max}$ on $A \odot B$).

(o) The set $\mathrm{CP}_f(A, B)$ of factorable c.p. maps $V\colon A \to B$ is a *convex* cone in the space $\mathcal{L}(A, B)$, – the bounded *linear* maps from A to B with finite-dimensional image in B –, but with the important additional property that the map $a \mapsto c^*\big(V \otimes \mathrm{id}_k(r^*(a)r)\big)c$ is in $\mathrm{CP}_f(A, B)$ for every $k \in \mathbb{N}$, every row matrix $r \in M_{1,k}(\mathcal{M}(A))$ and every column matrix $c \in M_{k,1}(\mathcal{M}(B))$:
Indeed: Let $k = 1, 2$, $V_k = T_k S_k$ with c.p. maps $S_k\colon A \to M_{n_k}$ and $T_k\colon M_{n_k} \to B$. Then $V_1 + V_2 = T_3 S_3$ with

$$S_3\colon A \ni a \mapsto S_1(a) \oplus S_2(a) \in M_{n_1} \oplus M_{n_2} \subseteq M_{n_3}\,,$$

and suitable c.p. maps $T_3\colon M_{n_3} \to B$ for $n_3 := n_1 + n_2$, e.g. for $T_3 := R \circ P$ where P denotes the conditional expectation from M_{n_3} onto $M_{n_1} \oplus M_{n_2} \subset M_{n_3}$, and $R(c \oplus d) := V_1(c) + V_2(d)$ for $c \oplus d \in M_{n_1} \oplus M_{n_2}$).

The set $\mathrm{CP}_f(A, B)$ is a matrix operator-convex cone in sense of Definition 3.2.2 and can be described as follows:
If $V = TS$, then $V \otimes \mathrm{id}_m = (T \otimes \mathrm{id}_m)(S \otimes \mathrm{id}_m)$ is in $\mathrm{CP}_f(A \otimes M_m, B \otimes M_m)$, $(S \otimes \mathrm{id}_m)(r^*(\cdot)r)$ is in $\mathrm{CP}(A, M_{nm})$ for $r \in A \otimes M_n$ and $c(T \otimes \mathrm{id}_m)(\cdot)c^*$ is in $\mathrm{CP}(M_{nm}, B)$ for $c \in B$.

Notice that $\mathrm{CP}_f(A, B)$ is contained in the set $\mathrm{CP}_{f.r.}(A, B) \subseteq \mathrm{CP}(A, B)$ of c.p. maps of finite rank inside $\mathrm{CP}(A, B)$. (But it is still not clear if all c.p. maps $V\colon A \to B$ of finite vector space dimension $\dim(V(A)) < \infty$ are contained in $\mathrm{CP}_f(A, B)$).

The operator-norm closure of $\mathrm{CP}_f(A, B)$ in the Banach space of general bounded linear maps $\mathcal{L}(A, B)$ ([5]) contains all c.p. maps $V\colon A \mapsto B$ of finite rank cf. Lemma 3.1.11(iv).

The finite rank self-adjoint linear maps from A to B are differences of c.p. maps of finite rank (using Lemma 3.1.4).

(i) Simple Hahn-Banach separation arguments (cf. proof of [**426**, lem. 2] or similar arguments in Section 7) show, that a completely positive map $V\colon A \to B$ is nuclear,

[5] This "uniform" closure of $\mathrm{CP}_f(A, B)$ is usually smaller than the point-norm closure, and therefore does not contain all nuclear maps in general.

if and only if, for every C^*-algebra D,

$$V \otimes^{\max} \mathrm{id}_D : \; A \otimes^{\max} D \to B \otimes^{\max} D$$

annihilates the kernel of the natural epimorphism $A \otimes^{\max} D \to A \otimes^{\min} D$, (cf. [**140**], [**726**, prop. 1.2]). I.e., $V \in \mathrm{CP}(A, B)$ *is nuclear, if and only if,* $V \otimes^{\max} \mathrm{id}_D$ *naturally factorizes over the spatial (= minmal) C^*-algebra tensor product* $A \otimes D$.

It suffices to consider unital singly generated D (e.g. by taking stabilization and unitization). If we let $G := C^*(F_2)$, the full group C^*-algebra of the free group F_2 on two generators. Then $D \cong G/J$ for a suitable closed ideal $J \triangleleft G$.

Every C^*-semi-norm μ on the algebraic tensor product $A \odot G$ with $\mu(a \otimes 1) > 0$ for $a \neq 0$ is given on $c \in A \odot G$ by $\mu(c) = \|(\mathrm{id} \otimes \pi_J)(c)\|_\beta$ for suitable $J \triangleleft G$ and some C^*-*norm* $\| \cdot \|_\beta$ on $A \odot (G/J)$ with $\| \cdot \|_{\min} \leq \| \cdot \|_\beta \leq \| \cdot \|_{\max}$, i.e., the kernel of $A \otimes^{\max} D \to A \otimes^{\beta} D$ is contained in the kernel of $A \otimes^{\max} D \to A \otimes^{\min} D$ for $D = G/J$ ([6]). Now notice that $B \otimes^{\max} (\cdot)$ is always a short exact functor, i.e., that $B \otimes^{\max} (J)$ is the kernel of $B \otimes^{\max} G \to B \otimes^{\max} (G/J)$. If we combine the above observations using a diagram check, then we can reformulate the tensorial nuclearity criterion equivalently by the following more flexible criterion:

A map $V \in \mathrm{CP}(A, B)$ *is nuclear, if and only if,* $(V \otimes^{\max} \mathrm{id})(I)$ *is contained in* $B \otimes^{\max} J$ *for every closed ideal* I *of* $A \otimes^{\max} C^*(F_2)$ *with* $I \cap (A \otimes 1) = \{0\}$ *and the largest ideal* J *of* $C^*(F_2)$ *with* $A \otimes^{\max} J \subseteq I$. ([7]).

In particular, *the identity map* id_A *on a C^*-algebra A is nuclear if and only if A is nuclear* ([**140**], [**426**], [**145**]).

(ii) A C^*-algebra $A \subseteq \mathcal{L}(H)$ is exact, if and only if, the inclusion map $A \hookrightarrow \mathcal{L}(H)$ is nuclear. We gave two different proofs in [**432**] and in [**438**]. The proof in [**432**] uses part(i), a simple intersection result (less general than the "distance" lemma [**438**, lem. 3.6]), an inspection of kernels in certain commutative diagrams, and the fact that on the algebraic tensor product $\mathcal{L}(H) \odot C^*(F)$ of $\mathcal{L}(H)$ with the full group C^*-algebra $C^*(F)$ of the free groups F on countably many generators there is only one C^*-norm. The authors proof in [**432**] of the uniqueness of the C^*-norm on $\mathcal{L}(H) \odot C^*(F)$ has been simplified considerably by G. Pisier [**640**].

(iii) The composition of a nuclear map with any other completely positive map is again nuclear. In particular:

(a) $W = b^* V(\cdot) b$ is nuclear if V is nuclear.
(b) The restriction $V|D \colon D \to C$ is nuclear if $D \subseteq A$ is a C^*-subalgebra and $V \colon A \to C$ is nuclear.

Therefore from (i) and (ii) we see, that *nuclear C^*-algebras are exact and that C^*-subalgebras of exact C^*-algebras are again exact.*

---

[6] It follows from [**766**]: A C^*-*semi*norm ν on $A \odot D$ satisfies $\nu(c) \geq \|c\|_{\min}$ for all $c \in A \odot D$, if and only if, $\nu(a \otimes d) > 0$ for each non-zero $a \in A_+$ and $d \in D_+$. See also [**767**, thm. IV.4.19].

[7] It is a special case of a characterization of point-norm closed m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ by its associated action $\Psi_{\mathcal{C}} \colon A \otimes^{\max} C^*(F_2) \mapsto A \otimes^{\max} C^*(F_2)$ cf. Section 7.

(c) Every completely positive map from A to B is nuclear if A or B is nuclear, cf. Part (i).

(d) *Every* completely positive map from A to B is nuclear if A is *exact* and B has the *weak expectation property* (i.e., equivalently, B is *weakly injective* in the sense: $\psi\otimes^{\max}\mathrm{id}_C\colon B\otimes^{\max}C \to D\otimes^{\max}C$ for every *-monomorphism $\psi\colon B \to D$ and every C^*-algebra C).
Indeed, let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(H)$ (for sufficiently large H and with embedding maps η_A and η_B), then a c.p. contractions $V\colon A \to B$ extends to a c.p. contraction $T\colon \mathcal{L}(H) \to \mathcal{L}(H)$ by the Arveson extension theorem (complete injectivity of $\mathcal{L}(H)$, cf. [**42**]), i.e., $T \circ \eta_A = \eta_B \circ V$ for the inclusion maps η_A and η_B. Then $T \circ \eta_A$ is nuclear by (ii). The nuclearity criteria in part (i) and the above given criteria for the weak injectivity of B show that $V\colon A \to B$ is nuclear.

(e) In general a completely positive map $V\colon A \to B \subseteq C$ is not nuclear as a map from A to B if $V \in \mathrm{CP}_{\mathrm{nuc}}(A, C)$, i.e., where V becomes nuclear if considered as a map from A to C.
(Use part (ii) and the existence of non-nuclear exact C^*-algebras A, as e.g. $V := \mathrm{id}_A$, $A = B = C^*_{\mathrm{red}}(G) \subseteq \mathcal{L}(\ell_2(G)) =: C$ for $G := \mathrm{SL}_2(\mathbb{Z})$.)

(f) Every c.p. map $V\colon A \to B$ of *finite rank*, i.e., with $\mathrm{Dim}(V(A)) < \infty$, is the limit in operator-norm of factorable maps, i.e., there exists a sequence of factorable maps $T_n\colon A \to B$ with $\lim_n \|T_n - V\| = 0$ in the norm of $\mathcal{L}(A, B)$, cf. [**426**], see also Lemma 3.1.11(iv).

(g) It follows from part (f) that the definition of nuclear c.p. maps given in [**426**] and that of M.D. Choi and E.G. Effros in [**140**] define the same class of maps.

(iv) There exist c.p. maps $W\colon A/J \to B$ such that $V := W\circ\pi_J\colon A \to B$ is nuclear, but the map W itself is *not* nuclear.
This follows from the existence of W*-algebras that are not weakly exact (compare Remark B.7.4, [**436**], [**597**]).

It is a *problem to find non-trivial sufficient criteria* on $(A, J \lhd A, B)$ that allow to conclude the nuclearity of $W\colon A/J \to B$ from the nuclearity of $W \circ \pi_J\colon A \to B$. We use non-exact subalgebras of the asymptotic corona $B = \mathrm{Q}(\mathbb{R}_+, D)$, – e.g. in our proofs on the classification of nuclear algebras. Those algebras are often *not locally reflexive*, because $\mathrm{Q}(\mathbb{R}_+, D)$ is locally reflexive, if and only if, D is sub-homogenous.

Local reflexivity is formally weaker than exactness of a C^*-algebra A, i.e., exactness implies that A is locally reflexive. But it was still unknown (June 2019) if local reflexivity implies exactness.

Therefore, we must *bypass* the problem concerning the nuclearity of $[V]_J = W\colon A/J \to B$ for nuclear $V\colon A \to B$ with $V(J) = \{0\}$, and have introduced the notion of Ψ-*residually nuclear* maps, cf. Definition 1.2.8, and recognize that it is of interest for us (!) to know when Ψ-equivariant *and* nuclear c.p. maps are automatically Ψ-*residually nuclear* maps.

(v) Here is a very special case, where one has a positive answer to the problem discussed in Part(iv):
Let $J \lhd A$ a closed ideal and suppose that $J \otimes D \to A \otimes D \to (A/J) \otimes D$ is exact for every unital separable C^*-algebra D ([8]), then it holds:

> (∗) *If* $W \in \mathrm{CP}(A/J, B)$ *and* $W \circ \pi_J$ *is nuclear – i.e.,* $W \circ \pi_J \in \mathrm{CP}_{\mathrm{nuc}}(A, B)$*, then* W *is nuclear.*

Local lifting results of [**238**] for locally reflexive C^*-algebras A show that the sequence

$$J \otimes D \to A \otimes D \to (A/J) \otimes D$$

is exact for every locally reflexive C^*-algebra A, every ideal $J \rhd A$ and every C^*-algebra D. Hence:

$W \in \mathrm{CP}(A/J, B)$ is nuclear if A is locally reflexive and $W \circ \pi_J : A \to B$ is nuclear.

One can see that exact C^*-algebras are locally reflexive, because the separable exact C^*-algebras are quotients of subalgebras of the nuclear CAR-algebra M_{2^∞} by [**438**, cor. 1.4(v)], and sub-quotients of locally reflexive C^*-algebras are locally reflexive by [**238**].

In the mean time the author has obtained the following ?????

Proof of (∗). If $W \circ \pi_J$ is nuclear, then, by Part(i),

$$(W \circ \pi_J) \otimes^{\max} \mathrm{id} \colon A \otimes^{\max} D \to B \otimes^{\max} D$$

(naturally) factorizes over $A \otimes D$, in the sense that $(W \circ \pi_J) \otimes^{\max} \mathrm{id}$ maps the kernel ideal of $A \otimes^{\max} D \to A \otimes D$ to zero. Thus, there is a c.p. map $V \colon A \otimes D \to B \otimes^{\max} D$ with $V \circ \lambda_{A,D} = (W \circ \pi_J) \otimes^{\max} \mathrm{id}$ for the natural *-epimorphism $\lambda_{A,D} \colon A \otimes^{\max} D \to A \otimes D$. It follows that $V(a \otimes d) = W(a + J) \otimes d$ for $a \in A$ and $d \in D$. It yields $V(J \otimes D) = \{0\}$. The exactness of $J \otimes D \to A \otimes D \to (A/J) \otimes D$ implies that there is a C^*-morphism $\beta \colon (A/J) \otimes D \to B \otimes^{\max} D$ with $\beta \circ (\pi_J \otimes \mathrm{id}_D) = \alpha$, i.e., $\beta((a + J) \otimes d) = W(a + J) \otimes d$ for $a \in A$, $d \in D$. Thus, the natural c.p. map $W \otimes^{\max} \mathrm{id} \colon (A/J) \otimes^{\max} D \to B \otimes^{\max} D$ factorizes over $(A/J) \otimes D$.

It implies that $W \colon A/J \to B$ is nuclear by Part(i). □

(vi) Quotients A/J of exact C^*-algebras A are exact.
Indeed: If $A/J \subseteq \mathcal{L}(K)$ and $A \subseteq \mathcal{L}(H)$, then, by the Arveson extension theorem, cf. [**42**], there is a complete contraction $T \colon \mathcal{L}(H) \to \mathcal{L}(K)$ with $T(a) = \pi_J(a)$, i.e., $T \circ \eta_A = \eta_{A/J} \circ \pi_J$, where η indicates the inclusion map. The map $\eta_A \colon A \hookrightarrow \mathcal{L}(H)$ is nuclear by (ii), the maps $\eta_{A/J} \circ \pi_J$ and $\eta_{A/J} \colon A/J \hookrightarrow \mathcal{L}(K)$ are nuclear by Remark (v). See [**436**] for a different proof of exactness for quotients of exact C^*-algebras.

(vii) A C^*-algebra B is nuclear, if and only if, A is exact and the natural *-epimorphism $A \otimes^{\max} C^*(F_2) \to A \otimes^{\min} C^*(F_2)$ is an isomorphism.

[8] The sequence $J \otimes D \to A \otimes D \to (A/J) \otimes D$ is *exact* for $D := \mathcal{L}(\ell_2)$ if and only if $\mathrm{id}_{A/J}$ is locally completely contractive liftable, cf. [**238**].

Indeed: It induces that $A \otimes^{\max} D \to A \otimes^{\min} D$ is an isomorphism for for every hereditary C^*-subalgebra D of $C^*(F_2)$. The exactness of $A \otimes^{\min} (\cdot)$ and the general exactness of $(\cdot) \otimes^{\max} (\cdot)$ imply that the natural C^*-morphisms $A \otimes^{\min} D/J \to A \otimes^{\max} D/J$ are isomorphisms for each closed ideal of D. It is not difficult to see that each separable C^*-algebra is isomorphic to a quotient of a hereditary C^*-subalgebra $C^*(F_2)$. Thus, $A \otimes^{\max} B \to A \otimes^{\min} B$ is an isomorphism for every separable C^*-algebra B. This carries over to all C^*-algebras B, i.e., A is nuclear.

REMARK 3.1.3. In in late 2015 the author has obtained the following characterization of *locally reflexive* separable C^*-algebras B by the following "co-exactness criterium" (unpublished but presented in talks of the author):

A separable C^*-algebra B is locally reflexive, if and only if, for every closed left ideal $L \subseteq \mathbb{K} \otimes B$ and every separable C^*-algebra C the sequence

$$L \otimes^{\min} C \to (\mathbb{K} \otimes B) \otimes^{\min} C \to ((\mathbb{K} \otimes B)/L) \otimes^{\min} C$$

is exact. That means equivalently that the natural linear contraction

$$((\mathbb{K} \otimes B) \otimes^{\min} C)/(L \otimes^{\min} C) \mapsto ((\mathbb{K} \otimes B)/L) \otimes^{\min} C$$

is an isomorphism of Banach spaces.

So far no example of a non-exact separable locally reflexive C^*-algebra has been found. But it is important to check this.

For completeness we give the almost obvious separation on finite sets of linear functionals on C^*-algebras.

LEMMA 3.1.4 (Separation lemma). *If $\rho_1, \ldots, \rho_n$ are continuous linear functionals on a C^*-algebra A, then there is a cyclic *-representation $d \colon A \to \mathcal{L}(\mathcal{H})$ with cyclic vector $\xi \in \mathcal{H}$ of norm $\|\xi\| = 1$, and operators $t_1, \ldots, t_n \in d(A)'$, such that $\rho_k(a) = \langle d(a)\xi, t_k\xi\rangle$ for $a \in A$ and $k = 1, \ldots, n$.*
The operators $t_k \in d(A)'$ are uniquely determined by (d, ξ, ρ_k).

If A is a W^*-algebra and if $\rho_1, \ldots, \rho_n$ are normal, then $d \colon A \to \mathcal{L}(H)$ and can be taken normal, i.e., such that the functionals $\langle d(\cdot)x, y\rangle$ are in the predual A_* of A.

PROOF. Since A^* is the linear span of the positive linear functionals on A, we may suppose that the ρ_k are positive and $\rho_k \neq 0$. Let $\lambda := \sum_k \rho_k$. The cyclic GNS-representation $d \colon A \to \mathcal{L}(\mathcal{H})$ of A with respect to λ has cyclic vector $y \in \mathcal{H}$ with $\|y\|^2 = \|\lambda\|$ and satisfies $\lambda(a) = \langle d(a)y, y\rangle$ for $a \in A$. In particular, $\rho_k(a^*a) \leq \|d(a)y\|^2$. It follows, that there are unique positive contractions $C_k \in \mathcal{L}(\mathcal{H})$ with $\rho_k(c^*b) = \langle d(b)y, C_k d(c)y\rangle$, and the uniqueness of the C_k implies that $C_k d(a) = d(a) C_k$ for $a \in A$, (cf. [**217**, prop. 2.5.1]). Now let $\xi := \|\lambda\|^{-1/2} y$ and $t_k := \|\lambda\| C_k$.

If A is a W*-algebra and the ρ_j are normal, then we can apply the same arguments to the $\rho_1, \ldots, \rho_n$ inside the predual A_* of A. □

The next lemma contains a useful criteria for the nuclearity of a c.p. map $V: A \to B$, that will be used later (e.g. in Chapter 6).

Say precisely where is it used and cited ? in chp.6 ? Give, or cite, or define here what means "*essential*" here.

LEMMA 3.1.5. *Suppose that $D \subseteq B$ is an* essential *hereditary C^*-subalgebra of B with (open) support projection $p_D \in B^{**}$, and that $V: A \to B$ is a completely positive map.*

If $(1-p_D)B^{**}(1-p_D)$ is an injective *W^*-algebra* and *if $a \in A \mapsto V_d(a) := dV(a)d \in D$ is a nuclear map for all $d \in D_+$, then $V: A \to B$ is nuclear.*

PROOF. We use that every c.p. map $W: A \to M$ from a C^*-algebra A to an injective W*-algebra M is weakly nuclear, because

??? the identity map id_M is weakly nuclear for all injective W*-algebras M, ??? only for all ???

and that a c.p. map $V: A \to B$ is nuclear if $V: A \to B^{**}$ is weakly nuclear.

The inclusion map for $A \subseteq L(H)$ is always weakly nuclear ??? Here is some misunderstanding ...

A better argument should be: If $(1-p_D)B^{**}(1-p_D)$ is injective (as vN-algebra), then there exists a central projection $Q \geq (1-p_D)$ in B^{**} such that $B^{**}Q$ is injective. Then the kernel of $b \mapsto bQ$ in a closed ideal J of B with $J \subseteq D$.

But it is not clear if ?????

Indeed, by Definition 3.1.1, each weakly nuclear map is in the point-$\sigma(B^{**}, B^*)$ topology the point-wise limit of factorable maps from A to B^{**} and those can be again point-wise approximated in $\sigma(B^{**}, B^*)$ topology by factorable maps from A to B. Thus, we can then use a Hahn-Banach separation argument to see that $V: A \to B$ can be approximated in point-norm by factorable maps from A to B.

We show the weak nuclearity of $V: A \to B^{**}$:
Let Q denote the smallest central projection of B^{**} with $(1-p_D) \leq Q$. Then $(1-Q) \leq p_D$ and $B^{**}Q$ is an injective W*-algebra.

Thus, $V(a) = (1-Q)V(a) + QV(a)$ is the sum of the weakly nuclear maps $a \mapsto QV(a)$ (by injectivity of $B^{**}Q$) and of $(1-Q)V(a) = (1-Q)p_D V(a) p_D$ (that is an weak limit of $a \mapsto (1-Q)dV(a)d$ with contractions $d \in D_+$).

Hence, V is weakly nuclear – if considered as a c.p. map from A to B^{**}. □

REMARK 3.1.6. Any positive linear map V from a unital complex C^*-algebra B into a C^*-algebra $C \subseteq \mathcal{L}(\mathcal{H})$ has norm $\|V\| = \|V(1_B)\|$, cf. [**402**, exercise 10.5.10] (using [**402**, exercise 10.5.4]), or use the original Russo-Dye theorem [**222**] for complex C^*-algebras to reduce the calculation of the norm $\|V\|$ to the case of the special complex C^*-algebra $B := \mathrm{C}(S^1)$, where all positive V are automatically completely positive. The proof of [**207**, thm. I.8.4] shows the stronger result that set of all convex combinations of finitely many unitaries in a unital C^*-algebra A contains all $a \in A$ with $\|a\| < 1$. This is also shown in [**402**, exercise 10.5.92].

In the case of real C^*-algebras C acting on real Hilbert spaces one has at least $\|V\| \leq \|V_2(1_B \otimes 1_2)\|$ in the case that $V_2 := V \otimes \mathrm{id}_{M_2(\mathbb{R})}$ is positive, because V is "real" 2-positive, and it suffices for real C^*-algebras B to consider the case $C := M_2(\mathbb{R})$.

If a map V is moreover completely positive then this implies that $\|V(1)\| = \|V\| = \|\mathrm{id}_n \otimes V\| = \|V\|_{cb} := \sup_n \|\mathrm{id}_n \otimes V\|$. Here id_n denotes the identity map on M_n and the norm on $M_n \otimes C \subseteq M_n \otimes \mathcal{L}(\mathcal{H}) \cong \mathcal{L}(\ell_2(n) \otimes \mathcal{H})$ is the operator norm.

The complete positivity of a map $V \in \mathrm{CP}(M_n, A)$ implies that $(\mathrm{id}_n \otimes V)(X) \geq 0$ for all $X \in (M_n \otimes A)_+$, where we naturally identify $M_n(A)$ with $M_n \otimes A$.

The *generalized Schwarz inequality* (of R.V. Kadison [**399**]) says that, if a linear map $\eta\colon A \to B$ is positive (i.e., if $\eta(A_+) \subseteq B_+$) and $\|\eta^{**}(1_A)\| \leq 1$, then $\eta(a)^2 \leq \eta(a^2)$ for all $a \in A_{s.a.}$.

`!!  Compare with other remarks above/below !!  e.g next.`

REMARK 3.1.7. We have often to do with positive maps $T\colon C \to A$ between C^*-algebras. This maps T are bounded:
Indeed, otherwise for each $n \in \mathbb{N}$ there would exist $c_n \in C_+$ with $\|c_n\| = 1$ and $\|T(c_n)\| \geq 4^n$, and the well-defined element $S := \sum_n 2^{-n} c_n \in C_+$ must have the – impossible – property $\|T(S)\| \geq 2^{-n}\|T(c_n)\| \geq 2^n$ for all $n \in \mathbb{N}$.

Another often used elementary fact for approximations of positive maps $T\colon C \to A$ used special case is the following property:

Let $C \subseteq \mathcal{M}(A)$ a C^*-subalgebra, $\mathcal{S}$ a set of positive maps from C into A with the property that $S(e(\cdot)e) \in \mathcal{S}$ for all $S \in \mathcal{S}$ and $e \in C_+$. If $T\colon C \to A$ can be approximated in point-norm topology by $S \in \mathcal{S}$, then T is positive and one finds also a point-norm approximation of T by $S \in \mathcal{S}$ with the additional property $\|S\| \leq \|T\|$, cf. Lemma 3.1.8.

The following Lemma 3.1.8 applies to to the particular case of the family of n-step approximately inner c.p. maps $V\colon C \to A$ and to the approximation of nuclear maps by more special factorable maps.

See also Lemma B.16.5(i) in Appendix B.

LEMMA 3.1.8. *Suppose that C is a C^*-algebra and B is a C^*-algebra (respectively that B is a von-Neumann algebra). Denote by $\mathcal{S}$ a* set *of positive linear maps from a C^*-algebra C into B.*

If $V\colon C \to B$ is a contraction *in the closure of $\mathcal{S}$ with respect to the point-norm topology (respectively with respect to the point-strong topology), then V is in the closure of the set of positive maps*

$$\mathcal{S}_1 := \{S(e(\cdot)e)\,;\ S \in \mathcal{S},\, e \in C_+\,,\ \|e\| < 1\,,\ \|S(e^2)\| \leq 1\}$$

with respect to the point-norm topology
(respectively then V is in the closure of the set of positive maps

$$\mathcal{S}_2 := \{fS(e(\cdot)e)f\,;\ S \in \mathcal{S},\ e \in C_+,\ \|e\| < 1,\ f \in B_+, \|f\| \leq 1\, \|fS(e^2)f\| \leq 1\,\}$$

*with respect to the point-*strong topology).*

Notice for later applications of Lemma 3.1.8 that the maps in $\mathcal{S}_1$ and $\mathcal{S}_2$ are *contractions*, but that we do not assume any sort *convexity* for the given set $\mathcal{S}$.

But in cases where we can consider convex sets $\mathcal{S}$, the following general fact in Functional analysis applies and improves Lemma 3.1.8 a bit:
If the point-norm closure of $\mathcal{S} \subseteq \mathcal{L}(C, B)$ is *convex*, then the point-$\sigma(B, B^*)$ closure of $\mathcal{S}$ coincides with its point-norm closure, (respectively the point-$\sigma(M, M_*)$ closure of bounded convex $\mathcal{S} \subseteq \mathcal{L}(C, M)$ coincides with its point-*strong closure of $\mathcal{S}$).

Proof. Case, where B is a C^*-algebra and $V \colon C \to B$ is in the point-norm closure of $\mathcal{S}$:
Let $c_1, \ldots, c_n \in C$ and $\varepsilon > 0$, $M := \max(\|c_1\|, \ldots, \|c_n\|)$, and $\eta := \varepsilon/(2 + M)$. There exists $d \in C_+$ with $\|d\| < 1$ such that $\|dc_jd - c_j\| < \eta$ for $j = 1, \ldots, n$. There is $S \in \mathcal{S}$ with $\|S(d^2)\| \leq \|V(d^2)\| + \eta$ and $\|S(dc_jd) - V(dc_jd)\| < \eta$. Let $t := (1 + \eta)^{-1/2}$, $e := td$ and $T(b) := S(ebe)$ for $b \in C + \mathbb{C} \cdot 1$. Then $\|T\| = \|T(1)\| = t^2\|S(e^2)\| \leq 1$, because T is positive. The map $T|C$ is in $\mathcal{S}_1$ and

$$\|T(c_j) - V(c_j)\| \leq 2t^2\eta + (1 - t^2)M \leq \varepsilon \qquad \text{for} \quad j = 1, \ldots, n\,.$$

Case, where $B \subseteq \mathcal{L}(\mathcal{H})$ is a von-Neumann algebra and $V \colon C \to B$ is in the point-strong closure of $\mathcal{S}$:
Suppose that $B \subseteq \mathcal{L}(\mathcal{H})$, $x_1, \ldots, x_m \in \mathcal{H}$, $c_1, \ldots, c_n \in C$, and $\varepsilon > 0$. Let $\mu := 2\max(1, \|x_1\|, \ldots, \|x_m\|)$, and $c_0 := 1 \in B$. There are $e \in C_+$ with $\|e\| < 1$ and $\|c_k - ec_ke\| < \varepsilon/\mu$. By assumption, there is a directed net (S_γ) of elements of $\mathcal{S}$ such that $S_\gamma(ec_ke)$ converges strongly to $V(ec_ke)$ for $k = 0, 1, \ldots, n$.

The function $h \colon t \in \mathbb{R} \to (0, \infty)$ with $h(0) := 1$ and $h(t) := \min(|t|^{-1/2}, 1)$ for $t \neq 0$ is in $\mathrm{C}_0(\mathbb{R})_+$. Thus, $f_\gamma := h(S_\gamma(e^2)) \leq 1$ converges strongly to $h(V_\gamma(e^2)) = 1$ by [**400**, thm. 5.3.4]. Moreover, $f_\gamma S_\gamma(e^2) f_\gamma \leq 1$. It follows that the map $T_\gamma(c) := f_\gamma S_\gamma(ece) f_\gamma$ (for $c \in C + \mathbb{C} \cdot 1$) is in the set $\mathcal{S}_2$, and has the property that $T_\gamma(c_k)$ converges strongly to $V(ec_ke)$ for $k = 1, \ldots, m$. Thus there is $T \in \mathcal{S}_2$ with $\|T(c_k)x_j - V(c_k)x_j\| \leq \varepsilon/2 + \|c_k - ec_ke\|\|x_j\| < \varepsilon$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. □

Proposition 3.1.9. *Let $e_{jk} \in M_n$ denote the matrix units of M_n for $n \in \mathbb{N}$ and let A denote a C^*-algebra.*

There is a bijective and additive relation between the cone of positive matrices $[a_{jk}] \in M_n(A)_+$ and the matrix operator-convex cone of all completely positive maps $V \colon M_n \to A$:

If $[a_{jk}] \in M_n(A)_+$ is given then a completely positive map $V \in \mathrm{CP}(M_n\,,\,A)$ is defined by

$$V(\beta) := \sum_{j,k=1}^{n} \beta_{jk} a_{jk} \quad \text{for all } \beta \in M_n\,.$$

If a completely positive map $V \in \mathrm{CP}(M_n\,,\,A)$ is given, then a positive matrix $[a_{j,k}] \in M_n(A)_+$ is defined by its entries $a_{jk} := V(e_{jk})$. Their norms estimate each other by

$$\|V\|_{cb} = \|V\| = \|a_{11} + a_{22} + \cdots + a_{nn}\| \quad \text{and} \quad \|[a_{jk}]\| \leq n\|V\|\,.$$

Following Parts (o)-(iv) show more details:

(o) *If $e_{jk} \in M_n$ denote the n^2 canonical matrix-units and if $V \colon M_n \to A$ is a completely positive map from M_n into a C^*-algebra A, then the linear map $\mathrm{id}_n \otimes V \in \mathcal{L}(M_n \otimes M_n, M_n \otimes A)$ is a positive map from $M_n \otimes M_n$ to $M_n \otimes A$. In particular, the matrix $[V(e_{jk})] = (\mathrm{id}_n \otimes V)([e_{jk}]) =: [a_{jk}]$ is positive in the C^*-algebra $M_n(A)$.*

(i) *If $[a_{jk}] \in M_n(A)$ is a positive element in the C^*-algebra $M_n(A)$, then there exists column matrices $c_1, \ldots, c_n \in M_{n,1}(A)$ such that the completely positive map $V \in \mathrm{CP}(M_n, A)$ defined by*

$$V(\beta) := \sum_{k=1}^{n} c_k^* \beta c_k \quad \text{for all } \beta \in M_n$$

satisfies $V(e_{jk}) = a_{jk}$ for $j,k \in \{1,\ldots,n\}$. Thus, $V(\beta) = \sum_{j,k} \beta_{j,k} a_{jk}$ and $\|V\| = \|V\|_{cb} = \|a_{11} + \cdots + a_{nn}\|$.

It implies that $\| \sum_{k=1}^{n} c_k^* c_k \| = \|V\|$ ([9]).

(ii) *Suppose that A has the property that for every positive contraction $a \in A_+$ and $\varepsilon > 0$ there exist elements $b_1, b_2 \in A$ with $\|b_j^* b_k - \delta_{jk} a\| < \varepsilon$ for $j,k \in \{1,2\}$ ([10]).*

Then each $V \in \mathrm{CP}(M_n\,, A)$ can be "orthogonal and elementary approximated in norm" – in the sense that for each $\gamma \in (0,1)$ there exist columns $d_0, d_1 \in M_{n,1}(A)$ with $d_0^* \beta d_1 = 0$ and $d_0^* \beta d_0 = d_1^* \beta d_1$ for all $\beta \in M_n$, and with the property that the completely positive map $U(\beta) := d_0^* \beta d_0$ ($\beta \in M_n$) satisfies that $V - U$ is a completely positive map with $\|V - U\| < \gamma$. In particular, $U(1_n) \leq V(1_n)$ and $\|d_0\| = \|d_1\| \leq \|V\|^{1/2}$.

(iii) *If $A = D/J$ for a C^*-algebra D and a closed ideal J of D, then there exists a c.p. map $T \colon M_n \to D$ with $\pi_J \circ T = V$ and $\|T\| = \|V\|$.*

(iv) *Let $A \subseteq N$ a C^*-subalgebra of a W^*-algebra N that is weakly dense in N and let $T \colon M_n \to N$ a c.p. contraction.*

*Then there exists a net $\{V_\rho\}$ of c.p. contractions $V_\rho \colon M_n \to A$ that converges in point-*ultrastrong topology to T.*

Part (ii) applies e.g. if the unit element of $\mathcal{M}(A)$ is properly infinite or if A is purely infinite, cf. Lemma 2.1.7(ii). Equivalent formulations of the additional

[9] Here we identify M_n naturally with $M_n(\mathbb{C} \cdot 1) \subseteq M_n(\mathcal{M}(A))$.

[10] We don't require here that $b_j b_j^* \in \overline{aAa}$.

assumption in Part (ii) can be found in Parts (b, i) and (b,iii) of Lemma 2.1.12. We use in our proof also the equivalence of Parts (a, i) and (a,ii) of Lemma 2.1.12.

PROOF. Let $\mathcal{M}(A)$ denote the C^*-algebra of two-sided multipliers of A. We identify naturally $M_n(A) \subseteq M_n(\mathcal{M}(A))$ with $M_n \otimes A \subseteq M_n \otimes \mathcal{M}(A)$ and $M_n = M_n(\mathbb{C})$ with $M_n \otimes 1 \subseteq M_n \otimes \mathcal{M}(A)$. ([11]).

(o): We identify sometimes $M_n(A)$ with $M_n \otimes A$ naturally. By Remark 3.1.6, the complete positivity of $V \in \mathrm{CP}(M_n, A)$ implies that $\|V(1_n)\| = \|V\| = \|\mathrm{id}_n \otimes V\|$ and $(\mathrm{id}_n \otimes V)(X) \geq 0$ for all $X \in (M_n \otimes A)_+$.

Recall that the matrix units $e_{ij} := e_i^* e_j \in M_n$, for $i, j = 1, \ldots, n$, are build from row-matrices

$$e_1 := [1, 0, 0, \ldots, 0]\,, \;\; e_2 := [0, 1, 0, \ldots, 0]\,, \;\ldots,\;\; e_n := [0, 0, \ldots, 0, 1]$$

that are the elements of the canonical basis of $M_{1,n}(\mathbb{C}) \cong \mathbb{C}^n$.

The $n^2 \times n^2$ matrix $E \in M_n(M_n)$ with the $n \times n$ matrices $e_{ij} \in M_n$ as its entries is given by $E := [e_{ij}] \in M_n(M_n)$ and can be expressed as $E = \sum_{ij} e_{ij} \otimes e_{ij} \in M_n \otimes M_n \cong M_n(M_n)$. It is positive in $M_{n^2} \cong M_n \otimes M_n$ because $E = R^* R$ for the row-matrix $R := [e_1, e_2, \ldots, e_n] \in M_{1,n^2}$, the norm of E is $\|E\| = n$ because $P := n^{-1}E$ is an orthogonal projection.

We define a matrix $F = [f_{jk}] \in M_n(A) \cong M_n \otimes A$ by taking its entries $f_{ij} := V(e_{ij})$, i.e., define

$$F := (\mathrm{id}_n \otimes V)(E) \;=\; \sum_{i,j} e_{ij} \otimes V(e_{ij}) \in M_n \otimes A\,.$$

Then $F \in M_n(A)_+$ is a positive matrix by the assumption of complete positivity of V, and $V(\beta) \;= \sum_{j,k=1}^n \beta_{jk} f_{j,k}$ for $\beta \in M_n$, because $P := n^{-1}E$ is an orthogonal projection in $M_n \otimes M_n$. The norm of F can be estimated roughly by

$$\|F\| \leq n\|\mathrm{id}_n \otimes V\| = n\|V(1_n)\| = n\|f_{11} + \cdots + f_{nn}\|\,.$$

(i): Let F any positive matrix in $M_n(A)_+$ with entries $f_{jk} \in A$, and let $G := F^{1/2} \in M_n(A)_+$ with entries $g_{ij} \in A$. We define a map $V \colon M_n \to A$ by $V(e_{jk}) := f_{j,k}$, i.e., $V(\beta) := \sum_{j,k=1}^n \beta_{j,k} f_{j,k}$. The complete positivity of the below shown alternative expression for the above defined map $V \colon M_n \to A$, – as sum of elementary c.p. maps $\beta \mapsto c^* \beta c$ with some $c \in M_{n,1}(A)$ –, implies then that V must be completely positive.

In conjunction with Part (i), this shows moreover:
Each completely positive map $V \in \mathrm{CP}(M_n, A)$ can be expressed as sum of n maps of the form $\beta \mapsto c^* \beta c$ with columns $c \in M_{n,1}(A)$.
(The $c_k \in M_{n,1}(A)$ are *not* uniquely determined by this property.)

We can rewrite the matrix $G \in M_n(A)_+$ in the notation for elements of $M_n \otimes A$ as $G = \sum_{i,j} e_{ij} \otimes g_{ij} \in M_n \otimes A$. The equations $G^* = G$ and $G^2 = F$ imply that

[11]Matrices $b = [b_{jk}] \in M_n(\mathcal{M}(A))$ with entries $b_{jk} \in \mathbb{C} \cdot 1$ can be written as $b = \beta \otimes 1$ with unique $\beta \in M_n$.

$g_{ij}^* = g_{ji}$ and $\sum_{k=1}^n e_{ij} \otimes g_{ik}g_{kj} = e_{ij} \otimes V(e_{ij})$, i.e.,

$$f_{ij} = V(e_{ij}) = \sum_{k=1}^{n} g_{ki}^* g_{kj} .$$

We define columns $c_1, c_2, \ldots, c_n \in M_{n,1}(A) \cong M_{n,1} \otimes A \subseteq M_n \otimes A$ by

$$c_k := [g_{k,1}, g_{k,2}, \ldots, g_{k,n}]^\top \in M_{n,1}(A) ,$$

i.e., take here the *transposes* of the rows $r_k = [g_{k,1}, g_{k,2}, \ldots, g_{k,n}]$ in $M_{1,n}(A)$ of the $n \times n$-matrix $G \in M_n(A)$ ([12]).

In the tensorial terminology with $a \in A$ identified with $e_{11} \otimes a$ it can be rewritten as $c_k = \sum_j e_{j1} \otimes g_{k,j}$.

It gives that $c_k^*(e_{ij} \otimes 1)c_k = e_{11} \otimes (g_{k,i}^* g_{k,j}) = e_{11} \otimes g_{ik}g_{k,j}$ for $k \in \{1, \ldots, n\}$, and implies $e_{11} \otimes V(e_{ij}) = \sum_{k=1}^n c_k^*(e_{ij} \otimes 1)c_k$ for $i, j \in \{1, \ldots, n\}$.

Notice that the products $c_j^* X c_k$ are in $e_{11} \otimes A$ for all $X \in M_n(A) \cong M_n \otimes A$ if we consider columns c_j as element of $M_{n,1} \otimes A$, i.e., as the element $\sum_j e_{j1} \otimes g_{k,j} \in M_n$. Then $c_k^*\big(e_{ij} \otimes 1_{\mathcal{M}(A)}\big)c_k = e_{11} \otimes g_{i,k}g_{k,j}$ and, therefore,

$$\sum_k c_k^*\big(\beta \otimes 1_{\mathcal{M}(A)}\big)c_k = e_{11} \otimes V(\beta) .$$

Hence $e_{11} \otimes V(\beta) = \sum_{k=1}^n c_k^*(\beta \otimes 1_{\mathcal{M}(A)})c_k$ for $\beta \in M_n$.

If we identify M_n with $M_n(\mathbb{C}1) \subseteq M_n(\mathcal{M}(A)) = \mathcal{M}(M_n(A))$, then $c_k^* e_{ij} c_k = g_{k,i}^* g_{k,j} = g_{i,k} g_{k,j}$. Thus, $\sum_k c_k^* e_{ij} c_k = \sum_k g_{i,k} g_{k,j} = V(e_{ij})$, which implies that $\sum_k c_k^* \beta c_k = V(\beta)$ for all $\beta \in M_n$, i.e., the conventional interpretation says that there are columns $c_1, \ldots, c_n \in M_{n,1}(A)$ that satisfy:

$$V(\beta) = \sum_{k=1}^{n} c_k^* \beta c_k \qquad \text{for all } \beta \in M_n .$$

(ii): We identify M_n with $M_n(\mathbb{C}1) = M_n \otimes 1 \subseteq M_n(\mathcal{M}(A))$, where $1 := 1_{\mathcal{M}(A)}$. By Part (i) there exists columns $c_1, \ldots, c_n \in M_{n,1}(A) \cong M_{n,1} \otimes A$ that satisfy $V(\beta) = \sum_k c_k^* \beta c_k$. The entries of c_k are given by $c_{\ell,1}^{(k)} := g_{k,\ell}$ i.e.,

$$c_k := [g_{k,1}, g_{k,2}, \ldots, g_{k,n}]^\top ,$$

and the $g_{j,k}$ are entries of a positive matrix $G = [g_{j,k}]^2 = [V(e_{jk})]$. It suffices to consider the case where $\|V\| = \|V(1_n)\| = 1$, because in case $V = 0$ nothing is to prove and non-zero V can be replaced by $\|V\|^{-1}V$ if necessary.

We get norm estimates for the $g_{k,\ell}$ ($k, \ell \in \{1, \ldots, n\}$) by

$$\|g_{k,\ell} g_{k,\ell}^*\| = \|g_{k,\ell}^* g_{k,\ell}\| \leq \|c_k^* c_k\| \leq \|V(1_n)\| \leq 1 .$$

Let $h := n^{-2} \sum_{k,\ell} g_{k,\ell} g_{k,\ell}^* \in A_+$. Then $\|h\| \leq 1$, and $g_{k,\ell} g_{k,\ell}^* \leq n^2 \cdot h$ implies that $g_{k,\ell} \in \overline{hA}$ for $k, \ell \in \{1, \ldots, n\}$.

[12] This are not the columns of the adjoint matrix. Even for positive complex 2×2-matrices, e.g. $[\alpha_{j,k}]$ with $\alpha_{j,k} := i^{j-k}$. Only for matrices with self-adjoint entries is this the same as the columns of the adjoint matrix.

Consider the functions $\psi_m(t) := \min\big(1, \max((m+1)t - 1/m, 0)\big)$ for $m \in \mathbb{N}$ and $t \in [0,\infty]$. Then $\psi_m(t) = 0$ for $t \in [0, 1/(m^2+m)]$, $\psi_m(t) = 1$ for all $t \geq 1/m$ and ψ_m is linear in between. Thus, $\lim_{m\to\infty} \|x^*(1-\psi_m(h))x\| = 0$ for all $x \in \overline{hA}$.

In particular, there exists $m \in \mathbb{N}$ with

$$\|(1-\psi_m(h))^{1/2} g_{k,\ell}\|^2 \,<\, \gamma/(1+n^2) \quad \text{for all } \ k,\ell \in \{1,\ldots,n\}\,.$$

Let $\varphi := \psi_m$ and $Q := 1 - \varphi(h) \geq 0$ from now on.

We write $1_n \otimes Q$ for the diagonal matrix in $M_n(\mathcal{M}(A))$ with entries Q in the main diagonal.

The function φ is an increasing non-negative function $\varphi \in \mathrm{C}_0(0,1]$ with $\varphi|[0,\delta] = 0$ for $\delta = (m^2+m)^{-1}$ and φ has the properties that $\|\varphi\| \leq 1$ and

$$\|Q^{1/2} g_{k,\ell}\|^2 = \|(1-\varphi(h))^{1/2} g_{k,\ell}\|^2 \,<\, \gamma/(1+n^2) \qquad \text{for all } \ k,\ell \in \{1,\ldots,n\}\,.$$

Since $\|c_k^*(1_n \otimes Q)c_k\| \leq \sum_\ell \|g_{k,\ell}^* Q g_{k,\ell}\|$ it implies that

$$\|\sum_{k=1}^n c_k^*(1_n \otimes Q)c_k\,\| \,<\, \gamma\,. \tag{1.1}$$

By assumptions in Part (ii), the C^*-algebra A has the property that for every positive contraction $h \in A_+$ and $\varepsilon > 0$ there exist contractions $b_1, b_2 \in A$ with $\|b_j^* b_k - \delta_{j,k} h\| < \varepsilon$ for $j,k \in \{1,2\}$. It causes that Lemma 2.1.12 applies to $h \in A_+$. The equivalences (b,i) $\iff$ (b,ii) and (a,i) $\iff$ (a,ii) in Lemma 2.1.12 imply together immediately that there exist $b_1, b_2, \ldots, b_{2n-1}, b_{2n} \in A$ with

$$b_i^* b_j = \delta_{i,j}\varphi(h) \qquad \text{for all} \quad i,j \in \{1,\ldots,2n\}\,.$$

The diagonal matrices $1_n \otimes b_j \in M_n(A)$ ($j \in \{1,\ldots,2n\}$) and all $\beta \in M_n$ satisfy $(1_n \otimes b_j)^*\beta(1_n \otimes b_k) = \delta_{jk} \cdot \beta \otimes \varphi(h)$ and

$$(1_n \otimes Q)^{1/2}\beta(1_n \otimes Q)^{1/2} = \beta \otimes Q = \beta \otimes (1 - \varphi(h))\,.$$

Thus, $\beta = (1_n \otimes b_j)^*\beta(1_n \otimes b_j) + (1_n \otimes Q)^{1/2}\beta(1_n \otimes Q)^{1/2}$ for $\beta \in M_n$ and $j \in \{1,\ldots,2n\}$. If we conjugate this with the columns $c_k \in M_{n,1}(A)$ then this shows that the completely positive maps $V_{k,j}$ and W_k defined by

$$V_{k,j}\colon\, M_n \ni \beta \mapsto ((1_n \otimes b_j)c_k)^*\beta((1_n \otimes b_j)c_k) \in A$$

and

$$W_k\colon\, M_n \ni \beta \mapsto ((1_n \otimes Q)^{1/2}c_k)^*\beta((1_n \otimes Q)^{1/2}c_k) \in A$$

have the property that $V_{k,j}(\beta) + W_k(\beta) \,=\, c_k^*\beta c_k$ for all $\beta \in M_n$ and $j \in \{1,\ldots,2n\}$

It implies $V_{k,i}(\beta) = V_{k,j}(\beta)$ for all $i,j \in \{1,2,\ldots,2n-1,2n\}$, and the estimate (1.1) implies for the completely positive map $W := \sum_{k=1}^n W_k$ the estimate

$$\|W\| = \|W(1_n)\| = \|\sum_{k=1}^n c_k^*(1_n \otimes Q)c_k\| \,<\, \gamma\,.$$

Calculation shows for $\mu,\nu \in \{1,\ldots,2n\}$ and $i,j,k,\ell \in \{1,\ldots,n\}$ that

$$((1_n \otimes b_\mu)c_k)^*(e_{i,j} \otimes 1)((1_n \otimes b_\nu)c_\ell) \,=\, \delta_{\mu,\nu} \cdot c_k^*(e_{i,j} \otimes \varphi(h))c_\ell\,.$$

We define for $\ell \in \{0, 1\}$ the columns

$$d_\ell := \sum_{k=1}^{n} (1_n \otimes b_{k+n\ell}) c_k = \sum_{k,j}^{n} e_{j,1} \otimes (b_{k+n\ell} g_{k,j}) \in M_{n,1}(A).$$

Above considerations show that the columns d_j, $j \in \{0, 1\}$ satisfy $(\sum_{k=1}^n c_k^* \beta c_k) - d_j^* \beta d_j = \sum_{k=1}^n c_k^* (1_n \otimes Q)^{1/2} \beta (1_n \otimes Q)^{1/2} c_k$, and that the c.b.-norm on the right side is $\leq \gamma \cdot \|\beta\|$.

The orthogonality $(1 \otimes b_j)^* \beta (1 \otimes b_k) = \delta_{jk} \beta \otimes \varphi(h)$ causes that $d_1^* \beta d_0 = 0$ for $\beta \in M_n$.

It follows that d_j $(j \in \{0, 1\})$ are column matrices in $M_{n,1}(A)$ with $d_0^* \beta d_1 = 0$ and $d_0^* \beta d_0 = d_1^* \beta d_1$ such that the c.p. map $U(\alpha) := d_0^* \alpha d_0$ satisfies that $W := V - U$ is c.p. and $\|W\| < \gamma$ and $U(1_n) \leq V(1_n)$.

(iii): It suffices to consider the case where $\|V\| = 1$. Let $[a_{jk}] \in M_n(A)_+$ associated to $V \in \mathrm{CP}(M_n, A)$ by $a_{jk} = V(e_{j,k})$. Denote by $\pi_J \colon D \to A = D/J$ the quotient map. By Parts (o) and (i) it suffices to find in $M_n(D)_+$ a matrix $[d_{jk}]$ with $\pi_J(d_{j,k}) = a_{j,k}$ and $\|d_{1,1} + d_{2,2} + \cdots + d_{n,n}\| \leq 1$.

It is possible to find $[c_{j,k}] \in M_n(D)_+$ with $\pi_J(c_{j,k}) = a_{j,k}$ because the map $M_n(D) \ni [d_{j,k}] \mapsto [\pi_J(d_{j,k})] \in M_n(A)$ is an C^*-algebra epimorphism onto $M_n(A)$.

Then let $e := c_{1,1} + \cdots + c_{n,n}$ and define $f := (1 + (e-1)_+)^{-1/2} \in \mathcal{M}(D)$ and the desired matrix $[d_{j,k}] \in M_n(D)$ by $d_{j,k} := f c_{j,k} f$. The corresponding map $T \colon M_n \to D$ has norm $\|T\| = \|fef\| \leq 1$ and satisfies $\pi_J \circ T = V$.

(iv): It suffices to consider the case where $\|V\| = \|\sum_k b_{k,k}\| = 1$ for completely positive $V \colon M_n \to N$ and the corresponding matrix $B := [b_{jk}] \in M_n(N)_+$ with $b_{j,k} := V(e_{j,k})$. Then $\|B\| = \|[b_{j,k}]\| \leq n$. Let $C = [c_{j,k}] \in M_n(N)_+$ the square root of B. Then $\|C\| \leq n^{1/2}$.

Since $A \subseteq N$ is weakly dense in N, the C^*-subalgebra $M_n(A)$ is weakly dense in the W*-algebra $M_n(N)$. By the Kaplansky density theorem [**616**, thm. 2.3.3], (or [**704**, thm. 1.9.1], [**400**, thm. 5.3.5]) the unit-ball of the C^*-algebra $M_n(A)$ is dense in the unit-ball of $M_n(N)$ with respect to the *-ultra-strong topology (i.e., is $\tau(M_n(N), M_n(N)_*)$-dense in the unit-ball of $M_n(N)$), because $M_n(A)$ is weakly dense (i.e., is $\sigma(M_n(N), M_n(N)_*)$-dense) in $M_n(N)$ if A is weakly dense in N. It follows that there exists a net of elements $\{C_\rho\} \subseteq M_n(A)$ with $\|C_\rho\| \leq n^{1/2}$ that converges to C with respect to the *-ultra-strong topology on $M_n(N)$.

Let $e_\rho := \sum_{j,k} c_{k,j}^* c_{k,j}$ and let $f^{(\rho)} \in M_n(\mathcal{M}(A))_+$ the positive diagonal matrix with entries $(f^{(\rho)})_{j,k} := \delta_{j,k}(1 + (e_\rho - 1)_+)^{-1/2} \in \mathcal{M}(D)$.

The obvious *-ultra-strong continuity of addition, multiplication and involution on bounded parts of $M_n(N)$, cf. [**616**, sec. 2], shows that the directed net $\{D_\rho\}$ defined by

$$D^{(\rho)} := f^{(\rho)} C_\rho^* C_\rho f^{(\rho)} \in M_n(A)_+$$

satisfies that $\|\sum_{k=1}^n D_{k,k}^{(\rho)}\| \leq 1$ and converges to $[b_{jk}]$ in $M_n(N)$ with respect to the *-ultra-strong topology. It implies for the related c.p. maps $V_\rho \in \mathrm{CP}(M_n, A)$

that $\|V_\rho\| \leq 1$ and that the net $\{V_\rho\}$ converges to $V\colon M_n \to N$ with respect to the point–*-ultra-strong topology. □

REMARKS 3.1.10. `The last here mentioned criteria seems to be the relevant issue! But check it again!`

The arguments in the proof of Part(ii) of Proposition 3.1.9 show that we could find moreover elements $b_1, b_2, \ldots, b_{2n-1}, b_{kn} \in A$ with $b_i^* b_j = \delta_{i,j}\varphi(h)$ and the other properties of the $b_j \in A$. Then we obtain from them columns $d_0, d_1, \ldots, d_k \in M_{n,1}$ with $d_j^* \beta d_k = \delta_{j,k} U(\beta)$ with U and $V - U$ c.p. with norm $\|V - U\| < \gamma$.

But the proof of Proposition 3.1.9 shows a bit more:

If A is a C^*-subalgebra of a C^*-algebra B such that for each element $a \in A_+$ and $\varepsilon > 0$ there exists $d_1, d_2 \in B$ with $\|d_i^* d_j - \delta_{i,j} a\| < \varepsilon$, then the columns $d_0, d_1, \ldots, d_k \in B$ can be found with above properties and such that the c.p. contraction $U \in \mathrm{CP}(M_n, B)$ – defined by the d_j via $\delta_{i,j} \cdot U(\beta) = d_i^* \beta d_j$ – is also contained in $\mathrm{CP}(M_n, A)$. It does not require that $\mathcal{M}(A)$ contains any properly infinite elements, and it does not require that all elements in B are properly infinite.

One could suppose that the elements of A are "large" in B in the sense that the multiplier algebra of the hereditary C^*-subalgebra $\overline{AB_\omega A}$ of B_ω has a properly infinite unit.

But the weakened condition on A and B simply is equivalent to the property that A is contained in a stably generated ideal of B_ω or likewise of B_∞.

LEMMA 3.1.11. *Let $V\colon A \to B$ a completely positive map, and $A \subseteq D$ where D is a unital C^*-algebra.*

(i) *If V is factorable, then there is always a factorization $V = T \circ S$ through some matrix algebra M_n such that $S^{**}\colon A^{**} \to M_n$ is unital, and (therefore) $\|T\| = \|T(1)\| = \|V\|$. There exists a u.c.p. map $S_e\colon D \to M_n$ with $S_e|A = S$* [13]*.*

(ii) *If V is nuclear, then V can be approximated in point norm by factorable maps TS where $S\colon A \to M_n$ and $T\colon M_n \to B$ are completely positive maps such that $S^{**}\colon A^{**} \to M_n$ is unital and $\|T\| \leq \|V\|$.*

If A, B and V are unital then S is unital and T can be chosen with $T(1) = 1$.

(iii) *The map $V\colon A \to B$ is nuclear, if and only if, $V\colon A \to B^{**}$ is weakly nuclear – considered as a map into the W^*-algebra B^{**}.*

(iv) *Every completely positive map $V\colon A \to B$ of finite rank can be approximated in operator-norm on $\mathrm{CP}(A, B)$ by factorable maps.*

PROOF. (i): This is evident if $x = S^{**}(1)$ is invertible, because we can replace S with $x^{-1/2} S(\cdot) x^{-1/2}$ and T with $T(x^{1/2}(\cdot)x^{1/2})$. If $S^{**}(1)$ is not invertible then,

[13] The extension S_e is an special case of the Arveson extension theorem [**42**]. Since the identity map of $\mathcal{L}(\ell_2)$ factorizes over $\ell_\infty(M_2, M_3, \ldots)$ this special cases imply conversely the general Arveson extension.

since $0 \leq a \leq 1$ implies $0 \leq S^{**}(a) \leq S^{**}(1)$, S^{**} must take its values in a corner of M_n. Replacing M_n with this corner and adjusting S, T accordingly, we may assume that $S^{**}(1)$ is invertible, and use the above transformations.

Since $T(S^{**})\colon A^{**} \to B \subseteq B^{**}$ is normal, $(TS)^{**} = T(S^{**})$, and thus $\|T(1)\| = \|(TS)^{**}(1)\|$. But $\|TS\| = \|(TS)^{**}\| = \|(TS)^{**}(1)\|$ and $\|T\| = \|T(1)\|$, because T and TS are positive.

It is easy to verify that a map $R\colon D \to M_n$ is completely positive, if and only if, the linear functional $\rho_e\colon D \otimes M_n \to \mathbb{C}$ with $\rho_e(d \otimes \alpha) = (1/n)\mathrm{Tr}(\alpha^\top R(d))$ is positive (Consider R as a map from D into $M_n \subseteq \mathcal{L}(L_2(M_n, (1/n)\mathrm{Tr})) \cong \mathcal{L}(\mathbb{C}^{n^2})$). Since $A \subseteq D$, we have $A^{**} \subseteq D^{**}$ and D is a unital C^*-subalgebra of D^{**}. Let $p \in D^{**}$ denote the unit element of A^{**}. We extend $S\colon A^{**} \to M_n$ to a unital c.p. map S_0 from $C := A^{**} + (1-p)D^{**}(1-p)$ into M_n by $S_0(c) := S^{**}(pcp)$ for $c \in C$.

The n^2-positivity of S_0 implies that the (well-defined) unital linear functional ρ on $C \otimes M_n$ with $\rho(c \otimes \alpha) = (1/n)\mathrm{Tr}(\alpha^\top S_0(c))$ is positive, i.e., is a state on $C \otimes M_n$. By Hahn–Banach extension theorem, it extends to a state ρ_e on $D^{**} \otimes M_n$ with $\rho_e(1 \otimes \alpha) = (1/n)\mathrm{Tr}(\alpha)$. Thus, the unique linear map $S_e\colon D \to M_n$ with $\rho_e(d \otimes \alpha) = (1/n)\mathrm{Tr}(\alpha^\top S_e(d))$ for all $d \in D$ and $\alpha \in M_n$ must be unital and extends S_0.

(ii): Let $\widetilde{A}$ the unitization of A. V is the point norm limit of completely positive maps $V_e|A$ where $V_e\colon \widetilde{A} \to B$ is defined by $V_e(a + \gamma 1) := V(eae + \gamma e^2)$ for $e \in A_+$, $\|e\| \leq 1$ (Take an approximate unit $\{e\}$ of A). V_e is completely positive and $\|V_e\| \leq \|V\|$. Every point-norm approximation of V by a net of factorable maps $W_\lambda = T_\lambda S_\lambda$ defines a point-norm approximation of V_e by the factorable maps $(W_\lambda)_e$. Thus it suffices to consider the case where A is unital. But then the approximation satisfies $\|V(1) - W_\lambda(1)\| \to 0$. Thus, eventually we find a small perturbation of the T_λ with the desired properties.

(iii): By Remark 3.1.2(o), the set $\mathrm{CP}_f(A, B)$ of factorable c.p. maps $W\colon A \to B$ is an operator-convex cone in $\mathrm{CP}(A, B)$. It follows that $V\colon A \to B$ is in the point-norm closure of $\mathrm{CP}_f(A, B)$ if and only if V is in the point-$\sigma(B, B^*)$ closure of $\mathrm{CP}_f(A, B)$.

Every factorable c.p. map $U = T \circ S\colon A \to B^{**}$, $T\colon M_n \to B^{**}$, $S\colon A \to M_n$ with $S^{**}(1) = 1$ and $\|T\| = \|T(1)\| = \|U\|$ is the point–*-strong limit of a net of maps $U_\mu = T_\mu \circ S$ by part (i) and Proposition 3.1.9(iv), where $T_\mu\colon M_n \to B$ is a contraction with $\|T_\mu\| \leq \|T\|$.

Thus, $V\colon A \to B$ is in the point-$\sigma(B, B^*)$ closure of $\mathrm{CP}_f(A, B)$, if and only if, V is in the point-$\sigma(B^{**}, B^*)$ closure of $\mathrm{CP}_f(A, B^{**})$.

The latter means that $V\colon A \to B \subseteq B^{**}$ is a weakly nuclear map from A into B^{**}.
(An alternative proof can be given by means of Remark 3.1.2(i) and its counterpart for weakly nuclear maps.)

(iv): We consider B as a C^*-subalgebra of $\mathcal{L}(\mathcal{H}_1)$ for some Hilbert space $\mathcal{H}_1$. Since $V\colon A \to B$ has finite rank, there are functionals $\rho_1, \ldots, \rho_n \in A^*$ and operators $b_1, \ldots, b_n \in B$ with $V(a) = \sum_k \rho_k(a) b_k$. By Lemma 3.1.4 there exists a cyclic representation $d\colon A \to \mathcal{L}(\mathcal{H}_0)$, with cyclic vector $\xi \in \mathcal{H}_0$ of norm $\|\xi\| = 1$, and operators $t_k \in d(A)'$ with $\rho_k(a) = \langle d(a)\xi, t_k\xi\rangle$ for $a \in A$.

Let $C := d(A)'$, $T := \sum_k t_k^* \otimes b_k \in C \odot B \subseteq C \otimes B$ and let λ_a denote the linear functional on C given by $\lambda_a(c) := \langle c d(a)\xi, \xi\rangle$ for $a \in A$ and $c \in C$. It has norm $\|\lambda_a\| \leq \|a\|$. Then $V(a) = (\lambda_a \otimes \mathrm{id})(T)$ for $a \in A$. Since the algebraic tensor product $d(A)\xi \odot \mathcal{H}_1$ is dense in the Hilbert-space tensor product $\mathcal{H}_0 \otimes \mathcal{H}_1$, a straight calculation shows that the complete positivity of V implies that T is positive in $\mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_1)$. It follows that T is also positive in $C \otimes B$ (by spectral permanence on the class of C^*-algebras).

Let $\varepsilon > 0$, then there is $S \in C \odot B$ with $\|T - S^*S\| < \varepsilon$ in $C \otimes B$. Let $W(a) := (\lambda_a \otimes \mathrm{id})(S^*S)$ then, for all $a \in A$,

$$\|W(a) - V(a)\| = \|(\lambda_a \otimes \mathrm{id})(T - S^*S)\| \leq \|\lambda_a\| \|T - S^*S\| < \varepsilon \|a\| \,,$$

i.e., $\|W - V\| < \varepsilon$ in $\mathcal{L}(A, B)$. Since $d\colon A \to \mathcal{L}(\mathcal{H}_0)$ has a cyclic vector ξ, the map $\gamma\colon C = d(A)' \to \mathcal{H}_0$ given by $\gamma\colon c \mapsto c\xi$ is injective, and we can write $S = \sum_j c_j \otimes d_j$ ($j = 1, \ldots, m$) with $\langle \gamma(c_i), \gamma(c_j)\rangle = \delta_{i,j}$. Let $u_n\colon \mathbb{C}^m \to \mathcal{H}_0$ the isometry with $u_n(e_j) = \gamma(c_j)$, $W_1\colon A \to M_m$ the c.p. contraction $W_1(a) := u_n^* d(a) u_n \in \mathcal{L}(\mathbb{C}^m)$, and $D \in M_{1,m}(B)$ the row matrix $D = [d_1, \ldots, d_m]$. We define $W_2\colon M_m \to B$ by $W_2(\alpha) := D\alpha D^*$, where M_m is considered as subalgebra of $M_m(\mathcal{M}(B)) \cong \mathcal{M}(M_m(B))$. Straight calculation shows that $W = W_2 \circ W_1$. Thus, V can be approximated in operator-norm by factorable maps. □

`Compare below Proof with that of Part(iv) above! Take the better one!  Notations have then to be adapted/changed.`

Details to Part (iii,f) of Remarks 3.1.2, see also Lemma 3.1.11(iv): Consider $V(a) := \sum_j f_j(a) b_j \in B \subseteq \mathcal{L}(\mathcal{H})$ and find $\varphi \in A_+^*$, $T_j \in D_\varphi(A)'$, such that $f_j(a) = \langle D_\varphi(a) x_0, T_j x_0\rangle$, cf. Lemma 3.1.4. Then $S := \sum T_j \otimes b_j \geq 0$ in the C^*-algebra $D_\varphi(A)' \otimes B$. Thus, $S^{1/2}$ can be approximated in operator norm by elements d_n in the algebraic tensor product $D_\varphi(A)' \odot B$. The maps

$$a \mapsto (\varphi \otimes \mathrm{id}_B)\big(d_n^*(D_\varphi(a) \otimes 1) d_n\big)$$

are factorable c.p. maps from A into B that approximate V with respect to the operator norm of $\mathcal{L}(A, B)$. □

We denote by $\mathrm{socle}(A)$ the algebraic ideal of A generated by the projections $p \in A$ with $pAp = \mathbb{C}p$. See [**210**], [**388**, p. 64] and [**661**, pp. 46,261–267] for notation and properties.

Compare

LEMMA 3.1.12. *Suppose that* $W : A \to M_m$ *is a completely positive map with* $W^{**}(1) = 1_m$, *and that* Γ *is a set of pure states on* A, *such that the irreducible representations* $\rho_j : A \to L_2(A, \mu_j)$ *corresponding to* $\mu_j \in \Gamma$ *are pairwise inequivalent and build a separating family for* A. *Then:*

(i) *For every compact subset* Ω *of* A *and every* $\varepsilon > 0$, *there exist pure states* $\mu_1, \dots \mu_k \in \Gamma$, $k, m_1, \dots, m_k \in \mathbb{N}$, *isometries* $I_j : \mathbb{C}^{m_j} \to L_2(A, \mu_j)$, *and a unital completely positive map* $T : F := M_{m_1} \oplus \cdots \oplus M_{m_k} \to M_m$, *such that*

$$\|T \circ S(a) - W(a)\| \le \varepsilon \qquad \forall \;\; a \in \Omega\,,$$

where $S : A \to M_{m_1} \oplus \cdots \oplus M_{m_k}$ *denotes the completely positive contraction* $S(a) := I_1^* \rho_1(a) I_1 \oplus \cdots \oplus I_k^* \rho_k(a) I_k$ *for* $a \in A$.

(ii) *If, in addition,* $W(\mathrm{socle}(A)) = 0$, *then one can manage that* $m_1, \dots, m_k \le m$, *and that* $T(f) = \sum_j d_j^* f_j d_j$ *for* $f = f_1 \oplus \cdots \oplus f_k \in F$, *for suitable* $d_j \in M_{m_j, m}$, *with* $\sum d_j^* d_j = 1_m$.

(iii) *If, in addition,* $W(\mathrm{socle}(A)) = 0$ *and the irreducible representations* d_μ *with* $\mu \in \Gamma$ *and* $d_\mu(\mathrm{socle}(A)) = 0$ *build a separating family for* $A/\overline{\mathrm{socle}(A)}$, *then in Part(i) the* $\mu_1, \dots, \mu_k \in \Gamma$ *can be taken such that* $\mu_j(\mathrm{socle}(A)) = 0$ *and* $S(\mathrm{socle}(A)) = 0$.

Note that S satisfies $S^{**}(1) = 1$. Part (iii) does *not* say that F, T and S can be found such that the properties of μ_j, I_j and T in (i), (ii) and (iii) hold at the same time (but this can be managed if $\mathrm{socle}(A) = 0$ or, at least, if $\mathrm{socle}(A/\overline{\mathrm{socle}(A)}) = 0$ and $W(\mathrm{socle}(A)) = 0$).

PROOF. (i): Let $H := \bigoplus_{\mu \in \Gamma} L_2(A, \mu)$ and $\rho : A \to \mathcal{L}(H)$ be the direct sum of irreducible representations which are defined by pure states in Γ. Then ρ is faithful. We denote by $R : A \to \mathcal{L}(\ell_2(H))$ the infinite repeat of ρ (i.e., $R(a) := \rho(a) \oplus \rho(a) \oplus \cdots$). Then $R(A) \cap \mathbb{K}(H) = 0$, and, by Lemma 2.1.22, there exists an isometry $v : \mathbb{C}^m \to H$ with $\|W(a) - v^* R(a) v\| < \varepsilon/3$ for $a \in \Omega$.

By definition of R, a small perturbation of v yields $q \in \mathbb{N}$, $\mu_1, \dots, \mu_q \in \Gamma$ (not necessarily different) and an isometry $t : \mathbb{C}^m \to L_2(A, \mu_1) \oplus \cdots \oplus L_2(A, \mu_q)$ with $\| v^* R(a) v - t^*(\rho_1(a) \oplus \cdots \oplus \rho_q(a)) t \| < \varepsilon/3$ for $a \in \Omega$, where $\rho_j : A \to L_2(A, \mu_j)$ are the representations corresponding to μ_j. We can arrange that $\mu_1, \dots, \mu_k$ are different and $\{\mu_{k+1}, \dots, \mu_q\} \subseteq \{\mu_1, \dots, \mu_k\}$ if $k \in \mathbb{N}$ denotes the maximal number of different elements of $\{\mu_1, \dots, \mu_q\}$. There are contractions $s_j : \mathbb{C}^m \to L_2(A, \mu_j)$ $(j = 1, \dots, q)$ such that $t(x) = s_1(x) \oplus \cdots \oplus s_q(x)$ for $x \in \mathbb{C}^n$, $\sum s_j^* s_j = 1_m$, and

$$\| v^* R(a) v - \sum_{j=1}^{q} s_j^* \rho_j(a) s_j \| < \varepsilon/3 \qquad \forall \;\; a \in \Omega\,.$$

For $1 \le j \le k$, we denote by L_j the linear subspace of $L_2(A, \mu_j)$ generated by the image of s_j and the images $s_i(\mathbb{C}^m)$ for $i \in \{k+1, \dots, q\}$ with $\mu_i = \mu_j$. Let $m_j := \mathrm{Dim}(L_j)$ and $I_j : \mathbb{C}^{m_j} \to L_j \subseteq L_2(A, \mu_j)$ an isometry $(j = 1, \dots, 0)$. Let $1 \le j \le k$ and $X(j) := \{1 \le i \le q\,;\; \mu_i = \mu_j\}$. We define $T_j : M_{m_j} \to M_m$ by

$T_j(g) := \sum_{i\in X(j)} s_i^* I_j g I_j^* s_i$ for $g \in M_{m_j}$ and $1 \leq j \leq k$, and then define $S\colon A \to F := M_{m_1} \oplus \cdots \oplus M_{m_k}$ and $T\colon F \to M_m$ by $S(a) := I_1^*\rho_1(a)I_1 \oplus \cdots \oplus I_k^*\rho_k(a)I_k$ respectively $T(f_1,\ldots,f_k) := T_1(f_1) + \cdots + T_k(f_k)$ for $a \in A$ and $f_j \in M_{m_j}$. Straight calculation shows that $\sum_{j=1}^q s_j^*\rho_j(a)s_j = T(S(a))$ for all $a \in A$.

(ii): Let H and $\rho\colon A \to \mathcal{L}(H)$ as above. Then $J := \rho^{-1}(\rho(A) \cap \mathbb{K}(H))$ is the closed ideal generated by the projections $p \in A$ with pAp of finite dimension, i.e., J is the closure of socle(A). Since $W(\text{socle}(A)) = 0$ by assumption, we get $W(J) = 0$. Now we can repeat the arguments for part (i) with ρ in place of R. Then, $k = q$ and $\text{Dim}(L_j) \leq m$ for $1 \leq j \leq k$. It follows that the map $t(x) := s_1(x) \oplus \cdots \oplus s_k(x)$ is an isometry from $\mathbb{C}^m$ into $I(\mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_k})$ and that $T_j(g) = s_j^* I_j g I_j^* s_j$ for all $g \in M_{m_j}$. Let $d_j := I_j^* s_j \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^{m_j}) \cong M_{m_j,m}$. Let $p_j\colon \mathbb{C}^n \to \mathbb{C}^{m_j}$ denote the orthogonal projection onto $\mathbb{C}^{m_j} \subseteq \mathbb{C}^n$. Then $T(f) = \sum d_j^* f_j d_j$ for $f := f_1 \oplus \cdots \oplus f_k \in F$, in particular, $\sum d_j^* d_j = 1_m$.

(iii): Apply part (i) to $A/\overline{\text{socle}(A)}$ and $[W]\colon A/\overline{\text{socle}(A)} \to M_m$. □

Lemma 3.1.13. *Let A a C^*-algebra and $W\colon A \to M_m$ a completely positive contraction with $W^{**}(1) = 1$, $\Omega \subseteq A_+$ a norm-compact subset, $f_0(t) := t$ ($t \in [0,1]$) and let $\varepsilon > 0$.*

- (I) *There exist*
  - (i) *a finite-dimensional C^*-algebra $F := M_{n_1} \oplus \cdots \oplus M_{n_k}$*
  - (ii) *a C^*-morphism $\psi\colon \mathrm{C}_0((0,1], F) \to A$,*
  - (iii) *a completely positive contraction $S\colon A \to F$ with $S(\psi(f)) = f(1)$ for all $f \in \mathrm{C}_0((0,1], F)$, and*
  - (iv) *a unital completely positive map $T\colon F \to M_m$,*

  such, that for $g := \psi(f_0 \otimes 1)$ and each $a \in \Omega$,

$$\|T \circ S(a) - W(a)\| < \varepsilon \quad \text{and} \quad \limsup_n \|g^{n+1} a g^{n+1} - g^n \psi\big(f_0^2 \otimes S(a)\big) g^n\| < \varepsilon .$$

- (II) *If $W(\text{socle}(A)) = \{0\}$ then F, ψ, S and T can be found such that, moreover, for every $\eta \in (0,1)$ the hereditary algebras*

$$(f_j - (1-\eta))_+ \cdot A \cdot (f_j - (1-\eta))_+$$

  contain n_j^2 non-zero pairwise orthogonal positive elements for $j = 1,\ldots,k$ (where $f_j := \psi(f_0 \otimes p_{11}^{(j)})$).
- (III) *If $W(\text{socle}(A)) = \{0\}$ then one can find F, ψ, S and T such that $n_j \leq m$ and that $S\colon F \to M_n$ is given by $S(f_1,\ldots,f_n) = \sum_j s_j^* f_j s_j$ for suitable $s_j \in M_{m_j,m}$.*

Proof. Apply Lemma 2.1.15. `To be filled in ??` □

Question 3.1.14.

Let $A \subseteq B$ a separable C^*-subalgebra of a unital C^*-algebra B.

Under what conditions is every nuclear c.p. map $T\colon A \to B$ approximately 1-step inner inside B?

I.e., when does there exists for each c.p. contractions $T \in \mathrm{CP}(M_n, B)$ and $S \in \mathrm{CP}(A, M_n)$ a sequence $s_1, s_2, \ldots$ of contractions in B such that $T \circ S(a) = \lim_{k\to\infty} s_k^* a s_k$ for all $a \in A$?

A *necessary* condition (for the case $n = 1$) is that for each nonzero $a \in A_+$ with $\|a\| = 1$ and every $c \in B_+$ there exists $d \in B$ – depending on c – with $d^*ad = (c - 1/2)_+$.

This implies that A is "relatively to B simple" (or $A = \{0\}$) and that the right ideal $R := AB$ satisfies $R^*R = B$.

Is this also sufficient?

Compare the result of Elliott and Kucerovsky [**264**] and its improvements in [**309**] in the case where B is a corona algebra $\mathcal{M}(C)/C$.

Following proposition is a non-commutative version of the Tietze extension theorem.

PROPOSITION 3.1.15. *Let A a σ-unital C^*-algebra and $J \subseteq A$ a closed ideal. Then the natural unital C^*-morphism $\mathcal{M}(\pi_J)\colon \mathcal{M}(A) \to \mathcal{M}(A/J)$ has kernel $\mathcal{M}(A, J) := \{T \in \mathcal{M}(A)\, ;\; AT \cup TA \subseteq J\}$.*

This natural C^*-morphism is surjective if J is σ-unital.

PROOF. Let $h\colon A \to \mathcal{M}(B)$, – e.g. for $B := A/J$ –, a C^*-morphism with the property that the hereditary C^*-subalgebra $h(A)Bh(A)$ of B contains an approximate unit of B.

This is equivalent to $B = \overline{h(A)B}$. Thus, there is a *unique unital* W*-morphism $H\colon A^{**} \to B^{**}$ that maps $\mathcal{M}(A) \subseteq A^{**}$ into $\mathcal{M}(B) \subseteq B^{**}$ and satisfies $H(a)b = h(a)b$ for all $a \in A$ and $b \in B$. The kernel of $H|\mathcal{M}(A)$ is easily seen as the operators $T \in \mathcal{M}(A)$ with $TA \cup T^*A \subseteq J$, where J is here the kernel of $h\colon A \to \mathcal{M}(B)$.

This covers the special case $B := A/J$ and $h := \pi_J$, where here simply $H = h^{**}$.

To get surjectivity for $\mathcal{M}(h) = h^{**}|\mathcal{M}(A)$ one can reduce the situation to suitable separable C^*-subalgebras if J and $B = A/J$ are σ-unital: We can find for given positive $T \in \mathcal{M}(B)$ a separable C^*-subalgebra $C \subseteq B$ that contains a strictly positive element f of B and $1_{\mathcal{M}(B)}, T \in \mathcal{M}(C) \subseteq \mathcal{M}(B)$. Then we can find a separable C^*-subalgebra D of A that contains a strictly positive element of J and satisfies $\pi_J(D) = C$. Thus, D contains a strictly positive element of A, satisfies $\mathcal{M}(D) \subseteq \mathcal{M}(A)$, and $\mathcal{M}(D)$ contains the unit element of $\mathcal{M}(A)$. Notice that under this conditions on D and C we get that

$$\mathcal{M}(\pi_J)|\mathcal{M}(D) = \mathcal{M}(\pi_{J\cap D})\colon \mathcal{M}(D) \to \mathcal{M}(C) \subseteq \mathcal{M}(B)\,.$$

This reduces in the case where J and $B = A/J$ are σ-unital the question on the surjectivity of $\mathcal{M}(\pi_J)$ to the case where B is separable. There the answer is affirmative, cf. [**616**, prop. 3.12.10]. □

Proposition 3.1.15 could be used to prove the following proposition:

PROPOSITION 3.1.16. *If A is a σ-unital C^*-algebra, then* $\mathrm{Q}(A) := \mathcal{M}(A)/A$ *is* σ-sub-Stonean, *i.e., for each separable C^*-subalgebra $B \subset \mathrm{Q}(A)$ and each $c, d \in (B' \cap \mathrm{Q}(A))_+$ with $cd = 0$ there exist contractions $e, f \in (B' \cap \mathrm{Q}(A))_+$ with $ef = 0$, $ec = c$ and $fd = d$.*

I.e., $\mathrm{Q}(A)$ *is σ-sub-Stonean in the sense of* [**448**, def. 1.4].

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Compare Chapter 5 for more general study
and compare with the -- possibly stronger -- Kasparov's technical
theorem:
```

$\mathcal{M}(A\otimes\mathbb{K})/(A\otimes\mathbb{K}) = \mathrm{Q}^{\mathrm{s}}(A)$ satisfies Kasparov's technical theorem. It implies that $\mathrm{Q}^{\mathrm{s}}(A)$ and $\mathrm{Q}(A) \cong P\,\mathrm{Q}^{\mathrm{s}}(A)P$ are sub-Stonean. Here $P := \pi_{A\otimes\mathbb{K}}(1_{\mathcal{M}(A)}\otimes p_{11})$.

Check below proof again. Compare this with Chapter 5.

PROOF. We can suppose that $1 \in B$ and $\|c-d\| = 1$ without loss of generality.

Let $g^* = g \in \mathcal{M}(A)$ with $\pi_A(g) = c-d$ and $\|g\| = 1$, and let $a_0 \in A_+$ a strictly positive contraction in A. We find a separable unital C^*-subalgebra $D \subset \mathcal{M}(A)$ with $\pi_A(D) = B$. Consider the separable unital C^*-subalgebra $E := C^*(a_0, g, D)$ of $\mathcal{M}(A)$ that is generated by $a_0, g \in A$ and the C^*-subalgebra $D \subseteq E$.

Take dense sequence $e_1, e_2, \ldots$ in the unit ball of E containing $e_1 := g_+$ and $e_2 := g_-$ in the first places.

Build from a_0 via functional calculus with increasing functions $f_1, f_2, \ldots$ in $\mathrm{C}_0(0,1]_+$ (with properties $\|f_n\| = 1$, $f_{n+1}f_n = f_n$ and $f_n([0,\alpha_n]) = \{0\}$ for some suitable zero sequence $\alpha_n > \alpha_{n+1} > 0$) a sequence of positive contractions $a_n = f_n(a_0) \in A_+$ that have the property

$$\|a_ne_k - e_ka_n\| + \|a_n^{1/2}e_k - e_ka_n^{1/2}\| \le 8^{-n}\,.$$

Parallel selection of $\|[g_+^{1/k_n}, e_k]\|$ and its commutator norms is needed:

We know that $[g_+^{1/k_n}, e_k] \in A$. The next k_{n+1} can be found after the $[(a_{n+1} - a_n)^{1/2}, e_k]$ and ??? becomes small enough and ????

Then we define a map V on $T \in \mathcal{M}(A)$ by

$$V(T) := a_1^{1/2}Ta_1^{1/2} + \sum_{n=1}^{\infty}(a_{n+1}-a_n)^{1/2}T(a_{n+1}-a_n)^{1/2}\,.$$

The right side is convergent with respect to the strict convergence in $\mathcal{M}(A)$. Then V is a unital completely positive unital map from $\mathcal{M}(A)$ into $\mathcal{M}(A)$.

It has the property that $V(e_n) - e_n \in A$ for all $n \in \mathbb{N}$. Thus $V(x) - x \in A$ for all $x \in E \subseteq \mathcal{M}(A)$.

We define a positive contraction $Y \in \mathcal{M}(A)_+$ by

$$Y := \sum_{n=1}^{\infty} := a_1^{1/2} g_+ a_1^{1/2} + \sum_{n=1}^{\infty} (a_{n+1} - a_n)^{1/2} g_+^{1/n} (a_{n+1} - a_n)^{1/2} .$$

`Here could be a problem with` $Yx - xY \in A$

`It could be that one has to select some of the` a_{k_n} `and` g_+^{1/ℓ_n} `more careful !!  :`

`We have` $[g_+^{1/n}, e_k] \in A$ `for all` $n, k \in \mathbb{N}$`, but need further control of the selection!!`

`The` n `has to go slowly enough to` ∞`??`

The positive contraction Y has the properties $Yg_+ - Y \in A$, $Yg_- \in A$ and $Yx - xY \in A$ for all $x \in E = C^*(a_0, g, D)$.

Thus, $e := \pi_A(Y) \in \mathrm{Q}(A)$ is a positive contraction with $ec = e$, $ed = 0$ and $e \in B' \cap \mathrm{Q}(A)$.

Now we can repeat the arguments, but starting now with e in place of former d and with d in place of former c and get a positive contraction $f \in B' \cap \mathrm{Q}(A)$ with $fe = 0$ and $fd = d$.

□

`Remarks/Conjectures:`

It is known that A_ω is σ-sub-Stonean for every non-zero C^*-algebra A.

Is A_∞ σ-sub-Stonean for non-zero A?

Suppose we could prove the following (1) and (2) then $\mathrm{Q}^s(A)$ is σ-sub-Stonean for each non-zero σ-unital C^*-algebra :

(1) A_∞ is σ-sub-Stonean.

(2) If A is σ-unital and stable, then, for every separable C^*-subalgebra B of $\mathrm{Q}(A)$ there exist a c.p. contraction $V \colon \mathrm{Q}(A) \to A_\infty$ that is orthogonality preserving on B and a c.p. contraction $W \colon A_\infty \to \mathrm{Q}(A)$ that is multiplicative on $C^*(V(B))$ such that $W \circ V(b) = b$ for all $b \in B$.

Indeed: Suppose that A is σ-unital and stable, $C \subset \mathrm{Q}(A)$ separable, and that $e, f \in C' \cap \mathrm{Q}(A)$ are positive contractions with $ef = 0$.

Suppose (2) is valid:

Let $B := C^*(C \cup \{e, f\})$, $V \colon \mathrm{Q}(A) \to A_\infty$ a c.p. contraction that is orthogonality preserving on B, and let $W \colon A_\infty \to \mathrm{Q}(A)$ that is multiplicative on $C^*(V(B))$ such that $W \circ V(b) = b$ for all $b \in B$.

Since V is orthogonality preserving, we get that $V(e) \cdot V(f) = 0$.

Perhaps V preserves commutativity? Leads to question (Let $T \colon \mathrm{C}_0(X) \to D$ an orthogonality preserving c.p. contraction with D a C^*-algebra, X a closed subset of $\mathbb{R}^2$, and T is "completely" isometric. Is $C^*(T(\mathrm{C}_0(X)))$ a commutative

C^*-algebra? We can not expect that T is multiplicative because $f \mapsto f \otimes g$ is c.i. and c.p. if $\|g\| = 1$ and $g \geq 0$.)

Now suppose that (2) implies that V respects commutativity and is orthogonality preserving.

Then $V(e)V(f) = 0$ and $V(e), V(f)$ commute with all elements from $C^*(V(B))$. (If comm. is respected!)

Now, if (1) is valid, then there exist $g, h \in C^*(V(B))' \cap A_\infty$ with $g \cdot h = 0$, $gV(f) = 0$ $gV(e) = V(e)$ and $hV(f) = V(f)$.

If we apply $W\colon A_\infty \to \mathrm{Q}(A)$, then we get that $W(g)e = e$, $W(g)f = 0$ $W(g)b = bW(g)$ for all $b \in C$, and $W(g)f = 0$. This implies then finally that $\mathrm{Q}(A)$ is σ-sub-Stonean.

2. On approximate inner nuclear maps

The following Lemma 3.2.1 gives a necessary and sufficient criteria for the containment $\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$ for point-norm closed subsets $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ of the c.p. maps from a C^*-algebra A into a C^*-algebra B.

Lemma 3.2.1. *Let $\mathcal{C}$ a point-norm closed* subset *of* $\mathrm{CP}(A, B)$ *with the property that $S(a^*(\cdot)a) \in \mathcal{C}$ for $S \in \mathcal{C}$ and $a \in A$, and denote by $f_0 \in \mathrm{C}_0(0, 1]$ the generator $f_0(t) := t$.*

Then the following are equivalent:

(i) *The set $\mathcal{C}$ contains all nuclear c.p. maps $U\colon A \to B$.*
(ii) *For every C^*-morphism $\psi\colon \mathrm{C}_0((0, 1], F) \to A$ of the cone $\mathrm{C}_0((0, 1], F)$ over a finite-dimensional C^*-algebra F with the property $\|\psi(f_0 \otimes p)\| = 1$ for all non-zero projections $p \in F$ and for every completely positive map $V\colon F \to B$, there exists a sequence $S_n \in \mathcal{C}$ with $\lim_n S_n(\psi(f)) = V(f(1))$ for all $f \in \mathrm{C}_0((0, 1], F)$.*

Proof. The condition on $\mathcal{C}$ implies that $t\mathcal{C} \subseteq \mathcal{C}$ for $t \in \mathbb{R}_+$ (use an approximate unit of A). Thus, it suffices to show that every factorable contraction U from A into B can be approximated in point-norm topology by elements of $\mathcal{C}$. Suppose that $U = V \circ W$ with c.p. contractions $W\colon A \to M_m$ and $V\colon M_m \to B$ with $W^{**}(1) = 1$, cf. Lemma 3.1.11(i).

Let $\varepsilon > 0$ and $\Omega \subseteq A$ a compact subset. We find F, $\psi\colon \mathrm{C}_0((0, 1], F) \to A$, $T\colon F \to M_m$, and $S\colon A \to F$ as in Lemma 3.1.13(I). Then $1 \geq \|\psi(f_0 \otimes p)\| \geq \|p\|$, because $p = T(\psi(f_0 \otimes p))$. Thus, by assumptions, there is a sequence $S_n \in \mathcal{C}$ with $\lim_n S_n(\psi(f)) = V \circ T(f(1))$ for all $f \in \mathrm{C}_0((0, 1], F)$. Let $G := \psi(\mathrm{C}_0((0, 1], F)) \subseteq A$ and let D denote the hereditary C^*-subalgebra of A that is generated by G. Lemma 3.1.8 applies to the set $\mathcal{C}|G$, because ψ is a C^*-morphism and $V(a^*(\cdot)a) \in \mathcal{C}$ for $V \in \mathcal{C}$ and $a \in A$. Hence, we can select the sequence $S_n \in \mathcal{C}$ such that, in addition, $\|S_n|G\| \leq \|V \circ T\| \leq 1$. But this implies that $\|S_n|D\| \leq 1$, because G contains an approximate unit of D. It follows $\lim_n S_n(g^k\psi(f_0\otimes S(a))g^k) = V(T(S(a)))$ and

$$\inf_{n,k} \|S_n(g^{k+1}ag^{k+1}) - V\circ W(a)\| < 3\varepsilon \qquad \forall\ \ a\in\Omega\,.$$

Conversely, suppose that $\mathrm{CP}_{\mathrm{nuc}}(A,B)\subseteq\mathcal{C}$. Let F a finite-dimensional algebra and $\psi\colon \mathrm{C}_0((0,1],F)\to A$ a *-morphism with $\|\psi(f_0\otimes p)\| = 1$ for all non-zero projections $p\in F$. The image $\psi\big(\mathrm{C}_0((0,1],F)\big)$ is contained in the normalizer algebra $\mathcal{N}(D)\subseteq A$ of the hereditary C^*-subalgebra $D\subseteq A$ that is generated by $\psi(f_0(1-f_0)\otimes 1)$. The condition $\|\psi(f_0\otimes p)\| = 1$ for non-zero projections $p\in F$ is equivalent to the injectivity of $[\pi_D\circ\psi]\colon F\to\mathcal{N}(D)/D$ where $\pi_D\colon\mathcal{N}(D)\to\mathcal{N}(D)/D$ is the quotient map. Since F is an injective algebra, we can use the Arveson extension to obtain completely positive contractions $W_1\colon\mathcal{N}(D)\to F$ with

???????

and then $W\colon A\to F$ with $W(\psi(f)) = W_1(\psi(f)) = f(1)$ for all $f\in\mathrm{C}_0((0,1],F)$. If $V\colon F\to B$ is completely positive, then $S_n := V\circ W$ is nuclear and is contained in $\mathcal{C}$ by assumption. □

Definition 3.2.2. Let A and B C^*-algebras and $\mathrm{CP}(A,B)$ the cone of completely positive maps from A into B.

A subset $\mathcal{C}$ of $\mathrm{CP}(A,B)$ is a **matrix operator-convex cone** of c.p. maps, if $\mathcal{C}$ has the following properties (i) and (ii):

(OC1) $d_1^*V_1(.)d_1 + d_2^*V_2(.)d_2\in\mathcal{C}$ for $V_1,V_2\in\mathcal{C}$, $d_1,d_2\in B$ ([14])

(OC2) The map $a\in A\mapsto c^*\big(V\otimes\mathrm{id}_n(r^*ar)\big)c$ is in $\mathcal{C}$ for every $V\in\mathcal{C}$, every row-matrix $r\in M_{1,n}(A)$ and every column-matrix $c\in M_{n,1}(B)$.
(I.e., for every $V\in\mathcal{C}$, $n\in\mathbb{N}$, $r_1,\ldots,r_n\in A$ and $c_1,\ldots,c_n\in B$, the completely positive map $W(a) := \sum_{ij}c_i^*V(r_i^*ar_j)c_j$ is again in $\mathcal{C}$.)

We abridge "matrix operator convex cone" as "**m.o.c. cone**" .

The m.o.c. cone $\mathcal{C}$ **closed** if $\mathcal{C}\subseteq\mathrm{CP}(A,B)\subseteq\mathcal{L}(A,B)$ is closed with respect to topology of point-norm convergence on $\mathcal{L}(A,B)$ (= strong operator topology on $\mathcal{L}(A,B)$).

An m.o.c. cone $\mathcal{C}$ is **non-degenerate** if, for every $b\in B_+$ and $\varepsilon>0$, there are $V\in\mathcal{C}$ and $a\in A_+$ with $\|V(a)-b\| < \varepsilon$.

The cone $\mathcal{C}$ is **faithful** if $a\in A_+$ and $V(a)=0$ for all $V\in\mathcal{C}$ implies $a=0$.

Let $\mathcal{S}$ be a subset of $\mathrm{CP}(A,B)$. We denote by $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ the smallest subset of $\mathrm{CP}(A,B)$ that is invariant under the operationsin (OC1) and (OC2), and by $\mathcal{C}(\mathcal{S})$ the point-norm closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ (i.e., the closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ in $\mathcal{L}(A,B)$ w.r.t. the strong operator topology). Then $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ and $\mathcal{C}(\mathcal{S})$ are operator-convex cones of completely positive maps.

We say that $\mathcal{S}$ **generates** a *closed* m.o.c. cone $\mathcal{C}$ if $\mathcal{C}=\mathcal{C}(\mathcal{S})$.

[14] Thus $\sum b_i^*V_i(.)b_i\in\mathcal{C}$ if $V_1,\ldots,V_n\in\mathcal{C}$ and $b_1,\ldots,b_n\in B$.

One can see, with help of approximate units in A or B, that every point-norm closed subset $\mathcal{V}$ of of $\mathrm{CP}(A,B)$ is a convex cone in the usual sense if it satisfies (OC1), and that $\mathcal{V}$ satisfies $t\mathcal{V} \subseteq \mathcal{V}$ for $t \in [0,\infty)$ if $\mathcal{V}$ satisfies (OC2).

REMARK 3.2.3. Let $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ any set of c.p. maps. We define the **(OC2)–hull** $\langle \mathcal{S} \rangle_2$ of $\mathcal{S}$ as smallest set $\langle \mathcal{S} \rangle_2 \subset \mathrm{CP}(A,B)$ of c.p. maps containing $\mathcal{S}$ and being invariant under the operations in (OC2) of Definition 3.2.2.

The point norm-closure $\mathcal{C}_2 := \overline{\langle \mathcal{S} \rangle_2}$ in $\mathrm{CP}(A,B)$ of the (OC2)-hull of $\mathcal{S}$ is in general *not convex*.

In particular such $\mathcal{C}_2$ does not satisfy the (operator convexity) property (OC1) of Definition 3.2.2 in general, and this even if $\mathcal{S}$ itself is *convex*:
Consider e.g. $\mathcal{S} := \mathbb{R}_+ \cdot \eta$ for $\eta \colon A := C^*(p_{11} - p_{22}) \hookrightarrow B := M_2$. Then $\langle \mathcal{S} \rangle_2 = \{X^* \eta(\cdot) X\,;\; X \in M_2\}$ has non-convex closure in $\mathrm{CP}(A,B)$, because the set $\{X^* p_{11} X\,;\; X \in M_2\}$ is the closed set that contains 0_2 and all positive matrices in M_2 of rank one. It is certainly not convex.

DEFINITION 3.2.4. A Hilbert (A,B)-module $(E, \phi \colon A \to \mathcal{L}(E))$, with right Hilbert B-module E, **generates** the *closed* m.o.c. cone $\mathcal{C}$, if $\mathcal{C} := \mathcal{C}(\mathcal{S})$ for the set $\mathcal{S}$ of completely positive maps $V_x \colon a \in A \mapsto \langle \phi(a)x, x \rangle \in B$.

We say that $(E, \phi \colon A \to \mathcal{L}(E))$ **defines** a closed m.o.c. cone $\mathcal{C}$ if (E,ϕ) generates $\mathcal{C}$ and, moreover, (E,ϕ) is unitarily equivalent to $(E \oplus_B E, \phi \oplus \phi)$.

(Does the latter mean that $\phi(A)' \cap \mathcal{L}(E)$ contains a copy of $\mathcal{O}_2$ unitally?)

`Need but have not the following:`

Let $V, W \in \mathrm{CP}(A,B)$ with separable C^*-algebra A.

To be shown: Then V and W are in the m.o.c. cone generated by $V + W$. (Could work with the tensor $V \otimes^{\max} \mathrm{id}_{C^*(F_\infty)}$ criterium ?)

Direct way ??

Question: Are V and W "coefficients" of the Hilbert A-B–module generated by $V + W$? Need an absorption theorem for Hilbert A-B-bi-module, like for Hilbert B-modules [**73**, thm. 13.6.2].

Suppose that E is a Hilbert B-module and that $\phi \colon A \to \mathcal{L}(E)$ a C^*-morphism.

Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ denote the point-norm closed m.o.c. cone generated by the "coefficients" $V_x \colon a \in A \mapsto \langle \phi(a)x, x \rangle \in B$ for $x \in E$.

Let E^∞ the Hilbert-B-module given by the sequences $(x_1, x_2, \ldots)$ with $x_k \in E$ and $\sum_{k=1}^{\infty} \langle x_k, x_k \rangle \in B$.

Define $\phi^\infty \colon A \to \mathcal{L}(E^\infty)$ by $\phi^\infty(a)(x_1, x_2, \ldots) := (\phi(a)x_1, \phi(a)x_2, \ldots)$

Then the set of "coefficient" maps $A \ni a \mapsto \langle \phi^\infty(a)y, y \rangle \in B$ for $y \in E^\infty$ is an m.o.c. cone $\mathcal{C}(\phi^\infty) \subseteq \mathrm{CP}(A,B)$.

This cone is in particular σ-additively and "homogenous ??" closed.

(not so likely) CONJECTURE (1):
The family of "coefficients" is *hereditary* in the following sense:

If $V: A \to B$ is a c.p. map and has the property that there exists a c.p. map $W: A \to B$ such that $V+W$ is a "coefficient" of a representation $\varphi: A \to \mathcal{L}(\mathcal{H}_B) \cong \mathcal{M}(B \otimes \mathbb{K})$, then V and W are "coefficients" of φ.

(very likely) CONJECTURE (2):

Suppose that A and B are stable C^*-algebras, where A is separable and B is σ-unital. $\phi_1, \phi_2: A \to \mathcal{M}(B)$ C^*-morphisms such that there exists a unitary $u \in \mathcal{M}(B)$ such that $\delta_\infty(\phi_1(a))u - u\delta_\infty(\phi_1(a)) \in B$ for all $a \in A$.

Then the "coefficients" $b^*\phi_1(\cdot)b$ and $b^*\phi_2(\cdot)b$ (for all $b \in B$ together!) generate the same point norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$.

(By a result in Chp. 5 one gets – conversely ??? – that $\delta_\infty \circ \phi_1$ and $\delta_\infty \circ \phi_2$ are unitarily homotopic in $\mathcal{M}(B)$.)

`Urgently needed / speculative Conjectures:`

If A is σ-unital then $\mathcal{C}(\phi^\infty)$ is closed with respect to the point-norm topology.

`This is not true?` Counterexample: Let $A := \mathrm{C}([0,1])$, $B := \mathbb{C}$ and $\phi: A \to \mathcal{L}(\ell_2) = \mathcal{M}(B \otimes \mathbb{K})$ the representation generated by $\phi(f) = \sum_n f(\rho(n))e_n$ for some map ρ from $\mathbb{N}$ onto the rational numbers in $[0,1]$. Then the point-norm closure of of the algebraic $\mathcal{C}(\phi^\infty)$ defined by the set of "coefficients" does not contain the c.p. map $f \in A \mapsto \int_0^1 f(t)\mathrm{d}t \in \mathbb{C}$. But this map is in the point norm closure of $\mathcal{C}(\phi^\infty) \subseteq \mathrm{CP}(A, \mathbb{C})$.

`Need here more precise definitions !!`

Perhaps use the following 3 lemmata (if true !):

1. Let K a convex cone in $\mathrm{C}(\mathbb{C}^n, B)_+$ that satisfies that $b^*Kb \subseteq K$ for all $b \in B$ and that K is σ-additive in the sense that if $V_1, V_2, \ldots \in K$ and $W(a) := \sum_n V_n(a) \in B$ for all $a \in \mathbb{C}^n$ then $W \in K$. Then K is point-norm closed.

2. Let A a separable C^*-algebra, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ an m.o.c. cone with the additional property that $\mathcal{C}$ is σ-additive – in the sense that if $V_1, V_2, \ldots \in \mathcal{C}$ and

$$W \otimes \mathrm{id}_2(a) := \sum_n V_n \otimes \mathrm{id}_2(a) \in B \otimes M_2 \quad \text{for all } a \in A \otimes M_2\,,$$

then $W \in \mathcal{C}$.

Then $\mathcal{C}$ is point-norm closed.

(Unfortunately it is not true in case $A = \mathrm{C}([0,1])$, $B := \mathbb{C}$???)

3. Something like this being "almost hereditary":

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a point-norm closed m.o.c. cone, $W \in \mathrm{CP}(A, B)$ and $V \in \mathcal{C}$ such that for each $T \in (A \otimes M_n)_+$ there exists $\gamma_{T,n} \in (0, \infty)$ with $(W \otimes \mathrm{id}_n)(T) \leq \gamma_{T,n} V(T)$

and ???? then $W \in \mathcal{C}$???? or?

REMARK 3.2.5. Suppose that A is separable and stable, B is σ-unital and stable and that $\mathcal{S} \subseteq \mathrm{CP}(A,B) \subseteq \mathcal{L}(A,B)$ is

a separable subset of $\mathcal{L}(A,B)$ with respect to point-norm topology ????, ???? or is "countably generated as m.o.c. cone"

in $\mathrm{CP}(A,B)$????

in the strong operator topology on $\mathcal{L}(A,B)$ ([15]).

Then, by Corollary 5.4.4, there is a C^*-morphism $H_0\colon A \to \mathcal{M}(B)$ such that

(i) H_0 is unitarily equivalent to its infinite repeat $\delta_\infty \circ H_0$,
(ii) the "coefficients" $A \ni a \mapsto b^* H_0(a) b \in B$ are all in the point-norm closed m.o.c. cone $\mathcal{C}(\mathcal{S}) \subseteq \mathrm{CP}(A,B)$ generated by $\mathcal{S}$, and
(iii) for each $V \in \mathcal{S}$ there exists $x \in B$ with $V(a) = \langle H_0(a)x, x\rangle$ for all $a \in A$.
more ????:
Need: (But have only weaker results !)
All $V \in \mathcal{C}(\mathcal{S})$ have this property !
Means:
The set of "coefficients" is closed in the topology of point-norm convergence.
(But this is not satisfied in general ... e.g. in case $\mathcal{S} \subseteq \mathrm{CP}(A,\mathbb{C})$ is generated by algebraical by a $\sigma(A^*, A)$ dense subset of the pure states of A.)

In particular, $H_0(A) \cap B = \{0\}$, and $V \in \mathcal{C}(\mathcal{S})$ if and only if there is a sequence $b_1, b_2, \ldots$ in B with $\|b_n\|^2 \le \|V\|$ such that $V(a) = \lim_n b_n^* H_0(a) b_n$ for all $a \in A$.

(This is a better property !!! But it does not say that V is itself a coefficient of of $\delta_\infty \circ H_0$. (The infinite repeat δ_∞ on $\mathcal{M}(B)$ is given by $\delta_\infty(a) := \sum_n s_n a (s_n)^*$ where $s_1, s_2, \ldots$ is a sequence of isometries in $\mathcal{M}(B)$ with $\sum_n s_n (s_n)^*$ strictly convergent to $1_{\mathcal{M}(B)}$, cf. Remark 5.1.1(8).)

H_0 is determined up to *unitary homotopy* in the sense of Definition 5.0.1, cf. Corollary 5.4.4.

(One has $H_0 = 0$ if $\mathcal{S} = \{0\}$.)

If A and B are separable then every m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A,B) \subset \mathcal{L}(A,B)$ is separable with respect to the point-norm operator topology (– given by the semi-norms $V \mapsto \|V(a)b\|$ on $\mathcal{L}(A,B)$ –), because $\mathcal{L}(A,B)$ is separable in the point-norm operator topology.

Question: Is the point-norm closure identical with the cone defined by the related kernel ideal of $A \otimes^{\max} C^*(F_\infty) \to B \otimes^{\max} C^*(F_\infty)$?

LEMMA 3.2.6. *Suppose that* $m_1, \ldots, m_s \in \mathbb{N}$ *and* $X_1, \ldots, X_s$ *are closed subsets of* $(0,1]$ *with* $1 \in X_j$, $j = 1, \ldots, s$.

[15] It is equal to the topology of point-norm convergence of $\mathcal{L}(A,B)$.

Let $A := \mathrm{C}_0(X_1, M_{m_1}) \oplus \cdots \oplus \mathrm{C}_0(X_n, M_{m_s})$, $F := M_{m_1} \oplus \cdots \oplus M_{m_s}$, *let* $\lambda \colon \mathrm{C}_0((0,1] \otimes F) \to A$ *the natural epimorphism given by*

$$\lambda(f \otimes (\alpha_1 \oplus \cdots \oplus \alpha_s)) := ((f|X_1) \otimes \alpha_1) \oplus \cdots \oplus ((f|X_n) \otimes \alpha_s)\,,$$

let $\pi_1 \colon A \to F$ *the epimorphism given by* $\pi_1(g_1 \oplus \cdots \oplus g_s) := g_1(1) \oplus \cdots \oplus g_s(1)$, *and let* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ *a subset with the property that*
for mutually orthogonal $a_1, \ldots, a_k \in A_+$ *with* $\|a_i\| = 1$, $b \in B_+$ *with* $\|b\| = 1$ *and* $\varepsilon > 0$, *there are* $d_1, \ldots, d_k \in B$ *and* $S \in \mathcal{S}$ *with*

$$\|d_i^* S(a_j) d_i - b\| < \varepsilon \qquad \text{for} \quad i = 1, \ldots, k\,.$$

Let $\mathcal{C}$ *denote the point-norm closure of the set of c.p. maps* $T \colon A \to B$ *such that there are* $S \in \mathcal{S}$, $n \in \mathbb{N}$, *a row* $r \in M_{1,n}(A)$ *and a column* $c \in M_{n,1}(B)$ *with* $T(\cdot) = c^* S(r^*(\cdot) r) c$.

- (o) $\mathcal{C}$ *contains the c.p. map* $T = V \circ \pi_1$ *if* $V \colon F \to B$ *is of* elementary *type, i.e.,* $V \circ \eta_j \colon M_j \to B$ *is given by a column* $b \in M_{n,1}(B)$ *in the sense* $V \circ \eta_j(\alpha) = b^* \alpha b$, *where* $\eta_j(\alpha) := 0 \oplus \cdots \oplus \alpha \oplus \cdots \oplus 0$ *with* α *at position* j.
- (i) *If, for every positive contraction* $b \in B_+$ *and* $\varepsilon > 0$, *there are contractions* $b_1, b_2 \in B$ *with* $b_2^* b_1 = 0$ *and* $\|b_j^* b_j b - b\| < \varepsilon$, *then* $\mathrm{CP}(F, B) \circ \pi_1 \subseteq \mathcal{C}$, *i.e.,* $\mathcal{C}$ *contains* all *c.p. maps* $T \colon A \to B$ *with* $T(\lambda(f_0(1 - f_0))) = 0$, *i.e.,* $V \circ \pi_1 \in \mathcal{C}$ *for all* $V \in \mathrm{CP}(F, B)$.
- (ii) *If* $\mathcal{C}$ *is convex, then* $\mathrm{CP}(F, B) \circ \pi_1 \subseteq \mathcal{C}$.
- (iii) *If* 1 *is* not *isolated in* X_j *for every* $j = 1, \ldots, k$, *then* $\mathrm{CP}(F, B) \circ \pi_1 \subseteq \mathcal{C}$.

For the parts (o)–(ii) one needs only the assumption on $\mathcal{S}$ with $k = 1$. The assumption with $k = \max(m_1, \ldots, m_s)$ is needed for the proof of part (iii).

PROOF. We consider the point-norm closed subset $\mathcal{C}_1$ of $\mathrm{CP}(F, B)$, consisting of the maps $V \in \mathrm{CP}(F, B)$ with the property that there is a sequence $T_n \in \mathcal{C}$ such that $\lim_n T_n(f) = V(f(1))$ for all $f \in \mathrm{C}_0((0,1], F)$.

The set $\mathcal{C}_1$ is invariant under the operations (OC2) of Definition 3.2.2, because $\mathcal{C}$ is invariant under the operations (OC2) and $\pi_1 \colon f \mapsto f(1)$ is an epimorphism. With other words, $V = c^* W(r^*(\cdot) r) c \in \mathcal{C}_1$ if $W \in \mathcal{C}_1$ and $c \in M_{n,1}(B)$ and $r \in M_{1,n}(F)$.

In particular, we have that $V(p(\cdot)) \in \mathcal{C}_1$ and $V_1(p_1(\cdot)) + V_2(p_2(\cdot)) \in \mathcal{C}_1$ for central projections $p, p_1, p_2 \in F$ with $p_1 p_2 = 0$, and for $V, V_1; V_2 \in \mathcal{C}_1$. (This reduces also the proof to the case where F is a full matrix algebra, i.e., to the case $s = 1$.)

(o): We denote the (upper left corner) minimal projection of $\eta_j(M_{m_j}) \cong M_{m_j}$ by $q_j \in F$, and let $\rho_j \colon F \to \mathbb{C}$ denote the (unique) pure state with $\rho_j(q_j) = 1$. Recall that $\rho_j(g) q_j = q_j g q_j$ for $g \in F$. Let $\rho = \rho_1 + \ldots + \rho_s$. (The q_j have pairwise orthogonal central supports p_j.)

If $V \colon F \to B$ is an

Def. of ''elementary'' in Part(o):
" $V \circ \eta_j \colon M_j \to B$ is given by a column $b \in M_{n,1}(B)$ in the sense $V \circ \eta_j(\alpha) = b^* \alpha b$, where $\eta_j(\alpha) := 0 \oplus \cdots \oplus \alpha \oplus \cdots \oplus 0$ with α at position j."

elementary c.p. map and $\delta > 0$, then there exists a contraction $e \in B_+$, such that $V = c^* W(r^*(\cdot)r)c$ for $W = \rho(\cdot)e$ some $c \in M_{t,1}(B)$ and $r \in M_{1,t}(F)$

Why? More precise? Part? (cf. Prop. 3.1.9).

Prop. 3.1.9 has been changed!!!

Thus, it suffices to show that, for $b \in B_+$ with $\|b\| = 1$, $\varepsilon > 0$ and $j \in \{1, \ldots, s\}$, there exists $V_j \in \mathcal{C}_1$ and $c_j \in B$ with $\|c_j^* V_j(p_j)c_j - e\| < \varepsilon$. Indeed, this implies that the point-norm closed $\mathcal{C}_1$ actually "contains" $\rho_j(\cdot)b$, and then $W, V \in \mathcal{C}_1$, because the central supports p_j of the q_j are orthogonal.

Define a c.p. map $W \colon F \to \mathrm{C}_0((0,1], F)$ by $W(g) := f_0 \otimes g$ for $g \in F$, where $f_0(t) = t$ for $t \in [0,1]$. Let $q := q_j$ and define

$$h_\delta := \delta^{-1}((f_0 - (1-\delta))_+) \otimes 1 \in \mathrm{C}_0((0,1], F)_+ \,.$$

Then $a_\delta := W(q)h_\delta$ is positive and $\|a_\delta\| = 1$, thus, there is $S_\delta \in \mathcal{S}$ and $d_\delta \in B$ with $\|d_\delta^* S_\delta(a_\delta^2)d_\delta - b\| < \delta$. Note that

$$T_\delta(g) := (d_\delta)^* S_\delta(a_\delta g a_\delta)d_\delta \qquad \text{for} \quad g \in \mathrm{C}_0((0,1], F)$$

defines a map in $\mathcal{C}$ and that $\lim_{\delta \to 0} \|T_\delta(g) - \rho_j(g(1))b\| = 0$ for all $g \in \mathrm{C}_0((0,1], F)$. Hence $\rho_j(\cdot)b \in \mathcal{C}_1$.

(i): By part (o), all

Why the Lemma applies? ??? Definition? of :

"elementary"

c.p. maps $V \colon F \to B$ are contained in $\mathcal{C}_1$. The supposed property of B implies that the elementary c.p. maps are dense in $\mathrm{CP}(F, B)$ by Proposition 3.1.9(ii). ???

Prop. 3.1.9(ii) = Old lem:3.3(ii) has been changed

(ii): This follows from part (o), because the set of $V \in \mathrm{CP}(F, B)$ with $V \circ \pi_1 \in \mathcal{C}$ contains all elementary maps V and is convex by the proposed convexity of $\mathcal{C}$.

(iii): If 1 is not isolated in X_j (for $j \in \{1, \ldots, s\}$), then we can find a strictly increasing sequence $0 =: t_0 < t_1 < t_2 < \cdots$ in X_j such that $\lim t_n = 1$. Let $q := q_j \in F$, $m := m_j$ and $p := p_j$ the central support of q.

Revise proof!
Prop. 3.1.9(i) has been changed!!

We show that our general assumptions (with $k \geq m$) implies that $\mathcal{C}_1$ contains $V(p(\cdot)) = V \circ \pi_j$ for every $V \in \mathrm{CP}(M_m, B)$:

By Proposition 3.1.9(i) there are columns $c_1 = [c_{11}, c_{12}, \ldots, c_{1n}]^\top, \ldots, c_n \in M_{n,1}(B)$ such that $V(\alpha) = \sum_{i=1}^n c_i^* \beta c_i$. There are contractions $e_n \in B_+$ with $\|c_i - (e_n \otimes 1_n)c_i\| < 1/n$.

We identify $C_0(0,1]$ naturally with the center $C_0(0,1] \otimes 1$ of $C_0((0,1], M_m) \subseteq C_0((0,1], F)$. We find (pice-wise linear) non-negative functions $f_n \in C_0(0,1]_+$ with support in $[(t_{n-1}+t_n)/2, (t_n+t_{n+1})/2]$ such that $\|f_n\| \leq 1$ and $f(t_n) = 1$.

Further there are functions $b_n \in C_0(0,1]_+$ with support in $[t_{n-1}, 1]$ such that $0 \leq b_n \leq 1$ and $b_n|[(t_{n-1}+t_n)/2, 1] = 1$. Then $b_n b_{n+1} = b_{n+1}$ and $b_n f_{n+j} = f_{n+j}$ for all $j \geq 0$.

Let $W(g) := f_0 \otimes g$

`Where/ What was Part (o) ???`

(as in proof of (o))

and $a_n := W(q) f_n \in C_0(0,1] \otimes q$. Then $\|a_n\| = t_n$ and the a_n are mutually orthogonal. Therefore (and by assumptions on $\mathcal{S}$), there exist $S_n \in \mathcal{S}$ and $d_{n,0}, \ldots, d_{n,m-1} \in B$ with

$$\|(d_{n,j})^* S_n\big((a_{n+j})^2\big) d_{n,j} - (t_{n+j})^2 e_n\| < 1/n \qquad \forall\ j \in \{0, \ldots, n-1\}.$$

Consider the c.p. maps $T_n(g) := Z_n^* S_n(R_n^* g R_n) Z_n$ with column $Z_n \in M_{m^2,1}(B)$ with transposed row $(Z_n)^\top$ given by

$$(Z_n)^\top := [d_{n,0} \cdot (c_1)^\top, \ldots, d_{n,m-1} \cdot (c_m)^\top] = [d_{n,0} c_{11}, d_{n,0} c_{21}, \ldots, d_{n,m-1} c_{mm}]$$

and row $R_n \in M_{1,m^2}(C_0((0,1], M_m))$ given by

$$R_n := [r_1^{(n)}, \ldots, r_m^{(n)}],$$

where $r_j^{(n)} := [(1 \otimes e_{1,1}) \cdot a_{n+j-1}, \ldots, (1 \otimes e_{m,1}) \cdot a_{n+j-1}]$ which is equal to

$f_{n+j-1} \cdot [(f_0 \otimes e_{1,1}), \ldots, f_0 \otimes e_{n,1}]$.

Here $e_{i,k}$ denote the matrix units of M_m. Note that $q = e_{1,1}$ and $[\alpha_{ij} q] = x^* \alpha x \in M_m(M_m)$ for $\alpha \in M_m$ and $x := [e_{1,1}, \ldots, e_{m,1}] \in M_{1,m}(M_m)$. It follows

$$T_n(f \otimes \alpha) = \sum_j \Big(\sum_{i,k} \alpha_{ik} c_{ij}^* d_{n+j-1}^* S_n(a_{n+j-1}(f \otimes q) a_{n+j-1}) d_{n+j-1} c_{kj}\Big)$$

for $f \in C_0(0,1]$ and $\alpha \in M_m$. `check? to be filled in`

??

The c.p. maps $d_n^* S_n(a_n(\cdot) a_n) d_n$ have norms $\leq 1 + 1/n \leq 2$. Since $0 \leq g \leq f \otimes 1_m \leq \|g\|$ for $f(t) = \|g(t)\|$ if $g \in C_0((0,1], F)_+$, we can see that $\|T_n\| \leq 2m\|V\|$. For $\varepsilon > 0$ there is n_0 with $\|f b_n - f(1) b_n\| < \varepsilon/(2m)^3$ for $n \geq n_0$. Since $T_n((f b_n) \otimes \alpha) = T_n(f \otimes \alpha)$ and $S_n(a_{n+j-1}(b_n \otimes q) a_{n+j-1}) = S_n(a_{n+j-1}^2)$, it follows that, for $n \geq n_0$,

$$\|T_n(f \otimes \alpha) - f(1) \sum_j \sum_{ik} \alpha_{ik} c_{ij}^* e_{n+j-1} c_{kj}\| \leq \max(|\alpha_{ik}|) \cdot m^3 (1/n + 2\varepsilon/(2m)^3)$$

Thus $\lim_n \|T_n(f \otimes \alpha) - f(1) V(\alpha)\| = 0$ for all $f \in C_0(0,1]$ and $\alpha \in M_m$. Since $\sup \|T_n\| < \infty$, it follows $\lim_n \|T_n(g) - V(g(1))\| = 0$ *for every* $g \in C_0((0,1], F)$. By our construction, $T_n \in \mathcal{C}$. Hence $V \in \mathcal{C}_1$. □

LEMMA 3.2.7. *Suppose that* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ *is a set of completely positive maps such that, for every* $k \in \mathbb{N}$, $a_1, \ldots, a_k \in A_+$ *with* $\|a_j\| = 1$ *and* $a_i a_j = 0$ *for* $i \neq j$, *every* $b \in B_+$ *with* $\|b\| \leq 1$ *and every* $\varepsilon > 0$, *there exist* $d_1, \ldots, d_k \in B$ *and* $S \in \mathcal{S}$ *with* $\|d_j^* S(a_j) d_j - b\| < \varepsilon$ *for* $j = 1, \ldots k$.

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ *denote the point-norm closure of the* (OC2)*-hull* $\langle \mathcal{S} \rangle_2$ *of* $\mathcal{S}$, *i.e., let* $\mathcal{C}$ *denote the point-norm closure of the set of c.p. maps* $T \colon A \to B$ *such that there are* $S \in \mathcal{S}$, $n \in \mathbb{N}$, *a row* $r \in M_{1,n}(A)$ *and a column* $c \in M_{n,1}(B)$ *with* $T(\cdot) = c^* S(r^*(\cdot) r) c$.

Then we get that ????

$\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$, *i.e., each nuclear c.p. map* $V : A \to B$ *can be approximated in point-norm by maps* $T := c^* S(r^*(\cdot) r) c$ *with* $S \in \mathcal{S}$.

in each of the following cases (i), (ii) *or* (iii)*:*

(i) *If, for every* $b \in B$ *and* $\varepsilon > 0$, *there exist* $e_1, e_2 \in B$ *with* $e_1^* e_2 = 0$ *and* $\|b - e_j^* e_j\| < \varepsilon$ *for* $j = 1, 2$.
(ii) *If* $\mathcal{C}$ *is convex.*
(iii) *If* $\mathrm{socle}(A) = \{0\}$.

The condition (i) is e.g. satisfied if B stable, or if B contains an approximate unit $\{e_\alpha\}$ of properly infinite contractions.

The proofs of the cases (i), (ii), and (iii) are rather different, e.g. in the proof of (i) and (ii), we use only the case $k = 1$. In a sense the assumption on $\mathcal{S}$ is almost necessary, because if $b \in B_+$ and orthogonal $a_1, \ldots, a_k \in A_+$ of norm one are given, then there are pure states $\rho_1, \ldots, \rho_k$ on A with $\rho_j(a_j) = 1$. Then $S(a) := (\sum_j \rho_j(a)) b^{1/3}$ and $d_j := b^{1/3}$ satisfy the assumptions on $\mathcal{S}$, e.g. in case $\mathcal{S} := \{ f(\cdot) b \,;\, f \in A_+^* \,, b \in B_+ \}$.

PROOF. (i, ii): We check the criteria for $\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$ of Lemma 3.2.1. Let F finite-dimensional and $\psi \colon \mathrm{C}_0((0, 1], F) \to A$ a *-morphism such that $\|\psi(f_0 \otimes p)\| = 1$ for all non-zero projections $p \in F$. Then the quotient $\psi(\mathrm{C}_0((0, 1], F))$ of $\mathrm{C}_0((0, 1], F)$ and the restrictions of $V \in \mathcal{S}$ satisfy the assumptions of Lemma 3.2.6, because (OC2)-hull of $\mathcal{S} \circ \psi$ is point-norm dense in $\mathcal{C} \circ \psi$ for the (OC2)-hull of $\mathcal{S} \subseteq \mathrm{CP}(A, B)$.

Thus the criteria is satisfied in the cases (i) and (ii) by Lemma 3.2.6(i,ii). Hence, $\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$ under the additional assumptions of parts (i) and (ii) by This proves (i) and (ii).

(iii): We use Lemma 3.2.6(iii) to show that the criteria of Lemma 3.2.1 is satisfied. Again let F finite-dimensional and $\psi \colon \mathrm{C}_0((0, 1], F) \to A$ a *-morphism such that $\|\psi(f_0 \otimes p)\| = 1$ for all non-zero projections $p \in F$. Unfortunately, 1 could be isolated in the spectrum of $\psi(f_0 \otimes p)$ for some minimal non-zero projection $p \in F$.

Let $F \cong M_{m_1} \oplus \cdots \oplus M_{m_k}$ and let $\psi_j \colon \mathrm{C}_0((0,1], M_{m_j}) \to A$ the *-morphisms with $\psi(g) = \psi_j \circ \pi_j(g)$ for $j = 1, \ldots, k$, where π_j denotes the natural epimorphism $\pi_j \colon F \to M_{m_j}$.

The hereditary C^*-subalgebras D_j of A generated by $\psi_j(\mathrm{C}_0((0,1], M_{m_j}))$ are mutually orthogonal. Since $\mathrm{socle}(A) = 0$ (by assumption), $\mathrm{socle}(D_j) = 0$ and Lemma A.10.3 says that there exist increasing continuous maps $\lambda_j \colon [0,1] \to [0,1]$ with $\lambda_j(0) = 0$ and $\lambda_j(1) = 1$, *-morphisms $\varphi_j \colon \mathrm{C}_0((0,1], M_{m_j}) \to D_j$ with $\varphi_j(f \circ \lambda_j) = \psi_j(f)$ for $f \in \mathrm{C}_0((0,1], M_{m_j})$, such that 1 is not isolated in the spectrum of $\varphi_j(f_0 \otimes 1)$ and $\varphi_j(f_0 \otimes 1)\varphi_i(f_0 \otimes 1) = 0$ for $i \neq j$. Then the image of $\varphi(g) := \sum \varphi_j(\pi_j(g))$ satisfies the assumptions of Lemma 3.2.6(iii), i.e., 1 is not isolated in $X_1, \ldots, X_n \subseteq (0,1]$ with $\varphi(\mathrm{C}_0((0,1] \otimes F)) \cong \mathrm{C}_0(X_1, M_{m_1}) \oplus \cdots \oplus \mathrm{C}_0(X_k, M_{m_k})$. Thus Lemma 3.2.6(iii) applies to φ, and for every $V \in \mathrm{CP}(F, B)$ there is a sequence $S_n \in \mathcal{C}$ with $\lim_n S_n(\varphi(g)) = V(g(1))$ for all $g \in \mathrm{C}_0((0,1], F)$. By Lemma 3.1.8, it follows from the (OC2)-invariance of $\mathcal{C}$ that sequence S_n can be chosen with $\|S_n\| \leq \|V\|$. Thus $S_n|D$ has norm $\leq \|V\|$, where D denotes the hereditary C^*-algebra generated by $\varphi(f_0 \otimes 1)$. In particular, $\lim_n S_n(\varphi((f_0(1 - f_0) \otimes 1)) = 0$. Since S_n is positive, it follows $\lim_n S_n(a) = 0$ for all $a \in E$, where E denote the hereditary C^*-subalgebra of A generated by $\varphi((f_0(1 - f_0) \otimes 1)$. By Lemma A.10.3, $\psi(g) - \varphi(g) = \sum_j \varphi_j(\pi_j(g \circ \lambda_j) - \pi_j(g)) \in E$ for $g \in \mathrm{C}_0((0,1], F)$. It follows $\lim_n S_n(\psi(g)) = \lim_n S_n(\varphi(g) + e) = V(g(1))$. Thus, the criteria of Lemma 3.2.1 is satisfied. □

Corollary 3.2.8. *The set* $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ *of nuclear c.p. maps is a point-norm closed m.o.c. cone in the sense of Definition 3.2.2.*

It is the point-norm closure of the smallest subset $\mathcal{S} \subseteq \mathrm{CP}(A, B)$*, that*

(a) *contains all maps* $a \mapsto \sum_{j=1}^{n} \rho_j(a) b_j$ *with pure sates* ρ_j *on* A ($j = 1, \ldots, n$) *and* $b_j \in B_+$*, and*
(b) *is closed under the operation* (OC2) *in Definition 3.2.2.*

If $\mathrm{socle}(A) = \{0\}$*, then it suffices to consider only pure states* ρ_j *on* A *that have pairwise non-equivalent irreducible representations* $D_j \colon A \to L_2(A, \rho_j)$.

Proof. The set of the maps $a \mapsto \sum_{j=1}^{n} \rho_j(a) b_j$ is convex and satisfies the assumptions of Lemma 3.2.7(ii). □

Remark 3.2.9. A first application of Lemma 3.2.7(ii) is the following observation:
Suppose that $C, D \subseteq \mathcal{M}(B)$ *are* C^**-subalgebras of the multiplier algebra* $\mathcal{M}(B)$ *of* B*, such that* DB *is dense in* B *and* $CD \subseteq D$*, and that* D *is simple. Then every nuclear map* $V \colon C \to B$ *is approximately inner*, i.e., can be approximated in point-norm topology by inner c.p. maps $c \in C \mapsto \sum_j b_j^* c b_j \in B$ (with $b_1, \ldots, b_n \in B$).

(Indeed, the cone $\mathrm{CP}_{\mathrm{in}}(C, B)$ of inner c.p. maps $c \mapsto \sum_j b_j^* c b_j$ satisfies (OC1) and (OC2). The assumptions of Lemma 3.2.7(ii) are satisfied, because D is a simple non-degenerate C^*-subalgebra of $\mathcal{M}(B)$ and $C \subseteq \mathcal{M}(D) \subseteq \mathcal{M}(B)$:

Indeed, if $a_1, \ldots, a_k \in C_+$ are non-zero, pair-wise orthogonal and have norms $\|a_j\| = 1$, $b \in B_+$, then, for $\varepsilon > 0$, there are $e_0, e_1, \ldots, e_k \in D$ with $e_i^* a_j e_i = \delta_{ij} e_0$, $\|e_0\| = 1$ and $f_1, \ldots, f_m \in C$ with $\|b - \sum b^{1/2} f_j^* e_0 f_j b^{1/2}\| < \varepsilon$ (because $C \subseteq \mathcal{M}(D) \subseteq \mathcal{M}(B)$, D is simple and DB is dense in B). Thus, $T(c) := \sum_{i,j} (e_i f_j b^{1/4})^* c (e_i f_j b^{1/2})$ and $d_i = b^{1/4}$ satisfy the assumptions of Lemma 3.2.7.)

Next related to WvN-Thm. form Elliott and Kucerovsky.
Give precise reference to it!!!

LEMMA 3.2.10. *Suppose that $C \subseteq \mathcal{M}(B)$ is a C^*-subalgebra, $J \lhd C$ is a closed ideal and that for every $c \in C_+$ with $\|c + J\| = 1$, every $b \in B_+$ and $\varepsilon > 0$ there exists $d = d(c, b, \varepsilon) \in B$ with $d^* c d \geq (b - \varepsilon)_+$.*

Let $V : C \to B$ a c.p. map with $V(J) = 0$ such that $[V]_J : C/J \to B$ is nuclear.

Then V is approximately one-step inner in $\mathcal{M}(B)$.

?...in the sense that there exists a net of elements $\{d_\tau\} \subseteq B$ with $V(c) = \lim_\tau d_\tau^* c d_\tau$ for each $c \in C$. In particular, $\lim_\tau d_\tau^* c d_\tau = 0$ for all $c \in J$....?

PROOF. We let $A := C/J$ and let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ denote the *set* of c.p. maps $V : A \to B$ with the property that $V \circ \pi_J : C \to B$ is one-step approximately inner, i.e., that for $c_1, \ldots, c_k \in C_+$ and $\varepsilon > 0$ there is $d \in B$ with $\|d\|^2 \leq \|V\|$ and $\|V(\pi_J(c_j)) - d^* c_j d\| < \varepsilon$ for $j = 1, \ldots, k$. Clearly, $\mathcal{C}$ is closed under the operations (OC2) of Definition 3.2.2.

If $A := C/J \cong \mathbb{C}$ then $A = \mathbb{C} \cdot q$. Let $V : A \to B$ completely positive, $W := V \circ \pi_J$, $b := V(q)$, and $f \in C_+$ a contraction with $\pi_J(f) = q$. For $c_1, \ldots, c_n \in C_+$ and $\delta > 0$ there is a contraction $e \in J_+$ with $\|(1 - e)(f c_k f - \alpha_k f^2)(1 - e)\| < \delta$ if $\alpha_k p = \pi_J(c_k)$, because $f c_k f - \alpha_k f^2$ is in J. Let $d \in B$ with $\|b - d^*(1-e) f^2 (1-e) d\| < \delta$, then $\|W(c_k) - g^* c_k g\| < 2\delta$ for $k = 1, \ldots, n$.

Suppose now that $A := C/J$ has (linear) dimension ≥ 2, then a maximal commutative C^*-subalgebra F of A has dimension ≥ 2. Thus there are $a_1, a_2 \in A_+$ with $a_1 a_2 = 0$ and $\|a_i\| = 1$ for $i = 1, 2$. There is $c \in C$ with $c^* = c$, $\|c\| = 1$ and $\pi(c) = a_1 - a_2$. Let $c_1 := c_+$ and $c_2 := c_-$. Then $\pi_J(c_i) = a_i$ for $i = 1, 2$.

Let $b \in B$ and $\varepsilon > 0$, then there are (by assumptions) $d_i \in B$ with $d_i^*(c_i)^2 d_i \geq (b - \delta)_+$ for $\delta := \varepsilon/3$. By Lemma 2.1.9 there are contractions $f_i \in B$ such that $e_i^* e_i = (b - 2\delta)_+$ for $e_i = c_i d_i f_i \in B$ $(i = 1, 2)$, and $e_1^* e_2 = 0$.

Thus suffices to show, that A, B and $\mathcal{S}$ satisfy the assumptions of Lemma 3.2.7(i) (where the case $k = 1$ is sufficient as the proof of Lemmas 3.2.6(i) and 3.2.7(i) shows), i.e., it suffices to show that, for (given) $a_0 \in A_+$ with $\|a_0\| = 1$, $b \in B_+$ with $\|b\| \leq 1$ and $\varepsilon > 0$, there is $S \in \mathcal{C}$ with $\|S(a_0) - b\| < \varepsilon$.

Let $c_0 \in C_+$ a contraction with $\pi_J(c_0) = a_0$, and let $\varphi \in A^*$ a pure state on A with $\varphi(a_0) = 1$. Then $\psi := \varphi \circ \pi_J$ is a pure state on C with $\psi(J) = 0$. Let $V(a) := \psi(a) b$. It suffices to show that $W := V \circ \pi_J$ is approximately inner. Note that $W(c) = \psi(c) b$.

By Lemma **??**, there exists a directed net $\mathcal{G} \subseteq C$ of positive contractions $g \in \mathcal{G}$ with $\psi(g) = 1$ such that $\lim_{g\to\mathcal{G}} \|gcg - \psi(c)g^2\| = 0$ for all $c \in C$. Thus, for $c_1, c_2, \ldots, c_n \in C$ and $\varepsilon > 0$ (and $\delta := \varepsilon/M$ for $M := (2+2\max_j \|c_j\|)$) there exists a contraction $g \in C_+$ with $\psi(g) = 1$ and $\|gc_jg - \psi(c_j)g^2\| < \delta$ for $j = 1, \ldots, n$. Since $\|\pi_J(g)\| \geq \psi(g) = 1$, there is $d \in B$ with $\delta^{-1}d^*(g^2 - (1-\delta))_+ d = (b-\delta)_+$ (by assumption and by Lemma 2.1.9). Let $f := \delta^{-1/2}(g^2 - (1-\delta))_+ d$. Then $\|f\| \leq 1$, and for $j = 1, \ldots, n$,

$$\|(gf)^* c_j (gf) - \psi(c_j)b\| \leq \delta(1 + 2\max_j \|c_j\|) < \varepsilon\,.$$

□

HERE in next lem.: ''compact'' and ''nuclear'' in what sense?
The point is the question what kind of approximation by factorable maps is required,
i.e., what kind of topology in the set of c.p. contractions has to be taken,
and what kind of c.p. maps S into M_n are accepted?
Strictly continuous S?,
uniformly continuous on which kind of subsets?
all compact subsets, ... norm-continuous anyway ...
Minimal: approx on finite subsets ...,
but ''uniformly'' continuous on the base sets of the topology.
Classic: point-wise uniform on finite sets.

LEMMA 3.2.11. *Let C be a separable C^*-subalgebra of the multiplier algebra $\mathcal{M}(D)$ of a C^*-algebra D (respectively of a von-Neumann algebra M) and $V\colon C \to B$ a nuclear completely positive contraction into a C^*-algebra B.*

Then for every compact subset $\Omega \subseteq C$, every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and completely positive contractions $S\colon \mathcal{M}(D) \to M_n$ (respectively $S\colon M \to M_n$), $T\colon M_n \to B$, such that

$$\|V(c) - TS(c)\| < \varepsilon \qquad \text{for} \quad c \in \Omega\,,$$

and S is unital and continuous w.r.t. the strict topology on $\mathcal{M}(D)$ (respectively S is unital and $\sigma(M, M_*)$-norm continuous).

PROOF. The multiplier algebra $\mathcal{M}(D)$ is unitally contained in the W*-algebra D^{**} with pre-dual $D^* = (D^{**})_*$. The natural embedding is continuous w.r.t. to the strict topology on $\mathcal{M}(D)$ on bounded parts and w.r.t. the $\sigma(D^{**}, D^*)$-topology on D^{**}. In particular, $S|\mathcal{M}(D)$ is unital and strictly continuous for every ultra-weakly continuous u.c.p. map $S\colon D^{**} \to M_n$. Thus, we can restrict to the case of $C \subseteq M$, where M is a von-Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Notice that $S_u(a) := u^*(a \otimes 1)u \in \mathcal{L}(\mathbb{C}^n) \cong M_n$ defines an ultra-weakly continuous u.c.p. map $S_u\colon M \to M_n$ if $u\colon \mathbb{C}^n \to \mathcal{H} \otimes \ell_2(\mathbb{N})$ is an isometry.

Let $\{c_1, \ldots, c_m\} \subseteq \Omega$ a δ-dense subset of Ω for $\delta := \varepsilon/4$. By Definition 3.1.1, Lemma 3.1.8 and Lemma 3.1.11(ii), there are $m \in \mathbb{N}$ and completely positive contractions $S_0 \colon C \to M_m$ and $T_0 \colon M_m \to B$, such that $\|V(c_j) - T_0(S_0(c_j))\| < \delta$ for $j = 1, \ldots, m$.

Since $C \subseteq M \subseteq \mathcal{L}(H)$ we can extend S_0 to a c.p. contraction $W \colon \mathcal{L}(H) \to M_n$ (e.g. by the Arveson extension theorem [**42**]).

The natural extensions of the irreducible representations of D to irreducible representations of $\mathcal{M}(D)$ are strictly continuous if one considers them with the *strong operator topology. The family of this particular irreducible representations of $\mathcal{M}(D)$ is separating for $\mathcal{M}(D)$. If we apply Lemma 3.1.12(i) to this class of irreducible representations of $\mathcal{M}(D)$, then we get a strictly continuous unital c.p. map $S_1 \colon \mathcal{M}(D) \to M_{m_1} \oplus \cdots \oplus M_{m_k} =: F$ and a c.p. contraction $T_1 \colon F \to M_n$ with $\|W(c_j) - T_1(S_1(c_j))\| < \delta$ for $j = 1, \ldots, n$. Thus $\|V(c) - T(S(c))\| < \varepsilon$ for $c \in \Omega$ and $S := T_1 \circ S_1$. □

`The following Lemma 3.2.12 is only` $1\times$ `really cited and used in the proof of Lemma ??. Formulations and proofs have to be improved in both cases.`

In the next proofs we use the property of non-elementary simple C^*-algebras stated in the following Lemma 3.2.12. Here $[\,\cdot\,]$ denotes matrices of operators, and $\|\cdot\|$ means their norms. In part (iv) we put two consecutive rows $[f_{ik}]_i$ ($i = 1, \ldots, k_n$, $k = n, n+1$) together to a row $[h_{in}]_i$ of length $k_n + k_{n+1}$.

Recall that a simple C^*-algebra A is "non-elementary" if A is not isomorphic to the algebra of compact operators on a Hilbert space (and is non-zero). The point of the following Lemma 3.2.12 is that the irreducible representation $D_\varphi \colon A \to \mathcal{L}(A/L_\varphi)$ is faithful on A but has the property that $A \cap \mathbb{K}(A/L_\varphi) = \{0\}$.

LEMMA 3.2.12. *Let A be a non-elementary simple C^*-algebra, φ be a pure state on A and $Y_1 \subseteq Y_2 \subseteq \ldots$ a sequence of subsets of the contractions in the multiplier algebra $\mathcal{M}(A)$ such that each Y_n is* compact *with respect to the* strict *topology on $\mathcal{M}(A)$.*

(This topology on $\mathcal{M}(A)$ is given by the semi-norms $T \in \mathcal{M}(A) \mapsto \|Ta\|$ for $a \in A$?

Check by Def.'s given in Chp. 5.

If A s unital then we could take $X_n = \{1\}$??? What is here X_n? Is it Y_n?)

Where the simplicity plays a role?

`Are ‘‘strictly’’ compact subsets of the unit ball of` $\mathcal{M}(A)$
(i.e., `compact with respect to the strict topology)`
`separable subsets of` $\mathcal{M}(A)$ `in norm topology?`
`It seems that this is supposed implicit below.`
`Have only that this sets are separable w.r.t. strict topology on` $\mathcal{M}(D)$ `if` D `is`σ`-unital.`

Is the unit ball of $\mathcal{L}(\ell_2)$ in the strong operator topology compact? It is at least weakly compact ...*

Let $S_n \colon \mathcal{M}(A) \to M_{k_n}$ a sequence of strictly continuous unital completely positive maps from $\mathcal{M}(A)$ into M_{k_n}.

Does it say that the S_n are ''supported'' in A?
And that the S_n are the unique strictly continuous extensions of $S_n|A$?
If ''yes'' then we find positive contractions $e_{m,n} \in A_+$ with
$\|1_{k_n} - S_n(e_{m,n})\| < 2^{-(m+n)}$,
This could the lead to the a reduction to the case
where A is σ-unital?

Then there exist contractions $f_{in} \in A$ ($n = 1, 2, \ldots$, $i = 1, \ldots, k_n$), $b \in A_+$ and a sequence $m_1 < m_2 < \ldots$ of positive integers such that

(i) $\|[\varphi(f_{in}^* a f_{jn})]_{ij} - S_n(a)\| < 4^{-n}$ *for* $a \in Y_n$,
(ii) $\|[\varphi(f_{im}^* a f_{jn})]_{ij}\| < 4^{-m}$ *for* $a \in Y_n$ *and* $m < n$, $i = 1, \ldots, k_m$, $j = 1, \ldots, k_n$,
(iii) $\|b\| = 1$ *and* $\varphi(b) = 1$,
(iv) $\|[b^\ell h_{in}^* a h_{jn} b^\ell]_{ij} - [b^{2\ell} \varphi(h_{in}^* a h_{jn})]_{ij}\| < 4^{-n}$ *for* $a \in Y_n$ *and* $m_n \leq \ell$ *where* $h_{jn} = f_{im}$ *with* $i = j$, $m = n$ *if* $j \leq k_n$, *and with* $i = j - k_n$, $m = n + 1$ *if* $k_n < j \leq k_{n+1}$.

Proof. (i) and (ii): We can suppose that $1 \in Y_n$.

Since the unital completely positive maps S_n are strictly continuous, there are sequences $d_{\ell,n} \in A_+$ such that $\|d_{\ell,n}\| = 1$ and $\lim_{\ell\to\infty} S_n(d_{\ell,n}) = 1_{k_n}$.

This implies that all S_n are supported on a σ-unital C^*-subalgebra $\overline{eAe}$ of A with $\{d_{\ell,n}\,;\ \ell, n \in \mathbb{N}\} \subseteq \overline{eAe}$.

Moreover, the S_n are restrictions to $\mathcal{M}(A) \subseteq A^{**}$ of its unique normalizations (i.e., weakly continuous extensions to A^{**}) of its restrictions $S_n|A$ to A.

The compactness of the sets Y_n with respect to the strict topology on $\mathcal{M}(A)$ implies that the set $Z := \{e\} \cup \{d_{\ell,n}\,;\ n, \ell \in \mathbb{N}\}$ and the union $\bigcup_{n,m\in\mathbb{N}} Y_n Z Y_m$ generates a separable C^*-subalgebra E of A, such that $Y_n E + E Y_n \subseteq E$ for $n = 1, 2, \ldots$.

Above we claimed that there is and invariant separable C^*-subalgebra E of A, such that $\bigcup_n Y_n$ contained in the two-sided normalizer $\mathcal{N}(E, \mathcal{M}(A)) \subseteq \mathcal{M}(A)$ of E in $\mathcal{M}(A)$.

At least we could find a σ-unital C^*-subalgebra E of A such that the S_n are supported on E, i.e., $\lim_\ell \|S_n(e^{1/\ell}) - 1_{k_n}\| = 0$ for each $n \in \mathbb{N}$ if $e \in E_+$ is a strictly positive contraction in E.

(That would be sufficiant for the rest of the proof.)

If $e \in E_+$ is a strictly positive contraction, then necessarily $\lim_{\ell\to\infty} S_n(e^{1/\ell}) = 1_{k_n}$ for $n = 1, 2, \ldots$ by strict continuity of the $S_n\colon A \to M_{k_n}$, i.e., $\lim_\ell \|S_n(e^{1/\ell}) - 1_{k_n})\| = 0$ in M_{k_n} for each $n \in \mathbb{N}$.

It follows that $\lim_{k\to\infty} S_n(e^{1/k}ae^{1/k}) = S_n(a)$ for $a \in \mathcal{M}(A)$ and $n = 1, 2, \ldots,$ because

$$\|S(fbf) - S(b)\|^2 \leq 8\|b\|^2\|1_C - S(f)\|$$

for every unital completely positive map $S\colon B \to C$ and $f \in B_+$ with $\|f\| \leq 1$.

The verification of the latter inequality can be reduced to the case $C := \mathbb{C}$:

For every state ρ on (general unital) C and contraction $c \in C_+$, holds $|1 - \rho(c)| \leq \|1 - c\|$ and there exists a pure state ρ on C with $|\rho(S(fbf) - S(b))| = \|S(fbf) - S(b)\|$ and such that the restriction of ρ to the Abelian C^*-subalgebra generated by $S(fbf) - S(b)$ is a character.

But if λ (e.g. $\lambda := \rho \circ S$) is a positive functional on B with norm $\|\lambda\| \leq 1$ and $b, f \in B_+$ with $\|f\| \leq 1$ then

$$|\lambda(fbf) - \lambda(b)|^2 \leq 8\|b\|^2|1 - \lambda(f)|\,.$$

Proof of latter inequality:

We can suppose that $B \subseteq \mathcal{L}(\mathcal{H})$ and λ is defined by $x \in \mathcal{H}$ as $\lambda(b) :=< bx, x >$. We let $y := f^{1/2}x$ and $z := fx = f^{1/2}y$. Have $\|x\| \leq 1$ and $\|z\| \leq 1$. $\lambda(f) =< y, y >$, $x - z = (1 - f)x$

$| < bx, x > - < bz, z > | \; = \; |\lambda(fbf) - \lambda(b)|$

$|(< bx, x > - < bx, z >) + (< x, bz > - < z, bz >)| \leq \|b\|\|x\|\|x - z\| + \|b\|\|z\|\|x - z\|$

$\leq 2\|b\|(\|(1 - f)x\|)\,.$

$\|x - fx\|^2 =< x, x > + < fx, fx > -2 < fx, x > = \|(1 - f)x\|^2 =< (1 - f)^2x, x >=< x, x > -2 < fx, x > + < f^2x, x >\leq 2(\|x\|^2 - < fx, x >)$.

$\leq 2(1- < fx, x >)$ if $0 \leq f \leq 1$ and $\|x\| \leq 1$.

Obtain estimate:

$$|\lambda(fbf) - \lambda(b)| \leq 2\|b\|(2(1 - \lambda(f)))^{1/2}$$

From the Banach-Steinhaus (uniform boundedness) theorem we get the existence of constants $C_n < \infty$ with $\|a\| \leq C_n$ for all $a \in Y_n$. Thus we find $g_n \in \mathbb{N}$ such that $\|S_n(e_nae_n) - S_n(a)\| \leq 8^{-n}$ for $e_n = e^{1/g_n}$, $a \in Y_n$.

`It seems that here a quasi-central approximate unit` e_n
`would do on all places a much better job!`

$\Omega_n := e_1Y_1e_1 \cup \cdots \cup e_nY_ne_n$ is a compact subset of $E \subseteq A$.

We consider M_{k_n} as naturally realized on $\mathbb{C}^{k_n}$. The irreducible representation R_φ defined by φ on $H := L_2(A, \varphi)$ (with canonical cyclic vector x) is faithful and

$R_\varphi(A) \cap \mathbb{K} = 0$, because A is simple. By Lemma 2.1.22, there is a sequence of isometries $v_n\colon \mathbb{C}^n \to H$ such that $\|S_n(a) - v_n^* a v_n\| \le 8^{-n}$ and $\|v_n^* a v_m\| < 8^{-n}$ for $a \in \Omega_n$, $m < n$. By the Kadison transitivity theorem, we get contractions $p_{in} \in A$ such that $R_\varphi(p_{in})x = v_n(e_{i,k_n})$. Thus $[\varphi(p_{jn}^* a p_{in})]_{ij} = v_n^* a v_n$ in M_{k_n} and $[\varphi(p_{jm}^* a p_{in})]_{ij} = v_n^* a v_m$ in $M_{k_n,k_m} = \mathcal{L}(\mathbb{C}^{k_m}, \mathbb{C}^{k_n})$ for $a \in \mathcal{M}(A)$ and $m < n$, where $i = 1, \ldots, k_n$, $j = 1, \ldots, k_m$. The elements $f_{in} := e_n p_{in}$ are as desired.

(iii) and (iv): The existence of $b \in A_+$ with (iii) and (iv) follows from Lemma **??**, because the union of the sets $f_{im}^* Y_k f_{jn}$ generate a separable C^*-subalgebra of A by compactness of the sets Y_k with respect to the strict topology. □

`next Prop. and others overlap with chp. 5?`

PROPOSITION 3.2.13. *Suppose that B is a C^*-algebra, and that $D \subseteq \mathcal{M}(B)$ is a simple,* purely infinite, *and non-degenerate C^*-subalgebra of the multiplier algebra $\mathcal{M}(B)$ of B, (i.e., $\overline{DB} = B$).*

Let C a separable C^*-subalgebra of $\mathcal{M}(B)$ that satisfies $CD \subseteq D$.

If $V : C \to B$ is a nuclear *completely positive contraction, then V can be approximated by 1-step inner c.p. maps in the sense of following property (i), respectively properties (i,ii) if B is unital:*

(i) *There exist sequences of contractions $d_n^{(i)} \in B$ ($i = 1, 2$, $n = 1, 2, \ldots$) with $\lim_{n\to\infty} (d_n^{(i)})^* a d_n^{(i)} = V(a)$ for $i = 1, 2$, $a \in C$, and $\lim_{n\to\infty} (d_n^{(i)})^* a d_n^{(j)} = 0$, for $i \neq j$, $a \in C + \mathbb{C}1$.*
(ii) *If, moreover, B is unital, $1_B \in C$ and $V(1) = 1$, then the sequences $(d_n^{(i)})$ ($i = 1, 2$) in (i) can be chosen as isometries with orthogonal ranges: $(d_n^{(i)})^* d_n^{(j)} = \delta_{ij} 1$.*

PROOF. The crucial role plays here that we have the intermediate simple purely infinite C^*-subalgebra $D \subseteq \mathcal{M}(B)$.

(i): We prove a more precise and general result: Let $C \subseteq \mathcal{M}(D)$, $D \subseteq \mathcal{M}(B)$ with $\overline{DB} = B$ (i.e., $1_{\mathcal{M}(B)} \in \mathcal{M}(D) \subseteq \mathcal{M}(B)$), D simple and purely infinite, and let $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq C$ be a sequence of norm compact subsets of C and $W_n\colon C \to B$ nuclear completely positive contractions, then there exist contractions $d_1, d_2, \ldots \in B$ such that $\|d_n^* d_{n+1}\| \le 2^{-n}$ and $\|d_n^* b d_n - W_n(b)\| < 2^{-n}$ for $b \in \Omega_n$ and $\|d_n^* b d_{n+1}\| < 2^{-n}$ for $b \in \Omega_n$, $n = 1, 2, \ldots$.

Let $V_1 = \ldots = V_8 := W_1$, $V_{n+8} := W_n$. At first we find by Lemma 3.2.11 completely positive contractions $S_n\colon \mathcal{M}(D) \to M_{k_n}$ and $T_n\colon M_{k_n} \to B$ such that S_n is unital and strictly continuous, and $\|T_n(S_n(b)) - V_n(b)\| < 8^{-n}$ for $b \in \Omega_n$.

`Prop. 3.1.9(i) (was old a Lemma) has been changed!!`

Moreover we can assume that T_n is of the form described in Proposition 3.1.9(i) (by passing, if necessary, to $M_{k_n} \otimes M_{k_n}$ and $S_n(.) \otimes 1$). That is, there exist

$e_1^{(n)} \ldots e_{k_n}^{(n)} \in B$ with $T_n(\alpha) = E_n \alpha E_n^*$ for $\alpha \in M_{k_n}$ where E_n is the row matrix $[(e_1^{(n)})^*, \ldots, (e_{k_n}^{(n)})^*]$ and $E_n E_n^* = \sum_{j=1}^{k_n} (e_j^{(n)})^* e_j^{(n)}$ is a contraction.

The Lemma 3.2.12 is imprecise formulated and is used only here!! Find a better source or an other proof here or there?

By Lemma 3.2.12 there exist row matrices $F_n = [f_1^{(n)}, \ldots, f_{k_n}^{(n)}]$ with elements in D, $b \in D_+$ and a sequence $m_1 < m_2 < \ldots$ of positive integers such that (with $b_n := b^{m_n}$)

$$\|(1_{k_n} \otimes b_n) F_n^* a F_n (1_{k_n} \otimes b_n) - S_n(a) \otimes b_n^2\| < 4^{-n} 2$$

and

$$\|(1_{k_n} \otimes b_n) F_n^* a F_{n+1} (1_{k_{n+1}} \otimes b_{n+1})\| < 4^{-n} 2 \qquad \forall \quad a \in \Omega_n \cup \{1\}.$$

Let $\delta_n := (\sup\{\|S_n(a)\| \colon a \in \Omega_n \cup \{1\}\})^{-1}$ and $\gamma_n := \delta_n/(5k_n)$. Consider the separable C^*-subalgebra B_1 of B which is generated by the $e_j^{(n)}$, $n = 1, 2, \ldots$, $j = 1, \ldots, k_n$. B_1 contains a strictly positive element $e \geq 0$ with $\|e\| = 1$. We find a sequence of positive integers $p_n < p_{n+1} < \ldots$ such that, with $e_n := e^{1/p_n}$, we get $\|e_n e_j^{(n)} - e_j^{(n)}\| < 16^{-n} \gamma_n$ for $j = 1, \ldots, k_n$, $n = 1, 2, \ldots$. Then from $B = \overline{DB}$ we get contractions $g_n \in D$ with $\|g_n e_n - e_n\| < 16^{-n} \gamma_n$.

The separable C^*-subalgebra D_1 of D generated by g_n contains a strictly positive element $h \in D_+$ with $\|h\| = 1$, and we find positive integers $r_n < r_{n+1} < \ldots$ such that (with $h_n := h^{1/r_n}$) $\|h_n g_n - g_n\| < 16^{-n} \gamma_n$.

It follows
$\|h_n e_j^{(n)} - e_j^{(n)}\| < 16^{-n} 5 \gamma_n = 16^{-n} \delta_n / k_n$ and $\|E_n^* - (1_{k_n} \otimes h_n) E_n^*\| < 16^{-n} \delta_n$.

Since D is purely infinite and simple, $b_n, h_n \in D_+$ and $\|b_n\| = \|h_n\| = 1$, we find (by Proposition 2.2.1(ii)) $c_n \in D$ with $\|c_n\| = 1$ and $\|c_n^* b_n^2 c_n - h_n\| < 16^{-n} \delta_n$. Then

$$\|E_n(\alpha \otimes c_n^* b_n^2 c_n) E_n^* - E_n \alpha E_n^*\| < 4^{-n} \|\alpha\| \delta_n$$

for $\alpha \in M_{k_n}$, because E_n is a contraction.

Here M_{k_n} is embedded in $M_{k_n}(\mathcal{M}(D)) \cong M_{k_n} \otimes \mathcal{M}(D)$ by $\alpha \mapsto \alpha \otimes 1$.

Now let $z_n := F_n(1_{k_n} \otimes (b_n c_n)) E_n^* \in B$, i.e. $z_n := \sum_{j=1}^{k_n} f_j^{(n)} b_n c_n e_j^{(n)}$. Then

$$\|z_n^* a z_n - T_n S_n(a)\| < 4^{-n} 2 + \|E_n(S_n(a) \otimes c_n^* b_n^2 c_n) E_n^* - E_n S_n(a) E_n^*\|.$$

$$\|z_n^* a z_{n+1}\| \leq \|(1_{k_n} \otimes b_n) F_n^* a F_{n+1} (1_{k_{n+1}} \otimes b_{n+1})\|,$$

because $\|(1 \otimes c_n) E_n^*\| = \|c_n\| \cdot \|E_n\| \leq 1$. It follows that

$$\|z_n^* a z_{n+1}\| < 4^{-n} 2 \qquad \text{and} \qquad \|z_n^* a z_n - T_n S_n(a)\| < 4^{-n} 3 \qquad \forall \quad a \in \Omega_n \cup \{1\}.$$

In particular $\|z_n\|^2 \leq 1 + 4^{-n} 3$.

Let $d_n := (\max(1, \|z_n\|))^{-1} z_n$, then d_n is as desired.

(ii): We have $\lim_{n\to\infty} (d_n^{(i)})^* d_n^{(j)} = 0$ and $\lim_{n\to\infty} (d_n^{(i)})^* d_n^{(i)} = 1$ ($i \neq j$ and $i, j = 1, 2$) from part (i). Thus, eventually small perturbations of the $d_n^{(i)}$'s (i=1,2) are as desired. □

Next remark sort of pre-version of the WvN-theorem in chp.5?

REMARK 3.2.14. There is a more delicate counterpart of Proposition 3.2.13 for the case $D \cong \mathbb{K}$ (which is implicitly contained in [**?**]):

Suppose that B is a C^*-algebra, and that $C, D \subseteq \mathcal{M}(B)$ are C^*-subalgebras of the multiplier algebra $\mathcal{M}(B)$ of B, such that D is non-degenerate, i.e., $D \cdot B = B$, and $D \cong \mathbb{K}$. Let C be a separable C^*-subalgebra of $\mathcal{M}(B)$ such that $CD \subseteq D$, and let $V \colon C \to B$ a completely positive map such that $V(C \cap D) = \{0\}$ and suppose that the c.p. map $[V] \colon C/(D \cap C) \to B$ on the "classes"

$$[c] := c \mod (D \cap C) = c + D \cap C$$

is a *nuclear* completely positive map.

Then V is approximately 1-step inner. More precisely, there exists a sequences of contractions $d_n \in B$ with $\lim_{n\to\infty} d_n^* a d_{n+k} = \delta_{0,k} V(a)$ for $a \in C$.

The *proof* uses Lemma 2.2.3 and modifications of arguments from the proof of Proposition 3.2.13.

In the following proposition $\mathrm{C_b}(X, B)$ means the C^*-algebra of continuous functions on a locally compact space X with values in B, and $\mathrm{C_0}(X, B)$ is the closed ideal in $\mathrm{C_b}(X, B)$ formed by the functions $f \in \mathrm{C_b}(X, B)$ with $x \in X \mapsto \|f(x)\|$ vanishing at infinity.

PROPOSITION 3.2.15. *Let $D \subseteq B$ be C^*-algebras, where D is simple and purely infinite and $\overline{DB} = B$. Let X be a locally compact and σ-compact space, C a separable C^*-subalgebra of $\mathrm{C_b}(X, D)$, and $x \in X \mapsto (V_x \colon D \to B) \in \mathrm{CP}(D, B)$ a point-norm continuous map from X into the* nuclear *completely positive contractions from D to B, i.e., $V_x \in \mathrm{CP_{nuc}}(D, B)$ for all $x \in X$.*

Consider the c.p. contraction $V\colon \mathrm{C_b}(X, D) \to \mathrm{C_b}(X, B)$ defined by $V(d)(x) := V_x(d(x))$ for $x \in X$ and $d \in \mathrm{C_b}(X, D)$.

(i) *There exists a contraction $d \in \mathrm{C_b}(X, B)$ such that $d^*cd - V(c) \in \mathrm{C_0}(X, B)$ for $c \in C$.*
(ii) *If moreover B and D are unital, then $1_D = 1_B$. If then $1_D \in C$, i.e., if C contains the unit element of the C^*-algebra $\mathrm{C_b}(X, D)$, and nuclear c.p. map $V_x \colon D \to B$ is unital for every $x \in X$, then the element d in Part(i) can be chosen such that d is moreover an isometry in $\mathrm{C_b}(X, B)$.*

Notice that the $\mathrm{C_b}(X)$-modular c.p. map $c \in \mathrm{C_b}(X, D) \to V(c) \in \mathrm{C_b}(X, B)$ is in general not nuclear, but it is easy to see with help of the "tensor product criterium" that the c.p. maps $c \mapsto f^*V(c)f \in \mathrm{C_0}(X, B)$ with $f \in \mathrm{C_0}(X, B)$ are all nuclear.

PROOF. We reduce the proof essentially to the proof of Part(i) by adding to C a special constant positive contraction $e \in D_+ \subseteq \mathrm{C_b}(X, D)_+$ that makes the larger $C^*(C, e) \supseteq C$ a separable σ-unital C^*-subalgebra of $\mathrm{C_b}(X, D)$, and then select the proposed contraction $d \in \mathrm{C_b}(X, B)$ such that it satisfies in addition the inequality $\|d^*ed - V(e)\| < 1/3$.

(i): By local compactness and σ-compactness of X, there exist an increasing sequence $X_1 \subseteq X_2 \ldots \subseteq X$ of open subsets of X with compact closures $\overline{X_n} \subseteq X_{n+1}$ such that $X = \bigcup X_n$.

If we use the Tietze-Urysohn extension theorem then we find continuous functions $f_n \colon X \to [0,1]$ with $f_n(x) = 1$ for $x \in \overline{X_n}$ and $f_n(x) = 0$ for $x \in X \setminus X_{n+1}$. Let $g_1 := f_1^{1/2}$, $g_n := (f_n - f_{n-1})^{1/2}$ for $n > 1$. Then $f_n f_{n+1} = f_n$, and therefore $g_n g_m = 0$ if $|n - m| > 1$.

Since C is separable by our assumptions, we find separable C^*-subalgebras $D_n \subseteq D$ such that the evaluation epimorphisms $\pi_x \colon \mathrm{C_b}(X_n, D) \to D$ for $x \in \overline{X_n}$ map the separable C^*-algebra $C|\overline{X_n}$ into the separable C^*-subalgebra D_n, i.e., $\pi_x(C) \subseteq D_n$. Thus, the separable C^*-subalgebra $D_\infty := \overline{\bigcup_n D_n}$ has the property that $C \subseteq \mathrm{C_b}(X, D_\infty)$. Let $e \in D_\infty$ a strictly positive contraction in D_∞. Consider it as constant element of $\mathrm{C_b}(X, D)$ and replace C by $C^*(e, C) \subseteq \mathrm{C_b}(X, D)$. Then $e \in C^*(e, C)$ is a strictly positive element of $C^*(e, C)$. If D is unital then we can likewise also take here 1_D in place of e. From now on we suppose that C contains a "distinguished" strictly positive contraction e. Likewise this is the unit of $\mathrm{C_b}(X, D)$ if D is unital.

Now let $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq C$ be finite-dimensional linear subspaces of C, such that C is the closure of the union of the Y_n. Since $C \subseteq \mathrm{C_b}(X, D)$, we find compact subsets $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq D$ such that $y(x) \in \Omega_n$ for points $x \in X_{n+1}$ and elements $y \in Y_n$ with $\|y\| \leq n + 1$. Consider the nuclear completely positive contractions $V_n \colon D \to \mathrm{C_0}(X, B)$ given by $V_n(b)(x) := (f_{n+1} V(b) f_{n+1})(x)$ for $x \in X$ and $b \in D$, where the $f_n \in \mathrm{C_0}(X)_+$ are as above defined.

Recall that $D \subseteq B \subseteq \mathrm{C_b}(X, B)$ and $\mathrm{C_b}(X, B) \subseteq \mathcal{M}(\mathrm{C_0}(X, B))$ naturally.

$\mathrm{C_0}(X, B)$ is the closure of $D\,\mathrm{C_0}(X, B)$, because is the closure of $B\,\mathrm{C_0}(X, B)$ and B is the closure of DB by the assumptions.

We have shown – under this assumptions of non-degeneracy – in the proof of Proposition 3.2.13 that there exist contractions $d_1, d_2, \dots \in \mathrm{C_0}(X, B)$ with $\|d_n^* b d_n - V_n(b)\| < 2^{-n-1}$, $\|d_n^* b d_{n+1}\| < 2^{-n-1}$ for $b \in \Omega_n$ and $\|d_n^* d_{n+1}\| < 2^{-n-1}$.

Let $h := \sum_{n=1}^{\infty} g_{n+1} d_n$. Now use the above listed properties of g_n, d_n, and the definitions of Ω_n and Y_n.

Then a simple calculation shows, that $h \in \mathrm{C_b}(X, B)$, $\|h(x)\| \leq 1 + 2^{-n}$ for $x \in X \setminus X_n$, $\|h(x)^* b(x) h(x) - V_x(b(x))\| \leq 2^{-n}\|b\|$ for $b \in Y_n$ and $x \in X \setminus X_n$ and $n > 1$, or $n = 1$ and $y \in X$.

Need that $\|d(x)^* e d(x) - e\| < 1/4$ ... ????

Thus $d(x) := (\max(1, \|h(x)\|))^{-1} h(x)$ is a contraction d in $\mathrm{C_b}(X, B)$ with the desired properties.

(ii): If B is unital, then $\overline{DB} = B$ implies $1_B \in D$.

If then $1_B \in C$ and the $V_x(\cdot)$ are unital, then the element we can assume that

$1 \in Y_1$ and have $d(x)^*d(x) \le 1$ and $1 - d^*d \in \mathrm{C}_0(X, B)$. Thus we can replace $d(x) := h(x)(h(x)^*h(x))^{-1/2}$ is an isometry in $\mathrm{C_b}(X, B)$ with the desired properties. □

COROLLARY 3.2.16. *Suppose that B is purely infinite and simple.*

(i) *Let C be a C^*-subalgebra of $\mathcal{M}(B)$. Then every nuclear completely positive contraction V from C to B can be approximated point-wise by maps of the form $c \in C \mapsto d^*cd \in B$, where $d \in B$ and $\|d\| \le 1$.*

(ii) *If C is a separable C^*-subalgebra of $\ell_\infty(B)$ and if $V_n : B \to B$ ($n = 1, 2, \ldots$) is a sequence of nuclear completely positive contractions, then there exist sequences $d_1, d_2, \ldots$ and $e_1, e_2, \ldots$ of contractions in B such that $\lim e_n^* d_n = 0$ for $n = 1, 2, \ldots$, $\lim \|d_n^* \pi_n(c) d_n - V_n(\pi_n(c))\| = 0$ and $\lim \|e_n^* \pi_n(c) e_n - V_n(\pi_n(c))\| = 0$.*

PROOF. (i): Let $Y \subseteq C$ a finite subset and $\varepsilon > 0$. By Proposition 3.2.13 (with $B = D$) there exists a contraction $d \in B$ with $\|d^*ad - V_n(a)\| < \varepsilon$ for $a \in Z$.

(ii): Let $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq C$ be a linear filtration of C. By Proposition 3.2.13 (with $B = D$) there exist contractions $d_n, e_n \in B$ with $\|e_n^* d_n\| \le 1/n$, $\|d_n^* a d_n - V_n(a)\| < 1/n$ and $\|e_n^* a e_n - V_n(a)\| < 1/n$ for every a in the compact set $\{\pi_n(c) : c \in Y_n, \|c\| \le n\}$. □

REMARK 3.2.17. Corollary 3.2.16(ii) has a converse, which is easily established (using Proposition 2.2.1(ii) and the nuclear contractions $a \mapsto \psi(a)b$ for suitable states ψ):
If every nuclear contraction from an arbitrary subalgebra B of $A \neq \{0\}$ into A can be approximated by maps $b \mapsto d^*bd$, $d \in A$ and $\|d\| \le 1$, then either A is simple and purely infinite or $A \cong \mathbb{C}$.

REMARK 3.2.18. As noted in Remark 3.2.17, the assumption that A is purely infinite is quite essential to approximate nuclear maps on subalgebras by maps $b \mapsto d^*bd$.

However, Haagerup and Zsido [**351**] published a proof of the Dixmier conjecture that every simple unital C^*-algebra A with a unique trace state ρ has the so-called "Dixmier property": There exists a net $\{V_\tau\}$ in the convex combination of the inner automorphisms on A that converges point-wise to the map $\rho(\cdot)1$.

REMARK 3.2.19. For contractions a and b in a C^*-algebra B and $\varepsilon \in [0, 1)$ with $\|a^*b\| < \varepsilon^2$ there exist contractions $c, d \in B$ such that $c^*d = 0$, $\|a - c\| < \varepsilon^{1/2}$ and $\|b - d\| < \varepsilon^{1/2}$.

In the case of positive contractions $a, b \in \mathrm{C}_0(X, M_2)_+$ with $\|ab\| < \varepsilon^2$ one can find contractions $c, d \in \mathrm{C}_0(X, M_2)_+$ with $c^*d = 0$ and $\|c - a\| < 2\varepsilon$ and $\|d - b\| < 2\varepsilon$. This is also the general conjecture for contractions $a, b \in C^*(a, b) \subseteq \mathcal{L}(\ell_2)$.

It is easy to see that in case of any functions $a, b \in \mathrm{C}_0(X)$ with $\|ab\| \le \varepsilon$ there exists functions $c, d \in \mathrm{C}_0(X)$ with $cd = 0$, $\|c - a\| \le \varepsilon$ and $\|d - b\| \le \varepsilon$.

In case of projections $a = p$ and $b = q$ we can take $c := p$ and $d := (1-p)q$ with $\|a-c\| = 0$ and $\|b-d\| = \|ab\|$.

The estimates allow us to show that the proofs of Proposition 3.2.13(i) and Corollary 3.2.16(i) can be arranged such that, moreover, $(d_n^{(i)})^* d_n^{(j)} = 0$ for $i \neq j$ (respectively that $e_n^* d_n = 0$).

More details for Remark 3.2.19 : We can replace the contractions a and b by the positive contractions $e := (aa^*)^{1/2}$ and $f := (bb^*)^{1/2}$. Then $a = ev = v(a^*a)^{1/2}$ and $b = fw = w(b^*b)^{1/2}$ for the polar decompositions. They satisfy $\|ef\| = \|a^*b\|$. Let $\gamma := \|ef\|$.

An estimate of the infimum of the distances $\max(\|e - ge^{1/2}\|, \|f - hf^{1/2}\|)$ to all contractions $g, h \in A$ with $g^*h = 0$ then we get also the desired estimate, because $c := ge^{1/2}v$ and $d := hf^{1/2}w$ satisfy $\|c-a\| = \|e - ge^{1/2}\|$ and $\|d-b\| = \|f - hf^{1/2}\|$.

Thus it suffices to consider the special case where $\|e - ge^{1/2}\|$ and $\|f - hf^{1/2}\|$ with $g := (e-f)_+^{1/2}$ and $h := (f-e)_+^{1/2} = (e-f)_-^{1/2}$. Then $(f-e)_+ = (e-f)_-$ implies $g^*h = 0$.

Recall $2(f-e)_+ = |f-e| + (f-e)$ and $2(e-f)_+ = |f-e| - (f-e) = 2(f-e)_-$.

It implies $e - (e-f)_+ = 2^{-1}(e + f - |e-f|) = f - (f-e)_+$.

Since $\|e\| \leq 1$ and the square root is operator convex, we get

$$\|e - ge^{1/2}\| \leq \|e^{1/2} - g\| \leq \|e - g^2\|^{1/2} = \|e - (e-f)_+\|^{1/2} = 2^{-1/2} \cdot \|e + f - |e-f|\|^{1/2} .$$

Now we use again that square root is operator convex and and that $|e-f|^2 = (e-f)^2$ to obtain $\|e + f - |e-f|\| \leq \|(e+f)^2 - (e-f)^2\|^{1/2}$. Using $(e+f)^2 - (e-f)^2 = 2(ef + fe)$ and $\|ef\| = \|fe\|$ we obtain $\|e + f - |e-f|\| \leq 2^{1/2}\|ef\|^{1/2}$ and finally that

$$\|e - ge^{1/2}\| \leq \|ef\|^{1/4} .$$

Similar calculation shows that

$$\|f - hf^{1/2}\| \leq \|ef\|^{1/4} .$$

COROLLARY 3.2.20. *Suppose that B is a simple C^*-algebra and A is a C^*-subalgebra of $\mathcal{M}(B)$.*

Then for every nuclear contraction $V : A \to B$, every compact subset Ω of A and every $\varepsilon > 0$ there are $b_1, \ldots, b_n$ in B such that $\|\sum b_i^* b_i\| \leq 1$ and

$$\Big\| V(a) - \sum b_i^* a b_i \Big\| < \varepsilon \qquad \text{for all } \ a \in \Omega .$$

It is an idea from ''3rd draft''.
It is true for all ideal-system preserving residually nuclear maps $V : A \to B$ from $A \subseteq \mathcal{M}(B)$ to B, i.e., if $V(A \cap \mathcal{M}(B,J)) \subseteq J$ and $[V]_J : A/(A \cap \mathcal{M}(B,J)) \to B/J$ is nuclear for all closed ideals $J \subseteq B$.
Thus, residually nuclear maps are approximately sums of inner c.p. maps ...

It comes from the generalized separation theorem for m.o.c. cones.

Derive it as a Corollary from the general result for those C^*-algebras where every ideal-respecting nuclear map is automatic residual nuclear??
E.g. where all ideals are ''locally semi-split''.
???Are all exact C^*-algebras locally semi-split ???

PROOF. Consider the nuclear map $V \otimes \mathrm{id}$ from $A \otimes \mathcal{O}_\infty \subseteq \mathcal{M}(B \otimes \mathcal{O}_\infty)$ into $B \otimes \mathcal{O}_\infty$ (that is again nuclear because $\mathcal{O}_\infty$ is a nuclear C^*-algebra; i.e., $V \otimes \mathrm{id}$ is nuclear). By Theorem E, the C^*-algebra $B \otimes \mathcal{O}_\infty$ is simple and purely infinite, because B is simple and $\mathcal{O}_\infty$ is simple and purely infinite, cf. [**169, 172**]. By Corollary 3.2.16, $V \otimes \mathrm{id}$ can be approximated by maps of the form $a \otimes \xi \mapsto x^*(a \otimes \xi)x$ for a suitable contraction $x \in B \otimes \mathcal{O}_\infty$. In particular, for a given $\varepsilon > 0$, we can find $x \in B \otimes \mathcal{O}_\infty$, $\|x\| \leq 1$ such that $\|V(b) \otimes 1 - x^*(b \otimes 1)x\| < \varepsilon/2$ for $b \in \Omega$. Approximating x by elementary tensors, we can achieve

$$\|V(b) \otimes 1 - (\sum a_i \otimes d_i)^*(b \otimes 1)(\sum a_i \otimes d_i)\| < \varepsilon$$

and

$$\|\sum a_i \otimes d_i\| \leq 1$$

for suitable elements $a_i \in B$ and $d_i \in \mathcal{O}_\infty$. Now let ρ be a pure state on $\mathcal{O}_\infty$. Applying the Gram-Schmidt orthogonalization procedure we may suppose that $\rho(d_i^* d_j) = \delta_{ij}$ for all i, j, and obtain suitable new $\sum a_i \otimes d_i$. But then (applying $\mathrm{id} \otimes \rho$) we get

$$\|V(b) - \sum a_i^* b a_i\| < \varepsilon$$

for $b \in \Omega$. Now $\|\sum a_i^* a_i\| \leq 1$ because $\|\sum a_i \otimes d_i\| \leq 1$. □

Next follows from s.p.i. permanences:
A s.p.i. and B exact
(e.g. B nuclear, $B = \mathrm{C}_0(X)$ for l.c. space X)
then $A \otimes B$ s.p.i.
Result and Proof should be given somewhere in Chp. 2.

COROLLARY 3.2.21. *If D is simple and purely infinite, Y a* compact *space, $\varepsilon > 0$ and $a, b \in \mathrm{C}(Y, D)_+$ such that $\|b\| \leq 1$ and $\|a(y)\| = 1$ for every $y \in Y$ then there exists a contraction $d \in \mathrm{C}(Y, D)$ such that $\|b - d^* a d\| < \varepsilon$.*

PROOF. For every $y \in Y$ we find a state λ_y on D such that $\lambda_y(a(y)) = 1$. In an open neighborhood $U(y)$ of y we have $|1 - \lambda_y(a(x))| < \varepsilon/2$ for $x \in U(y)$. Since Y is compact, we find a finite sequence $y_1, \ldots, y_n \in Y$ such that the union of the $U(y_k)$ equals Y. We may assume that the system is minimal with this property. Let $e_k \colon Y \to [0, 1]$ $(k = 1, \ldots, n)$ a finite decomposition of 1 such that the support of e_k is contained in $U(y_k)$.

Let $\lambda_k := \lambda_{y_k}$ and $T_y(c) := (\sum e_k(y)\lambda_k(c))b(y)$.

Then $\|T_y(a(y)) - b(y)\| \leq \varepsilon/2$ for every $y \in Y$ and $y \mapsto T_y$ is a strongly continuous map from Y into the nuclear completely positive contractions from D into D. Since $\{a(y) \colon y \in Y\}$ is norm compact, there exists a contraction $d \in \mathrm{C}(Y, D)$ with $\|T_y(a(x)) - d(y)^* a(x) d(y)\| < \varepsilon/2$ for all $x, y \in Y$, by Proposition 3.2.15. To obtain this estimate from Proposition 3.2.15, let $B := D$, $X := Y \times \mathbb{N}$, i.e. X is the countable disjoint union of copies of Y, define V by $V(a)(y, n) := T_y(a)$, and let C be the separable C^*-subalgebra of $D \subseteq \mathrm{C_b}(X, B) = \ell_\infty(\mathrm{C}(Y, D))$ which is generated by $\{a(x) \colon x \in Y\}$.

Thus $\|b - d^* a d\| < \varepsilon$. □

Lemma 3.2.22. *Let $A := \mathrm{C}(X_1, D_1) \oplus \ldots \oplus \mathrm{C}(X_n, D_n)$, where the X_k ($k = 1, \ldots, n$) are compact metric spaces and D_k ($k = 1, \ldots, n$) are simple purely infinite C^*-algebras and $\varepsilon > 0$.*

If $a, b \in A$, $b^* = b$, $0 \leq a$, $\|a\| \leq 1$ and there are contractions $c_1, \ldots, c_m \in A$ such that $\|b - \sum c_j^* a c_j\| < \varepsilon$ and $\|\sum c_j^* c_j\| \leq 1$, then there is a contraction $d \in A$ with $\|b - d^* a d\| < 3\varepsilon$.

Better use here:
Direct sums of s.p.i. algebras A_k are s.p.i.
and $\mathrm{C}(X, D) = \mathrm{C}(X) \otimes D$ is s.p.i. if D is s.p.i.
$\Leftarrow$ follows from s.p.i. permanences in Chapter 2.
Find -- give -- references!

In other words: A is a non-simple strongly purely infinite C^*-algebra in the sense of Definition 1.2.2.

Proof. We have $b = b_1 \oplus \ldots \oplus b_n$, $a = a_1 \oplus \ldots \oplus a_n$. Thus, it suffices to consider the case $A = \mathrm{C}(X, D)$ for simple purely infinite D and compact X. But then

$$\|b - \sum c_j^* a c_j\| < \varepsilon$$

implies $\|b - b_+\| < \varepsilon$ and

$$\|b(y)\| < \varepsilon + \|a(y)\|$$

for $y \in X$, because $a \to \sum c_j^* a c_j$ is a completely positive contraction.

Let $Y := \{y \in X \colon 2\varepsilon \leq \|b(y)\|\}$. By functional calculus, for $y \in Y$, we have $\|b(y)\| = \|b(y)_+\|$, $\varepsilon \leq \|(b(y) - \varepsilon)_+\| = \|b(y)\| - \varepsilon$ and $\|(b(y) - \varepsilon)_-\| \leq 2\varepsilon$, i.e., $\|b(y) - (b(y) - \varepsilon)_+\| \leq 2\varepsilon$.

On the other hand $\varepsilon < \|a(y)\|$ and $0 < \|a(y)\|^{-1}(\|b(y)\| - \varepsilon) < 1$ for $y \in Y$.

Let $c(y) := (\|b(y)\| - \varepsilon)^{-1}(b(y) - \varepsilon)_+$

By Corollary 3.2.21, there exists a contraction $d_1 \in \mathrm{C}(Y, D)$ such that

$$\|c(y) - d_1(y)^*(\|a(y)\|^{-1} a(y)) d_1(y)\| < \varepsilon \qquad \forall\, y \in Y\,.$$

$g(y) := ((\|b(y)\| - \varepsilon)/\|a(y)\|)^{1/2}$ has norm ≤ 1 on Y. The contraction $d_2(y) = g(y) d_1(y)$ in $\mathrm{C}(Y, D)$ satisfies $\|((b|Y) - \varepsilon)_+ - d_2^*(a|Y) d_2\| < \varepsilon$ and therefore $\|(b|Y) - d_2^*(a|Y) d_2\| < 3\varepsilon$.

Now let $d_3 \in \mathrm{C}(X, D)$ a contraction with $d_3(y) = d_2(y)$ for $y \in Y$. Then there is an open neighborhood U of Y with $\|b(x) - d_3(x)^* a(x) d_3(x)\| < 3\varepsilon$ for $x \in U$. We find $h \colon X \to [0,1]$ continuous with $h(y) = 1$ for $y \in Y$ and $h(x) = 0$ for $x \in X \setminus U$.

The map $d(x) := h(x)^{1/2} d_2(x)$ defines a contraction $d \in \mathrm{C}(X, D)$ such that

$$\|b(x) - d(x)^* a(x) d(x)\| \leq (1 - h(x))\|b(x)\| + h(x)\|b(x) - d_3(x)^* a(x) d_3(x)\| < 3\varepsilon .$$

□

B in next Corollary is s.p.i. because the A_n are s.p.i. and inductive limits of s.p.i. C^*-algebras are s.p.i.

Corollary 3.2.23. *Let $h_n \colon A_n \to A_{n+1}$ a sequence of C^*-algebra morphisms. Suppose that*

(i) *$B := \mathrm{indlim}(h_n \colon A_n \to A_{n+1})$ is non-zero, and*
(ii) *each A_n is a finite direct sum of algebras $\mathrm{C}(X_j, D_j)$ of continuous functions from compact spaces X_j into a simple purely infinite C^*-algebra D_j $(j = 1, \dots, k_n)$.*

Then B is strongly purely infinite.

Proof. By Lemma 3.2.22, the algebras A_n are strongly purely infinite in the sense of Definition 1.2.2. Therefore, B is strongly purely infinite by Proposition **??**(??) (See Proposition 2.15.5(iv) for the p.i. case.) □

3. From m.o.c. cones to morphisms in general position

We need a good metric on m.o.c. cones, that gives the point-norm convergence topology in case of separable A and countably generated $\mathcal{C} \subseteq \mathrm{CP}(A, B)$. And have to describe the natural transformations and extensions with respect to Morita equivalence.

Need

$$\mathrm{KK}(\mathcal{C};\, A, B) \cong \mathrm{KK}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{K}, \mathbb{K});\, A \otimes \mathbb{K}, B \otimes \mathbb{K}) .$$

Need

$$\mathrm{Ext}(\mathcal{C};\, A, B) \cong \mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{K}, \mathbb{K});\, A \otimes \mathbb{K}, B \otimes \mathbb{K}) .$$

Compare with related sections on m.o.c.c's versus bi-modules !!

Definition 3.3.1. Let B a stable σ-unital C^*-algebra and A a σ-unital C^*-algebra. A C^*-morphism $H \colon A \to \mathcal{M}(B)$ is **in general position** if H is *unitarily homotopic to its infinite repeat $\delta_\infty \circ H$*

By Definition 5.0.1 and Remark 5.1.1(8) this means that there exists a *norm-continuous* path of unitaries $t \in [0, \infty) \to U(t) \in \mathcal{M}(B)$ such that $U(t)^* H(a) U(t) - \delta_\infty(H(a)) \in B$ and, for all $a \in A$,

$$\lim_{t \to \infty} \|U(t)^* H(a) U(t) \,-\, \delta_\infty(H(a))\| \,=\, 0\,.$$

It is not clear if the condition $U(t)^* H(a) U(t) - \delta_\infty(H(a)) \in B$ (for all $a \in A$) in Definition 3.3.1 implies that for the norm-continuous map $t \in [0, \infty) \to X(t)$ given by $X(t) := U(t)^* H(a) U(t) - \delta_\infty(H(a)) \in B$ holds that $X(t)$ converges strictly to 0 if and only if $t \to \|X(t)\|$ converges to 0 for $t \to \infty$. If it is true, then it does not matter if we require that $\lim_{t \to \infty} X(t) = 0$ in the strict topology on $\mathcal{M}(B)$ or in norm topology on $\mathcal{M}(B)$. Here "strictly" says that $\lim_t \|X(t)b\| = 0$ for each $b \in B$, and "in norm-topology" means that $\lim_t \|X(t)\|^2 = 0$ for the bounded norm-continuous path $t \mapsto X(t) \in B$.

But it is not clear if one can here replace the strictly continuous path $t \in \mathbb{R}_+ \to U(t) \in \mathcal{U}(\mathcal{M}(B))$ by a norm-continuous path.

It is known ???? by definition ???? that each C^*-morphism $H \colon A \to \mathcal{M}(B)$ " in sufficiently general position", i.e., where H is unitary equivalent modulo B to $\delta_\infty \circ H$, is unitarily homotopic to its infinite repeat – in the sense of Definition 5.0.1, cf. Remark 5.1.1(8).

4. M.o.c. cones and Hilbert bi-modules

Remark 3.4.1. For every point-norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ there is a (right) Hilbert B-module $\mathcal{H}$ and a *-representation $d \colon A \to \mathcal{L}(\mathcal{H})$ such that for every $x \in \mathcal{H}$ the map $a \in A \mapsto \langle d(a)x, x\rangle \in B$ is in $\mathcal{C}$ and *for every* $V \in \mathcal{C}$ there exists $x \in \mathcal{H}$ such that $V(a) = \langle d(a)x, x\rangle$ for $a \in A$.

But even for separable A and B the needed "multiplicity" of $\mathcal{H}$ over B can be much bigger than the cardinality of a subset of $\mathcal{C}$ that is dense in $\mathcal{C}$ with respect to the matricial point-norm topology on $\mathrm{CP}(A, B)$. This can be seen even in the – for us not important – easy case where $A := \mathrm{C}([0,1])$, $B := \mathbb{C}$ and $\mathcal{C} := \mathrm{CP}(A, B) \cong \mathrm{C}([0,1])^*_+$.

Suppose, in addition, that A is separable, B is σ-unital and stable and that $\mathcal{C} \subseteq \mathrm{CP}(A, B) \subseteq \mathcal{L}(A, B)$ is countably generated, faithful and non-degenerate. Then an other module is given by a C^*-morphism $H_\mathcal{C} \colon A \to \mathcal{M}(B)$ such that $H_\mathcal{C}$ is unitarily equivalent to its infinite repeat $\delta_\infty \circ H_\mathcal{C}$, that $b^* H_\mathcal{C}(\cdot) b \in \mathcal{C}$ for each $b \in B$, and that for each $V \in \mathcal{C}$ with $\|V\| \leq 1$ there exist a sequence of contractions $b_1, b_2, \ldots \in B$ such that $V(a) := \lim_n b_n^* H_\mathcal{C}(a) b_n$ for all $a \in A$.

The $H_\mathcal{C}$ with this properties is uniquely determined by $\mathcal{C}$ up to unitary homotopy.

(Uses general W-vN-theorem: Sums of two of them will be asymptotic unitarily absorbed by each of them ...?).

(See Corollary 5.4.4 and Remark 5.4.5.)

Compare next with Parts (xvi) of Proposition 2.2.1 in Chp. 2 !
There it is the non-unital case.

LEMMA 3.4.2. *Suppose that $1_E \in A \subseteq E$ are C^*-algebras, where A is separable and for every $a \in A_+$ with $\|a\| = 1$ there exists a sequence of contractions $T_1, T_2, \ldots \in E$ with $\lim_n T_n^* a T_n = 1_E$. (In particular, each nonzero $a \in A_+$ is properly infinite inside E and generates E as closed ideal.)*

Then every factorable c.p. contraction $U \circ V$ with $V : A \to M_n$ and $U : M_n \to E$ is approximately 1-step inner in E, i.e., there exists a sequence of contractions $S_1, S_2, \ldots \in E$ such that $\lim_n S_n^* a S_n = U(V(a))$ for all $a \in A$.

PROOF. Should be part of Chapter 2 **??** □

More general alternative formulation for non-unital and not necessarily simple E:

LEMMA 3.4.3. *Suppose that $A \subseteq E$ are C^*-algebras, where A is separable and for every $a \in A_+$ with $\|a\| = 1$ and every contraction $b \in E_+$ in the closed ideal of E generated by a there exists a sequence of contractions $T_1, T_2, \ldots \in E$ with $\lim_n T_n^* a T_n = b$.*

Then every c.p. contraction $U \circ V$ given by c.p. contractions $V : A \to M_n$ and $U : M_n \to E$ such that $U \circ V$ is ideal-system preserving

Should work only for E with finitely many ideals???
We need case of simple A or simple E!

is approximately 1-step inner in E, i.e., there exists a sequence of contractions $S_1, S_2, \ldots \in E$ such that $\lim_n S_n^* a S_n = U(V(a))$ for all $a \in A$.

PROOF. **??** □

REMARKS 3.4.4. Let A a separable C^*-algebra and B a σ-unital stable C^*-algebra. Take any faithful non-degenerate *-representation $\varphi : A \to \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2(\mathbb{N}))$ and any non-degenerate *-representation $\lambda : \mathbb{K} \to \mathcal{M}(B)$.

Then the H_0 for $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ is up to unitary homotopy given by $H_0 := \delta_\infty \circ \mathcal{M}(\lambda) \circ \mathcal{M}(\varphi)$.

If D is non-unital, then $\pi_B \circ H_0$ dominates zero.

If a C^*-morphism $\psi : A \to \mathrm{Q}(B) := \mathcal{M}(B)/B$ has the property that for every $a \in A_+$ and $\varepsilon > 0$ there exists a contraction $T_a \in \mathrm{Q}(B)$ with $T_a^* \psi(a) T_a = \|a\|(1 - \varepsilon)1$, then ψ and $\psi \oplus \pi_B \circ H_0$ are unitarily equivalent by a unitary in $\mathcal{U}_0(\mathrm{Q}(B))$ if B is non-unital. If A and ψ are both unital then ψ and $\psi \oplus \pi_B \circ H_0$ are equivalent by unitary in $\mathcal{U}(\mathrm{Q}(B))$.

The existence of the T_a is equivalent to the property that for each $a \in A_+$, $\delta \in (0, \|a\|)$ and $c \in \mathcal{M}(B)$ with $\pi_B(c^* c) = \psi((a - \delta)_+)$ the hereditary *-subalgebra cBc^* of B contains a full stable C^*-subalgebra of B.

(Compare also [**264**] and [**310**].)

DEFINITION 3.4.5. Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a matrix o.c. cone. A Hilbert A-B-module $(\mathcal{H},\phi)$ is **$\mathcal{C}$-compatible** if the c.p. maps

$$A \ni a \mapsto \langle x, \phi(a)x\rangle \in B$$

are in $\mathcal{C}$ for all $x \in \mathcal{H}$.

DEFINITION 3.4.6. A point-norm closed matrix o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is **countably generated** if $\mathcal{C} = \mathcal{C}(\mathcal{S})$ for some countable subset $\mathcal{S} \subseteq \mathcal{C}$. We say that $\mathcal{C}$ is **singly generated** if $\mathcal{C} = \mathcal{C}(\{V\})$ for some $V \in \mathcal{C}$.

REMARK 3.4.7. If a point-norm closed m.o.c.c. $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is countably generated, say by c.p. contractions $\{V_1, V_2, \ldots\} \subseteq \mathcal{C}$, then $\mathcal{C}$ is also singly generated, e.g. by $V := \sum_n 2^{-n} V_n$, because $\mathcal{C}$ is hereditary in $\mathrm{CP}(A,B)$ by Corollary **??**.

If B is σ-unital (with strictly positive element $b_0 \in B_+$) and if A admits a faithful positive functional $f_0 \in A^*_+$, then $\mathrm{CP}_{\mathrm{nuc}}(A,B)$ is singly generated by $V = f_0(\cdot)b_0$.

If A and B are separable, then each point-norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is singly generated.

PROPOSITION 3.4.8. *Suppose that a compact metric group G acts on A and B (by α and β), where A is separable and B is σ-unital. Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a countably generated m.o.c. cone with $\beta(g) \circ \mathcal{C} \circ \alpha(g^{-1}) = \mathcal{C}$ for all $g \in G$.*

Then there is a "universal" $\mathcal{C}$-compatible Hilbert A-B-module $(\mathcal{H},\phi)$ with a G-action $\gamma\colon G \to \mathrm{Iso}(\mathcal{H})$, such that $\gamma(g)(\phi(a)xb) = \phi(\alpha(g)(a))\gamma(x)\beta(g)(b)$ for $a \in A$, $b \in B$, $x \in \mathcal{H}$, and that $G \ni g \mapsto \langle y, \gamma(g)(x)\rangle \in B$ is continuous for all $x, y \in \mathcal{H}$.

PROOF. Use G-invariant $V \in \mathcal{C}$ for construction. Then use that m.o.c. cones are hereditary in $\mathrm{CP}(A,B)$.

`to be filled in` **??** □

QUESTION 3.4.9. Remains Proposition 3.4.8 true if G is a second countable l.c. group?

5. Operator-convex cones of c.p. maps (2)

This section explains the reason for our study of commutative subalgebras in ultrapowers of w.p.i. and s.p.i. C^*-algebras, i.e., it shows that asymptotic and approximate versions of generalized Weyl–von-Neumann theorems for "residually nuclear" maps can be obtained from the corresponding results on c.p. maps from separable C^*-subalgebras of ultrapowers (or, more generally, corona algebras) into commutative C^*-subalgebras of the ultrapower or corona algebra in question. One has to apply the below outlined theory of (matrix) operator-convex cones of c.p. maps (in conjunction with Property (3) in Remark 3.11.3).

We introduce a useful duality between point-norm closed operator-convex cones $\mathcal{C}$ of completely positive maps in $\mathrm{CP}(A,B)$ and intersection-preserving maps ("actions") of the Hausdorff lattice $\mathcal{I}(B\otimes C^*(F_\infty))$ of closed ideals of $B\otimes C^*(F_\infty)$ on $A\otimes C^*(F_\infty)$.

This in an operator theoretic version of the Hahn-Banach separation theorem for m.o.c. cones.

It implies a duality between m.o.c. cones $\mathcal{C}$ of "residually nuclear" maps and actions Ψ of $\mathrm{Prim}(B)$ on A. Global operations with residually nuclear maps are functorial in a natural manner.

Definition 3.5.1. Suppose that A and B are C^*-algebras. Let $\mathrm{CP}(A,B)$ denote the cone of completely positive maps from A into B. A subset $\mathcal{C}$ of $\mathrm{CP}(A,B)$ is an *operator-convex cone* of c.p. maps if $\mathcal{C}$ has the following properties (i) and (ii):

(i) $d_1^*V_1(.)d_1+d_2^*V_2(.)d_2\in\mathcal{C}$ for $V_1,V_2\in\mathcal{C}$, $d_1,d_2\in B$.
(ii) $a\in A\mapsto c^*(V\otimes \mathrm{id}_n)(r^*ar)c$ is in $\mathcal{C}$ for every $V\in\mathcal{C}$, every row-matrix $r\in M_{1,n}(A)$ and every column-matrix $c\in M_{n,1}(B)$.

Let $\mathcal{S}$ be a subset of $\mathrm{CP}(A,B)$. We denote by $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ the smallest subset of $\mathrm{CP}(A,B)$ which is invariant under the operations in (i) and (ii), and by $\mathcal{C}(\mathcal{S})$ the point-norm closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ (i.e., the closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ in $\mathcal{L}(A,B)$ w.r.t. the strong operator topology). Then $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ and $\mathcal{C}(\mathcal{S})$ are operator-convex cones of completely positive maps.

We call $\mathcal{S}$ the *generating set* for the operator convex cone $\mathcal{C}$ if $\mathcal{C}=\mathcal{C}(\mathcal{S})$.

One can see with help of approximate units in A and B that every point-norm closed subset $\mathcal{C}$ of $\mathrm{CP}(A,B)$ is a convex cone in the usual sense if it satisfies Part(i) of Definition 3.5.1, and that $\mathcal{C}$ satisfies $t\mathcal{C}\subseteq\mathcal{C}$ for $t\in[0,\infty)$ if $\mathcal{C}$ satisfies Definition 3.5.1(ii).

We introduce a useful duality between closed m.o.c. cones $\mathcal{C}$ of completely positive maps in $\mathrm{CP}(A,B)$ and intersection-preserving maps ("actions") of the Hausdorff lattice $\mathcal{I}(B\otimes C^*(F_\infty))$ of closed ideals of $B\otimes C^*(F_\infty)$ on $A\otimes C^*(F_\infty)$. It leads to a duality of m.o.c. cones of "residually nuclear" maps and actions of $\mathrm{Prim}(B)$ on A. Operations with residually nuclear maps are functorial in a natural manner.

First we characterize the c.p. maps V in $\mathcal{C}(\mathcal{S})$ by the values $V\otimes^{\max}\mathrm{id}(c)$ of elements $c\in A\otimes^{\max}C^*(F_\infty)$. Here $C^*(F_\infty)$ denotes the full group C^*-algebra of the free group F_∞ on infinitely many generators, and $\otimes^{\max}$ denotes the maximal C^*-algebra tensor product. In the following theorem the *ideal-kernel* of a positive functional λ on $D:=B\otimes^{\max}C^*(F_\infty)$ means the maximal ideal of D in the kernel of λ, i.e., the set of elements $d\in D$ with $\lambda(edf)=0$ for all $e,f\in D$. This is also the kernel of the cyclic representation $\rho\colon D\to\mathcal{L}(L_2(D,\lambda))$ defined by λ on D.

Here is some relation between Part(v) of Theorem 3.5.2 to and the later considered *Separation* Theorem.

THEOREM 3.5.2. *Suppose that $\mathcal{S}$ is a subset of* $\mathrm{CP}(A,B)$ *and that* $V\colon A \to B$ *is completely positive. Then the following properties* (i)–(v) *of* V *are equivalent.*

(i) $V \in \mathcal{C}(\mathcal{S}) :=$ *the point-norm closed m.o.c. cone generated by* $\mathcal{S}$.

(ii) *For every* $c \in A \otimes^{\max} C^*(F_\infty)$, *the element* $V \otimes^{\max} \mathrm{id}(c)$ *is in the closed ideal of* $B \otimes^{\max} C^*(F_\infty)$ *that is generated by the c.p. maps*

$$\{\, W \otimes^{\max} \mathrm{id}((a^* \otimes 1)c(a \otimes 1))\,;\; W \in \mathcal{S}, a \in A\,\}\,.$$

(iii) *For every factorial representation* $\rho\colon B \to \mathcal{L}(\mathcal{H})$ *of* B, $\rho \circ V$ *is in the point-weak closure of* $\mathcal{C}(\{\,\rho \circ W\,;\; W \in \mathcal{S}\,\})$.

(iv) *For every pure state* λ *of* $B \otimes^{\max} C^*(F_\infty)$, *the kernel of* $\lambda \circ (V \otimes^{\max} \mathrm{id})$ *contains the intersection of all ideal-kernels of* $\lambda \circ (b^*Wb \otimes^{\max} \mathrm{id})$ *for* $W \in \mathcal{S}$, $b \in B$.

(v) *For every factorial cyclic representation* $\rho\colon B \to \mathcal{L}(\mathcal{H})$ *(with cyclic vector* $\eta \in \mathcal{H}$*) and every unital* C^**-morphism* $h\colon C^*(F_\infty) \to \rho(B)'$, *the kernel of the positive linear functional* $\lambda \circ (V \otimes^{\max} \mathrm{id})$ *contains the intersection of all ideal-kernels of* $\lambda \circ ((b^*W(\cdot)b) \otimes^{\max} \mathrm{id})$ *for* $W \in \mathcal{S}$, $b \in B$.
(Here λ *is the positive functional on* $B \otimes^{\max} C^*(F_\infty)$ *with* $\lambda(b \otimes f) = \langle \rho(b)h(f)\eta, \eta\rangle$*.)*

If one wants to apply the result easier for some sort of cone-related KK-theory, then one could replace $C^*(F_\infty) \subseteq C^*(F_2) = \mathrm{C}(S^1) * \mathrm{C}(S^1)$ by the unital free product C^*-algebra $\mathrm{C}([0,1]) * \mathrm{C}([0,1]) \supset C^*(F_2)$. The proofs are almost verbatim the same as the below given, but with $C^*(F_\infty)$ replaced by $\mathrm{C}([0,1]) * \mathrm{C}([0,1])$. The point is that this shows that no extra homotopy invariant is involved by this reduction to the study of ideals of tensor products.

Alternatively we can formulate the for applications important equivalence between Parts (i) and (ii) as follows:

Let $\mathcal{T} \subseteq \mathrm{CP}\big(A \otimes^{\max} C^*(F_2), B \otimes^{\max} C^*(F_2)\big)$ denote the set of c.p. maps

$$\mathcal{T} := \{\, W \otimes^{\max} \mathrm{id}\,;\; W \in \mathcal{S}\,\}\,.$$

And define a lower semi-continuous action

$$\Psi := \Psi[\mathcal{T}]\colon \mathcal{I}\big(B \otimes^{\max} C^*(F_2)\big) \to \mathcal{I}\big(A \otimes^{\max} C^*(F_2)\big)$$

by

$$\Psi(J) := \text{ biggest } I \in \mathcal{I}(A \otimes^{\max} C^*(F_2)) \text{ with } W \otimes^{\max} \mathrm{id}(I) \subseteq J \;\; \forall\, W \in \mathcal{S}\,.$$

I.e., Ψ is the lower semi-continuous action of $\mathrm{Prim}\big(B \otimes^{\max} C^*(F_2)\big)$ on $A \otimes^{\max} C^*(F_2)$ that attaches to a closed ideal $J \triangleleft B \otimes^{\max} C^*(F_2)$ the largest ideal $I \triangleleft A \otimes^{\max} C^*(F_2)$ with the property that $(W \otimes \mathrm{id})(I) \subseteq J$ for all $W \in \mathcal{S}$.

This is just the natural l.s.c. action $\Psi_{\mathcal{C}(\mathcal{T})}$ defined by the point-norm closed m.o.c. cone $\mathcal{C}(\mathcal{T})$ that is generated by $\mathcal{T} := \mathcal{S} \otimes^{\max} \mathrm{id}$.

The property of V in part (ii) is equivalent to:

$$V \otimes^{\max} \mathrm{id}(\Psi_{\mathcal{C}(\mathcal{T})}(J)) \subseteq J \quad \text{for all } J \in \mathcal{I}(B \otimes^{\max} C^*(F_2))\,.$$

Thus the equivalence of (i) and (ii) can be expressed as

$$V \in \mathcal{C}(\mathcal{S}) \iff V \otimes^{\max} \mathrm{id}(\Psi_{\mathcal{C}(\mathcal{S}\otimes^{\max}\mathrm{id})}(J)) \subseteq J \ \forall\, J \in \mathcal{I}(B \otimes^{\max} C^*(F_2)) .$$

If we let $\Psi := \Psi_{\mathcal{C}(\mathcal{S}\otimes^{\max}\mathrm{id})}$, where "id" means the identity automorphism of $C^*(F_\infty)$, and if we define, more generally, for C^*-algebras E, F and actions $\Psi\colon \mathcal{I}(F) \to \mathcal{I}(E)$ the point-norm closed m.o.c. cone $\mathcal{C}_\Psi \subseteq \mathrm{CP}(E,F)$ by

$$\mathcal{C}_\Psi := \{V \in \mathrm{CP}(E,F)\, ;\ V(\Psi(J)) \subseteq J \ \forall\, J \in \mathcal{I}(F)\} ,$$

then we can express the equivalence of (i) and (ii) by

$$\mathcal{C}(\mathcal{S}) \otimes^{\max} \mathrm{id} \ = \ (\mathrm{CP}(A,B) \otimes^{\max} \mathrm{id}) \cap \mathcal{C}_\Psi . \tag{5.1}$$

Proof. (i)⇒(ii): Let $c \in B \otimes^{\max} C^*(F_\infty)$ positive and let $I(\mathcal{S}, c)$ denote the closed ideal of $B \otimes^{\max} C^*(F_\infty)$ which is *generated* by

$$I(\mathcal{S}, c) := \{\, W \otimes^{\max} \mathrm{id}((a^* \otimes 1)c(a \otimes 1))\, ;\ W \in \mathcal{S}, a \in A \,\} .$$

If $T\colon A \to B$ is a completely positive map such that $T \otimes^{\max} \mathrm{id}((a^* \otimes 1)c(a \otimes 1))$ is in $I(\mathcal{S}, c)$ for all $a \in A$, then $T \otimes^{\max} \mathrm{id}((b^* \otimes 1)c(a \otimes 1))$ is in $I(\mathcal{S}, c)$ for all $a, b \in A$, as polar-decomposition of Hermitian forms shows. It follows that the set of all c.p. maps $T\colon A \to B$ with $T \otimes^{\max} \mathrm{id}((a^* \otimes 1)c(a \otimes 1)) \in I(\mathcal{S}, c)$ for all $a \in A$ contains $\mathcal{S}$ and is closed under the operations (OC1) and (OC2) of Definition 3.2.2, and is point-norm closed. Thus it contains $\mathcal{C}(\mathcal{S})$.

(ii)⇒(i): Suppose that V is not in $\mathcal{C}(\mathcal{S})$. Then there are

- (α) a cyclic representation $\rho\colon B \to \mathcal{L}(\mathcal{H})$ with cyclic vector $\eta \in \mathcal{H}$,
- (β) elements $e_1, \ldots, e_n \in \rho(B)'$,
- (γ) elements $a_1, \ldots, a_n \in A$, and
- (δ) a *-epimorphism h from $C^*(F_\infty)$ onto $C^*(1, e_1, \ldots, e_n) \in \rho(B)'$ and elements $f_1, \ldots, f_n \in C^*(F_\infty)$ with $h(f_k) = e_k$,

such that the complex number $\lambda((V \otimes^{\max} \mathrm{id})(c))$ is not in the closure of the set of complex numbers $\{\, \lambda((W \otimes^{\max} \mathrm{id})(c))\, ;\ W \in \mathcal{C}(\mathcal{S}) \,\}$, where

$$c \ := \ a_1 \otimes f_1 + \ldots + a_n \otimes f_n \ \in A \otimes^{\max} C^*(F_\infty)$$

and λ is the positive functional on $B \otimes^{\max} C^*(F_\infty)$ with $\lambda(b \otimes f) := \langle \rho(b)h(f)\eta, \eta\rangle$. (See proof of [**463**, lem. 7.18], and notice that the Part (δ) is obvious.)

Thus, the positive functional $\lambda_V := \lambda \circ (V \otimes^{\max} \mathrm{id})$ is not in the weak closure of the convex cone κ of positive functionals $\lambda_W := \lambda \circ (W \otimes^{\max} \mathrm{id})$ for $W \in \mathcal{C}(\mathcal{S})$.

Since η is cyclic for ρ, one can see (as in the proof of [**463**, lem. 7.18]) that the functionals $d^*(\lambda_W)d$ is in the weak closure of κ for $W \in \mathcal{C}(\mathcal{S})$ and $d \in A \otimes^{\max} C^*(F_\infty)$.

Let J denote the closed ideal of $d \in A \otimes^{\max} C^*(F_\infty)$ with $\lambda_W(d^*d) = 0$ for all $W \in \mathcal{C}(\mathcal{S})$.

It follows from [**463**, lem. 7.17(ii)] that there is $d_0 \in J$ with $\lambda_V(d_0^* d_0) > 0$. On the other hand, $I(\mathcal{S}, d_0^* d_0)$ is contained in the kernel of λ by definition of J. Hence $V \otimes^{\max} \mathrm{id}(d_0^* d_0)$ is not in $I(\mathcal{S}, d_0^* d_0)$.

(i)⇒(v): Let $J(\mathcal{S}) \subseteq A \otimes^{\max} C^*(F_\infty)$ denote the intersection of all ideal-kernels of all functionals $\lambda \circ ((b^* W b) \otimes^{\max} \mathrm{id})$ on $A \otimes^{\max} C^*(F_\infty)$ with $W \in \mathcal{S}$, $b \in B$. We show that $J(\mathcal{S})$ is the intersection of all ideal-kernels of all functionals $\lambda \circ (V \otimes^{\max} \mathrm{id})$ with $V \in \mathcal{C}(\mathcal{S})$:
Let $T \in \mathrm{CP}(A, B)$ with $J(\mathcal{S})$ in the kernel of $\lambda \circ ((b^* T b) \otimes^{\max} \mathrm{id})$ for all $b \in B$, then, using the polar-formula for Hermitian forms,

$$\lambda((b_2^* \otimes 1)T \otimes^{\max} \mathrm{id}((a_2^* \otimes 1)c(a_1 \otimes 1))(b_1 \otimes 1)) = 0$$

for $c \in J$, $a_1, a_2 \in A$ and $b_1, b_2 \in B$. It follows that the set of c.p. maps $V : A \to B$ with $J(\mathcal{S})$ in the kernel of $\lambda \circ ((b^* T b) \otimes^{\max} \mathrm{id})$ for all $b \in B$ is closed under the operations (OC1) and (OC2) of Definition 3.2.2 and under pont-norm convergence, i.e., is a closed operator-convex cone, which contains the set $\mathcal{S}$. Note that $b^* V(.) b$ is in $\mathcal{C}(\mathcal{S})$ for $V \in \mathcal{C}(\mathcal{S})$ and $b \in B$.

(v)⇒(iii): Suppose that $\rho \circ V : A \to \mathcal{L}(\mathcal{H})$ is not in the point-weak closure of the operator-convex cone $\mathcal{C}(\{\rho \circ W;\, W \in \mathcal{S}\})$.
Then there is $n \in \mathbb{N}$ and $\eta \in \mathcal{H}^n$, such that $\rho_1 \circ V : A \to \mathcal{L}(\mathcal{H}_1)$ is not in the point-weak closure of $\mathcal{C}(\{\rho_1 \circ W;\, W \in \mathcal{S}\})$ for the (again factorial) cyclic sub-representation ρ_1 of

$$\rho \otimes 1_n \colon\, b \in B \mapsto \rho(b) \otimes 1_n \,\in \mathcal{L}(\mathcal{H}^n) \cong \mathcal{L}(\mathcal{H}) \otimes M_n$$

obtained by restriction of $\rho \otimes 1_n$ to $\mathcal{H}_1 := \overline{(\rho(B) \otimes 1_n)\eta}$. The rest of the proof is similar to the proof of the implication (ii)⇒(i).

(iii)⇒(iv): Let $\iota \colon B \otimes^{\max} C^*(F_\infty) \to \mathcal{L}(\mathcal{H})$ the irreducible representation with cyclic state η corresponding to λ. Let $\rho(b) := \iota(b \otimes 1)$. Then $\rho \colon B \to \mathcal{L}(\mathcal{H})$ is factorial, and there is a representation $h \colon C^*(F_\infty) \to \rho(B)'$ such that $\iota(b \otimes f) = \rho(b) h(f)$.

The elements c in the intersection J of the ideal-kernels of $\lambda \circ \big((b^* W b) \otimes^{\max} \mathrm{id}\big)$ with $W \in \mathcal{S}$, $b \in B$ satisfy $\iota\big(W \otimes^{\max} \mathrm{id}(dce)\big) = 0$ for all $d, e \in A \otimes^{\max} C^*(F_\infty)$ and $W \in \mathcal{S}$.

If $c \in A \otimes^{\max} C^*(F_\infty)$ satisfies $\iota\big(W \otimes^{\max} \mathrm{id}(dce)\big) = 0$ for all $d, e \in A \otimes^{\max} C^*(F_\infty)$ and $W \in \mathcal{S}$, then $\iota\big(T \otimes^{\max} \mathrm{id}(c)\big) = 0$ for all $T \in \mathcal{C}(\mathcal{S})$. Since $\rho \circ V$ is in the point-weak closure of the maps $\{\, \rho \circ W \,;\; W \in \mathcal{C}(\mathcal{S}) \,\}$, we get that $\iota \circ (T \otimes^{\max} \mathrm{id})$ is in the point weak closure of the cone of maps $\iota \circ (W \otimes^{\max} \mathrm{id})$. Thus $\lambda\big(W \otimes^{\max} \mathrm{id}(c)\big) = 0$.

(iv)⇒(ii): Suppose that there is $c \in A \otimes^{\max} C^*(F_\infty)$, such that $V \otimes^{\max} \mathrm{id}(c)$ is not in $I(\mathcal{S}, c)$. Then there is a pure state λ on $B \otimes^{\max} C^*(F_\infty)$ with $\lambda(I(\mathcal{S}, c)) = 0$ and $\lambda(V \otimes^{\max} \mathrm{id}(c)) \neq 0$.

$(b^* \otimes 1)W \otimes^{\max} \mathrm{id}(gch)(b \otimes 1) \,\in I(\mathcal{S}, c)$ for $W \in \mathcal{S}$, $g, h \in A \otimes^{\max} C^*(F_\infty)$ and $b \in B$, because $W \otimes^{\max} \mathrm{id}\big((a^* \otimes 1)c(a \otimes 1)\big)$ is in the ideal $I(\mathcal{S}, c)$ for $a \in A$.

Thus, the closed ideal J of $A \otimes^{\max} C^*(F_\infty)$ generated by c is contained in all kernels of $\lambda \circ ((b^*Wb) \otimes^{\max} \mathrm{id})$ for $W \in \mathcal{S}$, $b \in B$. But c is not in the kernel of $\lambda \circ (V \otimes^{\max} \mathrm{id})$. □

We derive in good cases 1-step approximately inner approximation for the members of $\mathcal{C}(\mathcal{S})$ if the elements of $\mathcal{S}$ are 1-step-approximately inner.

PROPOSITION 3.5.3. *Suppose that A and B are C^*-algebras and that $\mathcal{S}$ is a set of c.p. maps $V: A \to B$.*

(i) *Suppose that, for $V_1, V_2 \in \mathcal{S}$, the multiplier algebra $\mathcal{M}(B)$ contains contractions e_1, e_2 with $e_1^* e_2 = 0$ such that $e_1 V_1(.) e_1^* + e_2 V_2(.) e_2^* \in \mathcal{S}$, $e_k^* e_k V_k(a) = V_k(a)$ for all $a \in A$ and $k = 1, 2$.*
(a local property of J.Cuntz)??
Then $\mathcal{C}(\mathcal{S})$ is the point-norm closure of the set of all maps

$$V: a \in A \mapsto c^*\big((W \otimes \mathrm{id}_n)(r^* a r)\big)c \in B$$

with $W \in \mathcal{S}$, rows $r \in M_{1,n}(A)$ and columns $c \in M_{n,1}(B)$, $n \in \mathbb{N}$.

(ii) *If, in addition to part (i), $A \subseteq \mathcal{M}(B)$ and there is (fixed) $m \in \mathbb{N}$ such that every $W \in \mathcal{S}$ is approximately m-step-inner, then every c.p. contraction $V \in \mathcal{C}(\mathcal{S})$ can be approximated in point-norm by maps*

$$T: a \in A \mapsto d_1^* a d_1 + d_2^* a d_2 + \ldots + d_m^* a d_m \in B$$

with $d_1, \ldots, d_m \in B$ and $\|d_1^* d_1 + \ldots + d_m^* d_m\| \leq 1$.

(iii) *If, in addition to (ii), A is separable and for every $W \in \mathcal{S}$ there exists a bounded sequence of elements $d_1, d_2, \dots \in B$ such that $\lim_{k\to\infty} d_k^* a d_{k+n} = \delta_{0,n} W(a)$ for $n = 0, 1, \ldots$, $a \in A$, then, for every $V \in \mathcal{C}(\mathcal{S})$, there exists a sequence $e_1, e_2, \ldots \in B$ with $\|e_k^* e_k\| \leq \|V\|$, $\lim_{k\to\infty} e_k^* a e_{k+n} = \delta_{0,n} V(a)$ for $n = 0, 1, \ldots$, $a \in A$, and $\lim_{k\to\infty} \|e_k^* e_{k+n}\| = 0$ for $n > 0$.*

PROOF. (i): For $V_1, V_2 \in \mathcal{S}$, $r_1 \in M_{1,n}(A)$, $r_2 \in M_{1,m}(A)$, $c_1 \in M_{n,1}(B)$, $c_2 \in M_{m,1}(B)$ and $d_1, d_2 \in B$, let $V(a) := e_1 V_1(a) e_1^* + e_2 V_2(a) e_2^*$ $(a \in A)$, $r := (r_1, r_2) \in M_{1,n+m}(A)$, and $d := (d_1^* c_1^* (e_1 \otimes 1_n), d_2^* c_2^* (e_2 \otimes 1_m))^* \in M_{n+m,1}(A)$, where $e_1, e_2 \in \mathcal{M}(B)$ are the elements as descibed in (i).

Then $d_1^*(c_1^* V_1 \otimes \mathrm{id}_n(r_1^* a r_1) c_1) d_1 + d_2^*(c_2^* V_2 \otimes \mathrm{id}_m(r_2^* a r_2) c_2) d_2 = c^* V \otimes \mathrm{id}_{n+m}(r^* a r) c$. Thus the minimal matrix operator-convex hull $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ of $\mathcal{S}$ is identical with the set of maps $c^* V \otimes \mathrm{id}_n(r^*(.) r) c$ with $V \in \mathcal{S}$, $r \in M_{1,n}(A)$, $c \in M_{n,1}(A)$, $n = 1, 2, \ldots$. $\mathcal{C}(\mathcal{S})$ is then the point-norm closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$.

(ii): By [**463**, lem. 7.2], a suitable modification of the maps T (with help of an approximate unit of A) leads to a net of m-step inner c.p. maps T with $\|d_1^* d_1 + \ldots + d_m^* d_m\| \leq 1$ (in addition) such that the net converges in point-norm to V, if $V \in CB(A, B)$ can be approximated in point-norm by m-step inner c.p. maps T.

If $V \in \mathcal{S}$ can be approximated in point-norm by a net of m-step inner c.p. maps $T = d_1^*(.)d_1 + \ldots + d_m^*(.)d_m$, then $c^*V \otimes \mathrm{id}_n(r^*(.)r)c$ can be approximated in point-norm by the net with elements $T' = c^*T \otimes \mathrm{id}_n(r^*(.)r)c = e_1^*(.)e_1 + \ldots + e_m^*(.)e_m$ where $e_j = r(d_j \otimes 1_n)c$ for $j = 1, \ldots, m$.

(iii): If $V = c^*W \otimes \mathrm{id}_n(r^*ar)c$ and $d_1, d_2, \ldots$ satisfies $\lim_{k\to\infty} d_k^*ad_{k+n} = \delta_{0,n}W(a)$ for $n = 0, 1, \ldots$, then $f_k := r(d_k \otimes 1_n)c$ satisfies $\lim_{k\to\infty} f_k^*af_{k+n} = \delta_{0,n}V(a)$ for $n = 0, 1, \ldots$.

If we multiply the sequence f_k from the left by the elements in a suitable countable approximate unit $a_1, a_2, \ldots$ of A then $e_k := a_kf_k$ satisfies $\|e_k^*e_k\| < \|V\|$, $\lim_{k\to\infty} e_k^*ae_{k+n} = \delta_{0,n}V(a)$ for $n = 0, 1, \ldots$, and $\lim_{k\to\infty} \|e_k^*e_{k+n}\| = 0$ for $n > 0$. □

REMARK 3.5.4. Suppose that A is separable and $A \subseteq B$, and that $D \subseteq \mathcal{M}(B)$ is a unital C^*-subalgebra of $\mathcal{M}(B)$ with $B \subseteq D$ such that for every $b \in B$ there is an isometry $t \in D$ with $bt = 0$, and that D contains two isometries with orthogonal ranges.

Then one gets for (ii) and (iii) of Proposition 3.5.3:

(α) for T in (i) holds $T(a) = f_1^*af_1 + \ldots + f_m^*af_m$ for all $a \in A$ with $f_1, \ldots, f_m \in D$ such that $f_1^*f_1 + \ldots + f_m^*f_m = \|V\| \cdot 1$ (in particular, f_1 is an isometry in case $m = 1$ and V contractive),

(β) the elements $e_1, e_2, \ldots$ in (iii) can be replaced by isometries $t_1, t_2, \ldots \in D$ with $t_j^*t_k = \delta_{j,k}1$ and $\lim_k \|V\|t_k^*at_{n+k} = \delta_{0,n}V(a)$ for $a \in A$.

PROOF. (α): Let $a_0 \in A_+$ a strictly positive element of A, $g := a_0 + d_1d_1^* + \ldots d_md_m^*$, $h := (\|V\| \cdot 1 - (d_1^*d_1 + \ldots + d_m^*d_m))^{1/2} \in D$, t an isometry in D with $gt = 0$. Then $f_k := d_k + (1/m)th \in D$ satisfies $d_k^*ad_k = f_k^*af_k$ for $a \in A$ and $f_1^*f_1 + \ldots + f_m^*f_m = \|V\|.1$.

(β): Let $g := a_0 + \sum_k 2^{-k}e_ke_k^*$ and $t \in D$ an isometry with $gt = 0$. If D contains two isometries with orthogonal ranges, then D contains a countable sequence $s_1, s_2, \ldots \in D$ of isometries with pair-wise orthogonal ranges: $s_j^*s_k = \delta_{j,k}$. If $V = 0$ then $t_k = ts_k$ is as desired. If $V \neq 0$, let $r_k := e_k + ts_k(\|V\| - e_k^*e_k)^{1/2}$, then $r_k \in D$, $r_j^*r_k = e_j^*e_k + \delta_{j,k}(\|V\| - e_k^*e_k)$ and $r_j^*ar_k = e_j^*ae_k$ for $a \in A$, $j, k = 1, 2, \ldots$. It follows that there is a sequence $u_1, u_2, \ldots$ of unitaries in D such that $\lim_k \|u_k - 1\| = 0$ and $s_k := \|V\|^{-1/2}u_kr_k$ is a sequence of isometries in D with $s_j^*s_k = \delta_{i,k}1$. It must satisfy $\lim_k \|V\|s_k^*as_{k+n} = \delta_{0,n}V(a)$ for $a \in A$. □

DEFINITION 3.5.5. Let Y a T_0 space (e.g. $Y \cong \mathrm{Prim}(D)$ for some algebra D), and $\mathbb{O}(Y)$ its lattice of closed ideals (e.g. $\mathbb{O}(Y) \cong \mathcal{I}(D)$).

We call a map $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{I}(A)$ an *action* or Ψ-*action* of Y on a C^*-algebra A, if $\Psi(U) \subseteq \Psi(V)$ for $U \subseteq V$. (More generally, one can consider increasing maps from down-ward directed partially ordered sets in place of $\mathbb{O}(Y)$ into the set $\mathcal{I}(A)$ of closed ideals of A.)

Ψ is by definition *lower semi-continuous* if $\Psi((\bigcap U_\tau)^\circ) = (\bigcap_\tau \Psi(U_\tau))^\circ$ for every family of open subsets of Y. (We use this name because the intersection property implies that Ψ transforms l.s.c. functions on $\mathrm{Prim}(A)$ into l.s.c. functions on Y in a natural manner.)

If $\Psi_A \colon \mathbb{O}(Y) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(Y) \to \mathcal{I}(B)$ are two actions, then a c.p. map $V \colon A \to B$ is Ψ-equivariant if $V(\Psi_A(U)) \subseteq \Psi_B(U)$ for all $U \in \mathbb{O}(Y)$. The set of Ψ-equivariant maps will be denoted by $\mathcal{C}(\Psi_A, \Psi_B)$. Clearly, $\mathcal{C}(\Psi_A, \Psi_B)$ is a point-norm closed operator-convex cone of c.p. maps, i.e., $\mathcal{C}(\Psi_A, \Psi_B)) = \mathcal{C}_{\mathrm{alg}}(\Psi_A, \Psi_B)$.

If $Y = \mathrm{Prim}(B)$ and Ψ_B is the identity map $\mathrm{id}_{\mathcal{I}(B)}$ of $\mathbb{O}(Y)$, then we write $\mathcal{C}(\Psi_A)$ for $\mathcal{C}(\Psi_A, \Psi_B)$. Clearly $V \in \mathcal{C}(\Psi_A)$ if and only if $V(a) \in J$ for $J \in \mathcal{I}(B)$ and $a \in \Psi_A(U_J)$.

If $\mathcal{S}$ is a subset of $\mathrm{CP}(A, B)$ let

$$\Psi^{\mathcal{S}}(J) := \{a \in A\,;\ V(b^*ab) \in J \ \ \forall V \in \mathcal{S},\ \ \forall b \in A\,\}$$

It is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, because $\mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B)$ naturally, and, obviously,

$$\Psi^{\mathcal{S}}(\bigcap_\tau J_\tau) = \bigcap_\tau \Psi^{\mathcal{S}}(J_\tau)$$

for every family $\{J_\tau\}_\tau$ of closed ideals of B. The action $\Psi^{\mathcal{S}}$ is possibly degenerate, e.g. if $\mathcal{S} = \{0\}$.

Obviously, $\Psi^{\mathcal{S}} = \Psi^{\mathcal{C}(\mathcal{S})}$ and $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}(\Psi^{\mathcal{S}})$. Clearly $V \in \mathcal{C}(\Psi^{\mathcal{S}})$ if and only if, for every $a \in A$, $V(a)$ is contained in the closed ideal of B generated by $W(a)$ for all $W \in \mathcal{S}$.

Example 3.5.6. If $B, C \subseteq A_\omega$, then $Y := \mathrm{Prim}(A_\omega)$ acts on B and C by the maps from Ψ_B and Ψ_C from $\mathbb{O}(Y) \cong \mathcal{I}(A_\omega)$ into $\mathcal{I}(B)$ (respectively $\mathcal{I}(C)$) by $\Psi_B(J) := B \cap J$ (respectively $\Psi_C(J) := C \cap J$). The corresponding action of $\mathrm{Prim}(C)$ on B is in general not l.s.c.

The equivalence of (i) and (ii) of Theorem 3.5.2 gives the following characterization of m.o.c. cones by actions:

Corollary 3.5.7. *Let $\mathcal{S} \subseteq \mathrm{CP}(A, B)$, $V \colon A \to B$ completely positive, and consider the set $\mathcal{T}$ of completely positive maps $W \otimes^{\max} \mathrm{id}$ from $A \otimes^{\max} C^*(F_\infty)$ into $B \otimes^{\max} C^*(F_\infty)$ for $W \in \mathcal{S}$.*

Then $V \in \mathcal{C}(\mathcal{S})$ if and only if $V \otimes^{\max} \mathrm{id} \in \mathcal{C}(\Psi^{\mathcal{S}})$.

`there is a similar result later on`

`compare results and notations ??`

Remark 3.5.8. An action Ψ_B naturally extends to stabilizations or to tensor-products: $\Psi_{B \otimes E}(Z) = \Psi_B(Z) \otimes E$.

If E is simple and nuclear then this extension is unique (as e.g. in the often considered cases where E is one of $\mathcal{O}_\infty$, $\mathcal{O}_2$, $\mathbb{K}$, M_n or the Jiang-Su algebra $\mathcal{Z}$).

A natural extension of Ψ_B to $C_0(X,B)$, e.g. for $X=[0,1]$ or $X=(0,1]$, is given by $\Psi_{C_0(X,B)}(Z) := C_0(X,\Psi_B(Z))$.

Also it extends naturally to Hilbert–B-modules $\mathcal{H}$ and to $\mathcal{L}(\mathcal{H})$ by $\Psi_{\mathcal{H}}(Z) := \overline{\mathcal{H}\Psi_B(Z)}$,

Next Def. o.k.? $\Psi_{\mathcal{L}(\mathcal{H})}(Z) := \{\, T\in\mathcal{L}(\mathcal{H})\,;\; T\mathcal{H}\subseteq \Psi_{\mathcal{H}}(Z)\,\}$ and so on ...

In particular, $\Psi_{M(B)}(Z) = M(B,\Psi_B(Z)) =$ strict closure of $\Psi_B(Z)$, that is $T\in\Psi_{M(B)}(Z)$ if and only if $d^*Td\in\Psi_B(Z)$. (Note that $\Psi_B(Z)$ is an ideal of B.)

Definition 3.5.9. If Ψ_A and Ψ_B are actions of Y on A and B, then a c.p. map $V\in \mathrm{CP}(A,B)$ is called **Ψ-residually nuclear** if

(i) V is Ψ-equivariant, i.e., $V(\Psi_A(U))\subseteq\Psi_B(U)$ for every $U\in\mathbb{O}(Y)$, and
(ii) the completely positive class map

$$[V]\colon A/\Psi_A(U)\to B/\Psi_B(U)$$

is nuclear for every $U\in\mathbb{O}(Y)$.

$\mathcal{C}_{rn}(\Psi_A,\Psi_B)$ denotes the **cone of Ψ-residually nuclear maps** . It is a point-norm closed (matrix) operator-convex cone of completely positive maps.

If $A\subseteq\mathcal{M}(B)$, then $Y=\mathrm{Prim}(B)$ acts naturally on A by

Same notation Ψ for Ψ_A and Ψ_B ???

$\Psi_A(J) := A\cap\mathcal{M}(B,J)$ and $\Psi_B(J)=J$

(????)

for $J\in\mathcal{I}(B)\cong\mathbb{O}(\mathrm{Prim}(B))$, then we write "residually nuclear" instead of "Ψ-residually nuclear".

We use the notation $\mathcal{C}_{rn}(A\subseteq\mathcal{M}(B))$ or $\mathcal{C}_{rn}(\Psi)$ for the operator-convex cone of residually nuclear maps $V\colon A\to B$.

That means $V(\Psi(J)\cap A)\subset J$ and $[V]_J\colon A/\Psi(J)\to B/J$ is nuclear for each $J\lhd B$.

Note, that, for the natural action of $\mathrm{Prim}(B)$ on $A\subseteq\mathcal{M}(B)$, a c.p. map $V\colon A\to B$ is equivariant if and only if $V(a)$ is contained in the closed ideal of B generated by BaB for every $a\in A_+$.

Remark 3.5.10. A Ψ-equivariant map $V\colon A\to B$ is Ψ-residually nuclear, if and only if, for every $U\in\mathbb{O}(Y)$ and every separable C^*-algebra F, the completely positive map

$$[V]\otimes^{\max}\mathrm{id}\colon (A/\Psi_A(U))\otimes^{\max}F\to(B/\Psi_B(U))\otimes^{\max}F$$

factorizes over $A/\Psi_A(U)\otimes^{\min}F$, i.e., the kernel of

$$(A/\Psi_A(U))\otimes^{\max}F\to(A/\Psi_A(U))\otimes^{\min}F$$

is contained in the kernel of $[V]\otimes^{\max}\mathrm{id}$.

The reason is that a c.p. map $T\colon D \to E$ is nuclear if and only if

$$T \otimes^{\max} \mathrm{id}\colon D \otimes^{\max} F \to E \otimes^{\max} F$$

factorizes over $D \otimes^{\min} F$ for every separable C^*-algebra F. The latter can be seen by a simple Hahn-Banach separation argument using [**463**, lem. 7.17(i)], the proof is implicitly contained in the proof of a less general result in [**426**].

Since the containment of ideals can be checked with help of primitive ideals and since the maximal tensor product on C^*-algebras is short-exact, we get the following equivalent characterizations of Ψ-residually nuclear maps:

$V \in CB(A, B)$ is Ψ-residually nuclear if and only if for every irreducible representation

$$\rho\colon B \otimes^{\max} C^*(F_\infty) \to \mathcal{L}(\mathcal{H})\,,$$

every $U \in \mathbb{O}(Y)$ with $\rho(\Psi_B(U) \otimes^{\max} C^*(F_\infty)) = 0$ and every closed ideal J of $C^*(F_\infty)$ with $\rho(B \otimes^{\max} J) = 0$, the completely positive map

$$\rho \circ (V \otimes^{\max} \mathrm{id})\colon A \otimes^{\max} C^*(F_\infty) \to \mathcal{L}(\mathcal{H})$$

factorizes over $(A/\Psi_A(U)) \otimes^{\min} (C^*(F_\infty)/J)$.

By the above cited separation arguments this can be easily seen to be equivalent to the following:

$V \in \mathrm{CP}(A, B)$ is Ψ-residually nuclear, if and only if, for every C^*-morphism h from B into a von-Neumann factor N with $h(B)'' = N$ and every $U \in \mathbb{O}(Y)$ with $h(\Psi_B(U)) = 0$, holds $h(V(\Psi_A(U))) = (h \circ V)(\Psi_A(U)) = 0$ and that the c.p. map $[h \circ V]\colon A/\Psi_A(U) \to N$ is weakly nuclear.

Question 3.5.11. Suppose that A is separable. Is $\mathcal{C}_{rn}(\Psi^{\mathcal{S}}) = \mathcal{C}(\mathcal{S})_{rn}$ for $\mathcal{S} = \mathcal{C}(\Psi_A, \Psi_B)$? I.e., $\mathcal{C}_{rn}(\Psi^{\mathcal{C}(\Psi_A,\Psi_B)}) = \mathcal{C}_{rn}(\Psi_A, \Psi_B)$? This would show that all operator-convex cones of Ψ-residually nuclear maps from A to B are cones of residually nuclear maps for suitable lower semi-continuous actions Ψ of $\mathrm{Prim}(B)$ on A.

Question has now (since 2015, unpublished) a positive answer for separable A and B,
because each stable separable $C*$-algebra B has
''Abelian'' factorization.

Lemma 3.5.12. *Suppose that A is a C^*-algebra, N a von-Neumann factor and $\mathcal{C} \subseteq \mathrm{CP}(A, N)$ an operator-convex cone of completely positive maps.*

If $\mathcal{C}$ is separating for A, i.e., if $W(a) = 0$ for all $W \in \mathcal{C}$ implies $a = 0$, then every nuclear c.p. map $V\colon A \to N$ is contained in the point-weak closure of $\mathcal{C}$.

Proof. One can reduce all to the particular case where N is a von-Neumann subalgebra of $\mathcal{L}(\mathcal{H})$, and $\mathcal{H}$ contains a cyclic vector η for N, in a way that it suffices to show that the positive linear functional $\lambda \circ (V \otimes^{\max} \mathrm{id})$ on $A \otimes^{\max} M$ is in the point-weak closure of the convex cone κ of linear functionals $\lambda \circ (W \otimes^{\max} \mathrm{id})$ with $W \in \mathcal{C}$ (compare the arguments in proofs of [**463**, lem. 7.17, 7.18]).

Here $M := N' \subseteq \mathcal{L}(\mathcal{H})$ is the commutant of N, and let λ the partially normal positive functional on $N \otimes^{\max} M$ given by $\lambda(n \otimes m) = \langle mn\eta, \eta \rangle$.

The functional $\lambda \circ (V \otimes^{\max} \mathrm{id})$ factorizes over $A \otimes^{min} M$, i.e., annihilates the elements in the kernel of the epimorphism $A \otimes^{\max} M \to A \otimes^{min} M$, because $V : A \to N$ is weakly nuclear.

Since η is an cyclic vector for N, one gets (as in the proof of the implication (ii)⇒(i) for Theorem 3.5.2) that [**463**, lem. 7.17(ii)] applies to weak closure of κ, i.e., every positive linear functional φ on $A \otimes^{\max} M$ which annihilates the intersection J of the ideal-kernels of the functionals in κ is in the point-weak closure of κ. J does not contain a non-zero elementary element $a \otimes m$, because M and N are factors and $\mathcal{C}$ is separating for A.

It follows that J is contained in in the kernel of $A \otimes^{\max} M \to A \otimes^{min} M$. Hence $\lambda \circ (V \otimes^{\max} \mathrm{id})$ is in the point-weak closure of κ. □

REMARK 3.5.13. Let $A \subseteq M(D) \subseteq D^{**}$ such that $A \hookrightarrow D^{**}$ is weakly nuclear. Then $V : A \to D$ is residually nuclear if and only if V is approximately inner.

In particular, the identity map of every nuclear C^*-subalgebra $A \subseteq \mathcal{M}(D)$ is residually nuclear for the natural action of $\mathrm{Prim}(D)$ on A.

The following corollaries are combinations of special cases of the above results combined with some of the results of Sections 4–8.

COROLLARY 3.5.14. *Suppose that A and B are C^*-algebras and that $\mathcal{S}$ is a set of c.p. maps $V : A \to B$.*

(i) *For every $J \in \mathcal{I}(B)$ holds*

$$\Psi^{\mathcal{S}}(J) = \bigcap_{V \in \mathcal{C}(\mathcal{S})} V^{-1}(J)$$

where $\Psi^{\mathcal{S}}$ is the lower semi-continuous action of $\mathrm{Prim}(B)$ on A as defined in Definition 3.5.5.

(ii) *Every $\Psi^{\mathcal{S}}$-residually nuclear map $W : A \to B$ is in $\mathcal{C}(\mathcal{S})$.*

PROOF. (i) follows from $b^* V(a^*(.)a)b \in \mathcal{C}(\mathcal{S})$ for $V \in \mathcal{C}(\mathcal{S})$.

(ii): Combine Theorem 3.5.2(iii), Remark 3.5.10 and Lemma 3.5.12. □

COROLLARY 3.5.15. *Suppose that $B, C \subseteq A$ are C^*-subalgebras of a C^*-algebra A and that $V : B \to A$ is a c.p. map with $V(B) \subseteq C$.*

(i) *V is a.i. in A, – i.e., V can be approximated inside $\mathrm{CP}(B, A)$ in point norm by restrictions to B of inside A inner completely positive maps $b \in A \mapsto \sum_j (a_j)^* b a_j$ –, if and only if, $V(B \cap J) \subseteq J$ for every closed ideal J of A and $[h \circ V] : B/(B \cap J) \cong h(B) \to h(C) \subseteq N$ is weakly a.i. in N for every von-Neumann factor representation $h : A \to N \subseteq \mathcal{L}(H)$ of A.*

(ii) *If B or C is nuclear then V is a.i. in A, if and only if, for every $b \in B_+$, $V(b)$ is in the closed ideal $\overline{\mathrm{span}(AbA)}$ of A generated by b.*

COROLLARY 3.5.16. *Suppose that A is an exact C^*-subalgebra of a C^*-algebra B, and that $V\colon A \to B$ is a c.p. map. We consider the natural action Ψ of $\mathrm{Prim}(B)$ on A given by*

$$\Psi\colon J \in \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \mapsto J \cap A \in \mathcal{I}(A)\,.$$

(i) *V is residually nuclear, if and only if, V is nuclear and Ψ-equivariant with respect to the natural action of $\mathrm{Prim}(B)$ on A, if and only if, V is nuclear and (finite-step-) a.i. in B.*
(ii) *Moreover, if B is s.p.i. and , V is 1-step-approximately inner, if V is residually nuclear.*
(iii) *Suppose that $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ satisfies that, for every $a \in A$ and $\varepsilon > 0$, there exists $W \in \mathcal{S}$, $u \in A$ and $v \in B$ such that $\|v^*W(u^*au)v - a\| < \varepsilon$. Then every Ψ-residually nuclear map is in $\mathcal{C}(\mathcal{S})$. In particular, every nuclear Ψ-equivariant map $V\colon A \to B$ is approximately one-step inner if the elements of $\mathcal{S}$ are approximately one-step inner.*
(iv) *If $B = D_\omega$ for some s.p.i. algebra D, V is a residually nuclear contraction and A is separable, then there is a contraction d in B with $V(b) = d^*bd$ for all $b \in A$.*
(v) *If, in addition to (iv), D is stable, then there is an isometry $S \in (M(D))_\omega \subseteq M(B)$ such that $V(b) = S^*bS$ for $b \in A$.*

PROOF. `To be filled in` ?? □

6. Approximate decomposition of residually nuclear maps

We derive now an important later used sufficient condition for approximate decompositions and for one-step innerness of residually nuclear maps inside ultra-powers B_ω. Recall that $D \subseteq \mathcal{M}(B)$ is "non-degenerate" if $\overline{D \cdot B} = B$.

PROPOSITION 3.6.1. *Suppose that D is a strongly purely infinite non-degenerate C^*-subalgebra of $\mathcal{M}(B)$ and that $C \subseteq \mathcal{M}(D) \subseteq \mathcal{M}(B)$ is a separable C^*-subalgebra. Let $X := \mathrm{Prim}(D)$, and define actions of X on C and B by*

$$\begin{aligned}\Psi_C &:= \Psi^{\mathrm{up}}_{D,C}\colon\ \mathcal{I}(D) \cong \mathbb{O}(X) \to \mathcal{I}(C)\\ &\textit{and}\\ \Psi_B &:= \Psi^{D,B}_{\mathrm{down}}\colon\ \mathcal{I}(D) \cong \mathbb{O}(X) \to \mathcal{I}(B)\,,\end{aligned}$$

as in Definition 1.2.7. Then:

(i) *A map $V\colon C \to B$ is $\Psi^{\mathrm{up}}_{B,C}$- $\mathrm{id}_{\mathcal{I}(B)}$–residually nuclear, if and only if, V is Ψ_C-Ψ_B–residually nuclear, if and only if, V is $\mathrm{id}_{\mathcal{I}(C)}$- $\Psi^{C,B}_{\mathrm{down}}$–residually nuclear.*

(ii) *Every* Ψ_C-Ψ_B*–residually nuclear contraction* $V: C \to B \subseteq B_\omega$ *can be approximated in point-norm topology by compositions* $T \circ S$*, where* $T: D_\omega \to B_\omega$ *is a one-step-approximately inner completely positive contraction, and* $S: C \to D_\omega$ *is a residually nuclear contraction with respect to the action* $\Psi^{\mathrm{up}}_{D_\omega, C}$ *of* $\mathrm{Prim}(D_\omega)$ *on* C*, where we consider the embeddings*

$$C \subseteq \mathcal{M}(D) \subseteq \mathcal{M}(D)_\omega \subseteq \mathcal{M}(D_\omega)\,.$$

(iii) *The maps* $S \circ T$ *of Part* (ii) *can be approximated by* 1*-step-inner completely positive contractions.*

In particular, $V: C \to B$ *is approximately inner in* $\mathcal{M}(B)$ *if* V *is* Ψ_C-Ψ_B*–residually nuclear.*

We use the following Lemma 3.6.2 for the proof of Proposition 3.6.1.

LEMMA 3.6.2. *Let* $A \subseteq \mathcal{M}(D)$ *and* $D \subseteq \mathcal{M}(B)$ *non-degenerate. Then:*

(i) $\mathcal{M}(B, J)$ *is the strict closure of* $J \lhd B$*, and, for every strictly closed ideal* I *of* $\mathcal{M}(B)$ *holds* $I = \mathcal{M}(B, J)$ *for* $J := I \cap B \lhd B$.
(ii) $\Psi^{\mathrm{up}}_{D,A}(\Psi^{\mathrm{up}}_{B,D}(J)) = \Psi^{\mathrm{up}}_{B,A}(J)$ *for all* $J \in \mathcal{I}(B)$.
(iii) $\Psi^{A,B}_{\mathrm{down}}(\Psi^{\mathrm{up}}_{D,A}(K)) \subseteq \Psi^{D,B}_{\mathrm{down}}(K)$ *for all* $K \in \mathcal{I}(D)$.
(iv) $\Psi^{A,B}_{\mathrm{down}}(\Psi^{\mathrm{up}}_{B,A}(J)) \subseteq \Psi^{D,B}_{\mathrm{down}}(\Psi^{\mathrm{up}}_{B,D}(J)) \subseteq J$ *for all* $J \in \mathcal{I}(B)$.
(v) $L \subseteq \Psi^{\mathrm{up}}_{B,A}(\Psi^{A,B}_{\mathrm{down}}(L))$ *for all* $L \in \mathcal{I}(A)$.

PROOF. (i): If $s \in \mathcal{M}(B)$ and $sB \subseteq J$, then $bs \in B$ and $bse \in J$ for all $e \in B_+$, which implies that $Bs \subseteq J$, i.e., $s \in \mathcal{M}(B, J)$. For $t \in \mathcal{M}(B)$, $s \in \mathcal{M}(B, J)$ holds $tsB \subseteq tJ = tBJ \subseteq J$ and $stB \subseteq sB \subseteq J$. Thus $\mathcal{M}(B, J)$ an ideal of $\mathcal{M}(B)$.

If $(t_\alpha) \subseteq \mathcal{M}(B, J)$ converges strictly to $s \in \mathcal{M}(B)$ then $sb = \lim t_\alpha b \in J$ for all $b \in B$, i.e., $s \in \mathcal{M}(B, J)$. Thus $\mathcal{M}(B, J)$ is a strictly closed ideal of $\mathcal{M}(B)$. In particular, $\mathcal{M}(B, J)$ is norm-closed in $\mathcal{M}(B)$. Therefore it is a *-ideal of $\mathcal{M}(B)$. For $s \in \mathcal{M}(B, J)$ and a bounded approximate unit $\{e_\alpha\}$ of B holds that $se_\alpha \in J$ converges strictly to s in $\mathcal{M}(B)$.

It holds $IB \subseteq B \cap I$ and $J := B \cap I$ is a closed ideal of B, thus $I \subseteq \mathcal{M}(B, J)$ and $J \subseteq I$. It follows $\mathcal{M}(B, J) = I$ because $\mathcal{M}(B, J)$ is the strict closure of J in $\mathcal{M}(B)$.

(ii): Since $D \subseteq \mathcal{M}(B)$ is non-degenerate, there is a unique strictly continuous and unital embedding $\iota: \mathcal{M}(D) \hookrightarrow \mathcal{M}(B)$ with $\iota(d) = d$ for all $d \in D$ ([16]), and we can identify $\mathcal{M}(D)$ with $\iota(\mathcal{M}(D))$.

Let $K := \Psi^{\mathrm{up}}_{B,D}(J) = D \cap \mathcal{M}(B, J)$. Then $\mathcal{M}(D, K) \subseteq \mathcal{M}(D) \cap \mathcal{M}(B, J)$, because $\mathcal{M}(D, K)$ is the strict closure of K in $\mathcal{M}(D)$ by (i), $\mathcal{M}(D) \hookrightarrow \mathcal{M}(B)$ is strictly continuous and $\mathcal{M}(D) \cap \mathcal{M}(B, J)$ is strictly closed in $\mathcal{M}(D)$. The same arguments show that $\mathcal{M}(D) \cap \mathcal{M}(B, J)$ is a strictly closed ideal of $\mathcal{M}(D)$. By (i), there is a closed ideal L of D with $\mathcal{M}(D, L) = \mathcal{M}(D) \cap \mathcal{M}(B, J)$. It follows

[16]In general, $\iota(\mathcal{M}(D))$ is not strictly closed in $\mathcal{M}(B)$, even if $D \hookrightarrow \mathcal{M}(B)$ is non-degenerate.

$L \subseteq K$ and $\mathcal{M}(D,K) \subseteq \mathcal{M}(D,L)$, ie $K = KD \subseteq L$. Thus, $A \cap \mathcal{M}(B,J) = A \cap \mathcal{M}(D) \cap \mathcal{M}(B,J) = A \cap \mathcal{M}(D,K)$, i.e., $\Psi^{\text{up}}_{B,A}(J) = \Psi^{\text{up}}_{D,A}(K)$ for $K = \Psi^{\text{up}}_{B,D}(J)$.

(iii): Let $K \triangleleft D$. Then $I := \Psi^{\text{up}}_{D,A}(K) \subseteq \mathcal{M}(D,K)$ Thus $BDIDB \subseteq BD\mathcal{M}(D,K)DB \subseteq BKB$. Since DB is dense in B, we get $\Psi^{A,B}_{\text{down}}(\Psi^{\text{up}}_{D,A}(K)) \subseteq \Psi^{D,B}_{\text{down}}(K)$.

(vi): Let $J \triangleleft B$. Consider $K := \Psi^{\text{up}}_{B,D}(J)$ and $L := \Psi^{\text{up}}_{D,A}(K)$. We have $\Psi^{D,B}_{\text{down}}(K) \subseteq J$, because $K = \Psi^{\text{up}}_{B,D}(J) \subseteq \mathcal{M}(B,J)$ and $B\mathcal{M}(B,J)B \subseteq J$. On the other hand, $\Psi^{\text{up}}_{B,A}(J) = L$ by (ii), and $DLD \subseteq D\mathcal{M}(D,K)D \subseteq K$. Thus $BDLDB \subseteq BKB$. Since $DB = B$, it follows $\Psi^{A,B}_{\text{down}}(L) \subseteq \Psi^{D,B}_{\text{down}}(K)$.

(v): Let $L \triangleleft A$, and let $J := \overline{\text{span}(BLB)} = \Psi^{A,B}_{\text{down}}(L))$. For $x \in L$, bx^*xb is in $J := \overline{\text{span}(BLB)}$, which implies $xb \in J$. Thus $LB \subseteq J$ and $BL = (LB)^* \subseteq J$, i.e., $L \subseteq \Psi^{\text{up}}_{B,A}(J)$. □

Proof of Proposition 3.6.1. Recall that $\Psi_C(K) := \Psi^{\text{up}}_{D,C}(K) := C \cap \mathcal{M}(D,K)$ and that $\Psi_B(K) = \Psi^{D,B}_{\text{down}}(K)$ is defined as the closure of span(BKB) for $K \in \mathcal{I}(D)$.

(i): Suppose that $V\colon C \to B$ is $\Psi^{\text{up}}_{B,C}$-$\text{id}_{\mathcal{I}(B)}$– residually nuclear, and let I a closed ideal of D. Then $\mathcal{M}(D,I) \subseteq \mathcal{M}(B,BIB)$ (because $IB + BI \subseteq BIB$ and $DB = B = BD$), and $C \cap \mathcal{M}(D,I) = \Psi_C(I) \subseteq \Psi^{\text{up}}_{B,C}(BIB)$. It follows $V(\Psi_C(I)) \subseteq BIB = \Psi_B(I)$. Furthermore, $[V]_I\colon C/\Psi_C(I) \to B/BIB$ satisfies $[V]_I((C \cap \mathcal{M}(B,BIB))/\Psi_C(I)) = 0$, because $V(C \cap \mathcal{M}(B,BIB)) \subseteq BIB$. Thus, $[V]_I$ factorizes $[V]_I = [V]_J \circ \pi$ for the nuclear map $[V]_J\colon C/\Psi^{\text{up}}_{B,C}(J) \to B/J$ where $J := BIB$. It follows that V is Ψ_C-Ψ_B–residually nuclear.

Suppose that V is Ψ_C-Ψ_B–residually nuclear, and let K a closed ideal of C. Then $\Psi^{C,B}_{\text{down}}(K) = \text{span}(BKB) = BDKDB = BIB$ for $I := DCD$. Then $K \subseteq C \cap \mathcal{M}(D,I) = \Psi_C(I)$ and

$$V(K) \subseteq V(\Psi_C(I)) \subseteq \Psi_B(I) = \Psi^{C,B}_{\text{down}}(K)\,,$$

and $[V]_K\colon C/K \to B/\text{span}(BKB) = B/\Psi_B(I)$ factorizes through the nuclear map $[V]_I\colon C/\Psi_C(I) \to B/\Psi_B(I)$ because $K \subseteq \Psi_C(I)$. It follows that V is $\text{id}_{\mathcal{I}(C)}$-$\Psi^{C,B}_{\text{down}}$–residually nuclear.

Suppose that V is $\text{id}_{\mathcal{I}(C)}$-$\Psi^{C,B}_{\text{down}}$– residually nuclear, and let J a closed ideal of B. Then $K := C \cap \mathcal{M}(B,J) = \Psi^{\text{up}}_{B,C}(J)$ is a closed ideal of C with

$$\Psi^{C,B}_{\text{down}}(K) = \overline{\text{span}(BKB)} \subseteq J\,,$$

because $cB + Bc \subseteq J$ for all $c \in \mathcal{M}(B,J)$. It follows that $V(K) \subseteq \overline{\text{span}(BKB)} \subseteq J$, and that $[V]\colon C/K \to C/\overline{\text{span}(BKB)}$ is nuclear. Thus $[V]\colon C/K \to C/J$ is also nuclear and $K = \Psi^{\text{up}}_{B,C}(J)$.

Thus, $V\colon C \to B$ is $\Psi^{\text{up}}_{B,C}$-$\text{id}_{\mathcal{I}(B)}$– residually nuclear.

(ii)+(iii): First we consider all maps $S\colon C \to D_\omega$ with the property that the C^*-subalgebra $A_S := C^*(S(C))$ of D_ω is abelian and that there exist contractions

$s_1, s_2, \ldots \in D_\omega$ with $s_n^*(C + \mathbb{C}1)s_m = \{0\}$ for $n \neq m$ and $s_n^* c s_n = S(c)$ for all $c \in C$ and $n \in \mathbb{N}$. Clearly, the maps S are $\Psi^{\mathrm{up}}_{D_\omega, C}$-residually nuclear, and, for all contraction $b \in B_\omega$, the maps $c \in C \mapsto b^* S(c) b \in B_\omega$ are $\Psi^{\mathrm{up}}_{B_\omega, C}$-residually nuclear in $C \subseteq \mathcal{M}(D_\omega) \subseteq \mathcal{M}(B_\omega)$. By definition, the maps $c \mapsto b^* S(c) b$ are one-step approximately inner in $\mathcal{M}(B)_\omega \subseteq \mathcal{M}(B_\omega)$. (Here we use that $D_\omega \subseteq \mathcal{M}(B)_\omega \subseteq \mathcal{M}(B_\omega)$ is non-degenerate, i.e., $D_\omega B_\omega = B_\omega$.)

We are going to show:

(α) The above considered maps $c \mapsto b^* S(c) b$ are point-norm dense in the set of contractions of a matrix operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}_{\mathrm{in}}(C, B_\omega)$.

(β) For every $c_0 \in C_+$ and $b_1, \ldots, b_n \in B_\omega$ with $\| \sum_j b_j^* b_j \| \leq 1$ there exist $S\colon C \to D_\omega$ as above descibed and a contraction $b \in B_\omega$ with $b^* S(c) b = \sum_j b_j^* c b_k$.

By (α) and (β), the operator-convex cone $\mathcal{C}$ is a sub-cone of the cone of approximately inner c.p. maps from $C \subseteq \mathcal{M}(B_\omega)$ to B_ω which has the property that the action $\Psi_{\mathcal{C}}$ of $\mathcal{I}(B_\omega) \cong \mathbb{O}(\mathrm{Prim}(B_\omega))$ on C defined by $\mathcal{C}$ is the same as $\Psi^{\mathrm{up}}_{B_\omega, C}$ (which is the action defined by the cone of approximately inner c.p. maps). It follows from Corollary 3.5.14(ii) that all $\Psi^{\mathrm{up}}_{B_\omega, C}$-residually nuclear contractions V from C into B_ω are in the point-norm closure of the set of maps $c \mapsto b^* S(c) b$ as described above. The $\Psi^{\mathrm{up}}_{B, C}$-residually nuclear contractions $V\colon C \to B$ define $\Psi^{\mathrm{up}}_{B_\omega, C}$-residually nuclear contractions $V\colon C \to B \subseteq B_\omega$. By (i) this shows that (α) and (β) imply (ii) and (iii).

For the proof of (α) and (β) we list the following facts (1)-(4) on ultrapowers D_ω of strongly purely infinite C^*-algebras D (taken from [**463**] and [**443**]):

(1) If X is a separable subset of $D_\omega + B_\omega$, then there exists positive contraction $e \in D_\omega$ with $ex = xe = x$ for all $x \in X$, and with $ec = ce$ and $\|ce\| = \|c\|$ for all $c \in C$.

(2) For every separable C^*-subalgebra $A \subseteq D_\omega$ and every commutative separable C^*-subalgebra $F \subseteq D_\omega$ there exists a contraction $d \in D_\omega$ with $df = fd$, $fd^*d = f$, $d^*adf = fd^*ad$, $d^*a^*dd^*ad = d^*add^*a^*d$ for all $f \in F$ and $a \in A$.

(3) For every separable commutative C^*-subalgebra $F \subseteq D_\omega$ there exists a *-morphism $h\colon F \otimes \mathcal{O}_\infty \to D_\omega$ with $h(f \otimes 1) = f$ for all $f \in F$.

(4) For every separable C^*-subalgebra $A \subseteq D_\omega$ and every approximately inner completely positive map $T\colon A \to D_\omega$ with commutative $C^*(T(A))$ there exists $d \in D_\omega$ with $\|d\|^2 \leq \|T\|$ and $d^* a d = T(a)$ for $a \in A$.

(α): `to be filled in` ??

(β): There are positive contractions $e, f \in C' \cap D_\omega$ with $eb_j = b_j$ and $fe = e$ for $j = 1, \ldots, n$ by (1), used twice. By (2), there is a contraction $d \in \{c_0 e, e, f\}' \cap D_\omega$ with $H := C^*(d^* f C f d \cup \{c_0 e, e, f\})$ commutative and $d^* d f = f$. Note $ed^* f c f d e = d^* e c e d = e d^* c d e$ for $c \in C$ and $d^* e (e c_0) e d = c_0 e^3$. There is *-monomorphism $k\colon H \otimes \mathcal{O}_\infty \to D_\omega$ with $k(h \otimes 1) = h$ for $h \in H$ by (3). Let $t_1, t_2, \ldots$ denote the canonical generators of $\mathcal{O}_\infty$ and let $S(c) := d^*eced$, $s_n := edk(f \otimes t_n)$, $b := \sum_j k(f \otimes t_j)b_j \in B_\omega$. Then $s_n^* c s_m = \delta_{n,m} S(c)$, $S(c_0) = c_0 e^3$, $C^*(S(C)) \subseteq H$ is commutative, $b^*b = \sum_j b_j^* b_j$ and $b^*S(c)b = \sum b_j^* S(c) b_j$. Thus S and $b \in B_\omega$ are as stipulated. □

Definition 3.6.3. Suppose that $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a point-norm closed matrix operator-convex sub-cone of $\mathrm{CP}(A,B)$, and that E, F are nuclear C^*-algebras.

We define $\mathrm{CP}(E,F) \otimes \mathcal{C} \subseteq \mathrm{CP}(E \otimes A, F \otimes B)$ as the point-norm closure $\mathcal{C}(\mathcal{S})$ of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$, where $\mathcal{S} := \{V \otimes S\,;\ S \in \mathcal{C}, V \in \mathrm{CP}(E,F)\}$.

Notice the m.o.c. cone $\mathrm{CP}(E,F) \otimes \mathcal{C}$ is *not* the completion of some algebraic tensor product.

Corollary 3.6.4. *Suppose that $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is a point-norm closed operator-convex cone of completely positive maps, that E and F are nuclear C^*-algebras, and $T \in \mathrm{CP}(E \otimes A, F \otimes B)$. Then:*

(i) *If $\mathcal{S}_2$ is a generating set for $\mathcal{C}$ and $\mathcal{S}_1$ is a generating set for $\mathrm{CP}(E,F)$, then $\mathcal{S} := \{U \otimes V\,;\ U \in \mathcal{S}_1,\ V \in \mathcal{S}_2\}$ is a generating set for $\mathrm{CP}(E,F) \otimes \mathcal{C}$. In particular, $\mathrm{CP}(M_n, M_n) \otimes \mathcal{C}$ is the point-norm closure of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ for $\mathcal{S} := \{\mathrm{id} \otimes V\,;\ V \in \mathcal{C}\}$.*

(ii) *$T_1 T_2 \in \mathrm{CP}(E,F) \otimes \mathcal{C}$, if $T_1 \in \mathrm{CP}(G,F) \otimes \mathcal{C}_1$ and $T_2 \in \mathrm{CP}(E,G) \otimes \mathcal{C}_2$ for a nuclear C^*-algebra G and operator-convex cones $\mathcal{C}_1 \subseteq \mathrm{CP}(A,C)$ and $\mathcal{C}_2 \subseteq \mathrm{CP}(C,B)$ with $\mathcal{S}_2 \circ \mathcal{S}_1 \subseteq \mathcal{C}$ for generating subsets $\mathcal{S}_1 \subseteq \mathcal{C}_1$ and $\mathcal{S}_2 \subseteq \mathcal{C}_2$.*

(iii) *The union (over $n = 1, 2, \ldots$) of the compositions $S_1 S_2 S_3$ with $S_2 \in \mathrm{CP}(M_n, M_n) \otimes \mathcal{C}$, $S_1 \in \mathrm{CP}(M_n, F) \otimes \mathrm{CP}_{\mathrm{in}}(B,B)$, and $S_3 \in \mathrm{CP}(E, M_n) \otimes \mathrm{CP}_{\mathrm{in}}(A,A)$ is point-norm dense in $\mathrm{CP}(E,F) \otimes \mathcal{C}$.*

(iv) *$T \in \mathrm{CP}(E,F) \otimes \mathcal{C}$, if and only if, the c.p. maps*

$$a \in A \mapsto (\chi \otimes \mathrm{id}_B)(T(e \otimes a)) \in B$$

are in $\mathcal{C}$ for every $e \in E_+$ and for every pure state χ on F.

(v) *The action $\Psi \colon \mathcal{I}(F \otimes B) \cong \mathbb{O}(\mathrm{Prim}(F \otimes B))$ of $\mathrm{Prim}(F \otimes B) \cong \mathrm{Prim}(F) \otimes \mathrm{Prim}(B)$ on $E \otimes A$ satisfies*

$$\Psi(J) \;=\; E \otimes \Psi_{\mathcal{C}}(J_B)$$

for $J \lhd F \otimes B$ in $\mathcal{I}(F \otimes B)$, where J_B is the biggest of the ideals $I \lhd B$ with $F \otimes I \subseteq J$.

(vi) *The natural map Φ from $\mathcal{I}((F \otimes B) \otimes^{\max} C^*(F_2))$ into $\mathcal{I}((F \otimes A) \otimes^{\max} C^*(F_2))$ defined by $\mathrm{CP}(E,F) \otimes \mathcal{C}$ satisfies $\Phi(J) = F \otimes \Phi_1(I(J))$, where Φ_1 denotes the natural map from $\mathcal{I}(B \otimes^{\max} C^*(F_2))$ into $\mathcal{I}(A \otimes^{\max} C^*(F_2))$ defined by $\mathcal{C}$ and $I(J)$ is the biggest ideal I of $B \otimes^{\max} C^*(F_2)$ with $F \otimes I \subseteq J$.*

Recall that the operator-convex cone $\mathrm{CP}_{\mathrm{in}}(A,A)$ of approximately inner c.p. maps from A into A is nothing else $\mathcal{C}(\mathrm{id}_A)$, and that every nuclear map $V \in \mathrm{CP}_{\mathrm{nuc}}(C,C)$ is approximately inner if C is simple, e.g. $\mathrm{CP}(M_n, M_n) = \mathcal{C}_{\mathrm{alg}}(\mathrm{id})$.

Proof. Let $\mathcal{H}_2$ a Hilbert A-B–module with infinite repeats, that is naturally related to $\mathcal{C}$ in the sense of Proposition **??**, i.e., $\mathcal{C}$ is the point-norm closure of the c.p. coefficient maps $a \in A \mapsto \langle x, ax\rangle \in B$ for $x \in \mathcal{H}_2$. Further let $\mathcal{H}_1$ a Hilbert E-F–module (with infinite repeats) that corresponds to $\mathrm{CP}(E,F)$ in the same way. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an Hilbert $E\otimes A$-$F\otimes B$–module (with infinite repeats), corresponding to the m.o.c. cone $\mathrm{CP}(E,F)\otimes\mathcal{C}$, cf. Proposition **??**.

(i): If $\mathcal{S}_2 \subseteq \mathcal{C}$ is a generating subset of $\mathcal{C}$ than a suitable Hilbert A-B–module $\mathcal{H}_2$ can be constructed by infinitely often repeated Hilbert A-B–module sums of the Stinespring dilations of the $V \in \mathcal{S}_1$. We can elaborate a suitable Hilbert E-F-module $\mathcal{H}_1$ from $\mathcal{S}_2$ in the same way. Then $\mathcal{H}_1\otimes\mathcal{H}_2$ is just the Hilbert $E\otimes A$-$F\otimes B$–module sum of the dilations of the maps $U\otimes V$ with $U \in \mathcal{S}_1$ and $V \in \mathcal{S}_2$ (infinitely often repeated). Thus $\mathrm{CP}(E,F)\otimes\mathcal{C}$ is generated by $\{U\otimes V\,;\; U\in\mathcal{S}_1\,, V\in\mathcal{S}_2\,\}$.

Here end of proof of (i)? **??**

Let $J\lhd(F\otimes B)\otimes^{\max}C^*(F_2)$ a closed ideal and let $\Psi(J) := I\lhd(E\otimes A)\otimes^{\max}C^*(F_2)$ the maximal closed ideal with $T\otimes^{\max}\mathrm{id}(I) \subseteq J$ for all generators $T = V\otimes S$ of $\mathrm{CP}(E,F)\otimes\mathcal{C}$, where $S\in\mathcal{C}$ and $V\in\mathrm{CP}(E,F) = \mathrm{CP}_{\mathrm{nuc}}(E,F)$. By Proposition **??** , then T is in the operator-convex cone $\mathcal{C}\otimes\mathrm{CP}(E,F)$, if and only if, $T\otimes^{\max}\mathrm{id}(\Psi(J)) \subseteq J$ for all $J\lhd(F\otimes B)\otimes^{\max}C^*(F_2)$. Recall here, that it is enough to consider primitive ideals J of $(F\otimes B)\otimes^{\max}C^*(F_2) = F\otimes(B\otimes^{\max}C^*(F_2))$. Since F is nuclear, they are sums

$$J = J_F\otimes(B\otimes^{\max}C^*(F_2)) + F\otimes J_1\,,$$

where J_1 is a primitive ideal of $B\otimes^{\max}C^*(F_2)$ and J_F is a primitive ideal of F.

Now note that $\mathrm{CP}(E,F) = \mathrm{CP}_{\mathrm{nuc}}(E,F)$ is generated as an operator-convex point-norm closed cone by maps $e\in E\mapsto \xi(e)f$, for $f\in F_+$ and ξ is a pure state on E. Thus, $\Psi_{\mathrm{CP}(E,F)}(J) = 0$ if $J\lhd F$ and $J\neq F$ and, trivially, $\Psi_{\mathrm{CP}(E,F)}(F) = E$.

(iv): Let $e\in E_+$ and $\chi\colon F\to\mathbb{C}$ a pure state on F. Let $\mathcal{C}_3 \subseteq \mathrm{CP}(A,A)$ and $\mathcal{C}_1\subseteq\mathrm{CP}(B,B)$ denote the approximately inner c.p. maps. Then $\mathcal{C}\circ\mathcal{C}_3 = \mathcal{C}$ and $\mathcal{C}_1\circ\mathcal{C} = \mathcal{C}$. Consider the maps $V_3\colon z\in\mathbb{C}\to ze\in E$ and $V_1 := \chi\colon F\to\mathbb{C}$. Then $V_3\otimes\mathrm{id}_A\in\mathrm{CP}(\mathbb{C},E)\otimes\mathcal{C}_3$ and $V_1\otimes\mathrm{id}_B\in\mathrm{CP}(F,\mathbb{C})\otimes\mathcal{C}_1$. It follows $(V_1\otimes\mathrm{id}_B)\circ T\circ(V_3\otimes\mathrm{id}_A)\in\mathcal{C} = \mathcal{C}\otimes\mathrm{CP}(\mathbb{C},\mathbb{C})$ by (ii).

If $U\in\mathrm{CP}(E,F)$, $V\in\mathcal{C}$, $h\in M_{1,m}(E\odot A)$, and $k\in M_{m,1}(F\odot B)$, then $S_0(a) := \chi\otimes\mathrm{id}_B\big(k^*(U\otimes V\otimes\mathrm{id}_m(h^*(e\otimes a)h))k\big)$ defines a completely positive map that is in $\mathcal{C}$:

`orthogonalize the` $e_{i,j}$`-components of` h_i
`and the` $f_{i,j}$`-components of` k_i

Now let $T\in\mathrm{CP}(E,F)\otimes\mathcal{C}$, $a_1,\ldots,a_m\in A$, and $\varepsilon>0$. Then there are $n\in\mathbb{N}$, $U_j\in\mathrm{CP}(E,F)$, $V_j\in\mathcal{C}$, $f_j\in M_{1,n}(E\odot A)$, $g_j\in M_{n,1}(F\odot B)$, $j=1,\ldots,n$ with

$$\|T(e\otimes a_k) - \sum_j g_j^*(U_j\otimes V_j)\otimes\mathrm{id}_n(f_j^*(e\otimes a_k)f_j)g_j\| < \varepsilon\,.$$

It follows that there is $S\in\mathcal{C}$ with

$$\|\chi\otimes\mathrm{id}_B(T(e\otimes a_k)) - S(a_k)\| < \varepsilon \qquad \text{for} \quad k = 1,\ldots,m\,.$$

Thus $(\chi \otimes \mathrm{id})T(e \otimes (\cdot)) \in \mathcal{C}$.

`try again?`

$(\mathrm{id}_B \otimes \chi) \circ T \circ ((\cdot) \otimes e) \in \mathcal{C}$, because this happens if T is one of the generators where $S \in \mathcal{C}$ and $V \in \mathrm{CP}(E, F) = \mathrm{CP}_{\mathrm{nuc}}(E, F)$, and because

??????????????????

The set of $T \in \mathrm{CP}(A \otimes E, B \otimes F)$ with $(\mathrm{id}_B \otimes \chi) \circ T \circ ((\cdot) \otimes e) \in \mathcal{C}$ for all $e \in E_+$ and pure states χ on F does not change the ???????

??

Conversely, suppose that $(\mathrm{id}_B \otimes \chi) \circ T \circ ((\cdot) \otimes e) \in \mathcal{C}$ for every pure state χ on F and $e \in E_+$. Since $\mathrm{id}_B \otimes \chi$ is in the point-norm closure of the cone generated by ???? It suffices to show that $(\mathrm{id} \otimes UW) \circ V \in \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, F)$ for c.p. contractions $W : M_n \to F$ and $U : F \to M_n$.

Now, $(\mathrm{id} \otimes U) \circ \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, M_n) \subseteq \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, F)$ as one can see on the generating sets.

`to be filled in` **??** □

`Next question has a positive answer:`

QUESTION 3.6.5. Suppose that $\mathcal{C}_j \subseteq \mathrm{CP}(B_j, B_{j+1})$ $(j = 1, 2)$ and $\mathcal{C}_3 \subseteq \mathrm{CP}(B_1, B_3)$ are point-norm closed matrix-operator convex cones, and that $\mathcal{S}_j \subseteq \mathcal{C}_j$ a generating set for $\mathcal{C}_j$ $(j = 1, 2)$ in sense of Definition **??**.

Let $V \in \mathrm{CP}(B_2, B_3)$ such that $V \circ (b^* W(\cdot) b) \in \mathcal{C}_3$ for all $W \in \mathcal{S}_1$ and $b \in B_2$. Is $V \circ \mathcal{C}_2 \subseteq \mathcal{C}_3$?

Let $W \in \mathrm{CP}(B_1, B_2)$ and $V \circ (b^* W(\cdot) b) \in \mathcal{C}_3$ for all $V \in \mathcal{S}_2$. Is $\mathcal{C}_1 \circ W \subseteq \mathcal{C}_3$?

`What about situation of ⊕-closed generating system?`

REMARK 3.6.6. **??** Suppose that X and Y are T_0-spaces, $\Psi^0 : \mathbb{O}(Y) \to \mathbb{O}(X)$ lower semi-continuous and $\Psi_0 : \mathbb{O}(X) \to \mathbb{O}(Y)$ upper semi-continuous.

There are maps $\Phi_0 : \mathbb{O}(X) \to \mathbb{O}(Y)$ and $\Phi^0 : \mathbb{O}(Y) \to \mathbb{O}(X)$, determined by

$$\Phi_0(U) \subseteq V \quad \Leftrightarrow \quad \Psi^0(V) \subseteq U\,,$$

respectively

$$\Phi^0(V) \subseteq U \quad \Leftrightarrow \quad \Psi_0(U) \subseteq V\,.$$

The map Φ_0 is upper semi-continuous, and Φ^0 is lower semi-continuous.

QUESTION 3.6.7. Suppose that A is separable, $J \triangleleft A$ is a closed ideal and that $\mathrm{id}_{A/J}$ is not locally liftable (i.e., that $J \otimes \mathcal{L}(\ell_2) \to A \otimes \mathcal{L}(\ell_2) \to (A/J) \otimes \mathcal{L}(\ell_2)$ is not exact), and let $B := \mathcal{L}(\ell_2)$ (or at least $B := \mathcal{L}(\ell_2)/\mathbb{K}$).
Does there exist a non-nuclear c.p. contraction $W : A/J \to B$ such that $W \circ \pi_J : A \to B$ is nuclear?
The question for $B := \mathcal{L}(\ell_2)/\mathbb{K}$ seems to be less difficult.

QUESTION 3.6.8. Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a point-norm closed operator-convex cone, A and B separable. Let $\Psi_{A,\mathcal{C}}$ denote the action of $\mathrm{Prim}(B)$ on A induced by $\mathcal{C}$. Recall, that $\Psi_{A,\mathcal{C}}(J) := \bigcap_{V\in\mathcal{C}} V^{-1}(J)$ for $J \in \mathcal{I}(B)$.

Denote by $\mathcal{C}_{rn} \subseteq \mathrm{CP}(A,B)$ the point-norm closed operator-convex cone of Ψ-residually nuclear maps. (Then $\mathcal{C}_{rn} \subseteq \mathcal{C}$.)

Is $\mathcal{C}_{rn} \subseteq (\mathcal{C}_{rn})_{rn}$?

(Note here that $\Psi_{A,\mathcal{C}}(J) \subseteq \Psi_{A,\mathcal{C}_{rn}}(J)$ for $J \in \mathcal{I}(B)$.)

Is the answer positive if A locally reflexive?

QUESTION 3.6.9. Suppose that $\mathcal{S}_1 \subseteq \mathrm{CP}(A,B)$ is point-norm closed and is invariant under the operations (OC1) of Definition 3.2.2. Let $\mathcal{S}_2$ denote the point-norm closure of the smallest subset of $\mathrm{CP}(A,B)$ that contains $\mathcal{S}_1$ and is invariant under the operations (OC2).
Is $\mathcal{S}_2$ konvex?
(Examples show that the convexity of $\mathcal{S}_1$ alone is not enough to get convexity of $\mathcal{S}_2$.)

LEMMA 3.6.10. *Suppose that $\mathcal{C} \subseteq CP(A,B)$ is the matrix operator-convex cone of completely positive maps that is generated by a subset $\mathcal{S} \subseteq \mathcal{C}$.*

Then every contraction $T \in \mathcal{C}$ can be approximated in point norm by maps $S := \sum_{j=1}^{n} c_j^* V_j(r_j^*(\cdot)r_j)c_j$ with suitable $m,n \in \mathbb{N}$, $V_j \in \mathcal{S}$, rows $r_j \in M_{1,m}(A)$, and columns $c_j \in M_{m,1}(B)$, such that $\| \sum_{j=1}^{n} c_j^* V_j(r_j^* r_j)c_j\| \leq 1$.

PROOF. `to be filled in` ?? □

LEMMA 3.6.11. `Separation from cone` $\mathcal{C}$??

DEFINITION 3.6.12. A subset $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ is **invariant under ε-generalized Cuntz addition**, if $d_1^* V(e(\cdot)e)d_1 + d_2^* W(e(\cdot)e)d_2 \in \mathcal{S}$ for every $V,W \in \mathrm{CP}(A,B)$, $e \in A_+$, $\varepsilon > 0$, and $d_1,d_2 \in B$ with $d_1^* d_2 = 0$ and $d_j^* d_j = (V(e^2) + W(e^2) - \varepsilon)_+$.

LEMMA 3.6.13. *Suppose that for every $b \in B_+$ and $\varepsilon > 0$ there exist $d_1,d_2 \in B$ with $d_1^* d_2 = 0$ and $\|b - d_j^* d_j\| < \varepsilon$ for $j = 1,2$ ([17]).*

If $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ is invariant under ε-generalized Cuntz addition, then the operator-convex cone generated by $\mathcal{S}$ is contained in the point-norm closure of the smallest subset of $\mathrm{CP}(A,B)$ *that invariant under the operations (OC2) of Definition 3.2.2* and *contains $\mathcal{S}$.*

PROOF. ?? □

DEFINITION 3.6.14. Let $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ a subset. Then the smallest subset $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ of $\mathrm{CP}(A,B)$ that contains $\mathcal{S}$ and is invariant under the operations (OC1) and (OC2) of Definition 3.2.2 will be called the **operator-convex cone algebraically generated by $\mathcal{S}$**.

[17] It does not say that b itself is infinite, e.g. every stable B has this property.

The point-norm closure $\mathcal{C}(\mathcal{S})$ of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ is called the **point-norm closed** *operator-convex cone generated by* $\mathcal{S}$.

LEMMA 3.6.15. *Let $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ a set of completely positive maps, and let $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ denote the the smallest subset of $\mathrm{CP}(A, B)$ that contains $\mathcal{S}$ and is invariant under the operations (OC1) and (OC2) of Definition 3.2.2.*

(i) *The point-norm closure $\mathcal{C}$ of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ is again a matrix operator-convex cone.*

(ii) *Every contraction in $\mathcal{C}$ can be approximated in point-norm topology by maps of the form $\sum_{k=1}^{n} c_k^*(V_k \otimes \mathrm{id}_{n_k})(r_k^*(\,\cdot\,)r_k)c_k$ with $V_k \in \mathcal{S}$, row-matrices $r_k \in M_{1,n_k}(A)$, and column-matrices $c_k \in M_{n_k,1}(B)$, $k = 1, \ldots, n$, that satisfy*

$$\| \sum_{k=1}^{n} c_k^*(V_k \otimes \mathrm{id}_{n_k})(r_k^* r_k)c_k \| \leq 1\,.$$

(iii) *In particular, $\mathcal{C}$ is the point-norm closure of the convex hull of the set of maps $c^*(V \otimes \mathrm{id}_n)(r^*(\,\cdot\,)r)c$ with $V \in \mathcal{S}$, row-matrix $r \in M_{1,n}(A)$ and column-matrix $c \in M_{n,1}(B)$.*

PROOF. (i): The operations $(x, y) \in B \oplus B \mapsto b_1^* x b_1 + b_2^* y b_2$, $a \in A \mapsto r^* a r \in A \otimes M_n$, $x \in B \otimes M_n \mapsto c^* x c \in B$ are bounded and $V \in \mathrm{CP}(A, B) \mapsto V \otimes \mathrm{id}_n \in \mathrm{CP}(A \otimes M_n, B \otimes M_n)$ is continuous with respect to point-norm topology. Thus, the point-norm closure of a subset $K \subseteq \mathrm{CP}(A, B)$ that is invariant under the operations (OC1) and (OC2) is again invariant under the operations (OC1) and (OC2) in Definition 3.2.2.

(iii): Let $\mathcal{X}$ denote the set of maps $c^*(V \otimes \mathrm{id}_n)(r^*(\cdot)r)c$ with $V \in \mathcal{S}$, $r \in M_{1,n}(A)$, $c \in M_{n,1}(B)$, $n \in \mathbb{N}$. Then $\mathcal{X}$ is invariant under the operations (OC2), because, if $W := c_1^*(V \otimes \mathrm{id}_m)(r_1^*(\cdot)r_1)c_1$, then $c_2^*(W \otimes \mathrm{id}_n)(r_2^*(\cdot)r_2)c_2$ for is equal the map $c_3^*(V \otimes \mathrm{id}_{mn})(r_3^*(\cdot)r_3)c_3$, with $r_3 \in M_{1,mn}(A)$ given by $(r_3)_{1,\ell} := (r_1)_{1,j}(r_2)_{1,k}$ for $\ell = (j-1)n + k \in \{1 \ldots mn\}$ and the entries $(r_i)_{1,p}$ of r_i $(i = 1, 2, 3)$, and, similar, with $c_3 \in M_{1,mn}(B)$ given by $(c_3)_{\ell,1} := (c_1)_{j,1}(c_2)_{k,1}$ for $\ell = (j-1)m + k$.

If a subset $\mathcal{X} \subseteq \mathrm{CP}(A, B)$ is invariant under the operations (OC2), then (in particular) $f^* b^* V(a^* e^*(\cdot)ea)bf \in \mathcal{X}$ for every $a \in \mathcal{M}(A)$, $e \in A$, $b \in \mathcal{M}(B)$ and $f \in B$. If we use approximate units $\{e_\alpha\}$ of A and $\{f_\beta\}$ of B, then this shows that the point-norm closure $\mathcal{Y} := \overline{\mathcal{X}}$ of $\mathcal{X}$ contains all maps $b^* V(\cdot)b$ with $b \in B$ and $V \in \mathcal{X}$. With the arguments in proof of part(i), we obtain that $\mathcal{S} \subseteq \mathcal{Y}$, $\mathbb{R}_+ \cdot \mathcal{Y} \subseteq \mathcal{Y}$ and that $\mathcal{Y}$ contains $c^*(W \otimes \mathrm{id}_n)(r^*(\cdot)r)c$ if $W \in \mathcal{Y}$, $r \in M_{1,n}(\mathcal{M}(A))$, $c \in M_{n,1}(\mathcal{M}(B))$, $n \in \mathbb{N}$.

Thus, the set $\mathcal{Z}$ of finite sums $W_1 + \cdots + W_m$ with $W_j \in \mathcal{Y}$ is a cone that is invariant under the operations (OC1) and contains $\mathcal{S} \cup \mathcal{X} \subseteq \mathcal{Y}$. Since $\mathcal{Y}$ is invariant under (OC1) and since $c^*((W_1 + \cdots + W_m) \otimes \mathrm{id}_n)(r^*(\cdot)r)c = U_1 + \cdots + U_m$ for $U_j := c^*(W_j \otimes \mathrm{id}_n)(r^*(\cdot)r)c$, we get moreover that $\mathcal{Z}$ is also invariant under the operations (OC2).

It follows that $\mathcal{C}_{\mathrm{alg}}(\mathcal{S}) \subseteq \mathcal{Z}$. On the other hand, $\mathcal{X} \subseteq \mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ and $\mathcal{Z}$ is contained in the point-norm closure of the set $\mathcal{W}$ of finite sums $V_1 + \cdots + V_m$ with $V_j \in \mathcal{X}$.

It follows that every element of the point-norm closure $\mathcal{C}(\mathcal{S})$ of $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ can be approximated in point-norm by maps $W \in \mathcal{W}$, where $W := \sum_{k=1}^{n} c_k^*(V_k \otimes \mathrm{id}_{n_k})(r_k^*(\,\cdot\,)r_k)c_k$ with $V_k \in \mathcal{S}$, row-matrices $r_k \in M_{1,n_k}(A)$, column-matrices $c_k \in M_{n_k,1}(B)$, $k = 1, \ldots, n$. Obviously, $\mathcal{W} \subseteq \mathcal{C}_{\mathrm{alg}}(\mathcal{S}) \subseteq \mathcal{C}$.

(ii): The norm $\|W\|$ of $W \in \mathcal{W}$ is given by $\sup\{\|W(e)\|\,;\ e \in A_+,\ \|e\| \le 1\}$ which is equal to $\|\sum_{k=1}^{n} c_k^*(V_k \otimes \mathrm{id}_{n_k})(r_k^* r_k)c_k\|$. Since $W(e(\cdot)e) \in \mathcal{W}$ for $W \in \mathcal{W}$, we get from Lemma 3.1.8 that the contractions in the point-norm closure $\mathcal{C}$ of $\mathcal{W}$ can be approximated in point-norm by contractions in $\mathcal{W}$. It proves also part(ii). □

Definition 3.6.16. Let $\mathcal{C}_1 \subseteq \mathrm{CP}(A, B)$ and $\mathcal{C}_2 \subseteq \mathrm{CP}(E, F)$ point-norm closed matrix operator-convex cones.

We define the point-norm closed m.o.c. cone $\mathcal{C}_1 \otimes \mathcal{C}_2 \subseteq \mathrm{CP}(A \otimes E, B \otimes F)$ as the smallest point-norm closed m.o.c. cone that contains all tensor products $S \otimes T \in \mathrm{CP}(A \otimes E, B \otimes F)$ of $S \in \mathcal{C}_1$ and $T \in \mathcal{C}_2$.

The m.o.c. cone $\mathcal{C}_1 \otimes \mathcal{C}_2 \subseteq \mathrm{CP}(A \otimes E, B \otimes F)$ is called the "**minimal tensor product**" of the m.o.c. cones $\mathcal{C}_1$ and $\mathcal{C}_2$.

If $A = \mathrm{C}_0(X)$ and $F = \mathbb{C}$, respectively $E = \mathbb{C}$ and $B = \mathrm{C}_0(Y)$, then we write sometimes also $[X]\mathcal{C}$, respectively $\mathcal{C}[Y]$, to simplify notation.

The m.o.c. cone $\mathcal{C}_1 \otimes \mathcal{C}_2$ is not a tensor product in the usual sense because it contains usually elements that can not be approximated in point-norm by convex combinations of elementary tensors $S \otimes T$.

Corollary 3.6.17. *Suppose that $\mathcal{S}_1 \subseteq \mathrm{CP}(A, B)$, $\mathcal{S}_2 \subseteq \mathrm{CP}(E, F)$ are subsets and that $\mathcal{C}_1 \subseteq \mathrm{CP}(A, B)$, $\mathcal{C}_2 \subseteq \mathrm{CP}(E, F)$ are the point-norm closed matrix operator-convex cones generated by $\mathcal{S}_1$ respectively $\mathcal{S}_2$.*

Then the point-norm closed matrix operator-convex cone $\mathcal{C}_1 \otimes \mathcal{C}_2 \subseteq \mathrm{CP}(A \otimes E, B \otimes F)$ that is generated by $\{V \otimes W\,;\ V \in \mathcal{C}_1\,,\ W \in \mathcal{C}_2\,\}$ is also generated by $\{S \otimes T\,;\ S \in \mathcal{S}_1\,,\ T \in \mathcal{S}_2\,\}$.

Proof. Let $\mathcal{C}_3 \subseteq \mathrm{CP}(A \otimes E, B \otimes F)$ denote the point-norm closure of the operator-convex cone generated by $\{S \otimes T\,;\ S \in \mathcal{S}_1\,,\ T \in \mathcal{S}_2\,\}$. Then $\mathcal{C}_3 \subseteq \mathcal{C}_1 \otimes \mathcal{C}_2$ and $\mathcal{C}_3$ is point-norm closed, is convex and is invariant under the operations (OC1) and (OC2) by Lemma 3.6.15(i). Thus, $\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathcal{C}_3$ if $\mathcal{C}_3$ contains every tensor product $V \otimes W$ with $V \in \mathcal{C}_1$ and $W \in \mathcal{C}_2$. By Lemma 3.6.15(ii,iii) and the convexity of the cone $\mathcal{C}_3$ is suffices to show that $(c_1^*(S \otimes \mathrm{id}_m)(r_1^*(\cdot)r_1)c_1) \otimes (c_2^*(T \otimes \mathrm{id}_n)(r_2^*(\cdot)r_2)c_2)$ is in $\mathcal{C}_3$ if $S \in \mathcal{S}_1$, $T \in \mathcal{S}_2$, $c_1 \in M_{m,1}(B)$, $r_1 \in M_{1,m}(A)$, $c_2 \in M_{n,1}(F)$ and $r_2 \in M_{1,m}(E)$. We have

$$(c_1^*(S \otimes \mathrm{id}_m)(r_1^*(\cdot)r_1)c_1) \otimes (c_2^*(T \otimes \mathrm{id}_n)(r_2^*(\cdot)r_2)c_2) = c_3^*((S \otimes T) \otimes \mathrm{id}_{mn})(r_3^*(\cdot)r_3)c_3$$

for $c_3 = c_1 \otimes c_2 \in M_{mn,1}(B \otimes F)$ and $r_3 := r_1 \otimes r_2 \in M_{1,mn}(A \otimes E)$ where we use the natural isomorphisms $M_{mn,1}(B \otimes F) \cong M_{m,1}(B) \otimes M_{n,1}(F)$, $M_{1,mn}(A \otimes E) \cong$

$M_{1,m}(A)\otimes M_{1,n}(E)$, $(A\otimes E)\otimes M_{mn}\cong(A\otimes M_m)\otimes(E\otimes M_n)$ and $(A\otimes E)\otimes M_{mn}\cong(A\otimes M_m)\otimes(E\otimes M_n)$ ([18]). Thus, $\mathcal{C}_3=\mathcal{C}_1\otimes\mathcal{C}_2$. □

Definition 3.6.18. Let A and B C^*-algebras. We define a **matrix-order** $S\leq T$ on the cone of completely positive maps $S,T\in \mathrm{CP}(A,B)$ by

$$S\leq T \quad \text{if and only if} \quad T-S\in \mathrm{CP}(A,B)\,.$$

Obviously, this definition says that: $S\leq T$ if and only if there exists $R\in \mathrm{CP}(A,B)$ with $T=S+R$. It is not enough to have $S(a)\leq T(a)$ for all $a\in A_+$. But $S\leq T$ is equivalent to the property that for each $n\in N$ and all positive matrices $a\in M_n(A)_+$ holds $S_n(a)\leq T_n(a)$, where S_n and T_n apply to $a=[a_{jk}]$ as $S_n(a)=[S(a_{jk})]$.

Corollary 3.6.19. *Every, with respect to the point-norm convergence topology closed, matrix operator-convex cone $\mathcal{C}\subseteq \mathrm{CP}(A,B)$ is* hereditary *with respect to the matrix-order on* $\mathrm{CP}(A,B)$ *in sense of Definition 3.6.18, i.e., if $S,T\in \mathrm{CP}(A,B)$ and $S+T\in\mathcal{C}$ then $S,T\in\mathcal{C}$.*

If $\mathcal{C}\subseteq \mathrm{CP}(A,B)$ is a point-norm closed hereditary convex sub-cone of $\mathrm{CP}(A,B)$, *then $\mathcal{C}$ is matrix operator-convex, if and only if, the maps $U_2^*V(U_1^*(\cdot)U_1)U_2$ are in $\mathcal{C}$ for every $V\in\mathcal{C}$, and for all unitaries $U_1:=\exp(ih)\in A+\mathbb{C}1$ and $U_2:=\exp(ik)\in B+\mathbb{C}1$ with $h^*=h\in A$, $\|h\|<\pi$, $k^*=k\in B$ and $\|k\|<\pi$.*

Proof. If $I\subseteq A\otimes^{\max}C^*(F_2)$ and $J\subseteq B\otimes^{\max}C^*(F_2)$ are closed ideals with $V\otimes^{\max}\mathrm{id}(I)\subseteq J$ for all $V\in\mathcal{C}$, then $((S+T)\otimes^{\max}\mathrm{id})(I)\subseteq J$ for $S,T\in \mathrm{CP}(A,B)$ with $S+T\in\mathcal{C}$. Let $e\in I_+$, $f:=S\otimes^{\max}\mathrm{id}(e)$ and $g:=T\otimes^{\max}\mathrm{id}(e)$ in $(B\otimes^{\max}C^*(F_2))_+$. Since $f+g\in J$ and $f,g\geq 0$, it follows that $f,g\in J_+$,i.e., $S\otimes^{\max}\mathrm{id}(I)\subseteq J$ and $T\otimes^{\max}\mathrm{id}(I)\subseteq J$ for all closed ideals $I\subseteq A\otimes^{\max}C^*(F_2)$ and $J\subseteq B\otimes^{\max}C^*(F_2)$ with the property $V\otimes^{\max}\mathrm{id}(I)\subseteq J$ for all $V\in\mathcal{C}$. It implies $S,T\in\mathcal{C}$ by Theorem 3.8.4.

Every point-norm closed matrix operator-convex cone $\mathcal{C}$ is convex, because V_1+V_2 and λV_1 are in $\mathcal{C}$ by condition (OC1), as one can see with help of an approximate unit of B. The maps $b^*V(a^*(\cdot)a)b$ are in $\mathcal{C}$ for every $V\in\mathcal{C}$, $a\in A$ and $b\in B$, by condition (OC2).

Suppose now that $\mathcal{C}$ is a hereditary point-norm closed convex sub-cone of $CP(A,B)$ such that the maps $U_2^*V(U_1^*(\cdot)U_1)U_2$ are in $\mathcal{C}$ for every $V\in\mathcal{C}$, and for all unitaries $U_1:=\exp(ih)\in A+\mathbb{C}1$ and $U_2:=\exp(ik)\in B+\mathbb{C}1$ with $h^*=h\in A$, $\|h\|<\pi$, $k^*=k\in B$ and $\|k\|<\pi$. Since $c+i(1-c^2)^{1/2}=\exp(ih)$ with $h^*=h$ of norm $\|h\|\leq 2\pi/3$ for $c^*=c$ with $\|c\|\leq 1/2$, and since $d^*xd+dxd^*=2(hxh+kxk)$ for $d=h+ik$ with $h^*=h$, $k^*=k$ in every C^*-algebra, we can use that the convex cone $\mathcal{C}$ is hereditary and that the c.p. maps $b^*V(a^*(\cdot)a)b$ are in $\mathcal{C}$ for every $V\in\mathcal{C}$, $a\in A$ and $b\in B$.

[18] The isomorphisms are (– up to the associativity transformation maps –) induced by the isomorphism $\mathbb{C}^{mn}\cong\mathbb{C}^m\otimes\mathbb{C}^n$ given by writing the rows of the tensors (on on the right side) as consecutive row (on the left side).

If $V \in \mathcal{C}$, $r = [a_1, \ldots, a_n] \in M_{1,n}(A)$ and $c = [b_1, \ldots, b_n]^\top \in M_{n,1}(B)$, then the map $Z(a) := (n(\sum_j a_j^* a a_j) \otimes 1_n) - r^* a r$ from A into $M_n(A)$ is completely positive by Lemma A.5.6. The maps $c^*(V \otimes \mathrm{id}_n)(a_j^*(\cdot)a_j \otimes 1_n)c = \sum_k b_k^* V(a_j(\cdots)a_j)b_k$ are in $\mathcal{C}$ (by additivity of $\mathcal{C}$). Since $a \mapsto n(\sum_j a_j^* a a_j) \otimes 1_n$ is the sum of the c.p. maps Z and $a \mapsto r^* a r$, we get that $c^*(V \otimes \mathrm{id}_n)(r^*(\cdot)r)c \in \mathcal{C}$, i.e., $\mathcal{C}$ is invariant under the operations (OC2) of Definition 3.2.2. It is also closed with respect to the operations (OC1), because $\mathcal{C}$ is point-norm closed, is convex and $b^* V(\cdot) b$ is in $\mathcal{C}$ for every $b \in B$. □

COROLLARY 3.6.20. *Suppose that* $\mathcal{C}_1 \subseteq \mathrm{CP}(A, B)$ *and* $\mathcal{C}_2 \subseteq \mathrm{CP}(B, C)$ *are point-norm closed operator convex cones, that* $\mathcal{S}_1 \subseteq \mathcal{C}_1$ *and* $\mathcal{S}_2 \subseteq \mathcal{C}_2$ *are generating sets and that* $X \subseteq B$ *is a subset of* B *with dense linear span.*

(i) *Then the set*

$$\{W(b^* V(\cdot) b)\,;\ V \in \mathcal{S}_1,\, W \in \mathcal{S}_1,\, b \in X\}$$

of c.p. maps generates the m.o.c. cone $\mathcal{C}_2 \circ \mathcal{C}_1 \subseteq \mathrm{CP}(A, C)$.

(ii) $\mathrm{CP}_{\mathrm{in}}(B) \circ \mathcal{C}_1 \circ \mathrm{CP}_{\mathrm{in}}(A) = \mathcal{C}_1$.

(iii) *If ???????????????????*

PROOF. `to be filled in` ?? □

LEMMA 3.6.21. *Let* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ *and let* $\mathcal{C}$ *denote the closure of the smallest subset of* $\mathrm{CP}(A, B)$ *that contains* $\mathcal{S}$ *and is invariant under the operations (OC2) of Definition 3.2.2.*

Suppose that, for every $V_1, V_2 \in \mathcal{S}$ *and* $b_1, b_2 \in B$, *there is a subset* $\mathcal{X} \in \mathcal{S}$, *such that*

(i) *the set of maps* $c^*(V \otimes \mathrm{id}_n)(r^*(.)r)c$ *with* $V \in \mathcal{X}$, $r \in M_{1,n}(A)$ *and* $c \in M_{n,1}(B)$ *has convex point-norm closure* $\mathcal{Y} := \overline{\mathcal{X}}$,

(ii) *there is* $W \in \mathrm{CP}(A, B)$ *such that* $b_1^* V_1(\cdot) b_1 + b_2^* V_2(\cdot) b_2 + W$ *is in* $\mathcal{Y}$.

Then $\mathcal{C}$ *satisfies automatic also (OC1) of Definition 3.2.2.*

PROOF. `to be filled in` ?? □

PROPOSITION 3.6.22. *Suppose that* $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ *is point-norm closed matrix operator-convex cone, and that* E *and* F *are* nuclear C^**-algebras.*

(i) *If* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ *generates* $\mathcal{C}$, *then the set of maps* $S \otimes T$ *for* $S \in \mathcal{S}$ *and* $T := \rho(\cdot) f$ *with* $f \in F_+$ *and* ρ *a pure state on* E *generate* $\mathcal{C} \otimes \mathrm{CP}(E, F)$.
`Check proof and statement of Part (i) !? ???????????` ??

(ii) *If* $R \in \mathcal{C} \otimes \mathrm{CP}(M_m, M_n)$ $W_1 \in \mathrm{CP}(E, M_m)$, *and* $W_2 \in \mathrm{CP}(M_n, F)$, *then* $V := (\mathrm{id}_B \otimes W_2) \circ R \circ (\mathrm{id}_A \otimes W_1)$ *is contained in* $\mathcal{C} \otimes \mathrm{CP}(E, F)$.

(iii) $T \in \mathrm{CP}(A \otimes E, B \otimes F)$ *is in* $\mathcal{C} \otimes \mathrm{CP}(E, F)$, *if and only if, the map* $a \in A \mapsto (\mathrm{id} \otimes \rho)(T(a \otimes e)) \in B$ *is in* $\mathcal{C}$ *for all* $e \in E_+$ *and all pure states* ρ *on* F.

PROOF. (i): The set of c.p. maps $\rho(\cdot)f$ with $f \in F_+$ and ρ a pure state on E generate $\mathrm{CP}_{\mathrm{nuc}}(E,F) = \mathrm{CP}(E,F)$, by Corollary **??**. Thus, the $S \otimes T$ with $S \in \mathcal{S}$ and those $T = \rho(\cdot)f$ generate $\mathcal{C} \otimes \mathrm{CP}(E,F)$ as an m.o.c. cone by Corollary 3.6.17.

(ii,iii): It suffices to compare the corresponding actions of $\mathrm{Prim}((B\otimes F)\otimes^{\max}C)$ on $(A \otimes E) \otimes^{\max} C$.

By part (i) and Lemma 3.6.15(ii), it suffices to consider the case where $R = c^*(S \otimes T \otimes \mathrm{id}_p)(r^*(\cdot)r)c$ with $S \in \mathcal{C}$, $T := \psi(\cdot)f_0$, $r \in M_{1,p}(A \otimes M_m)$ and $c \in M_{p,1}(B \otimes M_n)$, $f_0 \in (M_n)_+$, ψ pure state on M_m. If U is a suitable unitary in M_m, then $T(U^*\alpha U) = \alpha_{1,1} f_0$

???????????????????

By Lemma 3.1.9(i), there are columns $c_1, \ldots, c_n \in M_{n,1}(F)$ with $W_2(\beta) = \sum_{j=1}^n c_j^* \beta c_j$ for all $\beta \in M_n$.

Let $r = (r_1, \ldots, r_p)$ with ????

It follows that $id \otimes W_2(R(a \otimes d)) = \sum$????

and ???????????

`to be filled in` **??** □

LEMMA 3.6.23. *Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a point-norm closed matrix operator-convex cone, A, B, E, F C^*-algebras.*

(i) *Every $W \in \mathcal{C} \otimes \mathrm{CP}_{\mathrm{nuc}}(E,F)$ can be approximated in point-norm by sums of maps*

$$W' = (\mathrm{id}_B \otimes U_2) \circ I_2 \circ (V \otimes \mathrm{id}_n) \circ I_1 \circ (\mathrm{id}_A \otimes U_1)$$

where $V \in \mathcal{C}$, $U_1 \in \mathrm{CP}(E, M_n)$, $U_2 \in \mathrm{CP}(M_n, F)$, and the maps $I_1 \colon A \otimes M_n \to A \otimes M_n$ and $I_2 \colon B \otimes M_n \to B \otimes M_n$ are both 1-step-inner c.p. maps.

(ii) *If $W' \in \mathrm{CP}(A \otimes E, B \otimes F)$ is as in (i) then $W' \in \mathcal{C} \otimes \mathrm{CP}_{\mathrm{nuc}}(E,F)$.*

PROOF. (i): Let $T = S_2 \circ S_1 \colon E \to F$ with $S_1 \colon E \to \mathbb{C}$ $S_2 \colon \mathbb{C} \to F$ given by $S_1(e) := \rho(e)$ for some pure state ρ on E, and $S_2 \colon z \in \mathbb{C} \to zf_0 \in F$ for some $f_0 \in F$. We know from Remark **??**// from Corollary **??**// that $\mathrm{CP}_{\mathrm{nuc}}(E,F)$ is generated by maps T of this kind. It implies by Corollary 3.6.17 that the maps $V \otimes T$ with $V \in \mathcal{C}$ and $T = S_2 \circ S_1 \in \mathrm{CP}(E,F)$ generate $\mathcal{C} \otimes \mathrm{CP}(E,F)$.

Thus, by Lemma **??**, every $W \in \mathcal{C} \otimes \mathrm{CP}_{\mathrm{nuc}}(E,F)$ can be approximated by finite sums of maps

$$W'(a) := \sum_{1 \le j,k \le m} (c_j)^* \, (V \otimes T)((r_j)^* a r_k) \, c_k$$

where $a \in A \otimes M_m$, $V \in \mathcal{C}$, $T = S_2 \circ S_2$ is of the above type, $r_1, \ldots, r_m \in A \odot E$, $c_1, \ldots, c_m \in B \odot F$.

We have $r_j = \sum_{i=1}^{p_j} x_{ij} \otimes e_{ij}$ and

$$S_1((e_{ij})^* e e_{i'k}) = \rho((e_{ij})^* e e_{i'k}) = \langle D(e)(D(e_{i'k})x), D(e_{ij}))x \rangle$$

for the irreducible representation $D\colon E \to \mathcal{L}(L_2(E,\rho))$ with cyclic vector $x \in L_2(E,\rho)$ corresponding to ρ. Thus we may suppose that $p_j = p$, $e_{ij} = e_i$ and that $\{D(e_i)x\}_{1\leq i\leq p}$ is an orthonormal basis of of the linear span of $\{D(e_{ij})x\}$, i.e., it suffices to consider the case, where $r_j = \sum_{i=1}^{p} a_{ij} \otimes e_i$ (some of the a_{ij} can be zero). Then

$$(\mathrm{id}_A \otimes S_1 \otimes \mathrm{id}_m)(r^*(\cdot)r) = J_1 \circ (\mathrm{id}_A \otimes R_1)(\cdot)\,,$$

where $r := [r_1,\ldots,r_m] \in M_{1,m}(A\otimes E) \subseteq A\otimes E\otimes M_m$, $J_1\colon A\otimes M_p \to A\otimes M_m$ given by $J_1(\cdot) := X^*(\cdot)X$ with $X := [a_{ij}] \in M_{p,n}(A)$, and $R_1(e) := [\rho(e_i^* e e_{i'})]_{i,i'} = \kappa^* D(e)\kappa$ is defined by the isometry $\kappa\colon \mathbb{C}^p \hookrightarrow L_2(E,\rho)$ with $\kappa(z_1,\ldots,z_p) := \sum z_i e_i x$. The map $J_1\circ(\mathrm{id}_A\otimes R_1)$ maps $A\otimes E$ to $A\otimes M_m$.

Consider $c_j := \sum_{i=1}^{q_j} y_{ji}\otimes f_{ij} \in B\odot F$ for $j=1,\ldots,m$. Let $f_1,\ldots,f_q$ a basis of the linear span of $\{f_{ij}\}$, then there are unique $b_{ji}\in B$ with $c_j = \sum_{i=1}^{q} b_{ji}\otimes b_i$. Let $Y := [b_{ji}]_{j,i} \in M_{m,q}(B)$, then $J_2(\cdot) := Y^*(\cdot)Y$ maps $B\otimes M_m$ into $B\otimes M_q$. Now we define $R_2\colon M_q \to E$ by $R_2(\cdot) := g^*(\cdot)g$ for the column-matrix $g := [f_0^{1/2}f_1,\ldots,f_0^{1/2}f_q]^\top \in M_{q,1}(F)$.

We get

$$c^*((\mathrm{id}_B\otimes S_2)\otimes \mathrm{id}_p(\cdot))c = (\mathrm{id}_B\otimes R_2)\circ J_2\,,$$

thus

$$W' = c^*((V\otimes S_2\circ S_1)\otimes \mathrm{id}_m(r^*(\cdot)r)c = (\mathrm{id}_B\otimes R_2)\circ J_2\circ(V\otimes \mathrm{id}_m)\circ J_1\circ(\mathrm{id}_A\otimes R_1)\,.$$

If we let $n := \max(m,p,q)$ and consider M_m, M_p and M_q as corners of M_n, then $X\in A\otimes M_n$, $Y\in B\otimes M_n$, J_1 (respectively J_2) can be considered as 1-step-inner c.p. map of $A\otimes M_n$ (respectively $B\otimes M_n$) R_1 becomes a c.p. map $U_1\colon E\to M_n$, and R_2 extends naturally to a c.p. map $U_2\colon M_n\to F$. Hence, $W' = (\mathrm{id}_B\otimes U_2)\circ I_2\circ(V\otimes \mathrm{id}_n)\circ I_1\circ(\mathrm{id}_A\otimes U_2)$ is as desired.

(ii): We show that $W'\in \mathcal{C}\otimes \mathrm{CP}_{\mathrm{nuc}}(E,F)$:
If $I_1 = \mathrm{id}_A\otimes \mathrm{id}_n$ and $I_2 = \mathrm{id}_B\otimes\mathrm{id}_n$ then $W_0 = V\otimes(U_2\circ U_1)\in \mathcal{C}\otimes \mathrm{CP}_{\mathrm{nuc}}(E,F)$.

??????????????

Since, $W_1 := I_2\circ(V\otimes\mathrm{id}_n)\circ I_1 \in \mathcal{C}\otimes\mathrm{CP}(M_n)$ is suffices to show that $W_2 = W_1\circ(\mathrm{id}_A\otimes U_1) \in (\mathcal{C}\otimes\mathrm{CP}(A,M_n))\circ(\mathrm{CP}_{\mathrm{in}}(A)\otimes\mathrm{CP}(E,M_n))$ is in $\mathcal{C}\otimes\mathrm{CP}_{\mathrm{nuc}}(E,M_n)$, and that $W' = (\mathrm{id}_B\otimes U_2)\circ W_2 \in \mathrm{CP}_{\mathrm{in}}(B)\otimes\mathrm{CP}(M_n,F)\circ(\mathcal{C}\otimes\mathrm{CP}_{\mathrm{nuc}}(E,M_n))$ is in $\mathcal{C}\otimes\mathrm{CP}_{\mathrm{nuc}}(E,F)$.

`to be filled in` ?? □

`Next needed in Chapter 8:` ??

Suppose that $\mathcal{C}_1\subseteq \mathrm{CP}(A,B)$ and $\mathcal{C}_2\subseteq\mathrm{CP}(B,C)$ are point-norm closed matrix operator-convex cones, that (E_1,ϕ_1) is a $\mathcal{C}_1$-compatible Hilbert (A,B)-module and (E_2,ϕ_2) is a $\mathcal{C}_2$-compatible Hilbert (B,C)-module.

Then the Hilbert (A,C)-module $(E_3,\phi_3) := (E_1\otimes_B E_2, \phi_1(\cdot)\otimes_B 1)$ is $\mathcal{C}_2\circ\mathcal{C}_1$-compatible.

The module (E_3, ϕ_3) is generating for $\mathcal{C}_2 \circ \mathcal{C}_1$ if – moreover – (E_1, ϕ_1) and (E_2, ϕ_2) are generating for $\mathcal{C}_1$ respectively $\mathcal{C}_2$.

`next needed in chp:5 and chp:8,`
`Stinespring-Kasparov dilation inside` $\mathcal{C}$
`perhaps shift to the appendix:`

Lemma 3.6.24. *Suppose that E is a (right) Hilbert B-module and $V\colon A \to \mathcal{L}(E)$ is a completely positive contraction.*

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ denote the point-norm closed m.o.c. cone that is generated by the c.p. maps $V_e\colon a \in A \mapsto \langle V(a)e, e\rangle \in B$ with $e \in E$.

*Then there is a (left) Hilbert B-module F and a *-morphism $\phi\colon A \to \mathcal{L}(E \oplus_B F)$ such that for the natural projection $\pi_1\colon E \oplus_B F \to E$ holds*

(i) *$V(a)e = \pi_1(\phi(a)(e, 0))$ for all $e \in E$ and $a \in A$.*
(ii) *$\langle \phi(\cdot)(e, f), (e, f)\rangle \in \mathcal{C}$ for all $(e, f) \in E \oplus_B F$.*

If A is separable, B is σ-unital and E is countably generated over B then F and $\phi\colon A \to \mathcal{L}(E \oplus_B F)$ can be found such that F is again countably generated over B.

*If, furthermore, B is stable and $E = B$ (as right Hilbert B-module), then there is a *-morphisms $\psi\colon A \to \mathcal{M}(B)$ and isometries $s, t \in \mathcal{M}(B)$ such that $ss^* + tt^* = 1$, $V = s^*\phi(\cdot)s$, and $b^*\phi(\cdot)b \in \mathcal{C}$ for all $b \in B$.*

Proof. `to be filled in` ?? □

Definition 3.6.25. Let A and B graded C^*-algebras with gradings $\beta_A \in \mathrm{Aut}(A)$ and $\beta_B \in \mathrm{Aut}(B)$ and $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a m.o.c.cone with $\mathcal{C} \circ \beta_A \subseteq \mathcal{C}$ and $\beta_B \circ \mathcal{C} \subseteq \mathcal{C}$.

We say that a (graded) Hilbert (A, B)-module (E, ϕ) **generates** $\mathcal{C}$ if (E, ϕ) is $\mathcal{C}$-compatible and the vector states

$$V_x\colon a \in A \mapsto \langle \phi(a)x, x\rangle \in B$$

generate $\mathcal{C}$.

`Next: What about (OC1)?` ??

Remark 3.6.26. One can show that *a set $\mathcal{V}$ of c.p. maps that is closed under operations (OC2) and under “local” generalized Cuntz addition has a point-norm closure that also satisfies (OC1).*

Corollary 3.6.27. *Suppose that $A \neq \mathbb{C}$ is simple and purely infinite and that X is locally compact.*

Then for every $a, b \in \mathrm{C}_0(X, A)_+$ with $\|b(x)\| \leq \|a(x)\|$ for all $x \in X$ and $\varepsilon \in (0, 1)$ there exists a contraction $d \in \mathrm{C}_0(X, A)$ with $\|b - d^*ad\| < \varepsilon$.

PROOF. Since A is simple, it follows that $b(x)$ is in the ideal generated by $c(x)$, where $c(x) := (a(x) - \delta\|a(x)\|)_+ \neq 0$ for $\delta := \varepsilon/2$.

Since $A \neq \mathbb{C}$ is simple and purely infinite the algebra A is strongly purely infinite, by Proposition 2.2.1(v). The *strong* pure infiniteness of A implies that $C_0(X, A)$ is strongly purely infinite (cf. Corollary 2.17.5), in particular $C_0(X, A)$ is purely infinite. Thus, there is $e \in C_0(X, A)$ with $e^*ce = (b - \varepsilon)_+$. □

COROLLARY 3.6.28. *A point-norm closed convex subcone $\mathcal{C}$ of $\mathrm{CP}(A, B)$ is matrix operator-convex, if and only if, $\mathcal{C}$ is invariant under compositions with inner automorphisms $\mathrm{Ad}(U)$ for unitary elements $U \in \mathcal{M}(A)$ or $U \in \mathcal{M}(B)$, and $\mathcal{C}$ is a hereditary sub-cone of $\mathrm{CP}(A, B)$, i.e., $T_1 + T_2 \in \mathcal{C}$ implies $T_1, T_2 \in \mathcal{C}$.*

PROOF. ?? Use Hahn-Banach separation for $\mathcal{C}$
and for the point-norm closed m.o.c. cone
generated by $\mathcal{C}$. □

LEMMA 3.6.29. *Suppose that J is a closed ideal of B, (X, ρ) a separable metric space, and $\mathcal{S}$ is a set of continuous maps $f\colon X \to B$ that has the following properties (i) and (ii).*

(i) *There exists a continuous function $\mu(t)$ on $\mathbb{R}_+$ with $\mu(0) = 0$ such that $\|f(x) - f(y)\| \leq \mu(\rho(x, y))$ for all $x, y \in X$ and $f \in \mathcal{S}$.*
(ii) *For every $f \in \mathcal{S}$, every finite subset $Y \subseteq X$, every $\varepsilon > 0$ and $g \in \mathcal{S}$, there exists $h \in \mathcal{S}$ with $\|\pi_J(h(y)) - \pi_J(g(y))\| < \varepsilon$ and $\|f(y) - h(y)\| < \varepsilon + \|\pi_J(f(y)) - \pi_J(g(y))\|$ for all $y \in Y$.*

Then for every map $F\colon X \to B/J$ in the point-norm closure of $\pi_J \circ \mathcal{S}$ there is point-wise convergent sequence $f_n \in \mathcal{S}$ such that $F(x) = \pi_J(\lim_n f_n(x))$.

The condition (ii) is automatically satisfied if the map

$$h(x) := e^{1/2} f(x) e^{1/2} + (1 - e)^{1/2} g(x) (1 - e)^{1/2}$$

is in $\mathcal{S}$, for every positive contraction $e \in J_+$ with $\|e\| < 1$ and for every $f, g \in \mathcal{S}$.

PROOF. Let $(x_1, x_2, \ldots)$ a dense sequence in X. Suppose that $F\colon X \to B/J$ is in the point-norm closure of $\pi_J \circ \mathcal{S}$. Then, for fixed $x, y \in X$, $\|F(x) - F(y)\| \leq \sup_{f \in \mathcal{S}} \|f(x) - f(y)\| \leq \mu(\rho(x, y))$, and there is a sequence $f_n \in \mathcal{S}$ such that $\|F(x_k) - \pi_J(f_n(x_k))\| < 2^{-n-2}$ for $k \leq n$. By property (ii) and induction, we find $h_n \in \mathcal{S}$ with $h_1 := f_1$,

$$\|h_n(x_k) - h_{n+1}(x_k)\| \leq 2^{-n-2} + \|\pi_J(h_n(x_k)) - \pi_J(f_{n+1}(x_k))\|$$

and $\|\pi_J(h_{n+1}(x_k)) - \pi_J(f_{n+1}(x_k))\| < 2^{-n-3}$ for $k \leq n+1$. Then $\lim_n \pi_J(h_n(x_k)) = F(x_k)$ for all $k \in \mathbb{N}$, and $\|h_n(x_k) - h_{n+1}(x_k)\| \leq 2^{-n}$ for $k \leq n$. Thus, $H(x_k) = \lim_n h_n(x_k)$ exists for all $k \in \mathbb{N}$. It follows, $\|H(x_k) - H(x_j)\| \leq \mu(\rho(x_k, x_j))$ for $k, j \in \mathbb{N}$. Since $\{x_1, x_2, \ldots\}$ is dense in X, we get that H extends uniquely to a continuous map G from X into B with $\|G(x) - G(y)\| \leq \mu(\rho(x, y))$. For $x \in X$ and $\varepsilon > 0$ there exist x_k with $\mu(\rho(x, x_k)) < \varepsilon/3$. Then $\|G(x) - h_n(x)\| \leq \|H(x_k) - h_n(x_k)\| + 2\varepsilon/3$. Thus $G(x) = \lim_n h_n(x)$ for all $x \in X$. Since $\pi_J(G(x_k)) = F(x_k)$

and F is continuous, it follows $\pi_J \circ G = F$. In particular, $F(x) = \pi_J(\lim_n h_n(x))$ for all $x \in X$, and (h_n) is a sequence in $\mathcal{S}$ that converges point-wise to G.

We show that the condition (ii) is satisfied if the map

$$h(x) := e^{1/2} f(x) e^{1/2} + (1-e)^{1/2} g(x) (1-e)^{1/2}$$

is in $\mathcal{S}$, for every positive contraction $e \in J_+$ with $\|e\| < 1$ and for every $f, g \in \mathcal{S}$:

Consider the continuous function $\psi(\xi) := (1-\xi^2)^{1/2}$ for $\xi \in [0,1]$. Let $Y \subset X$ a finite subset, $\varepsilon \in (0,1)$ and $\delta := \varepsilon/4$.

If we let $e := d^2$ for $d \in J_+$ with $\|d\| < 1$ then $h(\cdot)$ becomes

$$h(\cdot) := h_d(\cdot) := d f(\cdot) d + \psi(d) g(\cdot) \psi(d)$$

depending on d. Then $\pi_J(h(x)) = \pi_J(g(x))$ for all $x \in X$. Notice here that $\pi_J(d) = 1$ because $\psi(d) \in C^*(d, 1)$. It follows

$$\|f(y) - h(y)\| \leq \|[f(y), d]\| + \|[f(y), \psi(d)]\| + \|\psi(d)(f(y) - g(y))\psi(d)\| .$$

Since B/J is a Banach quotient of B by J, there exists for each $y \in Y$ an element $c(y) \in J$ with

$$\|f(y) - g(y) + c(y)\| \leq \delta + \|\pi_J(f(y)) - \pi_J(g(y))\| .$$

Consider the finite sets $Q := \{c(y)\,;\ y \in Y\} \subset J$ and $R := \{f(y)\,;\ y \in Y\} \subset B$. The ideal J contains sequence of elements $d_n \in J_+$ with $\|d_n\| < 1$ and

$$\lim_n (\|q - d_n q\| + \|d_n q - q\|) = 0$$

for all $q \in Q$ and $\lim_n \|[r, d_n]\| = 0$ for all $r \in R$ by the existence of a quasi-central approximate unit in J_+ for B, cf. [**616**][thms. 1.4.2, 3.12.14]. It follows that $\lim_n \|\psi(d_n) q \psi(d_n)\| = 0$ for $q \in Q$ and $\lim_n \|[r, \psi(d_n)]\| = 0$ for $r \in R$. Thus, we find $n \in \mathbb{N}$ such that for $d := d_n$ the inequalities

$$\|\psi(d)(f(y) - g(y))\psi(d)\| \leq \delta + \|\psi(d)(f(y) - g(y) + c(y))\psi(d)\| .$$

and

$$\|[f(y), d]\| + \|[f(y), \psi(d)]\| < 2\delta .$$

Notice that

$$\|\psi(d)(f(y) - g(y) + c(y))\psi(d)\| \leq 2\delta + \|\pi_J(f(y)) - \pi_J(g(y))\| .$$

Summing up gives $\|f(y) - h(y)\| < \varepsilon + \|\pi_J(f(y)) - \pi_J(g(y))\|$ for all $y \in Y$. □

Proposition 3.6.30. *Suppose that A is separable, that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a matrix operator-convex cone in sense of Definition 3.2.2, and that J is a closed ideal of B.*

Then $\pi_J \circ \mathcal{C} := \{\pi_J \circ V\,;\ V \in \mathcal{C}\} \subseteq \mathrm{CP}(A, B/J)$ is a matrix operator-convex cone.

The m.o.c. cone $\pi_J \circ \mathcal{C}$ is closed with respect to the point-norm topology in $\mathcal{L}(A, B/J)$ if $\mathcal{C}$ is closed with respect to the point-norm topology in $\mathcal{L}(A, B)$.

Moreover, for every $W \in \pi_J \circ \mathcal{C}$ *there exists* $V \in \mathcal{C}$ *with* $\|V\| = \|W\|$ *and* $W = \pi_J \circ V$ *if* $\mathcal{C}$ *is point-norm closed.*

PROOF. Clearly, $\pi_J \circ \mathcal{C} \subseteq \mathrm{CP}(A, B/J)$. By Definition 3.2.2 of m.o.c. cones $\mathcal{C}$ we have to check that for $V_1, V_2, V \in \mathcal{C}$, elements $s, t \in B/J$, rows $r = [r_1, \ldots, r_n] \in M_{1,n}(A)$ and columns $c = [c_1, \ldots, c_n]^\top \in M_{n,1}(B/J)$ there exist $W_0, W \in \mathcal{C}$ such that $\pi_J(W_0(a)) = s^* \pi_J(V_1(a))s + t^* \pi_J(V_2(a))t$ and $\pi_J(W(a)) = c^*((\pi_J \circ V) \otimes \mathrm{id}_n)(r^* a r)c$ for $a \in A$.

Since π_J is surjective, we find $b_1, b_2, d_k \in B$ with $\pi_J(b_1) = s$, $\pi_J(b_2) = t$ and $\pi_J(d_k) = c_k$ for $k = 1, \ldots, n$. Let $W_0(a) := b_1^* V_1(a) b_1 + b_2^* V_1(a) b_2$ and $W(a) := d^*(V \otimes \mathrm{id}_n)(r^* a r)d$ for the column $d := [d_1, \ldots, d_n]^\top \in M_{n,1}(B)$ with $(\pi_J \otimes \mathrm{id}_n)(d) = c$.

Then W_0 and W are in in $\mathcal{C}$ and have the desired properties, i.e., $\pi_J \circ \mathcal{C}$ is a (not necessarily closed) m.o.c. cone.

It follows that the point-norm closure $\mathcal{C}'$ of $\pi_J \circ \mathcal{C}$ is again an m.o.c. cone, and that every element of of $T \in \mathcal{C}'$ is the limit in point-norm convergence of a sequence $\pi_J \circ V_n \in \pi_J \circ \mathcal{C}$ with $\|\pi_J \circ V_n\| \leq \|T\|$, cf. Lemma 3.1.8.

We show that $\pi_J \circ \mathcal{C}$ itself is closed in point-norm topology if $\mathcal{C}$ is closed in point norm topology, i.e., that the set $\pi_J \circ \mathcal{C}$ of images of c.p. maps $V \in \mathcal{C}$ under the map $\mathcal{C} \ni V \to \pi_J \circ V \in \mathrm{CP}(A, B/J)$ is closed under point-wise convergence.

Notice first that $V := e^{1/2} V_1(\cdot) e^{1/2} + (1-e)^{1/2} V_2(\cdot)(1-e)^{1/2}$ is in $\mathcal{C}$ for contractions $e \in J_+$ with $\|e\| \leq 1$ and $V_1, V_2 \in \mathcal{C}$, and that $\|V\| \leq \max(\|V_1\|, \|V_2\|)$.

To see that V is in the point-norm closure of $\mathcal{C}$ it suffices to observe that for an approximative unit $\{c_\tau\} \subset B_+$ of B the maps $a \in A \mapsto (1-e)^{1/2} c_\tau V_2(a) c_\tau (1-e)^{1/2}$ converge point-wise to $(1-e)^{1/2} V_2(\cdot)(1-e)^{1/2}$.

Thus the condition (ii) of Lemma 3.6.29 is satisfied for $X := A$, $\rho(x, y) = \|x - y\|$, the set $\mathcal{C}$ of c.p. maps $V \in \mathcal{C} \subset \mathrm{CP}(A, B)$ (in place of $\mathcal{S}$ there) and the map $f \colon V \mapsto \pi \circ V$. The condition (i) of Lemma 3.6.29 is also satisfied, with $\mu(t) := t$.

Thus, $\pi_J \circ \mathcal{C} \subseteq \mathrm{CP}(A, B/J)$ is point-norm closed by Lemma 3.6.29.

The same arguments apply if we apply Lemma 3.6.29 to the set $\mathcal{S}$ of $V \in \mathcal{C} \subset$ with $\|V\| \leq 1$: This $\mathcal{S}$ is closed in the point-norm topology (if $\mathcal{C}$ is closed in point-norm topology), $X := A$ is separable and

$$V := e^{1/2} V_1(\cdot) e^{1/2} + (1-e)^{1/2} V_2(\cdot)(1-e)^{1/2} \in \mathcal{S}$$

for $V_1, V_2 \in \mathcal{S}$ and $e \in J_+$ with $\|e\| \leq 1$.

Thus the image $\pi_J \circ \mathcal{S}$ is closed in the topology of point-norm convergence on A.

Since $\pi_J \circ \mathcal{S}$ is closed under point-wise convergence, it suffices to show that $\pi_J \circ \mathcal{S}$ is dense in the set of $W \in \pi_J \circ \mathcal{C}$ with $\|W\| \leq 1$, to get that each $W \in \pi_J \circ \mathcal{C}$ with $\|W\| \leq 1$ is in $\pi_J \circ \mathcal{S}$ itself.

Let $W \in \pi_J \circ \mathcal{S}$ with $\|W\| \leq 1$. Above we have shown that there exists $V \in \mathcal{C}$ with $\pi \circ V = W$. Since A is separable, there exists a strictly positive contraction $c \in A_+$. It satisfies for all $n \in \mathbb{N}$ that

$$\operatorname{dist}(V(c^{2/n}), J) = \|\pi_J(V(c^{2/n}))\| = \|W(c^{2/n})\| \leq \|c\|^{2/n} \leq 1 \,.$$

The sequence of c.p. contractions $a \in A \mapsto W(c^{1/n}ac^{1/n})$ converge point-wise to W, because $a = \lim_{n\to\infty} c^{1/n}ac^{1/n}$ for each $a \in A$.

Thus, it suffices to find for each contraction $c \in A_+$ a sequence of positive contractions $b_m \in B_+$ such that $\|b_m V(c^2) b_m\| \leq 1$ and

$$\lim_m \|\pi_J(b_m)W(cac)\pi_J(b_m) - W(cac)\| = 0 \qquad \text{for each } \ a \in A \ .$$

Since $W(cAc)$ is contained in the hereditary C^*-subalgebra of B/J generated by $W(c^2)$, it suffices to find contractions $b_m \in C^*(V(c^2))_+$ with $\pi_J(b_m) = W(c^2)^{1/m}$ and $\|b_m^2 V(c^2)\| \leq 1$, e.g. we can take $b_m := \varphi(V(c^2)^{1/m})$ for the function $\varphi(\xi) := \min(\xi, 1)$. □

Proposition 3.6.30 and the surjectivity in Proposition 3.1.9(iii) together imply the following equivalent formulation of the Choi-Effros lifting theorem for nuclear maps [**140**], cf. also [**43**].

Corollary 3.6.31. *If A is separable, and J is a closed ideal of B, then*

$$\pi_J \circ \mathrm{CP}_{\mathrm{nuc}}(A, B) = \mathrm{CP}_{\mathrm{nuc}}(A, B/J) \,.$$

Proof. Clearly, $\pi_J \circ \mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathrm{CP}_{\mathrm{nuc}}(A, B/J)$. Proposition 3.1.9(iii) says that $\pi_J \circ \mathrm{CP}_f(A, B) = \mathrm{CP}_f(A, B/J)$. The m.o.c. cones $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ and $\mathrm{CP}_{\mathrm{nuc}}(A, B/J)$ are the point-norm closures of $\mathrm{CP}_f(A, B)$ respectively of $\mathrm{CP}_f(A, B/J)$. It follows

$$\mathrm{CP}_f(A, B/J) \subseteq \pi_J \circ \mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathrm{CP}_{\mathrm{nuc}}(A, B/J) \,.$$

But $\pi_J \circ \mathrm{CP}_{\mathrm{nuc}}(A, B)$ is point-norm closed by Proposition 3.6.30. □

7. Operator-convex Cones versus Actions by topological Spaces

Lemma 3.7.1. *Suppose that $\mathcal{C} \subseteq \mathrm{CP}(B, A)$ is an m.o.c. cone and that $\Phi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is the related upper s.c. action defined by $\mathcal{C}$ (i.e., $\Phi(J) :=$ the closed ideal of A, that is generated by $\{V(b)\,;\ b \in J_+,\ V \in \mathcal{C}\}$).*

Let $J_b := \mathrm{span}(BbB)$. Then $\{V(b)\,;\ b \in J_+,\ V \in \mathcal{C}\} = \Phi(J)_+$, and, for every $b \in B_+$, $a \in \Phi(J_b)_+$ and $\varepsilon > 0$, there exist $V \in \mathcal{C}$ with $\|V(b) - a\| < \varepsilon$.

Conversely:
Let $\Phi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ an upper semi-continuous action, and $\Psi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ its lower semi-continuous Galois adjoint. Suppose that $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(\Psi; B, A)$ realizes Ψ, in the sense that $\Psi = \Psi_{\mathcal{C}}$. Then $\Phi_{\mathcal{C}} = \Phi$.

Proof. **??** □

Give (or refer to) definitions of $\mathrm{CP}_{\mathrm{rn}}(\Psi;\, B, A)$, $\mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), \Psi, \mathrm{id}; A, B)$, $\mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(A), \mathrm{id}, \Phi; A, B)$, $\mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), \mathrm{id}, \mathrm{id}; B, B)$, $\mathrm{CP}_{\mathrm{in}}(B, B) = \mathcal{C}(\mathrm{id}_B)$, $\mathrm{CP}(\mathrm{Prim}(B);\, B, B)$.

LEMMA 3.7.2. *Let $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ a lower s.c. action and $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ its upper s.c. Galois adjoint. Then*

$$\mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), \Psi, \mathrm{id}; A, B) = \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(A), \mathrm{id}, \Phi; A, B)$$

and

$$\mathrm{CP}(\mathrm{Prim}(B), \Psi, \mathrm{id}; A, B) = \mathrm{CP}(\mathrm{Prim}(A), \mathrm{id}, \Phi; A, B)\,.$$

PROOF. **??** □

REMARK 3.7.3. Let $\mathrm{CP}_{\mathrm{rn}}(B) := \mathrm{CP}_{\mathrm{rn}}(B, B) := \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), \mathrm{id}, \mathrm{id}; B, B)$. Then

$$\mathrm{CP}_{\mathrm{rn}}(B) \subseteq \mathrm{CP}_{\mathrm{in}}(B, B) = \mathcal{C}(\mathrm{id}_B) \subseteq \mathrm{CP}(\mathrm{Prim}(B);\, B, B)\,.$$

If B is separable and nuclear, then all this 3 m.o.c. cones are identical. (For the *proof* one can use that then $B \otimes \mathcal{O}_2$ contains a regular Abelian C^*-subalgebra, cf. [**359**], and – therefore – can apply Proposition **??** about non-commutative selection.)

`The content of ref. to missing Prop. should be:`
If B is separable and nuclear, and $B \otimes \mathcal{O}_2$ contains a "regular Abelian C^*-subalgebra" (here only cited) then this implies the existence of sufficient many "n.c. selections" that generate $\mathrm{CP}(\mathrm{Prim}(B);\, B, B)$???

If separable $B \neq \{0\}$ is simple and is *not* nuclear, then $\mathrm{CP}_{\mathrm{rn}}(B)$, $\mathrm{CP}_{\mathrm{in}}(B, B)$ and $\mathrm{CP}(B, B)$ are all different m.o.c. cones, but define the same lower s.c. action of $\mathrm{Prim}(B) = \{\{0\}\}$ on B. This happens e.g. for the exact simple C^*-algebra $B := C^*_{\mathrm{red}}(SL(2, \mathbb{Z}))$.

PROPOSITION 3.7.4. *If B is separable and $B \otimes \mathcal{O}_2$ contains a regular abelian C^*-subalgebra (cf. Definition B.4.1) $C \subseteq B \otimes \mathcal{O}_2$, then, for every lower semi-continuous action $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ of* $\mathrm{Prim}(B)$ *on a separable C^*-algebra A is realizable by* $\mathrm{CP}_{\mathrm{rn}}(\Psi;\, A, B)$, *i.e., $\Psi = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), \Psi, \mathrm{id}; A, B)$.*

PROOF. `to be filled in ??`

`transfer proof from chp. 12 to here !` □

Proposition 3.7.4 implies immediately:

COROLLARY 3.7.5. *If B is separable and $B \otimes \mathcal{O}_2$ contains a regular abelian C^*-subalgebra (cf. Definition B.4.1) $C \subseteq B \otimes \mathcal{O}_2$, then B has residual nuclear separation (cf. Definition 1.2.3).*

PROPOSITION 3.7.6. *Suppose that B has residually nuclear separation (cf. Definition 1.2.3). Then B has the Weyl–von-Neumann property (cf. Definition 1.2.3), if and only if, B is strongly purely infinite (cf. Definition 1.2.2).*

PROOF. To be filled in ??

Proof was somewhere outlined, perhaps in Chp. 1 ? □

8. Separation of m.o.c. cones by actions

There are proofs of some of the following corollaries, that are more elementary than given below. But we want to invite the reader to use (and think about) the conceptional ideas presented here. For possible future classification of some classes of stably finite algebras with non-trivial traces, one has to produce more refined tools, that give a "applicable" answers e.g. to the following more difficult question:

When is a given completely positive map V in a given cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ of completely positive maps compatible with a given map $T(B) \to T(A)$?

Here $T(B)$ means the set of lower semi-continuous 2-quasi-traces $\tau \colon B_+ \to [0, \infty]$ – possibly with values only in $\{0, \infty\}$. Our results together contain an answer in the special case where $\tau(B_+) \subseteq \{0, \infty\}$ for all $\tau \in T(B)$, because then the problem reduces to to the determination of $\mathrm{CP}_{\mathrm{nuc}}(\Psi;\, A, B)$ for a lower semi-continuous action $\Psi \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$, if we identify a $\{0, \infty\}$-valued lower semi-continuous 2-quasi-trace $\tau \colon B_+ \to [0, \infty]$ by its kernel $J := \mathrm{span}\{a \in B_+\,;\; \tau(a) = 0\}$. Notice that we do not require operator convexity in the following definition.

Compare for next Def. also Def. 3.12.1

DEFINITION 3.8.1. Let $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ any subset and $J \in \mathcal{I}(B)$.

We define the ideal $\Psi_{\mathcal{S}}(J) \in \mathcal{I}(A)$ by its positive part $\Psi_{\mathcal{S}}(J)_+$ as the set of all $a \in A_+$ with $V(\exp(-ih)a\exp(ih)) \in J$ for all $h^* = h \in A$ with $\|h\| < \pi$ and for all $V \in \mathcal{S}$. Then $\Psi_{\mathcal{S}}(J)_+$ is the positive part of a closed ideal $\Psi_{\mathcal{S}}(J)$ of A, and

$$\Psi_{\mathcal{S}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \cong \mathbb{O}(\mathrm{prime}(B)) \to \mathcal{I}(A)$$

is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, cf. Lemma 3.12.2(iv).

Let I any closed ideal of A. We denote by $\Psi^{\mathcal{S}}(I) \in \mathcal{I}(B)$ the smallest closed ideal of B that contains $\{V(a)\,;\; a \in I, V \in \mathcal{S}\}$. The map

$$\Psi^{\mathcal{S}} \colon \mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A)) \cong \mathbb{O}(\mathrm{prime}(B)) \to \mathcal{I}(B)$$

is an upper semi-continuous action, cf. Lemma 3.12.2 (xii).

Let X a topological space, and let $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(X) \to \mathcal{I}(B)$ increasing maps, i.e., *actions* of X on A and B. We define the **cone of Ψ-equivariant c.p. maps** as the set $\mathcal{C}_\Psi := \mathrm{CP}(\Psi_A, \Psi_B; A, B) \subseteq \mathrm{CP}(A, B)$ of maps $V \in \mathrm{CP}(A, B)$ with $V(\Psi_A(U)) \subseteq \Psi_B(U)$ for all open subsets $U \subseteq X$. In the special case, where $X := \mathrm{Prim}(B)$ and $\Psi_B \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(B)$ is given by the natural identification of the open subsets U of $\mathrm{Prim}(B)$ with the corresponding closed ideal J_U of B, where J_U denotes the intersection of all primitive ideals $I \in \mathrm{Prim}(B) \setminus U$, we write $\Psi := \Psi_A$ and drop Ψ_B, i.e., write $\mathcal{C}_\Psi := \mathrm{CP}(\Psi;\, A, B) \subseteq \mathrm{CP}(A, B)$ where $\Psi \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ is a lower semi-continuous action.

The use of the notation $\mathcal{C}_\Psi$, in place of $\mathcal{C}_{\Psi_A,\Psi_B}$, becomes justified in a different way in part (i) of Lemma 3.12.2.

`Compare next with Lemma 3.12.2 in eksec3-Part2 .tex`

LEMMA 3.8.2. *Let A and B C^*-algebras, $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ a set of completely positive maps, $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a m.o.c. cone, X a topological space, $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(X) \to \mathcal{I}(B)$ actions of X on A respectively B,*

Suppose that ?????????????

List of necessary assumptions? ??

Look for minimal m.o.c. cone assumptions!!!

Part (i) is only lattice stuff?

(i) *There exists a lower semi-continuous action $\Psi' \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \to \mathcal{I}(A)$ with $\Psi'(\Psi_B(U)) \supseteq \Psi_A(U)$), which is* minimal *in the sense that $\Phi(J) \supset \Psi'(J)$ for every $J \in \mathcal{I}(B)$ if $\Phi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is lower semi-continuous and $\Phi(\Psi_B(U)) \supset \Psi_A(U)$.*
This (minimal) action satisfies $\mathcal{C}_{\Psi'} = \mathcal{C}_\Psi$.

(ii) *The point-norm closure of an m.o.c. cone (in the algebraic sense) is a matrix operator-convex cone.*
The intersection of a family of m.o.c. cones is am m.o.c. cone.

(iii) *Let $\mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ denote the smallest "algebraic" m.o.c. cone that contains a subset $\mathcal{S} \subseteq \mathrm{CP}(A,B)$.*
Every contraction V in the point-norm closure $\mathcal{C}(\mathcal{S})$ of can be approximated by maps $W \in \mathcal{C}_{\mathrm{alg}}(\mathcal{S})$ of the particular form $W := \sum_k c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$ with $V_k \in \mathcal{S}$, $r_k \in M_{1,n}(A)$, $c_k \in M_{n,1}(B)$ and $\|\sum_k c_k^* V_k(r_k^* r_k)c_k\| \leq 1$.

(iv) *$\Psi_{\mathcal{S}}(J)_+$ is the positive part of the closed ideal $\Psi_{\mathcal{S}}(J)$ of A defined in Definition 3.8.1, and*

$$\Psi_{\mathcal{S}} \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \ni J \;\mapsto\; \Psi_{\mathcal{S}}(J) \in \mathcal{I}(A)$$

is a lower semi-continuous action of $\mathrm{Prim}(B)$ *on A.*
The action $\Psi_{\mathcal{S}}$ satisfies $\Psi_{\mathcal{S}}(B) = A$ and $V(\Psi_{\mathcal{S}}(J)) \subseteq J$ for all $J \in \mathcal{I}(B)$ and $V \in \mathcal{S}$.

(v) *$\Psi_{\mathcal{S}} = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathcal{C}(\mathcal{S})$.*

(vi) *If $\mathcal{S}_1 \subseteq \mathcal{S}$ then $\Psi_{\mathcal{S}}(J) \subseteq \Psi_{\mathcal{S}_1}(J)$ for all $J \in \mathcal{I}(B)$.*

(vii) *The set $\mathcal{C}_\Psi$ is a point norm-closed m.o.c. cone of completely positive maps.*

(viii) *$\mathcal{C}_{\Psi_1} \subseteq \mathcal{C}_\Psi$ if $\Psi(J) \subseteq \Psi_1(J)$ for all $J \in \mathcal{I}(B)$.*

(ix) *$\mathcal{C} \subseteq \mathcal{C}_{\Psi_{\mathcal{C}}}$.*

(x) *$\Psi_{\mathcal{C}_\Psi}(J) \supset \Psi(J)$ for $J \in \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$.*

(xi) *If $\mathcal{C}' \subseteq \mathrm{CP}(A \otimes^{\max} D, B \otimes^{\max} D)$ is a point-norm closed m.o.c. cone, then the set $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ of $T \in \mathrm{CP}(A,B)$ with $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ is a point-norm closed m.o.c. cone.*

(xii) *Let M a W^*-algebra and $B \subseteq M$ a $\sigma(M, M_*)$-dense C^*-subalgebra of M, and let $\mathcal{C} \subseteq \mathrm{CP}(A, B) \subseteq \mathrm{CP}(A, M)$ an m.o.c. cone. Then the point-*ultra-strong closure $\overline{\mathcal{C}} \subseteq \mathrm{CP}(A, M)$ of $\mathcal{C}$ is an m.o.c. cone that is also closed in the point-ultra-weak topology ($\sigma(M, M_*)$-topology on M).*

(xiii) *$\Psi^{\mathcal{S}} \colon \mathcal{I}(A) \to \mathcal{I}(B)$ is an upper semi-continuous action of $\mathrm{Prim}(A)$ on B. $I \subseteq \Psi_{\mathcal{S}}(\Psi^{\mathcal{S}}(I))$ for every closed ideal I of A and every subset $\mathcal{S} \subseteq \mathrm{CP}(A, B)$. It says that $\Psi^{\mathcal{S}}$ is the Galois adjoint of $\Psi_{\mathcal{S}}$ and vice-versa*

(ix) *What about $\Psi^{\mathcal{S}} \circ \Psi_{\mathcal{S}}$?*

PROOF. (iv): Let J a closed ideal of B and $T \colon A \to B$ a positive map. The set of $a \in A_+$ with $(\pi_J \circ T)(a) = 0$ is a *hereditary* closed convex sub-cone $C_{T,J}$ of A_+. Thus, the set $C_{\mathcal{S},J}$ of $a \in A_+$ with $T(a) \in J$ for all $T \in \mathcal{S}$ is a hereditary closed convex sub-cone of A_+, and $\Psi_{\mathcal{S}}(J)_+ \subseteq A_+$ of Definition 3.8.1 is the set of $a \in A_+$ with $\exp(ih)a\exp(-ih) \in C_{\mathcal{S},J}$.

By Lemma A.25.1, $\Psi_{\mathcal{S}}(J)_+$ is the positive part of a closed ideal $\Psi_{\mathcal{S}}(J)$ of A. Clearly, $\Psi_{\mathcal{S}}(B) = A$, and $V(\Psi_{\mathcal{S}}(J)_+) \subseteq J$ for $J \in \mathcal{I}(B)$ and $V \in \mathcal{S}$, by definition of $\Psi_{\mathcal{S}}$.

If $\{J_\sigma\}_{\sigma \in \Sigma}$ is a family of closed ideals and $a \in A_+$ and let $J := \bigcap_\sigma J_\sigma$. then $V(\exp(ih)a\exp(-ih)) \in J$ for all $V \in \mathcal{S}$ and $h^* = h \in A$ with $\|h\| < \pi$, if and only if, $V(\exp(ih)a\exp(-ih)) \in J_\sigma$ for all $V \in \mathcal{S}$ and $h^* = h \in A$ with $\|h\| < \pi$, i.e., $a \in \Psi_{\mathcal{S}}(J)_+$ if and only if $a \in \bigcap_\sigma \Psi_{\mathcal{S}}(J_\sigma)_+ = (\bigcap_\sigma \Psi_{\mathcal{S}}(J_\sigma))_+$. Thus,

$$\Psi_{\mathcal{S}} \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \ni J \mapsto \Psi_{\mathcal{S}}(J) \in \mathcal{I}(A)$$

is a lower semi-continuous map. The lower semi-continuity of the map $\Psi_{\mathcal{S}}$ implies that $\Psi_{\mathcal{S}}$ is monotone. Thus, $\Psi_{\mathcal{S}}$ is a lower s.c. action of $\mathrm{Prim}(B)$ on A.

(vii): Since $\Psi_B(U)$ is a closed ideal, the set of maps $V \in \mathrm{CP}(A, B)$ with $V(a) \in \Psi_B(U)$ for all $a \in \Psi_A(U)$ is point-norm closed. Thus, $\mathcal{C}_\Psi$ is point-norm closed. Similar arguments show:
Let $V_1, V_2 \in \mathrm{CP}(A, B)$ with $V_k(\Psi_A(U)) \subseteq \Psi_B(U)$ for all $U \in \mathbb{O}(X)$ ($k = 1, 2$), $t \in [0, \infty)$, and $r \in M_{1,n}(A)$, $c \in M_{n,1}(B)$. Then $(V_1 + tV_2)(a) \in \Psi_B(U)$ and $c^*(V_1 \otimes \mathrm{id}_n)(r^* a r)c \in \Psi_B(U)$ for $U \in \mathbb{O}(X)$ and $a \in \Psi_A(U)$. Thus $\mathcal{C}_\Psi$ is a point-norm closed m.o.c. cone of c.p. maps.

(i): If we use the natural isomorphisms $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ and $\mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$, and Lemma ??, then we get that there is a unique minimal lower semi-continuous action $\Psi' \colon \mathcal{I}(B) \to \mathcal{I}(A)$ of $\mathbb{O}(\mathrm{Prim}(B))$ on $\mathcal{I}(A)$ with $\Psi'(\Psi_B(U)) \supset \Psi_A(U)$.

If $V \in \mathcal{C}_{\Psi'}$, then $V(\Psi_A(U)) \subseteq V(\Psi'(\Psi_B(U))) \subseteq \Psi_B(U)$ for $U \in \mathbb{O}(X)$, because $\Psi_A(U) \subseteq \Psi'(\Psi_B(U))$. Thus, $\mathcal{C}_{\Psi'} \subseteq \mathcal{C}_\Psi = \mathcal{C}_{\Psi_A, \Psi_B}$.

Let $\mathcal{S} := \mathcal{C}_\Psi$, then $\Psi_{\mathcal{S}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ is a lower semi-continuous action by part (iv) with $V(\Psi_{\mathcal{S}}(J)) \subseteq J$ for all $V \in \mathcal{S}$.

Since $V(\exp(ih)a\exp(-ih)) \in \Psi_B(U)$ for all $a \in \Psi_A(U)_+$ and $h^* = h \in A$, $\Psi_{\mathcal{S}}(\Psi_B(U))_+ \supset \Psi_A(U)_+$ for $U \in \mathbb{O}(X)$. Thus $\Psi_{\mathcal{S}}(J) \supset \Psi'(J)$ for every $J \in \mathcal{I}(B)$

by minimality of Ψ'. It implies that $V(\Psi'(J)) \subseteq V(\Psi_{\mathcal{S}}(J)) \subseteq J$, i.e., $V \in \mathcal{C}_{\Psi'}$, for all $V \in \mathcal{C}_\Psi = \mathcal{C}_{\Psi_A,\Psi_B}$.

(ii): The point-norm closure $\mathcal{C}$ of an m.o.c. cone $\mathcal{C}_0$ is an m.o.c. cone, because the topology of point-norm convergence on $\mathcal{L}(A,B)$ coincides with the strong operator topology, and because $V_\alpha \otimes \mathrm{id}_n$ converges in point-norm to $V \otimes \mathrm{id}_n$ if $V_\alpha \in \mathcal{C}_0$ converges to V.

(iii): Let $n \in \mathbb{N}$ and let $\mathcal{C}^{(n)}$ denote the set of completely positive maps $W := \sum_k c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$ with $V_k \in \mathcal{S}$, $r_k \in M_{1,n}(A)$ and $c_k \in M_{n,1}(B)$, and let $\mathcal{C}_0 := \bigcup_n \mathcal{C}^{(n)}$. Clearly, $\mathcal{C}^{(n)} \subseteq \mathcal{C}$ for every matrix operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ with $\mathcal{S} \subseteq \mathcal{C}$, and the set $\mathcal{C}^{(n)}$ is closed under multiplication with positive scalars and under addition.

Moreover, $c^*(W \otimes \mathrm{id}_m)(r^*(\cdot)r)c \in \mathcal{C}^{(mn)}$ if $W \in \mathcal{C}^{(n)}$, $r \in M_{1,m}(A)$ and $c \in M_{n,1}(B)$, because

$$c^*(W_k \otimes \mathrm{id}_m)(r^*(\cdot)r)c = C^* V_k \otimes \mathrm{id}_{mn}(R^*(\cdot)R)C$$

for $W_k := c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$, $R = [R_1,\ldots,R_{mn}] \in M_{1,mn}(A)$ and $C = [C_1,\ldots,C_{mn}]^\top \in M_{mn,1}(B)$, where $R_{jm+y} := r_y r_j^{(k)}$ and $C_{jm+y} := c_j^{(k)} c_y$ with $r_k = [r_1^{(k)},\ldots,r_n^{(k)}] \in M_{1,n}(A)$, $r = [r_1,\ldots,r_m] \in M_{1,m}(A)$, $c_k = [c_1^{(k)},\ldots,c_n^{(k)}]^\top \in M_{n,1}(B)$, and $c = [c_1,\ldots,c_m]^\top \in M_{m,1}(B)$.

If we fill the rows $r_k \in M_{1,n}(A)$ and columns $c_k \in M_{n,1}(B)$ with zeros, then we see that $\mathcal{C}^{(n)} \subseteq \mathcal{C}^{(n+k)}$ for all $n,k \in \mathbb{N}$. Thus $\mathcal{C}_0 = \bigcup_n \mathcal{C}^{(n)}$ is a (not necessarily closed) m.o.c. cone that is contained in every matrix operator convex cone $\mathcal{C}$ with $\mathcal{S} \subseteq \mathcal{C}$.

Since A has an approximate unit $\{e_\tau\}$ of positive contractions, we get $\|W\| = \|\sum_k c_k^* V_k(r_k^* r_k)c_k\|$ for $W \in \mathcal{C}_0$, and that every element $V \in \mathcal{S}$ is in the point-norm closure of $\mathcal{C}^{(1)} \subseteq \mathcal{C}_0$.

By part (ii), the point-norm closure $\mathcal{C}_1$ of $\mathcal{C}_0$ is an m.o.c. cone that contains $\mathcal{S}$. Since $\mathcal{C}_0$ is contained in every point-norm closed m.o.c. cone $\mathcal{C}$ with $\mathcal{S} \subseteq \mathcal{C}$, we get $\mathcal{C}(\mathcal{S}) = \mathcal{C}_1$. The contractions T in the point-norm closure of $\mathcal{C}_0$ can be approximated by contractions $W \in \mathcal{C}_0$ with $\|W\| \le 1$ by Lemma 3.1.8.

(v): Since $\mathcal{S} \subseteq \mathcal{C}(\mathcal{S}) =: \mathcal{C}$, it holds $\Psi_{\mathcal{C}}(J) \subseteq \Psi_{\mathcal{S}}(J)$ for all $J \in \mathcal{I}(B)$, cf. part (vi).

If $a \in \Psi_{\mathcal{S}}(J)_+$ then $V(\exp(ih)a\exp(-ih)) \in J$ for all $h^* = h \in A$ with $\|h\| < \pi$ and all $V \in \mathcal{S}$.

By part (iv), this implies that $c_\ell^* V(r_\ell^* \exp(ih)a\exp(-ih)r_k)c_k \in J$ for $V \in \mathcal{S}$, $h^* = h \in A$, $r_1,\ldots,r_n \in A$, $c_1,\ldots,c_n \in B$. Thus, $W(\exp(ih)a\exp(-ih)) \in J$ for all $W \in \mathcal{C} := \mathcal{C}(\mathcal{S})$ and $h^* = h$ by part (iii), i.e., $a \in \Psi_{\mathcal{C}}(J)_+$, and $\Psi_{\mathcal{S}} = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathcal{C}(\mathcal{S})$.

(vi): Straight from definition of $\Psi_{\mathcal{S}}$.

(viii): If $V \in \mathcal{C}_{\Psi_1}$ and $\Psi(J) \subseteq \Psi_1(J)$, then $V(\Psi(J)) \subseteq V(\Psi_1(J)) \subseteq J$. Thus, $V \in \mathcal{C}_\Psi$ if $\Psi(J) \subseteq \Psi_1(J)$ for all $J \in \mathcal{I}(B)$.

(ix): For $J \in \mathcal{I}(B)$ and $a \in \Psi_{\mathcal{C}}(J)$ holds $V(a) \in J$ for all $V \in \mathcal{C}$ by definition of $\Psi_{\mathcal{C}}$.

(x): Let $J \in \mathcal{I}(B)$, and let $a \in \Psi(J)_+ \subseteq A_+$, $V \in \mathcal{C}_\Psi$ and $h^* = h \in A$. Then $V(\exp(ih)a\exp(-ih)) \in J$ by definition of $\mathcal{C}_\Psi$, i.e., $a \in \Psi_{\mathcal{C}_\Psi}(J)_+$ by definition of $\Psi_{\mathcal{C}}$. Thus, $\Psi(J) \subseteq \Psi_{\mathcal{C}_\Psi}(J)$.

(xi): If $\{T_\gamma\}_{\gamma\in\Gamma} \subseteq \mathrm{CP}(A, B)$ is a (norm-)bounded net that converges in point-norm to $T \in \mathcal{L}(A, B)$, then $T \in \mathrm{CP}(A, B)$ and the net

$$\{T_\gamma \otimes^{\max} \mathrm{id}\}_{\gamma\in\Gamma} \subseteq \mathrm{CP}(A \otimes^{\max} D, B \otimes^{\max} D)$$

converges point-wise to $T_\gamma \otimes^{\max} \mathrm{id}$, because $A \odot D$ is dense in $A \otimes^{\max} D$. Thus, if $T_\gamma \otimes^{\max} \mathrm{id} \in \mathcal{C}'$ for all $\gamma \in \Gamma$, then $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$. Clearly, the set $\mathcal{C}$ of $T \in \mathrm{CP}(A, B)$ with $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$ is convex. If $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$, $r \in M_{1n}(A)$, $c \in M_{n1}(B)$, $e \in D_+$, then $R := r \otimes e \in M_{1n}(A \otimes D)$, $C := c \otimes e \in M_{n1}(B \otimes D)$, and

$$R^*((T \otimes^{\max} \mathrm{id}) \otimes (\mathrm{id}_n))(C^*(\cdot)C)R = T' \otimes^{\max} S_e$$

for $T' := r^*(T \otimes \mathrm{id}_n)(c(\cdot)c^*)r$ $S_e(d) := e^2de^2$ $(d \in D)$. Since $\mathcal{C}'$ is matrix operator convex and is point-norm closed,it follows that $T' \otimes^{\max} \mathrm{id} \in \mathcal{C}'$. Thus, the set $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ of $T \in \mathrm{CP}(A, B)$ with $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ is a point-norm closed m.o.c. cone.

(xii): The point-norm closure $\mathcal{C}_0 := \overline{\mathcal{C}}^{\mathrm{norm}}$ of $\mathcal{C}$ is a convex cone with $c^*(V \otimes \mathrm{id}_n)(r^*(\cdot)r)c \in \mathcal{C}_0$ for $r \in M_{n,1}(A)$, $c \in M_{1,n}(B)$ and $V \in \mathcal{C}_0$. Since the point-*strong closures of $\mathcal{C}$ and $\mathcal{C}_0$ in $\mathcal{L}(A, M)$ coincide, and since $V_\gamma \otimes \mathrm{id}_n \to W \otimes \mathrm{id}_n$ point-*strongly if $V_\gamma \to W$ point-*strongly , we obtain that the point-*strong closure $\overline{\mathcal{C}} \subseteq \mathcal{L}(A, M)$ is a convex cone that satisfies $c^*(W \otimes \mathrm{id}_n)(r^*(\cdot)r)c \in \mathcal{C}_0$ for $r \in M_{n,1}(A)$, $c \in M_{1,n}(B)$ and $W \in \overline{\mathcal{C}}$. The unit-ball of $M_n(B)$ is *-ultra-strongly dense in the unit-ball of $M_n(M)$ by Kaplansky density theorem ([**616**, Thm. 2.3.3]). Thus, for $c \in M_{1,n}(M)$, there exists a net $c_\gamma \in M_{1,n}(B)$ with $\|c_\gamma\| \le \|c\|$ that *strongly converges to c. Hence, the point-*strong closure $\overline{\mathcal{C}}$ satisfies (OC2).

If $\{V_\gamma\} \subseteq \overline{\mathcal{C}}$ is a directed net that converges in point-$\sigma(M, M_*)$ topology to $W \in \mathrm{Lin}(A, M)$, then $W(e^2) \ge 0$ for every $e \in A_+$. In particular, W is positive and bounded, thus $\limsup_\gamma \|V_\gamma(e)\| =: \mu(e) < \infty$ for every $e \in A_+$, and $\mu := \limsup_\gamma \|V_\gamma\| < \infty$ by two-fold application of the Banach-Steinhaus theorem. It implies that W is also in the point-$\sigma(M, M_*)$ closure of the convex set of $V \in \overline{\mathcal{C}}$ with $\|V\| \le 1 + \nu$. Now a Hahn-Banach separation argument shows that $W \in \overline{\mathcal{C}}$.

(xiii): $\Psi^{\mathcal{S}}(\overline{\sum_\gamma I_\gamma}) = \overline{\sum_\gamma \Psi^{\mathcal{S}}(I_\gamma)}$ and $a \in \Psi_{\mathcal{S}}(\Psi^{\mathcal{S}}(I))$ for $a \in I \triangleleft A$ follow straight from the definitions of $\Psi_{\mathcal{S}}$ and $\Psi^{\mathcal{S}}$. □

`Compare also Example 3.12.3!!!`

EXAMPLE 3.8.3. We consider the action $\Psi_{nuc} \colon \mathcal{I}(B) \to \mathcal{I}(A)$ defined by $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ and examine some possible generating subsets $\mathcal{S}_0 \subseteq \mathrm{CP}(A, B)$ for $\mathrm{CP}_{\mathrm{nuc}}(A, B)$.

Let $S \subseteq A^*_+$ denote an "almost separating" set of positive functionals on A that is invariant under inner automorphisms of A, more precisely:

for each $a \in A_+$ there is $\varphi \in S$ and $h^* = h \in A$ with $\|h\| < \pi$ such that with $\varphi(\exp(ih)a\exp(-ih)) > 0$.

For example we can take as S the set of pure states of A, or any set S of pure-states φ of A such that the family $\{d_\varphi\}_{\varphi\in S}$ of irreducible representations of A is separating for A.
Furthermore, let $P \subseteq B_+$ a set of positive elements of B that generates B as a closed ideal of B, i.e., for each $c \in B_+$ and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and are $b_1, \ldots, b_n \in P$ and $d_1, \ldots, d_n \in B$ with $\|c - \sum_k d_k^* b_k d_k\| < \varepsilon$.

We consider the set of $\mathcal{S}_0 := \mathcal{S}_{S,P} \subseteq \mathrm{CP}(A,B)$ of c.p. maps $V_{\varphi,b}(a) := \varphi(a)b$, where $\varphi \in S$ and $b \in P$, and calculate the action $\Psi_0 \colon \mathcal{I}(B) \to \mathcal{I}(A)$ defined by $\mathcal{S}_0$:

Lemma 3.12.2(iv) shows that $\Psi_{\mathcal{S}_0}(B) = A$. Let $J \neq B$ a closed ideal of B. Then there is $b \in P$ that is not in J. If $0 \neq a \in A_+$ then there is $h^* = h \in A$ with $\|h\| < \pi$ and $\varphi \in S$ with $\varphi(\exp(ih)a\exp(-ih)) > 0$. It follows that $V_{\varphi,b}(\exp(ih)a\exp(-ih))$ is not contained in J.

Thus $\Psi_{\mathcal{S}_0}(J) = \{0\}$ for every closed ideal $J \neq B$ and $\Psi_{\mathcal{S}_0}(B) = A$.

Since $\mathcal{S}_0 \subseteq \mathrm{CP}_f(A,B) \subseteq \mathrm{CP}_{\mathrm{nuc}}(A,B) \subseteq \mathrm{CP}(A,B)$ we get

$$\Psi_{\mathrm{CP}(A,B)} = \Psi_{\mathrm{CP}_{\mathrm{nuc}}(A,B)}(J) \subseteq \Psi_{\mathcal{C}_0}(J)$$

for all closed ideals J of B. It says that the possibly different m.o.c. cones induce the same action

$$\Psi_{\mathrm{CP}(A,B)} = \Psi_{\mathrm{CP}_{\mathrm{nuc}}(A,B)} = \Psi_{\mathcal{S}_0}\,.$$

The special case where $A = B$ is not nuclear shows that,
in general, the lower semi-continuous action $\Psi_{\mathcal{C}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ *defined by an m.o.c. cone* $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ *does not identify* $\mathcal{C}$ *itself.*

In particular, different non-degenerate m.o.c. cones can define the same corresponding l.s.c. action.

It implies that there is not a unique way to define KK-theory corresponding to a given lattice action, e.g. as defined for $\mathbb{O}(X)$ on $\mathrm{C}(X)$-algebras.

The situation of example 3.8.3 changes considerably if we consider, instead of $\Psi_{\mathcal{C}}$, the action $\Psi' \colon \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ that is defined by the point-norm closed m.o.c.c. $\mathcal{C} \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C,C) \subseteq \mathrm{CP}(A \otimes^{\max} C, B \otimes^{\max} C)$, where $C := C^*(F_\infty)$ denotes the full group C^*-algebra over the free group F_∞ on countably many generators and where $\mathcal{C} \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C,C)$ is generated by the tensor products $V \otimes^{\max} \mathrm{id}_C$ with $V \in \mathcal{C}$. Then the following separation theorem of “Hahn-Banach type” holds for all point-norm closed m.o.c. cones.

Compare Thm. 3.8.4

Theorem 3.8.4 (Separation of m.o.c. cones). *Let* $C := C^*(F_\infty)$ *and* $\mathcal{S} \subseteq \mathrm{CP}(A,B)$ *a set of c.p. maps, and denote by* $\Psi' \colon \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ *the action on the ideal that is defined by the set* $\mathcal{S}'$ *of c.p. maps*

$$\{\, V \otimes^{\max} \mathrm{id}_C \colon A \otimes^{\max} C \to B \otimes^{\max} C\,;\;\; V \in \mathcal{S}\,\}\,.$$

Then a c.p. map $T\colon A \to B$ is in the point-norm closed matrix operator-convex cone $\mathcal{C}(\mathcal{S})$ generated by $\mathcal{S}$, if and only if, $T \otimes^{\max} \mathrm{id}_C$ is Ψ'-equivariant.

It implies that we can "essentially" define point-norm-closed m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ with help of suitable lower semi-continuous actions Ψ of prime($B \otimes^{\max} C^*_{\max}(F_2)$) on $A \otimes^{\max} C^*_{\max}(F_2)$. But this does not say (and is wrong) that a point-norm-closed m.o.c. cone $\mathcal{C}$ is defined by the action $\Psi_{\mathcal{C}}\colon \mathcal{I}(B) \to \mathcal{I}(A)$ defined by $\mathcal{C}$. Only for the later considered "residual nuclear" case there is really some sort of bijective duality.

Proof. Let $\Psi' := \Psi_{\mathcal{S}'}\colon \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ be the l.s.c. action of $\mathrm{Prim}(B \otimes^{\max} C)$ on $A \otimes^{\max} C$ defined by the set $\mathcal{S}'$ of c.p. maps

$$V \otimes^{\max} \mathrm{id}_C\colon A \otimes^{\max} C \to B \otimes^{\max} C$$

with $V \in \mathcal{S}$, cf. Lemma 3.12.2(iv). We denote by

$$\mathcal{C}' \subseteq \mathrm{CP}(A \otimes^{\max} C\,,\, B \otimes^{\max} C)$$

the m.o.c. cone

$$\mathcal{C}' := \mathcal{C}_{\Psi'} = \mathrm{CP}(\Psi'; A \otimes^{\max} C, B \otimes^{\max} C)$$

of Ψ'-equivariant maps.

Then $\mathcal{C}'$ is a point-norm closed matrix operator convex cone by Lemma 3.12.2(vii).

The set $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ of all $T \in \mathrm{CP}(A, B)$ with $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ is a point-norm closed matrix operator convex cone that contains $\mathcal{S}$, cf. Lemma 3.12.2(xi), i.e., $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}$ and

$$(V \otimes^{\max} \mathrm{id}_C)(\Psi'(J)) \subseteq J$$

for all closed ideals J of $B \otimes^{\max} C$.

Let $T \in \mathcal{C}$. We want to show that $T \in \mathcal{C}(\mathcal{S})$. Since $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{L}(A, B)$ is point-norm closed (by definition of $\mathcal{C}(\mathcal{S})$), it suffices to show that for (contractions) $a'_1, \ldots, a'_n \in A$ and $\varepsilon > 0$ there is $V \in \mathcal{C}(\mathcal{S})$ with $\max_k \|V(a'_k) - T(a'_k)\| < \varepsilon$. Since $\mathcal{C}(\mathcal{S})$ is convex, the Hahn-Banach separation applied to the convex cone $K := \{(V(a'_1), \ldots, V(a'_n))\,;\ V \in \mathcal{C}(\mathcal{S})\,\}$ in $B \otimes \ell_\infty(1, \ldots, n)$ says that $(T(a'_1), \ldots, T(a'_n))$ is in the closure of K, if and only if,

$$\Re \sum_k f_k(T(a'_k)) \geq 0$$

for every linear functionals $f_1, \ldots, f_n \in B^*$ on B with $\Re \sum_k f_k(V(a'_k)) \geq 0$ for all $V \in \mathcal{C}(\mathcal{S})$.

Now let $f_1, \ldots, f_n \in B^*$ such that $0 \leq \Re(\sum_k f_k(V(a'_k)))$ for all $V \in \mathcal{C}(\mathcal{S})$.

By Lemma 3.1.4 there is a cyclic representation $d\colon B \to \mathcal{L}(\mathcal{H})$ with cyclic vector $\xi \in \mathcal{H}$ (we can suppose that $\|\xi\| = 1$) and $r_1, \ldots, r_n \in d(B)'$ with $f_k(b) = \langle d(b)\xi, r_k^*\xi\rangle$. We find unitary operators in $v_1, \ldots, v_m \in C^*(1, r_1, \ldots, r_m) \subseteq d(B)'$ such that $\{r_1, \ldots, r_n\}$ is contained in the linear span of $\{v_1, \ldots, v_m\}$. Let $G :=$

$C^*(v_1,\dots v_m) = C^*(1, r_1,\dots,r_m) \subseteq d(B)'$. It implies, that there are $a_1,\dots,a_m \in A$ such that $\sum_k a'_k \otimes r_k = \sum_j a_j \otimes v_j$ in the *algebraic* tensor product $A \odot G$. In particular,

$$\sum_j \langle d(L(a_j))\xi, v_j^*\xi\rangle = \sum_k \langle d(L(a'_k))\xi, r_k^*\xi\rangle = \sum_k f_k(L(a'_k))$$

for *every linear map* $L\colon A \to B$.

Consider the unital *-epimorphism $\lambda\colon C \to G$ with $\lambda(u_j) = v_j$ for $j = 1,\dots,m$ and $\lambda(u_j) = 1$ for $j > m$. (Here the unitaries $u_1, u_2, \dots \in F_\infty$ are canonical generators of F_∞.)

It defines a C^*-morphism $D\colon B \otimes^{\max} C \to \mathcal{L}(\mathcal{H})$ with $D(b\otimes c) = d(b)\lambda(c)$ for $b \in B$, $c \in C$. We let $J \subseteq B \otimes^{\max} C$ denote the kernel of D. The cyclic vector $\xi \in \mathcal{H}$ defines a positive state $\rho(y) := \langle D(y)\xi,\xi\rangle$ with $\rho(J) = \{0\}$ and

$$\rho((W \otimes^{\max} \mathrm{id})(x)) = \sum_j \langle d(W(a_j))\xi, v_j^*\xi\rangle = \sum_k f_k(W(a'_k))$$

for $x := \sum_j a_j \otimes u_j \in A \otimes C$ and every completely positive map $W\colon A \to B$. In particular, we get for the real part $\Re$ the inequalities

$$\Re(\rho((V \otimes^{\max} \mathrm{id})(x)) \geq 0 \quad \text{for all } V \in \mathcal{C}(\mathcal{S})\,.$$

Next we show that the property $\Re(\rho((V \otimes^{\max} \mathrm{id})(x)) \geq 0$ for all $V \in \mathcal{C}(\mathcal{S})$ implies that $(x^* + x) + \Psi'(J)$ is positive in $(A \otimes^{\max} C)/\Psi'(J)$. It is the crucial point of the proof.

Let P denote the set of positive functionals $g_V \in (A \otimes^{\max} C)^*$ with $g_V := \rho((V \otimes^{\max} \mathrm{id}_C)(\cdot))$ $(V \in \mathcal{C}(\mathcal{S}))$. The set P of the positive linear functionals g_V has the following properties:

(α) $g_V(y) = 0$ for all $g_V \in P$ (i.e., for $V \in \mathcal{C}(\mathcal{S})$), if and only if, $y \in \Psi'(J)$.
(β) $g_V(z^*(\cdot)z)$ is in the point-$\sigma(A^*, A)$ closure of P for every $z \in A \otimes^{\max} C$
(γ) $(y^*+y)+\Psi'(J)$ is positive in $(A\otimes^{\max} C)/\Psi'(J)$, if and only if, $\Re(g_V(y)) \geq 0$ for all $V \in \mathcal{C}(\mathcal{S})$.

(α): If $V \in \mathcal{C}(\mathcal{S})$, $y \in \Psi'(J)$ then $V \otimes^{\max} \mathrm{id}(y) \in J$ and $g_V(y) = \rho(V \otimes^{\max} \mathrm{id}(y)) = 0$, i.e., $g_V(\Psi'(J)) = \{0\}$ for all $g_V \in P$.

If $y \in A \otimes^{\max} C$, $V \in \mathrm{CP}(A, B)$ and $\rho\big((b^* \otimes 1)(V \otimes^{\max} \mathrm{id})(y)(b \otimes 1)\big) = 0$ for all $b \in B$, then $(V \otimes^{\max} \mathrm{id})(y) \in J$. Indeed, $\rho((b\otimes 1)^* z(b\otimes 1)) = \langle D(z)d(b)\xi, d(b)\xi\rangle$ for $z \in B\otimes^{\max} C$, $d(B)\xi$ is dense in $\mathcal{H}$ and J is the kernel of $D\colon B\otimes^{\max} C \to \mathcal{L}(\mathcal{H})$.

We have that $W := b^*V(\cdot)b \in \mathcal{C}(\mathcal{S})$ and $g_W(y) = \rho((b^* \otimes 1)(V \otimes^{\max} \mathrm{id})(y)(b \otimes 1)$ for $V \in \mathcal{C}(\mathcal{S})$ and $b \in B$. Thus, $g_V(y) = 0$ for all $V \in \mathcal{C}(\mathcal{S})$ implies that $(V \otimes^{\max} \mathrm{id})(y) \in J$ for all $V \in \mathcal{C}(\mathcal{S})$. Conversely, $g_V(y) = \rho((V \otimes^{\max} \mathrm{id})(y)) = 0$ if $(V\otimes^{\max}\mathrm{id})(y) \in J$. Thus, $g_V(y) = 0$ for all $g_V \in P$, if and only if, $(V\otimes^{\max}\mathrm{id})(y) \in J$ for all $V \in \mathcal{C}(\mathcal{S})$.

For $y \in A \otimes^{\max} C$, $a \in A$, $c \in C$ and $V \in \mathrm{CP}(A, B)$ holds $(V \otimes^{\max} \mathrm{id})((a \otimes c)^* y(a\otimes c)) = (1\otimes c)^* W \otimes^{\max} \mathrm{id}(y)(1\otimes c)$ where $W = V(a^*(\cdot)a)$. Since $V(a^*(\cdot)a) \in \mathcal{C}(\mathcal{S})$ for $V \in \mathcal{C}(\mathcal{S})$, it follows that the closed linear subspace K of $y \in A \otimes^{\max} C$

with $(V \otimes^{\max} \mathrm{id})(y) \in J$ for all $V \in \mathcal{C}(\mathcal{S})$ (i.e., with $g_V(y) = 0$ for all $g_V \in P$) satisfies $(a \otimes c)^* K (a \otimes c) \subseteq K$. If we use the polar formula for bilinear maps (several times) and the linearity of K, we get that that the closed subspace K is a two-sided ideal of $A \otimes^{\max} C$. Since $(V \otimes^{\max} \mathrm{id})(K) \subseteq J$ for all $V \in \mathcal{S} \subseteq \mathcal{C}(\mathcal{S})$ and since $\Psi'(J) \subseteq K$, we get $K = \Psi'(J)$ (cf. definition of $\Psi'(J)$).

(β): Since the algebraic tensor product $A \odot C$ is dense in $A \otimes^{\max} C$, it suffices to consider $z = \sum_{k=1}^{m} a_k \otimes c_k \in A \odot C$ ([19]). Then

$$g_V(z^* y z) = \sum_{k,j=1}^{m} \langle D(V \otimes^{\max} \mathrm{id}((a_j^* \otimes 1) y (a_k \otimes 1))) \lambda(c_k)\xi, \lambda(c_j)\xi \rangle \, .$$

Let $F_n := [b_1^{(n)}, \ldots, b_m^{(n)}]^\top \in M_{m1}(B)$ such that $\lim_n d(b_k^{(n)})\xi = \lambda(c_k)\xi$ in $\mathcal{H}$ for every $k = 1, \ldots, m$. If we let $E := [a_1, \ldots, a_m] \in M_{1m}(A)$ and take $V_n := F_n^* \big(V \otimes \mathrm{id}_\ell\big)(E^*(\cdot)E\big) F_n \in \mathcal{C}(\mathcal{S})$, then $g_V(z^* y z) = \lim_n g_{V_n}(y)$ for every $y \in A \otimes^{\max} C$.

(γ): Let $y = (\ell - k) + ih$ with $h^* = h$, $k \geq 0$, $\ell \geq 0$, and $k\ell = 0$, $W \in \mathcal{C}(\mathcal{S})$. Then $\Re g_W(kyk) = -g_V(k^3)$ and $y^* + y = \ell - k$.

By (β), we get that $g_W(k(\cdot)k)$ is a point-wise limit of g_V $(V \in \mathcal{C}(\mathcal{S}))$. Thus, $\Re g_V(y) \geq 0$ for all $V \in \mathcal{C}(\mathcal{S})$ implies $g_W(k^3) = 0$ for all $W \in \mathcal{C}(\mathcal{S})$, which means $k^3 \in \Psi'(J)$ by (α). It follows that $(y^* + y) + \Psi'(J) = \ell + \Psi'(J)$ is positive in $(A \otimes^{\max} C)/\Psi'(J)$.

Conversely, we have $\Re(g_V(y)) = g_V(y^* + y) \geq 0$ if $(y^* + y) + \Psi'(J)$ is positive in $(A \otimes^{\max} C)/\Psi'(J)$, because $g_V(\Psi'(J)) = \{0\}$ by (α).

Above we have shown that there exist $a_1, \ldots, a_m \in A$, a closed ideal J of $B \otimes^{\max} C$, and a state ρ on $B \otimes^{\max} C$, such that ρ and the element $x := \sum_k a_k \otimes u_k \in A \otimes^{\max} C$ *satisfy the following conditions (i)–(iii):*

(i) $\rho((W \otimes^{\max} \mathrm{id}_C)(x)) = \sum_k f_k(W(a_k))$ for all $W \in \mathrm{CP}(A, B)$,
(ii) $\rho(J) = \{0\}$, and
(iii) $(x^* + x) + \Psi'(J)$ is positive in $(A \otimes^{\max} C)/\Psi'(J)$.

Notice that the property in (iii) follows from property (γ), because

$$\Re(g_V(x)) = \Re(\rho((V \otimes^{\max} \mathrm{id})(x))) \geq 0 \quad \text{for all } V \in \mathcal{C}(\mathcal{S})\,.$$

The properties (i)–(iii) imply, that $T \in \mathrm{CP}(A, B)$ with $T \otimes^{\max} \mathrm{id}(\Psi'(J)) \subseteq J$ satisfies

$$2\Re \sum_k f_k(T(a_k')) = \rho((T \otimes^{\max} \mathrm{id}_C)(x^* + x)) \geq 0\,,$$

because then $(T \otimes^{\max} \mathrm{id})(x^* + x) + J$ is positive in $(B \otimes^{\max} C)/J$. (Notice here, that if E and F are C^*-algebras, $I \triangleleft E$ and $J \triangleleft F$ are closed ideals, and if $L \colon E \to F$ a is positive map with $L(I) \subseteq J$, then $L(y) + J$ is positive in F/J if $y + I$ is positive in E/I.) □

[19] Here $m \in \mathbb{N}$ and $a_k \in A$, $c_k \in C$ are arbitrary, i.e., are not the specific ones for the definition of the element x above.

EXAMPLE 3.8.5. Let A, B and C denote C^*-algebras. We describe the action

$$\Psi' := \Psi_{\mathcal{C}'} : \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$$

that is induced from $\mathcal{C}' := \mathrm{CP}_{\mathrm{nuc}}(A,B) \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C,C)$:

More generally, we consider $\Psi' := \Psi_{\mathcal{C}'}$ induced from $\mathcal{C}' = \mathcal{C}(\mathcal{S}')$, where $\mathcal{S}' := \{V \otimes^{\max} \mathrm{id}_C\,;\ V \in \mathcal{S}_0\}$ with $\mathcal{S}_0 := \mathcal{S}_{S,P} \subseteq \mathrm{CP}(A,B)$ the set $\mathcal{S}_{S,P}$ of c.p. maps $V_{\varphi,b}(a) := \varphi(a)b$, where $\varphi \in S$ and $b \in P$, $S \subseteq A^*_+$ and $P \subseteq B_+$ as in Example 3.8.3.

Let $\pi_{A,C} : A \otimes^{\max} C \to A \otimes C := A \otimes^{\min} C$ the natural epimorphism from the maximal C^*-tensor product to the minimal C^*-tensor product of A with C, let $I_{A,C}$ denote its kernel, and let $S_\varphi : A \otimes^{\max} C \to C = \mathbb{C} \otimes C$ the map $S_\varphi := (\varphi \otimes \mathrm{id}_C) \circ \pi_{A,C}$. We denote by $K(J)$ the biggest closed ideal of C with $B \otimes^{\max} K(J) \subseteq J$. (It can happen that $K(J) = \{0\}$.)

Let $Y \subseteq C$ a closed linear subspace. The space $F(A,Y;A \otimes C) \subseteq A \otimes C$ is defined as the set of $z \in A \otimes C$ such that $\varphi \otimes \mathrm{id}(z) \in Y$ for all $\varphi \in A^*$. Since $M \overline{\otimes} N_1 = F_\sigma(M,N_1; M \overline{\otimes} N)$ (by the tensorial bi-commutation theorem for vN-algebras) for the weakly continuous W*-algebra analog and all W*-subalgebras $N_1 \subseteq N$, one has $F(A,Y;\, A \otimes C) = (A \otimes C) \cap (A^{**} \overline{\otimes} Y^{**})$ in $A^{**} \overline{\otimes} C^{**}$ if Y is a C^*-subalgebra of C. Thus $F(A,Y;\, A \otimes C)$ is a C^*-subalgebra (respectively is a closed ideal) of $A \otimes C$ if Y is a C^*-subalgebra (respectively a closed ideal) of C. It is not difficult to see that $z \in F(A,Y;A \otimes C)$ if $\varphi \otimes \mathrm{id}((a^* \otimes 1)z(a \otimes 1)) \in Y$ for all $a \in A$ and $\varphi \in S \subseteq A^*_+$, where $\mathrm{span}(a^*Sa)$ is $\sigma(A^*, A)$ dense in A^* (which is the case if S is separates all non-zero positive elements of A from zero).

An element $x \in (A \otimes^{\max} C)_+$ is in $\Psi'(J)_+$, if and only if, $(V_{\varphi,b} \otimes \mathrm{id})(y) = b \otimes S_\varphi(y) \in J$ for all $y := \exp(ih)x\exp(-ih)$ $(h^* = h \in A \odot C)$, $\varphi \in S$ and $b \in P$. Since $\Psi'(J)_+$ is the positive part of a closed ideal of $A \otimes^{\max} C$, this happens also for $(a \otimes c)^* x (a \otimes c)$ (in place of x, and for all $a \in A$ and $c \in C$). If we apply the property that $P \subseteq B_+$ generates B as a two-sided ideal, then we obtain – for $x \in (A \otimes^{\max} C)_+$ – that $x \in \Psi'(J)_+$ if and only if $S_\varphi((a \otimes 1)^* x (a \otimes 1)) \in K(J)$ for all $\varphi \in S$, $a \in A$ and , i.e., $\pi_{A,C}(x) \in F(A,K(J);A \otimes C)$.

We get $\Psi'(J) = \pi^{-1}_{A,B}(F(A,K(J);A \otimes C))$ for every closed ideal J of $B \otimes^{\max} C$. We see that $\Psi'(J)$ does not depend from from the special choice of $P \subseteq B_+$ and $S \subseteq A^*_+$.

If I is a closed ideal of C, then $K(B \otimes^{\max} I) = I$, in particular, $\Psi'(J) = \Psi'(B \otimes^{\max} K(J))$ for all closed ideals J of $B \otimes^{\max} C$.

It follows that $T \otimes^{\max} \mathrm{id}_C$ is Ψ'-residually equivariant, i.e., $T \otimes^{\max} \mathrm{id} \in \mathcal{C}_{\Psi'} = \mathrm{CP}(\Psi', A \otimes^{\max} C, B \otimes^{\max} C)$, if and only if, for every closed ideal I of C,

$$T \otimes^{\max} \mathrm{id}(\pi^{-1}_{A,B}(F(A,I;A \otimes C))) \subseteq B \otimes^{\max} I\,.$$

Every linear map $T : A \to B$ of finite rank has this property, because $T = \sum_k \alpha_k \varphi_k(\cdot) b_k$ with $\varphi_k \in A^*_+$, $b_k \in B_+$, $\alpha_k \in \mathbb{C}$. If we let $C = C^*(F_\infty)$, then

Theorem 3.8.4 implies

$$\mathcal{C}(\mathcal{S}_{S,P}) = \overline{\mathrm{CP}_f(A,B)} =: \mathrm{CP}_{\mathrm{nuc}}(A,B)\,,$$

for all $S \subseteq A^*_+$ and $P \subseteq B_+$ as in Example 3.8.3.

One can also say something in the case of $\mathcal{C} := \mathrm{CP}(A,B) \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C,C)$: We obtain from the above considered nuclear case, and from the trivial inclusions $V \cdot \mathrm{id}(A \cdot K(J)) \subseteq B \cdot K(J) \subseteq J$ that – for all closed ideals J of $B \otimes^{\max} C$ –,

$$A \otimes^{\max} K(J) \subseteq \Psi_{\mathcal{C}}(J) \subseteq \pi^{-1}_{A,B}(F(A, K(J); A \otimes C))\,.$$

The general study of actions $J \in \mathcal{I}(B) \mapsto \Psi_{\mathcal{C}}(J) \in \mathcal{I}(A)$ defined by $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ seems to be difficult, because related slice map problems for non-exact C^*-algebras have no suitable answer.

Corollary 3.8.6. *Each point-norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is* hereditary *inside* $\mathrm{CP}(A,B)$*, i.e., if $V, W \in \mathrm{CP}(A,B)$ and $V + W \in \mathcal{C}$ then $V, W \in \mathcal{C}$.*

Proof. By Theorem 3.8.4 the action

$$\Psi' \colon \mathcal{I}(B \otimes^{\max} C^*(F_2)) \to \mathcal{I}(A \otimes^{\max} C^*(F_2))$$

determines the elements of $\mathcal{C}$ by being Ψ'-equivariant in the sense that $T \in \mathcal{C}$ if and only if $(T \otimes \mathrm{id})(\Psi'(J)) \subseteq J$ for all $J \in \mathcal{I}(B \otimes^{\max} C^*(F_2))$. If we let here $T := V + W$ then this implies that the c.p. maps V and W satisfy this condition, because $((V + W) \otimes \mathrm{id}) = (V \otimes \mathrm{id}) + (W \otimes \mathrm{id})$ and all are completely positive maps on $A \otimes^{\max} C^*(F_2)$. Use here that $x + y \in J_+$ for $x \geq 0$ and $y \geq 0$ implies that $x, y \in J_+$. □

9. Temporary demands from other places

Prove and Move next blue to reasonable place !!!

Corollary 3.9.1. *Let A and B stable and separable C^*-algebras and let $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ a lower semi-continuous action of* $\mathrm{Prim}(B)$ *on A, such that $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$.*

*And let $\mathcal{C}(\Psi) \subseteq \mathrm{CP}(A,B)$ the point-norm closed m.o.c. cone of "all Ψ-*equivariant *c.p. maps $V \in \mathrm{CP}(A,B)$", i.e., of all that satisfy $V(\Psi(J)) \subseteq J$ for all $J \in \mathcal{I}(B)$ and $V \in \mathcal{C}$, and suppose that $\mathcal{C}(\Psi)$ is* separating *for the action $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$, – i.e., that for each $J \in \mathcal{I}(B)$,*

$$\Psi(J) \;=\; \{\, a \in A\,;\; V(a) \in J, \;\; \textit{for all } V \in \mathcal{C} \,\}\;.$$

If $\mathcal{C} \subseteq \mathcal{C}(\Psi)$ is a point-norm closed m.o.c. cone that defines Ψ.

Then $\mathcal{C}$ contains all Ψ-residually nuclear c.p. maps $V \colon A \to B$, i.e., all $V \in \mathrm{C_b}(A,B)$ with the property that $V(\Psi(J)) \subseteq J$ for all $J \in \mathcal{I}(B)$, and that all the quotient maps $[V] \colon A/\Psi(J) \to B/J$ are again nuclear.

I.e., $\mathcal{C}$ contains the m.o.c. cone $\mathcal{C}_{\mathrm{nuc}}(\Psi)$ of all Ψ-residually nuclear c.p. maps $V \colon A \to B$ with $V(\Psi(J)) \subseteq J$ for all $J \in \mathcal{I}(B)$ and $[V] \colon A/\Psi(J) \to B/J$ is nuclear.

PROOF. ??

Idea of proof:

Consider the closed ideal J of $A \otimes^{\max} C^*(F_\infty)$ that is defined by the intersection of the kernels of the c.p. maps $V \otimes^{\max} \mathrm{id}_{C^*(F_\infty)}$ into $B \otimes^{\max} C^*(F_\infty)$.

?????????? □

Citation: Corollary 3.9.2

For the following Corollary 3.9.2 we refere to Remark 5.1.1(8) for the existence – and uniqueness up to unitary equivalence – of the "*infinite repeat*" endomorphism $\delta_\infty \colon \mathcal{M}(B) \to \mathcal{M}(B)$ for stable C^*-algebras B.

COROLLARY 3.9.2. *Let A and B stable σ-unital C^*-algebras and let $H_k \colon A \to \mathcal{M}(B)$, $k \in \{1, 2\}$, two non-degenerate C^*-morphisms, i.e., $H_k(A)B$ is dense in B.*

If A is separable and if $\mathcal{C}(H_1) = \mathcal{C}(H_2)$ for the point-norm closed m.o.c. cones $\mathcal{C}(H_k) \subseteq \mathrm{CP}(A, B)$, generated by the coefficient maps $A \ni a \mapsto b^* H_k(a) b \in B$, then there exists a norm-continuous path $t \in [0, \infty) \to U(t)$ of unitaries in $\mathcal{M}(B)$ such that, for all $a \in A$, $\delta_\infty \circ H_1(a)U(t) - U(t)\delta_\infty \circ H_2(a) \in B$ for all $t \in [0, \infty)$ and $\lim_{t\to\infty} \|\delta_\infty \circ H_1(a)U(t) - U(t)\delta_\infty \circ H_2(a)\| = 0$, i.e., $\delta_\infty \circ H_1$ and $\delta_\infty \circ H_2$ are unitary homotopic in the sense of Definition 5.0.1.

PROOF. ?? □

HERE ENDS eksec3-Part.1.tex

10. Non-simple algebras and the WvN-property

HERE STARTS eksec3-Part2.tex !!

We generalize some parts of the above results on simple C^*-algebras to non-simple C^*-algebras. In fact, we replace some of the above considered general properties of simple p.i. algebras by more useful suitable *definitions* in the non-simple situation. See, for example, the WvN-property in Definition 1.2.3.

DEFINITION 3.10.1. Let C be a C^*-subalgebra of the multiplier algebra $\mathcal{M}(B)$ of a C^*-algebra B and $V \colon C \to B$ a completely positive map. We call the map V **inner** if there are $m \in \mathbb{N}$ and $d_i \in B$, $i = 1, \ldots, m$, such that

$$V(a) = \sum_{i=1}^{m} d_i^* a d_i \,.$$

The c.p. map V is **1-step inner** if we can take here $m = 1$, i.e., $V := d^*(\cdot)d$.

A completely positive map $V \colon C \subseteq M(B) \to B$ is **approximately inner** if V is the point-norm limit of inner completely positive maps. The map V is **1-step approximately inner** if V can be approximated by 1-step inner c.p. maps.

Let J denote a closed ideal of a C^*-algebra B, then define

$$\mathcal{M}(B, J) := \{T \in \mathcal{M}(B) \colon TB + BT \subseteq J\} \,.$$

Let $C \subseteq \mathcal{M}(B)$ a C^*-subalgebra of the multiplier algebra of a C^*-algebra B. We call a completely positive map $V: C \to B$ **residually nuclear** if $V(C \cap \mathcal{M}(B,J)) \subseteq J$ and $[V]: C/(C \cap \mathcal{M}(B,J)) \to B/J$ is nuclear for every closed ideal J of B.

Let $h: C \to M$ a C^*-morphism into a von-Neumann algebra. The morphism h **weak-approximately majorizes** a c.p. contraction $W: C \to M$ if there is a net (T_γ) of inner completely positive maps from M into M such that W is the point-weak limit of the net $(T_\gamma \circ h)$.

A C^*-morphism $h: D \to \mathcal{M}(B)$ **approximately majorizes** a completely positive map $V: D \to B$ if there exists a net (d_γ) of elements $d_\gamma \in B$, such that V is the point-norm limit of the net of maps $d_\gamma^* h(\cdot) d_\gamma$.

The morphism h **majorizes** a completely positive *contraction* $V: D \to B$ if there is a *contraction* $d \in B$ with $V = d^* h(\cdot) d$.

The homomorphism $h: D \to \mathcal{M}(B)$ **approximately-inner majorizes** a completely positive contraction $V: D \to B$ if h weak-approximately majorizes V in the von-Neumann algebra $M := B^{**} \supset \mathcal{M}(B)$.

If one uses an approximate unit of C, then one can see that one can choose the d_j and T_γ in Definition 3.10.1 such that $\| \sum d_j^* d_j \| \leq \|V\|$ respectively $\|T_\gamma\| \leq \|W\|$, cf. Lemma 3.1.8. Recall Definition 3.1.1

LEMMA 3.10.2. *Suppose that $A \subseteq M$ a C^*-subalgebra of a von-Neumann factor M, and that $W: A \to M$ a weakly nuclear contraction, i.e., $\|W\| \leq 1$ and W can be approximated in the point-$\sigma(M, M_*)$ topology by a net of maps*

$$(W_\gamma : A \to M)_{\gamma \in \Gamma},$$

that are factorable maps in the sense of Definition 3.1.1. Then there is a net $(V_\lambda : M \to M)_{\lambda \in L}$ of inner completely positive contractions on M, such that W is the point-$\sigma(M, M_*)$-limit of the net $(V_\lambda | A : A \to M)$.

PROOF. There is even a net $(W_\gamma : A \to M)$ of factorable maps that converges in point-strong topology to W, because the set of factorable maps from A into M is convex by Remark 3.1.2(o). The Lemma 3.1.8 shows that we can suppose that the net $(W_\gamma : A \to M)$ consists of contractions. Therefore, again by Lemma 3.1.8, it suffices to show that every factorable contraction $W = T \circ S : A \to M$ can be approximated in point-$\sigma(M, M_*)$ topology by inner c.p. maps on M.

If M is of type I_n, II_1 or if M is *countably decomposable* and of type III, then M is simple, and Corollary 3.2.20 gives that W can be approximated by inner completely positive contractions (even in point norm topology).

If M is a semi-finite II_∞-factor, then the Breuer ideal D of M is simple. (Recall that the Breuer ideal of M is generated as C^*-algebra by the finite projections in M.) If M is of type III, then the ideal D generated by the countably decomposable projections is simple.

In both cases M is the multiplier algebra $\mathcal{M}(D)$ of a simple C^*-algebra D which contains an approximating unit consisting of projections.

For every projection $q \in D$, the factorable c.p. contraction $W_q\colon a \in A \subseteq \mathcal{M}(D) \mapsto qW(a)q \in D$ is the point-norm limit of a net of inner completely positive contractions by Corollary 3.2.20, i.e., W_q is approximately inner on D in the sense of Definition 3.10.1.

Since W_q converges point-strongly to W for $q \nearrow 1$ strongly, also W is the point-strong limit of a net of inner completely positive contractions. □

The following Lemma 3.10.3 shows that we can replace in the above definition of the residually nuclear completely positive maps the set $\mathcal{I}(B)$ of all closed ideals J of B by the *factorial* ideals I of B.

Recall that $I \triangleleft B$ is **factorial** if it is the kernel of a factorial representation of B. All primitive ideals are factorial ideals, and factorial ideals are prime (with respect to intersection in the lattice of closed ideals), as one can easily see, cf. e.g. proof of [**616**, prop. 3.13.10]. Since $\mathrm{Prim}(B)$ has the Baire property for C^*-algebras B, all prime ideals of *separable* C^*-algebras B are primitive if B is separable, cf. [**616**, prop. 4.3.6]. (This does not hold for *non-separable* C^*-algebras B, cf. [**815**].)

LEMMA 3.10.3. *Let C a C^*-subalgebra of $\mathcal{M}(B)$.*

(i) *For every family $\{J_\gamma\}$ of closed ideals of B holds* $\mathcal{M}(B, \bigcap J_\gamma) = \bigcap \mathcal{M}(B, J_\gamma)$ *and*
(ii) $\mathrm{dist}(a, \mathcal{M}(B, \bigcap J_\gamma)) = \sup_\gamma \mathrm{dist}(a, \mathcal{M}(B, J_\gamma))$ *for $a \in \mathcal{M}(B)$.*
(iii) *If B is σ-unital then $\mathcal{M}(B, J_1) + \mathcal{M}(B, J_2) = \mathcal{M}(B, J_1 + J_2)$ for every pair of closed ideals J_1 and J_2 of B.*
(iv) *Suppose that $J_0 \subseteq B$ is a closed ideal and that $V\colon C \to B$ is a completely positive map, such that $V(C \cap \mathcal{M}(B, J)) \subseteq J$ and*

$$[V]_J := [\pi_J \circ V]\colon\ C/(C \cap \mathcal{M}(B, J)) \ \to\ B/J$$

is nuclear for every factorial *ideal J of B with $J_0 \subseteq J$, then*

$$[V]_{J_0}\colon C/(C \cap \mathcal{M}(B, J_0)) \to B/J_0$$

is nuclear.
(v) *If $V\colon C \to B$ is residually nuclear, then for every non-degenerate* factorial **-representation $\rho\colon B \to N$ of B holds:* $\mathcal{M}(B, \ker(\rho)) = \ker \mathcal{M}(\rho)$, $V(C \cap \ker \mathcal{M}(\rho)) \subseteq \ker(\rho)$ *and the restricted map $\mathcal{M}(\rho)|C\colon C \to N$ majorizes $\rho \circ V$ approximately with respect to the weak topology.*

PROOF. The easy proofs of (i)-(iii) are left to the reader: Use an approximate unit of B for the straight calculations.

(iv): We have $V(C \cap \mathcal{M}(B, J_0)) \subseteq V(C \cap \mathcal{M}(B, J)) \subseteq J$ for all primitive ideals J of B with $J \supseteq J_0$. The closed ideal J_0 is the intersection of the primitive ideals J that contain J_0 (cf. [**616**, cor. 3.13.8]). Hence $V(C \cap \mathcal{M}(B, J_0)) \subseteq J_0$.

By Remark 3.1.2(i), a map $T\colon D \to E$ is nuclear, if and only if,

$$T \otimes^{\max} \mathrm{id}\colon D \otimes^{\max} C^*(F) \to E \otimes^{\max} C^*(F)$$

annihilates the kernel I of the natural epimorphism $D \otimes^{\max} C^*(F) \to D \otimes C^*(F)$.

It is easy to see that an irreducible representations ρ of $E \otimes^{\max} C^*(F)$ factorizes over $(E/K) \otimes^{\max} C^*(F)$ for some factorial ideal K of E.

In particular, *the family of quotient maps* $\pi_K \otimes^{\max} C^*(F)$ *from* $E \otimes^{\max} C^*(F)$ *to* $(E/K) \otimes^{\max} C^*(F)$ *is separating for* $E \otimes^{\max} C^*(F)$ ([20]). Hence, T is nuclear, if and only if, $(\pi_K T) \otimes^{\max} \mathrm{id}(I) = 0$ for all factorial ideals K of E, if and only if, $\pi_K \circ T\colon D \to E/K$ is nuclear for all factorial ideals J of E.

Now apply this to $D := C/(C \cap \mathcal{M}(B, J_0))$, $E := B/J_0$, $T := [V]_{J_0}$, and use that $\pi_{J/J_0} \circ [V]_{J_0} = [V]_J \circ \eta_J$ for $\eta_J\colon D \to C/(C \cap \mathcal{M}(B, J_0))$, and that $K := J/J_0$ runs through all factorial ideals of E if J runs through all factorial ideals $J \supset J_0$ of B.

(v): Let $E := \rho(B) \subseteq N \subseteq \mathcal{L}(\mathcal{H})$. Since $\rho\colon B \to \mathcal{L}(\mathcal{H})$ is non-degenerate, the normalizer algebra $\mathcal{N}(E) := \{t \in \mathcal{L}(\mathcal{H})\,;\ tE + Et \subseteq E\}$ of E in $\mathcal{L}(\mathcal{H})$ is unital and can be identified with the multiplier algebra $\mathcal{M}(E)$ of E in a natural way. It is easy to see that $\mathcal{N}(E) \subseteq E'' \subseteq N$. The representation ρ defines a *-epimorphism from B onto E. Thus, ρ uniquely extends to a unital strictly continuous map $\mathcal{M}(\rho)\colon \mathcal{M}(B) \to \mathcal{M}(E) = \mathcal{N}(E) \subseteq N$ such that $\mathcal{M}(\rho)(c)\rho(b) = \rho(cb)$ for $c \in \mathcal{M}(B)$ and $b \in B$. It follows that the kernel $\ker(\mathcal{M}(\rho))$ of $\mathcal{M}(\rho)$ is $\mathcal{M}(B, J) := \{c \in \mathcal{M}(B)\,;\ cB + Bc \subseteq J\}$ for $J := \ker(\rho)$. Let $\Psi(J) := C \cap \mathcal{M}(B, J) = C \cap \ker(\mathcal{M}(\rho))$.

If we let $A := \mathcal{M}(\rho)(C) \subseteq N$, then $\mathcal{M}(\rho)$ defines a natural isomorphism γ from $C/\Psi(J)$ onto A with $h := \mathcal{M}(\rho)|C = \gamma \circ \pi_{\Psi(J)}$, i.e., $\gamma(c + \Psi(J)) = \mathcal{M}(\rho)(c)$ for $c \in C$.

We define $W\colon A \to N$ by $W := [\rho]_J \circ [V]_J \circ \gamma^{-1}\colon A \to E \subseteq N$. Here $[\rho]_J$ denotes the natural isomorphism from $B/\ker(\rho) = B/J$ onto $E = \rho(B) \subseteq N$ with $[\rho]_J(b + J) = \rho(b)$ for $b \in B$, i.e., $[\rho]_J \circ \pi_J = \rho$. The map $[V]_J\colon C/\Psi(J) \to B/J$ with $[V]_J \circ \pi_{\Psi(J)} = \pi_J \circ V$ is nuclear, because V is residually nuclear. Therefore $W\colon A \to N$ is nuclear. By Lemma 3.10.2, there is a net of inner c.p. maps $(V_\lambda\colon N \to N)$ with $\|V_\lambda\| \leq 1$ such that $(V_\lambda|A)$ converges in point-$\sigma(N, N_*)$ topology to W.

Thus the net $V_\lambda \circ h$ tends in point-$\sigma(N, N_*)$ topology to $W \circ h = [\rho]_J \circ [V]_J \circ \pi_{\Psi(J)} = \rho \circ V$. It means that h weak-approximately majorizes $\rho \circ V$ (in the sense of Definition 3.10.1). □

Proposition 3.10.4. *Suppose that* C *is a* C^**-subalgebra of* $\mathcal{M}(B)$ *and that* $V\colon C \to B$ *is a completely positive contraction. Then the following are equivalent:*

(i) V *is approximately inner (in the sense of Def. 3.10.1).*

---

[20] Moreover, by an reduction to certain separable C^*-subalgebras of E with an argument in Chapter 12, one can show that the family of maps $\pi_K \otimes^{\max} \mathrm{id}$ is separating for $E \otimes^{\max} C^*(F)$ if K runs through all primitive ideals of E. Therefore one can replace the factorial or prime ideals by *primitive* ideals in Lemma 3.10.3(iv) and in Proposition 3.10.4(ii).

(ii) *For every prime ideal J of B, $V(\Psi(J)) \subseteq J$ and $[V]\colon C/\Psi(J) \to B/J$ is approximately inner (cf. Def. 3.10.1), where $\Psi(J) := C \cap \mathcal{M}(B,J)$ and $C/\Psi(J) \subseteq \mathcal{M}(B)/\mathcal{M}(B,J) \subseteq \mathcal{M}(B/J)$.* ([21])

(iii) *For every non-degenerate factorial representation $\rho\colon B \to N$, the natural extension $h := \mathcal{M}(\rho)|C\colon C \to N$ of ρ to C weak-approximately majorizes $W := \rho \circ V$, where $N := \rho(B)''$ in the sense of Definition 3.10.1.*

Proof. From Definition 3.10.1 we can see that (i) implies (ii) and that (ii) implies (iii). The latter, because the kernel of each factorial representation of B is a prime ideal J of B.

(iii)⇒(i): Suppose that (iii) holds, but that V is not approximately inner.

Let $\eta\colon C \hookrightarrow \mathcal{M}(B)$ denote the inclusion map. The Hahn-Banach separation argument in the proofs of [**426**, lem.2,lem.3] shows that there exist $c_1, \ldots, c_n \in C$ and a cyclic representation $d\colon B \to \mathcal{L}(H)$ with cyclic vector $x \in H$ and $f_1, \ldots, f_n \in d(B)'$, such that:

(*) The positive linear functional $\psi \circ d_1 \circ (V \otimes^{\max} \mathrm{id}_F)$ is not weakly contained in the representation $k := d_1 \circ (\eta \otimes^{\max} \mathrm{id}_F)$ of $C \otimes^{\max} F$ on H.

Here $F := C^*(1, f_1, \ldots, f_n) \subseteq \mathcal{L}(H)$, d_1 denotes the *natural extension of* d to a unital *-representation of $\mathcal{M}(B) \otimes^{\max} F$ into $\mathcal{L}(H)$ such that $d_1(c \otimes f)d(b) = d(cb)f$ for $c \in \mathcal{M}(B)$, $b \in B$, $f \in F$, and where ψ is the vector state on $\mathcal{L}(H)$ given by x.

Since there is $p \in (C \otimes^{\max} F)_+$ with $0 = k(p) = d_1((\eta \otimes^{\max} \mathrm{id}_F)(p))$ and $0 \neq d((V \otimes^{\max} \mathrm{id}_F)(p)) \in d(B \otimes^{\max} F)$, there is a pure state φ on $d(B \otimes^{\max} F)$ with $\varphi(d((V \otimes^{\max} \mathrm{id}_F)(p))) > 0$, and we obtain:

(**) *There is a pure state φ on $d(B \otimes^{\max} F)$ which is weakly contained in $d\colon B \otimes^{\max} F \to H$ such that the positive functional $\varphi \circ d \circ (V \otimes^{\max} \mathrm{id}_F)$ on $C \otimes^{\max} F$ is not weakly contained in k.*

There is a unique extension λ of the pure state φ to a pure state of $d_1(\mathcal{M}(B) \otimes^{\max} F)$, because $d(B \otimes^{\max} F)$ is an ideal of $d_1(\mathcal{M}(B) \otimes^{\max} F)$. Thus, $\lambda \circ k$ is weakly contained in k.

But, φ is the cyclic vector of an irreducible representation d_2 of $B \otimes^{\max} F$ over a Hilbert space H_1 with corresponding cyclic vector y such that $\varphi d_1 = \langle d_2(.)y, y \rangle$.

The irreducible representation d_2 decomposes into a commuting pair of factorial representations ρ and d_3 of B and F respectively.

If d_4 denotes the natural extension of d_2 to $\mathcal{M}(B) \otimes^{\max} F$ then $\lambda(.) = \langle d_4(.)y, y \rangle$, and $c \mapsto d_4(c \otimes 1)$ is the natural extension $\mathcal{M}(\rho)$ of ρ to $\mathcal{M}(B)$.

The representation d_4 is weakly contained in d_1, because d_2 is weakly contained in d and is non-degenerate. Thus $d_4(\eta \otimes^{\max} \mathrm{id}_F)$ is weakly contained in k.

[21] The inclusions are defined by the natural monomorphisms.

By the assumptions of (iii), $h := \mathcal{M}(\rho)|C$ weak-approximately majorizes ρV in $N := \rho(B)''$. Thus, $\varphi \circ d \circ V \otimes^{\max} \mathrm{id}_F)$ is weakly contained in $d_4(\eta \otimes^{\max} \mathrm{id}_F)$, and, therefore, it is weakly contained in k.

We have derived a contradiction to (**), which was implied from the assumption that V is not approximately inner. □

Suppose that $h_1, h_2 \colon A \to B$ are C^*-morphisms and that $s, t \in \mathcal{M}(B)$ are two isometries with orthogonal ranges. Then we define for this chapter the Cuntz addition by

$$(h_1 \oplus h_2)(a) := (h_1 \oplus_{s,t} h_2)(a) := sh_1(a)s^* + th_2(a)t^*, \qquad \text{for} \quad a \in A$$

defines a C^*-morphism from A into B ([22]). The C^*-algebra generated by s and t contains a unital copy of $\mathcal{O}_\infty$. Therefore one can define in a similar way the k-fold sum $h \oplus h \oplus \cdots \oplus h$ of k copies of h, $k = 2, 3, \ldots$. (Compare Chapter 4 for more details in the particular situation where B contains a copy of $\mathcal{O}_2$ unitally.)

LEMMA 3.10.5. *Suppose that A is a separable C^*-algebra, E is a C^*-algebra such that $\mathcal{M}(E)$ contains isometries s, t with ortogonal ranges (i.e., $s^*t = 0$), $h \colon A \to E$ is a C^*-morphism, and that $(V_k \colon E \to E)_k$ is a sequence of approximately inner c.p. contraction.*

(i) *If h majorizes (respectively approximately majorizes) the C^*-morphism*
$$h \oplus h := sh(\cdot)s^* + th(\cdot)t^*,$$
then there exists a C^*-morphism g from $A \otimes \mathcal{O}_\infty$ into E, – respectively into $E_\infty := \ell_\infty(E)/c_0(E)$ –, with $g(a \otimes 1) = h(a)$ for $a \in A$.

(ii) *If there is a C^*-morphism g from $A \otimes \mathcal{O}_\infty$ into E (respectively into E_∞) with $g(a \otimes 1) = h(a)$ for $a \in A$, then there exists a sequence d_n of contractions in E such with the properties $d_n^* b d_{n+m} = 0$ (respectively $\lim_{n\to\infty} d_n^* b d_{n+m} = 0$) for $b \in h(A) + \mathbb{C}1$, $m > 0$, and $\lim_{n\to\infty} \| d_n^* h(a) d_n - V_n(h(a)) \| = 0$ for $a \in A$.*

Note that in general (i) does *not* imply the existence of isometries $s_1, s_2, \ldots \in h(A)' \cap E$ with orthogonal ranges. But this could happen modulo the two-sided annihilator of $h(A)$ in E.

PROOF. (i): If there is a contraction $d \in E$ with $d^* h(\cdot) d = h \oplus h$, then the proof of Proposition 4.3.5(v) works and gives that there is a C^*-morphism $g \colon A \otimes \mathcal{O}_\infty \to E$ with $g(a \otimes 1) = h(a)$ for $a \in A$.

Suppose that h approximately majorizes $h \oplus h$. The C^*-morphism $h \oplus h$ is a contraction. By Lemma 3.1.8 $h \oplus h$ is in the point-norm closure of the set of maps $d^* h(\cdot) d$ with contractions $d \in E$. Since A is separable, we can select a sequence $(d_1, d_2, \ldots)$ of contractions in E such that $h(a) \oplus h(a) = \lim_n d_n^* h(a) d_n$ for $a \in A$. Thus $h \oplus h = d_\infty^* h(\cdot) d_\infty$ in E_∞ for the contraction $d_\infty := (d_1, d_2, \ldots) + c_0(E) \in E_\infty$.

[22] The here given definition of $h_1 \oplus h_2$ is a bit more general than the definition of the Cuntz Addition in Chapters 1 and 4, where we require that moreover $ss^* + tt^* = 1$.

(ii): We fix a strictly positive contraction $c \in A_+$. Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq A$ an approximate filtration of A by finite-dimensional linear subspaces X_n, and let $s_1, s_2, \ldots$ the canonical generators of $\mathcal{O}_\infty$. By Lemma 3.1.8, for each $n \in \mathbb{N}$ we find $k(n) \in \mathbb{N}$ and $f_1^{(n)}, \ldots, f_{k(n)}^{(n)} \in E$ with

$$\| \sum_{j=1}^{k(n)} (f_j^{(n)})^* f_j^{(n)} \| \leq 1$$

and $\|V_n(h(a)) - W_n(h(a))\| \leq 2^{-n}\|a\|$ for $a \in X_n$, where W_n is the inner c.p. contraction on E given by

$$W_n(b) := \sum_j (f_j^{(n)})^* b (f_j^{(n)}) \qquad \text{for} \quad b \in E\,.$$

We find $p(n) \in \mathbb{N}$ such that $\|c^{1/p(n)} a c^{1/p(n)} - a\| \leq 2^{-n}\|a\|$ for $a \in X_n$. Now let $c_n := c^{1/p(n)}$, $r(n) := k(1) + k(2) + \cdots + k(n-1)$ for $n > 1$ and $r(1) := 0$. We define $e_n := \sum_j g(c_n \otimes s_{r(n)+j}) f_j^{(n)}$ for $n = 1, 2, \ldots$. Then $e_n \in E_\infty$, $e_n^* b e_m = 0$ for $m \neq n$ and $e_n^* b e_n = W_n(c_n b c_n)$ for $b \in h(A) + \mathbb{C} \cdot 1$. In particular, $\|e_n\| \leq 1$ and, for $a \in X_n$,

$$\|e_n^* h(a) e_n - V_n(h(a))\| \leq 3 \cdot 2^{-n} \|a\|\,.$$

If $g(A \otimes \mathcal{O}_\infty) \subseteq E$, then the elements $d_n := e_n$ are contractions in E and have the desired properties.

In the case where $g(A \otimes \mathcal{O}_\infty) \subseteq E_\infty$, we can select from the representing sequences of contractions for e_n suitable elements d_n that have the in part (ii) desired properties. □

Corollary 3.10.6.

(I) *If $C \subseteq \mathcal{M}(B)$ and $T\colon C \to B$ is residually nuclear, then T is approximately inner in the sense of Definition 3.10.1.*

(II) *Suppose $h\colon A \to \mathcal{M}(B)$ is a C^*-morphism and $V\colon A \to B$ is a completely positive contraction that satisfy the following conditions:*
  - (α) *$h(A)$ commutes elementwise with a unital copy of $\mathcal{O}_\infty$ in $\mathcal{M}(B)$,*
  - (β) *$V(\Psi(J)) \subseteq J$, and*
  - (γ) *$[V]\colon A/\Psi(J) \to B/J$ is nuclear for every factorial ideal $J \subseteq B$, where $\Psi(J) := h^{-1}(h(A) \cap \mathcal{M}(B, J))$.*

  Then h approximately majorizes V in the sense of Definition 3.10.1.

Proof. (I) follows from Proposition 3.10.4(iii) and Lemma 3.10.3(v).

(II): Let $C := h(A)$. For $J = 0$, J_A is the kernel of h. Thus there is a completely positive map $T\colon C \to B$ such that $Th = V$. Then (β) and (γ) say that T is residually nuclear. By (I), T is approximately inner. But since C commutes with a copy of $\mathcal{O}_\infty$, we get that T is moreover one-step approximately inner. This means that h approximately majorizes V. □

By Remark 3.1.2(iv), Corollary 3.10.6 implies:

Corollary 3.10.7. *Suppose that $C \subseteq \mathcal{M}(B)$, C is exact, $V: C \to B$ is nuclear, and $V(C \cap \mathcal{M}(B, J)) \subseteq J$ for every closed ideal J of B, then V is approximately inner in the sense of Definition 3.10.1.*

Corollary 3.10.8. *Suppose that $\mathcal{M}(B)$ contains two isometries with orthogonal ranges, A is separable, $h: A \to B$ is a C^*-morphism that approximately majorizes $h \oplus h$ and that $T: A \to B$ is a completely positive contraction.*

Assume that T is nuclear and A is exact, or that $[T]: A/J \to B/J_T$ is nuclear for every closed ideal J of A, where J_T denotes the closed ideal of B which is generated by $T(J)$.

Then the following are equivalent:

(i) *h approximately majorizes T.*
(ii) *$T(J)$ is contained in the closure of $Bh(J)B$ for every closed ideal J of A.*
(iii) *$T(h^{-1}(h(A) \cap K)) \subseteq K$, i.e., $a \mapsto V(a) + K$ annihilates the kernel of $a \mapsto h(a) + K$, for every primitive ideal K of B.*

Proof. The implication (i)⇒(ii) is trivial. (ii)⇒(iii), because, with $J := h^{-1}(h(A) \cap K)$, $h(J) \subseteq K$ and, therefore, $T(J) \subseteq K$.

(iii) ⇒ (i): By induction, h approximately majorizes the k-fold sum of h for $k = 1, 2, \ldots$. Therefore, and since A is separable, it suffices to show that T is in the point norm closure of the convex set of maps $S \circ h$, where the S runs through the inner complete contractions on B. For the kernel J of h we have $T(J) = 0$, because $J \subseteq h^{-1}(h(A) \cap K$ and, therefore, $T(J) \subseteq K$ for every primitive ideal K of B.

Let $C := h(A)$ and let V be the unique completely positive contraction with $Vh = T$. Then, for every primitive ideal K of B, $J := C \cap K$ is a closed ideal of C and $V(J)$ is contained in K. If K_0 is any closed ideal, then $V(C \cap K_0) \subseteq V(C \cap K) \subseteq K$ for every primitive ideal K of B with $K_0 \subseteq K$. Thus $V(C \cap K_0) \subseteq K_0$.

If A is exact and T is nuclear, then C is exact and V is nuclear by Remark 3.1.2(iv). Thus V is approximately inner by Corollary 3.10.7.

If A is not exact, but $[T]: A/J \to B/J_T$ is nuclear for every closed ideal J of A, then with $J := h^{-1}(C \cap K)$ we get $V(C \cap K) \subseteq T(J) \subseteq K$ and, therefore, $T(J) \subseteq J_T \subseteq K$, $[T]: A/J \to B/K$ is nuclear. Under the natural isomorphism $A/J \cong C/(C \cap K)$, $[V]: C \cap K \to B/K$ is the same as $[T]$. Thus V is residually nuclear, and is approximately inner by Corollary 3.10.6(I).

Since V is a contraction, it is then, moreover, in the point-norm closure of the convex set of restrictions $S|C$ of inner complete contractions on B. □

Let us recall here the Definition 1.2.3 of : ??????

Remark 3.10.9.

(i) We shall see in Chapter 5 that σ-unital C^*-algebras B with the WvN–property satisfy a generalized Weyl–von-Neumann–Voiculescu type theorem for weakly residually nuclear completely positive contractions.

(ii) Proposition 3.2.13 says that simple purely infinite C^*-algebras have the WvN–property.

(iii) C^*-algebras B with the WvN-property are p.i. in the sense of Definition 1.2.1:

Let $a \in B_+$. We consider $C = C^*(\mathrm{diag}(a, 0))$ and the *-monomorphism V from C into $M_2(B)$ which sends $\mathrm{diag}(a, 0)$ to $\mathrm{diag}(a, a)$. V is residually nuclear. Therefore, by the WvN-property, for every $\varepsilon > 0$, there exists a contraction $d \in M_2(B)$ with $d^* \,\mathrm{diag}(a, 0)d = (\mathrm{diag}(a, a) - \varepsilon)_+$. This implies that B is p.i.

(iv) It follows from joint work with M. Rørdam and from [**443**] that all strongly p.i. algebras (in the sense of Definition 1.2.2) have the WvN–property, and that in several particular cases p.i. algebras are strongly p.i., see the Remarks 2.15.12, 3.11.1, 3.11.6 and 12.2.9.

(v) Residually nuclear separation passes to hereditary C^*-subalgebras.

(vi) It is easy to see that the non-commutative Michael selection principle for $B \otimes \mathbb{K}$ in Conjecture 12.3.6 implies residually nuclear separation for B in the sense of Definition 1.2.3, cf. Remarks 12.3.7, 12.3.8.
(Every simple stable C^*-algebra trivially satisfies the non-commutative Michael selection principle.)

(vii) It follows from [**463**] and [**443**] that C^*-algebras with the WvN–property and residually nuclear separation are strongly p.i. in the sense of Definition 1.2.2.

(viii) Since all separable C^*-algebras have Abelian factorization, each C^*-algebra has residually nuclear separation.

`Since every separable C*-algebra has abelian factorization, we get that they all also have residually nuclear separation`

`??`

Corollary 3.10.10. *Suppose that B has the WvN–property.*

(i) *If C is an exact subalgebra of $\mathcal{M}(B)$ such that that $V : C \to B$ is nuclear and $V(C \cap \mathcal{M}(B, J)) \subseteq J$ for every primitive ideal of B, then V is 1-step approximately inner, i.e. if V is approximately majorized by id_C in the sense of Definition 3.10.1.*
(ii) *If A is exact, $h : A \to B$ is a nuclear C^*-morphism, and $\mathcal{M}(B)$ contains two isometries s, t with orthogonal ranges, then h approximately majorizes $h \oplus_{s,t} h$.*

Proof. (i) follows from Corollary 3.10.7 and Definition 1.2.3.

(ii): Let $C := h(A)$. C is exact and $\varepsilon_C : C \hookrightarrow B$ is nuclear by Remark 3.1.2(iv). Since h and $h \oplus h$ have the same kernel, there is a C^*-morphism $V : C \to B$ such that $Vh = h \oplus h$, in fact $V = \varepsilon_C \oplus \varepsilon_C$, which is nuclear with ε_C. Thus (i) applies. □

Corollary 3.10.11. *Let $c \in (\mathcal{O}_2)_+$ be an element with $Spec(c) = [0, 1]$. If $a, b \in A_+$ and $\varrho \geq 0$ satisfy $\|b + J\| \leq \|a + J\| + \varrho$ for all primitive ideals J*

of A, then, for every $\delta > 0$ there exist an isometry $s \in \mathcal{M}(A \otimes \mathcal{O}_2)$ such that $\| s^*(a \otimes c)^k s - ((b \otimes c) - \varrho)_+^k \| < \delta$ for $k = 1, 2$.

Thus $\| s^*(a \otimes c)^k s - (b \otimes c)^k \| < \delta + \varrho \max(1, 2\|b\|)$ for $k = 1, 2$.

Proof. The assumptions imply that $\|b \otimes c\| = \|b\|$ and, for every closed ideal I of A,

$$\|b + I\| = \sup\{ \|b + J\| : \ J \in \mathrm{Prim}(A), \quad I \subseteq J \} \ \leq \ \|a + I\| + \varrho \, .$$

In particular, $\|d\| \leq \|a\|$ where $d := ((b \otimes c) - \varrho)_+$.

Let $h_0 \colon \mathrm{C}([0,1]) \to \mathcal{O}_2$ denote the unital *-monomorphism with $h_0(f) := f(c)$ for $f \in \mathrm{C}([0,1])$. By Corollary 3.10.10, h_0 approximately majorizes $h_0 \oplus h_0$, because $\mathrm{C}([0,1])$ is nuclear, $\mathcal{O}_2$ is a simple p.i. algebra and, therefore, by Proposition 3.2.13, $\mathcal{O}_2$ has the WvN–property.

Let $C := \mathrm{C}([0, \|a\|])$, and $\gamma \colon C \to C \otimes \mathrm{C}([0,1])$ the unital *-monomorphism given by $\gamma(f)(x,y) := f(xy)$. We define C^*-morphisms h and k from C into the unitization of $A \otimes \mathcal{O}_2$, and h_1 from C into the unitization of A, by $h(f) := f(a \otimes c)$, $k(f) := f(d)$ and $h_1(f) := f(a)$.

Then $h_1 \otimes h_0$ and, therefore, $h = (h_1 \otimes h_0)\gamma$ approximately majorizes their two-fold sums $(h_1 \otimes h_0) \oplus (h_1 \oplus h_0) = h_1 \otimes (h_0 \oplus h_0)$ and $h \oplus h$.

Note that, for every closed ideal J of A and $e \in A_+$, the spectrum of $\pi_J(e) \otimes c$ is the closed interval $[0, \|e + J\|]$. By Proposition B.4.2, for every closed ideal K of $A \otimes \mathcal{O}_2$ there exist a closed ideal J of A with $K = J \otimes \mathcal{O}_2$.

Thus, for every factorial ideal $K = J \otimes \mathcal{O}_2$ of $A \otimes \mathcal{O}_2$, the spectrum of $\pi_K(d)$ is $[0, (\|b + J\| - \varrho)_+]$, which is contained in the spectrum $[0, \|a + J\|]$ of $\pi_K(a \otimes c)$.

Hence, by Corollary 3.10.8, h approximately majorizes k. This implies easily the stated result. □

Corollary 3.10.12. *Let $\kappa(a)$ denote the convex set of elements $c = \sum d_k^* a d_k$ in A_+, where the d_k are in the unitization of A with $\sum d_k^* d_k = 1$ in $\mathcal{M}(A)$, $k = 1, \ldots, n$, $n \in \mathbb{N}$.*

Then, for $a, b \in A_+$,

$$\mathrm{dist}(b, \kappa(a)) = \max(0, \sup\{ \|b + J\| - \|a + J\| \colon J \in \mathrm{Prim}(A)\}) \, .$$

Proof. Let $\delta > 0$.

There exists $V = \sum d_k^*(\,\cdot\,) d_k$, such that $\|b - V(a)\| < \delta + \mathrm{dist}(b, \kappa(a))$.

For every closed ideal J, $\|V(a) + J\| \leq \|a + J\|$ and $\|b - V(a) + J\| \leq \|b - V(a)\|$.

Thus, $\|b + J\| - \|a + J\| \leq \delta + \mathrm{dist}(b, \kappa(a))$.

The other direction, i.e.,
$\mathrm{dist}(b, \kappa(a)) \leq \varrho$ for every $\varrho > 0$ with $\|b + J\| - \|a + J\| \leq \varrho$ for all primitive ideals J of A, follows from Corollary 3.10.11 by the method in the proof of Corollary 3.2.20,

because there exists by Proposition 3.2.13 a sequence of isometries $d_n \in \mathcal{O}_2$ with $d_n^* c d_n \to 1$. □

REMARK 3.10.13. There are strictly continuous versions of Corollaries 3.10.11 and 3.10.12 for $a, b \in \mathcal{M}(A)_+$ and weakly continuous versions for a, b in the positive part of a von Neumann algebra N. Of course, one has to modify all what is needed, beginning with Definition 3.10.1, but in a *very* straightforward way. The outcome is, e.g., the following modifications of Corollary 3.10.12:

Let $\kappa(a)_{st}$ and $\kappa(a)_s$ denote the strict closure of $\kappa(a)$ in $\mathcal{M}(A)$ and the strong closure of $\kappa(a)$ in N, respectively. Then, for $a, b \in \mathcal{M}(A)_+$,

$$\mathrm{dist}(b, \kappa(a)_{st}) \;=\; \max\big(0, \sup\{\|b + \mathcal{M}(A,J)\| - \|a + \mathcal{M}(A,J)\|\,;\; J \in \mathrm{Prim}(A)\}\big)\,.$$

Let $\mathcal{PZ}$ denote the set of projections in the center of N. Then, for $a, b \in N_+$,

$$\mathrm{dist}(b, \kappa(a)_s) = \max(0, \sup\{\|bp\| - \|ap\|\,;\; p \in \mathcal{PZ}\}).$$

COROLLARY 3.10.14. *Suppose that B and $D \subseteq \mathcal{M}(B)$ are σ-unital and stable, where D has the WvN–property and DB is dense in B.*

*Let $A \subseteq \mathcal{M}(D) \subseteq \mathcal{M}(B)$ a separable C^*-subalgebra. Suppose that there exists a *-monomorphism $h\colon A \hookrightarrow \mathcal{M}(D)$ such that, for every closed ideal J of D,*

$$h(A \cap \mathcal{M}(D,J)) = h(A) \cap \mathcal{M}(D,J)$$

and, for every $d \in D$, the c.p. map $a \in A \mapsto d^* h(a) d$ is residually nuclear.

with respect to which action? l.s.c. from $\mathcal{I}(B)$ to $\mathcal{I}(D)$ and then from $\mathcal{I}(D)$ to A???

Furthermore, let $V\colon A \to B$ a completely positive contraction such that

(i) *$V(A \cap \mathcal{M}(D,J))$ is contained in the closed linear span of BJB, and*
(ii) *$[V]\colon A/(A \cap \mathcal{M}(D,J)) \to B/J_B$ is nuclear for every closed ideal J of D.*

Then V is 1-step approximately inner in the sense of Definition 3.10.1, i.e., there exists a sequence of contractions $b_n \in B$ such that $V(a) = \lim b_n^* a b_n$ for every $a \in A$.

PROOF. First we show, that V is approximately majorized by an infinite repeat of h, – cf. Remark 5.1.1(8) and Lemma 5.1.2(i,ii) in Chapter 5 concerning infinite repeats $\delta_\infty \circ h$.

Let $s_1, s_2, \ldots$ be a sequence of isometries in $\mathcal{M}(D)$ such that $s_n^* s_m = \delta_{nm}$ and $\sum s_n s_n^*$ strictly converges to the unit element of $\mathcal{M}(D)$. We define an **infinite repeat** δ_∞ on $\mathcal{M}(B)$ by $\delta_\infty(a) := \sum s_n a s_n^*$.

Since $\delta_\infty(\mathcal{M}(D,J)) = \delta_\infty(\mathcal{M}(D)) \cap \mathcal{M}(D,J)$ for every closed ideal J of D, the infinite repeat $\delta_\infty \circ h$ of h satisfies that $(\delta_\infty \circ h)(A \cap \mathcal{M}(D,J))$ coincides with $((\delta_\infty \circ h)(A)) \cap \mathcal{M}(D,J)$ $h\colon A \hookrightarrow \mathcal{M}(D)$ for every closed ideal J of D,

and that $\delta_\infty \circ h$ is again residually nuclear ???? For which action of what lattice?

Therefore, we can assume that the image $C := h(A)$ of A in $\mathcal{M}(D)$ commutes elementwise with a unital copy of $\mathcal{O}_\infty$ in $\mathcal{M}(D)$. (It is *not* the copy that is defined by the isometries $s_1, s_2, \ldots$.)

Let us denote by $\Psi^{D,B}_{\mathrm{down}}(J)$ the closed linear span of BJB in B for ideals J of D.

We also use the notations $\Psi^{\mathrm{up}}_{D,A}(J) := A \cap \mathcal{M}(D,J)$, and, similarly $\Psi^{\mathrm{up}}_{D,C}(J)$, for closed ideals $J \subseteq D$, and $\Psi^{\mathrm{up}}_{B,C}(I) := C \cap \mathcal{M}(B,I)$, $\Psi^{\mathrm{up}}_{B,D}(I)$, for closed ideals $I \subseteq B$, as introduced at the end of Chapter 1.

Let $I_A := h^{-1}(\Psi^{\mathrm{up}}_{B,C}(I))$ and $J_A := h^{-1}(\Psi^{\mathrm{up}}_{D,C}(J))$.

Our assumptions say that, in particular, $\Psi^{\mathrm{up}}_{D,A}(J) =: J_A$, V maps $\Psi^{\mathrm{up}}_{D,A}(J)$ into $\Psi^{D,B}_{\mathrm{down}}(J)$ and that $[V]\colon A/J_A \to B/\Psi^{D,B}_{\mathrm{down}}(J)$ is nuclear.

Suppose that I is an ideal of B, then

$$\Psi^{\mathrm{up}}_{B,C}(I) \subseteq \Psi^{\mathrm{up}}_{D,C}(\Psi^{\mathrm{up}}_{B,D}(I))\,,$$

because $t \in \mathcal{M}(D)$ and $tB \subseteq I$ imply $tDB \subseteq I$ and, therefore, $tD \subseteq \mathcal{M}(B,I) \cap D$. Further $\Psi^{D,B}_{\mathrm{down}}(\Psi^{\mathrm{up}}_{B,D}(I)) \subseteq I$, because $B(D \cap \mathcal{M}(B,I))B \subseteq I$.

Thus we get $h^{-1}(\Psi^{\mathrm{up}}_{B,C}(I)) \subseteq J_A$ and $\Psi^{D,B}_{\mathrm{down}}(J) \subseteq I$ for $J := D \cap \mathcal{M}(B,I)$.

Therefore, the conditions (α)-(γ) of Corollary 3.10.6(ii) are satisfied for A, h and V. Thus, there exists a sequence of contractions d_n in B such that $d_n^* h(.) d_n$ converges in point-norm to V.

Since DB is dense in B, we find a net e_n of contractions in D_+ such that $e_n d_n - d_n$ converges to zero.

But, by assumption, $e_n h(\cdot) e_n$ is residually nuclear on $A \subseteq \mathcal{M}(D)$, and D has the WvN–property. Thus, there is a sequence f_n of contractions in D such that $f_n^* a f_n - e_n h(a) e_n$ tends to zero for every $a \in A$.

Let $b_n := f_n d_n$, then $V(a) = \lim b_n^* a b_n$ for every $a \in A$. The b_n are contractions in B. □

Proposition 3.10.15. *For every σ-unital C^*-algebra B, $F := B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$ and $\mathrm{C_b}(\mathbb{R}_+, F)/\,\mathrm{C_0}(\mathbb{R}_+, F)$ have both the WvN–property.*

Proof. Let D be a stable hereditary σ-unital C^*-subalgebra of $F \otimes \mathbb{K}$ (respectively of $E \otimes \mathbb{K}$).

It is not difficult to see that such D have the property that we can find, for given $\eta > 0$, $f_1, \ldots, f_n \in D$ and $m \in \mathbb{N}$, isometries $s_1, \ldots, s_m$ in the multiplier algebra $\mathcal{M}(D)$ of D such that $\|[s_i, f_j]\| < \eta$ and $s_k^* s_i = \delta_{i,k}$ for $j = 1, \ldots, n$, $i, k = 1, \ldots, m$.

To see this, e.g. for $D \subseteq E \otimes \mathbb{K}$, one can at first show that the unital *-monomorphisms $b \mapsto 1 \otimes b$ and $b \mapsto 1 \otimes b$ from $\mathcal{O}_\infty$ to $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ are homotopic in $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$. This can be seen by examination of finite subsets $t_1, \ldots, t_n$ of the generators of $\mathcal{O}_\infty$ by Theorem E, [**172**] and Künneth theorem applied to $\mathrm{K}_*(\mathcal{O}_\infty \otimes \mathcal{O}_\infty)$.

This can be used to show that there is a unital copy of $\mathcal{O}_\infty$ in the multiplier algebra of $(F \otimes \mathbb{K})_\omega$ (respectively of $\mathrm{Q}(\mathbb{R}_+, F \otimes \mathbb{K})$) that commutes with D. Since D is hereditary, we get such a copy of $\mathcal{O}_\infty$ also in $\mathcal{M}(D_\omega)$.

Let $C \subseteq \mathcal{M}(D)$ and suppose that $V \colon C \to D$ is residually nuclear. By Corollary 3.10.6, for $1 > \varepsilon > 0$ and $c_1, \ldots, c_n \in C$, there exist $d_1, \ldots, d_m \in D$ such that $\|V(c_j) - \sum d_i^* c_j d_i\| < \varepsilon/3$ for $j = 1, \ldots, n$.

Let $\gamma := 1 + \|\sum d_i^* d_i\| + \max\{\|c_j\|\}^2$ and $\eta := \varepsilon/(3m^2\gamma)$.

There is a contraction $e \in D_+$ with $\|ed_i - d_i\| < \eta$ for $i = 1, \ldots, m$. We find isometries $s_1, \ldots, s_m$ in the multiplier algebra of D such that $\|[s_i, ec_je]\| < \eta$ and $s_k^* s_i = \delta_{i,k}$ for $j = 1, \ldots, n$, $i, k = 1, \ldots, m$. Then $\|V(c_j) - d^* c_j d\| < \varepsilon$ for $d := e \sum s_i d_i$, $j = 1, \ldots, n$. □

11. More on strongly purely infinite algebras

The following remarks give an outline of some of the results of joint work with M. Rørdam and E. Blanchard on purely infinite algebras. See also end of Chapter 2.

REMARK 3.11.1. Let (A_n) be a sequence of unital C^*-algebras that contain a copy of $\mathcal{O}_\infty$ unitally.

Let $F := B \otimes A_1 \otimes A_2 \otimes \ldots$, then the same arguments as in the proof of Proposition 3.10.15 show that F, F_ω and $\mathrm{Q}(\mathbb{R}_+, F)$ have the WvN–property. An approximately commuting copy of $\mathcal{O}_\infty$ can also be used to show that F, F_ω and $\mathrm{Q}(\mathbb{R}_+, F)$ are strongly purely infinite.

Since in the middle of Chapter 10 we finish the proof of Theorem B, we can then use Corollary H and see from Proposition 3.10.15 and its proof that $B \otimes \mathcal{O}_\infty$ always has the WvN–property and is strongly purely infinite. This yields a proof that $B \otimes C$ has the WvN–property and is strongly purely infinite if C is a simple purely infinite separable nuclear C^*-algebra.

Conversely, $B \cong B \otimes \mathcal{O}_\infty$ if B has the WvN–property and is separable, stable and nuclear, by Corollary 10.3.8.

It is an open question whether or not (non-simple) purely infinite separable stable *nuclear* C^*-algebras B tensorial absorb $\mathcal{O}_\infty$. (It is the case if and only if B is s.p.i.).

Lemma 3.2.22 shows, that $\mathrm{C}(X, D)$ has the WvN–property for compact X and simple purely infinite D.

REMARK 3.11.2. *A stable C^*-algebra D has the WvN–property, if for every separable subalgebra A of the ultrapower D_ω of D, every approximately inner completely positive contraction $V \colon A \to D_\omega$ into a commutative subalgebra $C^*(V(A))$ of D_ω is 1-step approximately inner.*

PROOF. The following observations (1)–(6) together prove the sufficiency criterium for the WvN-property:

(1) The assumption immediately implies that D_ω is purely infinite, by the argument in Definition 1.2.3.

(2) This implies that D is purely infinite.

(3) For every separable C^*-subalgebra A of the ultrapower D_ω of a p.i. algebra D and every separable commutative C^*-subalgebra C of A there is a completely positive residually nuclear contraction V from A into a commutative C^*-subalgebra $C^*(V(A))$ of D_ω, such that $V|C = \mathrm{id}\,|C$. (see [**443**], compare also Lemma 12.1.6 for the first step in this direction, and use the semigroup structure argument of the proof of Lemma 12.1.6).

(4) The set C_A of approximately inner completely positive contractions $V\colon A \to D_\omega$ with commutative $C^*(V(A))$ is closed under Cuntz addition (see Chapter 4 or the remark before Corollary 3.10.8), and, therefore, by the observation (3) and the below given generalization of Lemma 3.2.7, C_A generates a dense subset of the convex set X of the residually nuclear completely positive contractions from A into D_ω, where one gets X if one applies to the set C_A only Cuntz addition and the operations in part (OC2) of Definition 3.2.2.

(5) The set of 1-step approximately inner completely positive maps is stable under point-norm limits and under the operations in part (OC2) of Definition 3.2.2, and, therefore, the assumptions and observation (4) together imply that all residually nuclear completely positive maps from A into D_ω are 1-step approximately inner.

(6) Stable D (and moreover D_ω) has the WvN–property if, for every separable C^*-subalgebra A of D_ω, every residually nuclear completely positive contraction V from A into D_ω is 1-step approximately inner:
if C is a separable C^*-subalgebra of $\mathcal{M}(D)$ and $V\colon C \to D$ is a residually nuclear completely positive contraction, then V is approximately inner by Corollary 3.10.6(I). If we use a suitable, approximately with C commuting, approximate unit e_n of D, we get a contraction $e \in (D_\omega)_+$ with $ec = ce$ for $c \in C$ and an approximately inner completely positive contraction $W\colon A \to D_\omega$ from the C^*-subalgebra A of D_ω, which is generated by $\{e^n c\colon c \in C, n \in \mathbb{N}\}$, such that $V(c) = W(a^n c)$ for $c \in C$ and $n \in \mathbb{N}$. The residual nuclearity of V then implies the residual nuclearity of W, see Proposition 2.12.8 for a step in this direction. If (d_k) is a sequence of contractions in D_ω such that $d_k^*(.)d_k$ converges to W in point norm, then one can pick up from the representing sequences $(e_n d_{kn})$ of ed_k elements f_m such that $f_m^* c f_m$ converges to $V(c)$ for every $c \in C$. □

A generalization of the sixth observation shows that D_ω has the WvN–property if every approximately inner map is

locally 1-step approximately inner.

Since D_ω has residually nuclear separation in the sense of Definition 1.2.3, it follows that D_ω is strongly purely infinite.

Therefore, D is strongly purely infinite if, for every separable C^*-subalgebra A of D_ω, every residually nuclear contraction $V: A \to D_\omega$ is 1-step approximately inner.

We replace the assumption of Lemma 3.2.7 and its conclusion in the following manner to get a generalization of Lemma 3.2.7 :

We suppose that $A \subseteq \mathcal{M}(B)$ and replace the assumption of Lemma 3.2.7 by the following *strict non-degeneracy* criteria where $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ denotes a point-norm closed m.o.c. cone:

Property (OC-snd):
Let $A \subseteq \mathcal{M}(B)$. Consider the following property of a subset $\mathcal{C} \subseteq \mathrm{CP}(A, B)$:

For every $a \in A_+ \subseteq \mathcal{M}(B)_+$, and every $b \in B_+$ with the property that

$$\|b + J\| \leq \|a + \mathcal{M}(B, J)\|$$

holds for every closed ideal J of B, there exists for every $\varepsilon > 0$ a c.p. contraction $T \in \mathcal{C}$, that satisfies $\|T(a) - b\| < \varepsilon$ and has the property that $T(A \cap \mathcal{M}(B, J)) \subseteq J$ for every closed ideal J of B.

If $A \cdot B$ is dense in B and $\mathcal{C}$ contains all inner c.p. maps $a \mapsto c^*ac$ with $c \in B$, then condition (OC-snd) is satisfied:

Indeed, if $\|b+J\| \leq \|a+\mathcal{M}(B, J)\|$ for all $J \in \mathcal{I}(B)$, then $b \in J_0 := \overline{\mathrm{span}(BaB)}$, because $a \in \mathcal{M}(B, J_0)$.

Thus, for given $\varepsilon \in (0, 1)$, let $\gamma := \gamma(\varepsilon) := (2\|b\|^{1/2} + 1)^{-1}\varepsilon$ there are $c_1, \ldots, c_n, d_1, \ldots, d_n \in B$ with $\|b^{1/2} - e\| < \gamma$ for $e := \sum_k d_k a^{1/2} c_k$. This implies $\|b - e^*e\| < \varepsilon$.

Notice that

$$e = [d_1, \ldots, d_n] \cdot \mathrm{diag}(a, \ldots, a)^{1/2} \cdot [c_1, \ldots, c_n]^\top .$$

Therefore,

$$e^*e \leq \| \sum_k d_k d_k^* \| \cdot \sum_k c_k^* a c_k .$$

It follows that there is $f \in B$ such that $V(a) := \sum_k (c_k f)^* a (c_k f)$ satisfies $(b-\varepsilon)_+ = V(a)$.

The question is, if property (OC-snd) also implies that $\mathcal{C}$ contains all c.p. maps $V(a) := c^*ac$ with $c \in B$. This *seems to be* the case if $A \subseteq \mathcal{M}(B)$, A is *exact* and B is *weakly injective*.

But it would be sufficient that B has residual nuclear separation and $\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$.

We would get that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ satisfies property (OC-snd) in case $A \subseteq \mathcal{M}(B)$, if and only if, $\mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathcal{C}$.

Modified version of Lemma 3.2.7:
If $A \subseteq \mathcal{M}(B)$ satisfies (OC-snd), and if $\Psi_A(J) = A \cap \mathcal{M}(B, J)$ denotes the l.s.c. action of $\mathrm{Prim}(B)$ *on A, then every Ψ-residually nuclear contraction $V\colon A \to B$ can be approximated in point norm by contractions in $\mathcal{C} :=$* $\mathrm{CP}_{\mathrm{in}}(A, B)$.

The proof is similar to the proof of Lemma 3.2.7. But one has to use the arguments in the proof of Lemma 3.10.3 and of Proposition 3.10.4. In fact, it is easier to use Stinespring-Kasparov dilation directly. Then, by (OC1), (OC2) and (OC-snd), for an residually nuclear contraction $V\colon A \to B$, one finds a σ-unital hereditary subalgebra $E \subseteq B$ and a *-monomorphism $h\colon A \hookrightarrow \mathcal{M}(E \otimes \mathbb{K})$ such that $A \cup V(A) \subseteq E$, and, for every contraction $d \in E \otimes \mathbb{K}$, $(1 \otimes p_{11})d^* h(.)d(1 \otimes p_{11})$ is of form $T(\cdot) \otimes p_{11}$ for some

`approximately inner???`

T

`in the point norm closure of` $K =$?????,

and $h(A \cap J) = h(A) \cap \mathcal{M}(E \otimes \mathbb{K}, J \otimes \mathbb{K})$ for every closed ideal J of E. Moreover, we can replace h by its infinite repeat $\delta_\infty \circ h$. Thus, we may assume that $h(A)' \cap \mathcal{M}(E \otimes \mathbb{K})$ contains a copy of $\mathcal{O}_2$ unitally.

Now we are in position to apply Corollary 3.10.6(II) on A, h and $a \in A \mapsto V(a) \otimes p_{11}$ to get a proof of the above given generalization of Lemma 3.2.7.

The general idea of the application of the above modified Lemma 3.2.7 is the following observation:

Suppose that, for every separable C^*-subalgebra A of the ultrapower $B = D_\omega$ of a stable σ-unital C^*-algebra D, there exists a set $\mathcal{S}$ of 1-step approximately inner completely positive contractions $V\colon A \to B$ into commutative C^*-subalgebras of B, such that $\mathcal{S}$ is closed under Cuntz addition, and for every $a \in A_+$ there exist V in $\mathcal{S}$ and $d \in B$ such that $\|a - d^* V(a) d\| < 1$.

Then, by the *modified version* of Lemma 3.2.7, every residually nuclear $T\colon A \to B$ is 1-step approximately inner.

As we have seen above, this implies that D and its ultrapower D_ω have the WvN–property and are strongly p.i.

The above given third, fourth and fifth observations also show that the modified version of Lemma 3.2.7 yields the following result:

Suppose that A is a separable C^*-subalgebra of D_ω, and that for every residually nuclear contraction $V\colon A \to D_\omega$ with commutative image $C^*(V(A))$, there is a sequence of contractions $d_n \in D_\omega$ such that, for $a \in A_+$ and $k > 0$,

$$\begin{aligned} \lim d_n^* a d_n &= V(a), \qquad \text{and} \\ \lim d_n^* a d_{n+k} &= 0. \end{aligned}$$

Then such a sequence (d_n) exists also for every *residually nuclear contractions $V\colon A \to D_\omega$ with non-commutative image $V(A)$.*

Note that, with help of a countable approximate unit of A, we can modify the sequence d_n such that it also satisfies $\lim d_n^* d_{n+k} = 0$ for $k > 0$.

Proof. Consider the set I_A of all contractions $V : A \to D_\omega$ such that there exist a sequence d_n with the above properties.

It is easy to see, that the operations (OC2) in Definition 3.2.2 let I_A invariant.

The assumption says that C_A of the above fourth observation is contained in I_A.

By the third and fourth observation, the smallest subset of I_A, that contains C_A and is invariant under the operations (OC2) in Definition 3.2.2, satisfies the assumptions of the above given modified version of Lemma 3.2.7.

It is not very hard to check that I_A is closed under point norm limits. Thus, by the modified version of Lemma 3.2.7, every residually nuclear contraction V from A into D_ω is in I_A. □

FIND END OF NEXT REMARK !!!

Remark 3.11.3. We list here some later needed properties of strongly purely infinite C^*–algebras. (See [**463**] for details of the here outlined proofs.)

In the following let $[x, y]$ denote the commutator $xy - yx$.

Suppose that A is strongly purely infinite.

Let $b, a_1, \ldots, a_n \in A_+$ *commuting* elements, $c \in A_+$, $F := [f_{jk}] \in M_m(A)_+$ a positive $m \times m$ matrix, and let $\varepsilon > 0$.

(i) A_ω and $A \otimes \mathbb{K}$ are strongly purely infinite.
    Every quotient and every hereditary C^*-subalgebra of A is strongly purely infinite.
(ii) There exist contractions $d_1, \ldots, d_m \in A$, such that

$$\| D^* F D - \operatorname{diag}(f_{11}, \ldots, f_{mm}) \| < \varepsilon \,,$$

where $D := \operatorname{diag}(d_1, \ldots, d_m)$.
(iii) There exist contractions $v_1, v_2 \in A$ with
$\|[v_i, a_k]\| < \varepsilon$ and $\|v_i^* b v_j - \delta_{i,j} b\| < \varepsilon$ for $i, j \in \{1, 2\}$, $k = 1, \ldots, n$.
(iv) There exist contractions $s_1, s_2 \in A$ with
$\|s_i^* a_k s_i - a_k\| < \varepsilon$, $\|[s_i, a_k]\| < \varepsilon$, $\|s_1^* s_2\| < \varepsilon$, $\|s_1^* c s_2\| < \varepsilon$, and $\|[s_i^* c s_i, a_k]\| < \varepsilon$, for $i = 1, 2$, $k = 1, \ldots, n$.

Property (ii) is an inductive repeat of the defining property of 'strongly p.i.' in Definition 1.2.2. (ii) implies (iii) and (iv). The proof of this implication is not trivial. It uses appropriate decompositions of the unit of the unitization of $C^*(a_1, \ldots, a_n, b)$ respectively of $C^*(a_1, \ldots, a_n)$, to translate (iii) and (iv) to problems, which are solvable by property (ii). The proofs can be found in [**463**].

It follows from [**463**] that p.i. algebras with (iii) and (iv) have the WvN-property of Definition 1.2.3.

In fact we show that properties (iii) and (iv), applied to the strongly p.i. algebra A_ω, yield the following stronger results (v)-(vii).

Suppose that A is strongly p.i., that C is a separable commutative C^*-subalgebra of A_ω, that B is a separable C^*-subalgebra of A_ω, and that V is a residually nuclear contraction from B into C. Then:

(v) There is a *-monomorphism h from $C\otimes\mathcal{O}_\infty$ into A_ω with $h((.)\otimes 1) = \mathrm{id}\,|C$,
(vi) For pairwise commuting $a_1,\ldots,a_n \in (A_\omega)_+$, arbitrary $y_1,\ldots,y_m \in A_\omega$ and $\varepsilon > 0$, there is a contraction $d \in A_\omega$, such that $\|d^*a_id - a_i\| < \varepsilon$ for $1 \le i \le n$ and that $\{d^*a_1d,\ldots,d^*a_nd, d^*y_1d,\ldots,d^*y_md\}$ generates a commutative C^*-subalgebra of A_ω.
(vii) There is a sequence of contractions $d_n \in A_\omega$ with $\lim d_n^*bd_n = V(b)$ and $\lim d_n^*bd_{n+k} = \{0\}$ for $b \in B$, $k > 0$.

Here (vii) follows from (v) and (vi).

Now we can apply the ideas of Remark 3.11.2:

In [**463**] it is shown that for a C^*-algebra A the following properties (1)–(4) and (8) are equivalent. The equivalence of (1) and (5)–(7) follows from [**443**] and the results of Chapter 7.

(1) *A is strongly purely infinite in the sense of Definition 1.2.2.*
(2) *The ultrapower A_ω of A is strongly purely infinite.*
(3) *A admits no non-zero character, and, for every separable C^*-subalgebra B of A_ω and every approximately inner completely positive contraction $V\colon B \to A_\omega$ into a commutative C^*-subalgebra C of A_ω there is a contraction $d \in A_\omega$ with $V(b) = d^*bd$ for $b \in B$.*
(4) *For every residually nuclear contraction V from a separable C^*-subalgebra C of A_ω into A_ω there is a sequence of contractions $d_n \in A_\omega$ with $d_n^*cd_n = V(c)$ for $c \in C$, $d_m^*d_n = 0$, and $d_m^*Cd_n = \{0\}$ for $m \ne n$, $m,n = 1,2,\ldots$.*
(5) *The asymptotic corona $\mathrm{Q}(\mathbb{R}_+,A)$ of A is strongly purely infinite.*
(6) $\mathrm{Q}(\mathbb{R}_+,A)$ *has the WvN-property of Definition 1.2.3.*
(7) *A admits no non-zero character, and for every* separable *C^*-subalgebra B of $\mathrm{Q}(\mathbb{R}_+,A)$ the following property holds:*

Let $t \in \mathbb{R}_+ \mapsto V(t)$ a strongly continuous map from $\mathbb{R}_+$ into the approximately inner completely positive contractions from A into A and let T the completely positive contraction from $\mathrm{Q}(\mathbb{R}_+,A)$ into $\mathrm{Q}(\mathbb{R}_+,A)$ given by

$$T(a + \mathrm{C}_0(\mathbb{R}_+,A)) := V(a) + \mathrm{C}_0(\mathbb{R}_+,A)\,,$$

where $a \in \mathrm{C_b}(\mathbb{R}_+,A)$ and $V(a)(t) := V(t)(a(t))\,.$

The property of A in question is:

If $T|B$ is residually nuclear, then there exists a contraction $d \in \mathrm{Q}(\mathbb{R}_+,A)$ such that $T(b) = d^*bd$ for $b \in B$.
(8) *For every $a,b \in A_+$ and $\varepsilon > 0$ there exist* contractions *$s,t \in A$ such that $\|a^2 - s^*a^2s\|$, $\|b^2 - t^*b^2t\|$ and $\|s^*abt\|$ are all less than ε.*

The property in (8) implies that *the class of strongly purely infinite algebras is closed under inductive limits.*

REMARK 3.11.4. Propositions 3.2.13 and 3.2.15 are used to prove a Weyl-von Neumann–Voiculescu type theorem in Chapter 5 and its asymptotic version in Chapter 7.

The reader can observe that our proofs of Propositions 3.2.13 and 3.2.15 works also in the following more general and non-simple case:

Suppose that D is σ-unital, stable and strongly purely infinite, and that for every separable C^*-subalgebra A of D there exists a *-monomorphism $h\colon A \hookrightarrow \mathcal{M}(D)$, such that:

(1) $h(A \cap J) = h(A) \cap \mathcal{M}(D, J)$, and
(2) $[h]_J\colon A/(A \cap J) \to \mathcal{M}(D/J)$ is weakly nuclear for every closed ideal J of D.
(3) The closures of DKD and $Dh(K)D$ in D coincide for every closed ideal K of A.
(4) For every sequence Ω_n of compact subsets of A, and for every approximately commutative unit of D for $h(A)$ with $e_n e_{n+1} = e_n$, there exists a sequence f_n of contractions in D such that $\|f_n^* a f_n - e_n h(a) e_n\| < 2^{-n}$, $\|f_n^* a f_{n+k}\| < 2^{-n}$ and $\|f_n^* f_{n+k}\| < 2^{-n}$ for $a \in \Omega_n$, $k > 0$.

From Corollary 3.10.6(I) we get, that if we assume (1) and (2), then (3) is equivalent to:

(3') $Dh(A)D$ is dense in DAD.

It follows from a result with M. Rørdam [**463**] that the property (4) in Remark 3.11.3 implies, that a σ-unital stable purely infinite C^*-algebra D has residually nuclear separation in the sense of definition 1.2.3, if and only if, for every separable C^*-subalgebra $A \subseteq D$, there exists a *-monomorphism $h\colon A \hookrightarrow \mathcal{M}(D)$ with properties (1)-(4) of Remark 3.11.4.

The modifications in the assumptions of Propositions 3.2.13 and 3.2.15 are then: B is σ-unital, $D \subseteq \mathcal{M}(B)$ with $D \cdot B$ dense in B, and D has the above considered property.

In the general version of Proposition 3.2.13, C has to be a separable C^*-subalgebra of $\mathcal{M}(D) \subseteq \mathcal{M}(B)$, and $V\colon C \to B$ is a completely positive contraction such that $V(\Psi^{\mathrm{up}}_{D,C}(J)) \subseteq \Psi^{D,B}_{\mathrm{down}}(J)$ and $[V]\colon C/\Psi^{\mathrm{up}}_{D,C}(J) \to B/\Psi^{D,B}_{\mathrm{down}}(J)$ is nuclear for every closed ideal $J \subseteq D$. See the last part of the introduction or the proof of Corollary 3.10.14 for the definition of the Ψ's.

In the modified version of Proposition 3.2.15, C has to be a separable C^*-subalgebra of $\mathrm{C_b}(X, D)$ and the point-norm continuous family of complete contractions V_y from D into B must consist of residually nuclear maps.

Note that (1)-(4) imply also the assumptions of Corollary 3.10.14.

PROPOSITION 3.11.5. *Let B a stable σ-unital strongly p.i. algebra, A a separable exact algebra, and $h\colon A \to \mathrm{Q}(B) := \mathcal{M}(B)/B$ a nuclear C^*-morphism. Then $A' \cap \mathrm{Q}(B)$ is strongly p.i.*

PROOF. ?? `to be filled in` □

REMARK 3.11.6. In [**462**], [**463**], [**92**], [**93**] and [**443**] it is shown, that p.i. algebras in several cases are strongly p.i. (and thus have the WvN–property). We list some related results:

(i) Locally purely infinite C^*-algebras of real rank zero are strongly purely infinite, cf. [**463**] and [**93**].

(ii) The algebra A of sections of a continuous field of C^*-algebras over a locally compact space is strongly purely infinite, if it is purely infinite and if the field has simple fibers. This is just the case where A is purely infinite and $\mathrm{Prim}(A)$ is Hausdorff. (Joint work with E. Blanchard, [**92**], [**93**].)

In particular, $\mathrm{C}_0(X, D)$ is strongly purely infinite, if D is simple and purely infinite.

(iii) $\mathrm{C}_0(X, D)$ is strongly purely infinite if D is strongly purely infinite and X is locally compact ([**443**]).

(iv) $\mathcal{M}(D)$ is strongly purely infinite if D is strongly purely infinite and σ-unital ([**443**]).

(v) Extension of strongly purely infinite algebras are strongly purely infinite ([**443**]).

It follows that $\mathrm{Q}(\mathbb{R}_+, B)$ is strongly purely infinite if B is strongly purely infinite.

12. Collection of some topics

`Next: from old proof of Proposition 3.6.1`

????? (i):

Let $J \triangleleft B$ and $C \subseteq \mathcal{M}(B)$ a C^*-subalgebra of $\mathcal{M}(B)$. Recall that $\Psi^{\mathrm{up}}_{B,C}(J) := C \cap \mathcal{M}(B, J)$, that $\mathcal{M}(B, J) := \{t \in \mathcal{M}(B)\,;\, tB + Bt \subseteq J\}$ is the same as the closure of $J \triangleleft B$ in the strict topology on $\mathcal{M}(B)$, and that every strictly closed ideal I of $\mathcal{M}(D)$ coincides with $\mathcal{M}(D, I \cap D)$.

Since DB and BD are dense in B (by assumption), we get that the natural unital *-monomorphism $\mathcal{M}(D) \hookrightarrow \mathcal{M}(B)$ is strictly continuous. Thus, $\mathcal{M}(B, J) \cap \mathcal{M}(D) = \mathcal{M}(D, D \cap \mathcal{M}(B, J))$ and $\Psi^{\mathrm{up}}_{B,C}(J) = \Psi^{\mathrm{up}}_{C,D}(K) =: \Psi_C(K)$ for $K := D \cap \mathcal{M}(B, J)$. The ideal $J_1 := \Psi_B(K) := \Psi^{D,B}_{\mathrm{down}}(K)$ of B is defined as the closure of $\mathrm{span}(BKB)$ and is contained in $B \cap \mathcal{M}(B, J) = J$. Indeed, $\mathrm{span}(BKB) \subseteq \mathcal{M}(B, J)$, because $K \subseteq \mathcal{M}(B, J)$.

$\Psi^{\mathrm{up}}_{D,B}(\Psi_D(I)) =$??????

something missing ???

Suppose that V is Ψ_C-Ψ_B–residually nuclear, and $J \in \mathcal{I}(B)$. Let $K := \Psi^{\mathrm{up}}_{B,D}(J)$. Then $L := \Psi_C(K) = \Psi^{\mathrm{up}}_{B,C}(J)$ by Lemma 3.6.2(ii).

Thus, $V(L) = V(\Psi_C(K)) \subseteq \Psi_B(K)$, and $[V] \colon C/\Psi_C(K) \to B/\Psi_B(K)$ is nuclear. Since $\Psi_B(K) \subseteq J$ (by Lemma 3.6.2(iv)), it follows that $V(\Psi^{\mathrm{up}}_{B,C}(J)) \subseteq J$, and that $[V]_J \colon C/\Psi^{\mathrm{up}}_{B,C}(J) \to B/J$ is nuclear. Hence, V is $\Psi^{\mathrm{up}}_{B,C}$-residually nuclear.

Conversely, suppose that V is $\Psi^{\mathrm{up}}_{B,C}$–residually nuclear, and let $N \triangleleft D$. Consider the ideal $J := \Psi_B(N) := \overline{\mathrm{span}(BNB)}$. We get $N \subseteq K := D \cap \mathcal{M}(B,J)$ and $J = BNB \subseteq BKB \subseteq J$. Thus $\mathcal{M}(D,N) \subseteq \mathcal{M}(D,K) = \mathcal{M}(D) \cap \mathcal{M}(B,J)$ and $\Psi_C(N) \subseteq \Psi^{\mathrm{up}}_{B,C}(J)$.

Since $V(\Psi^{\mathrm{up}}_{B,C}(J)) \subseteq J$ and $[V]_J \colon C/\Psi^{\mathrm{up}}_{B,C}(J) \to B/J$ is nuclear, we get that $V(\Psi_C(N)) \subseteq J = \Psi_B(N)$ and $[V] = [V]_J \circ \pi \colon C/\Psi_C(N) \to B/J$ is nuclear, where $\pi \colon C/\Psi_C(N) \to C/\Psi^{\mathrm{up}}_{B,C}(J)$ is the natural epimorphism. Hence, V is Ψ_C-Ψ_B–residually nuclear.

End of old proof-parts. Missing somewhere? Remove above??

Used in Remark **??** and quoted to Chapter 3:

Suppose that B is strongly purely infinite. Then, for every finitely generated Abelian C-subalgebra $A \subseteq B_\omega$ with at most 1-dimensional maximal ideal space* $\mathrm{Prim}(A)$*, there exists a C*-morphism*

$$h \colon A \otimes \mathcal{O}_\infty \to B_\omega \quad \text{with} \quad h(a \otimes 1) = a \ \ \forall\, a \in B\,.$$

Overlap with Chp.3 Part 2 beginning here

Compare below with Section 7

See also begin of Section 8 for next:

There are proofs of some the following corollaries, that are more elementary than given below. But we want to invite the reader to use (and think about) the more conceptional ideas presented here. For the classification of stably finite algebras with non-trivial traces, one has to produce refined tools, that answer e.g. the more difficult question (by a *non-trivial* and *applicable* answer):

When there is a given completely positive map V in a given cone $\mathcal{C}$ of completely positive maps $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ that are compatible with a given map $T(B) \to T(A)$?

Here $T(B)$ means the set of lower semi-continuous (unbounded) 2-quasi-traces $\tau \colon B_+ \to [0\infty]$. Our results together contain an answer in a special case where $\tau(B_+) \subseteq \{0,\infty\}$ for all $\tau \in T(B)$, because then the problem reduces to the study of $\mathrm{CP}_{\mathrm{nuc}}(\Psi;\, A, B)$ for a lower semi-continuous action $\Psi \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim})(B) \to \mathcal{I}(A)$, if we identify a $\{0,\infty\}$-valued lower semi-continuous 2-quasi-trace $\tau \colon B_+ \to [0,\infty]$ by its kernel $J := \mathrm{span}\{a \in B_+\,;\ \tau(a) = 0\}$.

Compare for next Def. also Def. 3.8.1

DEFINITION 3.12.1. Let $\mathcal{S} \subseteq \mathrm{CP}(A, B)$.

We define $\Psi_{\mathcal{S}}(J)_+$ as the set of $a \in A_+$ with $V(\exp(-ih)a\exp(ih)) \in J$ for all $h^* = h \in A$ with $\|h\| < \pi$ and for all $V \in \mathcal{S}$. Then $\Psi_{\mathcal{S}}(J)_+$ is the positive part of a closed ideal $\Psi_{\mathcal{S}}(J)$ of A, and

$$\Psi_{\mathcal{S}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$$

is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, cf. Lemma 3.12.2(iv).

Let I a closed ideal of A. We denote by $\Psi^{\mathcal{S}}(I))$ the smallest closed ideal of B that is contains $\{V(a)\,;\; a \in I, V \in \mathcal{S}\}$. The map

$$\Psi^{\mathcal{S}} \colon \mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A)) \to \mathcal{I}(B)$$

is an upper semi-continuous action, cf. Lemma 3.12.2(xii).

Let X a topological space, and let $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(X) \to \mathcal{I}(B)$ increasing maps, i.e., *actions* of X on A and B. We define the **cone of Ψ-equivariant c.p. maps** as the set $\mathcal{C}_\Psi := \mathrm{CP}(\Psi_A, \Psi_B; A, B) \subseteq \mathrm{CP}(A, B)$ of maps $V \in \mathrm{CP}(A, B)$ with $V(\Psi_A(U)) \subseteq \Psi_B(U)$ for all open subsets $U \subseteq X$. In the special case, where $X := \mathrm{Prim}(B)$ and $\Psi_B \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(B)$ is given by the natural identification of the open subsets U of $\mathrm{Prim}(B)$ with the corresponding closed ideal J_U of B, where J_U denotes the intersection of all primitive ideals $I \in \mathrm{Prim}(B) \setminus U$, we write $\Psi := \Psi_A$ and drop Ψ_B, i.e., write $\mathcal{C}_\Psi := \mathrm{CP}(\Psi; A, B) \subseteq \mathrm{CP}(A, B)$ for $\Psi \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$.

The use of the notation $\mathcal{C}_\Psi$, in place of $\mathcal{C}_{\Psi_A, \Psi_B}$, becomes justified in a different way in part (i) of Lemma 3.12.2.

`Compare with Lemma 3.8.2`

LEMMA 3.12.2. *Let A and B C^*–algebras, $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ a set of completely positive maps, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a matrix operator convex cone, X a topological space, $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(X) \to \mathcal{I}(B)$ actions of X on A respectively B,*

Suppose that ???????????? **??**

(i) *There exists a lower semi-continuous action $\Psi' \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \to \mathcal{I}(A)$ with $\Psi'(\Psi_B(U)) \supset \Psi_A(U))$, which is* minimal *in the sense that $\Phi(J) \supset \Psi'(J)$ for every $J \in \mathcal{I}(B)$ if $\Phi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is lower semi-continuous and $\Phi(\Psi_B(U)) \supset \Psi_A(U)$.*
It holds $\mathcal{C}_{\Psi'} = \mathcal{C}_\Psi$.

(ii) *The point-norm closure of a matrix operator-convex cone (in the algebraic sense) is a matrix operator-convex cone.*
The intersection of a family of matrix operator-convex cones is a matrix operator-convex cone.

(iii) *Let $\mathcal{C}(\mathcal{S})$ denote the smallest point-norm closed matrix operator-convex cone that contains a subset $\mathcal{S} \subseteq \mathrm{CP}(A, B)$. Then every contraction V in the point-norm closure $\mathcal{C}(\mathcal{S})$ of can be approximated by maps $W \in \mathcal{C}_{\mathrm{alg}}(\mathcal{S})$*

of the particular form $W := \sum_k c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$ *with* $V_k \in \mathcal{S}$, $r_k \in M_{1,n}(A)$, $c_k \in M_{n,1}(B)$ *and* $\|\sum_k c_k^* V_k(r_k^* r_k)c_k\| \leq 1$.

(iv) $\Psi_{\mathcal{S}}(J)_+$ *is the positive part of a closed ideal* $\Psi_{\mathcal{S}}(J)$ *of* A, *and*

$$\Psi_{\mathcal{S}} \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \ni J \mapsto \Psi_{\mathcal{S}}(J) \in \mathcal{I}(A)$$

is a lower semi-continuous action of $\mathrm{Prim}(B)$ *on* A.
The action $\Psi_{\mathcal{S}}$ *satisfies* $\Psi_{\mathcal{S}}(B) = A$ *and* $V(\Psi_{\mathcal{S}}(J)) \subseteq J$ *for all* $J \in \mathcal{I}(B)$ *and* $V \in \mathcal{S}$.

(v) $\Psi_{\mathcal{S}} = \Psi_{\mathcal{C}}$ *for* $\mathcal{C} := \mathcal{C}(\mathcal{S})$.

(vi) $\Psi_{\mathcal{S}}(J) \subseteq \Psi_{\mathcal{S}_1}(J)$ *for all* $J \in \mathcal{I}(B)$, *if* $\mathcal{S}_1 \subseteq \mathcal{S}$.

(vii) *The set* $\mathcal{C}_\Psi$ *is a point norm-closed matrix operator convex cone of completely positive maps.*

(viii) $\mathcal{C}_{\Psi_1} \subseteq \mathcal{C}_\Psi$ *if* $\Psi(J) \subseteq \Psi_1(J)$ *for all* $J \in \mathcal{I}(B)$.

(ix) $\mathcal{C} \subseteq \mathcal{C}_{\Psi_{\mathcal{C}}}$.

(x) $\Psi_{\mathcal{C}_\Psi}(J) \supset \Psi(J)$ *for* $J \in \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$.

(xi) *If* $\mathcal{C}' \subseteq \mathrm{CP}(A \otimes^{\max} D, B \otimes^{\max} D)$ *is a point-norm closed matrix operator convex cone, then the set* $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ *of* $T \in \mathrm{CP}(A, B)$ *with* $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ *is a point-norm closed matrix operator convex cone.*

(xii) *Let* M *a* W^**-algebra and* $B \subseteq M$ *a* $\sigma(M, M_*)$*-dense* C^**-subalgebra of* M, *and let* $\mathcal{C} \subseteq \mathrm{CP}(A, B) \subseteq \mathrm{CP}(A, M)$ *a matrix operator-convex cone. Then the point-*ultra-strong closure* $\overline{\mathcal{C}} \subseteq \mathrm{CP}(A, M)$ *of* $\mathcal{C}$ *is a point-*$\sigma(M, M_*)$ *closed matrix operator convex cone.*

(xiii) $\Psi^{\mathcal{S}} \colon \mathcal{I}(A) \to \mathcal{I}(B)$ *is an upper semi-continuous action of* $\mathrm{Prim}(A)$ *on* B. $I \subseteq \Psi_{\mathcal{S}}(\Psi^{\mathcal{S}}(I))$ *for every closed ideal* I *of* A *and every subset* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$.

PROOF. (iv): Let J a closed ideal of B and $T \colon A \to B$ a positive map. The set of $a \in A_+$ with $(\pi_J \circ T)(a) = 0$ is a *hereditary* closed convex sub-cone $C_{T,J}$ of A_+. Thus, the set $C_{\mathcal{S},J}$ of $a \in A_+$ with $T(a) \in J$ for all $T \in \mathcal{S}$ is a hereditary closed convex sub-cone of A_+, and $\Psi_{\mathcal{S}}(J)_+ \subseteq A_+$ of Definition 3.12.1 is the set of $a \in A_+$ with $\exp(ih)a\exp(-ih) \in C_{\mathcal{S},J}$. By Lemma **??**, $\Psi_{\mathcal{S}}(J)_+$ is the positive part of a closed ideal $\Psi_{\mathcal{S}}(J)$ of A. Clearly, $\Psi_{\mathcal{S}}(B) = A$, and $V(\Psi_{\mathcal{S}}(J)_+) \subseteq J$ for $J \in \mathcal{I}(B)$ and $V \in \mathcal{S}$, by definition of $\Psi_{\mathcal{S}}$.

If $\{J_\sigma\}_{\sigma \in \Sigma}$ is a family of closed ideals and $a \in A_+$ and let $J := \bigcap_\sigma J_\sigma$. then $V(\exp(ih)a\exp(-ih)) \in J$ for all $V \in \mathcal{S}$ and $h^* = h \in A$ with $\|h\| < \pi$, if and only if, $V(\exp(ih)a\exp(-ih)) \in J_\sigma$ for all $V \in \mathcal{S}$ and $h^* = h \in A$ with $\|h\| < \pi$, i.e., $a \in \Psi_{\mathcal{S}}(J)_+$ if and only if $a \in \bigcap_\sigma \Psi_{\mathcal{S}}(J_\sigma)_+ = (\bigcap_\sigma \Psi_{\mathcal{S}}(J_\sigma))_+$. Thus,

$$\Psi_{\mathcal{S}} \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \ni J \mapsto \Psi_{\mathcal{S}}(J) \in \mathcal{I}(A)$$

is a lower semi-continuous map. The lower semi-continuity of the map $\Psi_{\mathcal{S}}$ implies that $\Psi_{\mathcal{S}}$ is monotone. Thus, $\Psi_{\mathcal{S}}$ is a lower s.c. action of $\mathrm{Prim}(B)$ on A.

(vii): Since $\Psi_B(U)$ is a closed ideal, the set of maps $V \in \mathrm{CP}(A, B)$ with $V(a) \in \Psi_B(U)$ for all $a \in \Psi_A(U)$ is point-norm closed. Thus, $\mathcal{C}_\Psi$ is point-norm closed. Similar arguments show:

Let $V_1, V_2 \in \mathrm{CP}(A,B)$ with $V_k(\Psi_A(U)) \subseteq \Psi_B(U)$ for all $U \in \mathbb{O}(X)$ $(k=1,2)$, $t \in [0,\infty)$, and $r \in M_{1,n}(A)$, $c \in M_{n,1}(B)$. Then $(V_1 + tV_2)(a) \in \Psi_B(U)$ and $c^*(V_1 \otimes \mathrm{id}_n)(r^*ar)c \in \Psi_B(U)$ for $U \in \mathbb{O}(X)$ and $a \in \Psi_A(U)$. Thus $\mathcal{C}_\Psi$ is a point-norm closed matrix operator convex cone of c.p. maps.

(i): If we use the natural isomorphisms $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ and $\mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$, and Lemma **??**, then we get that there is a minimal lower semi-continuous action $\Psi' \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ with $\Psi'(\Psi_B(U)) \supset \Psi_A(U)$.

If $V \in \mathcal{C}_{\Psi'}$, then $V(\Psi_A(U)) \subseteq V(\Psi'(\Psi_B(U))) \subseteq \Psi_B(U)$ for $U \in \mathbb{O}(X)$, because $\Psi_A(U) \subseteq \Psi'(\Psi_B(U))$. Thus, $\mathcal{C}_{\Psi'} \subseteq \mathcal{C}_\Psi = \mathcal{C}_{\Psi_A,\Psi_B}$.

Let $\mathcal{S} := \mathcal{C}_\Psi$, then $\Psi_{\mathcal{S}} \colon \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ is a lower semi-continuous action by part (iv) with $V(\Psi_{\mathcal{S}}(J)) \subseteq J$ for all $V \in \mathcal{S}$.

We have $\Psi_{\mathcal{S}}(\Psi_B(U))_+ \supset \Psi_A(U)_+$ for $U \in \mathbb{O}(X)$, because $V(e{-}ihae^{ih}) \in \Psi_B(U)$ for $a \in \Psi_A(U)_+$ and $h^* = h \in A$. Thus, $\Psi_{\mathcal{S}}(J) \supset \Psi'(J)$ for every $J \in \mathcal{I}(B)$ by minimality of Ψ'. It implies that $V(\Psi'(J)) \subseteq V(\Psi_{\mathcal{S}}(J)) \subseteq J$, i.e., $V \in \mathcal{C}_{\Psi'}$, for all $V \in \mathcal{C}_\Psi = \mathcal{C}_{\Psi_A,\Psi_B}$.

(ii): The point-norm closure $\mathcal{C}$ of a matrix operator-convex cone $\mathcal{C}_0$ is a matrix operator-convex cone, because the topology of point-norm convergence on $\mathcal{L}(A,B)$ coincides with the strong operator topology, and because $V_\alpha \otimes \mathrm{id}_n$ converges in point-norm to $V \otimes \mathrm{id}_n$ if $V_\alpha \in \mathcal{C}_0$ converges to V.

(iii): Let $n \in \mathbb{N}$ and let $\mathcal{C}^{(n)}$ denote the set of completely positive maps $W := \sum_k c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$ with $V_k \in \mathcal{S}$, $r_k \in M_{1,n}(A)$ and $c_k \in M_{n,1}(B)$, and let $\mathcal{C}_0 = \bigcap_n \mathcal{C}^{(n)}$. Clearly, $\mathcal{C}^{(n)} \subseteq \mathcal{C}$ for every matrix operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ with $\mathcal{S} \subseteq \mathcal{C}$, and the set $\mathcal{C}^{(n)}$ is closed under multiplication with positive scalars and under addition.

Moreover, $c^*(W \otimes \mathrm{id}_m)(r^*(\cdot)r)c \in \mathcal{C}^{(mn)}$ if $W \in \mathcal{C}^{(n)}$, $r \in M_{1,m}(A)$ and $c \in M_{n,1}(B)$, because

$$c^*(W_k \otimes \mathrm{id}_m)(r^*(\cdot)r)c = C^*V_k \otimes \mathrm{id}_{mn}(R^*(\cdot)R)C$$

for $W_k := c_k^*(V_k \otimes \mathrm{id}_n)(r_k^*(\cdot)r_k)c_k$, $R = [R_1, \ldots, R_{mn}] \in M_{1,mn}(A)$ and $C = [C_1, \ldots, C_{mn}]^\top \in M_{mn,1}(B)$, where $R_{jm+y} := r_y r_j^{(k)}$ and $C_{jm+y} := c_j^{(k)} c_y$ with $r_k = [r_1^{(k)}, \ldots, r_n^{(k)}] \in M_{1,n}(A)$, $r = [r_1, \ldots, r_m] \in M_{1,m}(A)$, $c_k = [c_1^{(k)}, \ldots, c_n^{(k)}]^\top \in M_{n,1}(B)$, and $c = [c_1, \ldots, c_m]^\top \in M_{m,1}(B)$.

If we fill the rows $r_k \in M_{1,n}(A)$ and columns $c_k \in M_{n,1}(B)$ with zeros, then we see that $\mathcal{C}^{(n)} \subseteq \mathcal{C}^{(n+k)}$ for all $n, k \in \mathbb{N}$. Thus $\mathcal{C}_0 := \bigcup_n \mathcal{C}^{(n)}$ is a (not necessarily closed) matrix operator convex cone that is contained in every matrix operator convex cone $\mathcal{C}$ with $\mathcal{S} \subseteq \mathcal{C}$.

Since A has an approximate unit $\{e_\tau\}$ of positive contractions, we get $\|W\| = \|\sum_k c_k^* V_k(r_k^* r_k)c_k\|$ for $W \in \mathcal{C}_0$, and that every element $V \in \mathcal{S}$ is in the point-norm closure of $\mathcal{C}^{(1)} \subseteq \mathcal{C}_0$.

By part (ii), the point-norm closure $\mathcal{C}_1$ of $\mathcal{C}_0$ is a m.o.c. cone that contains $\mathcal{S}$. Since $\mathcal{C}_0$ is contained in every point-norm closed m.o.c. cone $\mathcal{C}$ with $\mathcal{S} \subseteq \mathcal{C}$, we get $\mathcal{C}(\mathcal{S}) = \mathcal{C}_1$. The contractions T in the point-norm closure of $\mathcal{C}_0$ can be approximated by contractions $W \in \mathcal{C}_0$ with $\|W\| \leq 1$ by Lemma 3.1.8.

(v): Since $\mathcal{S} \subseteq \mathcal{C}(\mathcal{S}) =: \mathcal{C}$, it holds $\Psi_{\mathcal{C}}(J) \subseteq \Psi_{\mathcal{S}}(J)$ for all $J \in \mathcal{I}(B)$, cf. part (vi).

If $a \in \Psi_{\mathcal{S}}(J)_+$ then $V(\exp(ih)a\exp(-ih)) \in J$ for all $h^* = h \in A$ with $\|h\| < \pi$ and for all $V \in \mathcal{S}$.

By part (iv), this implies that $c_\ell^* V(r_\ell^* \exp(ih)a\exp(-ih)r_k)c_k \in J$ for $V \in \mathcal{S}$, $h^* = h \in A$, $r_1, \ldots, r_n \in A$ and $c_1, \ldots, c_n \in B$. Thus, $W(\exp(ih)a\exp(-ih)) \in J$ for all $W \in \mathcal{C} := \mathcal{C}(\mathcal{S})$ and $h^* = h$ by part (iii), i.e., $a \in \Psi_{\mathcal{C}}(J)_+$, and $\Psi_{\mathcal{S}} = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathcal{C}(\mathcal{S})$.

(vi): Straight from definition of $\Psi_{\mathcal{S}}$.

(viii): If $V \in \mathcal{C}_{\Psi_1}$ and $\Psi(J) \subseteq \Psi_1(J)$, then $V(\Psi(J)) \subseteq V(\Psi_1(J)) \subseteq J$. Thus, $V \in \mathcal{C}_\Psi$ if $\Psi(J) \subseteq \Psi_1(J)$ for all $J \in \mathcal{I}(B)$.

(ix): For $J \in \mathcal{I}(B)$ and $a \in \Psi_{\mathcal{C}}(J)$ holds $V(a) \in J$ for all $V \in \mathcal{C}$ by definition of $\Psi_{\mathcal{C}}$.

(x): Let $J \in \mathcal{I}(B)$, and let $a \in \Psi(J)_+ \subseteq A_+$, $V \in \mathcal{C}_\Psi$ and $h^* = h \in A$. Then $V(\exp(ih)a\exp(-ih)) \in J$ by definition of $\mathcal{C}_\Psi$, i.e., $a \in \Psi_{\mathcal{C}_\Psi}(J)_+$ by definition of $\Psi_{\mathcal{C}}$. Thus, $\Psi(J) \subseteq \Psi_{\mathcal{C}_\Psi}(J)$.

(xi): If $\{T_\gamma\}_{\gamma\in\Gamma} \subseteq \mathrm{CP}(A,B)$ is a (norm-)bounded net that converges in point-norm to $T \in \mathcal{L}(A,B)$, then $T \in \mathrm{CP}(A,B)$ and the net

$$\{T_\gamma \otimes^{\max} \mathrm{id}\}_{\gamma\in\Gamma} \subseteq \mathrm{CP}(A \otimes^{\max} D, B \otimes^{\max} D)$$

converges point-wise to $T_\gamma \otimes^{\max} \mathrm{id}$, because $A \odot D$ is dense in $A \otimes^{\max} D$. Thus, if $T_\gamma \otimes^{\max} \mathrm{id} \in \mathcal{C}'$ for all $\gamma \in \Gamma$, then $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$. Clearly, the set $\mathcal{C}$ of $T \in \mathrm{CP}(A,B)$ with $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$ is convex. If $T \otimes^{\max} \mathrm{id} \in \mathcal{C}'$, $r \in M_{1n}(A)$, $c \in M_{n1}(B)$, $e \in D_+$, then $R := r \otimes e \in M_{1n}(A \otimes D)$, $C := c \otimes e \in M_{n1}(B \otimes D)$, and

$$R^*((T \otimes^{\max} \mathrm{id}) \otimes (\mathrm{id}_n))(C^*(\cdot)C)R = T' \otimes^{\max} S_e$$

for $T' := r^*(T \otimes \mathrm{id}_n)(c(\cdot)c^*)r$ and $S_e(d) := e^2 d e^2$ $(d \in D)$. Since $\mathcal{C}'$ is matrix operator convex and is point-norm closed,it follows that $T' \otimes^{\max} \mathrm{id} \in \mathcal{C}'$. Thus, the set $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ of $T \in \mathrm{CP}(A,B)$ with $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ is a point-norm closed matrix operator convex cone.

(xii): The point-norm closure $\mathcal{C}_0 := \overline{\mathcal{C}}^{\mathrm{norm}}$ closure of $\mathcal{C}$ is a convex cone with $c^*(V \otimes \mathrm{id}_n)(r^*(\cdot)r)c \in \mathcal{C}_0$ for $r \in M_{n,1}(A)$, $c \in M_{1,n}(B)$ and $V \in \mathcal{C}_0$. Since the point-*strong closures of $\mathcal{C}$ and $\mathcal{C}_0$ in $\mathcal{L}(A,M)$ coincide, and since $V_\gamma \otimes \mathrm{id}_n \to W \otimes \mathrm{id}_n$ point-*strongly if $V_\gamma \to W$ point-*strongly , we obtain that the point-*strong closure $\overline{\mathcal{C}} \subseteq \mathcal{L}(A,M)$ is a convex cone that satisfies $c^*(W \otimes \mathrm{id}_n)(r^*(\cdot)r)c \in \mathcal{C}_0$ for $r \in M_{n,1}(A)$, $c \in M_{1,n}(B)$ and $W \in \overline{\mathcal{C}}$. The unit-ball of $M_n(B)$ is *-ultrastrongly dense in the unit-ball of $M_n(M)$ by Kaplansky density theorem ([**616**, Thm. 2.3.3]). Thus, for $c \in M_{1,n}(M)$ there exists a net $c_\gamma \in M_{1,n}(B)$ with $\|c_\gamma\| \leq \|c\|$ that *strongly converges to c. Thus, the point-*strong closure $\overline{\mathcal{C}}$ satisfies (OC2).

`check next argument` ??(xii) If $V_\gamma \in \overline{\mathcal{C}}$ converges in point-$\sigma(M, M_*)$ topology to $W \in \mathrm{Lin}(A, M)$, then $W(e^2) \geq 0$ for every $e \in A_+$. In particular, W is positive and bounded, thus $\limsup_\gamma \|V_\gamma(e^2)\| =: \mu < \infty$ by uniform boundedness theorem (Banach-Steinhaus). It implies that W is also in the point-$\sigma(M, M_*)$ closure of the convex set of $V \in \overline{\mathcal{C}}$ with $\|V\| \leq \mu$. An Hahn-Banach separation argument shows now that $W \in \overline{\mathcal{C}}$.

(xiii): $\Psi^{\mathcal{S}}(\overline{\sum_\gamma I_\gamma}) = \overline{\sum_\gamma \Psi^{\mathcal{S}}(I_\gamma)}$ and $a \in \Psi_{\mathcal{S}}(\Psi^{\mathcal{S}}(I))$ for $a \in I \triangleleft A$ follow straight from the definitions of $\Psi_{\mathcal{S}}$ and $\Psi^{\mathcal{S}}$. □

`Compare also Example 3.8.3!!!`

EXAMPLE 3.12.3. We consider the action $\Psi_{nuc} \colon \mathcal{I}(B) \to \mathcal{I}(A)$ defined by $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ and examine some possible generating subsets $\mathcal{S}_0 \subseteq \mathrm{CP}(A, B)$ for $\mathrm{CP}_{\mathrm{nuc}}(A, B)$.

Let $S \subseteq A_+^*$ denote an "almost separating" set of positive functionals on A that is invariant under inner automorphisms of A, more precisely:
for each $a \in A_+$ there is $\varphi \in S$ and $h^* = h \in A$ with $\|h\| < \pi$ such that with $\varphi(\exp(ih)a\exp(-ih)) > 0$. For example we can take as S the set of pure states of A, or any set S of pure-states φ of A such that the family $\{d_\varphi\}_{\varphi \in S}$ of irreducible representations of A is separating for A.
Furthermore, let $P \subseteq B_+$ a set of positive elements of B that generates B as a closed ideal of B, i.e., for each $c \in B_+$ and $\varepsilon > 0$, there are $b_1, \ldots, b_n \in P$ and $d_1, \ldots, d_n \in B$ with $\|c - \sum_k d_k^* b_k d_k\| < \varepsilon$.

We consider the set of $\mathcal{S}_0 := \mathcal{S}_{S,P} \subseteq \mathrm{CP}(A, B)$ of c.p. maps $V_{\varphi,b}(a) := \varphi(a)b$, where $\varphi \in S$ and $b \in P$, and calculate the action $\Psi_0 \colon \mathcal{I}(B) \to \mathcal{I}(A)$ defined by $\mathcal{S}_0$: We have $\Psi_(\mathcal{S}_0)(B) = A$ by Lemma 3.12.2(iv). Let $J \neq B$ a close ideal of B. Then there is $b \in P$ that is not in J. If $0 \neq a \in A_+$ then there is $h^* = h \in A$ with $\|h\| < \pi$ and $\varphi \in S$ with $\varphi(\exp(ih)a\exp(-ih)) > 0$. It follows that $V_{\varphi,b}(\exp(ih)a\exp(-ih))$ is not contained in J.
Thus $\Psi_{\mathcal{S}_0}(J) = \{0\}$ for every closed ideal $J \neq B$ and $\Psi_{\mathcal{S}_0}(B) = A$.

Since $\mathcal{S}_0 \subseteq \mathrm{CP}_f(A, B) \subseteq \mathrm{CP}_{\mathrm{nuc}}(A, B) \subseteq \mathrm{CP}(A, B)$ we get $\Psi_{\mathrm{CP}(A,B)} = \Psi_{\mathrm{CP}_{\mathrm{nuc}}(A,B)}(J) \subseteq \Psi_{\mathcal{C}_0}(J)$ for all closed ideals J of B. Thus,

$$\Psi_{\mathrm{CP}(A,B)} = \Psi_{\mathrm{CP}_{\mathrm{nuc}}(A,B)} = \Psi_{\mathcal{S}_0}\,.$$

The special case where $A = B$ is not nuclear shows that,
in general, the lower semi-continuous action $\Psi_{\mathcal{C}} \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ defined by a m.o.c.c. $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ does identify $\mathcal{C}$ itself.

The situation of example 3.12.3 changes considerably if we consider, instead of $\Psi_{\mathcal{C}}$, the action $\Psi' \colon \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ that is defined by the point-norm closed m.o.c.c. $\mathcal{C} \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C, C) \subseteq \mathrm{CP}(A \otimes^{\max} C, B \otimes^{\max} C)$, where $C := C^*(F_\infty)$ denotes the full group C^*-algebra over the free group F_∞ on countably many generators and where $\mathcal{C} \otimes^{\max} \mathrm{CP}_{\mathrm{in}}(C, C)$ is generated by the tensor products $V \otimes^{\max} \mathrm{id}_C$ with $V \in \mathcal{C}$. Then the following theorem holds:

THEOREM 3.12.4 (Separation of m.o.c.c.'s). *Let* $\mathcal{S} \subseteq \mathrm{CP}(A, B)$ *a set of c.p. maps, and denote by* $\Psi' : \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ *the action that is defined by the set* $\mathcal{S}'$ *of c.p. maps* $V \otimes^{\max} \mathrm{id}_C : A \otimes^{\max} C \to B \otimes^{\max} C$.

Then a c.p. map $T : A \to B$ *is in the point-norm closed matrix operator-convex cone* $\mathcal{C}(\mathcal{S})$ *generated by* $\mathcal{S}$*, if and only if,* $T \otimes^{\max} \mathrm{id}_C$ *is* Ψ'*-equivariant.*

Compare also Thm. 3.8.4

PROOF. Let $\Psi' := \Psi_{\mathcal{S}'} : \mathcal{I}(B \otimes^{\max} C) \to \mathcal{I}(A \otimes^{\max} C)$ be the l.s.c. action of $\mathrm{Prim}(B \otimes^{\max} C)$ on $A \otimes^{\max} C$ defined by the set $\mathcal{S}'$ of c.p. maps $V \otimes^{\max} \mathrm{id}_C : A \otimes^{\max} C \to B \otimes^{\max} C$ for $V \in \mathcal{S}$, cf. Lemma 3.12.2(iv).

We denote by $\mathcal{C}' \subseteq \mathrm{CP}(A \otimes^{\max} C, B \otimes^{\max} C)$ the cone $\mathcal{C}' := \mathcal{C}_{\Psi'} = \mathrm{CP}(\Psi'; A \otimes^{\max} C, B \otimes^{\max} C)$ of Ψ'-equivariant maps.

Then $\mathcal{C}'$ is a point-norm closed matrix operator convex cone by Lemma 3.12.2(vii).

The set $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ of all $T \in \mathrm{CP}(A, B)$ with $T \otimes^{\max} \mathrm{id}_D \in \mathcal{C}'$ is a point-norm closed matrix operator convex cone that contains $\mathcal{S}$, cf. Lemma 3.12.2(xi), i.e., $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}$ and $(V \otimes^{\max} \mathrm{id}_C)(\Psi'(J)) \subseteq J$ for all closed ideals J of $B \otimes^{\max} C$.

?????????

□

13. Graded m.o.c. cones

Needed topics:

Graded Hilbert B-modules E and interplay with grading on B.

Opposite E^{op} module: Interchange of the $\mathbb{Z}_2$ "eigen" spaces.

Opposite grading on B for E^{op}?

Gradings on E^{op}. Induced grading on $\mathcal{L}(E)$ is given by conjugation with the grading operator α on E:

$T \in \mathcal{L}(E)^{(0)}$ if and only if T commutes with α.

$T \in \mathcal{L}(E)^{(1)}$ if and only if T anti-commutes with $\alpha T \alpha = -T$, i.e., $\alpha T + T\alpha = 0$.

Notation on graded C^*-algebras A:

α or α_A is a C^*-automorphism of order 2.

$A^{(0)} := \{a \in A;\ \alpha_A(a) = a\}$, $A^{(1)} := \{a \in A;\ \alpha_A(a) = -a\}$,

(also denoted by A_0, A_1)

$a \in A^{(0)} \cup A^{(1)}$ are "homogeneous". Depending if $a \in A^{(0)}$ or $a \in A^{(1)}$ for homogeneous $a \in A$ is defined a *degree* $\deg(a) := j =: \partial a$ if $a \in A^{(j)}$.

The graded commutator $[a, b]_{\mathrm{gr}}$ in a graded algebra A with grading automorphism $\alpha \in \mathrm{Aut}(A)$ is defined by $[a, b] = ab - ba$ if $\alpha(a) = a$ or $\alpha(b) = b$ i.e., if

$a, b \in A_0$. Defined in general by:

$$[a,b]_{\text{gr}} := ab - (-1)^{\deg(a)\cdot\deg(b)} ba$$

It gives $[a,b]_{\text{gr}} = ab - ba$ if at least one of a, b has degree zero. if both have degree one, i.e., $\deg(a) = 1 = \deg(b)$, then $[a,b]_{\text{gr}} := ab + ba \in A^{(0)}$.

Gradings on m.o.c. cones $\mathcal{C}$. $\widehat{\mathcal{H}_B}$ for graded B. Effect to passage to opposite grading on B, $\mathcal{H}_B$ and $\widehat{\mathcal{H}_B} := \mathcal{H}_B \oplus \mathcal{H}_B^{op}$.

Related grading on $H_{\mathcal{C}} \colon A \to \mathcal{L}(\mathcal{H}_B)$???

Interplay of grading with on A and B with $\mathrm{CP}(A,B) \subset \mathcal{L}(A,B)$? What kind of compatibility of m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is necessary to get that $H_{\mathcal{C}} \colon A \to \mathcal{L}(\widehat{\mathcal{H}_B})$ becomes grading preserving?

Change on E the "eigen" spaces but not the grading on $\mathcal{L}(E)$?

$E = E^{(0)} \oplus E^{(1)}$ (Is this really the orthogonal sum ?)

The action of graded B should be (all calculated modulo 2):

$E^{(m)} B^{(n)} \subseteq E^{(m+n)}$

and

$E^{(m)}, E^{(n)} \subseteq B^{(m+n)}$

$E^{op} = (E^{op})^{(0)} \oplus (E^{op})^{(1)}$ is same E, but with $(E^{op})^{(0)} := E^{(1)}$ and $(E^{op})^{(1)} := E^{(0)}$.

Means replacing the order-2 grading isomorphism α by $(-1)\cdot\alpha$.

(Don't know what happens if $E^{(1)} = 0$, then only 0 is fixed?)

See sec. 14.2 of Blackadar K-theory.

Standard *odd* grading on $A \oplus A$ is given by $(A \oplus A)^{(0)} = \{(a,a)\,;\ a \in A\}$ and $(A \oplus A)^{(1)} = \{(a,-a)\,;\ a \in A\}$.

If $A = \mathbb{C}$ then $\mathbb{C}_1$ is a Clifford algebra, isomorphic to the group C^*-algebra of $\mathbb{Z}_2$.

Standard *even* grading on $M_2(A)$ is given by conjugation with $\mathrm{diag}(1,-1)$, i.e., $M_2(A)^{(0)}$ are the diagonal matrices, and $M_2(A)^{(1)}$ are the matrices with zero diagonal.

If α is grading on A and β grading on Hilbert B-module E and $\phi \colon A \to \mathcal{L}(B)$ is grading preserving. Then:

(0.) $\beta := -\alpha$ is the opposite grading on E.

(1.) $\psi := \phi \circ \beta \colon A \to \mathcal{L}(E)$ is grading preserving for E^{op} if ϕ was grading preserving for for the grading on $\mathcal{L}(E)$.

(2.) If (E, ϕ, F) is a Kasparov module then $(E^{op}, \psi, -F)$, $(E \oplus E^{op}, \phi \oplus \psi, G(t))$ are Kasparov module for the 2×2-matrix $G(t)$ with entries $G(t)_{11} := \cos(t\pi/2)F$, $G(t)_{22} := -\cos(t\pi/2)F$, $G(t)_{12} := \sin(t\pi/2)$ and $G(t)_{12} := \sin(t\pi/2)$.

The $G(t)$ is a linear combination of $\mathrm{diag}(F,-F) = G(1)$ and $G(0)$. Both have degree 1 under the grading $\mathrm{diag}(\alpha,-\alpha)(\cdot)\,\mathrm{diag}(\alpha,-\alpha)$ in $\mathrm{Lin}(E\oplus E^{op})$. The $G(0)$ commutes with $\phi(a)\oplus\psi(a) = \mathrm{diag}(\phi(a),\psi(a))$ for all $a\in A$ with $\alpha_A(a)=a$, and "anti-commutes" for all $a\in A$ with $\alpha_A(a)=-a$. This is because $\deg(G(0))=1$, $A\in a\mapsto \mathrm{diag}(\phi(a),\psi(a))$ is grading preserving, i.e., $\deg(\mathrm{diag}(\phi(a),\psi(a)))=1$ in $M_2(\mathcal{L}(E))$ if $\deg(\mathrm{diag}(\phi(a),\psi(a)))=1$ and

$$[\,G(0),\mathrm{diag}(\phi(a),\psi(a))\,]_{\mathrm{gr}} = G(0)\cdot\mathrm{diag}(\phi(a),\psi(a)) + \mathrm{diag}(\phi(a),\psi(a))\cdot G(0) = 0$$

In a the graded C^*-algebra $\mathcal{L}(E\oplus E^{op})$ we have the grading automorphism $\alpha(\cdot)\alpha$ for the grading $\alpha := \alpha_E\oplus(-\alpha_E)$ coming from the grading on $E\oplus E^{op}$:

And $\alpha\cdot G(1)\cdot\alpha = -G(1)$???

CHAPTER 4

Comparison and Addition of some C^*-morphisms

Actual plan:

(A) Remove 2-fold explanations/definitions

(B) Finish proof of Thm. 4.4.6.

(C) Outline ''germ'' of un-suspended but stable variant of E-theory. In particular:

(C1) Derive central sequences of copies of $\mathcal{O}_2$ for $\mathcal{O}_2$

(C2) Derive approximate unitary equivalence of all unital endomorphisms of $\mathcal{O}_2$ and of $\mathcal{O}_\infty$.

(C3) Derive needed central sequences from this.

(C4) Also quasi-central asymptotic paths of copies of $\mathcal{O}_2$ and of $\mathcal{O}_\infty$ ... ??

(C?? from Into 1.)

"Squeezing Property" implies K_1-injectivity. See Chp. 4. for definitions!

Here a question remains:

When K_1-injectivity implies the Squeezing Property?

At least in unital separable nuclear case ??

(Would be very nice ! But have no idea about this)

We study here some properties of **Cuntz addition**, that is, a "sum" of morphisms in the category of C^*-algebras, more generally we consider "sums" of general maps between C^*-algebras. This idea of an addition has been used implicitly in the topological K-theory of C^*-algebras since a long time, but without precise description e.g. of the "almost ridiculous " relationship between the isomorphisms of $\mathbb{K}$ with $M_2(\mathbb{K})$ and unital representations of the Cuntz algebra $\mathcal{O}_2$ in $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\mathcal{H})$, needed for precise algebraic definition of equivalence classes of asymptotic and in some sense "equivariant" C^*-morphisms. It was J. Cuntz ([**172**], [**169**]) who pointed out in a more general context that a unital copy of $\mathcal{O}_2$ or at least of $\mathcal{O}_\infty$ is needed to adjust them carefully. The natural transformations between those copies of $\mathcal{O}_2$ are present for our definitions of sorts of equivariant theories by actions of groups, ideal-lattices or categories of matrix-convex cones.

The point is that we use them later to describe the constructive aspects of the used Ext-, KK-theories and unsuspended variants of cone-depending $\mathcal{E}$-theory that corresponds to them.

The general notion of Cuntz addition is related to K_*-theory (cf. Lemma 4.2.6 and Proposition 4.4.3). It is taylor-made for our study of nuclear asymptotic C^*-morphisms into non-simple purely infinite algebras in Chapters 3, 5 and 7. And it helps to describe the construction of our more general $\mathrm{KK}(\mathcal{C}\,;\cdot,\cdot)$-theory for operator-convex cones $\mathcal{C}$ in Chapter 8. The Cuntz addition is neither a direct product nor a direct sum in the usual sense of category theory – here on the category of C^*-algebras. This restricts "straight borrowing" from category theory analogies.

The considered situations and constructed objects appear only modulo various ideals in its applications in Chapters 5, 7, 8 and 9. We apply the here proved results, after dividing by suitable ideals in question. But this has to be done with some care:

Two C^*-morphisms $h_1, h_2 \colon D \to E$ into a unital C^*-algebra E can become unitarily equivalent after dividing out an ideal J of E. In general, then there is no unitary u in the original algebra with $h_1(a) - u^* h_2(a) u \in J$ for all $a \in D$. But in the particular situation where this unitary equivalence in E/J can be realized by a unitary in the connected component $\mathcal{U}_0(E/J)$ of 1 in $\mathcal{U}(E/J)$, then this unitary can be lifted to a unitary in E. This is, for example, the case if there exists an isometry $r \in E/J$ with $r^* \pi_J(h_1(d) + h_2(d)) r = 0$ for $d \in D$, cf. Proposition 4.3.6(iv). It applies e.g. to equivalence of stable extensions, cf. Chapter 5, and to asymptotic C^*-morphisms into stable σ-unital algebras, cf. Chapter 7 and Corollary 4.6.7.

Recall here that $\mathcal{U}(E)$ denotes the unitary group of a unital C^*-algebra E and $\mathcal{U}_0(E) \subseteq \mathcal{U}(E)$ the connected component of $1 \in \mathcal{U}(E)$.

We consider also some real C^*-algebras and real versions $E = E_{\mathbb{R}}$ of C^*-algebras E, e.g. the real versions $(\mathcal{O}_2)_{\mathbb{R}}$, $(\mathcal{E}_2)_{\mathbb{R}}$ and $C^*(P,Q,1)_{\mathbb{R}}$ of $\mathcal{O}_2$, $\mathcal{E}_2$ and the universal unital (complex) C^*-algebra $C^*(P,Q,1)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ generated by two projections.

We do not discuss KK-theory of real C^*-algebras here, but it seems natural to use the real basic algebras, because the complex versions are often just natural complexification of the real version. In the real cases $E = E_{\mathbb{R}}$, $(\mathcal{O}_2)_{\mathbb{R}}$ and $(\mathcal{E}_2)_{\mathbb{R}}$ we use also the notations "unitary", $\mathcal{U}(E)$ and $\mathcal{U}_0(E)$ instead of "orthogonal operator", $O(E)$ and $SO(E)$, for the orthogonal operators $u \in E$, respectively for $u \in E$ in the connected component of 1 in $O(E)$.

Formulas that cover also the "real" case look sometimes more complicate.

If the reader wants simpler formulas for complex C^*-algebras E then he can replace e.g. the order 4 unitary

$$(1 - zz^* - z^*z) + (z^* - z) = \exp((\pi/2)(z^* - z)) \in \mathcal{U}_0(E)$$

for partial isometries z with $z^2 = 0$ by the symmetry $(1 - zz^* - z^*z) + (z^* + z)$. Symmetries are not necessarily in $\mathcal{U}_0(E)$ if E is a real C^*-algebra. But there are also real C^*-algebras where every symmetry is connected inside the orthogonal operators with 1 by a path of orthogonal operators. Examples are all real unital

C^*-algebras that have a central sequence of unital copies of $M_2(\mathbb{R})$, i.e., that absorb the real UHF algebra $M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes \cdots$ tensorial.

This is one of the reasons that we try to express canonical transformations as products of Halmos unitaries (defined in Remark 4.2.4), because those are also for real unital C^*-algebras A connected to 1 in $\mathcal{U}(A)$.

We need the verification of the homotopy invariance for definitions and constructions, those "soft" homotopy invariances will be later used to prove "hard" *unitary* homotopy in the sense of Definition 5.0.1. The passage to unitary homotopy is a basic tool in our proofs of classification results from stable homotopy coming from KK_X-equivalences or from the more general $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$-equivalences.

The results of Lemmata 4.2.6, 4.2.10 and Propositions 4.2.11, 4.2.15, 4.3.6 and 4.4.3 will be used in the proofs in Chapters 5, 7, 8, 9 and 11. Theorem 4.4.6 plays a crucial role in the proof of the first parts of Theorems B and M, that are given in Chapters 6 and 9 – in case of Theorem M with an additional separation assumption that will be removed later in Chapter 12 by the last step in the proof of Theorem K that uses the very special case of Theorem M proven before in Chapter 6.

DEFINITION 4.0.1. Let E a C^*-algebra such that its multiplier algebra $\mathcal{M}(E)$ contains a copy of $\mathcal{O}_2$ unitally. For a fixed pair of generators s_1, s_2 of $\mathcal{O}_2$ in $\mathcal{M}(E)$, we define the **Cuntz addition** $\oplus_{s_1,s_2}$ in E as follows:

$$a \oplus_{s_1,s_2} b \;=\; s_1 a s_1^* + s_2 b s_2^* \,; \quad a, b \in E\,.$$

We define and use in the "overview" Lemma 4.2.6 also a generalization of this addition, that is more natural, up to certain transformations and up to uniqueness depending from K_1-injectivity. A careful study of all transformations is in particular necessary if one wants to go into further research concerning equivariant actions of lattices, groups, quantum groups, groupoids, etc. After discussing some basic material on absorbing elements in Abelian semigroups and compatibility of projections, we study some less trivial properties of the generalized Cuntz addition of C^*-morphisms. This is not new, but gives a constructive "algebraic" base for our in applications used versions of K-theory, cone-depending KK-theory and unsuspended "continuous" $\mathcal{E}$-theory.

REMARK 4.0.2. The above defined *Cuntz addition* is a unital *-monomorphism from $E \oplus E$ into E. It is, up to unitary equivalence in E, independent of the choice of copies of $\mathcal{O}_2$: If s_1, s_2 and t_1, t_2 are generators of two copies of $\mathcal{O}_2$ in E, then

$$a \oplus_{s_1,s_2} b \;:=\; u(a \oplus_{t_1,t_2} b)u^* \qquad \text{for all} \;\; a, b \in E$$

where $u \;:=\; s_1 t_1^* + s_2 t_2^* \in E$ is a unitary.

More generally, one can add n elements of E along generators of a unitally contained copy of $\mathcal{O}_{\mathrm{n}}$ in E with generators $s_1, \ldots, s_n$ in the natural way:

$$(a_1, a_2, \ldots, a_n) \mapsto \sum s_i a_i s_i^*\,.$$

As before the sum is independent of the choice of generators – up to unitary equivalence by a fixed unitary. Notice that $\mathcal{O}_2$ contains all Cuntz algebras $\mathcal{O}_n$, $n \in \mathbb{N}$, unitally, e.g. $\mathcal{O}_3 \cong C^*(s_1, s_2 s_1, s_2^2)$. (In fact every separable unital exact C^*-algebra is isomorphic to a unital C^*-subalgebra of $\mathcal{O}_n$ by – cite: Kirchberg-Phillips.)

Computation shows that adding along $\mathcal{O}_n$ is the same as successively adding along $\mathcal{O}_2$ – up to unitary equivalence. In particular, Cuntz addition is associative and commutative up to (global) unitary equivalence.

The uniqueness up to unitary equivalence allows us to write sometimes simply $\oplus$ instead of $\oplus_{s_1,s_2}$ if we consider unitary equivalence classes and different unital copies of $\mathcal{O}_2$ in a given C^*-subalgebra of E.

The generators of a copy of $\mathcal{O}_2$ define an "almost inner" *-isomorphism ψ from $M_2(E)$ onto E, by

$$\psi\colon [a_{ik}] \in M_2(E) \mapsto v[a_{ik}]v^\top = \sum_{ik} s_i a_{ik} s_k^* \in E\,,$$

where v is the row matrix $v := [s_1, s_2] \in M_{1,2}(E)$.

There exists a unitary matrix $V \in M_4(E)$ (a Halmos unitary as in Remark 4.2.4) such that $V^* \operatorname{diag}(p;\, 0, 0)V = \operatorname{diag}(\psi(p), 0;\, 0, 0)$ for all projections $p \in M_2(E)$ and $V^* \operatorname{diag}(u; 1, 1)V = \operatorname{diag}(\psi(u), 1; 1, 1)$ for all unitaries $u \in M_2(E)$, cf. proof of Lemma 4.2.6(v,2).

Therefore, equivalence and homotopy in $M_{2^n}(E)$ can be over-carried to equivalence and homotopy in E itself.

In some calculations it plays a role if this can be made only up to unitary equivalence or, moreover, up to unitary homotopy. This difference has usually to do with problems related to K_1-injectivity and has forced us to show at least all (not necessarily simple) purely infinite unital C^*-algebras are K_1-injective, cf. Proposition 4.2.15(b), respectively Lemma 4.2.13 for the "squeezing property" of Definition 4.2.14 that induces K_1-injectivity by Proposition 4.2.15(?).

discover what???

!!! HIER BIN ICH !!!!

KUERZE DAS ZEUG ODER GEBE GENAUE DEF'S, FORMELN AN ...

A detailed discussion of the required transformation and precise conditions for uniqueness – up to unitary homotopy – will be given in the proofs of Lemma 4.2.6, that considers a generalized version of Cuntz addition on A in the case where here $E := \mathcal{M}(A)$ is properly infinite.

The below given Lemmata 4.1.3, 4.2.3 and 4.2.6 collect some basic facts concerning the description of $\mathrm{K}_*(E)$ for properly infinite unital C^*-algebras E by Murray–von Neumann equivalence classes of properly infinite projections in E (respectively by homotopy classes of unitaries in E), and they rephrase some technics appearing in proofs of [**172**] and of [**180**]. In particular the notions of absorption and domination will be discussed in connection with some technics to surround a possibly *missing* K_1-injectivity for some properly infinite C^*-algebra.

We show that all purely infinite unital C^*-algebras have the more general "squeezing property" of Definition 4.2.14 that induces K_1-injectivity by ???????

1. Two projections in C^*-algebras

We have often, but sometimes only "implicit", to do with two projections and their relative positions, e.g. with the open support projections of two closed left ideals, or with positions of two hereditary C^*-subalgebras of a C^*-algebra A. We consider in this Section 4.1 only selfadjoint projections, i.e. in the following a "*projection*" p in a C^*-algebra $A \subseteq \mathcal{L}(\mathcal{H})$ means always an orthogonal projection on a real or complex Hilbert space $\mathcal{H}$, i.e., $p = p^*p$. Then $0 \leq p = p^2$ and we denote the set of projections in A by $\mathrm{Proj}(A)$. The group of unitary elements in the unitization $\widetilde{A}$ will be denoted by

Or if we study orthogonality of positive elements of A ...

This ??????

Projections $p, q \in A$ (respectively unitaries u, v in the unitization $\widetilde{A}$ of A) are *homotopic* — denoted by $p \sim_h q$ (respectively $u \sim_h v$) —, if there is a (continuous) path $t \in [0,1] \mapsto p(t) \in \mathrm{Proj}(A)$ (respectively $t \in [0,1] \mapsto u(t) \in \mathcal{U}(\widetilde{A})$) such that $p(0) = p$ and $p(1) = q$ (respectively $u(0) = u$ and $u(1) = v$).

Here $a \sim_{MvN} b$ for $a, b \in A_+$ means **Murray–von-Neumann equivalence** – also denoted by $[a]_{MvN} = [b]_{MvN}$ or simply by $a \sim b$ – and is defined as the existence of $d \in A$ with $d^*d = a$ and $dd^* = b$.

The notation $p \sim q$ (more precisely $p \sim_{MvN} q$) for projections $p, q \in A$ denotes Murray–von-Neumann equivalence, i.e., the existence of $v \in A$ with $v^*v = p$ and $vv^* = q$. Recall that this implies $[p] = [q]$ in $\mathrm{K}_0(A)$.

`Beginning from HERE some text of old Appendix has to be integrated !!!`
`Ends up ????`

Integrate next into the further below considerations!!!

LEMMA 4.1.1. *Suppose that A is a C^*-algebra, and let $p, q \in A$ projections.*

(i) *If $\|pq\| < 1$, then 0 is isolated in $\mathrm{Spec}(p+q)$. And $q \sim_h (1_B - p)$ inside the* unital *C^*-subalgebra $B := C^*(p, q) \subseteq \overline{(p+q)A(p+q)}$.*
(Is identical with $(p+q) - \xi 1_{\mathcal{M}(A)}$ invertible in $\mathcal{M}(A)$ for all non-zero $\xi \in \mathbb{C}$ with $0 < |\xi| <$???? for small ????. Explicit bound ????)

(ii ???) *There is a (norm-continuous) path $[0,1] \ni t \mapsto u(t) \in \mathcal{U}(\widetilde{A})$ with $u(0) = 1$ and $u(1)^*pu(1) = q$, if and only if, $p \sim_h q$ in A.*
(If A is unital, then a path $t \mapsto u(t)$ with $u(0) = 1$ and $u(1)^*pu(1) = q$. can be found in $\mathcal{U}(A)$ itself.)

(ii,iii): See [**692**, prop. 2.2.6]*, or* [**73**, prop. 4.3.3, prop. 4.6.3]*, or* [**207**, prop. IV.1.2, lem.IV.1.4].

(iii ???) *If* $\|pq\| < 1$*, and if there exists a partial isometry* $z \in A$ *with* $z^*z = p$ *and* $zz^* = q$*, then* $p \sim_h q$*.*

Proof: (ii,iii): See [**692**, prop. 2.2.6], or [**73**, prop. 4.3.3, prop. 4.6.3], or [**207**, prop. IV.1.2, lem.IV.1.4].

(iii): By (i), there is a path $t \mapsto q(t) \in \operatorname{Proj}(A)$ with $q(0) = q$ and $q(1)p = 0$, and by (ii), there is a unitary $u \in \mathcal{U}_0(\widetilde{A})$ with $u^*qu = q(1)$. Then $w := u^*z$ satisfies $w^*w = p$ and $ww^* = q(1)$. Thus, $s := w^* + w + (1 - p - q(1))$ is a self-adjoint unitary in $\widetilde{A}$ with $sps = q(1)$. Hence, $(us)p(us)^* = q$ and $us \in \mathcal{U}_0(\widetilde{A})$. If $t \mapsto U(t) \in \mathcal{U}(\widetilde{A})$ is a path with $U(0) = 1$ and $U(1) = (us)^*$, then $t \to p(t) := U(t)^*pU(t)$ is the desired path, and $p \sim_h q$.

Recall that a hereditary C^*-subalgebra $D \subseteq A$ is *full* in A if the ideal of A generated by D is dense in A. An element $b \in A$ is *full* in A if the linear span of AbA is dense in A, i.e., if $A = J(b)$ for the closed ideal $J(b)$ of A generated by $\{b\}$. In particular, a projection $p^*p = p \in A$ is a full projection of A if the hereditary C^*-subalgebra pAp of A is full in A.

REMARK 4.1.2. Let $D \subseteq A$ a *full* hereditary C^*-subalgebra of A, then the inclusion map $\eta\colon a \in D \mapsto a \in A$ defines an isomorphism η_* from $\mathrm{K}_*(D)$ onto $\mathrm{K}_*(A)$.

In particular, $[q_1] = [q_2]$ in $\mathrm{K}_0(D)$ if $q_1, q_2 \in M_n(D)$ and there are projections $p_1, p_2 \in A$ with $[p_1] = [p_2]$ in $\mathrm{K}_0(A)$ and $p_k \sim_{MvN} q_k$ in $M_n(A)$ for $k = 1, 2$.

The study of sufficient and necessary criteria for K_1-injectivity of a given properly infinite unital C^*-algebra A requires some basics on pairs of projections $p, q \in A$ that are "almost compatible" in the sense that $\min(\|(1-q)p\|, \|(1-p)q\|) < 1$. The notation M_2 means in the following lemma the algebras $M_2(\mathbb{R})$ or $M_2(\mathbb{C})$ – depending if we consider a real or complex C^*-algebra $C^*(p, q, 1)$ generated by projections p, q. The Lemma itself should be "folklore", but since we use it in several proofs for estimates and check of "controlled" homotopy we give a careful detailed and elementary proof. Some of it can be also obtained by observing first that the "universal" unital C^*-algebra $C^*(P, Q, 1) := C^*(P, Q, 1;\ P^*P = P,\ Q^*Q = Q)$ is naturally isomorphic to the group C^*-algebra $C^*(\mathbb{Z} \rtimes \mathbb{Z}_2)$. The viewpoint of Lemma 4.1.3 is to derive algebraic and geometric informations from the behaviour of the norms (i.e., the "distance") about the kind of algebra they generate and how looks a unitary that transforms the symmetries into each other. We extract rules for moving near elements with help of a path in the inner automorphisms. A mainly operator theoretic and algebraic big overview is contained in [**101**], but there with some different terminology.

LEMMA 4.1.3. *Let* A *a real or complex unital* C^**-algebra and* $p, q \in A$ *projections.*

(i) *The C^*-algebra $C^*(p,q,1)$ has a character χ with $|\chi(p-q)| = 1$ and $\chi(pq) = 0$, if and only if, $\|p-q\| = 1$, if and only if, $\|(1-q)p\| = 1$ or $\|(1-p)q\| = 1$.*

If $\|p-q\| < 1$ then $\|p-q\| = \|(1-q)p\| = \|(1-p)q\|$. In particular, always $\|p-q\| = \max(\|(1-q)p\|, \|(1-p)q\|)$.

In more detail: If $\|(1-q)p\| < 1$ then the following properties (a)–(d) *are equivalent:*

(a) $\|(1-p)q\| = 1$.

(b) $\|p-q\| = 1$.

(c) *There exist a projection $p_0 \neq 0$ in $C^*(p,q)$ with $q_0 \leq 1-p$, $q_0 \leq q$.*

(d) *There exists a character χ on $C^*(p,q,1)$ with $\chi(q) = 1$ and $\chi(p) = 0$.*

(ii) *Always* $\mathrm{Spec}(p+q) \subseteq \{0\} \cup [1-\|pq\|, 1+\|pq\|]$.

In particular, if $\|pq\| < 1$ then 0 is isolated in $\mathrm{Spec}(p+q)$*, and $q \sim_h (1_B - p)$ inside the* unital *C^*-subalgebra $B := C^*(p,q) \subseteq \overline{(p+q)A(p+q)}$. The hereditary C^*-subalgebra $\overline{(p+q)A(p+q)}$ of A – generated by $p+q$ – is unital and coincides with the algebraic *-algebra $(p+q)A(p+q)$ with unit $g_\delta(p+q) \leq \delta^{-1} \cdot (p+q)$, where $\delta := (1-\|pq\|)/2$ and $g_\delta(t) :=$* $\min(1, \max(0, t-\delta))$.

(iii) *The* universal *unital real (or complex) C^*-algebra $C^*(P,Q,1)$ generated by two projections P, Q is naturally isomorphic to the C^*-subalgebra of* $\mathrm{C}([0,\pi/2], M_2)$*, that is generated by $1 := 1_2 \in M_2$, $P =: e_{11} = [\eta_{jk}] \in M_2$ with $\eta_{11} = 1$ and $\eta_{jk} = 0$ for $(j,k) \neq (1,1)$, and $Q := \exp(-H)P\exp(H)$, where $H \in \mathrm{C}([0,\pi/2], M_2)$ is given by $H(\varphi) := \varphi \cdot Z$ ($\varphi \in [0,\pi/2]$) for $Z := [\zeta_{jk}] \in M_2$ with $\zeta_{jk} := j-k$, $j,k \in \{1,2\}$.*

(iv) *If A is unital and $\|(1-q)p\| < 1$, then exist $h \in C^*(p,q) \subseteq A$ and a projection $q_0 \in C^*(p,q)$ with $q_0 p = 0$, $php = 0 = (1-p)h(1-p)$, $h^* = -h$ ([1]), and $\|h\| = \arcsin \|(1-q)p\| < \pi/2$, such that $hq_0 = q_0 h$ and*

$$\exp(-h)p\exp(h) + q_0 = \exp(-h)(p+q_0)\exp(h) = q\,.$$

The projection q_0 is non-zero if and only if $\|(1-p)q\| = 1$.

(v) *If A is unital and $\|p-q\| < 1$ then there exists $h \in C^*(p,q) \subseteq \overline{(p+q)A(p+q)}$ with $h^* = -h$ such that $\exp(-h)p\exp(h) = q$, $\|h\| =$* $\arcsin\|p-q\| < \pi/2$ *and $php = 0 = (1-p)h(1-p)$.*

(vi) *If $\|pq\| < 1$ and $\|p-q\| < 1$, then there exists $h \in (p+q)A(p+q)$ with $h^* = -h$, $php = 0 = (1-p)h(1-p)$ and $\|h\| = \arcsin\|p-q\|$ such that $\exp(h) \in (p+q)A(p+q)$ and $q = \exp(-h)p\exp(h)$.*

(old i) *If $\|pq\| < 1$, then 0 is isolated in* $\mathrm{Spec}(p+q)$*. And $q \sim_h (1_B - p)$ inside the* unital *C^*-subalgebra $B := C^*(p,q) \subseteq \overline{(p+q)A(p+q)}$.*

(Is identical with $(p+q) - \xi 1_{\mathcal{M}(A)}$ invertible in $\mathcal{M}(A)$ for all non-zero $\xi \in \mathbb{C}$ with $0 < |\xi| <$???? for small ????. Explicit bound????)

(old i and ii ???) *There is a (norm-continuous) path $[0,1] \ni t \mapsto u(t) \in \mathcal{U}(\widetilde{A})$ with $u(0) = 1$ and $u(1)^* p u(1) = q$, if and only if, $p \sim_h q$ in A.*

[1] Thus, if A is complex then $(ih)^* = ih$.

(If A is unital, then a path $t \mapsto u(t)$ with $u(0) = 1$ and $u(1)^* p u(1) = q$. can be found in $\mathcal{U}(A)$ itself.)

The elementary realization $C^*(P, Q, 1) \subset \mathrm{C}([0, \pi/2], M_2)$ of the universal unital C^*-algebra generated by two projections – as defined in Part (iii) – and its properties will be used in the proofs of Parts (iv) and (vi) and on other places, e.g. Remark 4.5.7. Compare Remark 4.5.8 for an alternative approach if A is complex.

Proof of Lemma 4.1.3. (i): The C^*-subalgebra $C^*(p, q) \subseteq A$ generated by p and q is an ideal of $C^*(p, q, 1)$. If $p+q$ is not invertible in $C^*(p, q, 1)$ then $C^*(p, q)$ is the kernel of the character χ_0 on $C^*(p, q, 1)$ with $\chi_0(p) = \chi_0(q) = 0$. Notice that $(p + q)/2$ is a strictly positive contraction of the C^*-algebra $C^*(p, q)$.

The W*-algebras generated by two projections $p^* = p = p^2$ and $q^* = q = q^2$ are described in [**767**, chp.V, thm. 1.41]. It implies in particular the below listed observations concerning the characters and the irreducible representations of any unital C^*-algebras $C^*(p, q, 1)$ generated by two projections p, q.

It does not matter here if we consider here real or complex C^*-algebras.

Real or complex unital C^*-algebras $C^*(p, q, 1)$ generated by (self-adjoint) projections p, q have only 1- or 2-dimensional irreducible *-representations.

The algebra $C^*(p, q, 1)$ admits at most 4 different characters.

Alternatively, one can obtain the above and below listed properties of $C^*(p, q, 1)$ by straight calculation from the identities $(p - q)^2 p = p(p - q)^2$ and $(p - q)^2 q = q(p - q)^2$, i.e., from the fact that the positive self-adjoint contraction $(p - q)^2$ is always in the center of $C^*(p, q, 1)$.

The selfadjoint contraction $(D(p) - D(q))^2$ is necessarily a non-negative scalar for each irreducible representation $D\colon C^*(p, q, 1) \to \mathcal{L}(\mathcal{H})$, which implies immediately that the Hilbert space $\mathcal{H}$ has dimension ≤ 2. It has dimension $= 2$ if and only if the projections $D(p)$ and $D(q)$ do not commute. The latter can only happen if the projections $D(p)$ and $D(p)$ have rank one and $D(pqp) = \alpha D(p)$ for some $\alpha \in (0, 1)$. The scalar α defines our parameter $\varphi := \arccos(\alpha^{1/2}) = \arcsin \|D(p - q)\|$ that determines D up to unitary equivalence (respectively *orthogonal* equivalence in the real case).

The parameter

$$\varphi := \arcsin \|D(p(1 - q))\| = \arccos \|D(pq)\| \ \in \ (0, \pi/2)$$

determines all *unitary* equivalence classes of 2-dimensional irreducible *-representations $D\colon C^*(p, q, 1) \to M_2$ uniquely, because $\arccos \|rs\|$ determines all pairs (r, s) of *non-commuting* orthogonal projections $r, s \in M_2$ (necessarily both of rank one) up to unitary equivalence in M_2:

If $r, s \in M_2$ are projections that do not commute, there exists $u \in \mathrm{O}(2)$ (respectively $u \in \mathcal{U}(2)$) such that $u^* r u = P := e_{11} := [\eta_{jk}]$ with $\eta_{11} = 1$ and $\eta_{jk} = 0$ for $(j, k) \neq (1, 1)$ and that $u^* s u = [\alpha_{jk}]$ with real $\alpha_{12} < 0$.

It follows that there is unique $\varphi \in (0, \pi/2)$ such that

$$u^* s u = Q(\varphi) := [\cos(\varphi), -\sin(\varphi)]^\top \cdot [\cos(\varphi), -\sin(\varphi)] .$$

Thus, $P - Q(\varphi) = \sin(\varphi) \cdot T$ where $T^* = T = [t_{jk}]$ with $t_{11} = \sin(\varphi) = -t_{22}$, $t_{1,2} = t_{2,1} = \cos(\varphi)$ for some $\varphi \in (0, \pi/2)$. It follows $T^2 = 1_2$ and $(P - Q(\varphi))^2 = \sin(\varphi)^2 1_2$, $u^*(r - rsr)u = P - PQ(\varphi)P = \sin(\varphi)^2 P$ and $u^*(1-r)s(1-r)u = \sin(\varphi)^2(1-P)$. Thus, if $r, s \in M_2$ are projections with $rs \neq sr$ then $0 < \|r - s\| < 1$ and $\varphi := \arcsin(\|r - s\|) \in (0, \pi/2)$ satisfies

$$\sin(\varphi) = \|r - s\| = \|(1-s)r\| = \|(1-r)s\| .$$

Let $p, q \in A$ projections with $p \neq q$. Since e.g. $(1-q)(p-q) = (1-q)p$, we get that $\|p - q\| \geq \max\big(\|(1-q)p\| , \|(1-p)q\| \big)$. If $\|(1-q)p\| = 0 = \|(1-p)q\|$, then $p \leq q$ and $q \leq p$, i.e., $\|p - q\| = 0$.

If $0 < \|p-q\| < 1$, then there exists a pure state ρ on $C^*(p, q, 1)$ with $|\rho(p-q)| = \|p - q\|$.

Since $0 < |\rho(p - q)| < 1$, the state ρ can not be multiplicative on $C^*(p, q)$, i.e., the corresponding irreducible representation is necessarily 2-dimensional.

Let $D \colon C^*(p, q, 1) \to M_2$ the irreducible GNS representation of $C^*(p, q, 1)$ defined by ρ and let λ a state on M_2 with $\rho = \lambda \circ D$. Then we get that $r := D(p)$ and $s := D(q)$ satisfy

$$|\lambda(r - s)| \leq \|r - s\| \leq \|p - q\| = |\rho(p - q)| = |\lambda(r - s)| ,$$

i.e., $\|r - s\| = \|p - q\| < 1$. It leads to

$$\|(1-q)p\| \leq \|p - q\| = \|r - s\| = \|(1-s)r\| \leq \|(1-q)p\|$$

and

$$\|(1-p)q\| \leq \|p - q\| = \|r - s\| = \|(1-r)s\| \leq \|(1-p)q\|$$

by the above mentioned properties of non-commuting projections (r, s) in M_2.

If $\|p - q\| = 1$ then $C^*(p - q, 1)$ has a character ρ with $|\rho(p - q)| = 1$. Since $\chi(1) = 1$ and $\|\chi\| \leq 1$, the character ρ extends to a state χ on $C^*(p, q, 1)$ by Hahn-Banach extension. The state χ satisfies $\chi(p), \chi(q) \in [0, 1]$ and $|\chi(p) - \chi(q)| = 1$. It follows that $\chi(p), \chi(q) \in \{0, 1\}$ and $\chi(p)\chi(q) = 0$. In particular, 1, p and q are in the multiplicative domain of the unital completely positive map χ from $C^*(p, q, 1)$ into $\mathbb{R}$ (respectively into $\mathbb{C}$), because in both cases $\chi(p)^2 = \chi(p) = \chi(p^2)$ and $\chi(q)^2 = \chi(q^2)$.

It follows that χ is a character on $C^*(p, q, 1)$ with $|\chi(p) - \chi(q)| = 1$.

Thus, χ is a character on $C^*(p, q, 1)$ with $\chi(p - q) \in \{+1, -1\}$. This implies that $(\chi(p), \chi(q))$ is equal to $(1, 0)$ or to $(0, 1)$. In particular, $1 = \chi(p - pqp)$ or $1 = \chi(p - qpq)$. The latter implies $1 = \|p - pqp\| = \|(1-q)p\|^2$ or $1 = \|q - qpq\| = \|(1-p)q\|^2$.

It yields that $\|p - q\| = \max(\|(1-p)q\|, \|(1-q)p\|)$ and that $\|(1-p)q\| = \|(1-q)p\|$ if $\|p - q\| < 1$.

We show the equivalence of the properties (a)–(d) in Part (i) under the pre-assumption $\|(1-q)p\| < 1$, in row (a)→(b)→(d)→(a)→(c)→(a), where (b)→(d) is a consequence of the pre-assumption, and only (a)→(c) is non-trivial.

(a)⇒(b): Since $(1-p)q = -(1-p)(p-q)$, $1 \geq \|p-q\| \geq \|(1-p)q\|$. Thus $\|(1-p)q\| = 1$ implies $\|p-q\| = 1$.

(b)⇒(d): Above we have seen that $\|p-q\| = 1$ implies the existence of a character χ on $C^*(p,q,1)$ with $|\chi(p) - \chi(q)| = 1$. It follows that $(\chi(p), \chi(q))$ is equal to $(0,1)$ or to $(1,0)$ and implies that $\max(\chi(p), \chi(q)) = 1$ and $\chi(qp) = 0$. Since

$$1 > \|(1-q)p\| \geq |\chi((1-q)p)| = |\chi(p)|\,,$$

only the cases $\chi(p) = 0$ and $\chi(q) = 1$ remain.

(d)⇒(a): Suppose that there exists a character χ on $C^*(p,q)$ with $\chi(p) = 0$ and $\chi(q) = 1$. Then $\|(1-p)q\| = 1$, because

$$1 = 1 - \chi(p) = \chi(q - qpq) \leq \|q - qpq\| = \|(1-p)q\|^2 \leq 1\,.$$

(a)⇒(c): Suppose that $\|(1-p)q\| = 1$. Then $\|q - qpq\| = \|(1-p)q\|^2 = 1$ and the positive element qpq of the C^*-algebra $qC^*(p,q,1)q$ has necessarily 0 in its spectrum:

$$0 \in \operatorname{Spec}(qpq, qC^*(p,q,1)q)\,.$$

Since our pre-assumption is equivalent to $\|1 - (pqp + (1-p))\| = \|(1-q)p\|^2 < 1$, it follows that the positive elements $T := pqp + (1-p)$ and $Tp = pTp$ are invertible in $C^*(p,q,1)$ respectively in $pC^*(p,q,1)p$. Then $S := T^{-1/2} \in C^*(p,q,1)$ is a (bounded) positive invertible element with

$$\|T\|^{-1} \cdot 1 \leq S^2 \leq \|T^{-1}\| \cdot 1\,.$$

Since $T(1-p) = (1-p) = (1-p)T$, the positive operator S commutes with p and $S(1-p) = 1-p$.

The element $v := qpS$ satisfies $v^*v = p$ and $r := vv^* \leq q$, i.e., v is a partial isometry and r a projection in $qC^*(p,q,1)q$.

Moreover, using that $\|T\|^{-1} \leq S^2 \leq \|T^{-1}\|$ by definition of S,

$$\|T\|^{-1} qpq \leq vv^* \leq \|T^{-1}\| qpq\,.$$

Thus $qpq \leq r$ and qpq is invertible in $rC^*(p,q,0)r$ and 0 is isolated in the spectrum of qpq in $qC^*(p,q,1)q$ with non-zero support projection $q_0 := q - r$. The projection q_0 satisfies $q_0 p = 0$, because $T^{-1}pqp = p$ and $v = qpS$ satisfy

$$pq_0p = p(q - vv^*)p = pqp - pqpS^2pqp = pqp - pqpT^{-1}pqp = 0\,.$$

(c)⇒(a): If $p_0 \in C^*(p,q)$ is a non-zero projection with $q_0 \leq 1-p$ and $q_0 \leq q$, then $q_0(1-p)q = q_0$. It follows that $1 \geq \|(1-p)q\| \geq \|q_0\| = 1$.

(ii): $\operatorname{Spec}(p+q) \subseteq \{0\} \cup [1 - \|pq\|, 1 + \|pq\|]$ can be seen from the spectrum of the Murray–von-Neumann equivalent 2×2-matrix $X := [p,q]^\top \cdot [p,q]$, because $[p,q] \cdot [p,q]^\top = (p+q) \oplus 0$ in $M_2(C^*(p,q))$. The projection $Y := \operatorname{diag}(1-p, 1-q)$

satisfies $YX = 0 = XY$ and $X + Y = 1_2 + T$ for the self-adjoint 2×2-matrix $T := [\beta_{jk}]$ with $\beta_{11} = \beta_{22} = 0$ and off-diagonal entries $\beta_{12} := pq$, $\beta_{21} := qp$. Thus $p+q$ has – up to the possible spectral values $\{0\,,\,1\}$ – the same spectrum as 1_2+T. Clearly $\operatorname{Spec}(1_2 + T) \subseteq [1 - \|T\|, 1 + \|T\|]$ and $\|T\| = \|pq\|$.

(iii): Each unital C^*-algebra $C^*(p, q, 1)$ generated by projections p, q is in a natural way a quotient of the *universal* unital C^*-algebra $C^*(P_u, Q_u, 1)$ with defining relations $P_u^* = P_u = P_u^2$ and $Q_u^* = Q_u = Q_u^2$.

In particular, there is a unique unital C^*-morphism Φ with $\Phi(P_u) = P$ and $\Phi(Q_u) = Q$ from the universal unital C^*-algebra $C^*(P_u, Q_u, 1)$ onto the unital C^*-subalgebra $C^*(P, Q, 1)$ of $\mathrm{C}([0, \pi/2], M_2)$ generated by $1, P \in M_2 \subset \mathrm{C}([0, \pi/2], M_2)$ and $Q := \exp(-H)P\exp(H) \in \mathrm{C}([0, \pi/2], M_2)$, with $H \in \mathrm{C}([0, \pi/2], M_2)$ as defined in Part (iii).

The natural *-epimorphism Φ from $C^*(P_u, Q_u, 1)$ onto $C^*(P, Q, 1) = \Phi(C^*(P_u, Q_u, 1))$ is *faithful* because $C^*(P, Q, 1)$ has 4 different characters and every irreducible 2-dimensional representation D factorizes over Φ. The factorization property for the irreducible representations D of $C^*(P_u, Q_u, 1)$ can be seen by using the parameter φ of the unitary equivalence class of D:

$$\arcsin \|(P - Q)|_{\{\varphi\}}\| = \arcsin \|P - Q(\varphi)\| = \varphi := \arcsin \|D(P_u) - D(Q_u)\|$$

for $Q(\varphi) := \exp(-H(\varphi))P\exp(H(\varphi)) = \exp(-\varphi Z)P\exp(\varphi Z)$. Notice that $\exp(\varphi Z) =: [\alpha_{jk}(\varphi)]$ is an orientation preserving rotation with angle φ, i.e., $\alpha_{11}(\varphi) = \alpha_{22}(\varphi) = \cos(\varphi)$ and $\alpha_{21}(\varphi) = \sin(\varphi) = -\alpha_{12}(\varphi)$.

Since $C^*(P, Q, 1)$ is naturally isomorphic to the universal unital C^*-algebra $C^*(P_u, Q_u, 1)$ generated by two projections, each unital C^*-algebra $C^*(p, q, 1)$ generated by two self-adjoint projections p, q is in a natural way a *quotient* of $C^*(P, Q, 1)$ by the unique unital *-epimorphism $\eta\colon C^*(P, Q, 1) \to C^*(p, q, 1)$ that satisfies $\eta(P) = p$ and $\eta(Q) = q$. This completes the proof of Part (iii).

We add here some additional information on the realisation $C^*(P, Q, 1) \subset \mathrm{C}([0, \pi/2], M_2)$ that can be useful to get alternative and less abstract proofs of Parts (iv)–(vi):

There is no essential difference in the properties of the real or complex version of the universal algebra $C^*(P, Q, 1)$, i.e., we can define it as the real C^*-subalgebra of $\mathrm{C}([0, \pi/2], M_2(\mathbb{R}))$ or as the complex C^*-subalgebra of $\mathrm{C}([0, \pi/2], M_2(\mathbb{C}))$ generated by $\{P, Q, 1\}$: Simply replace $\mathbb{R}$ by $\mathbb{C}$ and the in Part (iii) defined order 4 orthogonal operator Z with $Z = -Z^*$ by the self-adjoint (unitary) symmetry $-iZ = (-iZ)^* \in M_2(\mathbb{C})$ to get the complex case.

The definition of $(P, Q, 1)$ shows that $C^*(P, Q, 1)$ is the C^*-subalgebra of $\mathrm{C}([0, \pi/2], M_2)$ given by all continuous sections of the continuous field on $[0, \pi/2]$ with full fibers M_2 at $\varphi \in (0, \pi/2)$, “diagonal” fibers $C^*(1_2, P) = C^*(P, 1_2 - P) \subset M_2$ at $\varphi = 0$ and the fiber generated by $\{P, Q(\pi/2)\} = \{P, 1_2 - P\} \subset M_2$ at $\varphi = \pi/2$. I.e., the C^*-algebra $C^*(P, Q, 1)$ consists of all continuous matrix-valued functions $f\colon [0,\pi/2] \to M_2$ in $\mathrm{C}([0,\pi/2],M_2)$ that take values $f(0)$ and $f(\pi/2)$ in the diagonal matrices of M_2

An element $f \in \mathrm{C}([0,\pi/2],M_2)$ is in $C^*(P,Q)$, if and only if, $f \in C^*(P,Q,1)$ and $f(0) = \alpha P$ for some $\alpha \in \mathbb{R}$ (respectively $\alpha \in \mathbb{C}$). Therefore the in Part (iii) above defined elements $Z \in M_2$, $H, \exp(H) \in \mathrm{C}([0,\pi/2],M_2)$ are *not contained* in $C^*(P,Q,1)$, but the restrictions $H|[0,\psi]$ are in $C^*(P,Q)|[0,\psi]$ for each $\psi \in [0,\pi/2)$. Moreover, $\exp(H)|[\psi_1,\psi_2]$ is in $C^*(P,Q)|[\psi_1,\psi_2]$ for $0 < \psi_1 \leq \psi_2 < \pi/2$.

The quotient $C^*(P,Q)|[\psi,\pi/2] \subset \mathrm{C}([\psi,\pi/2],M_2)$ of $C^*(P,Q)$ (given by restriction to $[\psi,\pi/2]$) is *unital* for $0 < \psi \leq \pi/2$, because $(P+Q)|[\psi,\pi/2]$ is invertible in $\mathrm{C}([\psi,\pi/2],M_2)$.

The ideal $\mathrm{C}_0((0,\pi/2),M_2)$ of $\mathrm{C}([0,\pi/2],M_2)$ is the intersection of the kernels of the 4 characters of $C^*(P,Q,1)$ and is strictly contained in $C^*(P,Q)$, because the C^*-algebra $C^*(P,Q)$ has 3 characters.

Let $\psi \in (0,\pi/2)$ and define a function

$$f_\psi(\varphi) := (\varphi-\psi)_+ := \max(0,\varphi-\psi) \in \mathrm{C}_0(0,\pi/2]_+ \,.$$

The element $f_\psi \cdot P \in C^*(P,Q)$ generates the closed ideal J_ψ of $C^*(P,Q) \subset \mathrm{C}([0,\pi/2],M_2)$. The closed subspace J_ψ is also an ideal of $C^*(P,Q,1)$, because $C^*(P,Q)$ is an ideal of $C^*(P,Q,1)$.

If $0 < \psi < \pi/2$, then the quotient $C^*(P,Q,1)/J_\psi$ of $C^*(P,Q,1)$ is naturally isomorphic to

$$C^*(P,Q,1)|[0,\psi] \;\oplus\; (\mathbb{R}\cdot Q|\{\pi/2\}) \;\subset\; \mathrm{C}([0,\psi]\cup\{\pi/2\}\,,M_2)\,,$$

and the algebra $C^*(P,Q,1)|[0,\psi]$ is equal to

$$\mathbb{R}\cdot P + \mathbb{R}\cdot(1-P) + \mathrm{C}_0((0,\psi],M_2) \subset \mathrm{C}([0,\psi],M_2)\,.$$

The parameter φ for $Q(\varphi)$ is *natural* in the sense that the restrictions $P = P|_{\{\varphi\}} = P|_{\{0\}}$ and $Q(\varphi) := Q|_{\{\varphi\}}$ satisfy $\|(1-Q(\varphi))P\| = \sin(\varphi)$, $\|(1-P)Q(\varphi)\| = \sin(\varphi)$, $\|Q(\varphi)P\| = \cos(\varphi)$ and $\|P-Q(\varphi)\| = \sin(\varphi)$. Thus, for all $\varphi \in [0,\pi/2]$,

$$\|P(1-Q(\varphi))P\| + \|PQ(\varphi)P\| = 1\,.$$

E.g., the matrix $(P-Q)|_{\{\varphi\}} \in M_2$ is given by $[\alpha_{jk}] := P - \exp(-\varphi Z)P\exp(\varphi Z)$ and has entries $\alpha_{11} := \sin^2(\varphi)$, $\alpha_{12} := \alpha_{21} := -\sin(\varphi)\cos(\varphi)$ and $\alpha_{22} := -\sin^2(\varphi)$. It is $\sin(\varphi)$-times an orthogonal matrix, thus $\|\,(P-Q)|_{\{\varphi\}}\,\| = \sin(\varphi)$.

Since the norms of $(1-Q(\varphi))P$, $(1-P)Q(\varphi)$ and $P-Q(\varphi)$ are increasing and since the norms of $Q(\varphi)P$ are decreasing with respect to the parameter φ, we get the equal value $\sin(\varphi)$ for the norms of restrictions of $(1-Q)P$, $(1-P)Q$ and $P-Q$ to $[0,\varphi]$, and that $\|PQ|[\varphi,\pi/2]\| = \cos(\varphi)$.

(iv): Recall that $\widetilde{A} := A + \mathbb{R}\cdot 1 \subseteq \mathcal{M}(A)$ (respectively $\widetilde{A} := A + \mathbb{C}\cdot 1$) if A is not unital and $\widetilde{A} := A$ if A is unital.

Let $\gamma\colon C^*(P,Q,1) \to \widetilde{A}$ denote the canonical C^*-morphism from the universal C^*-algebra $C^*(P,Q,1)$ onto $C^*(p,q,1) \subseteq \widetilde{A}$. The kernel of γ is the intersection of the kernels of $\rho\circ\gamma$, where ρ runs through all irreducible representations of $C^*(p,q,1)$.

The natural C^*-morphism

$$\gamma\colon C^*(P,Q,1) \to C^*(p,q,1) \subseteq A$$

satisfies $\gamma((P(1-Q)P-\|(1-q)p\|^2)_+) = 0$ if $\|(1-q)p\| < 1$. Since $P(1-Q(\varphi))P = \sin(\varphi)^2 P$, this implies that the element $(P(1-Q)P - \|(1-q)p\|^2)_+ \in C^*(P,Q)$ and the in proof of Part (iii) defined element $f_\psi \cdot P \in C^*(P,Q) \subset \mathrm{C}([0,\pi/2], M_2)$ generate the same ideal of $C^*(P,Q,1)$ for $\psi := \arcsin\|(1-q)p\|$.

Thus, γ factorizes over the restriction $C^*(P,Q,1)|([0,\varphi_1]\cup\{\pi/2\})$ with $\varphi_1 := \arcsin\|(1-q)p\| < \pi/2$ and maps $P|\{\pi/2\} = P(1-Q)P|\{\pi/2\}$ to zero.

The projection $Q_0 \in C^*(P,Q,1)|([0,\varphi_1]\cup\{\pi/2\})$ defined by $Q_0(\varphi) = 0$ on $[0,\varphi_1]$ and by $Q_0(\pi/2) = 1 - P = Q|\{\pi/2\}$ at $\{\pi/2\}$ is *not* in the kernel of γ, if and only if, $C^*(p,q,1)$ has a character χ with $\chi(p) = 0$ and $\chi(q) = 1$. Compare the equivalence of (c) and (d) in Part(i).

This character χ is "isolated" in the sense that the image $C^*(p,q)$ of the C^*-morphism $\gamma\colon C^*(P,Q) \to A$ is isomorphic to a direct sum $C^*(p,q) \cong B \oplus \mathbb{C}$ such that q corresponds to $q_1 \oplus 1$, q_0 to $0 \oplus 1$ and p to $p \oplus 0$. It shows that the character χ is given by the projection of $B \oplus \mathbb{C}$ onto $0 \oplus \mathbb{C} \cong \mathbb{C}$.

The restriction $H' := H|[0,\varphi_1]$ is contained in the non-unital C^*-subalgebra $C^*(P,Q)|[0,\varphi_1]$, the restriction $\exp(H')|[0,\varphi_1]$ of the unitary exponential $\exp(H')$ to $[0,\varphi_1]$ is contained in $C^*(P,Q,1)|[0,\varphi_1]$ and

$$\exp(-H')P|[0,\varphi_1]\exp(H') \;=\; Q|[0,\varphi_1] \;\le\; Q|([0,\varphi_1]\cup\{\pi/2\}).$$

If $C^*(p,q,1)$ has a character χ with $\chi(p) = 0$ and $\chi(q) = 1$, then the character $\chi\circ\gamma$ on $C^*(P,Q,1)|[0,\varphi_1]\cup\{\pi/2\}$ is necessarily supported at the isolated one-point set $\{\pi/2\}$, and is given on $C^*(P,Q,1)|\{\pi/2\} \cong \mathrm{diag}(M_2)$ by $\chi\circ\gamma(\mathrm{diag}(1,0)) = 0$ and $\chi\circ\gamma(\mathrm{diag}(0,1)) = 1$. It follows that in this case, and only in this case, γ contains in its support $\subseteq [0,\pi/2]$ also the isolated point $\pi/2$, and $P|\{\pi/2\}$ is in the kernel of γ.

Thus $h := \gamma(H')$ is a skew adjoint element that is contained in $C^*(p,q)$.

The elements h and $q_0 := \gamma(Q_0) \le q := \gamma(Q)$ fulfill the quoted properties in Part (iv), because $q_0\exp(h) = \exp(h)q_0 = q_0$, $q_0 \le q$ and $q_0 p = 0$.

This *completes* the proof of Part (iv).

(v): By Part (i), $\|p-q\| < 1$ implies that $1 > \|p-q\| = \|(1-p)q\| = \|(1-q)p\|$. Thus, Part (iv) applies with $\|h\| = \arcsin\|p-q\|$ and $q_0 = 0$.

(vi): By Part (ii), the inequality $\|pq\| < 1$ implies that the algebraic *-algebra $(p+q)A(p+q)$ is a unital C^*-algebra. Since we require also that $\|p-q\| < 1$, the Part (v) applies to $p,q \in (p+q)A(p+q)$ and the, in this case unital, C^*-algebra $(p+q)A(p+q)$ in place of A. Then h $(= -h^*)$ and the unitary $\exp(h)$ can be defined inside the unital C^*-algebra $(p+q)A(p+q)$.

(old ?? ii,iii ??): See [**692**, prop. 2.2.6], or [**73**, prop. 4.3.3, prop. 4.6.3], or [**207**, prop. IV.1.2, lem.IV.1.4].

□

REMARKS 4.1.4. Following remarks about the natural $\mathbb{Z}_2$-grading α of the universal algebra $C^*(P, Q, 1)$ can be easily deduced from Lemma 4.1.3:

There is a unique grading automorphism $\beta \colon C^*(P, Q, 1) \to C^*(P, Q, 1)$ with $\beta(P) := 1-P$ and $\beta(Q) := 1-Q$ that is well-defined by universality of $C^*(P, Q, 1)$.

If we use the canonical embedding $C^*(P, Q, 1) \subseteq \mathrm{C}([0, 1], M_2)$ then β extends to $\mathrm{C}([0, 1], M_2)$ by the "odd", not "inner" in $C^*(P, Q, 1)$, grading of M_2, here by the order 2 outer automorphism $\beta(a)(t) := -Za(t)Z = Z^{-1}a(t)Z$ for the orthogonal matrix $Z = \exp((\pi/2)Z)$ in $M_2(\mathbb{R})$ of order 4, as defined in Part (iii) of Lemma 4.1.3, i.e., with $Z := [\zeta_{j,k}] \in M_2$ where $\zeta_{j,k} := j - k$, $j, k \in \{1, 2\}$.

The elements of degree $= 0$ are given by $f \cdot 1_2 + g \cdot Z$ $(f, g \in \mathrm{C}[0, 1])$ and the elements of degree $= 1$ by $f \cdot X + g \cdot ZX$ $(f, g \in \mathrm{C}[0, 1])$ with $X := \mathrm{diag}(1, -1)$.

Thus, in the case of real $M_2(\mathbb{R})$ and $f, g \in \mathrm{C}([0, 1], \mathbb{R})$ the elements of degree $= 0$ are just the natural images of complex-valued continuous functions $t \mapsto f_1(t)+if_2(t)$ in its "real" interpretation.

Let B a unital C^*-algebra, $\alpha \in \mathrm{Aut}(B)$ a grading of B, i.e., $\alpha^2 = \mathrm{id}_B$ and let $p, q \in B$ projections. Then $1 - 2p$ is of degree $= 1$, i.e., $\alpha(1 - 2p) = 2p - 1$, if and only if, $\alpha(p) = 1 - p$.

Notice that $\|(1 - 2q) - (1 - 2p)\| < 2$ if and only if $\|p - q\| < 1$.

If $A := C^*(p, q, 1) \subset B$ is the unital C^*-subalgebra of B generated by p, q that satisfy $\alpha(1 - 2p) = -(1 - 2p)$ and $\alpha(1 - 2q) = -(1 - 2q)$, then the natural C^*-morphism φ from $C^*(P, Q, 1)$ onto $C^*(p, q, 1)$ satisfies $\varphi \circ \beta = \alpha \circ \varphi$.

In particular, $\alpha(C^*(p, q, 1)) = C^*(p, q, 1)$.

If $\|(1 - 2q) - (1 - 2p)\| < 2$ then $\|p - q\| < 1$ and there exists $H \in C^*(p, q)$ with $\alpha(H) = H$, $H^* = -H$, $\|H\| < \pi/2$ and $q = \exp(-H)p\exp(H)$.

The grading $\alpha|C^*(p, q, 1)$ itself is given by an inner automorphism of $C^*(p, q, 1)$, if and only if, $\|p - q\| < 1$ and $\|pq\| < 1$, i.e., *both* inequalities must be valid.

REMARK 4.1.5. Let A a not-necessarily unital C^*-algebra. Consider in the non-unital case $A + \mathbb{C} \cdot 1_{\mathcal{M}(A)} \subseteq \mathcal{M}(A)$. Each $u \in \mathcal{U}_0(A)$ with $\pi_A(u) = 1$ is a product $u = \exp(h_1) \cdot \exp(h_2) \cdot \ldots \exp(h_n)$ of exponentials of elements $-h_k^* = h_k \in A$. We can here suppose that $\|h_k\| < \pi/2$. If $p, q \in A$ projections, then $p \sim_h q$ in A, i.e., p and q are (norm-) homotopic inside the metric space of of projections in A, then Lemma 4.1.3(v) shows that there exists a unitary $u \in \mathcal{U}_0(A + \mathbb{C} \cdot 1) \cap (A + 1)$ with $u^*pu = q$. Clearly, for each $u \in \mathcal{U}_0(A + \mathbb{C} \cdot 1)$ the projection $q := u^*pu \sim_h p$ in the projections of A.

2. On K-theory of properly infinite C^*-algebras

Recall that a projection $p \neq 0$ in a (not necessarily unital) C^*-algebra A is **properly infinite** if there are projections $r \leq p$ and $s \leq p$ such that $r + s \leq p$ and r, s are both Murray–von-Neumann equivalent to p.

A unital C^*-algebra E is called **properly infinite** if its unit element 1_E is properly infinite.

We can reduce *our* calculations related to K_*-theory and $\mathrm{KK}(\mathcal{C}; \cdot, \cdot)$ to the case of properly infinite unital C^*-algebras E or to C^*-algebras A with properly infinite $E := \mathcal{M}(A)$ by using – among others – the following lemmata and remarks.

```
Compare other places with next topic !?
Rewrite shorter on places with repeats.  Observations (C0), (Cu) ?
Better explanation?  ...
```

This observation ???? has a partial converse in the class of full properly infinite projections, as the following lemma of J. Cuntz [**172**] shows.

Recall that projections $p \in A$ are *full* in A if the linear span of ApA is dense in A. Some elementary observations on properly infinite projections – considered first by J. Cuntz [**172**] – are collected in the following Lemma:

LEMMA 4.2.1. *Let $p \in A$ a* properly *infinite projection in a C^*-algebra A, i.e., there exist partial isometries $u, v \in A$ with*

$$u^*u = v^*v = p \quad \textit{and} \quad uu^* + vv^* \leq p\,.$$

Let $B := \overline{\mathrm{span}(ApA)}$ (the closed ideal of A generated by p) and $q \in B \otimes M_n \cong M_n(B)$ a projection, then there exist a partial isometry $z \in M_n(B) \subseteq M_n(A)$ with $zz^* \leq p \otimes e_{11}$ and $z^*z = q$.

Let $r := p - vv^*$. There exist partial isometries $s, t \in rAr$ with $s^*s = t^*t = r$ and $s^*t = 0$.

The projection $r := p - vv^*$ is a full projection in B and the elements of $\mathrm{K}_0(B)$ are the classes $[q] \in \mathrm{K}_0(B)$ given by properly infinite projections $q \leq r$ that are full in B and have the property that $r - q$ is also properly infinite and full in B.

The addition can be carried out inside the classes of Murray–von-Neumann equivalent projections of $rBr = rAr$ itself via the operation $[q_1] + [q_2] = [q_1 \oplus q_2]$ (where $\oplus = \oplus_{s,t}$, i.e., $q_1 \oplus q_2 := sq_1s^* + tq_2t^*$).

If, in addition, $q \in A$ is also properly infinite, $[q] = [p] \in \mathrm{K}_0(A)$ and $p \in \overline{\mathrm{span}(AqA)}$, then there exists a partial isometry $w \in \overline{\mathrm{span}(ApA)}$ with $w^*w = p$ and $ww^* = q$.

In particular, full properly infinite projections $p, q \in A$ satisfy $[p] = [q] \in \mathrm{K}_0(A)$ if and only if p and q are M-vN-equivalent, i.e., if there exists a partial isometry $v \in A$ with $v^*v = p$ and $vv^* = q$.

If A contains a full properly infinite projection $p \in A$ then for each $x \in \mathrm{K}_0(A)$ there exists a properly infinite full projection $q \in A$ with $q \leq p$ and $x = [q] \in \mathrm{K}_0(A)$.

We omit the proof of Lemma 4.2.1. It is easy to see, except that for each unital C^*-algebra A with properly infinite unit element $1 \in A$ and $[1] = 0$ in $\mathrm{K}_0(A)$ there exist isometries $s, t \in A$ with $s^*t = 0$ and $ss^* + tt^* = 1_A$, see the "basic property" (Cu) of full properly infinite projections in the proof of Lemma 4.2.6(i,ii). Combined with Lemma 4.2.3 it delivers all other statements. Plus$^+$ some extra service for the reader concerning of pairs of near odd graded isometries and others.

If A is unital (with properly infinite unit), then a projection $p \in A$ is (called) **splitting** if p and $1 - p$ are both full and properly infinite.

Recall that A is K_1**-injective**, if $u \in \mathcal{U}(\widetilde{A})$ and $0 = [u] \in \mathrm{K}_1(\widetilde{A})$ together always imply that $u \in \mathcal{U}_0(\widetilde{A})$, i.e., $u \sim_h 1$ in $\mathcal{U}(\widetilde{A})$. It implies that unitaries u_1 and u_2 in $\widetilde{A}$ are homotopic inside $\mathcal{U}(\widetilde{A})$, if and only if, $[u_1] = [u_2]$ in $\mathrm{K}_1(\widetilde{A})$.

There is a natural group-morphism $\mathcal{U}(E) \ni u \mapsto [u] \in \mathrm{K}_1(E)$ from the unitary group $\mathcal{U}(E)$ of E into $\mathrm{K}_1(E)$ if E is unital. Its kernel contains the connected component $\mathcal{U}_0(E)$ of the identity element $1 \in E$.

Definition 4.2.2. A unital C^*-algebra E is called K_1**-surjective** if the natural group-morphism $u \mapsto [u]$ is a surjective map from $\mathcal{U}(E)$ onto $\mathrm{K}_1(E)$.

The C^*-algebra E is K_1**-injective** if the connected component $\mathcal{U}_0(E)$ of 1_E in $\mathcal{U}(E)$ is equal to the kernel of $\mathcal{U}(E) \to \mathrm{K}_1(E)$, i.e., the induced map from $\mathcal{U}(E)/\mathcal{U}_0(E)$ to $\mathrm{K}_1(E)$ is *faithful* (= injective).

The C^*-algebra E is K_1**-bijective** if E is K_1-surjective and K_1-injective, i.e., $\mathcal{U}(E)/\mathcal{U}_0(E) \cong \mathrm{K}_1(E)$ by the natural group morphism.

The K_1-bijectivity of stable coronas obtained in Proposition 4.2.15 with help of the "squeezing" property defined in Definition 4.2.14 plays later a role in some of our computations.

It is (Feb 2021) not known if the unital (full) free product C^*-algebras $\mathcal{O}_\infty * \mathcal{O}_\infty$, $\mathrm{C}(S^1) * \mathcal{O}_2 \cong \mathcal{O}_2 * \mathcal{O}_2$ or $\mathrm{C}(S^1) * \mathcal{E}_2$ are K_1-injective.

Where is a proof of $\mathrm{C}(S^1) * \mathcal{O}_2 \cong \mathcal{O}_2 * \mathcal{O}_2$?

The proof of Lemma 4.2.10(i) shows that the question *"Is every unital C^*-algebra with properly infinite unit K_1-bijective?"* is equivalent to the question – if the unitary $U^* \cdot (1 - s_1 s_1^* + s_1 U s_1^*)$ in the unital full free product $A := \mathrm{C}(S^1) * \mathcal{E}_2$ is in $\mathcal{U}_0(A)$, where U denotes the canonical unitary generator of $\mathrm{C}(S^1)$ and the $s_1, s_2 \in \mathcal{E}_2$ denote the generating isometries of $\mathcal{E}_2$ ([2])

If S is an Abelian semigroup then we denote by $\mathrm{Gr}(S)$ the Grothendieck group of S, i.e., the formally defined group of "formal differences" of elements of the *cancellation semi-group* build by stable equivalence classes of elements in S. The

[2] This was also mentioned in a talk of B. Blackadar in Barcelona (2007). Compare also Lemma 4.2.10(iii).

natural semigroup morphism from S into $\mathrm{Gr}(S)$ is given by $x \mapsto [x]$ and maps onto the cancellation semi-group $[S] \subseteq \mathrm{Gr}(S)$, where $[x] \subseteq S$ denotes the stable equivalence class of x in S given by

$$[x] := \{ y \in S ; \ \exists \ s \in S \text{ with } x + s = y + s \} .$$

In general the natural injective additive morphism from the Abelian cancellation semigroup $[S]$ of S into the Grothendieck group $\mathrm{Gr}(S)$ is not surjective, $[S]$ is not identical with $\mathrm{Gr}(S)$, i.e., $[S]$ is not itself a group.

The proof of the following Lemma 4.2.3 is straight calculation and is left to the reader. It gives some sufficient conditions for cases where $[S] = \mathrm{Gr}(S)$, i.e., where $[S]$ is itself is a group .

LEMMA 4.2.3. *Let S a commutative semigroup and denote, for $z \in S$, by $S(z) \subseteq S$ the set of all elements $x \in S$ with the property that there exists $y \in S$ with $x + y = z$.*

(i) *If $e \in S$ satisfies $e + e = e$, and that for every $x \in S$ there exists some $y \in S$ with $x + y = e$, i.e., $S(e) = S$, then e is the only element of S with this property and the sub-semigroup $G := S + e$ is a sub*group *of S.*

An element $x \in S$ is in G, if and only if, x absorbs *e, i.e., if $x = x+e$.*

The map $x \in S \mapsto x + e \in S$ is a semigroup homomorphism that extends to a natural isomorphism from G onto the Grothendieck group $\mathrm{Gr}(S)$ of S in the following manner:

By universality of $\mathrm{Gr}(S)$ there is a well-defined natural group homomorphism φ from $\mathrm{Gr}(S)$ into G, given by $\varphi([x]) := x + e$ on the classes $[x] \in \mathrm{Gr}(S)$ for $x \in S$.

This homomorphism is a group isomorphism *from $\mathrm{Gr}(S)$ onto G with inverse isomorphism $x + e \mapsto [x + e] \in \mathrm{Gr}(S)$.*

(ii) *More generally, if S contains a element $z_0 \in S$ such that for each $x \in S$ there exists $y \in S$ with $x + y = z_0$, i.e., $S = S(z_0)$, then there exists an element $y_0 \in S$ with $2z_0 + y_0 = z_0$.*

The element $e := z_0+y_0$ and pair (S, e) have the in Part (i) considered properties.

(iii) *If $z_0 \in S$ is an element that satisfies only $2z_0 \in S(z_0)$, i.e., if there exists $y_0 \in S$ with $2z_0 + y_0 = z_0$, then $e := e(z_0) := z_0 + y_0$ satisfies $S(e(z_0)) = S(z_0)$ and $e + e = e$.*

The subset *$S(z_0)$ of S is a sub-semigroup of S such that $G(e(z_0)) := S(z_0) + e(z_0) \subseteq S(z_0)$ and $G(e(z_0))$ is a sub*group *of S, that is natural isomorphic to the Grothendieck group of $S(z_0)$.*

(iv) *If $z_0, z_1 \in S$ satisfy $z_0 \in S(z_1)$ and $2z_j \in S(z_j)$ for $j = 0, 1$, then $e(z_1) = e(z_0) + e(z_1)$ and the mapping*

$$y \in S(z_0) + e(z_0) \mapsto y + e(z_1) \in S(z_1) + e(z_1)$$

is a group homomorphism from $G(e(z_0))$ into $G(e(z_1))$.

□

There are examples of group homomorphism of the type described in Part (iv) of Lemma 4.2.3 that are neither injective nor surjective.

An element $a \in A_+$ of a (not-necessarily unital) C^*-algebra A will be called **full** in A if span(AaA) is dense in A, i.e., if $\overline{aAa}$ is full hereditary C^*-subalgebra of A. (For example, all strictly positive elements of A are full in A.)

Recall that the unit element 1_E of a unital C^*-algebra E is properly infinite in E, if and only if, there are isometries $s_1, s_2 \in E$ with $s_1^* s_2 = 0$. Then we say that E is a **properly infinite** unital C^*-algebra.

We let $p_{[-1]} := 1 - s_1 s_1^* - s_2 s_2^*$. This projection becomes zero if and only if s_1, s_2 are the canonical generators of a unital copy of $\mathcal{O}_2$ in E. It holds $[p_{[-1]}] = -[1]$ in $\mathrm{K}_0(E)$.

Then a ***generalized*** *variant of the* ***Cuntz addition*** using isometries s_1, s_2 with $s_1^* s_2 = 0$ is defined by

$$a \oplus b \; := \; a \oplus_{s_1,s_2} b \; := \; s_1 a s_1^* + s_2 b s_2^* \quad \text{for } \; a, b \in E \, .$$

If we consider unitaries $u, v \in \mathcal{U}(E)$ then we can use the **corrected version** of generalized Cuntz addition given with $p_{[-1]} := 1 - s_1 s_1^* - s_2 s_2^*$ by

$$u \oplus' v := \; u \oplus'_{s_1,s_2} v := \; s_1 u s_1^* + s_2 v s_2^* + p_{[-1]} \, .$$

We say that a projection $p \in E$ in a unital C^*-algebra E is **splitting** if p and $1-p$ are both properly infinite projections and are full elements of E.

The definitions of proper infiniteness and fullness yield that this property is equivalent to the property that there exist isometries $t_1, t_2 \in E$ with $t_1 t_1^* \leq p$ and $t_2^* t_2 \leq 1 - p$.

The notion of splitting projections generalizes the notion of a **proper projection** p defined by the property that p and $1-p$ are both MvN-equivalent to 1. Most of the following results on splitting projections have been obtained before by J. Cuntz and N. Higson for the special case of proper projections in place of the more general "splitting" projections (introduced by B. Blackadar, 2007).

It is easy to see, e.g. using Lemma 4.1.3(v), that projections $p, q \in E$ are homotopic inside the subset of projections in E, if and only if, there is a unitary $u \in \mathcal{U}_0(E)$ with $u^* p u = q$.

REMARK 4.2.4. The **Halmos unitary** $U(c)$ in $M_2(E)$, build from a contraction $c \in E$, is an "orientation-preserving rotation" given by the unitary matrix (respectively by the "special orthogonal" matrix if E is a *real* C^*-algebra) $U(c) := [a_{jk}(c)]$ with entries $a_{11}(c) := c$, $a_{12}(c) := -(1 - cc^*)^{1/2}$, $a_{21}(c) := (1 - c^* c)^{1/2}$ and $a_{22}(c) := c^*$.

Each Halmos unitary $U(c)$ is in $\mathcal{U}_0(M_2(E))$ because $U(1) = 1_2$ and $c \mapsto U(c)$ is uniformly Hölder continuous on the closed unit ball of E with very rough estimate

(?????? see commented lines !!!?)

$$\|U(c_1) - U(c_2)\| \leq (2\|c_1 - c_2\|)^{1/2} + \|c_1 - c_2\| \leq 3\|c_1 - c_2\|^{1/2} \tag{2.1}$$

It shows that each Halmos unitary $U(c)$ is in $\mathcal{U}_0\big(M_2(C^*(c,1))\big)$ and has an exponential length $\text{cel}(U(c))$, – i.e., the "geodesic" distance of $U(c)$ from 1_2 inside the metric space $\mathcal{U}_0(M_2(E))$ –, of value $\text{cel}(U(c)) < 3\pi/2$.

If one considers the very special case $E := \mathrm{C}([0,1])$ then it cuts down to the case of $E := \mathbb{C}$. And in this case one can use the estimates of $\text{cel}(U(c))$ for $c \in \mathbb{C}$ with $|c| \leq 1$.

check this estimate $3\pi/2$ again!!
How long is the path $t \in [0,1] \mapsto U(tc + (1-t))$ in $\mathcal{U}_0(E)$
for fixed $c \in E$ with $c \in E$ with $\|c\| < 1$?
How good is the estimate for positive contraction?
Very important:
Give precise citation/reference to definition of cel of a unitary
or change and DEFINE notation !!!

In the special case of $U(T)$ for an isometry $T \in E$ we can take a new parametrization $\varphi \in [0, \pi/2] \mapsto U(\cos(\varphi) \cdot T) \in M_2(E)$ and get a continuously differentiable curve of length $\leq \pi/2$.

For $h^* = h \in E$ with $0 \leq h \leq \pi \cdot 1$, the matrix $Z = [z_{ij}] \in M_2$ with $z_{ij} := i - j$, and $X_h := h \otimes Z \in M_2(E)$ holds $\exp(X_h) = U(\cos(h))$. Therefore we have changed the original definition of P. Halmos [**352**] into our formula for $U(c)$ to make it fitting to other terminology used for the terms in our estimates.

The original definition given by Halmos was $\text{diag}(1, -1) \cdot U(c) \cdot \text{diag}(1, -1)$, if we express it with our definition of $U(c)$. Some authors define the matrix $U(c) \cdot \text{diag}(1, -1)$ as Halmos unitary.

Compare above text about estimates and arc-length
with below blue - including blue text in proof of Lemma 4.2.6!!!

The estimate (2.1) follows from

$$\|U(c_1) - U(c_2)\| \leq \|c_1 - c_2\| + \big(\max\{\|c_1^*c_1 - c_2^*c_2\|, \|c_1c_1^* - c_2c_2^*\|\}\big)^{1/2} \leq 3\|c_1 - c_2\|^{1/2},$$

where we have used that $\|a^{1/2} - b^{1/2}\| \leq \|a - b\|^{1/2}$ for $a, b \in E_+$. More precise is $\|U(c_1) - U(c_2)\| \leq \gamma(\|c_1 - c_2\|)$ with the function $\gamma(\xi) := \xi + (2\xi)^{1/2}$ for $\xi \in [0, 2]$. This estimate can be applied to the special pairs of contractions $(c_1, c_2) := ((1/2)c, 0)$ and $(c, (1/2)c)$. It shows that $U(c) = U(0)\exp(H_1)\exp(H_2)$ for $H_1, H_2 \in E$ with $H_k^* = -H_k$ and $\|H_k\| < \pi/2$. Since $U(0)$ is a 90° rotation, it implies that $U(c)$ has exponential length bounded by $3\pi/2$.

But this estimate (with this parametrization) does not imply that the arc $[0,1] \ni t \mapsto U(tc)$ is a rectifiable path (of finite length), that requires Lipschitz continuity (and not only this obtained Hölder continuity). In more detail: $\|U(tc) - U(sc)\| \leq (t - s) + (t^2 - s^2)^{1/2}$ for $0 \leq s < t \leq 1$.

Recall here $|\exp(i\varphi)-1| \leq x < 2$ for $\varphi \in [-\pi, \pi]$ if and only if $2-x^2 \leq 2\cos(\varphi)$, i.e., $|\varphi| \leq \arccos(1-x^2/2) < \pi$.

Thus, $\|U(tc) - U(0)\| \leq 2t < 2$ for $t \in [0,1)$, implies that there is a unique $H \in E$ with $U(tc) = U(0)\exp(H)$, $H^* = -H$, and $\|H\| \leq \arccos(1-2t^2) < \pi$ and $U(tc) = U(0)\exp(H)$ if $t \in [0,1)$, i.e., $\|H\| \leq \arccos(1/2) < \pi/2$ if $t = 1/2$. If $\delta \in [0,1)$, then

$$\|U(c) - U((1-\delta)c)\| \leq \delta + (1-(1-\delta)^2)^{1/2} \leq 3\delta < \pi\,,$$

and we get for $\delta := 1/2$, that $\|U(c) - U((1/2)c)\| \leq x$ with $1 < x := (1+\sqrt{3})/2 < \sqrt{2}$ and $1 - x^2/2 = (2-\sqrt{3})/4 > 0$ and there exists $K \in E$ with $K^* = -K$, $U(c) = U((1/2)c)\exp(K)$ and $\|K\| \leq \arccos((2-\sqrt{3})/4) < \pi/2$.

(It seems that the Hölder continuous map $t \in [0,1] \to U(t\cdot c) \in \mathcal{U}_0(M_2(E))$ itself is not always rectifiable, because it is not Lipschitz continuous for arbitrary contractions $c \in E$.)

In case of $U(T)$ for an isometry $T \in E$ we can take a parametrization $\varphi \in [0, \pi/2] \mapsto U(\cos(\varphi)\cdot T) \in M_2(E)$ and get a continuously differentiable curve of length $\leq \pi/2$, because, for $s \in [0,1]$,

$$U(s\cdot T) = \operatorname{diag}(T,1)U(s\cdot 1)\operatorname{diag}(1,T^*) + X\,,$$

where $X = [x_{jk}]$ is given by $x_{12} := 1 - TT^*$ and $x_{11} = x_{21} = x_{22} := 0$.

We use often the following general equation for isometries $t \in E$ and unitaries $v \in \mathcal{U}(E)$:

$$[(1-tt^*) + tvt^*] = [v] \quad \text{in} \quad \mathrm{K}_1(E)\,. \tag{2.2}$$

Indeed, the diagonal matrices $\operatorname{diag}(v,1)$ and $\operatorname{diag}\big((1-tt^*) + tvt^*\,,\,1\big)$ are unitary and are unitary equivalent in $M_2(E)$ by the Halmos unitary $U(t) \in \mathcal{U}_0(M_2(E))$, i.e.,

$$U(t)\operatorname{diag}(v,1) = \operatorname{diag}\big((1-tt^*) + tvt^*\,,\,1\big)U(t)\,.$$

Remark 4.2.5. Recall that C^*-algebra E has a properly infinite unit, if and only if, E contains two isometries t_1, t_2 with orthogonal ranges, i.e., with relations $t_i^* t_j = \delta_{i,j} 1_E$.

Obviously the elements $s_n := (t_1)^{n-1} t_2$ satisfy $s_k^* s_\ell = \delta_{k,\ell} 1_E$, i.e., generate a copy of $\mathcal{O}_\infty$ that is unitally contained in E, cf. the proof of Proposition 4.2.11.

It implies that for every contraction $a \in E$ there exists a unitary $u \in \mathcal{U}_0(E)$ such that $s_1^* u s_2 = a$ with the isometries s_1, s_2 are taken from this fixed copy of $\mathcal{E}_3$.

It shows that any separable C^*-algebra E with a properly infinite unit element is a quotient of the unital free product $\mathrm{C}(S^1) * \mathcal{E}_2$, because all separable C^*-algebras with properly infinite unit are singly generated by Proposition B.17.2.

Indeed, we can use the Halmos unitary $U(a) \in M_2(a)$ and the projection $p := 1 - s_1 s_1^* - s_2 s_2^* \in E$. Then the unitary $u := p + [s_1, s_2]U(a)[s_2, -s_1]^* \in E$ has this properties, because $U(a) \in \mathcal{U}_0(A)$ is connected in $\mathcal{U}_0(A)$ to $U(0)$ and $[s_1, s_2]U(0)[s_2, -s_1]^* = 1 - p$.

LEMMA 4.2.6. *Let E a unital C^*-algebra with properly infinite unit 1_E. Then there exist isometries $s_1, s_2 \in E$ with orthogonal ranges, i.e., with $s_1^* s_2 = 0$, such that s_1 and s_2 satisfy the following non-degeneracy condition* (ND):

(ND) *The projection*

$$p_{[-1]} := 1 - s_1 s_1^* - s_2 s_2^* = 1 - (1 \oplus 1)$$

is either *full and properly infinite* or $p_{[-1]} = 0$.

The s_1, s_2 can be always found such that $p_{[-1]}$ is full and properly infinite, s_1, s_2 with $p_{[-1]} = 0$ exists if and only if $[1] = 0$ in $\mathrm{K}_0(E)$.

The (generalized) Cuntz addition $\oplus_{s_1,s_2}$ defined by isometries $s_1, s_2 \in E$, that satisfy $s_1^* s_2 = 0$ and condition (ND), *has following properties and connections to unitary operators and splitting projections:*

(o) Commutativity and Associativity of $\oplus$ up to unitary homotopy:

There exist $U_c, U_d \in \mathcal{U}_0(C^*(s_1, s_2)) \subseteq \mathcal{U}_0(E)$ with exponential length $\mathrm{cel}(U_c) \leq \pi/2$ and $\mathrm{cel}(U_d) \leq \pi$

`Look up definition of ''cel'' in published papers, change notation?`

`Give Citation!!!`

that satisfy $p_{[-1]} U_c = p_{[-1]} = p_{[-1]} U_d$, $U_c^* (a \oplus b) U_c = (b \oplus a)$, for $a, b \in E$, and

$$U_d^* \big(a \oplus_{s_1,s_2} (b \oplus_{s_1,s_2} c)\big) U_d = \big((a \oplus_{s_1,s_2} b) \oplus_{s_1,s_2} c\big) \qquad \text{for } a, b, c \in E\,.$$

(i) *The map $(p, q) \mapsto p \oplus_{s_1,s_2} q$ is a commutative and associative operation up to unitary equivalence by unitary elements $u \in \mathcal{U}_0(E)$, i.e., up to homotopy in the projections of E. It is compatible with unitary equivalence, homotopy and Murray–von Neumann equivalence.*

If $p, q \in E$ are projections then $\mathrm{diag}(p \oplus q, 0)$ is MvN-equivalent to $\mathrm{diag}(p, q)$ in $M_2(E)$. The projection $p \oplus q$ is MvN-equivalent to $p + q$ if $pq = 0$.

In particular, $[p \oplus q] = [p] + [q] \in \mathrm{K}_0(E)$ for projections $p, q \in E$, and $[p \oplus q] = [p + q]$ if $pq = 0$.

(ii) *Suppose that $p, q \in E$ are projections and that there exist isometries $s, t \in E$ with $ss^* \leq p$ and $tt^* \leq q$, i.e., p and q are full and properly infinite.*

Then p and q are Murray–von Neumann equivalent in E if and only if $[p] = [q] \in \mathrm{K}_0(E)$.

If p and q are splitting *projections, – i.e., if each of the projections p, $1 - p$, q and $1 - q$ are full and properly infinite –, then p and q are unitary equivalent in E if and only if $[p] = [q] \in \mathrm{K}_0(E)$.*

If $p, q \in E$ are unitary equivalent projections and p is full and properly infinite then there exists $u \in \mathcal{U}(E)$ with $0 = [u] \in \mathrm{K}_1(E)$ and $u^* p u = q$.

Any projection $q \in M_n(E)$ is MvN-equivalent to a projection $q' \in E$ such that $q' \leq p$ for some splitting projection $p \in E$ ([3]).

(iii) *The* unitary *equivalence classes $[p]_u$ of splitting projections $p \in E$ build a group with generalized Cuntz addition $[p]_u + [q]_u := [p \oplus q]_u$, and the natural map $[p]_u \to [p] \in \mathrm{K}_0(E)$ defines an* isomorphism *from this group onto $\mathrm{K}_0(E)$.*

(iv) (a) (*cf.* [**94**, prop. 2.5]) *If p and q are full and properly infinite projections with $[p] = [q] \in \mathrm{K}_0(E)$, and if there exists an isometry $t \in E$ with $t^*(p+q)t = 0$, then there exists a unitary $u \in \mathcal{U}_0(E)$ such that $u^*pu = q$, i.e., p and q are homotopic in the projections of E.*
(b) *Special case: Two projections $p, q \in E$ represent the same element $[p] = [q]$ of $\mathrm{K}_0(E)$, if and only if, there exists a unitary $u \in \mathcal{U}_0(E)$ such that $u^*(p \oplus 1 \oplus 0)u = q \oplus 1 \oplus 0$.* ([4])

(v) *Let $u \sim_h v$ denote homotopy of $u, v \in \mathcal{U}(E)$ inside the unitary group $\mathcal{U}(E)$ of E, and let $\mathcal{U}_0(E)$ the connected component of 1 in $\mathcal{U}(E)$.*

The natural map $u \mapsto [u]$ from the group of unitary operators $\mathcal{U}(E)$ of E into $\mathrm{K}_1(E)$ is an epimorphism, *i.e., E is* K_1-surjective. *It has following properties – expressed with the "corrected" Cuntz sum $u \oplus' v := (u \oplus v) + p_{[-1]}$ on the unitary group:*

(1) $[u \oplus' v] = [u] + [v] = [uv]$ *in* $\mathrm{K}_1(E)$,
(2) $[u] = [v] \in \mathrm{K}_1(E)$, *if and only if,* $u \oplus' 1 \sim_h v \oplus' 1$.
In particular, $u \oplus' 1 \in \mathcal{U}_0(E)$ *if* $[u] = 0 \in \mathrm{K}_1(E)$.
(Recall $u \oplus' 1 := s_1 u s_1^* + (1 - s_1 s_1^*)$ *.)*
(3) *The unitary operators* $u^* \oplus' u$ *are in* $\mathcal{U}_0(E)$.
(4) $u \oplus' v \sim_h vu \oplus' 1 \sim_h v \oplus' u$.
(5) *Let $p \in E$ a projection such that $1 - p$ is full and properly infinite.*
If $u \in \mathcal{U}(pEp)$ is a unitary with $[u + (1-p)] = 0$ in $\mathrm{K}_1(\mathcal{U}(E))$ then $u + (1-p) \in \mathcal{U}_0(E)$.
If $p \in E$ is moreover a splitting *projection, then $\eta \colon pEp \hookrightarrow E$ defines an isomorphism η_* from $\mathrm{K}_*(pEp)$ onto $\mathrm{K}_*(E)$.*
It satisfies $\eta_1([u]) = [u + (1-p)]$ for $u \in \mathcal{U}(pEp)$ and $\eta_0([q]) = [q]$ for projections $q \in pEp$.
In particular, for splitting $p \in E$ and $u \in \mathcal{U}(pEp)$ holds:

$$0 = [u] \in \mathrm{K}_1(pEp) \iff u + (1-p) \in \mathcal{U}_0(E) \,.$$

(6) *If $z \in E$ is a partial isometry with $z^2 = 0$ then the "rotation" unitary*

$$R(z) := z - z^* + (1 - z^*z - zz^*) = \exp((\pi/2)(z - z^*))$$

is in $\mathcal{U}_0(E)$. If E is a complex *C^*-algebra, then the "symmetry"*

$$S(z) := z + z^* + (1 - z^*z - zz^*)$$

[3] Identify $p \in E$ with $p \otimes e_{11} \in E \otimes M_n \cong M_n(E)$.
[4] $(1 \oplus 1) \oplus 0$ and $1 \oplus 1$ are MvN-equivalent but are not unitarily equivalent.

is in $\mathcal{U}_0(E)$.

It holds $R(z)^* a R(z) = zaz^* = S(z)aS(z)$ *for all* $a \in z^*zEz^*z$.

(vi) (*cf.* [**94**, lem. 2.4(ii)].) *Let* $p \in E$ *a splitting projection in* E.

If $u_1, u_2 \in \mathcal{U}(E)$ *satisfy* $[u_1] = [u_2] \in \mathrm{K}_1(E)$ *and* $\|u_k p - p u_k\| < 1$ *for* $k = 1, 2$, *then* $u_1 \sim_h u_2$ *in* $\mathcal{U}(E)$.

Special cases are:

(α) $u \in \mathcal{U}_0(E)$ *if* $\|up - pu\| < 1$ *and* $0 = [u] \in \mathrm{K}_1(E)$.

Thus, $u \in \mathcal{U}_0(E)$ *if* $up = pu$ *and* $[u] = 0$ *in* $\mathrm{K}_1(E)$.

`COMPARE, also clever:`

(iv,old): If $u \in \mathcal{U}(A)$, $p \in A$ *is a* splitting *(!!!) projection, and* $\|up - pu\| < 1$, *then there exists a unitary* $v \in \mathcal{U}(pAp)$ *such that* u *is homotopic to* $v + (1 - p)$ *in* $\mathcal{U}(A)$.

(Proof: iv,old): If w *is unitary with* $wp = pw$, *and* $z \in A$ *with* $zz^* = (1 - p)$ *and* $z^*z \le p$, *then* $p + (1 - p)w$ *is unitary and is homotopic to* $z^*wz + (p - z^*z) + (1 - p)$ *(using that* $C^*(z, 1-p) \cong M_2$ *has unit* $z^*z + (1 - p)$*).*

Thus, $w \sim_h v + (1 - p)$ *with* $v := pw(z^*uz + (p - z^*z)) \in \mathcal{U}(pAp)$.

If $\|up - pu\| < 1$, *then* $\|u - a\| < 1$ *for* $a := pup + (1 - p)u(1 - p)$. *The polar decomposition* $a = w(a^*a)^{1/2}$ *in* $pAp + (1 - p)A(1 - p)$ *gives a unitary* $w \in A$ *with* $-1 \notin \mathrm{Spec}(w^*u)$ *and* $wp = pw$. *Thus,* $u \sim_h w \sim_h v + (1 - p)$ *for suitable* $v \in \mathcal{U}(pAp)$.

(β) *If* $t_1, t_2 \in E$ *are isometries with* $\|t_1^* t_2\| < 1$ *and if* $u \in \mathcal{U}(E)$ *satisfies* $\|t_1^* u t_2\| < 1$ *and* $[u] = 0 \in \mathrm{K}_1(E)$, *then* $u \in \mathcal{U}_0(E)$.

(v,old) `FROM: lem:old.A.K-basic1:`

If $u \in \mathcal{U}(A)$ *with* $0 = [u] \in \mathrm{K}_1(A)$, *and if there exists a splitting projection* $p \in A$ *with* $\|up - pu\| < 1$, *then* $u \in \mathcal{U}_0(A)$.

In particular, A *is* K_1*-injective if this is the case for every* $u \in \mathcal{U}(A)$ *with* $0 = [u] \in \mathrm{K}_1(A)$.

[Above is now contained in in new (vi,β)!!!]

(v,old): Let $u \in \mathcal{U}(A)$ *with* $0 = [u] \in \mathrm{K}_1(A)$, *and suppose that there is a splitting projection* $p \in A$ *with* $\|pu - up\| < 1$. *By (iv),* $u \sim_h v + (1-p)$ *for some unitary* $v \in \mathcal{U}(pAp)$. *It follows* $[v + (1-p)] = [u] = 0$ *in* $\mathrm{K}_1(A)$. *Since* $[v + (1 - p)] = 0$, *there exists* $n \in \mathbb{N}$ *such that* $(v + (1 - p)) \oplus 1_n \in \mathcal{U}_0(M_{n+1}(A))$, *i.e., there is a path* $W(t) \in \mathcal{U}(M_{n+1}(A))$ *with* $W(0) = 1_{n+1}$ *and* $W(1) = (v + (1-p)) \oplus 1_n$. *Since* $1-p$ *is full and properly infinite in* A, *the projection* $(1-p) \oplus 0_n$ *is full and properly infinite in* $M_{n+1}(A)$. *Thus, there is a partial isometry* $Z \in M_{n+1}(A)$ *with* $Z^*Z = (1 - p) \oplus 1_n$ *and* $ZZ^* \le (1 - p) \oplus 0_n$. *Let* $w(t) := (p + Z)W(t)(Z^* + p) + ((1-p) \oplus 0_n - ZZ^*)$. *Then* $w(t)^*w(t) = 1 \oplus 0_n = w(t)w(t)^*$, $w(0) = 1 \oplus 0_n$ *and* $w(1) = (v + (1 - p)) \oplus 0_n$. *If we naturally identify* A *with the corner* $A \oplus 0_n$ *of* $M_{n+1}(A)$, *then* $w(t)$ *becomes a continuous path in* $\mathcal{U}(A)$ *that connects* $v + (1-p)$ *with* 1. *Thus,* $u \sim_h 1$.

(vii) Independence of $\oplus$ from the isometries $s_1, s_2 \in E$ with $\oplus := \oplus_{s_1,s_2}$ *and non-degeneracy property* (NP) *up to unitary equivalence:*

Suppose that $t_1, t_2 \in E$ are isometries with orthogonal ranges such that $q_{[-1]} := 1 - t_1t_1^* - t_2t_2^*$ is full and properly infinite if $p_{[-1]} \neq 0$ and is $q_{[-1]} = 0$ if $p_{[-1]} = 0$.

Then there exists a unitary $U \in \mathcal{U}(E)$ such that $Us_k = t_k$ $(k = 1, 2)$, and $U(a \oplus_{s_1,s_2} b)U^* = a \oplus_{t_1,t_2} b$ and $Up_{[-1]}U^* = q_{[-1]}$ for $a, b \in E$.

If $p_{[-1]}$ and $q_{[-1]}$ are full and properly infinite, then the unitary $U \in \mathcal{U}(E)$ can be chosen such that $[U] = 0$ in $\mathrm{K}_1(E)$.

In the cases where $p_{[-1]} \neq 0$ and E is K_1-injective, or where there exists a full and properly infinite projection $r \in E$ with $r \leq p_{[-1]}$ and $r \leq q_{[-1]}$, one can find $U \in \mathcal{U}_0(E)$ with $Us_k = t_k$.

Before we start with the proofs of the part ???? of Lemma **??** here hint of one of its limited applicability: ?????

`This comes from old appendix!!! Transfer old Lemma partly into other Lemmata. Trash remains.`

LEMMA 4.2.7. *(v) FROM: lem:old.A.K-basic1:*

If $u \in \mathcal{U}(A)$ with $0 = [u] \in \mathrm{K}_1(A)$, and if there exists a splitting projection $p \in A$ with $\|up - pu\| < 1$, then $u \in \mathcal{U}_0(A)$.

In particular, A is K_1-injective if for every $u \in \mathcal{U}(A)$ with $0 = [u] \in \mathrm{K}_1(A)$ there exists a splitting projection $p \in A$ with $\|up - pu\| < 1$, then $u \in \mathcal{U}_0(A)$.

(v): Let $u \in \mathcal{U}(A)$ with $0 = [u] \in \mathrm{K}_1(A)$, and suppose that there is a splitting projection $p \in A$ with $\|pu - up\| < 1$. By (iv), $u \sim_h v + (1 - p)$ for some unitary $v \in \mathcal{U}(pAp)$. It follows $[v + (1 - p)] = [u] = 0$ in $\mathrm{K}_1(A)$. Since $[v + (1 - p)] = 0$, there exists $n \in \mathbb{N}$ such that $(v + (1 - p)) \oplus 1_n \in \mathcal{U}_0(M_{n+1}(A))$, i.e., there is a path $W(t) \in \mathcal{U}(M_{n+1}(A))$ with $W(0) = 1_{n+1}$ and $W(1) = (v + (1 - p)) \oplus 1_n$. Since $1 - p$ is full and properly infinite in A, the projection $(1 - p) \oplus 0_n$ is full and properly infinite in $M_{n+1}(A)$. Thus, there is a partial isometry $Z \in M_{n+1}(A)$ with $Z^*Z = (1 - p) \oplus 1_n$ and $ZZ^* \leq (1 - p) \oplus 0_n$. Let $w(t) := (p + Z)W(t)(Z^* + p) + ((1 - p) \oplus 0_n - ZZ^*)$. Then $w(t)^*w(t) = 1 \oplus 0_n = w(t)w(t)^*$, $w(0) = 1 \oplus 0_n$ and $w(1) = (v + (1 - p)) \oplus 0_n$. If we naturally identify A with the corner $A \oplus 0_n$ of $M_{n+1}(A)$, then $w(t)$ becomes a continuous path in $\mathcal{U}(A)$ that connects $v + (1 - p)$ with 1. Thus, $u \sim_h 1$.

(vi) *If, for any two splitting projection $p, q \in A$, there are splitting projections $r, s \in A$ with $r \leq p$, $s \leq q$ and $\|rs\| < 1$, then all splitting projections with same class $[p] = [q]$ in $\mathrm{K}_0(A)$ are homotopic.*

(vi): If A does not contain splitting projections then (vi) is true. Suppose that A contains splitting projections. (Then the unit of A is properly infinite, and if $S, T \in A$ are isometries with $S^*T = 0$, then SS^* and TT^* are splitting projections.)

Let $p, q \in A$ splitting projections with $[p] = [q]$ in $\mathrm{K}_0(A)$.

By assumptions, there are splitting projections $r \leq p$ and $s \leq q$ with $\|rs\| < 1$. Since, r and s are properly infinite and full, there exists partial isometries $y, z \in A$ with $y^*y = p$, $r_1 := yy^* \leq r$, $z^*z = q$, and $s_1 := zz^* \leq s$. It implies, that r_1 and s_1 are splitting projections, and that $\|r_1 s_1\| \leq \|rs\| < 1$.

*Since $[p] = [q]$, we get $[r_1] = [s_1]$. Thus, the properly infinite projections r_1 and s_1 are MvN-equivalent (cf. [**172**]), i.e., there is $x \in A$ with $x^*x = r_1$ and $xx^* = s_1$. Now parts (iii) gives that $r_1 \sim_h s_1$.*

Since p is splitting, $1 - p$ is full and properly infinite, and there is $w \in A$ with $w^*w = p$ and $ww^* \leq 1 - p$. It follows $ww^* \sim_h p$ by part (iii). Part (iii) applies also to $ww^* = (yw^*)^*(yw^*)$ and $r_1 = (yw^*)(yw^*)^*$, and gives $ww^* \sim_h r_1$. Thus, $p \sim_h r_1$. Similar arguments show $q \sim_h s_1$. It follows $p \sim_h q$.

(vii) *If A has a properly infinite unit, then A is K_1-injective, if and only if, all splitting projections with same class $[p] = [q]$ in $\mathrm{K}_0(A)$ are homotopic.*

(viii) *????? For every isometries s and t with orthogonal ranges (i.e., $s^*t = 0$) and every unitary $u \in \mathcal{U}(A)$ the projections tt^* and u^*tt^*u are homotopic inside the projections of A, noted by $tt^* \sim_h u^*tt^*u$.*

(Requires K_1-injectivity !!!???)

(ix) *For every isometries s and t with orthogonal ranges (i.e., $s^*t = 0$) and every unitary $u \in \mathcal{U}(A)$ the projections tt^* and u^*tt^*u are homotopic inside the projections of A, – noted by $tt^* \sim_h u^*tt^*u$ –, for every unitary $u \in A$, if and only if, A is K_1-injective.*

Remark 4.2.8. The Part (v,5) of Lemma 4.2.6 has to be considered with some care, because it can happen for some $u \in \mathcal{U}(pEp)$ that $u + (1 - p) \in \mathcal{U}_0(E)$ but $0 \neq [u] \in \mathrm{K}_1(pEp)$ if the projection p is *not* full in E, e.g. :

Let $E = \mathcal{M}(\mathrm{C}(S^1) \otimes \mathbb{K})$, $u_0 \in \mathcal{U}(\mathrm{C}(S^1))$ the canonical generator of $\mathrm{C}(S^1)$, $p = 1 \otimes e_{11} \in \mathrm{C}(S^1) \otimes \mathbb{K}$. Then $pEp \cong \mathrm{C}(S^1)$. If $u := u_0 \otimes e_{11}$, then $u + (1 - p) \in \mathcal{U}_0(E) = \mathcal{U}(E)$ by the generalized Kuiper theorem [**180**], but $0 \neq [u] \in \mathrm{K}_1(pEp)$.

Proof. (vii): By assumption, the unit 1 is properly infinite in A, i.e., there are isometries $s, t \in A$ with $s^*t = 0$. Thus A contains a splitting projection p, e.g. $p := ss^*$.

Suppose that *all splitting projections with same class in* $\mathrm{K}_0(A)$ *are homotopic*, and let $u \in \mathcal{U}(A)$ with $[u] = 0$ in $\mathrm{K}_1(A)$. The projection u^*pu is again splitting, and is MvN-equivalent to p. Thus, $[p] = [u^*pu]$ in $\mathrm{K}_0(A)$, and $p \sim_h u^*pu$. By part(ii), there is $v \in \mathcal{U}_0(A)$ with $v^*u^*puv = p$. We have $[uv] = [u] + [v] = 0$. The arguments in the proof of part (vi) show that $uv \in \mathcal{U}_0(A)$. Thus $u \sim_h 1$.

Suppose, conversely, that A is K_1-injective. Let $p, q \in A$ splitting projections with $[p] = [q]$ in $\mathrm{K}_0(A)$. It follows $[1-p] = [1]-[p] = [1-q]$. Since p, q, $1-p$ and $1-q$ are all full and properly infinite (by definition of splitting projections), it follows from “an observation in [**172**]”, that $p \sim q$ and $1 - p \sim 1 - q$ (WvN-equivalence).

Thus, there exists a unitary $W \in A$ with $W^*pW = q$. Since $1-p$ is full and properly infinite, there exists an isometry $s \in A$ with $ss^* \leq 1-p$. The projection $1-ss^*$ satisfies $[1-ss^*]=0$. Thus, the properly infinite full projections $0 \oplus 1$ and $(1-ss^*)\oplus 1$ are MvN-equivalent in $M_2(A)$, and there is a partial isometry $Z \in M_2(A)$ with $Z^*Z = 0 \oplus 1$ and $ZZ^* = (1-ss^*)\oplus 1$. Let $V := (s\oplus 0)+Z$, then V is unitary and $((1-ss^*)+sWs^*)\oplus 1 = V(W\oplus 1)V^*$. In particular, $[(1-ss^*)+sWs^*] = [W]$ in $\mathrm{K}_1(A)$. The operator $u := ((1-ss^*)+sW^*s^*)W$ is unitary, satisfies $u^*pu = q$, and $[u] = [W]-[(1-ss^*)+sWs^*] = 0$ in $\mathrm{K}_1(A)$. Since A is K_1-injective, we get $u \in \mathcal{U}_0(A)$, i.e., $p \sim_h q$.

(ix): If $s,t \in A$ are isometries with $s^*t = 0$, then $p := tt^*$ is a splitting projection. Now suppose that $u \in \mathcal{U}(A)$ satisfies $0 = [u] \in \mathrm{K}_1(A)$, and that $p \sim_h u^*pu$. Then, by part (ii), there is $w \in \mathcal{U}_0(A)$ with $p = w^*(u^*pu)w$, i.e., $vp = pv$ for the unitary $v = uw$. We get $[v] = [u]+[w] = 0$, because, $[u] = 0 = [w]$. It follows $v \in \mathcal{U}_0(A)$ by part (iv). Hence, $u = vw^* \in \mathcal{U}_0(A)$.

If A is K_1-injective and $u \in A$ unitary, then $u^*tt^*u = v^*tt^*v \sim_h tt^*$ for $v := ((1-ss^*)+su^*s^*)u$, because $[v] = 0$ in $\mathrm{K}_1(A)$, which implies $v \in \mathcal{U}_0(A)$. □

Proof. The Lemmata 4.2.1 and 4.2.6 are also valid for *real* C^*-algebras E. Some definitions have to be done with some care, see e.g. proof of Part (v,6) or proofs in preliminary Observation (o).

Suppose that 1_E is properly infinite, then there are isometries $t_1, t_2 \in E$ with $t_1^*t_2 = 0$. We can take the isometries $s_1 := t_1$ and $s_2 := t_2t_1$, then $s_1^*s_2 = 0$ and

$$p_{[-1]} := 1-(s_1s_1^* + s_2s_2^*) \geq t_2^2(t_2^2)^*$$

is full and properly infinite with $[p_{[-1]}] = [1]-2[1] = -[1]$ in $\mathrm{K}_0(E)$. Thus here $s_1 := \psi(T_1)$ and $s_2 := \psi(T_2)$ come from from a unital C^*-morphism from $\mathcal{E}_3 := C^*(T_1,T_2,T_3\,;\ T_j^*T_k = \delta_{j,k}1)$ into E.

The more special case is where we can find isometries $t_1, t_2 \in E$ with $t_1t_1^* + t_2t_2^* = 1$ then $[1] = [t_1t_1^*]+[t_2t_2^*] = 2[1]$ in $\mathrm{K}_0(E)$, i.e., $[1]=0$ in $\mathrm{K}_0(E)$, and the $s_1 := t_1, s_2 := t_2$ come from a unital C^*-morphism of $\mathcal{O}_2 := C^*(T_1,T_2\,;\ T_j^*T_k = \delta_{j,k}1,\ T_1T_1^*+T_2T_2^* = 1)$. This is exactly the case where $[1]=0$ in $\mathrm{K}_0(E)$: Then $1-t_2t_2^*$ is a properly infinite and full projection in E with $0 = [1] = [1-t_2t_2^*] \in \mathrm{K}_0(E)$. By the below proved Part (ii), or by [**172**, thm. 1.4], there is an isometry $s_1 \in E$ with $s_1s_1^* + t_2^*t_2 = 1$. Let $s_2 := t_2$ and get $p_{[-1]} = 0$ for this s_1, s_2, that generate copy of $\mathcal{O}_2$.

Recall that *MvN-equivalence of projections* $p, q \in M_n(A)$ means that there is a partial isometry $v \in M_n(A)$ such that $v^*v = p$ and $vv^* = q$. The equivalence definition does not require that A is unital.

Part (i) can be seen from the definitions regarding the arguments in preliminary Part (o) of our proof and Lemmata 4.1.3 and 4.2.1.

Part (ii) is contained essentially in [**172**, thm. 1.4], which says – in our terminology – that the Murray–von-Neumann equivalence classes of full properly infinite projections in a properly infinite unital C^*-algebra E build a group with (generalized) Cuntz addition and that this group is naturally isomorphic to $\mathrm{K}_0(E)$.

If we combine this with the below considered observation (C0) and Remark 4.1.2 then we get the following basic property (Cu) that generalizes [**172**, thm. 1.4] (for simple unital C^*-algebras A) to the case of general C^*-algebras A:

Basic property of full properly infinite projections:

(Cu) Full and properly infinite projections p, q in a C^*-algebra A are Murray–von-Neumann equivalent if and only if $[p] = [q] \in \mathrm{K}_0(A)$. (Here A is not necessarily unital.)

If A contains a full and properly infinite projection $e \in A$, then each element $x \in \mathrm{K}_0(A)$ is the class $[p] = x$ of a properly infinite and full projection $p \in A$ with $p \leq e$.

The observation (Cu) of J. Cuntz is almost equivalent to Part (ii). Our proof uses the Parts (i) and (ii) of the "innocent" abstract Lemma 4.2.3. It shows at first that the sub-semigroup of MvN-equivalence classes of full and properly infinite projections build a group under Cuntz sum and that the natural morphism of this group into $\mathrm{K}_0(A)$ is surjective. Then we compare the defining relations and obtain from Lemma 4.2.3(ii) that this morphism must be a group isomorphism. This proof of observation (Cu) is only slightly different from that of [**172**, thm. 1.4]. It is useful to prove the below considered Observation (o) and Parts (i)–(iii) together.

The below stated basic observation (C0) in the proof of (i,ii,iii) reduces the proof of observation (Cu) to the case of the *unital* properly infinite $E = eAe$, using the natural isomorphism $\mathrm{K}_0(E) \cong \mathrm{K}_0(A)$ of Remark 4.1.2, cf. also Part (v,5).

(o): Let

$$U_c := s_1 s_2^* - s_2 s_1^* + p_{[-1]} = \exp\big((\pi/2)(s_1 s_2^* - s_2 s_1^*)\big)\,. \tag{2.3}$$

Then $U_c \in \mathcal{U}_0(C^*_{\mathbb{R}}(s_1, s_2))$, $U_c^2 = p_{[-1]} - (s_1 s_1^* + s_2 s_2^*)$, U_c has "geodesic" distance

$$\mathrm{cel}(U_c) \leq (\pi/2)\|s_1 s_2^* - s_2 s_1^*\| = \pi/2$$

to 1 inside the metric space $\mathcal{U}_0(E)$, and

$$U_c s_1 = -s_2\,, \quad U_c s_2 = s_1\,, \quad U_c p_{[-1]} = p_{[-1]}\,.$$

Hence, $U_c^*(a \oplus_{s_1,s_2} b) U_c = b \oplus_{s_1,s_2} a$ for $a, b \in E$ and $U_c \in \mathcal{U}_0(E)$ for real or complex C^*-algebras E.

We define a partial unitary Z with $Z^*Z = 1 - p_{[-1]} = ZZ^*$ by

$$Z := s_1^2 s_1^* + s_2 (s_2^*)^2 + (s_1 s_2)(s_2 s_1)^* - s_1 p_{[-1]} s_2^* \in C^*(s_1, s_2) \subseteq E\,.$$

Straight calculation gives $Z\big(a \oplus (b \oplus c)\big) = \big((a \oplus b) \oplus c\big) Z$ for $a, b, c \in E$, and $Z s_2 p_{[-1]} s_2^* = s_1 p_{[-1]} s_1^* Z$.

If $U(s_1) \in \mathcal{U}_0\big(M_2(C^*(s_1, s_2))\big) \subseteq \mathcal{U}_0(M_2(E))$ denotes the Halmos unitary of the isometry s_1 (cf. Remark 4.2.4), then

$$Z = [s_1, s_2]\,\big(U(s_2) \cdot \mathrm{diag}(1, U_c)\big)\,[s_1, s_2]^*\,.$$

Thus $Z \in \mathcal{U}_0\big((1-p_{[-1]})C^*(s_1,s_2)(1-p_{[-1]})\big)$, and the unitary

$$U_d := Z + p_{[-1]} = s_1^2 s_1^* + (s_1 s_2)(s_2 s_1)^* + s_2(s_2^*)^2 - s_1 p_{[-1]} s_2^* + p_{[-1]} \tag{2.4}$$

is in $\mathcal{U}_0(C^*(s_1,s_2)) \subseteq \mathcal{U}_0(E)$, and that U_d "realizes" the distributive law for $\oplus_{s_1,s_2}$ as unitary equivalence via U_d.

By Remark 4.2.4, $\mathrm{cel}(U(s_2)) \leq \pi/2$. If we apply this to the above given decompositions of Z and U_d, then we get that the unitary U_d has "exponential length", i.e., "geodesic" distance

$$\mathrm{cel}(U_d) \leq \mathrm{cel}(U(s_2)) + \mathrm{cel}(U_c) \leq \pi$$

from 1 inside the metric space $\mathcal{U}_0(E)$.

Calculation shows that the unitary equivalence by U_c and U_d includes also the "un-truncated" or "corrected" version $\oplus'$ of $\oplus_{s_1,s_2}$ given by

$$u \oplus' v := s_1 u s_1^* + s_2 v s_2^* + p_{[-1]}$$

on the group of unitary operators or the semi-group of isometries $u, v, w \in E$:

$$U_c^*(u \oplus' v)U_c = v \oplus' u$$

and

$$U_d^*((u \oplus' v) \oplus' w))U_d = u \oplus' (v \oplus' w)\,.$$

This can be seen from $U_c p_{[-1]} = p_{[-1]}$ and $Z s_2 p_{[-1]} s_2^* = s_1 p_{[-1]} s_1^* Z$.

(i,ii,iii): The commutative and associative law for $\oplus = \oplus_{s_1,s_2}$, up to unitary equivalence with a unitary in $\mathcal{U}_0(E)$, follows from Part (o). The other formulae by simple calculations, e.g. $u^*(a \oplus b)u = (u_1^* a u_1) \oplus (u_2^* a u_2)$ for $u := (u_1 \oplus u_2) + p_{[-1]}$, and is in $\mathcal{U}_0(E)$ if $u_1, u_2 \in \mathcal{U}_0(E)$, ...

`Old Lemma form Appendix A, now somewhere above contained:`

Suppose that $p, q \in A$ are full properly infinite projections.

If $[p] = [q]$ in $\mathrm{K}_0(A)$ then there exists $v \in A$ with $v^*v = p$ and $vv^* = q$.

If A contains a full and properly infinite projection p then for each $x \in \mathrm{K}_0(A)$ there exists a full properly infinite projection $q \leq p$ with $x = [q]$.

If $p, r \in A$ are full and properly infinite projections that have the same class $[p] = [q] \in \mathrm{K}_0(A)$ then there exists a partial isometry $z \in A$ with $z^*z = p$ and $zz^* = q$.

The proof of the observation (Cu) of J. Cuntz uses following elementary observation (C0), given below, but the proofs of (Cu), (ii) and (iii) follows not immediately from (C0):

(C0) If A is a (not necessarily unital) C^*-algebra and $e \in A$ is a full and properly infinite projection, then for every $n \in \mathbb{N}$ and every projection $p \in M_n(A)$ there is a projection $p' \in A$ such that p and $p' \oplus 0_{n-1} = \mathrm{diag}(p', 0, \ldots, 0)$ are MvN-equivalent in $M_n(A)$, $p' \leq e$ and $e - p'$ is full and properly infinite in A.

The observation (C0) reduces the proof of observation (Cu) to the case of properly infinite *unital* algebras.

In fact it is formally more general, and is easy to see, that the following holds: If e, p are projections in a C^*-algebra B such that e is properly infinite and p is in the closed ideal $\overline{BeB}$ generated by e, then there exists a partial isometry $z \in B$ with $z^*z = p$ and $zz^* \leq e$. Below we outline this in case $B = M_n(eAe)$.

The *proof of* (C0) goes as follows: $E := eAe$ has a properly infinite unit $1_E := e$, i.e., there are isometries $s_1, s_2 \in E$ with $s_1^* s_2 = 0$. Define isometries $t_k := s_2^{k-1} s_1 \in E$ for $k = 1, 2, \ldots$ with $t_k^* t_j = \delta_{j,k} 1_E$. Let $z_k := t_k \oplus 0_{n-1} = \mathrm{diag}(t_k, 0, \ldots, 0) \in M_n(E) \subseteq M_n(A)$. Then $z_j^* z_k = \delta_{jk}\, \mathrm{diag}(e, 0, \ldots, 0)$.

Since the projection $e \in A$ is full in A and is properly infinite and since the direct sum $A \oplus \{0_{n-1}\} \cong A \otimes p_{11}$ is full in $M_n(A) \cong A \otimes M_n$, the projection $p := \mathrm{diag}(e, 0, \ldots, 0) = e \oplus 0_{n-1}$ is full and properly infinite in $M_n(A)$.

It follows that, for each projection $q \in M_n(A)$ and $\varepsilon \in (0, 1/4]$, there are $m \in \mathbb{N}$ and elements $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$ such that $\| q - \sum_k a_k p b_k \| \leq \varepsilon$. We get $\| q - x^* p y \| \leq \varepsilon$ for $x := \sum_j z_j a_j$ and $y := \sum_k z_k b_k$. It implies the existence of an element $c \in M_n(A)$ with $c^* p c = q$. The projection $r = pcc^*p \leq p = (e, 0, \ldots, 0)$ has form $r = (p'', 0, \ldots, 0)$ for a projection $p'' \leq e$. The projection $p' := s_1 p'' s_1^* \leq e$ has the quoted property that $e - p' \geq s_2 s_2^*$ is full and properly infinite in $E = eAe$ and the given projection $r \in M_n(A)$ is Murray–von-Neumann equivalent to $p' \in E \cong E \oplus 0_{n-1}$ inside $M_n(A)$.

Therefore it suffices to consider only the MvN-equivalence classes $[p]$ of projections $p \in E := eAe$, because each MvN-equivalence class $[q]$ of a projection q in $\bigcup_n M_n(A)$ is identical with the MvN-class $[p]$ of some projection $p \in E$, and we restrict later our considerations to C^*-algebras E with properly infinite unit element.

The projections in E build a commutative semi-group S under Cuntz addition $[p]_{MvN} + [q]_{MvN} := [p \oplus_{s_1,s_2} q]_{MvN}$. See Parts (o) and (i) for the additive and associative law modulo unitary homotopy, i.e., modulo unitary equivalence by unitaries in $\mathcal{U}_0(E)$.

If we consider E as a full corner of $M_n(E)$ by $a \in E \mapsto \mathrm{diag}(a, 0, \ldots, 0)$, then the sum $p \oplus q$ is MvN-equivalent to $p' + q'$ if $p', q' \in M_n(E)$ are projections with $p'q' = 0$, and are MvN-equivalent in $M_n(E)$ to p respectively q. It follows that the natural map $[p]_{MvN} \mapsto [p] \in \mathrm{K}_0(E)$ becomes a semi-group *epimorphism* from the semi-group S of the MvN-equivalence classes of projections $p \in E$ onto $\mathrm{K}_0(E)$.

Since, by definition of K_0, $[p] = [q] \in \mathrm{K}_0(E)$ if and only if there exists $n \in \mathbb{N}$ and a projection $r' \in M_n(E)$ and such that $\mathrm{diag}(p, r')$ and $\mathrm{diag}(q, r')$ are MvN-equivalent in $M_{n+1}(E)$, we can find a projection $r \in E$ that is MvN-equivalent to r' and such that $p \oplus_{s_1,s_2} r$ and $q \oplus_{s_1,s_2} r$ are MvN-equivalent. It follows that the map from the semi-group S of MvN-classes of projections in E into $\mathrm{K}_0(E)$ defines just the same equivalence relations on S as the canonical semi-group morphism from S into its Grothendieck group $\mathrm{Gr}(S)$.

In conclusion, the natural group epimorphism from $\mathrm{Gr}(S)$ onto $\mathrm{K}_0(E)$ is an isomorphism.

By observations (C0), the element $z_0 := [1_E]_{MvN} \in S$ can play the role of z_0 in Lemma 4.2.3(ii) for the semi-group S of MvN-equivalence classes of projections in E with Cuntz addition, because for every $p \in E$, $p \oplus (1-p)$ is MvN-equivalent to 1_E, cf. rules in Part (i), i.e., $[p] + [1-p] = [1] = z_0$ in S. If we let $y_0 := [1 - (1 \oplus 1)] = [p_{[-1]}]$, then $2z_0 + y_0 = z_0$ in S, and $e := z_0 + y_0 = [1 \oplus p_{[-1]}]$ is a dominate zero element of S by Lemma 4.2.3. Thus, the "splitting" projection $1 \oplus p_{[-1]} \oplus 0$ represents the "absorbing" zero element of $e + S \cong \mathrm{Gr}(S)$ via $e + s \mapsto [s]_{\mathrm{Gr}}$ by Lemma 4.2.3. Since $e + e = e$ in S, the natural map from S onto $\mathrm{K}_0(E)$ maps e into the zero element and defines an isomorphism from the subgroup $e + S$ of S onto $\mathrm{K}_0(E)$, because it induces an isomorphism of the Grothendieck group $\mathrm{Gr}(S)$ with $\mathrm{K}_0(E)$ by Lemma 4.2.3(i).

It implies that p and q are MvN-equivalent, if $p, q \in E$ if $[p] = [q]$ in $\mathrm{K}_0(E)$ and are MvN-equivalent to projections in $E \oplus (1 \oplus p_{[-1]})$, i.e., have MvN-class in $S + e$. Since $1 \oplus p_{[-1]} \geq s_1 s_1^*$ is full and properly infinite, each projection that is MvN-equivalent to projections in $E \oplus (1 \oplus p_{[-1]})$ is necessarily full and properly infinite.

The observation (C0) gives that full and properly infinite projections $p \in E$ are MvN-equivalent to projections in $E \oplus (1 \oplus p_{[-1]})$, because there exist a partial isometry $z \in E$ with $z^* z = 1 \oplus p_{[-1]}$ and $zz^* \leq p$, thus p is MvN-equivalent to $(p - zz^*) \oplus (1 \oplus p_{[-1]})$. The projections in $E \oplus (1 \oplus p_{[-1]})$ itself are all *splitting* because its complements majorize $s_2(1 - p_{[-1]})s_2^* + p_{[-1]} \sim_{MvN} 1$. This finishes the proof of observation (Cu), and shows that the elements of $S + e$ can be realized all as MvN-equivalence classes of splitting projections in E.

It is clear that the full and properly infinite projections p are all MvN-equivalent to splitting projections, e.g. take $s_1 p s_1^*$ in the MvN-equivalence class of p.

Sums $p \oplus q$ with full and properly infinite p are again full and properly infinite, and if at least one of $1 - p$ or $1 - q$ is full and properly infinite then $p \oplus q$ becomes a splitting projection.

The observation (Cu) implies that full and properly infinite projections $p, q \in E$ are MvN-equivalent if and only if $[p] = [q]$ in $\mathrm{K}_0(E)$.

If $[p] = [q]$ then also $[1 - p] = [1] - [p] = [1] - [q] = [1 - q]$. If p and q are splitting then $1 - p$ and $1 - q$ are again full and properly infinite. Thus, there exists a partial isometry z with $z^* z = 1 - p$ and $zz^* = 1 - q$. The unitary $u := v + z$ satisfies $up = qu$ if $vv^* = q$ and $v^* v = p$.

If $p, q \in E$ are properly infinite full projections then there exists an isometry $t \in E$ with $tt^* \leq p$. With this isometry it holds: If $v \in E$ is a unitary with $v^* p v = q$

then $u := (tv^*t^* + (1 - tt^*))v$ is a unitary with $0 = [u] \in \mathrm{K}_1(E)$ and $u^*pu = q$, because $tv^*t^* + (1 - tt^*)$ commutes with p.

(iv,a): Let $p, q \in E$ full and properly infinite projections with $[p] = [q] \in \mathrm{K}_0(E)$, and suppose that there exists an isometry $t \in E$ with $t^*(p + q)t = 0$. Then p, q and $r := tpt^*$ are splitting projections and $[p] = [q] = [r]$ in $\mathrm{K}_0(E)$. Since this projections are full and properly infinite, there exist by Part (ii) partial isometries $z_1, z_2 \in \mathcal{U}(E)$ with $z_1^*z_1 = r = z_2^*z_2$ and $z_1z_1^* = p$ and $z_2z_2^* = q$. We get from $t^*(p + q)t = 0$ that $t^*(p + q) = 0$ and $r(p + q) = 0$. Thus $z_k^2 = 0$ for $k = 1, 2$.

The unitary elements

$$u_k := (1 - z_k^*z_k - z_kz_k^*) + (z_k^* - z_k) = \exp((\pi/2)(z_k^* - z_k))\,,$$

are in $\mathcal{U}_0(E)$ and satisfy $u_k^*(z_kz_k^*)u_k = z_k^*z_k = r$ for $k = 1, 2$ by Part (v,6). Hence $u_1^*pu_1 = r = u_2^*qu_2$, and the unitary $u := u_1u_2^*$ with $u^*pu = q$ is in $\mathcal{U}_0(E)$.

(iv,b): Let $p, q \in E$ projections, and define $P := (p \oplus 1) \oplus 0 = s_1^2p(s_1^*)^2 + s_1s_2(s_1s_2)^*$ and $Q := (q \oplus 1) \oplus 0 = s_1^2q(s_1^*)^2 + s_1s_2(s_1s_2)^*$. Then $[p] = [q]$ in $\mathrm{K}_0(E)$, if and only if,

$$[P] = [(p \oplus 1) \oplus 0] = [p] + [1] = [q] + [1] = [(q \oplus 1) \oplus 0] = [Q]\,.$$

The projections P and Q are full and properly infinite because $s_1s_2(s_1s_2)^* \leq P$ and $s_1s_2(s_1s_2)^* \leq Q$. They are also splitting projections and satisfy the assumptions of Part (iv,a), because $s_2^*(P + Q)s_2 = 0$.

Thus, $[p] = [q]$, if and only if, $[P] = [Q]$, if and only if, there exist $u \in \mathcal{U}_0(E)$ with $u^*Pu = Q$.

(v): We show first (v,2) and that $u \in \mathcal{U}(E) \mapsto [u] \in \mathrm{K}_1(E)$ is surjective. Since (v,6) is obvious, this allows to verify (v,1) and (v,4) easily by using (v,3) and (v,6). Therefore we display only the proofs of (v,2), (v,5) and (v,3) in more detail.

(v,2): Recall that $\mathrm{K}_1(E)$ for unital C^*-algebras E can be described as the inductive limit of the discrete groups $\mathcal{U}(M_n(E))/\mathcal{U}_0(M_n(E))$ by the group morphisms

$$u \in \mathcal{U}(M_n(E)) \mapsto u \oplus 1 \in \mathcal{U}(M_{n+1}(E))\,,$$

where on this place $\oplus$ means the *direct* sum (despite that the original definition of the topological $\mathrm{K}_1(B)$ for unital Banach algebras is formally different).

Let $s_1, s_2 \in E$ isometries with $s_1^*s_2 = 0$ (e.g. as above chosen). We define isometries $t_k := s_2^{k-1}s_1 \in E$ for $k = 1, 2, \ldots$.

The t_n satisfy $t_n^*t_m = \delta_{n,m}1$. Let T (more precisely T_n) denote the row $[t_1, t_2, \ldots, t_n, t_n] \in M_{1,n}(E)$. The row T defines a partial isometry in $M_n(E)$ with $T^*T = 1_n$ and $TT^* \leq \mathrm{diag}(1, 0, \ldots, 0)$. We define a C^*-morphism

$$\varphi\colon\ M_n(E) \ni [a_{jk}] \mapsto T[a_{jk}]_{nn}T^* = \sum_{jk} t_ja_{jk}t_k^*$$

from $M_n(E)$ onto e_nEe_n, where

$$e_n := TT^* = t_1t_1^* + \cdots + t_nt_n^* \leq 1\,.$$

The restriction to diagonal elements is given by

$$\varphi\big(\mathrm{diag}(a_1, a_2, \ldots, a_n)\big) \;=\; T_n\;\mathrm{diag}(a_1, a_2, \ldots, a_n)\, T_n^* \;=\; \sum_{1\le j\le n} t_j a_j t_j^*\,.$$

That $u \in \mathcal{U}(E) \mapsto [u] \in \mathrm{K}_1(E)$ is *surjective* follows from the fact that $V \oplus 1_n$ and

$$\mathrm{diag}(\varphi(V) + 1 - \varphi(1_n)\,,\; 1, \ldots, 1) := (\varphi(V) + 1 - \varphi(1_n)) \oplus 1_{2n-1}$$

are unitarily equivalent in $M_{2n}(E)$ for each unitary $V \in M_n(E)$ by the *Halmos* unitary $U(T_n) \in M_2(M_n(E)) \cong M_{2n}(E)$ of the contraction T_n (cf. Remark 4.2.4), i.e., $U(T_n) = [a_{jk}]$ is given by $a_{11} := T_n$ (considered as matrix in $M_n(E)$ with 2-nd to n-th rows equal to zero), $a_{21} := 0$, $a_{22} := a_{11}^*$ and $a_{12} := -(1 - a_{11}a_{11}^*)$. The matrix $U(T_n)$ is a Halmos unitary in $\mathcal{U}_0(M_{2n}(E))$, and

$$U(T_n)(\,V \oplus 1_n\,)U(T_n)^* = a_{11}\, V\, a_{11}^* \;+\; 1_n - a_{11}a_{11}^*$$

The right side of the latter formula is the same as $\mathrm{diag}(\,\varphi(V) + 1 - \varphi(1)\,,\, 1,\, \ldots\,, 1)\,.$ Thus, $[\varphi(V) + 1 - \varphi(1)] = [V]$ in $\mathrm{K}_1(E)$ and the natural map $u \in \mathcal{U}(E) \mapsto [u] \in \mathrm{K}_1(E)$ is *surjective*.

Let $u_0, u_1 \in \mathcal{U}(E)$ and $u \in \mathcal{U}(pEp)$ with $[u_0] = [u_1]$ in $\mathrm{K}_1(E)$.

By (topological) definition of K_1-groups, cf. [**692**, def. 8.1.3], $[u_0] = [u_1]$ in $\mathrm{K}_1(E)$, if and only if, there is $n \in \mathbb{N}$ such that the unitaries $\mathrm{diag}(u_0, 1, \ldots, 1)$ and $\mathrm{diag}(u_1, 1, \ldots, 1)$ are homotopic in $\mathcal{U}(M_n(E))$, i.e., $W := \mathrm{diag}(u_1^* u_0, 1, \ldots, 1) \in \mathcal{U}_0(M_n(E))$ for sufficiently big $n \in \mathbb{N}$. It follows that

$$s_1(u_1^* u_0)s_1^* + (1 - s_1 s_1^*) \;=\; \varphi(W) \;+\; (1 - e_n) \;\in \mathcal{U}_0(E)\,.$$

It gives the desired homotopy in $\mathcal{U}(E)$:

$$(u_1 \oplus 1) + p_{[-1]} = s_1 u_1 s_1^* + (1 - s_1 s_1^*) \sim_h s_1 u_0 s_1^* + (1 - s_1 s_1^*) = (u_0 \oplus 1) + p_{[-1]}\,.$$

(v,5): The difference between the case where p is not full and the here considered case where p is full can be seen in Example 4.2.9.

Let $p \in E$ a projection such that $1 - p$ is full and properly infinite in E and let $u \in \mathcal{U}(pEp)$ a unitary that satisfies $[u + (1 - p)] = 0$ in $\mathrm{K}_1(E)$.

Since $1 - p$ is full and properly infinite, the unit of E must be properly infinite. Thus, there are isometries $s_1, s_2 \in E$ that satisfy the non-degeneracy property (ND) on $p_{[-1]}$.

We define below unitary operators $W, v \in E$ that satisfy

$$(v \oplus_{s_1,s_2} 1) + p_{[-1]} \;=\; s_1 v s_1^* + (1 - s_1 s_1^*) \;=\; W^*(u + (1 - p)W\,.$$

It follows $[(v \oplus 1) + p_{[-1]}] = [u + (1 - p)] = 0$ in $\mathrm{K}_1(E)$. We get $(v \oplus 1) + p_{[-1]} \in \mathcal{U}_0(E)$ by Part (v,2), and can conclude that $u + (1 - p) = W((v \oplus 1) + p_{[-1]})W^* \in \mathcal{U}_0(E)$.

There is an isometry $t \in E$ with $tt^* \le (1-p)$, because $(1-p)$ is full and properly infinite. The projection $r := t(s_2 s_2^* + p_{[-1]})t^*$ is splitting, because $ts_2(ts_2)^* \le r$ and $ts_1(ts_1)^* \le 1 - r$. Obviously, $s_2 s_2^* + p_{[-1]}$ is a splitting projection.

The Weyl–von-Neumann equivalence $r \sim_{WvN} s_2s_2^* + p_{[-1]}$, and the unitary equivalence of splitting projections with same class in $K_0(E)$ – as established in Part (iii) – imply the existence of a unitary $W \in \mathcal{U}(E)$ with $W^*rW = s_2s_2^* + p_{[-1]} = 1 - s_1s_1^*$.

Since $r \leq (1-p)$, the element $e := 1-(p+r) \leq 1-p$ is a projection in E with $p+e = 1-r$ and $u+e$ is a unitary in $(p+e)E(p+e) = (1-r)E(1-r)$.

For every $u \in \mathcal{U}(pEp)$, we can define an element $v \in E$ by

$$v := s_1^*W^*(u+e)Ws_1 = s_1^*W^*(u+(1-p))Ws_1 \,.$$

The element $v \in E$ is unitary, because $u+e \in \mathcal{U}((p+e)E(p+e))$, $Ws_1s_1^* = (1-r)W = (p+e)W$ and $s_1^*W^*Ws_1 = 1$. Then $1-p = e+r$ and

$$(v \oplus 1) + p_{[-1]} = s_1vs_1^* + (1 - s_1s_1^*) = W^*((u+e)+r)W = W^*(u+(1-p))W \,.$$

We consider now the case where the projection $p \in E$ is splitting in E:

If $p \in E$ is a splitting projection, then the inclusion map $\eta\colon pEp \hookrightarrow E$ defines an isomorphism η_* from $K_*(pEp)$ onto $K_*(E)$, because pEp and E are in a natural way stably equivalent: There exists an isometry T in $\mathcal{M}(E \otimes \mathbb{K})$ with $TT^* = p \otimes 1$.

Explicitly the group homomorphisms η_* are given for the classes of splitting projections $e \in pEp$ by $\eta_0([e]) := [e]$ and by $\eta_1([u]) := [u+(1-p)]$ for unitary $u \in \mathcal{U}(pEp)$.

The (generalized) Cuntz addition $\oplus_{s_1,s_2}$ can be chosen such that η_* is compatible with the splitting projection, e.g. such that $s_1s_1^* + s_2s_2^* \leq p$ and and $p - s_1s_1^* - s_2s_2^*$ is a splitting projection in pEp.

This isometries s_1, s_2 can be used to describe the inverse $\theta_* := (\eta_*)^{-1}$ of the η_* on classes $[e] \in K_0(E)$ and $[u] \in K_1(E)$ by $\theta_0([e]) := [s_1es_1^*]$ and $\theta_1([u]) := [s_1us_1^* + (p - s_1s_1^*)]$. Elementary verifications of $\eta_1 \circ \theta_1 = \mathrm{id}$ and $\theta_1 \circ \eta_1 = \mathrm{id}$ follow then from the identities

$$s_1us_1^* + (p - s_1s_1^*) + (1-p) = s_1us_1^* + (1 - s_1s_1^*)$$

and $[u] = [s_1us_1^* + (1 - s_1s_1^*)]$ in $K_1(E)$, and of $[s_1(u+(1-p))s_1^* + (p - s_1s_1^*)] = [u]$ in $K_1(pEp)$. The identities $\eta_0 \circ \theta_0 = \mathrm{id}$ and $\theta_0 \circ \eta_0 = \mathrm{id}$ come from $[s_1es_1^*] = [e]$ for $e \in E$ in $K_0(E)$ and for $e \in pEp$ inside $K_0(pEp)$.

One of the consequences for splitting $p \in E$ and $u \in \mathcal{U}(pEp)$ is:
$[u + (1-p)] = 0$ in $K_1(E)$, if and only if, $[u] = 0$ in $K_1(pEp)$, if and only if, $u + (1-p) \in \mathcal{U}_0(E)$.

(v,3): $(u^* \oplus u) + p_{[-1]} = V^*U_c^*VU_c$, for $V := s_1us_1^* + (1 - s_1s_1) = u \oplus' 1$ and with $U_c \in \mathcal{U}_0(E)$ defined by Equation (2.3) in the proof of Part (o).

(vi): We prove first the result for the special case in sub-part (vi, α), where u commutes with p:
If p is splitting, i.e., if p and $1-p$ are both full and properly infinite projections in

E then there exist isometries $t_1, t_2 \in E$ with $t_1t_1^* \leq p$ and $t_2t_2^* \leq 1-p$. The unitary

$$R := t_1t_2^* - t_2t_1^* + (1 - t_1t_1^* - t_2t_2^*) = \exp((\pi/2)(t_1t_2^* - t_2t_1^*))$$

is in $\mathcal{U}_0(E)$. Obviously $t_1^*R = t_2^*$, $pt_1t_1^* = t_1t_1^*$, $(1-p)t_2t_2^* = t_2t_2^*$ and $R^*t_1t_1^*R = t_2t_2^*$. Thus,

$$R^*pRt_2t_2^* = R^*pt_1t_1^*R = R^*t_1t_1^*R = t_2t_2^*\,.$$

It implies that $R^*(1-p)R\,t_2t_2^* = 0$, $u(1-p)R\,t_2t_2^* = 0$ and that

$$R^*(p + u(1-p))R\,t_2t_2^* = t_2t_2^*$$

Since $(1-p)t_2t_2^* = t_2t_2^*$, we get finally that

$$(up + (1-p))R^*(p + u(1-p))Rt_2t_2^* = t_2t_2^*\,.$$

We let $q := 1 - t_2t_2^*$ and $V := (up + (1-p))R^*(p + u(1-p))R$. Then $1 - q = t_2t_2^*$ is full and properly infinite in E and $V(1-q) = (1-q)$. Hence, there is a unitary $w \in qEq$ such that $w + (1-q) = V$. Since $R \in \mathcal{U}_0(E)$, the unitary V is homotopic in $\mathcal{U}(E)$ to $u = (up + (1-p))(p + u(1-p))$. By assumption, $[u] = 0$ in $\mathrm{K}_1(E)$. It follows $[w + (1-q)] = [V] = [u] = 0$.

We can apply Part (v,5) to $w + (1-q)$ (in place of $u + (1-p)$ of Part (v,5)), because $(1-q) = s_2s_2^*$ is full and properly infinite in E, and $w \in \mathcal{U}(qEq)$ satisfies $[w + (1-q)] = 0$ in $\mathrm{K}_1(E)$. Part (v,5) implies that $V = w + (1-q)$ is in $\mathcal{U}_0(E)$.

Since $u \sim_h V$, it follows that $u \in \mathcal{U}_0(E)$. This proves case (α).

Now suppose, *more generally*, that $u_1, u_2 \in \mathcal{U}(E)$ and a splitting projection $p \in E$ satisfy $\|u_kp - pu_k\| < 1$, for $k = 1, 2$ and that $[u_1] = [u_2] \in \mathrm{K}_1(E)$. Then $q_k := u_kp_ku_k^*$ satisfy $\|q_k - p\| < 1$ for $k = 1, 2$.

By Lemma 4.1.3(v), if $p, q \in E$ are two projections with $\|p - q\| < 1$, then there exists a unitary $v \in \mathcal{U}_0(A)$ with $v^*pv = q$ in the connected component $\mathcal{U}_0(A)$ of 1 in $\mathcal{U}(A)$, where $A := C^*(1, p, q) \subseteq E$.

This applies to the projections p and $q_k := u_kpu_k^*$ $(k = 1, 2)$, and gives $v_1, v_2 \in \mathcal{U}_0(E)$ with $v_kq_kv_k^* = p$. Thus, $v_1u_1pu_1^*v_1^* = v_2u_2pu_2^*v_2^*$ and $u := u_2^*v_2^*v_1u_1$ commutes with $p = t_1t_1^*$. Since $[u_1] = [u_2]$ and $[v_k] = 0$ we get $[u] = [v_1] + [u_1] - [v_2] - [u_2] = 0$.

By the above considered special case (α), we get that $u \in \mathcal{U}_0(E)$. The unitary u is homotopic to $u_2^*u_1$ in $\mathcal{U}(E)$ because $v_2^*v_1 \in \mathcal{U}_0(E)$. Hence, $u_2^*u_1 \in \mathcal{U}_0(E)$ and $u_1 \sim_h u_2$ in $\mathcal{U}(E)$.

(vi,β): Let $t_1, t_2 \in E$ isometries with $\|t_1^*t_2\| < 1$ and $u \in U(E)$ such that $\|t_1^*ut_2\| < 1$ and $[u] = 0 \in \mathrm{K}_1(E)$.

We consider the projection $q := 1 - t_2t_2^*$ and unitarily equivalent properly infinite full projections $p_1 := t_1t_1^*$ and $p_2 := u^*p_1u$. Then p_1 and p_2 are unitarily equivalent and, for $k = 1, 2$,

$$\|(1-q)p_k\| = \|p_kt_2\| \leq \max\big(\|t_1^*t_2\|, \|t_1^*ut_2\|\big) < 1\,.$$

Hence, Lemma 4.1.3(iv) applies and gives skew-adjoint operators $h_k = -h_k^* \in E$ with $\|h_k\| \leq \arcsin \|(1-q)p_k\| < \pi/2$ such that $r_k := \exp(-h_k)p_k \exp(h_k) \leq q$.

The definition of the projections p_k shows that p_1 and p_2 are unitarily equivalent properly infinite and full projections in E. The same follows for the r_k. Moreover the r_k satisfy $t_2^*(r_1 + r_2)t_2 = 0$ because $r_k \leq q = 1 - tt^*$. By Part (iv,a), there exists a unitary $v \in \mathcal{U}_0(E)$ such that $v^* r_1 v = r_2$. It follows that

$$v^* \exp(-h_1)p_1 \exp(h_1)v = v^* r_1 v = r_2 = \exp(-h_2)u^* p_1 u \exp(h_2)\,,$$

i.e., the unitary $w = u \exp(h_2)v^* \exp(-h_1)$ commutes with the projection p_1. The projection $p_1 = \exp(h_1)r_1 \exp(-h_1)$ is splitting, because $p_1 \leq 1 - SS^*$ for the isometry $S := \exp(h_1)t_2$.

The $\mathrm{K}_1(E)$-class $[w]$ of w is zero, because $[u] = 0$ by assumption, $v, \exp(h_k) \in \mathcal{U}_0(E)$ and $[w] = [u] + [\exp(h_1)] - [v] - [\exp(h_2)] = 0$. Thus, Part (vi,α) applies to w and p_1. It proves that $w \in \mathcal{U}_0(E)$, and that $u = w \exp(h_1)v \exp(-h_2)$ is in $\mathcal{U}_0(E)$.

(vii): The observation (Cu), applied to $p := 1 - s_1 s_1^*$ and $q := 1 - t_1 t_1^*$, gives immediately that for two pairs $s_1, s_2 \in E$ and $t_1, t_2 \in E$ of isometries with orthogonal ranges there exist a *unitary* $u \in E$ such that $us_1 = t_1$ and $[u] = 0 \in \mathrm{K}_1(E)$. It can appear that there does not exist any unitary u with $us_1 = t_1$ that satisfies also $us_2 = t_2$. This is e.g. the case if $p_{[-1]}$ and $q_{[-1]}$ are not both zero or are not both properly infinite and full projections.

If $p_{[-1]} = q_{[-1]} = 0$, then $u := t_2 s_2^* + t_1 s_1^*$ is the unique unitary in E with $us_k = t_k$. Therefore, e.g. if $\mathrm{K}_1(E) \neq \{0\}$ and $0 = [1] \in \mathrm{K}_0(E)$, it can happen that $u \in \mathcal{U}(E)$ and $us_k = t_k$ imply $[u] \neq 0$.

But one can always find a unitary v that satisfies $vs_1 = t_1$ and $0 = [u] \in \mathrm{K}_1(E)$ (but not necessarily $vs_2 = t_2$), e.g. let $v := u(1 \oplus_{s_1,s_2} u^*)$ in case $p_{[-1]} = 0$.

In the general case of given pairs of isometries (s_1, s_2) and (t_1, t_2) with orthogonal ranges let $v := t_1 s_1^* + z$ for some partial isometry z with $z^* z = p := 1 - s_1 s_1^*$ and $zz^* = q := 1 - t_1 t_1^*$, existing by Part (ii), because p and q are full and properly infinite. The unitary $u := w^* v$ with $w := (1 - t_2 t_2^*) + t_2 v t_2^*$ is a unitary with $us_1 = t_1$ and $[u] = 0 \in \mathrm{K}_1(E)$.

We can proceed as in the above consideration for the finding of a unitary if $p_{[-1]}$ and $q_{[-1]}$ are both full and properly infinite: Use that $[p_{[-1]}] = -[1_E] = [q_{[-1]}]$ and get from Part (ii) that there is a partial isometry z with $z^* z = p_{[-1]}$ and $zz^* = q_{[-1]}$. Then $v := t_1 s_1^* + t_2 s_2^* + z$ is a unitary with $vs_k = t_k$.

Since $p_{[-1]}$ is full and properly infinite, we get that there exists an isometry $S \in E$ with $SS^* \leq p_{[-1]}$. Then $u := v \cdot \big((1 - SS^*) + Sv^* S\big)$ is a unitary with $us_k = t_k$ and $[u] = 0$ in $\mathrm{K}_1(E)$.

The general observation is the following: Let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ isometries in E with $s_j^* s_k = \delta_{jk} 1 = t_j^* t_k$, i.e., the s_k and t_k correspond to unital C^*-morphisms

h_0 and h_1 from $\mathcal{E}_n$ into E. Let $p_{[1-n]} := 1-\sum_{k=1}^n s_k s_k^*$ and $q_{[1-n]} := 1-\sum_{k=1}^n t_k t_k^*$. Clearly $[p_{[1-n]}] = (1-n)[1] = [q_{[1-n]}]$ in $\mathrm{K}_0(E)$.

If $p_{[1-n]} = q_{[1-n]} = 0$ then $u := \sum_{k=1}^n t_k s_k^*$ is the *unique* unitary with $us_k = t_k$ and it can happen that $[u] \neq 0$.

The projections $p_{[1-n]}$ and $q_{[1-n]}$ are MvN-equivalent in E, if and only if, there exists a $u \in \mathcal{U}(E)$ with $us_k = t_k$. This unitary induces a unitary equivalence of $p_{[1-n]}$ and $q_{[1-n]}$ by $up_{[1-n]}u^* = q_{[1-n]}$. In particular, $p_{[1-n]} \sim_{MvN} q_{[1-n]}$.

The latter is always the case if $p_{[1-n]}$ and $q_{[1-n]}$ are both full and properly infinite:

If $p_{[1-n]}$ and $q_{[1-n]}$ are both full and properly infinite, then they are MvN-equivalent by Part (ii), and there exist isometries s_{n+k} $(k = 1, 2, \ldots)$ with $s_{n+k}^* s_{n+\ell} = 0$ for $k \neq \ell$ and $s_{n+k}s_{n+k}^* \leq p_{[1-n]}$.

The unitary $u \in \mathcal{U}(E)$ with $us_k = t_k$ is determined up to multiplication with elements in $G := (1 - p_{[1-n]}) + \mathcal{U}(p_{[1-n]}Ep_{[1-n]})$ from the right side. It is the same as the multiplication with elements in $(1 - q_{[1-n]}) + \mathcal{U}(q_{[1-n]}Eq_{[1-n]})$ from the left side – because $up_{[1-n]} = q_{[1-n]}u$. Since $v := s_{n+1}u^*s_{n+1}^* + (1 - s_{n+1}s_{n+1}^*)$ is in the group G, it follows that uG contains unitaries w with $[w] = 0$ in $\mathrm{K}_1(E)$, e.g. $w := uv$. Our below explained method and Part (vi,β) indicates that it suffices to find an isometry $t \in E$ such that $\|(1 - p_{[1-n]})t\| < 1$ and $\|(1 - p_{[1-n]})u^*t\| < 1$ to show that $uG \cap \mathcal{U}_0(E)$ is not empty.

The consideration of the case $n = 2$ explains the ideas and appearing problems in the general case of *missing* K_1*-injectivity*. We discuss here only the case $n = 2$ in any detail:

The case – where a properly infinite and full projection $r \in E$ with $r \leq p_{[-1]}$ and $r \leq q_{[-1]}$ exists – uses Part (v,5), or likewise Part (vi,α). (Notice that the above given proofs of Parts (v) and (vi) are independent from our considerations given here.)

Suppose that a properly infinite and full projection $r \in E$ exists with $r \leq p_{[-1]}$ and $r \leq q_{[-1]}$. We find an isometry $T \in E$ such that $TT^* \leq r$. Define $s_3 := t_3 := Ts_1$, and the partial isometry $V := t_1s_1^* + t_2s_2^* + t_3s_3^*$. Then $1 - V^*V =: p_{[-2]} = p_{[-1]} - Ts_1s_1^*T$ and $1 - VV^* =: q_{[-2]} = q_{[-1]} - Ts_1s_1^*T$ are full and properly infinite projections, because each of it dominates the range $Ts_2s_2^*T^*$ of the isometry Ts_2 and E is properly infinite. The projections $p_{[-2]}$ and $q_{[-2]}$ are properly infinite and $[p_{[-2]}] = [1] - [V^*V] = [1] - [VV^*] = [q_{[-2]}]$. Thus, there exists a partial isometry Z with $Z^*Z = p_{[-2]}$ and $ZZ^* = q_{[-2]}$. We obtain that

$$U := (T + Z)(1 - Ts_2s_2^*T^* + Ts_2(T + Z)^*s_2^*T^*)$$

satisfies $Us_k = t_k$ $(k = 1, 2, 3)$ and $[U] = 0$ in $\mathrm{K}_1(E)$. Since $s_3 = t_3$, we get $U = W + (1 - P)$ with $P := 1 - Ts_1s_1^*T^* = 1 - s_3s_3^*$ and $W := UP \in \mathcal{U}(PEP)$. The facts that $(1 - P)$ is full and properly infinite and $[W + (1 - P)] = 0$ in $\mathrm{K}_1(E)$ imply that $U = W + (1 - P) \in \mathcal{U}_0(E)$ by Part (v,5) or Part (vi,α). Clearly $Us_k = t_k$ and $Up_{[-1]} = q_{[-1]}U$. □

Example 4.2.9. The difference between the use of splitting or non-splitting properly infinite projection p in the formulation of Lemma 4.2.6(v,5) can be seen from the following example of a strongly purely infinite C^*-algebra E with $\mathcal{U}_0(E) = \mathcal{U}(E)$ and a non-splitting properly infinite projection $p \in E$ that satisfies the first assumption in Lemma 4.2.6(v,5) such that $1-p$ properly infinite and full in E, but $\mathrm{K}_1(pEp) \neq \{0\}$:

Consider $E := \mathcal{M}(\mathcal{P}_\infty \otimes \mathbb{K})$ and $p := 1 \otimes p_{11}$, where $\mathcal{P}_\infty$ denotes here the (up to isomorphisms) unique unital pi-sun algebra in the UCT-class with $\mathrm{K}_0(\mathcal{P}_\infty) = 0$ and $\mathrm{K}_1(\mathcal{P}_\infty) \cong \mathbb{Z}$. Thus, $[u] = 0$ implies in this case $[u + (1-p)] = 0$, but the opposite direction is wrong.

Recall that all properly infinite unital C^*-algebras are K_1-surjective by Lemma 4.2.6(ii). The following Lemma 4.2.10 collects necessary and sufficient conditions for the K_1-injectivity of properly infinite unital C^*-algebras. (But the squizing property of Definition allowes more sufficiant cases for K_1-injectivity.)

Lemma 4.2.10. *Let E a* properly infinite *unital C^*-algebra. Each of the following conditions* (i)–(iv) *is equivalent to the* K_1-injectivity *of E:*

(i) *For every unitary $u \in E$ with $[u] = 0$ in $\mathrm{K}_1(E)$ there exist isometries $t_1, t_2 \in E$ and a unitary $v \in \mathcal{U}_0(E)$ with $\|t_1^* u t_2\| < 1$ and $\|t_1^* v t_2\| < 1$.*

(ii) *For every unitary $u \in E$ with $[u] = 0$ in $\mathrm{K}_0(E)$, there exist isometries $r_1, r_2 \in E$ and a unitary $v \in \mathcal{U}_0(E)$ with $r_1^* v r_2 = 0$ and $r_1^* u r_2 = 0$.*

(iii) *For every isometry $s \in E$ with full and properly infinite $1 - ss^*$ and for every unitary $u \in E$ one can find isometries $t_1, t_2 \in E$ and a unitary $v \in \mathcal{U}_0(E)$ with $ust_1 = vst_2$.*

(iv) *Splitting projections $p, q \in E$ with $[p] = [q] \in \mathrm{K}_0(E)$ are unitarily equivalent by a unitary in $\mathcal{U}_0(E)$.*

(v) *For every isometries s and t with orthogonal ranges (i.e., $s^*t = 0$) and every unitary $u \in \mathcal{U}(E)$ the projections tt^* and u^*tt^*u are homotopic inside the projections of E (denoted by $tt^* \sim_h u^*tt^*u$).*

A sufficient condition for the K_1-injectivity *of E is the following (possibly stronger) property (vi):*

(vi) *For every $u \in \mathcal{U}(E)$ with $0 = [u] \in \mathrm{K}_1(E)$ there exists a splitting projection $p \in E$ with $\|up - pu\| < 1$.*

For example, this is the case for each fixed given $u \in \mathcal{U}(E)$, if there exists a C^*-morphism

$$H_u \colon C^*(u) \otimes \mathcal{O}_\infty \to E^\infty := \ell_\infty(E)/c_0(E)$$

with $H_u(u \otimes 1) = (u, u, \ldots) + c_0(E) \in E^\infty$.

Those C^*-morphisms $H_u \colon C^*(u) \otimes \mathcal{O}_\infty \to E^\infty$ exist for example if E is *strongly* purely infinite, cf. Chapter 3. It is not known if such morphisms H_u exist for all unital purely infinite C^*-algebras E and $u \in \mathcal{U}(E)$. An answer to this question could be interesting for the open question if every purely infinite C^*-algebra is strongly purely infinite. But notice here that all unital p.i. C^*-algebras are K_1-injective because they satisfy the formally stronger "squeezing" property (sq) defined in Definition 4.2.14.

PROOF. The equivalence of each of (i)–(v) with K_1-injectivity:

K_1-injectivity $\Rightarrow$ (iii): Let $s \in E$ an isometry with full and properly infinite $1 - ss^*$ and $u \in E$ unitary. There exists an isometry $t \in E$ with $t^*s = 0$. Let $v := u(tu^*t^* + 1 - tt^*)$ and $t_1 := t_2 := t$. Then $ust_1 = vst_2$ and v is a unitary with $[v] = 0$ in $\mathrm{K}_1(E)$. The K_1-injectivity implies $v \in \mathcal{U}_0(E)$.

(iii) $\Rightarrow$ (ii): Let $u \in \mathcal{U}(E)$ with $[u] = 0$ in $\mathrm{K}_0(E)$. Since 1_E is properly infinite, there exists an isometry $s \in E$ with full and properly infinite $1 - ss^*$. By assumptions of (iii), there exist isometries $t_1, t_2 \in E$ and a unitary $u_0 \in \mathcal{U}_0(E)$ such that $ust_1 = u_0st_2$.

There exists an isometry $t \in E$ with $t^*s = 0$, because $1 - ss^*$ is full and properly infinite by assumptions on s.

The element

$$w := 1 - ss^* - tt^* + st_2s^* + tt_2^*t^* - s(1 - t_2t_2^*)t^* = (1 - ss^* - tt^*) + [s, t]U(t_2)[s, t]^*$$

is a unitary in $\mathcal{U}_0(E)$, because the Halmos unitary $U(t_2)$ is in $\mathcal{U}_0(M_2(E))$, cf. Remark 4.2.4.

Obviously, $ws = st_2$. Since $ust_1 = u_0st_2$, we obtain that $(u_0t)^*(u_0w)st_1 = t^*st_2t_1 = 0$ and $(u_0t)^*u(st_1) = (u_0t)^*u_0st_2 = 0$.

If we take $r_1 := u_0t$, $r_2 := st_1$ and $v := u_0w$ then $v \in \mathcal{U}_0(E)$, and r_1, r_2 are isometries with $r_1^*vr_2 = 0$ and $r_1^*ur_2 = 0$, i.e., the elements v, r_1, r_2 fulfill condition (ii) for the given unitary u with $[u] = 0$.

The implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ K_1-injectivity: Let $u \in \mathcal{U}(E)$ with $0 = [u] \in \mathrm{K}_1(E)$ and suppose that (i) holds, i.e., that there exist isometries $t_1, t_2 \in E$ and a unitary $v \in \mathcal{U}_0(E)$ with $\|t_1^*vt_2\| < 1$ and $\|t_1^*ut_2\| < 1$.

We can apply Part (vi,β) of Lemma 4.2.6 to (t_1, vt_2, uv^*) (in place of the triple (t_1, t_2, u) in Part (vi,β) of Lemma 4.2.6), because $[uv^*] = [u] - [v]$ and $[u] = 0$, $v \in \mathcal{U}_0(E)$, and get $uv^* \in \mathcal{U}_0(E)$ and finally $u \in \mathcal{U}_0(E)$. Thus *every* unitary $u \in E$ with $0 = [u] \in \mathrm{K}_1(E)$ is in $\mathcal{U}_0(E)$.

K_1-injectivity $\Rightarrow$(iv): Let $p, q \in E$ splitting projections with $[p] = [q] \in \mathrm{K}_0(E)$. By Part (ii) of Lemma 4.2.6, there exists a unitary $u \in E$ with $u^*pu = q$ and $0 = [u] \in \mathrm{K}_1(E)$. If E is K_1-injective this implies $u \in \mathcal{U}_0(E)$.

(iv)$\Rightarrow$ K_1-injectivity: Let $v \in E$ a unitary with $0 = [v] \in \mathrm{K}_1(E)$. The projection $p := s_1s_1^* \in E$ is splitting, because $s_2s_2^* \leq 1 - p$. Clearly $q := v^*pv$ is also a splitting projection that has same class $[q] = [p]$ in $\mathrm{K}_0(E)$.

By assumption of (iv) there exists a unitary $u \in \mathcal{U}_0(E)$ with $p = u^*qu$. Then vu commutes with the splitting projection $p = s_1 s_1^*$ and $0 = [v] + [u] = [vu]$. The Part (vi,α) of Lemma 4.2.6 shows that $vu \in \mathcal{U}_0(E)$. Thus $v \in \mathcal{U}_0(E)$.

K_1-injectivity $\Rightarrow$(v): Suppose that E is K_1-injective. Let $s, t \in E$ isometries with $t^*s = 0$ and $u \in \mathcal{U}(E)$. Define $v := ((1 - ss^*) + su^*s^*)u$, then $[v] = 0$ in $\mathrm{K}_1(A)$. It implies $v \in \mathcal{U}_0(A)$, because E is K_1-injective. It follows $u^*tt^*u = v^*tt^*v \sim_h tt^*$.

(v)$\Rightarrow$ K_1-injectivity: If $s, t \in E$ are isometries with $s^*t = 0$, then $p := tt^*$ is a splitting projection. Let $u \in \mathcal{U}(A)$ with $0 = [u] \in \mathrm{K}_1(E)$. By assumption of (v), $p \sim_h u^*pu$. Remark 4.1.5 says that there exists $w \in \mathcal{U}_0(A)$ with $p = w^*(u^*pu)w$, i.e., $vp = pv$ for the unitary $v = uw$. We have $[v] = [u] + [w] = 0$, because $[u] = 0 = [w]$. It follows $v \in \mathcal{U}_0(A)$ by Lemma 4.2.6(vi,α). Hence, $u = vw^* \in \mathcal{U}_0(A)$.

This finish the proof of the equivalences of (i)–(v).

(vi)$\Rightarrow$(i): Suppose that for each $u \in \mathcal{U}(E)$ there exists a C^*-morphisms $H_u : C^*(u) \otimes \mathcal{O}_\infty \to E^\infty$ with $H_u(u \otimes 1) = u \in E \subseteq E^\infty$.

Let $u \in \mathcal{U}(E)$ with $[u] = 0$ in $\mathrm{K}_1(E)$ and

$$H_u : C^*(u) \otimes \mathcal{O}_\infty \to E^\infty := \ell_\infty(E)/c_0(E)$$

a C^*-morphism with $H_u(u \otimes 1) = u \in E \subseteq E^\infty$.

Let $s_1, s_2, \ldots$ canonical generators of $\mathcal{O}_\infty$. $H_u(1 \otimes s_k) \subseteq E^\infty$ for $k \in \{1, 2\}$ lift to a sequence of contractions $(s_{k,1}, s_{k,2}, \ldots) \in \ell_\infty(E)$ that satisfy, for $j, k \in \{1, 2\}$, the equations $\lim_n \|s_{j,n}^* s_{k,n} - \delta_{j,k}1\| = 0$ and $\lim_n \|[u, s_{k,n}]\| = 0$.

Replace, for sufficiently big $n \in \mathbb{N}$, the $s_{k,n}$ ($k \in \{1, 2\}$) by its small perturbations $t_{k,n} \in E$ that are isometries and satisfy $\|[u, t_{k,n}]\| < 1/4$ and $\|t_{1,n}^* t_{2,n}\| < 1/4$. If we let $v := 1$, $t_k := t_{k,n}$, then u, v, t_1 and t_2 satisfy the condition in Part (i). Thus, E is K_1-injective if it has the property in Part (v). □

In the following Proposition 4.2.11 the elements $s_1, s_2, \ldots$ denote the canonical generators of $\mathcal{O}_\infty := C^*(s_1, s_2, \ldots\, ;\ s_j^* s_k = \delta_{jk}1)$. Recall that C^*-morphisms $h_0, h_1 : A \to E$ are **homotopic**, if there exists a point-norm continuous path $\xi \in [0, 1] \mapsto H_\xi \in \mathrm{Hom}(A, E)$ of C^*-morphisms with $h_0 = H_0$ and $h_1 = H_1$.

Proposition 4.2.11. *Suppose that E is a C^*-algebra with properly infinite unit element.*

Then, for any unital C^*-morphisms $h_0, h_2 : \mathcal{O}_\infty \to E$, there exists a unitary $u_0 \in \mathcal{U}(E)$ with $u_0 h_2(s_1) = h_0(s_1)$ and $0 = [u_0] \in \mathrm{K}_1(E)$ such that h_0 is homotopic to h_1 in $\mathrm{Hom}(\mathcal{O}_\infty, E)$ with point-norm topology, where we build $h_1 : \mathcal{O}_\infty \to E$ from h_2 by $h_1(s_n) := u_0 h_2(s_n)$ for $n \in \mathbb{N}$.

The algebra E is K_1-injective if and only if any two unital C^*-morphisms from $\mathcal{O}_\infty$ into E are homotopic.

Proof. The unital C^*-morphisms $h_0, h_2 \colon \mathcal{O}_\infty \to E$ are injective because $\mathcal{O}_\infty$ is simple (and purely infinite) by Corollary 2.2.7, cf. also [**169**, thm. 1.13, thm. 3.4] and [**172**, prop. 1.6].

We use this to simplify notations, because we can now suppose that $\mathcal{O}_\infty$ is unitally contained in E (via h_0) by identifying the canonical generators s_n of $\mathcal{O}_\infty$ with its images $h_0(s_n)$, and get a reference copy of $\mathcal{O}_\infty$ inside E. Hence, we can suppose without loss of generality that $\mathcal{O}_\infty \subseteq E$ and that $h_0(s_n) = s_n$, i.e., that h_0 is the identity map of $\mathcal{O}_\infty$.

Let $Z := h_2(s_1)s_1^*$. Then $Zs_1 = h_2(s_1)$, and the projections $Z^*Z = s_1s_1^*$, $ZZ^* = h_2(s_1s_1^*)$, $1-Z^*Z = 1-s_1s_1^*$ and $1-ZZ^* = h_2(1-s_1s_1^*)$ are full and properly infinite projections with $[1-Z^*Z] = [1-ZZ^*]$ in $\mathrm{K}_0(E)$. By Lemma 4.2.6(ii), cf. also [**172**, thm. 1.4], there exists a partial isometry $Y \in E$ with $Y^*Y = 1 - Z^*Z$ and $YY^* = 1 - ZZ^*$. Since $Y^*Z = 0 = ZY^*$, the sum $W := Y + Z$ is a unitary with $Ws_1 = h_2(s_1)$. The element $V := W((1 - s_2s_2^*) + s_2W^*s_2^*)$ is again a unitary with $Vs_1 = h_2(s_1)$, but now with class $[V] = 0$ in $\mathrm{K}_1(E)$.

Let $u_0 := V^*$. It satisfies $[u_0] = 0$ and $u_0h_2(s_1) = s_1 = h_0(s_1)$. We define a new C^*-morphism $h_1 \colon \mathcal{O}_\infty \to E$ by $h_1(s_n) := u_0h_2(s_n)$ for $n = 1, 2, \ldots$. It has the property $h_1(s_1) = s_1$.

The arguments used above (for h_2, Z and construction of V) work also for h_1 in place of h_2 and the partial isometries $Z_n := \sum_{k=1}^n h_1(s_k)s_k^*$:

The Z_n satisfy $P_n := Z_n^*Z_n = \sum_{k=1}^n s_ks_k^*$ and

$$Z_nZ_n^* \;=\; h_1(\sum_{k=1}^n s_ks_k^*) \;=\; h_1(P_n)\,.$$

The full and properly infinite projections $1 - Z_n^*Z_n$ and $1 - Z_nZ_n^* = h_1(1 - Z_n^*Z_n)$ are MvN-equivalent because they have have same $\mathrm{K}_0(E)$-class. Let $Y_n \in E$ a partial isometry with $Y_n^*Y_n = 1 - Z_n^*Z_n$ and $Y_nY_n^* = 1 - Z_nZ_n^*$. The sum $W_n := Y_n + Z_n$ is unitary and the unitary

$$V_n := W_n(1 - s_{n+1}s_{n+1}^* + s_{n+1}W_n^*s_{n+1}^*)$$

satisfies $0 = [V_n] \in \mathrm{K}_1(E)$ and $V_ns_k = h_1(s_k)$ for $k \le n$. In particular, $V_1P_1 = P_1$ and $V_1(1 - s_1s_1^*)$ is a unitary in $\mathcal{U}((1 - s_1s_1^*)E(1 - s_1s_1^*))$. Since $s_1s_1^*$ is a splitting projection and $[V_1] = 0$, this implies by Lemma 4.2.6(v,5) that $V_1(1 - s_1s_1^*)$ has class $[V_1(1 - s_1s_1^*)] = 0$ in $\mathrm{K}_1((1 - s_1s_1^*)E(1 - s_1s_1^*))$, and that $V_1 \in \mathcal{U}_0(E)$.

It follows that there is a continuous path $\lambda \in [0,1] \mapsto V_\lambda \in \mathcal{U}_0(E)$ that connects $V_0 := 1$ and the above defined unitary $V_1 \in \mathcal{U}(E)$.

Similar arguments apply for $n \ge 1$ to the unitary $Q_n := V_n^*V_{n+1}$ and the projection $P_n = Z_n^*Z_n$:

We get that $Q_nP_n = P_n$ and that P_n is splitting. Hence, $[Q_n] = [V_{n+1}] - [V_n] = 0$ in $\mathrm{K}_1(E)$ implies that $[Q_n(1 - P_n)] = 0$ in $\mathrm{K}_1((1 - P_n)E(1 - P_n))$ because $1 - P_n$ is a full projection in E, see Lemma 4.2.6(v,5) with $Q_n(1 - P_n)$ and $1 - P_n$ in place of u and p.

If we let $P_0 := 0$ and $P_1 = s_1 s_1^*$, then by Lemma 4.2.6(v,5),

$$Q_n(1 - P_{n-1}) = Q_n(1 - P_n) + s_n s_n^* \in \mathcal{U}_0((1 - P_{n-1})E(1 - P_{n-1}))\,.$$

We obtain a continuous path $\lambda \in [0,1] \mapsto R_\lambda^{(n)} \in \mathcal{U}((1 - P_{n-1})E(1 - P_{n-1}))$ with $R_0^{(n)} = (1 - P_{n-1})$ and

$$R_1^{n+1} = Q_n(1 - P_n) + s_n s_n^*\,.$$

Notice that $V_{n+1} = V_n Q_n = V_n \cdot (R_1^{(n)} + P_{n-1})$.

We combine the above defined maps $\lambda \in [0,1] \mapsto V_\lambda$ and $n \in \mathbb{N} \mapsto V_n$ to a continuos path $\lambda \in [0,\infty) \mapsto V_\lambda \in \mathcal{U}_0(E)$ for $\lambda \in [n, n+1]$ $(n \geq 1)$ by

$$V_\lambda := V_n \cdot (R_{\lambda - n}^{(n)} + P_{n-1})\,.$$

Since $V_n s_k = h_1(s_k)$ for $k \leq n$ and $P_{n-1} s_k = s_k$ for $k \leq n-1$, it follows that $V_\lambda s_k = h_1(s_k)$ for all $k \leq \lambda - 1$.

The above defined path $\lambda \in [0,\infty) \to V_\lambda \in \mathcal{U}_0(E)$ defines a point-norm continuous path $\xi \in [0,1] \mapsto H_\xi$ in the unital C^*-morphisms from $\mathcal{O}_\infty$ into E by assigning to the generators $s_1, s_2, \ldots \in h_0(\mathcal{O}_\infty)$ and $\xi \in [0,1)$ the values

$$H_\xi(s_n) := V_{(1-\xi)^{-1}\xi} \cdot s_n \;=\; u_\xi \cdot s_n$$

where we let $u_\xi := V_{(1-\xi)^{-1}\xi}$ and $H_1 := h_1$. Then $H_0 = h_0$, and the path is continuous at $\xi = 1$, because

$$\lim_{\xi \to 1} u_\xi s_n = h_1(s_n) \qquad \text{for all } \; n \in \mathbb{N}\,.$$

Moreover, $H_\xi(s_k) = u_\xi s_k = s_k$ if $k + 1 \leq (1 - \xi)^{-1}$ by our construction.

Suppose that E is K_1-injective, i.e., that $[u] = 0 \in \mathrm{K}_1(E)$ implies $u \in \mathcal{U}_0(E)$. Then above considered u_0 is in $\mathcal{U}_0(E)$ and we find a path $\xi \in [1,2] \mapsto u(\xi) \in \mathcal{U}(E)$ with $u(1) = u_0$ and $u(2) = 1$. We can define a point-norm continuous path

$$\xi \in [0,2] \mapsto H_\xi \colon \mathcal{O}_\infty \to E$$

by $H_\xi := h_\xi$ for $\xi \in [0,1]$, and by $H_\xi(s_n) := u(\xi) h_1(s_n)$ for $\xi \in [1,2]$. This path satisfies $H_0 = h_0$ and $H_2 = h_2$.

Suppose that 1_E is properly infinite and that any two unital C^*-morphisms $h_1, h_2 \colon \mathcal{O}_\infty \to E$ are homotopic. Let $u \in \mathcal{U}(E)$ with $0 = [u]$ in $\mathrm{K}_1(E)$.

Since 1 is properly infinite there exist isometries $t_1, t_2 \in E$ with orthogonal ranges, i.e., $t_j^* t_k = \delta_{jk} 1$ for $j, k \in \{1, 2\}$. The sequence $s_n := t_2^{n-1} t_1$ with $t_2^0 := 1$ defines a unital C^*-morphism $H^{(0)}$ from $\mathcal{O}_\infty$ into E, because $s_j^* s_k = \delta_{jk} 1$ for $j, k \in \mathbb{N}$, and $\mathcal{O}_\infty$ is the universal C^*-algebra with this defining relations. Since $\mathcal{O}_\infty$ is simple, cf. Corollary 2.2.7, the C^*-morphisms are injective.

We get another *-monomorphism $H^{(1)}$ from $\mathcal{O}_\infty$ into E that maps the canonical generators $s_1, s_2, \ldots$ of $\mathcal{O}_\infty \cong H^{(0)}(\mathcal{O}_\infty)$ to $H^{(1)}(s_n) := u s_n$ for $n \in \mathbb{N}$.

By assumption there exists a point-norm continuous path $\xi \in [0,1] \mapsto H_\xi$ into the unital C^*-morphisms $H_\xi \colon \mathcal{O}_\infty \to E$, such that $H_k = H^{(k)}$ for $k \in \{0, 1\}$.

Then $p_\xi := H_\xi(s_1 s_1^*)$ is a continuous path in the projections with $p_0 = s_1 s_1^*$ and $p_1 = H^{(1)}(p_0)$. Thus, there exists $V \in \mathcal{U}_0(E)$ with $V^* p_0 V = p_1 = u p_0 u^*$, cf. Lemma 4.1.3(v).

We get $V u p_0 = p_0 V u$ for the splitting projection $p_0 = s_1 s_1^*$. Now $V \in \mathcal{U}_0(E)$ implies that $[Vu] = [u] = 0$ in $\mathrm{K}_1(E)$. It follows that $Vu \in \mathcal{U}_0(E)$ by Part (vi,β) of Lemma 4.2.6. Using again $V \in \mathcal{U}_0(E)$, we obtain $u \in V^* \mathcal{U}_0(E) = \mathcal{U}_0(E)$. □

Remark 4.2.12. The proof of Proposition 4.2.11 shows that two unital C^*-morphisms $h_1, h_2 \colon \mathcal{O}_\infty \to E$ are homotopic if $h_1(s_1) = h_2(s_1)$.

Notice that following Lemma 4.2.13 is not trivial, because it does not require that A is simple.

Lemma 4.2.13. *Let A a purely infinite unital C^*-algebra, let $e, f \in A_+$ with $0 \le f \le e \le 1$ and $S \in A$ an isometry such that $S^* f S$ is contained in the closed ideal $J := I(1-e)$ generated by $(1-e)$.*

Then there exists an isometry $T \in A$ and $\delta \in (0,1]$ that satisfy $\|T^* f T\| = 1 - \delta < 1$.

Proof. The inequality $0 \le (1-e) \le (1-f)$ implies that $I(1-e) \subseteq I(1-f)$. Thus, $S^* f S \in I(1-f)$, and for each $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in A$ such that

$$\|S^* f S - \sum_{k=1}^{n} g_k^* (1-f) g_k\| < \varepsilon$$

Combined with the equation $S^* f S = 1 - S^*(1-f)S$ it shows that 1_A is contained in the closed ideal $I(1-f)$ generated by $1-f$.

If A is purely infinite then this is equivalent to the existence of $d_\varepsilon \in A$ with $\|1 - d_\varepsilon^* (1-f) d_\varepsilon\| < \varepsilon$ for each $\varepsilon \in (0, 1/2)$.

CHECK AGAIN THE ESTIMATES!! :

There exists $\delta \in (0,1)$, – depending on the norm $\|d_{1/4}\|$ –, with

$$\|1 - d_{1/4}^* (1 - \delta - f)_+ d_{1/4}\| < 1/4 .$$

The element $T := C(C^*C)^{-1/2}$ with $C := (1-\delta-f)_+^{1/2} d_{1/4}$ is an isometry in A with $\|T^* f T\| \le 1 - \delta < 1$, because $\|T^* f T\| \le \sup_n \|(1-\delta-f)_+^{1/n} f\|$. □

Definition 4.2.14. We say that a unital C^*-algebra A has the “squeezing” property if A satisfies following **Property (sq)**:

(sq) For each $a \in A$ there exist isometries $r_1, r_2 \in A$ (depending on a) with $\|r_1^* a r_2\| \le 2\|a\|/3$.

It is still not known if C^*-algebras with properly infinite units are K_1-injective, even in case of weakly purely infinite C^*-algebras. Almost all cases where one could prove ([5]) the K_1-injectivity have implicitly to do with the now defined “squeezing” Property (sq) that implies K_1-bijectivity.

[5] – up to August 2018 –

PROPOSITION 4.2.15. *If a unital C^*-algebra A has the "squeezing" Property* (sq) *of Definition 4.2.14 then A is* K_1*-bijective in sense of Definition 4.2.2.*

Following examples (a - c) *of C^*-algebras satisfy Property* (sq)*:*

(a) $\mathcal{E}_n := C^*(s_1, \ldots, s_n\,;\ s_i^* s_j = \delta_{ij} 1)$, *for* $n = 2, \ldots, \infty$.
(b) *each unital purely infinite C^*-algebra A, and*
(c) *the multiplier algebra $A := \mathcal{M}(B)$ of every stable σ-unital C^*-algebra B.*

The class of C^*-algebras $A, A_1, A_2, \ldots$ with Property (sq) *is invariant under following constructions* (i-vii)*:*

(i) *Infinite direct sums* $\ell_\infty(A_1, A_2, \cdots)$,
(ii) *tensor products* $A \otimes^{\max} B$ *with any unital C^*-algebra B,*
(iii) $\mathrm{C_b}(X, A)$ *for locally compact σ-compact Hausdorff spaces X,*
(iv) *quotients A/J, for closed ideals $J \neq A$ of A,*
(v) *inductive limits with unital C^*-morphisms,*
(vi) *fix-point algebras of permutation actions on the n-times tensor product with $n \geq 2$:* $A^{\otimes^n} := A \otimes A \otimes \cdots \otimes A$, *and*
(vii) $B + \varphi(\mathcal{O}_\infty)$, *for every unital C^*-morphism $\varphi\colon \mathcal{O}_\infty \to \mathcal{M}(B)$ if B is a σ-unital C^*-algebra and its multiplier algebra $\mathcal{M}(B)$ has the Property* (sq) *(e.g. if B is stable – by Part (c)).*

In particular it follows:

If a unital C^*-algebra A satisfies Property (sq) *then the asymptotic corona* $\mathrm{C_b}([0,\infty), A)/\,\mathrm{C_0}([0,\infty), A)$ *and the C^*-algebras $E := (A \otimes^{\max} C)/J$, for every unital C^*-algebra C and every closed ideal $J \not\ni 1$ of $A \otimes^{\max} C$, have Property (sq) and are all* K_1*-bijective.*

Stable coronas $\mathrm{Q^s}(B)$ *of σ-unital C^*-algebras B have Property* (sq) *and are* K_1*-bijective.*

Examples (a) and permanence property (v) show that $\mathcal{O}_\infty = \operatorname{indlim}_n \mathcal{E}_n$ *has Property (sq) and is* K_1*-bijective.*

PROOF. We divide the proof into 3 groups.

(1st) shows that unital A with Property (sq) of Definition 4.2.14 contains isometries $s, t \in A$ with $s^*t = 0$ – which implies K_1-surjectivity by Part (v) of Lemma 4.2.6 –, and that Property (sq) implies that A satisfies the necessary and sufficient condition (i) of Lemma 4.2.10 for K_1-bijectivity. Moreover we reformulate Property (sq) in a more applicable form.

(2nd) the Property (sq) has the quoted permanence properties (i-vii), and

(3rd) the under (a,b,c) listed classes of C^*-algebras have Property (sq).

Ad(1st): The Property (sq) implies that A contains two isometries s_1, s_2 with "orthogonal ranges", i.e., with $s_1^* s_2 = 0$. Indeed, for $a := 1$ exist by Property (sq) isometries $r_1, r_2 \in A$ with $\|r_1^* r_2\| \leq 2/3$. Let $p := r_1 r_1^*$ and $q := 1 - r_2 r_2^*$. The projections p, q satisfy $\|p(1-q)\| = \|r_1^* r_2\| \leq 2/3$. By Lemma 4.1.3(iv) there exists a unitary $w \in \mathcal{U}_0(A)$ with $p \leq wqw^* = 1 - (wr_2)(wr_2)^*$, i.e., $r_1^*(wr_2) = 0$. Thus, $s_1^* s_2 = 0$ for the isometries $s_1 := r_1$ and $s_2 := wr_2$, and A has a properly infinite unit element. It implies that A is K_1-surjective by Lemma 4.2.6(v).

We derive a stronger and more flexible formulation of Property (sq):
If there exists $\delta_0 \in (0,1)$ with the property that, for each $a \in A$, there exist isometries $r_1, r_2 \in A$ – depending on a – with $\|r_1^* a r_2\| \leq (1-\delta_0) \cdot \|a\|$, then we can replace a by $r_1^* a r_2$ and get isometries $t_1, t_2 \in A$ – depending on $r_1^* a r_2$ – with $\|(r_1 t_1)^* a (r_2 t_2)\| \leq \|a\| \cdot (1-\delta_0)^2$.

Repeats of this argument show that the Property (sq) is equivalent to the property that for each element $a \in A$ and given $\varepsilon > 0$ there exists isometries $r_1, r_2 \in A$ with $\|r_1^* a r_2\| < \varepsilon$.

If we replace in Definition 4.2.14 the element a by $s_1^* a s_2$ with isometries $s_1, s_2 \in A$ with orthogonal ranges, i.e., $s_j^* s_k = \delta_{jk} 1$, then we find isometries $r_1, r_2 \in A$ with $\|r_1^*(s_1^* a s_2) r_2\| < \varepsilon$. The element $t_1 := s_1 r_1$ and $t_2 := s_2 r_2$ are *isometries* in A that satisfy

$$\|t_1^* a t_2\| < \varepsilon, \quad \text{and} \quad t_j^* t_k = \delta_{jk} 1\,. \tag{2.5}$$

It implies immediately that A fulfills the sufficient condition for K_1-injectivity in Lemma 4.2.10(i) with $v := 1$ for each $u \in \mathcal{U}(A)$. Indeed, take $a := u$ and $t_k := r_k$ ($k \in \{1,2\}$) for some isometries $r_1, r_2 \in A$ with $\|r_1^* a r_2\| < 1/2$ and $r_1^* r_2 = 0$.

Induction over n in Inequality (2.5) shows that, for any finite subset $F = \{a_1, \ldots, a_n\} \subset A$ and $\varepsilon > 0$, there exist isometries $r_1, r_2 \in A$ with $\|r_1^* x r_2\| < \varepsilon$ for all $x \in F$, and $r_1^* r_2 = 0$.

The induction step from n to $n+1$ uses that isometries $s, t \in A$ exist with $\|s^*(r_1^* a_{n+1} r_2) t\| < \varepsilon$. Then $(r_1 s)^*(r_2 t) = 0$ and $\|(r_1 s)^* a_k (r_2 t)\| < \varepsilon$ for $k = 1, \ldots, n$.

Ad(2nd): C^*-algebras A with the above derived formally stronger property, – but that is equivalent to Property (sq) –, have the permanence properties (i-vi):

(i): If $A_1, A_2, \ldots$ are unital C^*-algebras with Property (sq), then the algebra $\ell_\infty(A_1, A_2, \cdots)$ of bounded sequences $a := (a_1, a_2, \ldots)$ with $a_k \in A_k$ is a unital C^*-algebra. If $\varepsilon > 0$ is given then we find isometries $j \in \{1,2\}$, $r_{k,j} \in A_k$ such that $\|r_{n,1}^* a_n r_{n,2}\| < \varepsilon/2$. The $r_j := (r_{1,j}, r_{2,j}, \ldots)$ are isometries in $\ell_\infty(A_1, A_2, \ldots)$ and satisfy $\|r_1^* a r_2\| < \varepsilon$.

(ii): If A has Property (sq) and B is a unital C^*-algebra, then the tensor product $A \otimes^{\max} B$ has Property (sq).

Indeed, let $c \in A \otimes^{\max} B$ and $\varepsilon > 0$. There exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ with $\|b_k\| \leq 1$ such that in $A \otimes^{\max} B$ holds

$$\|\, c - (a_1 \otimes b_1 + \ldots + a_n \otimes b_n)\,\| \,<\, \varepsilon/2\,.$$

By the (2nd) observation there exist isometries $t_1, t_2 \in A$ with $t_1^* t_2 = 0$ and $\|t_1^* a_k t_2\| < \varepsilon/2n$ for $k = 1, \ldots, n$. Then $r_\ell := t_\ell \otimes 1 \in A \otimes^{\max} B$ ($\ell = 1,2$) are isometries that satisfy $\|r_1^* c r_2\| < \varepsilon$.

Induction shows that moreover for every finite sequence $c_1, \ldots, c_n \in A \otimes^{\max} B$ and $\varepsilon > 0$ there exists isometries $t_1, t_2 \in A$ with $t_1^* t_2 = 0$ that satisfy the inequalities

$$\|(t_1^* \otimes 1) c_k (t_2 \otimes 1)\| < \varepsilon \qquad \text{for } \ 1 \leq k \leq n\,.$$

(iii): Suppose that A is unital and has property (sq), and that X is a locally compact σ-compact Hausdorff space. The σ-compactness of X is equivalent to the property that $\mathrm{C}_0(X)_+$ contains a function $x \mapsto e(x) \in [0,1]$ that defines a strictly positive element e of the C^*-algebra $\mathrm{C}_0(X)$ with $\|e\| = 1$. We fix such a function $e(x)$.

Let $X \ni x \mapsto a(x) \in A$ a continuous map with $\|a(x)\| \leq 1$ for all $x \in X$. We show the existence of isometries $S, T \in \mathrm{C_b}(X, A)$ with $T^* S = 0$, and

$$\|T(x)^* a(x) S(x)\| \leq e(x)/2 \qquad \text{for all } \ x \in X\,.$$

In particular, $\|T^* a S\| \leq 1/2$ and $T^* a S \in \mathrm{C}_0(X, A)$.

We consider A as the C^*-subalgebra of $\mathrm{C_b}(X, A)$ consisting of constant A-valued functions from X to A. The subsets

$$K_n := e^{-1}[2^{-n}, 1] = \{x \in X\,;\ e(x) \geq 2^{-n}\}$$

build an increasing family of compact "reference" subsets of X with the properties:

$$K_{n-1} \subseteq K_n^\circ \quad \text{and} \quad \bigcup_n K_n = X\,,$$

that follow from

$$K_{n-1} = e^{-1}([2^{-(n-1)}, 1]) \subseteq e^{-1}\big((2^{-n}, 1]\big) \subseteq K_n^\circ \subseteq K_n = e^{-1}([2^{-n}, 1])\,.$$

The arguments need some care because we can not suppose that the interior of $e^{-1}(2^{-n})$ is empty, – different to the case e.g. of the function $f_0(\tau) := \tau$ on $(0,1]$ – in place of e for $(0,1]$ in place of X.

We use the functions $h_n \in \mathrm{C}_0(0,1]_+$ considered in Section 22 of Appendix A to build a suitable decomposition of $1 \in \mathrm{C_b}(X, A)$ inside $\mathrm{C_b}(X)_+$:

The increasing functions $h_n \in \mathrm{C}_0(0,1]_+$ are given by

$$h_n(\tau) := \min\big(1, \max(2^n \tau - 1, 0)\big)\,, \quad \text{for } \ \tau \in [0,1]\,,\ n = 0, 1, \ldots\,.$$

The set $(2^{-n}, 1]$ is the open support of h_n, contained in $h_{n+1}^{-1}(1) = [2^{-n}, 1]$. Thus, $h_n h_{n+1} = h_n$ and the non-negative function $h_{n+1} - h_n$ has support in the open interval $(2^{-(n+1)}, 2^{-(n-1)})$. It implies that the below used functions $\psi_n := (h_n - h_{n-1})^{1/2}$ (applied to the values $e(x)$ of e) are well-defined and have the property that $\psi_n \psi_m = 0$ for $|n - m| > 1$. Notice that $h_0 := 0\,,\ h_1 = (2f_0 - 1)_+\,, \ldots$, and in general $h_n = (2^n f_0 - 1)_+ - (2^{n-1} f_0 - 1)_+$ in the algebra $\mathrm{C}_0(0,1]$ and for $n \in \mathbb{N}$.

It follows that for each $\gamma \in (0,1]$ only finitely many functions $h_n - h_{n-1}$ have support in $[\gamma, 1]$. In particular, $\sum_{n \geq 1} (h_n - h_{n-1})$ converges on each interval $[\gamma, 1]$ uniformly to 1. Notice that

$$(h_n \circ e)^{-1}(0,1] = e^{-1}(h_n^{-1}(0,1]) = e^{-1}\big((2^{-n}, 1]\big) \subseteq K_n^\circ \subseteq K_n\,.$$

It shows that $(h_{n+1} \circ e)(x) = 1$ if and only if $x \in e^{-1}([2^{-n}, 1]) = K_n$. The support of $h_n \circ e$ is contained in the interior K_n° of K_n because $h_{n+1}h_n = h_n$.

Hence, the open support of the non-negative function $h_n(e(x)) - h_{n-1}(e(x))$ is contained in $V_n := K_n^\circ \setminus K_{n-2}$. It says that the support of $(h_n \circ e) - (h_{n-1} \circ e)$ is an open subset of X that is contained in the open subset V_n of X. Obviously V_n is contained in the closed subset $F_n := K_n \setminus K_{n-2}^\circ$ of X. Summing up, the functions $\varphi_n(x) := \big(h_n(e(x)) - h_{n-1}(e(x))\big)^{1/2} := \psi_n(e(x))$ with $\psi_n := (h_n - h_{n-1})^{1/2}$ for $n = 1, 2, \dots$ earn from the $h_n - h_{n-1} \geq 0$ the properties (I), (II) and (III):

(I) $\varphi_n \varphi_m = 0$ if $|n - m| > 1$,

(II) The open supports $U_n := \varphi_n^{-1}((0, 1])$ of φ_n are contained in the open subsets $V_n := K_n^\circ \setminus K_{n-2}$. And V_n is contained in the closed subset $F_n := K_n \setminus (K_{n-2}^\circ)$.

(III) $\sum_{n=1}^\infty \varphi_n(x)^2$ converges on each compact subset Y of X uniformly to 1. More precisely: There exists $\nu(Y) \in \mathbb{N}$ such that $\varphi_n | Y = 0$ for all $n > \nu(Y)$.

Parts (II) of this property list shows that $\sum_n \varphi_n(x) a_n$ is a well-defined element in $\mathrm{C_b}(X, A)$ of norm ≤ 2 if $a_1, a_2, \ldots \in A$ is an arbitrary sequence of contractions.

We define isometries $S, T \in \mathrm{C_b}(X, A)$ by the sums

$$S(x) := \sum_{n=1}^{\infty} \varphi_n(x) s_n \quad \text{and} \quad T(x) := \sum_{n=1}^{\infty} \varphi_n(x) t_n\,,$$

Here s_n, t_n are the below constructed isometries in A with mutually orthogonal ranges, i.e., with $s_m^* t_n = 0$ and $s_m^* s_n = \delta_{m,n} 1 = s_m^* s_n$ for all $m, n \in \mathbb{N}$. The in this way defined S and T are always in $\mathrm{C_b}(X, A)$, and satisfy $S^*S = 1 = T^*T$, $S^*T = 0$.

We complete the proof by an inductive selection of isometries $s_n, t_n \in A$ with above orthogonality properties and such that the corresponding isometries T and S satisfy

$$\|T(x)^* a(x) S(x)\| \leq 4^{-n} \qquad \text{for all} \quad x \in K_n^\circ \setminus K_{n-2}\,.$$

This implies $\|T(x)^* a(x) S(x)\| \leq e(x)/2$ for all $x \in X$ because $e(x) \in [2^{-n}, 4 \cdot 2^{-n})$ on $K_n \setminus K_{n-2}$.

In the (1st) step it was shown that A has a properly infinite unit if A has Property (sq). Let $r_1, r_2 \in A$ isometries in A with $r_1^* r_2 = 0$, and let $y_n := r_1^n r_2$ and $z_n := r_2^n r_1$. Then $y_m^* y_n = \delta_{m,n} 1 = z_m^* z_n$ and $y_m^* z_n = 0$ for all $m, n \in \mathbb{N}$.

We define by induction sequences of isometries $(g_1, g_2, \ldots)$ and $(h_1, h_2, \ldots)$ with $g_n, h_n \in A$, and build from them and from y_n, z_n new sequences $(s_1, s_2, \ldots)$ and $(t_1, t_2, \ldots)$ of isometries in A with the desired property by the following rules:

(α) $s_1 := r_1 r_2 \cdot g_1$ and $t_1 := r_2 r_1 \cdot h_1$,

(β) $s_{n+1} := r_1 s_n g_{n+1}$ and $t_{n+1} := r_2 t_n h_{n+1}$.

Then $s_1 = y_1 g_1$, $s_2 = y_2 g_1 g_2$, $s_n = y_n g_1 \cdot \ldots \cdot g_n$ and $t_n = z_n h_1 \cdot \ldots \cdot h_n$. To get the desired estimate we have to require that the isometries $g_n, h_n \in A$ satisfy the following inequalities, where $a_n := a|K_n$. We must consider also the overlap for the $x \in X$ with $\varphi_n(x)\varphi_{n+1}(x) \neq 0$.

It implies that we must find in each step pairs of isometries (g_1, h_1), respectively (g_{n+1}, h_{n+1}), in A that fulfill the three inequalities $\|h_1^* c_{1,k} g_1\| < 4^{-1}$ $(k = 1, 2, 3)$ for given elements $c_{1,1}, c_{1,2}, c_{1,3} \in \mathrm{C}(K_2, A)$, respectively $\|h_{n+1}^* c_{n,k} g_{n+1}\| < 4^{-1}$ for – in induction process new defined – given elements $c_{n,1}, c_{n,2}, c_{n,3} \in \mathrm{C}(K_{n+2}, A)$.

In our case here we have to take in $\mathrm{C}(K_2, A)$ the 3 elements

$$c_{1,1} := r_1^* r_2^* a_2 r_1 r_2 \,, \quad c_{1,2} := r_1^* (r_2^*)^2 a_2 r_1 r_2 \quad \text{and} \quad c_{1,3} := r_1^* r_2^* a_2 (r_1)^2 r_2 \,.$$

In the induction step we have to consider in $\mathrm{C}(K_{n+2}, A)$ the elements

$$c_{n,1} := t_n^* r_2^* a_{n+2} r_1 s_n \,, \quad c_{n,2} := t_n^* (r_2^*)^2 a_{n+2} r_1 s_n \quad \text{and} \quad c_{n,3} := t_n^* r_2^* a_{n+2} (r_1)^2 s_n \,.$$

If A has Property (sq) and $B := \mathrm{C}(K)$ for a compact space K then the proof of Part (2nd,ii) shows moreover that – for $c_1, \ldots, c_n \in \mathrm{C}(K, A)$ and $\varepsilon > 0$ – there exist isometries $s, t \in A \subseteq \mathrm{C}(K, A)$ with $t^* s = 0$ and $\|t^* c_k s\| < \varepsilon$ for $k = 1, \ldots, n$.

This implies the existence of the desired $g_{n+1}, h_{n+1} \in A$:
Simply put here $\varepsilon := 4^{-n}$ and take for c_1, c_2 and c_3 the above considered elements in $\mathrm{C}(K_{n+2}, A)$. Then we can take $g_{n+1} := s$ and $h_{n+1} := t$. This finishes the proof of the existence of the proposed isometries g_{n+1} and h_{n+1} in A, that define together with r_1, r_2 the desired new isometries s_{n+1} and t_{n+1} in A by composition rules (α) and (β).

(iv): If A is unital and has property (sq) and $J \neq A$ is a closed ideal of A, then A/J is unital. Let $b \in A/J$ and $\varepsilon > 0$. There exists $a \in A$ with $\pi_J(a) = b$ and isometries $r_1, r_2 \in A$ with $\|r_1^* a r_2\| < \varepsilon$. The isometries $t_j := \pi_J(r_j) \in A/J$ satisfy $\|t_1^* b t_2\| < \varepsilon$.

(v): If A is the inductive limit of a directed net of unital morphisms between C^*-algebras, then A has the "local" property that for each contraction $a \in A$ and $\nu \in (0, 1/4)$ there exists a unital C^*-algebra B with property (sq), an element $b \in B$ with $\|b\| \leq 1$, isometries $r_1, r_2 \in B$ that satisfy $r_1^* r_2 = 0$ and $\|r_1^* b r_2\| \leq \nu$, and a unital C^*-morphism $\psi \colon B \to A$ with $\|\psi(b) - a\| < \nu$. Then $t_1 := \psi(s_1)$ and $t_2 := \psi(s_2)$ are isometries in A with orthogonal ranges that satisfy $\|t_1^* a t_2\| \leq 2\nu < 1/2$.

(vi): Suppose that the unital C^*-algebra A has property (sq), and let $\phi \colon A^{\otimes^n} \to A^{\otimes^n}$ a permutation *-automorphism that is defined via

$$\phi(a_1 \otimes a_2 \otimes \cdots \otimes a_n) := a_{\pi(1)} \otimes a_{\pi(2)} \otimes \cdots \otimes a_{\pi(n)}$$

by a permutation π of $\{1, \ldots, n\}$.

As we have seen in (1st) Part and Part (ii) that for every $b \in A^{\otimes^n}$ and $\varepsilon > 0$ there exist isometries $t_1, t_2 \in A$ such that $t_1^* t_2 = 0$ and

$$\| \, (t_1^* \otimes 1 \otimes \cdots \otimes 1) b (t_2 \otimes 1 \otimes \cdots \otimes 1) \, \| \, < \, \varepsilon \,.$$

Thus $s_k := t_k \otimes t_k \otimes \cdots \otimes t_k$, $k \in \{1, 2\}$ are isometries with $\phi(s_k) = s_k$, $s_1^* s_2 = 0$ and $\| s_1^* b s_2 \| < \varepsilon$. The s_1 and s_2 are contained in the fix-point algebra of all actions of permutation automorphisms of $A^{\otimes^n}$.

(vii): The C^*-algebra $\mathcal{M}(B)$ has a properly infinite 1, i.e., contains isometries $S, T \in \mathcal{M}(B)$ with $T^* S = 0$, and $\mathcal{M}(B)$ is K_1-bijective by (1st) Part. For each $b \in B$ and $\varepsilon \in (0, 1)$ there exists isometries $r_1, r_2 \in \mathcal{M}(B)$ with $r_2^* r_1 = 0$ and $\| r_2^* b r_1 \| < \varepsilon$ by proof of (1st) Part. We may suppose here that $1 - r_1 r_1^* - r_2 r_2^*$ is a full and properly infinite projection, otherwise replace r_2 by $r_2 r_1$.

Let $\varphi \colon \mathcal{O}_\infty \to \mathcal{M}(B)$ a unital C^*-morphism, $X \in B + \varphi(\mathcal{O}_\infty)$ and $\varepsilon \in (0, 1/8)$.

Then $X = c + \varphi(d)$ with $c \in B$ and $d \in \mathcal{O}_\infty$. $\mathcal{O}_\infty$ has Property (sq) because it is the inductive limit of the $\mathcal{E}_n$ in Part (a) and Part (v) applies (or use Part (b) because $\mathcal{O}_\infty$ is purely infinite by Corollary 2.2.7). If follows that there exists isometries $t_1, t_2 \in \mathcal{O}_\infty$ with $t_2^* t_1 = 0$ and $\| t_2^* d t_1 \| < \varepsilon$. Here $1 - (t_1 t_1^* + t_2 t_2^*)$ is a full and properly infinite projection ([6]). Define $b := \varphi(t_2)^* d \varphi(t_1) \in B$. Then $\| (\varphi(t_2) S_2)^* X (\varphi(t_1) S_1) \| < \varepsilon + \| S_2^* b S_1 \|$ for all isometries $S_1, S_2 \in \mathcal{M}(B)$. The Property (sq) of $\mathcal{M}(B)$ implies that we find isometries $S_1, S_2 \in \mathcal{M}(B)$ with $\| S_2^* b S_1 \| < \varepsilon$ and $S_2^* S_1 = 0$. Here we can replace S_2 by $S_2 S_1$ to make sure that $1 - S_1 S_1^* - S_2 S_2^*$ is full and properly infinite. Then we find a unitary $v \in \mathcal{U}(\mathcal{M}(B))$ with $v \varphi(t_1) = S_1$, $v \varphi(t_2) = S_2$ and $[v] = 0 \in \mathrm{K}_1(\mathcal{M}(B))$. The Property (sq) of $\mathcal{M}(B)$ causes the K_1-injectivity of $\mathcal{M}(B)$, by (1st) Part. Thus, $v \in \mathcal{U}_0(\mathcal{M}(B))$ and there exist $f_1, \ldots, f_n \in \mathcal{M}(B)$ with $f_k^* = -f_k$ and $v = \exp(f_1) \cdot \ldots \cdot \exp(f_n)$. Let $\{e_j\} \subset B_+$ $(j \in \mathbb{N})$ a quasi-central approximate unit in B, build e.g. by functional calculus from a strictly positive contraction in B_+. The unitaries $U_j := \exp(e_j f_1 e_j) \cdot \ldots \cdot \exp(e_j f_n e_j))$ are in $1_{\mathcal{M}(B)} + B$ and $\lim_{j \to \infty} \| U_j^* x U_j - v^x v \| = 0$ for each $x \in B$. Thus, there exists $U \in \mathcal{U}(B + \mathbb{C} \cdot 1) \cap (1 + B)$ with $\| U^* b U - v^* b v \| < \varepsilon$. We get $\| (U \varphi(t_1))^* b (U \varphi(t_2)) \| \leq 2\varepsilon$. Then $T_k := \varphi(t_k) U \varphi(t_k)$ $(k = 1, 2)$ are isometries in $B + \varphi(\mathcal{O}_\infty)$ with $\| T_2^* X T_1 \| < 3\varepsilon < 1/2$.

(Ad 3rd) Some algebras with Property (sq):

The argument at the beginning of the proof of (1st) shows also that is suffices to require for all elements a in a dense subset X of A that there exists a constant $\gamma \in (0, 1)$ such that for each $a \in X$ there exist isometries r_1, r_2 (depending from a) with $\| r_1^* a r_2 \| < \gamma \| a \|$, e.g. we could start with the property that there exist for each $a \in X$ isometries $r_1, r_2 \in A$ with $3 \| r_1^* a r_2 \| \leq 2 \| a \|$.

(a): $\mathcal{E}_n := C^*(s_1, \ldots, s_n \, ; \, s_i^* s_j = \delta_{ij})$, $n = 2, \ldots, \infty$, satisfy Property (sq):

Indeed, let $w := k_1 k_2 \ldots k_p$, $v := \ell_1 \ell_2 \ldots \ell_q$, “words” of length $p, q \in \mathbb{N}$ with “letters” $k_j, \ell_i \in \{1, \ldots, n\}$ from the “alphabet” $\{1, \ldots, n\}$. Define $T_0 := 1$ and an isometry by the product

$$T_p(w) := s_{k_1} s_{k_2} \cdot \ldots \cdot s_{k_p} \, .$$

[6] Use here that $[1] = 1$ by $\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, or simply replace t_2 by $t_2 t_1$ to make sure that $1 - t_1 t_1^* - t_2 t_2^*$ is full and properly infinite.

It is easy to see that the linear span of the element 1 and of the "elementary" products $T_q(w)$, $T_q(w)^*$ and $T_q(w)(T_r(v)^*)$ is dense in the C^*-algebra $\mathcal{E}_n$ (in both of real or complex case), because the product $T_q(v)^*T_p(w)$ is equal to 1 if $v = w$, is equal to zero if there exists $g \leq r := \min(p, q)$ with $k_g \neq \ell_g$, is equal to $T_{(p-q)}(u_1)$ if $q < p$ and $w = vu_1$ and is equal to $T_{(q-p)}(u_2)^*$ if $p < q$ and $v = wu_2$.

We denote by G one of this elementary products or let $G = 1$. A sort of "length" $L(G)$ is defined by $L(1) := 0$, $L(T_q) := q$, $L((T_q)^*) := q$ and $L(T_q \cdot (T_r)^*) := q + r$. Clearly, $s_k^*T_p(w) = \delta_{k,k_1}T_{p-1}(v)$ for all $k = 1, \ldots, n$ with word $v := k_2k_3 \ldots k_n$ for $w = k_1k_2 \ldots k_n$.

Let $G = T_p(w)T_q(v)^*$ with $w = k_1 \ldots k_p$, $v = \ell_1 \ldots \ell_q$ $(p, q \geq 1)$, then, $L(s_k^*Gs_\ell) = L(G) - 2$ if and only if $k_1 = k$ *and* $\ell_1 = \ell$, otherwise only $s_k^*Gs_\ell = 0$ can happen.

In case $G = T_p(w)$, $w = k_1 \ldots k_p$, we get $s_k^*Gs_\ell \neq 0$ if and only if $k = k_1$ and then $L(s_k^*Gs_\ell) = L(G)$, because then $G = T_p(u)$ with $u = k_2 \ldots k_p\ell$. Since $L(G^*) = L(G)$ the same holds for $G = T_p(w)^*$ with the role of k and ℓ interchanged.

Therefore, one can find, for each linear combination $C = \alpha_0 1 + \sum_{j=1}^{m} \alpha_j G_j$ (with scalars α_j), an index $1 \leq j_0 \leq m$ and suitable $k, \ell \in \{1, \ldots, n\}$ such that $k \neq \ell$, $s_k^*G_{j_0}s_\ell = 0$ for at least one $j_0 \in \{1, \ldots, m\}$ and $L(s_k^*G_js_\ell) \leq L(G_j)$ for $j \neq j_0$. Thus $s_k^*Cs_\ell = \sum_{j\neq 0, j\neq j_0} \alpha_j s_k^*G_js_\ell$ with elementary products $s_k^*G_js_\ell$ (it can be equal to 1).

Iteration – say m-times – of this operation leads to words w and v in the alphabet $\{1, \ldots, n\}$ such that for the isometries $T_m(w)$ and $T_m(v)$ holds $T_m(w)^*T_m(v) = 0$ and $T_m(w)^*CT_m(v) = 0$. It says that $\mathcal{E}_n$ has Property (sq).

(b): The proof for purely infinite unital C^*-algebras is much more engaged than the proof for strongly purely infinite unital C^*-algebras. The latter class contains all *simple* unital purely infinite C^*-algebras by Proposition 2.2.1(v).

The question whether or not pure infiniteness implies the – for our considerations important – strong pure infiniteness is still unsolved (June 2017).

Let us give first, for the readers convenience, the obvious and easy proof for strongly purely infinite C^*-algebras. This algebras have the "matrix diagonalization" property, cf. Definition 2.16.3 and Proposition 2.16.4:

If $a \in A$ is a contraction then we can build the positive 2×2-matrix $b = [b_{jk}] \in M_2(A)_+$ with diagonal entries $b_{11} := 1 =: b_{22}$ and $b_{21}^* := b_{12} := a$. By the matrix diagonalization property of A there exists a diagonal matrix $D := \mathrm{diag}(d_1, d_2)$ with $\|D^*bD - 1_2\| < 1/4$, i.e., $\|d_1^*ad_2\| < 1/4$ and $\|d_j^*d_j - 1\| < 1/4$ for $j = 1, 2$. Thus, the positive elements $d_j^*d_j$ are invertible with spectrum $\mathrm{Spec}(d_j^*d_j) \subseteq (3/4, 5/4)$ and $s_j := d_j(d_j^*d_j)^{-1/2}$ are isometries with $\|s_1^*as_2\| \leq (1/4)(4/3) = 1/3$.

The different and more engaged proof of Part (b) for the general case of purely infinite unital C^*-algebras A goes as follows:

Since $1 \in A$ is properly infinite for purely infinite A there exists isometries $S_1, S_2 \in A$ with orthogonal ranges, i.e., $S_j^* S_k = \delta_{jk} 1$:

The isometries $T_k := S_2^k S_1 \in A$ satisfy $T_k^* T_\ell = \delta_{jk} 1$, and $C^*(T_1, T_2, \dots) \subset A$ is a copy $\mathcal{O}_\infty$ in A.

Let $a \in A$ an element of norm $\|a\| = 1$. We are going to show that there exist isometries $s, t \in A$ such that $\|t^* a s\| < 1$:

If $\|T_2^* a T_1\| < 1$, then we can take $s := T_1$ and $t := T_2$. If $\|T_2^* a T_1\| = 1$, then we consider (from now on) the closed ideal $J := I(1 - T_1^* a^* T_2 T_2^* a T_1)$ of A generated by $1 - T_1^* a^* T_2 T_2^* a T_1 \in A_+$:

If 1_A is contained in J, then Lemma 4.2.13 applies with $e := f := T_1^* a^* T_2 T_2^* a T_1$ and $S := 1$, and implies that there exists an isometry $T \in A$ and $\delta \in (0, 1]$ with $\|T^* f T\| = 1 - \delta < 1$. Thus, $\|T_2^* a (T_1 T)\| = (1 - \delta)^{1/2} < 1$, and we can take $s := T_1 T$ and $t := T_2$.

If 1_A is not contained in J, then $\pi_J(T_1^* a^* T_2 T_2^* a T_1) = 1$. Thus $\pi_J(T_2^* a T_1)$ is an isometry in A/J. Then $\pi_J(T_2^* a T_1 T_3) = \pi_J(T_2^* a T_1) \pi_J(T_3)$ is again an isometry in A/J. The contraction $Z := T_2^* a T_1 T_2 \in A$ satisfies $\pi_J(Z^*) \pi_J(T_2^* a T_1 T_3) = 0$ and $\pi_J(Z^* Z) = 1$.

We define a closed left ideal of A by

$$L := \; \{ \, b \in A \, ; \; b \cdot T_2^* a T_1 T_3 \in J \, \} \, .$$

It contains Z^* and all elements of J, i.e., $J \cup \{Z^*\} \subseteq L$.

Let K denote the closed ideal of A that is generated by the hereditary C^*-subalgebra $D := L^* \cap L$. Then $L^* \cdot L \subseteq D$, and D contains J and the contraction ZZ^*. It follows that the closed ideal K of A contains J and $Z^*(ZZ^*)Z$. The equation $\pi_J(Z^*) \pi_J(ZZ^*) \pi_J(Z) = \pi_J(1)$ shows that $1 - Z^*(ZZ^*)Z \in J$. Hence, $1 \in K$ and $K = A$.

The C^*-algebra A and its full hereditary C^*-subalgebra D are purely infinite, and the ideal generated by D contains 1_A. It implies the existence of elements $g \in A$ and $d \in D_+$ such that $g^* d g = 1$.

The isometry $V := d^{1/2} g \in A$ satisfies $V^* \in L$ because $VV^* \in D$. Thus, $V^* T_2^* a T_1 T_3 \in J$ by definitions of L and $D = L^* \cap L$.

The positive contractions $f := T_1^* a^* T_2 V V^* T_2^* a T_1$, $e := T_1^* a^* T_2 T_2^* a T_1$ and isometry $S := T_3$ satisfy the assumptions of Lemma 4.2.13, i.e., $0 \leq f \leq e$ and $S^* f S \in I(1 - e) = J$.

By Lemma 4.2.13 there exists an isometry $T \in A$ with $\|T^* f T\| < 1$. Thus, $\|V^* T_2^* a T_1 T\| = \|T^* f T\|^{1/2} < 1$. It says that the isometries $s := T_1 T$ and $t := T_2 V$ satisfy $\|t^* a s\| < 1$.

We have seen above that for each unital purely infinite A and $a \in A$ with $\|a\| = 1$ there exists isometries $s, t \in A$ such that $\|t^* a s\| < 1$ and $t^* s = 0$. This argument does *not* give immediately a fixed general constant $\gamma_0 \in (0, 1)$ with the property that we for every $a \in A$ there exist isometries $s, t \in A$ (depending on a) with $\|t^*as\| \leq (1-\gamma_0) \cdot \|a\|$. So that the iteration argument in the (1st) Part of the proof does not apply here.

But we can give an indirect argument that proves that for each $a \in A$ and $\varepsilon > 0$ there are isometries $s, t \in A$ (depending on a and ε) with $\|t^*as\| < \varepsilon$:

Define for $a \in A$ the number

$$\gamma(a) := \inf\{ \|t_2^* a t_1\| \,;\; t_1, t_2 \in A\,,\; t_j^* t_k = \delta_{j,k} 1\,, \}\,.$$

Obviously $0 \leq \gamma(a) \leq \|a\|$ and $\gamma(\xi a) = |\xi| \cdot \gamma(a)$. We show that $\gamma(a) = 0$ for all $a \in A$:

Suppose − to derive a contradiction − that $\gamma(a) > 0$ for some $a \in A$ with $\|a\| = 1$. Then we can find sequences $t_{k,1}, t_{k,2}, \ldots, t_{k,n}, \ldots \in A$ ($k \in \{1, 2\}$) with $t_{j,n}^* t_{k,n} = \delta_{j,k} 1$ and $\lim_n \|t_{2,n}^* a t_{1,n}\| = \gamma(a)$. Let

$$b_n := \|t_{2,n}^* a t_{1,n}\|^{-1} t_{2,n}^* a t_{1,n}\,.$$

Then $b := (b_1, b_2, \ldots)$ is a contraction in $\ell_\infty(A)$ that has the property that for every isometries $S, T \in \ell_\infty(A)$ holds $\|T^* b S\| = 1$.

It is easy to see that C^*-algebra $\ell_\infty(A)$ is again purely infinite if A is purely infinite in sense of Definition 1.2.1, cf. the permanence properties of purely infinite C^*-algebras listed in Proposition 2.5.19. Hence, our above "individual" squeezing result is also true for elements $b \in \ell_\infty(A)$.

But this general fact contradicts the existence of a contraction $b \in \ell_\infty(A)$ with the property that $\|T^* b S\| \geq 1$ for all isometries $S, T \in \ell_\infty(A)$.

It follows that the above defined number $\gamma(a) \in [0, \|a\|]$ is zero for all $a \in A$ if A is a unital purely infinite C^*-algebra. In particular, we find then for each $a \in A$ isometries $s, t \in A$ with $3\|t^*as\| \leq 2$.

– (c): Let B a stable σ-unital C^*-algebra and $A := \mathcal{M}(B)$. By Lemma A.23.1, for every separable C^*-subalgebra $C \subset \mathcal{M}(B)$ there exist isometries $S, T \in \mathcal{M}(B)$ with the property that $T^*CS \subseteq B$ and $T^*S = 0$.

By Remark 5.1.1(8) we find a sequence of isometries $s_k \in \mathcal{M}(B)$ with the property that $\sum_k s_k s_k^*$ converges strictly to 1 with respect of the strict topology on $\mathcal{M}(B)$. If we select suitable s_{k_n} from the sequence $s_1, s_2, \ldots$ then we can manage that $T_n := T s_{k_n}$ and $T_n := T s_{k_n}$ satisfy $\lim_n \|T_n^* c S_n\| = 0$ and $T_n^* S_n = 0$ for all $c \in C$. Thus, $A := \mathcal{M}(B)$ has Property (sq). □

Now we turn to the generalization of K_1-injectivity and K_1-surjectivity to the case of non-unital σ-unital C^*-algebras:

Definition 4.2.16. If A is a *not necessarily unital* C^*-algebra, then we can consider the natural **unitization** $\widetilde{A} := A + \mathbb{C}1 \subseteq \mathcal{M}(A)$ of A, and then use the definitions of K_1-surjectivity and K_1-injectivity for $\widetilde{A}$ in place of E in Definition 4.2.2, where we notice that $\mathrm{K}_1(\widetilde{A}) = \mathrm{K}_1(A)$ and each unitary $u \in A + \mathbb{C}1$ is homotopic in $\mathcal{U}(\widetilde{A})$ to a unitary $v \in \mathcal{U}(\widetilde{A}) \cap (1+A)$. Notice that our definition of $\widetilde{A}$ entails that $\widetilde{A} = A$ if A is unital.

I.e., A is K_1**-surjective** (respectively is K_1**-injective**) if the natural map

$$u = 1 + a \in \mathcal{U}(\widetilde{A}) \mapsto [u] \in \mathrm{K}_1(A)$$

from $(1+A) \cap \mathcal{U}(A + \mathbb{C} \cdot 1)$ to $\mathrm{K}_1(A)$ is surjective (respectively has kernel equal to $(1+A) \cap \mathcal{U}_0(\widetilde{A})$).

The C^*-algebra A is K_1**-bijective** if A is K_1-surjective and K_1-injective.

Following Remark 4.2.17 lists some sufficient criteria for K_1-surjectivity and K_1-injectivity of some special non-unital C^*-algebras. It will be used for our "constructive" version of K_*-theory and the study of Cuntz semi-groups of C^*-morphisms.

Remark 4.2.17. C^*-algebras A with the property that the unit element of $\mathcal{M}(A)$ is properly infinite appear often in our considerations.

Suppose that A is a σ-unital non-unital C^*-algebra. The multiplier algebra $\mathcal{M}(A)$ has a properly infinite unit, if and only if, there exists a unital C^*-morphism $\mathcal{O}_\infty \to \mathcal{M}(A)$, and this is the case if and only if A is isomorphic to an ideal of some C^*-algebra E with properly infinite unit.

The class of C^*-algebras A with properly infinite $1 \in \mathcal{M}(A)$ is invariant under passage to non-zero ideals and non-zero quotients.

If $\mathcal{M}(A)$ has a properly infinite unit, then A is K_1*-surjective* in the sense of Definition 4.2.16.

If the multiplier algebra $\mathcal{M}(A)$ has a properly infinite unit, then the algebra A is K_1*-injective* in the sense of Definition 4.2.16, if and only if, $A + \mathcal{O}_\infty \subseteq \mathcal{M}(A)$ is a K_1-injective C^*-algebra.

Every σ-unital stable C^*-algebra A is K_1-bijective, because $\mathcal{M}(A)$ has the "squeezing" Property (sq), and this property carries over to $A + \psi(\mathcal{O}_\infty)$ for every unital C^*-morphism $\psi \colon \mathcal{O}_\infty \to \mathcal{M}(A)$, cf. Parts (c) and (vii) of Proposition 4.2.15.

Details for Remark 4.2.17. Let E a C^*-algebra, J a closed ideal of E and $\psi \colon J \to A$ an isomorphism from J onto A. Then $\mathcal{M}(\psi)$ is an isomorphism from $\mathcal{M}(J)$ onto $\mathcal{M}(A)$ and there exists a unital C^*-morphism $\varphi \colon E \to \mathcal{M}(A)$. If the unit of E is properly infinite then the unit $1_{\mathcal{M}(A)} = \varphi(1_E)$ of $\mathcal{M}(A)$ is properly infinite. Conversely, if $1_{\mathcal{M}(A)}$ is properly infinite in $\mathcal{M}(A)$, then one can take $E := \mathcal{M}(A)$ and $J = A$.

If A is a non-zero ideal of a C^*-algebra E with properly infinite unit 1_E then every closed ideal $\{0\} \neq J$ of A is a closed ideal E. If $J \neq A$, then A/J is a non-zero closed ideal of E/J and E/J has again a properly infinite unit.

If B is unital and $\mathcal{O}_\infty \subseteq \mathcal{M}(B) = B$, then $B = 1 + B$ is K_1*-surjective*, by Lemma 4.2.6(v).

If A is non-unital and $1 \in \mathcal{M}(A)$ is properly infinite, then there is copy of $\mathcal{O}_\infty$ unitally contained in $\mathcal{M}(A)$ with $\mathcal{O}_\infty \cap A = \{0\}$. With this copy there are natural isomorphisms $\mathrm{K}_1(A) = \mathrm{K}_1(A+\mathbb{C}\cdot 1) \cong \mathrm{K}_1(A+\mathcal{O}_\infty)$, and $\mathcal{U}(A+\mathbb{C}\cdot 1)/\mathcal{U}_0(A+\mathbb{C}\cdot 1) \cong \mathcal{U}(A+\mathcal{O}_\infty)/\mathcal{U}_0(A+\mathcal{O}_\infty)$ and $\big((A+1)\cap\mathcal{U}(A+\mathbb{C}\cdot 1)\big)/\big((A+1)\cap\mathcal{U}_0(A+\mathbb{C}\cdot 1)\big) \cong \mathcal{U}(A+\mathbb{C}\cdot 1)/\mathcal{U}_0(A+\mathbb{C}\cdot 1)$.

The proposed K_1-injectivity of $\mathcal{M}(A)$ implies that unital C^*-morphisms $\varphi_1, \varphi_2 \colon \mathcal{O}_\infty \to \mathcal{M}(A)$ are point-norm homotopic, cf. Proposition 4.2.11 i.e., there is a norm-continuous path of unitaries $u(\xi) \in \mathcal{U}(\mathcal{M}(A))$ $\xi \in [0,\infty)$ with $u(1) = 1$ and $\lim_{\xi\to\infty} u(\xi)\varphi_1(s_n) = \varphi_2(s_n)$ for all $n \in \mathbb{N}$. Such unitaries can be in strict topology approximated by products of of exponentials with exponents $-x^* = x \in A$. That shows that the interaction with elements in A can be done by correction with products by $u_1, \ldots, u_n \in (A+1)\cap\mathcal{U}_0(A+\mathbb{C}\cdot 1)$. This gives hope for over carry of Property (sq).

If A is stable and σ-unital, then the multiplier algebra $\mathcal{M}(A)$ has Property (sq) and $\mathrm{K}_*(\mathcal{M}(A)) = 0$, which implies $\mathcal{U}(\mathcal{M}(A)) = \mathcal{U}_0(\mathcal{M}(A))$ by K_1-bijectivity.

If A is *not* unital, but $1_{\mathcal{M}(A)}$ is properly infinite in $\mathcal{M}(A)$, then $A + \mathcal{O}_\infty \subset \mathcal{M}(A)$ is unital and K_1-surjective and the inclusion $A + \mathbb{C}\cdot 1 \hookrightarrow A + \mathcal{O}_\infty$ defines a natural isomorphism $\mathrm{K}_1(A) = \mathrm{K}_1(A+\mathbb{C}\cdot 1) \cong \mathrm{K}_1(A+\mathcal{O}_\infty)$ that coincides with the isomorphism $\mathrm{K}_1(A) \cong \mathrm{K}_1(A+\mathcal{O}_\infty)$ given by splitting of the 6-term exact K_*-sequence for the (right-)split exact sequence $0 \to A \to A+\mathcal{O}_\infty \to \mathcal{O}_\infty \to 0$, because $A \cap \mathcal{O}_\infty = \{0\}$ if A is not unital, and $\mathrm{K}_1(\mathcal{O}_\infty) = \mathrm{K}_1(\mathbb{C}) = 0$.

Since $A+\mathcal{O}_\infty$ is K_1-surjective by Lemma 4.2.6(v), for each $x \in \mathrm{K}_1(A+\mathcal{O}_\infty) \cong \mathrm{K}_1(A+\mathbb{C}\cdot 1)$ there is a unitary $u = a+v \in \mathcal{U}(A+\mathcal{O}_\infty)$ with $a \in A$ and $v \in \mathcal{U}(\mathcal{O}_\infty)$ such that $x = [u] = [v]+[v^*a+1]$. But $\mathcal{U}(\mathcal{O}_\infty) = \mathcal{U}_0(\mathcal{O}_\infty) \subset \mathcal{U}_0(A+\mathcal{O}_\infty)$ by [**172**, cor. 3.12], cf. also Section 1 of Appendix A. Thus $[v] = [1] = 0 \in \mathrm{K}_1(A+\mathcal{O}_\infty)$ and $x = [v^*a+1] \in \mathrm{K}_1(A+\mathcal{O}_\infty) \cong \mathrm{K}_1(A+\mathbb{C}\cdot 1)$ is the class of the unitary $v^*a+1 \in A+1$. This establishes the K_1-*surjectivity* of A, i.e., shows the surjectivity of the map

$$u := a+1 \in \mathcal{U}(A+\mathbb{C}\cdot 1)\cap(A+1) \mapsto [u] \in \mathrm{K}_1(A)\,.$$

Suppose that $A+\mathcal{O}_\infty$ ($\subseteq \mathcal{M}(A)$) is K_1-*injective.* If $0 = [a+1] \in \mathrm{K}_1(A)$ for $a+1 \in \mathcal{U}(A+\mathbb{C}\cdot 1)$, then $0 = [a+1] \in \mathrm{K}_1(A+\mathcal{O}_\infty)$ and $a+1 \in \mathcal{U}_0(A+\mathcal{O}_\infty)$. Let $\pi \colon A+\mathcal{O}_\infty \to \mathcal{O}_\infty$ the natural *-epimorphism $\pi(a+b) = b$ for $a \in A$ and $b \in \mathcal{O}_\infty$.

By the proposed K_1-injectivity of $A+\mathcal{O}_\infty$, there exist a continuous path $\xi \in [0,1] \mapsto v(\xi) \in \mathcal{U}(A+\mathcal{O}_\infty)$ such that $v(0) = 1$ and $v(1) = a+1$. Then $u(\xi) := \pi(v(\xi)^*)\cdot v(\xi)$ is a continuous path in $\mathcal{U}(A+\mathbb{C}\cdot 1)\cap(A+1)$ with $u(0) = 1$ and $u(1) = a+1$, i.e., $a+1 \in \mathcal{U}_0(A+\mathbb{C}\cdot 1)$. Thus, $A+\mathbb{C}\cdot 1$ is K_1-injective.

Now suppose conversely that $A+\mathbb{C}\cdot 1$ is K_1-injective and $u = a+v \in \mathcal{U}(A+\mathcal{O}_\infty)$ satisfies $0 = [u] \in \mathrm{K}_1(A+\mathcal{O}_\infty)$. Notice $\pi(u) = v \in \mathcal{U}_0(\mathcal{O}_\infty) = \mathcal{U}(\mathcal{O}_\infty)$. Then $\pi(u^*)u = v^*a+1$ satisfies $0 = [v^*a+1] \in \mathrm{K}_1(A+\mathbb{C}\cdot 1) \cong \mathrm{K}_1(A+\mathcal{O}_\infty)$. Thus, $v^*a+1 \in \mathcal{U}_0(A+\mathbb{C}1) \subset \mathcal{U}_0(A+\mathcal{O}_\infty)$ and $u = v\cdot(v^*a+1) \in \mathcal{U}_0(A+\mathcal{O}_\infty)$.

Every σ-unital stable C^*-algebra A is K_1-bijective:

If A is stable, then there is a non-degenerate C^*-morphism $\psi\colon \mathbb{K} \to \mathcal{M}(A)$ with $\psi(\mathbb{K}) \cap A = \{0\}$. It extends to a strictly continuous unital *-monomorphism $\mathcal{M}(\psi)\colon \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K}) \to \mathcal{M}(A)$. Let $T_1, T_2, \ldots \in \mathcal{L}(\ell_2)$ isometries such that $\sum_n T_n T_n^*$ converge to id_{ℓ_2} in $\mathcal{L}(\ell_2)$ with respect to the *strong operator topology. Then $s_n := \mathcal{M}(\psi)(T_n)$ have the property that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(A)$.

If $\mathcal{O}_\infty = C^*(s_1, s_2, \ldots) \subseteq \mathcal{M}(A)$ is a copy of $\mathcal{O}_\infty$ with the additional property that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(A)$, then $A + \mathcal{O}_\infty \subseteq \mathcal{M}(A)$ satisfies the squeezing condition (sq) of Definition 4.2.14 and is K_1-injective by Proposition 4.2.15 :

Indeed, let $a + X \in A + \mathcal{O}_\infty$ with $a \in A$ and $X \in \mathcal{O}_\infty$, and $\varepsilon > 0$. We find $s_{n_k} \in \mathcal{O}_\infty$ with $\|s_{n_k}^* a s_{n_{k+1}}\| < \varepsilon/2$, by strict convergence to $1_{\mathcal{M}(A)}$ of $\sum s_n^* s_n$. Proposition 4.2.15(a,iii) provide isometries $S, T \in \mathcal{O}_\infty$ that satisfy

$$\|S^*(s_{n_k}^* X s_{n_{k+1}})T\| < \varepsilon/2\,.$$

The isometries $(s_{n_k}S, s_{n_{k+1}}T)$ are ε-squeezing for $a + X$.

Thus if $\psi\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ is a unital C^*-morphism with the additional property that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(A)$, then $A+\psi(\mathcal{O}_\infty)$ has the "squeezing" Property (sq) of Definition 4.2.14. It implies by Proposition 4.2.15 that $A+\psi(\mathcal{O}_\infty)$ is K_1-bijective.

If $\varphi\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ is any unital C^*-morphism with the property that there exists a unitary $u \in \mathcal{M}(A$ such that $u^*\varphi(b)u - \psi(b) \in A$ for all $b \in \mathcal{O}_\infty$, then the C^*-subalgebra $A + \varphi(\mathcal{O}_\infty) \subseteq \mathcal{M}(A)$ satisfies again the "squeezing" Property (sq) of Definition 4.2.14 and is therefore K_1-bijective. □

Notice here that the proof of Proposition 4.2.15(vii) shows that $A+\varphi_1(\mathcal{O}_\infty)$ has Property (sq), if and only if, $A + \varphi_2(\mathcal{O}_\infty)$ has Property (sq) under the assumption that $\mathcal{M}(A)$ is K_1-injective.

If $\varphi\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ is any unital C^*-morphism then the C^*-subalgebra $A + \varphi(\mathcal{O}_\infty) \subseteq \mathcal{M}(A)$ satisfies again the "squeezing" Property (sq) of Definition 4.2.14.

REMARK 4.2.18. Let A be a σ-unital non-unital C^*-algebra such that $\mathcal{M}(A)$ is K_1-injective, as e.g. in case where A is stable, and let $\varphi_k\colon \mathcal{O}_\infty \to \mathcal{M}(A)$, $k \in \{1, 2\}$, unital C^*-morphisms.

(i) $A + \varphi_1(\mathcal{O}_\infty)$ has Property (sq), if and only if, $A + \varphi_2(\mathcal{O}_\infty)$ has Property (sq).
(ii) If $\varphi\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ is any unital C^*-morphism then the C^*-subalgebra $A + \varphi(\mathcal{O}_\infty) \subseteq \mathcal{M}(A)$ satisfies again the "squeezing" Property (sq) of Definition 4.2.14.

DETAILS FOR REMARK 4.2.18. Then φ_1 and φ_2 are homotopic by Proposition 4.2.11.

(i): Is true and provable, because arguments in the proof of Proposition 4.2.15(vii) apply here.

(ii): It follows from Proposition 4.2.15(vii), because $\mathcal{M}(A)$ satisfies the "squeezing" Property (sq) of Definition 4.2.14 if A is σ-unital and stable. □

It would be useful for the above considerations if the following Conjectures 4.2.19 can be verified. Are there similar results for $\mathcal{O}_2$ in place of $\mathcal{O}_\infty$?

Conjecture 4.2.19. *Let A a separable* stable *C^*-algebra and $\varphi_k\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ $(k = 1, 2)$ unital C^*-morphisms.*

(C. 1) *The C^*-subalgebras $A + \varphi_1(\mathcal{O}_\infty)$ and $A + \varphi_2(\mathcal{O}_\infty)$ are isomorphic.*
(C. 2) *There exist a unitary $U \in \mathcal{M}(A)$ such that $U^*\varphi_1(b)U - \varphi_2(b) \in A$ for all $b \in \mathcal{O}_\infty$.*
(C. 3) *The C^*-morphisms $\varphi_1, \varphi_2\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ are approximately unitary equivalent by a sequence of unitaries $u_1, u_2, \ldots \in \mathcal{M}(A)$ with the additional property $u_k^*\varphi_2(b)u_k - \varphi_1(b) \in A$ for all $b \in \mathcal{O}_\infty$.*

Clearly (C. 3) *implies* (C. 2)*, and* (C. 2) *implies* (C. 1)*.*

Is (C. 1) *also true for non-stable σ-unital non-unital separable A if it is true for all stable separable A?*

The following lemma explains the equivalence of rather different looking defining relations that are used for definitions of Ext-groups or KK-groups (in particular that here in this book versus those given by other authors).

Lemma 4.2.20. *Suppose that E is unital, D is a* full *hereditary C^*-subalgebra of E, and that A is a unital C^*-subalgebra of E such that D is an ideal of A.*

(o) *There are isomorphisms*

$$\mathrm{K}_*(A/D) \cong \mathrm{kernel}(\mathrm{K}_*(A) \to \mathrm{K}_*(E))\,,$$

which split the 6-term exact sequence into split exact sequences

$$0 \to \mathrm{K}_*(D) \to \mathrm{K}_*(A) \to \mathrm{K}_*(A/D) \to 0\,.$$

(i) *Let p_1 and p_2 projections in A, such that $p_2 - p_1 \in D$ and $[p_1]_E = [p_2]_E$ in $\mathrm{K}_0(E)$. Then $[p_1]_A = [p_2]_A$ in $\mathrm{K}_0(A)$.*
(ii) *Let u_1 and u_2 unitaries in A, such that $u_2 - u_1 \in D$ and $[u_1]_E = [u_2]_E$ in $\mathrm{K}_1(E)$. Then $[u_1]_A = [u_2]_A$ in $\mathrm{K}_1(A)$.*
(iii) *If A is K_1-surjective, then A/D is K_1-surjective.*

The equations $[p_1]_A = [p_2]_A$ in (i), or $[u_1]_A = [u_2]_A$ in (ii), can fail if $D \triangleleft A$ is not hereditary in E or is hereditary in E but is not full in E.

Give example!

This obstruction plays a role for the classifications of non-stable and non-unital extensions, or unital extensions that do not dominate a unital "zero" extension. See the ????

`Give example for the latter !! Gabe's remarks ???`

Is A/D K_1-bijective if A and E (hence D ??) are K_1-bijective?

Proof. (o): The inclusion maps $\eta_D\colon D \hookrightarrow A$ and $\eta_A\colon A \hookrightarrow E$ define group homomorphisms $[\eta_D]_*\colon \mathrm{K}_*(D) \to \mathrm{K}_*(A)$ and $[\eta_A]_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(E)$. The composition $\alpha := \eta_A \circ \eta_D\colon d \in D \mapsto d \in E$ of this homomorphisms defines isomorphisms $[\alpha]_* = [\eta_A]_* \circ [\eta_D]_*$ from $\mathrm{K}_*(D)$ onto $\mathrm{K}_*(E)$, because D is full and hereditary in E.

It follows that $[\eta_D]_*$ must be *injective* and that $[\eta_A]_*$ is a surjective group homomorphism from $\mathrm{K}_*(A)$ onto $\mathrm{K}_*(E)$. Thus, the boundary maps

$$\partial\colon\ \mathrm{K}_{(*+1) \bmod 2}(A/D) \to \mathrm{K}_*(D)$$

of the natural 6-term exact sequence must be trivial, i.e.,

$$0 \to \mathrm{K}_*(D) \to \mathrm{K}_*(A) \to \mathrm{K}_*(A/D) \to 0$$

are exact sequences with group morphisms $[\eta_D]_*$ and $[\pi_D]_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(A/D)$ defined by the quotient map $\pi_D\colon A \to A/D$.

In particular, the maps $[\pi_D]_*$ are epimorphisms with kernels $[\eta_D]_*(\mathrm{K}_*(D))$.

Natural "splitting" epimorphisms from $\mathrm{K}_*(A)$ onto $\mathrm{K}_*(D)$ can be defined by

$$\beta_* := [\alpha]_*^{-1} \circ [\eta_A]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(D)\,.$$

We define a group endomorphisms $\gamma_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(A)$ with $(\gamma_*)^2 = \gamma_*$ by

$$\gamma_* := [\mathrm{id}_A]_* - [\eta_D]_* \circ \beta_*\,.$$

The endomorphism γ_* maps $\mathrm{K}_*(A)$ onto the kernel of $[\eta_A]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(E)$. The "orthogonal" idempotent group endomorphism $[\mathrm{id}_A]_* - \gamma_* = [\eta_D]_* \circ [\alpha]_*^{-1} \circ [\eta_A]_*$ of $\mathrm{K}_*(A)$ maps $\mathrm{K}_*(A)$ onto $[\eta_D]_*(\mathrm{K}_*(D))$.

The restriction of $[\pi_D]_*$ onto the kernel $\gamma_*(\mathrm{K}_*(A))$ of $[\eta_A]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(E)$ defines an isomorphism $[\pi_D]_* \circ \gamma_*$ from the kernel of $[\eta_A]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(E)$ onto $\mathrm{K}_*(A/D)$.

Thus, the kernel of $[\eta_A]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(E)$ is isomorphic to $\mathrm{K}_*(A/D)$ in a natural way by $[\pi_D]_* \circ \gamma_*$.

We get in particular that

$$[\eta_A]_* \oplus [\pi_D]_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_*(E) \oplus \mathrm{K}_*(A/D)$$

are group isomorphisms from $\mathrm{K}_*(A)$ onto $\mathrm{K}_*(E) \oplus \mathrm{K}_*(A/D)$ that realize the splitting of $\mathrm{K}_*(D) \to \mathrm{K}_*(A) \to \mathrm{K}_*(A/D)$.

(i,ii): Let $p_1, p_2 \in A$ projections (respectively $u_1, u_2 \in \mathcal{U}(A)$). Consider $x_0 := [p_2] - [p_1] \in \mathrm{K}_0(A)$ (respectively $x_1 := [u_2] - [u_1] \in \mathrm{K}_1(A)$).

If $p_2 - p_1 \in D$ (respectively $u_2 - u_1 \in D$) then $[\pi_D]_*(x_*) = 0$. If $[p_1]_E = [p_2]_E$ in $\mathrm{K}_0(E)$ (respectively $[u_1]_E = [u_2]_E$ in $\mathrm{K}_1(E)$), then $[\eta_A]_*(x_*) = 0$ in $\mathrm{K}_*(E)$. Above we have seen that $[\eta_A]_* \oplus [\pi_D]_*$ is faithful on $\mathrm{K}_*(A)$. Thus, the assumptions of Parts (ii) or (iii) imply that $x_* = 0$ in $\mathrm{K}_*(A)$.

(iii): If $u \in \mathcal{U}(A) \mapsto [u] \in \mathrm{K}_1(A)$ and $[\pi_D]_1 \colon \mathrm{K}_1(A) \to \mathrm{K}_1(A/D)$ are surjective, then for each $x \in \mathrm{K}_1(A/D)$ there exist $u \in \mathcal{U}(A)$ with $x = [\pi_D]_1([u]) = [\pi_D(u)]$. Thus, Part (iii) follows from $\pi_D(\mathcal{U}(A)) \subseteq \mathcal{U}(A/D)$. □

3. Addition of C^*-morphisms

We apply the above definition of $\oplus_{s_1,s_2}$ to C^*-morphisms and completely positive maps, and study later associated commutative semigroups and the corresponding Grothendieck groups. Throughout this section we consider two C^*-algebras D and E such that E is unital and contains a copy of $\mathcal{O}_2$ unitally.

Definition 4.3.1. Let $h_1, h_2 \colon D \to E$ be two C^*-morphisms (or completely positive maps) and let s_1, s_2 be generators of a unital copy of $\mathcal{O}_2$ in E. Then the **Cuntz addition** of h_1 and h_2 (with respect to s_1, s_2) is defined by:

$$(h_1 \oplus_{s_1,s_2} h_2)(b) := s_1 h_1(b) s_1^* + s_2 h_2(b) s_2^*, \quad \forall\, b \in D\,.$$

Two C^*-morphisms $h, k \colon D \to E$ are said to be **unitarily equivalent** (notation: $[h] = [k]$) if there exists a unitary u in E such that $k(b) = u^* h(b) u$ for all $b \in D$ (notation: $k = u^* h(\cdot) u$).

Sometimes one has to consider only unitary equivalences $k = u^* h(\cdot) u$ with unitaries u in a subgroup $\mathcal{G}$ of the unitary group $\mathcal{U}(E)$ of E, where $\mathcal{G}$ often contains the connected component $\mathcal{U}_0(E)$ of the unit element 1_E in $\mathcal{U}(E)$ (notation: $h \approx_{\mathcal{G}} k$). We write $h \approx k$ or $[h]_\approx = [k]_\approx$ if here $\mathcal{G} := \mathcal{U}_0(E)$, i.e.,

$$h \approx k \quad \Leftrightarrow \quad k \in [h]_\approx \quad \Leftrightarrow \quad \exists\, u \in \mathcal{U}_0(E) \text{ with } k = u^* h(\cdot) u\,.$$

Cuntz addition defines the structure of a commutative semigroup on the set $[\mathrm{Hom}(D, E)]$ of unitary equivalence classes of C^*-morphisms from D into E, by setting

$$[h_1] + [h_2] := [h_1 \oplus_{s_1,s_2} h_2]$$

for arbitrary choice of generators $\{s_1, s_2\}$ of $\mathcal{O}_2$ in E. The correctness of this definition can be verified using the following Proposition 4.3.2.

It shows *moreover* that the more refined addition

$$[h_1]_{\mathcal{G}} + [h_2]_{\mathcal{G}} := [h_1 \oplus_{s_1,s_2} h_2]_{\mathcal{G}}$$

is well-defined if $\mathcal{U}_0(E) \subseteq \mathcal{G} \subseteq \mathcal{U}(E)$ and defines the structure of a commutative semigroup on the set of all equivalence classes $[h]_{\mathcal{G}}$. We write $[\mathrm{Hom}(D, E)]_\approx$ in case $\mathcal{G} = \mathcal{U}_0(E)$.

PROPOSITION 4.3.2. *Let $h_i \colon D \to E$, $i = 1,2,3$ be C^*-morphisms and let $\{s_1, s_2\}$ and $\{t_1, t_2\}$ be canonical generators of copies of $\mathcal{O}_2$ in E. The Cuntz-addition $[h_1] + [h_2] := n[h_1 \oplus_{s_1,s_1} h_2]$ on unitary equivalence* classes *$[h]$ of C^*-morphisms $h \colon D \to E$ is well-defined, independent from the chosen canonical generators s_1, s_2 of $\mathcal{O}_2 \subseteq E$, commutative and associative:*

(i) ($\oplus$ is well-defined) *For any unitary operators v and w in E,*

$$vh_1v^* \oplus_{s_1,s_2} wh_2w^* = u(h_1 \oplus_{s_1,s_2} h_2)u^*,$$

where $u := s_1vs_1^* + s_2ws_2^*$ is a unitary operator in E.

(ii) (Consistency)

$$u^*((h_1 \oplus_{s_1,s_2} h_2)(b))u = (h_1 \oplus_{t_1,t_2} h_2)(b), \quad b \in D,$$

where $u := s_1t_1^* + s_2t_2^*$ is a unitary operator in E;

(iii) (Commutativity)

$$U_c(h_1 \oplus_{s_1,s_2} h_2)U_c^* = h_2 \oplus_{s_1,s_2} h_1,$$

where $U_c := s_2s_1^* - s_1s_2^*$ is a unitary operator in the connected component $\mathcal{U}_0(E)$ of 1_E in $\mathcal{U}(E)$;

(iv) (Associativity)

$$U_d(h_1 \oplus (h_2 \oplus h_3))U_d^* = (h_1 \oplus h_2) \oplus h_3,$$

where $\oplus$ means $\oplus_{s_1,s_2}$ and $U_d := s_1^2s_1^* + s_2(s_2^*)^2 + s_1s_2(s_2s_1)^*$ is in $\mathcal{U}_0(E)$.

PROOF. Each of Parts (i)–(iv) can be seen by straight calculation – compare introductory remarks in the

proof of Lemma 4.2.6 and the

proofs of Parts (o) and (vii) of Lemma 4.2.6. The unitaries U_c and U_d are in $\mathcal{U}_0(C^*(s_1, s_2))$, cf. proof of Part (o) of Lemma 4.2.6. □

The observation that $U_c, U_d \in \mathcal{U}_0(C^*(s_1, s_2))$ follows also from the less elementary observation that $\mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$, cf. [**172**, thm. 1.9, thm. 3.7] in case of complex C^*-algebras. It is possible to show it also for the "real" version $(\mathcal{O}_2)_\mathbb{R}$ of $\mathcal{O}_2$ that each "orthogonal" operator in $(\mathcal{O}_2)_\mathbb{R}$ is inside the orthogonal operators connected to 1 by a continuous path, but the proofs of [**172**, thms. 1.9, 3.7] have to be modified slightly, cf. Section 1 of Appendix A. We say this here because it implies that some parts of our results remain valid in the case of real C^*-algebras.

DEFINITION 4.3.3. Let $h_1, h_2 \colon D \to E$ be C^*-morphisms. Then h_1 will be said to n-**dominate** h_2 if there exist $a_1, \ldots, a_n$ in E such that $a_1^*a_1 + \ldots + a_n^*a_n = 1$ and $h_2(b) = a_1^*h_1(b)a_1 + \ldots + a_n^*h_1(b)a_n$ for all $b \in D$. We shall say that h_1 **dominates** h_2 if h_1 1-dominates h_2, i.e., if $h_2(b) = s^*h_1(b)s$, $b \in D$, for some isometry s in E.

Note that for unital D a non-unital h_1 can dominate a unital h_2. But a unital h_1 can not dominate a non-unital h_2 by (iii) of the following Lemma 4.3.4.

LEMMA 4.3.4. *Suppose that $h_1\colon D \to E$ is a C^*-morphism, that $s, t \in E$ are isometries and that $g \in E$ is a contraction. Define completely positive contractions $h_2, h_3, T\colon D \to E$ by $h_2 := s^* h_1(\cdot)s$, $h_3 := t^* h_1(\cdot)t$ and $T := g^* h_1(\cdot)g$.*

(i) *The c.p. map $T\colon D \to E$ is a C^*-morphism if and only if $gg^* \in h_1(D)' \cap E$* and $(gg^* - gg^*gg^*)h_1(D) = \{0\}$.
Moreover, $T = h_1$ if and only if $g \in h_1(D)' \cap E$ and $(1 - g^*g)h_1(D) = \{0\}$.
In particular, h_2 is a C^*-morphism if and only if $ss^* \in h_1(D)' \cap E$. It holds $h_2 = h_1$ if and only if $s \in h_1(D)' \cap E$.

(ii) *If h_2 is a C^*-morphism, then h_3 is unitarily equivalent to h_2, if and only if, tt^* is Murray–von Neumann equivalent to ss^* in $h_1(D)' \cap E$.*
Moreover, $h_2 = h_3$ if and only if $ts^* \in h_1(D)' \cap E$.

(iii) *h_2 is unital if h_1 is unital.*

(iv) *h_1 dominates zero if and only if there is an isometry $v \in E$ which range vv^* orthogonal to the image of h_1, i.e., $vv^*h_1(D) = 0$.*

(v) *Suppose that there exists a unital C^*-morphism from $\mathcal{O}_2$ into E that is given by isometries $s_1, s_2 \in E$ as canonical generators, that $h_1 \oplus_{s_1,s_2} h_1 = r^* h_1(\cdot) r$ for an isometry $r \in E$, and that $h_2 := s^* h_1(\cdot)s$ and $h_3 := t^* h_1(\cdot)t$. Then $h_2 \oplus_{s_1,s_2} h_3 = q^* h_1(\cdot) q$ for the isometry $q := r(s \oplus_{s_1,s_2} t)$.*

PROOF. (i): Let $g \in E$ with $\|g\| \leq 1$ and $a^* = a \in E$ (likewise $a = h_1(d)$ for some $d^* = d \in D$).

Clearly, $g^*agg^*bg = g^*abg$ and $g^*bgg^*ag = g^*bag$ for all $b \in E$, if gg^* commutes with a and if $x = 0$ for $x := g^*(1 - gg^*)a$. That means $gg^*a = agg^*$ and $0 = x^*x = (gg^* - gg^*gg^*)a(gg^*a) = 0$.

Conversely, if $g^*a^2g = g^*agg^*ag$, then $(1 - gg^*)^{1/2}ag = 0$. It implies $gg^*a = gg^*agg^* = agg^*$, because $a^* = a$ and $(1 - gg^*)agg^* = 0$. It follows that gg^* commutes with a and $(gg^* - gg^*gg^*)a = 0$.

Hence, $gg^* \in h_1(D)' \cap E$ and $(gg^* - gg^*gg^*)h_1(D) = \{0\}$, if and only if, the map $T := g^*h_1(\cdot)g$ from D into E satisfies $T(dc) = T(d)T(c)$ for all $d^* = d, c \in D$, i.e., if and only if T is a C^*-morphism.

If $g^*ag = a$ and $g^*a^2g = a^2$, then $0 \leq (ag - ga)^*(ag - ga) = -a(1 - g^*g)a \leq 0$. Thus, $ga = ag$ and $(1 - g^*g)a = 0$ for all $a^* = a \in h_0(D)$ if $h_1 = g^*h_1(\cdot)g =: T$. The converse is obvious.

If g is an isometry $g = s$ then only the necessary and sufficient condition $ss^* \in h_0(D)' \cap E$ remains, and the condition $s \in h_0(D)' \cap E$ for the case $h_0 = h_2$.

(ii): If $u \in E$ is a unitary in E with $u^*h_2(\cdot)u = h_3(\cdot)$, then $v = sut^*$ is a partial isometry in E with $vv^* = ss^*$, $v^*v = tt^*$ and $v^*h_1(\cdot)v = tt^*h_1(\cdot)tt^* = tt^*h_1(\cdot)$. Thus $h_1(b)v = ss^*h_1(b)v = vv^*h_1(b)v = vtt^*h_1(b) = vh_1(b)$ for every $b \in D$.

Conversely, if there is a partial isometry $v \in h_1(D)' \cap E$ with $vv^* = ss^*$ and $v^*v = tt^*$, then $u = s^*vt$ is a unitary in E such that $u^*h_2(\cdot)u = h_3$.

(iii): $s^*h_1(1)s = s^*s = 1$.

(iv): Suppose that h_1 dominates zero. Then there is an isometry $v \in E$ with $v^* h_1(a) v = 0$ for all $a \in D$. Thus $vv^* h_1(a)(vv^* h_1(a))^* = 0$ for all $a \in D$. Conversely, $v^* h_1(a) v = v^*(vv^* h_1(a)) v = 0$ if $v \in E$ is an isometry with range orthogonal to $h_1(D)$.

(v): by straight calculation. □

PROPOSITION 4.3.5. *Let D and E be C^*-algebras and suppose that E contains a copy of $\mathcal{O}_2 = C^*(s_1, s_2)$ unitally. Let $h_i \colon D \to E$, $i = 1, 2$ be C^*-morphisms.*

- (i) *If h_1 dominates h_2 and $h_2(D)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally, then $h_1 \oplus h_2$ and h_1 are unitarily equivalent.*
- (ii) *If h_1 n-dominates h_2 and $h_1(D)' \cap E$ contains two isometries with orthogonal range, then h_1 dominates h_2.*
- (iii) *$[h_1] + [h_1] = [h_1]$ if and only if $h_1(D)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally.*
- (iv) *h_1 dominates $h_1 \oplus h_1$ if and only if $h_1(D)' \cap E$ contains a copy of $\mathcal{O}_\infty$ unitally.*

  For every non-zero projection p in this copy of $\mathcal{O}_\infty$ in $h_1(D)' \cap E$ with $[p] = 0$ in $\mathrm{K}_0(\mathcal{O}_\infty)$ there exists an isometry $s \in E$ with $ss^* = p$. The c.p. map $h_0 := s^* h_1(\cdot) s$ is a C^*-morphism from D into E with $[h_0] + [h_0] = [h_0]$, and h_0 dominates h_1.
- (v) *If there is a contraction $f \in E$ such that $h_1 \oplus h_1 = f^* h_1(\cdot) f$, then there exists a C^*-morphism k from $D \otimes \mathcal{O}_\infty$ into E such that $h_1 = k((\cdot) \otimes 1)$.*

The copies of $\mathcal{O}_2$ considered in the assumptions and in Parts (i) and (iii), or the copies of $\mathcal{O}_\infty$ appearing in Parts (ii), (iv) and (v) can be different, are even not necessarily homotopic. In the situation of Part (v) it can happen that there is no copy of $\mathcal{O}_\infty$ unitally contained in E itself that commutes with the elements of $h_1(D)$.

PROOF. (i): Suppose $h_2(\cdot) = t^* h_1(\cdot) t$, for $t \in E$ with $t^* t = 1$. Then tt^* commutes with the images of h_1 by Lemma 4.3.4(i). If $r_1, r_2 \in h_2(D)' \cap E$ are isometrics with $r_1 r_1^* + r_2 r_2^* = 1$, then $t_1 := (1 - tt^*) + t r_1 t^*$ and $t_2 := t r_2$ are isometries that satisfy $t_1 t_1^* + t_2 t_2^* = 1$. Thus, $V := t_1 r_1^* + t_2 r_2^*$ is a unitary. The unitary V realizes the unitary equivalence $V(h_1 \oplus_{r_1, r_2} h_2) V^* = h_1 \oplus_{t_1, t_2} h_2$. Then $t h_2(\cdot) t^* = t(t^* h_1(\cdot) t) t^* = tt^* h_1(\cdot)$ implies $h_1 \oplus_{t_1, t_2} h_2 = h_1$ because

$$(1 - tt^*) h_1(\cdot) + t r_1 h_2(\cdot) r_1^* t^* + t r_2 h_2(\cdot) r_2^* t^* = (1 - tt^*) h_1(\cdot) + t h_2(\cdot) t^* = h_1 .$$

The unitary equivalence of h_1 and $h_1 \oplus_{s_1, s_2} h_2$ with the before given isometries s_1, s_2 can be realized by $u^* h_1(\cdot) u = h_1 \oplus_{s_1, s_2} h_2$ where $u := s_1 t_1^* + s_2 t_2^*$. This unitary u can be replaced here by any unitary $v_0 \cdot u \cdot (v_1 \oplus_{s_1, s_2} v_2)$ with unitaries $v_0, v_1 \in h_1(D)' \cap E$ and $v_2 \in h_2(D)' \cap E$.

(ii): By assumption, there are isometries $t_1, t_2 \in h_1(D)' \cap E$ satisfying $t_1^* t_2 = 0$. The isometries $s_j := t_2^j t_1$, $j = 1, \ldots, n$, are in $h_1(D)' \cap E$ and satisfy $s_i^* s_j = \delta_{ij}$, $1 \le i, j \le n$. Now suppose $h_2 = \sum_{j=1}^n a_j^* h_1(\cdot) a_j$, with $a_j \in E$ and $\sum a_j^* a_j = 1$.

Then, by taking $b := \sum_j s_j a_j$, we have

$$h_2 = \sum a_j^* h_1(\cdot) a_j = \sum\nolimits_{i,k} a_i^* s_i^* h_1(\cdot) s_k a_k = b^* h_1(\cdot) b.$$

Moreover,

$$b^* b = \sum\nolimits_{i,k} a_i^* s_i^* s_k a_k = \sum a_j^* a_j = 1.$$

(iii): By definition $[h_1] + [h_1] = [h_1]$ means that there is a unitary $u \in E$ such that $u^* h_1(\cdot) u = s_1 h_1(\cdot) s_1^* + s_2 h_1(\cdot) s_2^*$ for isometries $s_1, s_2 \in E$ which are the canonical generators of a copy of $\mathcal{O}_2$. Then $t_i := u s_i$, $i = 1, 2$, are generators of $\mathcal{O}_2$ in $h_1(D)' \cap E$, because, for $a \in D$ and $i \in \{1, 2\}$,

$$h_1(a) t_i \;=\; t_1 h_1(a) t_1^* t_i + t_2 h_1(a) t_2^* t_i \;=\; t_i h(a)\,.$$

Conversely, $h_1 \oplus_{t_1,t_2} h_1 = h_1$ if $t_1, t_2 \in h_1(D)' \cap E$ are isometries and generators of a copy of $\mathcal{O}_2$. Now apply Proposition 4.3.2(i).

(iv): If h_1 dominates $h_1 \oplus h_1$, then there is an isometry $t \in E$ such that $t^* h_1(\cdot) t = s_1 h_1(\cdot) s_1^* + s_2 h_1(\cdot) s_2^*$. The elements $t_i := t s_i$, $i = 1, 2$, are isometries with orthogonal ranges such that $h_1 = t_i^* h_1(\cdot) t_i$ for $i = 1, 2$. We get $t_1, t_2 \in h(D)' \cap E$ by Lemma 4.3.4(i). Thus $C^*(t_1, t_2)$ is unitally contained in $h_1(D)' \cap E$ and contains the copy $C^*(t_2^n t_1\,;\, n = 1, 2, \ldots)$ of $\mathcal{O}_\infty$ unitally.

Conversely, suppose that $h_1(D)' \cap E$ contains a copy of $\mathcal{O}_\infty$ unitally. Then there are isometries t_1 and t_2 in $h_1(D)' \cap E$ with orthogonal ranges. The element $t := t_1 s_1^* + t_2 s_2^* \in E$ is an isometry and satisfies $t^* h_1(\cdot) t = s_1 h_1(\cdot) s_1^* + s_2 h_1(\cdot) s_2^*$.

Suppose that $p \neq 0$ is a projection in a given copy of $\mathcal{O}_\infty \subseteq h_1(D)' \cap E$ with $[p] = 0$ in $\mathrm{K}_0(\mathcal{O}_\infty)$ and $1_E \in \mathcal{O}_\infty$, e.g. $p = 1 - (t_2 t_1)(t_2 t_1)^*$ if $\mathcal{O}_\infty \cong C^*(t_2^n t_1\,;\, n = 1, 2, \ldots) \subseteq E$. It implies that also $[p] = 0$ in $\mathrm{K}_0(E)$ and that p is properly infinite in $\mathcal{O}_\infty$ and E, because $\mathcal{O}_\infty$ is simple and purely infinite by Corollary 2.2.7, (cf. also [**169**, thm. 3.4]).

Thus, by Lemma 4.2.6(ii), p is Murray–von-Neumann equivalent to 1 in E, i.e., there exists an isometry $s \in E$ with $ss^* = p$. The map $h_0 := s^* h_1(\cdot) s$ is a C^*-morphism from D into E, because $p = ss^*$ commutes with $h_1(D)$. The commutant $h_0(D)' \cap E$ of $h_0(D)$ contains $s^* p \mathcal{O}_\infty p s$ unitally, and $s^* p \mathcal{O}_\infty p s$ is isomorphic to $p \mathcal{O}_\infty p$. By [**172**], $\mathcal{O}_\infty$ is a simple purely infinite C^*-algebra. It follows for non-zero projections $p \in \mathcal{O}_\infty$ with $[p] = 0$ in $\mathrm{K}_0(\mathcal{O}_\infty)$ that the hereditary C^*-subalgebra $p \mathcal{O}_\infty p$ of $\mathcal{O}_\infty$ (with unit p) contains a copy of $\mathcal{O}_2$ unitally. Thus $[h_0] + [h_0] = [h_0]$ by Part (iii).

The C^*-morphism h_0 dominates h_1, because there exists an isometry $t \in \mathcal{O}_\infty \subseteq h_1(D)' \cap E$ with $tt^* \leq p = ss^*$. Then $ss^* t = t$, $s^* t$ is an isometry in E, $h_1 = t^* h_1(\cdot) t$ and $t^* s h_0(\cdot) s^* t = t^* s s^* h_1(\cdot) s s^* t$.

(v): Let $\{s_1, s_2\}$ denote the canonical generators of the given copy of $\mathcal{O}_2$ in E. We define C^*-subalgebras $C, N \subseteq E$ by

$$N := \mathrm{Ann}(h_1(D)) := \{\, e \in E\,;\; h_1(D) e \,\cup\, e h_1(D) \,= \{0\}\,\}$$

and by $C := h_1(D)' \cap E$. Then N is an ideal of C and C/N is a *unital* C^*-algebra. It is not difficult to see that there exists a unique C^*-morphism $\lambda\colon D \otimes^{\max} C/N \to E$ with the property

$$\lambda(d \otimes (g + N)) = dg \quad \text{for all } d \in D \text{ and all } g \in C\,.$$

If $f^* h_1(\cdot) f = h_1 \oplus_{s_1,s_2} h_1$ for a contraction $f \in E$, then the elements fs_1 and fs_2 are in C and $t_j := fs_j + N$ $(j = 1,2)$ are isometries in C/N with $t_1^* t_2 = 0$, i.e., $h_1(a) fs_k = fs_k h_1(a)$, $(1 - s_k^* f^* f s_k) h_1(a) = 0$ for $a \in D$ for $k = 1,2$, and $s_1^* f^* f s_1 + s_2^* f^* f s_2 = f^* f \oplus f^* f \leq 1$. The latter is obvious, and the first two equations follow from $s_k^* f h_1(\cdot) f s_k = h_1$ by Lemma 4.3.4(i).

The isometries $T_n := t_2^n t_1$ for $n \in \mathbb{N}$ build a sequence of isometries in C/N with $T_n^* T_m = 0$ for $m \neq n$. Thus, there is a unital *-monomorphism $\gamma\colon \mathcal{O}_\infty \to C/N$. Then $k := \lambda \circ (\mathrm{id}_D \otimes \gamma)$ is a C^*-morphism from $D \otimes \mathcal{O}_\infty$ into E with $k(d \otimes 1) = h_1(d)$ for $d \in D$. □

The Cuntz addition $[h] + [k]$ of unitary equivalence classes of morphisms $h, k\colon D \to E$ is far away from the sum of linear maps, e.g. $[h] + [0] \neq [h]$ if h is unital.

If we consider the zero-absorbing maps $h \oplus 0$ then the following proposition gives additional later useful sufficient conditions on domination and unitary equivalence by unitaries in $\mathcal{U}_0(E)$. Here $\mathcal{U}_0(E)$ denotes the connected component of 1_E in the group $\mathcal{U}(E)$ of unitaries in E with operator-norm topology.

PROPOSITION 4.3.6. *Suppose that E is unital and contains a copy of $\mathcal{O}_2$ unitally with canonical generators s_1, $s_2 \in \mathcal{O}_2 \subseteq E$. Let $h_1\colon D \to E$ be a C^*-morphism and $h_2\colon D \to E$ a contractive linear map.*

(i) *h_1 dominates zero, if and only if, $[h_1] + [0] = [h_1]$.*
This is the case, if and only if, there exists $u \in \mathcal{U}(E)$ such that $t_1 := us_1$, $t_2 := us_2$ and u satisfy $t_1 h_1(a) = h_1(a) = h_1(a) t_1$, $h_1(a) t_2 = 0$ for $a \in D$.
The unitary u can be taken such that $[u] = 0 \in \mathrm{K}_1(E)$ in addition.

(ii) *If h_1 dominates zero and $x \in E$ is a contraction, then the element $s := t_1 x + t_2 (1 - x^* x)^{1/2}$ – with t_1, t_2 as in* Part(i) *– is an isometry in E that satisfies $h_1(\cdot) x = h_1(\cdot) s$ and $x^* h_1(\cdot) = s^* h_1(\cdot)$.*
In particular, $s^* h_1(\cdot) s = h_2$ for an isometry $s \in E$ if h_1 dominates zero and $x^* h_1(\cdot) x = h_2$ for some contraction $x \in E$.

(iii) *Suppose that*
 (α) *h_1 dominates zero,*
 (β) *the inclusion map $C^*(h_2(D)) \hookrightarrow E$ dominates zero,* and
 (γ) *there are contractions $x, y \in E$ such that $h_2(\cdot) = x^* h_1(\cdot) y$ and $x x^* h_1(\cdot) y y^* = h_1$, i.e., also $x h_2(\cdot) y^* = h_1$.*

Then there exist $U, V \in \mathcal{U}(E)$ such that $h_2(\cdot) = U^* h_1(\cdot) V$.
If, moreover, $x = 1$, then one can find U and V such that $U = 1$.

In particular, h_1 is unitarily equivalent to h_2, if h_1 and h_2 dominate both zero and there exists $x \in E$ with $\|x\| \leq 1$, $x^* h_1(\cdot)x = h_2$ and $xh_2(\cdot)x^* = h_1$.

(iv) *Suppose that h_1 dominates zero and that h_2 is unitarily equivalent to h_1 in E.*

(a) *There exists $v \in \mathcal{U}(E)$ with $h_2 = v^* h_1(\cdot)v$ and $[v] = 0$ in $\mathrm{K}_1(E)$.*

(b) *If E is K_1-injective, then there is a unitary $v \in \mathcal{U}_0(E)$ with $h_2 = v^* h_1(\cdot)v$.*

(c) *If there exists an isometry $r \in E$ with $r^*(h_1(a) + h_2(a))r = 0$ for all $a \in D$, then there exists $v \in \mathcal{U}_0(E)$ with $h_2 = v^* h_1(\cdot)v$.*

(d) *If E has moreover the property that for every unitary $u \in E$ and every isometry $t \in E$ there are isometries $r, s \in E$ and $w \in \mathcal{U}_0(E)$ with $utr = wts$ then there is $v \in \mathcal{U}_0(E)$ with $h_2 = v^* h_1(\cdot)v$.*

(v) *If $R \in E$ is an isometry with $R^* h_1(\cdot)R = 0$ and $p \in E$ is a projection with $pR = 0$, then the $u \in \mathcal{U}(E)$ in* Part(i) *can be chosen such that $0 = [u] \in \mathrm{K}_1(E)$ and $t_1 = us_1$ satisfies $t_1 h_1(a) = h_1(a) = h_1(a)t_1$ for $a \in D$ and $pt_1 = p = t_1 p$.*

Proof. (i): If h_1 dominates zero, then $[h_1]+[0] = [h_1]$ by Proposition 4.3.5(i), because the copy of $\mathcal{O}_2$ in E commutes with the zero morphism $0(a) := 0$.

Conversely, if $[h_1] + [0] = [h_1]$, then, by definition of Cuntz Addition, there is a unitary $v \in E$ with $vs_1 h_1(\cdot)s_1^* v^* = h_1$. Then $r_j := vs_j$ $(j = 1, 2)$ are isometries with $r_1 h_1(\cdot)r_1^* = h_1$ and $r_1 r_1^* + r_2 r_2^* = 1$. In particular, $h_1(\cdot)r_2 = 0$, i.e., h_1 dominates zero. It follows $h_1(\cdot)r_1 r_1^* = h_1(\cdot)$ and $h_1(D)r_2 E r_2^* = \{0\}$.

Let $q_2 := r_2 r_1$ and $q_1 := r_1 r_1^* + r_2 r_1 r_2^*$, then q_1 and q_2 are isometries in E with $q_1 q_1^* + q_2 q_2^* = 1$, $h_1(\cdot)q_2 = 0$ and $q_1 h_1(\cdot) = h_1(\cdot) = h_1(\cdot)q_1$. The unitary $v := q_1 s_1^* + q_2 s_2^* \in \mathcal{U}(E)$ satisfies $vs_j = q_j$. Let $w := q_1 q_1^* + q_2 v^* q_2^*$, $u := wv$, $t_1 := us_1 = q_1$ and $t_2 := us_2 = q_2 v^*$. Then u, t_1, t_2 have the desired properties, because $h_1(\cdot)q_2 = 0$ and $[u] = [w] + [v] = [1] + [v^*] + [v] = [1] = 0$ in $\mathrm{K}_1(E)$.

(ii): Obviously $s := t_1 x + t_2(1 - x^* x)^{1/2}$ is an isometry and $h_1(\cdot)s = h_1(\cdot)t_1 x = h_1(\cdot)x$ by Part (i).

If $h_2 = x^* h_1(\cdot)x$ then $h_2 = s^* h_1(\cdot)s$, because $x^* h_1(a) = (h_1(a^*)x)^*$ and $D^2 = D$.

(iii): Let $a \in D_+$, $d_1 := xx^*$, $d_2 := yy^*$ and $b := h_1(a)$.

Then $b \geq 0$, $0 \leq d_i \leq 1$ $(i = 1, 2)$ and $d_1 b^{1/n} d_2 = b^{1/n}$ $(n = 1, 2, \ldots)$. This implies $d_1 p d_2 = p$ for the support p of b in E^{**}, because p is the weak limit of $b^{1/n}$.

Thus $pd_1 pd_2 p = p$. Since $0 \leq d_i \leq 1$, we get $pd_i p = p$, $p(1 - d_i)p = 0$, $(1 - d_i)^{1/2} p = 0 = p(1 - d_i)^{1/2}$, and $(1 - d_1)^{1/2} b = 0 = b(1 - d_2)^{1/2}$, because $pb = b = bp$. for $i = 1, 2$. That means

$$(1 - d_1)^{1/2} h_1(\cdot) = 0 = h_1(\cdot)(1 - d_2)^{1/2}.$$

Let $k\colon C^*(h_2(D)) \to E$ the inclusion map. By Part (i) there exist isometries $\tau_1, \tau_2 \in E$ such that $1 = \tau_1\tau_1^* + \tau_2\tau_2^*$, $k(\cdot)\tau_1 = \tau_1 k(\cdot) = k(\cdot)$ and $k(\cdot)\tau_2 = 0$. Let $t_1, t_2 \in E$ for h_1 as in (i). It follows from Part (i) and from the above equations for d_1, d_2 and h_1 that

$$U := t_1 x \tau_1^* - t_2 x^* \tau_2^* + t_1(1-d_1)^{1/2}\tau_2^* + t_2(1-x^*x)^{1/2}\tau_1^*$$

and

$$V := t_1 y \tau_1^* - t_2 y^* \tau_2^* + t_1(1-d_2)^{1/2}\tau_2^* + t_2(1-y^*y)^{1/2}\tau_1^*$$

are unitaries in E with

$$U^*h_1(\cdot)V = \tau_1(x^*h_1(\cdot)y)\tau_1^* = \tau_1 h_2(\cdot)\tau_1^* = h_2.$$

If $x = 1$, then, with $a := c^{1/2}$ for $c \in D_+$,

$$U^*h_1(c)U = h_2(a)(h_2(a)^*) = h_1(a)d_2h_1(a) = h_1(a^2) = h_1(c).$$

Thus, U commutes with the elements of $h_1(D)$, and $h_2(\cdot) = h_1(\cdot)U^*V$.

Suppose that h_1 and h_2 both dominate zero and that there is a contraction $x \in E$ with $x^*h_1(\cdot)x = h_2$ and $xh_2(\cdot)x^* = h_1$. Then there is an isometry $t \in E$ with $t^*h_2(\cdot)t = 0$, i.e., $h_2(D) = \mathrm{span}(h_2(D_+))$ is contained $(1-tt^*)E(1-tt^*)$. Thus, $t^*bt = 0$ for all $b \in C^*(h_2(D))$, and the above construction of U and V works for x and $y := x$. It gives $d_2 = d_1$ and $V = U$. Hence, $h_2 = U^*h_1(\cdot)U$.

(iv): Let $U \in \mathcal{U}(E)$ with $U^*h_1(\cdot)U = h_2$. In particular, $[h_2] = [h_1] = [h_1] + [0] = [h_2] + [0]$, i.e., h_2 dominates zero.

(iv,a): Let $t_1, t_2 \in E$ the isometries with the properties given in (i), then $V := (1 \oplus_{t_1,t_2} U^*)U$ is unitary and satisfies $V^*h_1(\cdot)V = U^*h_1(\cdot)U = h_2\,.$ $[V] = [1] + [U^*] + [U] = 0$ in $\mathrm{K}_1(E)$, cf. Lemma 4.2.6(v,1).

(iv,b): If $\mathcal{U}_0(E)$ is the kernel of $\mathcal{U}(E) \to \mathrm{K}_1(E)$ (i.e., if E is K_1-injective) then $V \in \mathcal{U}_0(E)$ for the V as in Part (iv, a).

(iv,c): Suppose that there is an isometry $r \in E$ with $r^*(h_1(a) + h_2(a))r = 0$ for all $a \in D$. Then $h_k(D) \subseteq (1-rr^*)E(1-rr^*)$ for $k = 1, 2$. We consider the C^*-subalgebra $F := C^*(h_1(D) + h_2(D)) \subseteq E$ generated by

$$h_1(D) + h_2(D) \subseteq (1-rr^*)E(1-rr^*)\,.$$

The inclusion morphism $f \in F \mapsto f \in E$ dominates zero, because $r^*Fr = 0$. By Part (i) there are canonical generators t_1, t_2 of a copy of $\mathcal{O}_2$ in E such that $t_1 f = f = f t_1$ and $f t_2 = 0 = t_2^* f$ for $f \in F$. Let $V = t_1 U t_1^* + t_2 U^* t_2^*$. Then V is in $\mathcal{U}_0(E)$ by Lemma 4.2.6(v,3), and $V^*h_1(\cdot)V = t_1U^*h_1(\cdot)Ut_1^* = t_1h_2(\cdot)t_1^* = h_2\,.$

(iv,d): The property in Part (iv,d) implies that $\mathcal{U}_0(E)$ is the kernel of $\mathcal{U}(E) \ni u \mapsto [u] \in \mathrm{K}_1(E)$, cf. Lemma 4.2.10(iii). Thus Part (iv, b) applies.

(v): Since $pR = 0$ and $R^*h_1(D)R = \{0\}$, the C^*-subalgebra $F := C^*(h_1(D), p) \subseteq E$ is contained in $(1-RR^*)E(1-RR^*)$. The identity map $f \in F \mapsto f \in E$ dominates zero because $R^*fR = 0$ for all $f \in (1-RR^*)E(1-RR^*)$.

Application of Part (i) to (F, id_F) – in place of (D, h_1) – gives that there is a unitary $u \in \mathcal{U}(E)$ with $0 = [u] \in \mathrm{K}_0(E)$ and $t_1 f = f = f t_1$ for $t_1 := us_1$ and $f \in F$. In particular $t_1 p = p = pt_1$ and $t_1 h_1(a) = h_1(a) = h_1(a)t_1$ for $a \in D$. □

COROLLARY 4.3.7. *Suppose that A and E are unital C^*-algebras, $h_1, h_2 \colon A \to E$, unital C^*-morphisms, and that unital C^*-morphisms $\psi_1, \psi_2 \colon \mathcal{O}_2 \to E$ exist with $\psi_k(\mathcal{O}_2) \in h_k(A)' \cap E$ for $k = 1, 2$.*

If h_1 and h_2 one-step dominate each other, i.e., if there exist contractions $S, T \in E$ with $h_2 := S^* h_1(\cdot) S$ and $h_1 := T^* h_2(\cdot) T$, then h_1 and h_2 are unitary equivalent in E.

PROOF. By Proposition 4.3.5(i), h_1 is unitarily equivalent to $h_1 \oplus h_2$ and h_2 is unitarily equivalent to $h_2 \oplus h_1$.

The symmetry $T := s_2 s_1^* + s_1 s_2^*$ satisfies $T^*(h_1 \oplus_{s_1,s_2} h_2)T = h_2 \oplus_{s_1,s_2} h_1$. □

COROLLARY 4.3.8. *Two unital C^*-morphisms $h_0, h_1 \colon \mathcal{O}_2 \to E$ are approximately unitary equivalent.*

They are homotopic if and only if the unitary $u := h_1(s_1)h_0(s_1^*) + h_1(s_2)h_0(s_2^*)$ is in $\mathcal{U}_0(E)$.

PROOF. In fact h_1 and h_2 are unitary equivalent in $E_\infty := \ell_\infty(E)/c_0(E)$ if we use here the later in Chapter 5 proven result that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is unitally contained in $\mathcal{O}_2$, (or could use here that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ by the later proven Corollary F(iii)), and that this unital imbedding $\psi \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ has the property that $a \mapsto \psi(a \otimes 1)$ is approximately unitarily equivalent to the identity map of $\mathcal{O}_2$ by the approximate innerness of $\delta_2 \colon \mathcal{O}_2 \to \mathcal{O}_2$. gives the ψ (In fact, the morphisms $\delta_2 \colon \mathcal{O}_2 \to \mathcal{O}_2$ and $a \mapsto \psi(a \otimes 1)$ are moreover unitarily homotopic to the identity map of $\mathcal{O}_2$.) This and the nuclearity of $\mathcal{O}_2$ ensures that that any two unital C^*-morphisms from $\mathcal{O}_2$ into E_∞ 1-step dominate each other. Hence Corollary 4.3.7 applies to $h_1, h_2 \colon \mathcal{O}_2 \to E_\infty$.

The homotopy of h_1 and h_2 with $h_1(s_k) = u(1)h_0(s_k)$ for a continuous map $[0,1] \ni t \mapsto u(t) \in \mathcal{U}(A)$ with $u(0) = 1$ is given by $h_t \colon \mathcal{O}_2 \to \mathcal{U}(A)$ with $h_t(s_k) := u(t)s_k$. □

REMARK 4.3.9. If $h_0, h_1 \colon \mathcal{O}_2 \to E$ are homotopic, then one can show that they are moreover "unitarily homotopic", cf. Corollary 4.6.4, i.e., are unitary equivalent in $\mathrm{C_b}([0,\infty), E)/\mathrm{C}_0([0,\infty), E)$. (It does not imply that, conversely, h_0 and h_1 itself are unitarily equivalent or are homotopic in E.)

4. Groups defined by absorbing C^*-morphisms

Suppose that $h_0 \colon D \to E$ is a C^*-morphism such that $h_0(D)' \cap E$ contains $\mathcal{O}_2$ unitally. Proposition 4.3.5(iii) says that this is equivalent to the existence of a unital C^*-morphism from $\mathcal{O}_2$ into E and the equation $[h_0] + [h_0] = [h_0]$.

DEFINITION 4.4.1. We define then the semigroup

$$S(h_0\,;\, D, E) := \{\, [h]\,;\;\, h \in \mathrm{Hom}(D, E)\,,\;\, h \text{ is dominated by } h_0 \,\}$$

and

$$G(h_0\,;\,D,E) := \{\,[h]+[h_0]\,;\ [h]\in S(h_0\,;\,D,E)\,\}\,.$$

Notice that $G(h_0\,;\,D,E) = [h_0] + S(h_0\,;\,D,E)\ \subseteq S(h_0\,;\,D,E)\,.$

Due to the terminology used in the abstract Lemma 4.2.3, we call sometimes the morphisms $h \in \operatorname{Hom}(D,E)$ with the property $[h] \in G(h_0\,;\,D,E)$ the **absorbing** morphisms in the class of those morphisms $k \in \operatorname{Hom}(D,E)$, that are dominated by h_0 .

G. Kasparov has introduced a similar notion of absorption in his early paper [**405**] on Ext-groups. By Proposition 4.3.5(i), a unitary equivalence class $[h] \in S(h_0\,;\,D,E)$ is in $G(h_0\,;\,D,E)$ if and only if h dominates h_0.

The definitions, the above listed properties and Remark 3.1.2(iii) together immediately imply the following consequences of "$[k] \in S(h_0\,;\,D,E)$":

(i) $k(D) \subseteq J$ if $[k] \in S(h_0\,;\,D,E)$, $h_0(D) \subseteq J$ and J is an ideal of E.
(ii) k is nuclear if $[k] \in S(h_0\,;\,D,E)$ and h_0 is nuclear.

More general, it is obvious that $k \in \mathcal{C}$ if $h_0 \in \mathcal{C}$ for some matrix operator-convex cone $\mathcal{C} \subseteq \operatorname{CP}(D,E)$. This applies in particular to the m.o.c. cone $\mathcal{C}(h_0)$ that is generated by h_0.

PROPOSITION 4.4.2. *Suppose that E contains a copy of $\mathcal{O}_2$ unitally and $h_0\colon D \to E$ is a C^*-morphism with $[h_0] + [h_0] = [h_0]$.*

(i) *$S(h_0\,;\,D,E)$ is a commutative semigroup under Cuntz addition of unitary equivalence classes, and*
(ii) *$G(h_0\,;\,D,E)$ is a subgroup of $S(h_0\,;\,D,E)$, and $[h] \to [h] + [h_0]$ is a semigroup epimorphism from $S(h_0\,;\,D,E)$ onto $G(h_0\,;\,D,E)$. It identifies $G(h_0\,;\,D,E)$ naturally with the Grothendieck group of $S(h_0\,;\,D,E)$, in the sense that every semi-group morphism $\varphi\colon S(h_0\,;\,D,E) \to \Gamma$ into a group Γ satisfies $\varphi([h]) = \varphi([h]+[h_0])$ for all $[h] \in S(h_0\,;\,D,E)$.*

PROOF. Let $s_1, s_2 \in h_0(D)' \cap E$ isometries with $s_1s_1^* + s_2s_2^* = 1$, as given by Proposition 4.3.5(iii).

(i): If h_1 is dominated by h_0 then the same property holds for every map in the unitary equivalence class $[h_1]$. Suppose that $[h_1]$ and $[h_2]$ are in $S(h_0; D, E)$. Then $[h_1]+[h_2] \in S(h_0\,;\,D,E)$, because $h_1 \oplus_{s_1,s_2} h_2$ is dominated by h_0, cf. Lemma 4.3.4(v). The semigroup structure follows now from Proposition 4.3.2(i).

(ii): Since $[h_0]+[h_0] = [h_0]$, the subset $G(h_0\,;\,D,E)$ is a subgroup of the semigroup $S(h_0\,;\,D,E)$, and $[h_0]$ is the neutral element (zero element) of $G(h_0\,;\,D,E)$. The inverse element of $[h]$ in $G(h_0\,;\,D,E)$ can be detected as follows: Since $h \in S(h_0\,;\,D,E)$, there is an isometry $t \in E$ with $h = t^*h_0t$. Then $p = tt^*$ is a projection in $h_0(D)' \cap E$ by Lemma 4.3.4(i), and $(1-p)t = 0 = t^*(1-p)$. Hence,

$$k := (1-p)h_0(\cdot) + th_0(\cdot)t^*$$

is a C^*-morphism. The morphism h_0 dominates k by Proposition 4.3.5(ii) because h_0 2-dominates k.

Now it is easy to verify that $k \oplus_{s_1,s_2} h = u^* h_0(\cdot) u$, where $\{s_1, s_2\}$ are generators of $\mathcal{O}_2$ in $h_0(D)' \cap E$, and u is the unitary operator $s_1 t s_2^* + s_1(1-p)s_1^* + s_2 t^* s_1^*$. Then $[(h \oplus h_0) \oplus (k \oplus h_0)] = [h_0 \oplus h_0 \oplus h_0] = [h_0]$. Hence $[k \oplus h_0]$ is the inverse of $[h \oplus h_0]$ in $G(h_0\,;\,D, E)$. This shows that Lemma 4.2.3 applies to the semigroup $S(h_0\,;\,D, E)$ and $G(h_0\,;\,D, E) = [h_0] + S(h_0\,;\,D, E)$. □

The following Proposition 4.4.3 identifies $G(h_0\,;\,D, E)$ with a subgroup of the K_0 group $\mathrm{K}_0(h_0(D)' \cap E)$ and gives the later used characterizations of the elements of this subgroup, and it describes the K_*-theoretic picture of our $\mathrm{Ext}(\mathcal{C};\,\cdot,\cdot)$- and $\mathrm{R}(\mathcal{C};\,\cdot,\cdot)$-groups studied in Chapters 5 and 7. Here the Cuntz addition $\oplus$ in Parts (ii) and (iii) has to be defined with help of the canonical generators of a fixed copy of $\mathcal{O}_2$ that is unitally contained in $h_0(D)' \cap E$.

PROPOSITION 4.4.3. *Let $h_0 \colon D \to E$ a C^*-morphism, such that $h_0(D)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally, and let p and p' projections in $h_0(D)' \cap E$.*

(i) *The group $G(h_0\,;\,D, E)$ is naturally isomorphic to the kernel of the natural K_*-theory map*

$$i_0 \colon \mathrm{K}_0(h_0(D)' \cap E) \to \mathrm{K}_0(E),$$

where $i_0 := \mathrm{K}_0(i)$ for the inclusion map $i \colon h_0(D)' \cap E \hookrightarrow E$.

(ii) *The element $[p] \in \mathrm{K}_0(h_0(D)' \cap E)$ is in the kernel of i_0, if and only if, there is a unitary v in the connected component $\mathcal{U}_0(E)$ of 1_E in $\mathcal{U}(E)$ such that $v^*(p \oplus 1 \oplus 0)v = 1 \oplus 0$.*

The equation $[p] = [p']$ holds in $\mathrm{K}_0(h_0(D)' \cap E)$, if and only if, there is a unitary $u \in \mathcal{U}_0(h_0(D)' \cap E)$ such that $u^*(p \oplus 1 \oplus 0)u = p' \oplus 1 \oplus 0$.

(iii) *If h_0 dominates zero, then projections p and p' in $h_0(D)' \cap E$ define the same element of $\mathrm{K}_0(h_0(D)' \cap E)$, if and only if, $[p] = [p']$ in $\mathrm{K}_0(E)$ and there is a unitary $u \in h_0(D)' \cap E$ such that $\Delta := (p' \oplus 1 \oplus 0) - u^*(p \oplus 1 \oplus 0)u$ is in the (two-sided) annihilator of $h_0(D)$, i.e., $h_0(a)\Delta = 0$ for all $a \in D$.*

PROOF. We let $C := h_0(D)' \cap E$ and define the Cuntz addition $\oplus$ by isometries $s_1, s_2 \in C$ with $s_1 s_1^* + s_2 s_2^* = 1_E$, i.e., $\oplus = \oplus_{s_1,s_2}$.

(i): A C^*-morphism $h \colon D \to E$ has unitary equivalence class $[h]$ in $S(h_0\,;\,D, E)$ if and only if there is an isometry $t \in E$ with $h = t^* h_0(\cdot) t$ (cf. Definition 4.3.3). Then $tt^* \in C$ by Lemma 4.3.4(i). If $s \in E$ is an other isometry with $s^* h_0(\cdot) s = h$, then $z = ts^*$ is a partial isometry in C by Lemma 4.3.4(ii). We have $tt^* = zz^*$ and $z^* z = ss^*$, i.e., tt^* and ss^* are Murray–von-Neumann equivalent in C. Let $k \colon D \to E$ a C^*-morphism in the unitary equivalence class $[h]$ of h. Then there is a unitary $u \in E$ with $k = u^* h(\cdot) u = u^* t^* h_0(\cdot) t u$. Thus k is a C^*-morphism that is dominated by h_0, and $[k] = [h]$. Note that $(tu)(tu)^* = tt^*$.

It follows that the map θ from $S(h_0\,;\,D, E)$ into $\mathrm{K}_0(C)$, defined by

$$\theta([h]) := [tt^*] \in \mathrm{K}_0(C)$$

for $h = t^*h_0(\cdot)t$, with $t^*t = 1$ is well-defined, because $[s^*h_0(\cdot)s] = [t^*h_0(\cdot)t] = [h]$ implies that the projections tt^* and ss^* are Murray–von Neumann equivalent in C.

By Lemma 4.3.4(v), $h_1 \oplus h_2 = r^*h_0(\cdot)r$ with $r = t_1 \oplus t_2$ where $h_i(\cdot) = t_i^*h_0(\cdot)t_i$ for $i = 1, 2$. Thus, $\theta([h_1] + [h_2]) = [t_1t_1^* \oplus t_2t_2^*] \in \mathrm{K}_0(C)$. This is the same as $[t_1t_1^*] + [t_2t_2^*] = \theta([h_1]) + \theta([h_2])$ by Lemma 4.2.6(i). In particular $\theta([h_0]) = 0$, because $2\theta([h_0]) = \theta([h_0])$. Since $\mathrm{K}_0(C)$ is a group, $[h] + [h_0] \mapsto \theta([h])$ is a well-defined group homomorphism from $G(h_0\,; D, E)$ into $\mathrm{K}_0(C)$.

As $1_E \in \mathcal{O}_2 \subseteq E$, we get for $h = t^*h_0(\cdot)t$ with $t^*t = 1$,

$$[0]_E = [1]_E = [t^*t]_E = [tt^*]_E \in \mathrm{K}_0(E),$$

so that θ maps $S(h_0\,; D, E)$ into the kernel of i_0.

Let $h_1 = s^*h_0(\cdot)s$, $h_2 = t^*h_0(\cdot)t$ and $\theta([h_1]) = \theta([h_2])$, i.e., $[ss^*] = [tt^*]$ in $\mathrm{K}_0(C)$. The projections $1 \oplus ss^*$ $(= (1 \oplus s)(1 \oplus s)^*)$ and $1 \oplus tt^*$ are full and properly infinite in C, thus are MvN-equivalent in C by Lemma 4.2.6(ii). But this means that $h_0 \oplus h_1$ and $h_0 \oplus h_2$ are unitarily equivalent, by Lemma 4.3.4(ii). Thus $[h_1] + [h_0] = [h_2] + [h_0]$ if $\theta([h_1]) = \theta([h_2])$, and $\theta | G(h_0\,; D, E)$ is a *monomorphism*.

Now let $x \in \mathrm{K}_0(C)$ with $i_0(x) = 0$. Since C is properly infinite, it follows from Lemma 4.2.6(iii), that there exists a projection $p \in C$, and isometries $s, t \in C \subseteq E$, such that $x = [p]$, $ss^* \leq p$ and $tt^* \leq 1 - p$. Then $i_0(x) = [p] = 0 = [1]$ in $\mathrm{K}_0(E)$, and, by Lemma 4.2.6(ii), there is $r \in E$ with $rr^* = p \in C$ and $r^*r = 1$. The completely positive map $h := r^*h_0(\cdot)r$ is a C^*-morphism from D into E by Lemma 4.3.4(i) and h_0 dominates h, i.e., $[h] \in S(h_0\,; D, E)$. Moreover, $\theta([h] + [h_0]) = \theta([h]) = [rr^*] = [p] = x$. Hence, $[h] + [h_0] \mapsto \theta([h])$ is a group *isomorphism* from $G(h_0\,; D, E)$ *onto* $\ker(i_0) \subseteq \mathrm{K}_0(C)$.

(ii): Recall that $1 \oplus 0 = s_1s_1^*$ is unitarily equivalent to $1 \oplus 1 \oplus 0$ by a unitary in $\mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$, cf. Proposition 4.3.2. Thus, (ii) is a special case of Lemma 4.2.6(iv,b) (with C in place of E for the equation $[p] = [p'] \in \mathrm{K}_0(C)$).

(iii): Let p, p' are projections in C. If $[p] = [p']$ in $\mathrm{K}_0(C)$, then $[p] = [p']$ in $\mathrm{K}_0(E)$ and, by (ii), there is a unitary $u \in C$ with $u^*(p \oplus 1 \oplus 0)u = p' \oplus 1 \oplus 0$.

Conversely, suppose that $[p] = [p']$ in $\mathrm{K}_0(E)$ and that there is a unitary $u \in C$ such that $\Delta := (p' \oplus 1 \oplus 0) - u^*(p \oplus 1 \oplus 0)u$ satisfies $\Delta \cdot h_0(D) = \{0\}$.

Let $p_1 := u^*(p \oplus 1 \oplus 0)u$ and $p_2 := p' \oplus 1 \oplus 0$. Then $[p_1] = [p] + [1] + [0] = [p]$ and $[p_2] = [p']$ as well in $\mathrm{K}_0(C)$ as in $\mathrm{K}_0(E)$. Therefore, p_1 and p_2 are projections in C with $[p_1] = [p_2]$ in $\mathrm{K}_0(E)$ and $p_2 - p_1 \in N$, where $N \subseteq E$ denotes the (two-sided) annihilator of $h_0(D)$ in E.

The two-sided annihilator N of $h_0(D)$ is a hereditary C^*-subalgebra of E and is an ideal of $C = h_0(D)' \cap E$. The hereditary C^*-subalgebra N of E is *full* in E, because h_0 dominates zero by assumptions of Part (iii), i.e., there is an isometry $t \in E$ with $tt^* \in N$. By Lemma 4.2.20(i), we get $[p_1] = [p_2] \in \mathrm{K}_0(C)$ from $p_2 - p_1 \in N$ and $[p_1] = [p_2] \in \mathrm{K}_0(E)$. □

REVISE HISTORIC REMARKS:

The assumptions in the following Theorem 4.4.6 are the same as of Lemma [**442**, lem. 3.8], but with assumption [**442**, lem. 3.8(vi)] removed, because it can be derived from the other assumptions, cf. Lemma 4.4.7(vii). An old outline of the proof of Lemma [**442**, lem. 3.8] – displayed in beamer presentations – incorrectly claimed that the natural surjective group homomorphism $\varphi\colon G(h_0\,;\,D,E) \to G(H_0\,;\,D,E)$ is always *injective*. This is still unknown, and Professor Claire Anantharaman-Delaroche mentioned her concerns about the injectivity claim for the natural group homomorphism φ to the author (likely around 1996). And she suggested that the homotopy invariance of the "unsuspended" but stable $\mathcal{E}$-theory (respectively the slightly different $\mathcal{O}_2$-unital $\mathcal{E}$-theory in the approach of N. Ch. Phillips) should show the equivalence to KK-theory.

It turns out that the Grothendieck groups of some kinds of unsuspended but stable $\mathcal{E}$-theories are homotopy invariant, but that gives not a proof for the coincidence with the related sort of KK-theories if equipped with certain additional "equivariant" behavior.

At the actual state we have two different proofs of the homotopy invariance of the continuous Rørdam groups $\mathrm{R}(\mathcal{C}\,;\,A,B)$ that are defined in Chapter 7 for stable separable A and B and non-degenerate m.o.c. cone $\mathcal{C} := \mathcal{C}(h_0)$, – which is in this special case isomorphic to the below considered group $G(h_0\,;\,D,E)$:

One way is to prove first independently that the groups $\mathrm{R}(\mathcal{C}\,;A,B)$ are homotopy invariant itself by a generalization and modification of the ideas of N.Ch. Phillips in his proof of the homotopy invariance of the generalized unsuspended E-theory for pi-sun algebras in [**627**].

See Section **??** concerning a similar but more general sort of stable, but unsuspended, E-semigroups and an outline of the homotopy invariance of its Grothendieck group. It is not clear if one can use a "controlled" homotopy invariance of $\mathrm{R}(\mathcal{C}\,;\,A,B)$ to establish the injectivity of the natural epimorphism $\varphi\colon G(h_0\,;\,D,E) \to G(H_0\,;\,D,E)$, in the special case where naturally $\mathrm{R}(\mathcal{C}\,;\,A,B) = G(h_0\,;\,D,E)$ with $D := A \otimes \mathbb{K}$, $E := \mathcal{M}(\mathrm{C}_0(\mathbb{R},B))/\,\mathrm{C}_0(\mathbb{R},B)$ and

$$h_0\colon D \to B \subset J := \mathrm{C}_\mathrm{b}(\mathbb{R}_+,B)/\,\mathrm{C}_0(\mathbb{R}_+,B) \subset E\,,$$

a generating homomorphism for the given m.o.c. cone $\mathcal{C} \subset \mathrm{CP}(A,B)$, cf. Chp. 7. Notice that $G(h_0\,;\,D,E)$ is in this particular case the same as $\mathrm{Ext}(\mathcal{C}\,;\,A,SB) \cong \mathrm{KK}(\mathcal{C}\,;\,A,B)$.

```
Until now:
We have only that (DC) is valid for our special applications,
that (DC) implies functorial equivalence of R(C;.,.) with KK(C;.,.).
The injectivity criteria of the following Theorem 4.4.6
and arguments in Chapters 7,
8 and 9 show that the injectivity of the
group epimorphism φ implies also homotopy invariance of R(C; A,B).
```

An ''independent proof'' from the natural isomorphism
between $\mathrm{R}(\mathcal{C}\,;\,A,B[0,1])$ and $\mathrm{R}(\mathcal{C}\,;\,A,B)$
to the injectivity of φ is not found or written up!!!
So far, until now.
An isomorphism $\mathrm{R}(\mathrm{CP}_{\mathrm{nuc}}\,;A,B[0,1])\cong\mathrm{R}(\mathrm{CP}_{\mathrm{nuc}}\,;A,B))$
was shown first by N.Ch. Phillips in [627] (for A and B in Cuntz standard form?).
Then he used the characterization of N. Higson [364].
We do not know if we can also go this way
but the *general problem* is that it is not known if $U(0)=1$ and

$$U^*(k\oplus H_0)(a)U-H_0(a)\in \mathrm{C}_0([0,\infty),B)\ \text{ for all } a\in A$$

always implies that $h_0\otimes k$ and h_0
are asymptotical homotopic,
i.e., if there exists $g_{s,t}\in S(h_0;\,A,B)$ with
$s,t\in[0,1]\times[0,\infty)$ point-norm continuous
and $g_{0,t}=h_0\oplus k$ and $g_{1,t}=h_0$.

We generalize in Section **??** an idea of N.Ch. Phillips that shows that the *Grothendieck* groups of certain semigroups of asymptotic morphisms are homotopy invariant, in a way that covers also the case of our (!) more general types of Rørdam groups.

But we underline that in our special case the injectivity criteria (DC) of Theorem 4.4.6 follows from the homotopy invariance of $\mathrm{KK}(\mathcal{C}\,;\,A,B)$, where it is very important that this holds for *arbitrary* m.o.c. cones. It allows a precise explanation for the sketched and a bit imprecise arguments outlined at the end of the Preprint [**434**], cf. also [**442**].

We describe in Theorem 4.4.6 the hypothetical elements in the kernel of the homomorphism φ, and give an applicable sufficient criteria (DC) for the injectivity of φ.

We show in Section 3 of Chapter 9 that in the later used special case of the canonical surjection $\mathrm{R}(\mathcal{C}\,;\,A,B)\to\mathrm{Ext}(\mathcal{C}\,;\,A,\,SB)\cong\mathrm{KK}(\mathcal{C}\,;\,A,B)$ the decomposition condition (DC) of Theorem 4.4.6 is a consequence of

$$\mathrm{KK}(\mathcal{C}(0,1]\,;\,A;\mathrm{C}_0((0,1],B))=0\,,$$

i.e., the homotopy invariance of $\mathrm{KK}(\mathcal{C}\,,A,B)$ allows to proof for our applications the property (DC) that implies finally the needed bijectivity.

([7]).

It shows that the homotopy invariance of the Kasparov groups alone imply that of $\mathcal{C}$-related Rørdam groups and KK-groups are isomorphic. As claimed in the appendix of the Preprint [**434**], there not really proved.

[7] The author regrets for his remarks “Es ist ein Gradeaus-Beweis.” below [**442**, lem. 3.8], and ”... should be able to work out the details” in the appendix of [**434**].

We need for the description of the *kernel* of the group homomorphism $\varphi\colon G(h_0\,; D,E) \to G(H_0\,; D,E)$ in the following Theorem 4.4.6 and in Lemma 4.4.15, some basic notations and definitions of algebraic nature:

DEFINITION 4.4.4. Let $A \subseteq E$ a C^*-subalgebra, $I \triangleleft E$ an closed ideal of E, $C := A' \cap E$ the commutant (or "centralizer") of A in E, and define by

$$\mathrm{Ann}(A) := \mathrm{Ann}(A,E) := \{\, e \in E\,;\ eA \cup Ae = \{0\}\,\}\,. \tag{4.1}$$

the *annihilator* of A in E. We consider following C^*-subalgebras of E:

$$\mathrm{Der}(A,I) := \{\, e \in E\,;\ ea - ae \in I,\ \forall\, a \in A\,\} \;=\; \pi_I^{-1}\big(\,\pi_I(A)' \cap (E/I)\,\big)\,, \tag{4.2}$$

$$\mathcal{N}(A,I) := \{\, e \in E\,;\ ea, ae \in I,\ \forall\, a \in A\,\} \;=\; \pi_I^{-1}\big(\,\mathrm{Ann}(\pi_I(A), E/I)\,\big)\,. \tag{4.3}$$

Compare Remarks 6.1.2 for further details on $\mathrm{Ann}(\,\cdot\,)$ and $\mathcal{N}(\,\cdot\,,\,\cdot\,)$. Notice that $C = \mathrm{Der}(A,\{0\}) \subseteq \mathrm{Der}(A,I)$ and that $\mathcal{N}(A,I)$ is a closed ideal of $\mathrm{Der}(A,I)$.

DEFINITION 4.4.5. We define a discrete abelian group $\Gamma\big(A,I,E\big)$ by

$$\Gamma\big(A,I,E\big) := \mathrm{K}_1\big(\mathrm{Der}(A,I)\big)\,/\,\mu\big(\mathrm{K}_1(C)\big) \tag{4.4}$$

where the group morphism $\mu := [\eta]_1\colon \mathrm{K}_1(C) \to \mathrm{K}_1(\mathrm{Der}(A,I))$ comes from the inclusion map $\eta\colon C \hookrightarrow \mathrm{Der}(A,I)\,$.

The formulation "later specified" used in Parts (iii) and (iv) of the following Theorem means that different possible choices of the (non-unital) C^*-subalgebra $F \subseteq C$ with the properties in Part (iii) and of isometries $s_0, t_0 \in E$ with the properties in Part (iv) are *not* uniquely determined, but have to exist and can be be chosen/adjusted such that they interplay in the manner that is described in Parts (iii,iv). Their existence with the required interplay is a property of the system $(J \subseteq E, \beta, h_0, H_0)\,$.

THEOREM 4.4.6. *Suppose that $h_0\colon D \to E$ and $H_0\colon D \to E$ are *-monomorphisms, where D is a separable C^*-algebra and E is a unital C^*-algebra. Further let J be a closed ideal of E and β a *-automorphism of E, such that $h_0(D) \subseteq J$, $\beta^2 = \mathrm{id}$, $\beta H_0 = H_0$ and $\beta(J) \cap J = 0$.*

We require that E, J, D, H_0, h_0, and β satisfy the following assumptions (i)-(vi)*, with $C := H_0(D)' \cap E$.*

- (i) *The algebra $h_0(D)' \cap H_0(D)' \cap E \subseteq C$ contains β-invariant isometries s, t with $ss^* + tt^* = 1$.*
- (ii) *If H_0 dominates $k\colon D \to E$, then there exists a unitary $u \in E$ such that $u^*(k \oplus H_0)(a)u - H_0(a) \in J$ for all $a \in D$.*
- (iii) *There exists a (later specified) C^*-subalgebra $F \subseteq C$ with following properties:*
  - (a) *$F \cdot H_0(D) \subseteq J + \beta(J)$,*
  - (b) *$\beta(F) = F$, and*
  - (c) *for every $f \in J + \beta(J)$ and every projection $q \in F$, there is a projection $p \in F$ with $pf = fp = f$, $q \le p$ and $q \neq p$.*

(iv) *There are (later specified) β-invariant isometries $s_0, t_0 \in E$ that interplay with F, C, H_0 and h_0 in the following manner*

$$p_0 := s_0 s_0^* \in F\,, \ \ t_0 \in C\,, \ \ s_0 s_0^* + t_0 t_0^* = 1$$

and

$$s_0^* H_0(a) s_0 = h_0(a) + \beta(h_0(a)) \quad \textit{for all } a \in D\,.$$

(v) *Every non-zero projection in F is unitarily equivalent to p_0 by a unitary in $1 + F$.*

(vi) *There exists a unitary u_1 in $J' \cap E$ such that $u_1 \beta h_0(\cdot) s s^* = t t^* \beta h_0(\cdot) u_1$.*

If assumptions (i)-(vi) *are satisfied, then $h_0 \oplus H_0$ is unitarily equivalent to H_0 and the mapping $\varphi\colon [k] + [h_0] \mapsto [k] + [H_0]$ is a (well-defined) group epimorphism from $G(h_0\,;\, D, E)$ onto $G(H_0\,;\, D, E)$.*

The kernel *of this epimorphism is naturally isomorphic to $\Gamma(H_0(D), J, E)$.*

The natural group epimorphism $\varphi\colon [k] + [h_0] \mapsto [k] + [H_0]$ from $G(h_0\,;\, D, E)$ onto $G(H_0\,;\, D, E)$ is injective, *if and only if, E, J, β, D, and H_0 satisfy the following* decomposition condition (DC)*:*

(DC) *For each $u \in \mathcal{U}_0(E) \cap \mathrm{Der}(H_0(D), J)$ there exists $v \in \mathcal{U}(C)$ such that*

$$v(u \oplus_{s,t} 1) \in \mathcal{N}(H_0(D), J + \beta(J)) + \mathbb{C} \cdot 1\,.$$

The condition (DC) *is equivalent to the following (formally weaker) property* (wDC)*:*

(wDC) *If $u \in \mathcal{U}_0(E) \cap \mathrm{Der}(H_0(D), J)$ and $k \in S(h_0\,;\, D, E)$ satisfy*

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot) u\,,$$

then

$$[k] + [h_0] = [h_0]\,.$$

`Compare below with above concerning` R(...) `to` KK(...).

The group homomorphism $G(h_0\,;\, D, E) \;\to\; G(H_0\,;\, D, E)$ is a "minimal" abstract version of the natural group epimorphism from the generalized Rørdam groups $\mathrm{R}(\mathcal{C}\,;\, D, B)$, cf. Chapter 7, to the $\mathcal{C}$-equivariant extension groups $\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB) \cong \mathrm{KK}(\mathcal{C};\, D, B)$ introduced in Chapter 5 and compared with KK-groups in Chapter 8.

The criteria (wDC) shows that the injectivity of this group epimorphism is `equivalent ???`
to a suitable version of homotopy invariance for the groups $\mathrm{R}(\mathcal{C}\,;\, D, B)$.

`Not proved until now !!! Imprecise idea exists?`

The Rørdam groups are unsuspended – but stable – versions of, from a cone $\mathcal{C}$ depending, E-theory groups. The definition of $\mathrm{R}(\mathcal{C}\,;\, D, B)$ is entirely algebraic and does not require any sort of homotopy invariance in its formulation, even not implicitly.

`A cited condition (vwDC) is not defined somewhere!!!`
In fact we show that a below given formally weaker condition (vwDC) implies that $\Gamma(H_0(D), J, E) = 0$.

We have used in the formulation of Parts (iii) and (iv) in Theorem 4.4.6 the terminus "specified", that should say the following (together with Part (v)): We demand that our "given" natural system $(E, J, D, H_0, h_0, \beta)$ has the property that F, s_0 and t_0 exist with the properties listed in Parts (iii). But then we fix one of the possible systems (F, s_0, t_0) and add it to the structure given by $(E, J, D, H_0, h_0, \beta)$ simply for further references in the proof. The additional (F, s_0, t_0) are not uniquely determined by $(E, J, D, H_0, h_0, \beta)$. The F in most of our applications satisfies in addition that $\mathrm{K}_1(F) = 0$, that can not be derived from the assumptions of Theorem 4.4.6.

The difference to Part (v) is that this is an additional requirement on F, and we need from Part (vi) only that such u_0 *exists* in later proofs it not necessary to use always the same u_0 with the in (vi) quoted property.

The Parts (i)–(v) of following Lemma 4.4.7 are not obvious because $[\varphi] = [\varphi + 0] \neq [\varphi] + [0] = [\varphi \oplus 0]$ for C^*-morphisms φ that do *not* dominate zero. The used isometry s_0 and $p_0 := s_0 s_0^*$ are given by assumption (iv) of Theorem 4.4.6.

Recall that $C := H_0(D)' \cap E$ in Theorem 4.4.6.

LEMMA 4.4.7. *The assumptions* (i)–(vi) *on* h_0 *and* H_0 *in Theorem 4.4.6 imply the following properties* (i)-(xii)*:*

(i) *Let B_j, $j = 1, 2$, separable C^*-algebras, $k_j \colon B_j \to J \subseteq E$ C^*-morphisms, and use $u_1 \in \mathcal{U}(E)$ from assumption (vi) in Theorem 4.4.6. Then, for all $b \in B_1$, $c \in B_2$ and $d \in D$,*

$$u_1\big((k_1(b) + \beta h_0(d)) \oplus_{s,t} k_2(c)\big)u_1^* = k_1(b) \oplus_{s,t} (k_2(c) + \beta h_0(d)) .$$

In particular, for all C^*-morphisms $k_1, k_2 \colon D \to J \subseteq E$,

$$[k_1 + \beta h_0] + [k_2] = [k_1] + [k_2 + \beta h_0] .$$

For each σ-unital C^*-algebra B and C^*-morphism $k \colon B \to J + \beta J$ holds $[k] = [k] + [0]$. Moreover, for every isometry $T \in E$ there exists a unitary $u \in \mathcal{U}_0(E)$ such that $k(b)u = k(b)T^*$ for all $b \in B$. In particular, $u^* k(\cdot) u = T k(\cdot) T^*$.

Special cases are $[k_1 + \beta k_3] = [k_1 + \beta k_3] + [0]$, for all C^*-morphisms $k_1, k_3 \colon D \to J \subseteq E$, and – with $k_2 := 0$, $k_3 := h_0$ –,

$$[k_1 + \beta h_0] = [k_1 + \beta h_0] + [0] = [k_1] + [\beta h_0] .$$

(ii) $[h_0] + [\beta h_0] = [h_0 + \beta h_0] = [h_0 + \beta h_0] + [h_0] = [h_0 + \beta h_0] + [0]$.

(iii) $[H_0] + [h_0 + \beta h_0] = [H_0]$.

(iv) $[H_0] + [h_0] = [H_0] = [H_0] + [\beta h_0]$.

(v) *Each of H_0, $h_0 + \beta h_0$, βh_0 and h_0 dominates zero.*

(vi) *There exist isometries* $s_1, t_1 \in E$ *with* $s_1 s_1^* + t_1 t_1^* = 1$, $p_0 t_1 = p_0 = t_1 p_0$ *and* $t_1 H_0(\cdot) = H_0(\cdot) = H_0(\cdot) t_1$. *In particular,* $t_1 \in C$, $s_1 s_1^* \leq 1 - p_0$, $s_1 s_1^* H_0(\cdot) = 0$ *and* $s_1 s_1^* E s_1 s_1^* = s_1 E s_1^* \subseteq \mathrm{Ann}(H_0(D), E) \subseteq C$.

(vii) *If* $[h] \in S(H_0\,; D, E)$ *and* $h(D) \subseteq J + \beta J$, *then* h *has a unique decomposition* $h = k_1 + \beta k_2$ *with* C^**-morphisms* $k_1, k_2 \colon D \to J$ *that satisfy* $[k_1], [k_2] \in S(h_0\,; D, E)$.

In particular, $[k] \in S(h_0\,; D, E)$ *if* $[k + \beta h_0] \in S(H_0\,; D, E)$ *and* $k(D) \subseteq J$.

If $x \in E$ *is a contraction, such that* $(1 - xx^*) H_0(D) x = \{0\}$ *and* $x^* H_0(D) x \subset J + \beta J$, *then there are unique* $k_1, k_2 \in S(h_0\,; D, E)$ *with* $k_1 + \beta k_2 = x^* H_0(\cdot) x$.

This applies to projections $x := P \in C$ *with* $H_0(D) P \subseteq J + \beta J$.

(viii) *Let* B *a* σ*-unital* C^**-algebra and* $k_1, k_2 \colon B \to J$ C^**-morphisms. Then* $[k_1] = [k_2]$, *if and only if, there exists a unitary* $u \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ *with* $\beta h_0(d) u = \beta h_0(d)$ *for all* $d \in D$ *and and* $k_2(b) = u^* k_1(b) u$ *for all* $b \in B$.

In particular, then $u \in \mathcal{U}_0(E)$ *and* $\beta(u) \in h_0(D)' \cap E$.

If $B := D$ *then* $[k_1 + \beta h_0] = [k_2 + \beta h_0]$ *implies* $[k_1] = [k_2]$.

If B *is a* σ*-unital* C^**-algebra,* $k \colon B \to J$ *a* C^**-morphism and* $T \in E$ *is an isometry, then there exists* $u \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ *that satisfies* $u^* k(\cdot) u = T k(\cdot) T^*$ *and* $\beta h_0(d) u = \beta h_0(d)$ *for all* $d \in D$.

(ix) *Let* $q_1, q_2 \in F$ *denote non-zero* commuting *projections in* F. *Then there exists a unitary* $u \in (F + 1) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1) \subseteq C$ *with* $u^* q_1 u = q_2$.

For every separable subset $X \subseteq J + \beta J$ *there exists a projection* $p \in F$ *and an isometry* $R \in C$ *with* $pR = 0$ *and* $pxp = x$ *for all* $x \in X$.

For each isometry $V \in E$ *(or unitary* V*) there exists* $U \in \mathcal{U}_0(E)$ *with* $xV^* = xU$ *for all* $x \in X$.

(x) *There exist isometries* $s_2, s_3 \in E$ *that satisfy*

$$s_2^* H_0(\cdot) s_2 = h_0 \ , \quad s_3^* H_0(\cdot) s_3 = \beta h_0 \ , \quad \text{and} \quad s_2 s_2^* + s_3 s_3^* = p_0 \, .$$

The projection $p_2 := s_2 s_2^*$ *is in* C, *and and* p_2, s_2 *and* s_3 *satisfy* $s_2 h_0 s_2^* = p_2 H_0(\cdot) = s_0 h_0 s_0^*$ *and*

$$s_3 \beta h_0 s_3^* = (p_0 - p_2) H_0(\cdot) = s_0 \beta h_0 s_0^* = \beta(s_0 h_0 s_0^*) \, .$$

In particular, $p_2 H_0(D) \subseteq J$ *and* $(p_0 - p_2) H_0(D) \subseteq \beta J$.

If $s, t \in C$ *are the isometries in Theorem 4.4.6(i), then the partial isometries* $s_2 s s_2^*$, $s_2 t s_2^*$, $s_3 s s_3^*$, $s_3 t s_3^*$, $s_0 s s_0^*$ *and* $s_0 t s_0^*$ *are in* C.

In particular, the projections $p_2 = s_2 s_2^*$, $p_0 - p_2 = s_3 s_3^*$ *and* p_0 *are properly infinite in* C, *satisfy* $0 = [p_2] \in \mathrm{K}_0(p_2 C p_2)$,

??? $0 = [p_0 - p_2] \in \mathrm{K}_0((p_0 - p_2) C (p_0 - p_2))$ *??? or* $= 0$ *in* $\mathrm{K}_0(p_0 C p_0)$ *???*

and $[p_0] = 0$ *in* $\mathrm{K}_0(p_0 C p_0)$.

There exists an isometry $t_2 \in C$ *with* $t_2 t_2^* = 1 - p_2$ *and unitary elements* $u, v \in \mathcal{U}(C)$ *such that*

$$u^* p_2 u = p_2 + t_2 p_2 t_2^* \qquad \text{and} \qquad v^* p_2 v = p_2 \oplus_{s,t} p_2 \, .$$

The elements $s_0^* s_2,\, s_0^* s_3 \in E$ *are isometries with* $(s_2^* s_0)(s_0^* s_3) = 0$.

If B *is a* σ*-unital* C^**-algebra and* $h\colon B \to J$ *a* C^**-morphism then there exists* $u_h \in \mathcal{U}_0((\beta h_0)(D)' \cap E)$ *such that* $\beta h_0(\cdot) u_h = \beta h_0$ *and* $e h(b) u_h = e h(b) s_2^* s_0$ *for all* $b \in B$ *and* $e \in E$.

(xi) *For every non-zero projection* $p \in F$ *there exists properly infinite projections* $q, r \in C$ *with* $q + r = p$, $qr = 0$, $q \in \mathcal{N}(H_0(D), J)$ *and* $r \in \mathcal{N}(H_0(D), \beta J)$. *Each decomposition* $p = q + r$ *of this kind satisfies* $[qH_0(\cdot)] = [h_0]$ *and* $[rH_0(\cdot)] = [\beta h_0]$.

(xii) *If* $p \in F$ *is a projection with* $p \geq p_0$ *and* $p \neq p_0$, *then there exist* $V \in \mathcal{U}(C)$ *and* $W \in \mathcal{U}_0(E)$ *such that* $W^*(h_0 + \beta h_0) W = p_0 H_0(\cdot)$, $V p_0 = p_0 = p_0 V$ *and*

$$V^*(p - p_0) H_0(\cdot) V = t_0 W^*(h_0 + \beta h_0) W t_0^* .$$

Proof. We give a pre-information in Point (o) and prove Part (ix) before the other Parts of Lemma 4.4.7.

(o): Assumptions (iii) and (iv) of Theorem 4.4.6 together imply $p_0 H_0(d) = s_0 h_0(d) s_0^* + s_0 \beta h_0(d) s_0^*$ and $t_0 H_0(d) t_0^* = (1 - p_0) H_0(d)$ for $d \in D$. Thus

$$H_0 = (h_0 + \beta h_0) \oplus_{s_0, t_0} H_0 .$$

(ix): Let $q_1, q_2 \in F$ non-zero *commuting* projections. We show the existence of $u \in \mathcal{U}_0(F + \mathbb{C} \cdot 1) \cap (1 + F)$ with $u^* q_1 u = q_2$:
If $q_1, q_2 \in F$ are projections with $q_1 q_2 = q_2 q_1$, then

$$q_j \leq q_3 := (q_1 + q_2) - (q_1 q_2) \in F \quad \text{for } j \in \{1, 2\} ,$$

and q_3 is a projection. The unitaries in $(1 + F) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ build a group. Thus, it suffices to consider the special cases where $0 \neq q_1 \leq q_2$:

By assumption (iii) of Theorem 4.4.6 with $f = 0$, there is a projection $p \in F \subseteq C := H_0(D)' \cap E$ such that $p \geq q_2$ and $p \neq q_2$.

Any non-zero projection $q \in F$ is unitary equivalent to p_0 by a unitary $w \in 1 + F$ by assumption (v) of Theorem 4.4.6. Thus, there are unitaries $w_k, w \in 1 + F$, $k \in \{1, 2\}$, with $w_k^* p_0 w_k = q_k$ and $w^* p_0 w = p - q_2$. The partial isometries $v_k := w_k^* p_0 w \in F$, satisfy $v_k v_k^* = q_k$, $v_k^* v_k = p - q_2$ and $v_k^2 = 0$. It follows that $q_k + (p - q_2) = v_k v_k^* + v_k^* v_k$ and

$$U_k := v_k^* + v_k + (1 - v_k v_k^* + v_k^* v_k)$$

are selfadjoint unitaries in $\mathcal{U}_0(F + \mathbb{C} \cdot 1)$ with $U_k q_k = (p - q_2) U_k$ for $k = 1, 2$. Hence, $u^* q_2 u = q_1$ and $u \in (F + 1) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ for $u := U_2^* U_1$.

(If E is a real C^*-algebra E then we can take here $u := V_2^* V_1$ with unitaries

$$V_k := \exp((\pi/2)(v_k^* - v_k)) = v_k^* - v_k + (1 - v_k v_k^* - v_k^* v_k)$$

because also $V_k q_k = (p - q_2) V_k$.)

Let $p \in F$ is a projection with $p \geq p_0$. We show the existence of isometries $R, T \in C$, that satisfy $pR = 0$, $pT = Tp = p$ and $TT^* + SS^* = 1$ for $S := Rs_0 \in E$ (not necessarily in C):

Let $u \in \mathcal{U}_0(F + \mathbb{C} \cdot 1) \subseteq \mathcal{U}_0(C)$ with $u^* p_0 u = p \geq p_0$.

If $p := p_0$, then we can take the isometries $R_0, T_0 \in C$ and $S_0 \in E$ defined by $R_0 := t_0$, $T_0 := p_0 + t_0^2 t_0^*$ and $S_0 := R_0 s_0 = t_0 s_0$.

The assumptions $s_0 s_0^* = p_0$, $s_0 s_0^* + t_0 t_0^* = 1$, $p_0 \in F \subseteq C$, $t_0 \in C$ in parts (iii,iv) of Theorem 4.4.6, show that $S_0 \in E$ and $R_0, T_0 \in C$ are isometries with $R_0 R_0^* = 1 - p_0$, $p_0 R_0 = 0$, $p_0 S_0 = 0$, and $T_0 T_0^* + S_0 S_0^* = 1$.

It follows that $R := u^* R_0$, $S := Rs_0 = u^* S_0$ and $T := u^* T_0 u$ have the desired properties with respect to $p = u^* p_0 u$.

For every separable subset $X \subseteq J + \beta J$ there exists a projection $p \in F$ and an isometry $R \in C$ with $pxp = x$ and $xR = 0$ for all $x \in X$:

Let $A := C^*(X) \subseteq J + \beta J$ the separable C^*-subalgebra generated by X and $f \in A_+$ a strictly positive contraction for A. By assumptions (iii,c) and (iv) of Theorem 4.4.6 there exists a projection $p \in F$ with $pf = f = fp$ and $p \geq p_0 = s_0 s_0^* \in F$. Since fAf is dense in A, we get $pxp = x$ for all $x \in X \subseteq A$.

Let $p \in F$ with $p \geq p_0$ and $pxp = x$ for all $x \in X$.

There exists $u \in \mathcal{U}_0(C)$ with $u^* p_0 u = p$ as shown above. The isometry $R = u^* t_0$ is in C and satisfies $pR = 0$. It follows that $xR = xpR = 0$ for all $x \in X$.

Let $X \subseteq J + \beta J$ a separable subset, and $V \in E$ an isometry. We show the existence of $U \in \mathcal{U}_0(E)$ with $xV^* = xU$ for all $x \in X$:

If we replace the above considered separable subset $X \subseteq J + \beta J$ by the separable subset $XV^* \cup X \subseteq J + \beta J$, then the above observations lead to a projection $p \in F$ with $p \geq p_0$, $xp = x$ and $xV^* p = xV^*$ for all $x \in X$.

Take a unitary $u \in \mathcal{U}_0(F + \mathbb{C} \cdot 1) \cap (1 + F)$ with $u^* p_0 u = p$. Let $R := u^* t_0$, $S := Rs_0$ and $T := p + Rt_0 R^*$ as above considered with $pS = 0$, $pT = p = Tp$ and $SS^* + TT^* = 1$.

Thus, for $x \in X$, $xS = xpS = 0$, $xT = xpT = xp = x$ and

$$xV^* T^* = xV^* pT^* = xV^* p = xV^* .$$

It implies that $xTV^* T^* = xV^*$ and $xSe = 0$ for all $x \in X$ and $e \in E$. We define $U \in E$ by

$$U := TV^* T^* + SVS^* + S(1 - VV^*)T^* .$$

Since $xS = 0$ we get

$$xU = xTV^* T = xV^* \quad \text{for all } x \in X .$$

It remains to prove that $U \in \mathcal{U}_0(E)$:

The operator $Z := [T, S] \in M_{1,2}(E) \subset M_2(E)$ satisfies $ZZ^* = 1$ and $Z^*Z = 1_2 := \mathrm{diag}(1, 1)$, i.e.,

$$\psi\colon a \in M_2(E) \mapsto ZaZ^* \in E$$

is a *-algebra isomorphism ψ from $M_2(E)$ onto E. Obviously, the above defined unitary $U \in \mathcal{U}(E)$ is the image $U = \psi(U(V^*))$ of the *Halmos unitary* $U(V^*) \in M_2(E)$ built from the contraction V^*, as defined in Remark 4.2.4. The Halmos unitaries are all in $\mathcal{U}_0(M_2(E))$. Hence, $U = \psi(U(V^*)) \in \mathcal{U}_0(E)$.

(i): Let $k_j\colon B_j \to J$ C^*-morphisms of C^*-algebras B_j, $j \in \{1, 2, 3, 4\}$. The property $\beta(J) \cap J = \{0\}$ shows that the linear maps $(a, b) \mapsto k_i(b) + \beta k_j(c)$ from $B_i \oplus B_j$ into $J + \beta J$ are C^*-morphisms.

If $B_j := B$ for $j \in \{1, 2, 3, 4\}$ then this implies that the linear maps $b \in B \mapsto k_j(b) + \beta k_{j+2}(b)$ $(j \in \{1, 2\})$ are C^*-morphism form B into $J + \beta J$. The equality $[k_1 + \beta k_3] = [k_2 + \beta k_4]$ in $[\mathrm{Hom}(B, J + \beta J)]$ implies $[k_1] = [k_2]$ and $[k_3] = [k_4]$, because there exists a unitary $u \in E$ that satisfies for $v := \beta(u)$ and all $b \in B$ that

$$J \ni (u^*k_1(b)u - k_2(b)) = -\beta(vk_3(b)v^* - k_4(b)) \in \beta J\,,$$

which implies that $u^*k_1(\cdot)u = k_2$ and $v^*k_3(\cdot)v = k_4$ by assumption $J \cap \beta J = \{0\}$ of Theorem 4.4.6.

In particular $k_j + \beta h_0$ for $j = 1, 2$ are C^*-morphisms from D into $J + \beta J$ if $B_1 := B_2 := D$ and $k_3 := h_0$, because $h_0(D) \subseteq J$.

If we use the Cuntz sum $\oplus_{s,t}$ with s, t in Theorem 4.4.6(i), then

$$u_1\big((e + \beta h_0(d)) \oplus_{s,t} f\big)u_1^* = e \oplus_{s,t} (f + \beta h_0(d)) \quad \text{for all } d \in D,\ e, f \in J\,, \tag{4.5}$$

where $u_1 \in \mathcal{U}(E)$ comes from assumption (vi) of Theorem 4.4.6. We use here that $u_1 \in \mathcal{U}(J' \cap E)$ by assumption 4.4.6(vi), that $\beta(s) = s$ and $\beta(t) = t$ are in $h_0(D)' \cap E$ by assumption (i) of Theorem 4.4.6, and that – therefore – assumption 4.4.6(vi) implies moreover

$$u_1 s\beta h_0(\cdot)s^*u_1^* = t\beta h_0(\cdot)t^*\,. \tag{4.6}$$

Equation (4.5) says that $(k_1(b) + \beta h_0(d)) \oplus_{s,t} k_2(c)$ is unitarily equivalent to $k_1(b) \oplus_{s,t} (\beta h_0(d) + k_2(c))$ by the unitary u_1 for all $b \in B_1$, $c \in B_2$ and $d \in D$.

If $B_1 = B_2 = D$ we get especially

$$[k_1 + \beta h_0] + [k_2] = [k_1] + [k_2 + \beta h_0]\,.$$

An other special case is $k_2 := 0$, $k_1 := k$ and $B_1 := B$ for σ-unital C^*-algebras B and C^*-morphisms $k\colon B \to J$. We get that the C^*-morphisms $(b, d) \mapsto (k(b) + \beta h_0(d)) \oplus_{s,t} 0$ and $(b, d) \mapsto k(b) \oplus_{s,t} \beta h_0(d)$ are unitary equivalent.

It implies $[k + \beta h_0] + [0] = [k] + [\beta h_0]$ for $k\colon D \to J$.

Let B any σ-unital C^*-algebra, $k\colon B \to J + \beta J$ a C^*-morphism and $T \in E$ an isometry. There exists a strictly positive contraction $b_0 \in B_+$ for B if B is σ-unital.

If we apply the above proven Part (ix) to the subset $X := \{k(b_0)\}$ of $J + \beta J$, then we see that there exists $u \in \mathcal{U}_0(E)$ with $k(b)T^* = k(b)u$ for all $b \in Bb_0$. We get

$k(b)T = k(b)u$ for all $b \in B = \overline{Bb_0}$. Thus $Tk(b^*a)T^* = u^*k(b^*a)u$ for all $a, b \in B$. It yields $Tk(\cdot)T^* = u^*k(\cdot)u$, because both sides are bounded linear maps and the linear span of $B^* \cdot B$ is dense in B (actually $B \cdot B = B$ by [**616**, prop. 1.4.5]).

If we take $T := s \in C$ with s from assumption (i) of Theorem 4.4.6, then we get $[k] + [0] = [k \oplus_{s,t} 0] = [Tk(\cdot)T^*]$ and $[sk(\cdot)s^*] = [k]$.

(ii): We get $[h_0] + [\beta h_0] = [h_0 + \beta h_0] + [0] = [h_0 + \beta h_0]$, if we let $k_1 := h_0$, $k_2 := 0$ and $k_3 := h_0$ in Part (i). By Theorem 4.4.6(i), $s, t \in h_0(D)' \cap E$ and $ss^* + tt^* = 1$. It implies that $[h_0] + [h_0] = [h_0]$. We obtain:

$$[h_0 + \beta h_0] + [h_0] = [h_0] + [\beta h_0] + [h_0] = [h_0] + [\beta h_0]\,.$$

(iii): Info (o) shows that $[H_0] = [h_0 + \beta h_0] + [H_0]$ as elements of $S(H_0\,;\,D, E)$.

(iv): $[H_0] + [h_0] = [H_0] + [h_0 + \beta h_0] + [h_0] = [H_0] + [h_0 + \beta h_0] = [H_0]$ in $S(H_0\,;\,D, E)$, by Parts (iii) and (ii).

$[H_0] + [\beta h_0] = [H_0] + [h_0] + [\beta h_0] = [H_0] + [h_0 + \beta h_0] = [H_0]$ by Parts (ii, iii).

(v): By Part (i), with $k_3 := h_0$ and $k_1 = 0$, we get $[\beta h_0] = [\beta h_0] + [0]$, and, with $k_3 := 0$ and $k_1 = h_0$, $[h_0] = [h_0] + [0]$, i.e., that each of βh_0 and h_0 dominate zero. Part (ii) implies that $h_0 + \beta h_0$ dominates zero. This and Part (iv) together imply that H_0 dominates zero, i.e., that $[H_0] = [H_0] + [0]$.

(vi): By Proposition 4.3.6(v), it suffices to show that there exists an isometry $R \in E$ with $p_0 R = 0$ and $R^* H_0(\cdot) R = 0$.

Part (v), – i.e., $[H_0] = [0] + [H_0]$ by Proposition 4.3.5(i) –, and assumptions (i,iv) of Theorem 4.4.6 imply the existence of a unitary $u \in E$ with

$$u^* H_0(\cdot) u \;=\; 0 \oplus_{s_0, t_0} H_0(\cdot) \;=\; t_0 H_0(\cdot) t_0^* = (1 - p_0) H_0(\cdot)\,.$$

Let $R := t_0 u s_0$. Then $R^*R = 1$, $p_0 R = 0$ and $t_0^* H_0(\cdot) R = H_0(\cdot) u s_0$, because s_0 and t_0 are isometries with $p_0 = s_0 s_0^*$, $s_0^* t_0 = 0$ and $t_0 \in C$ – by assumptions (iv) in Theorem 4.4.6. It follows for $a \in D$ that

$$R^* H_0(a) R = s_0^* u^* H_0(a) u s_0 = s_0^* t_0 H_0(a) t_0^* s_0 = 0\,.$$

(vii): If $[h] \in S(H_0\,;\,D, E)$ then there exists an isometry $T \in E$ with $h = T^* H_0(\cdot) T$ by definition of $S(H_0\,;\,D, E)$.

If $h(D) \subseteq J + \beta J$, then the bounded linear map $\lambda(d) := H_0(d)T$ $(d \in D)$ maps D into $J + \beta J$. The C^*-subalgebra $A = C^*(\lambda(D))$ of $J + \beta J$ generated by the image $\lambda(D)$ of the separable D by the linear contraction λ is separable. Let $f \in A_+$ a strictly positive contraction. By assumptions (iii) and (iv) of Theorem 4.4.6, there is a projection $p \in F \subseteq C := H_0(D)' \cap E$ such that $pf = f$ and $p \geq p_0$. In particular $T^* p H_0(d) T = T^* H_0(d) T = h(a)$ for $d \in D$. The above proven Part (ix) provides a unitary $u \in (F + 1) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1) \subseteq \mathcal{U}_0(C)$ with $upu^* = p_0$. It follows that $h = T_0^* p_0 H_0(\cdot) T_0$ for the isometry $T_0 := uT$. Let $k_1 := T_0^* s_0 h_0(\cdot) s_0^* T_0$ and $k_2 := \beta(T_0)^* s_0 h_0(\cdot) s_0^* \beta(T_0)$. Then $k_j(D) \subseteq J$ for $j = 1, 2$ and k_1, k_2 are

C^*-morphisms from D into J with $h = k_1 + \beta k_2$, because $p_0 H_0 = s_0(h_0 + \beta h_0)s_0^*$, $h_0(D) \subseteq J$, $\beta(s_0) = s_0$ and $J \cap \beta J = \{0\}$.

By Part (v), h_0 dominates zero, i.e., $[h_0] = [0] + [h_0]$ and there exists a unitary $v \in E$ such that such that $v^* h_0(\cdot) v = s_0 h_0(\cdot) s_0^*$. Then $T_1 := vT_0 = vuT$ and $T_2 := v\beta(T_0)$ are isometries in E such that $h = k_1 + \beta k_2$ for $k_j := T_j^* h_0(\cdot) T_j$ $(j = 1, 2)$. In particular, $[k_1], [k_2] \in S(h_0\,;\, D, E)$.

The property $J \cdot \beta(J) = J \cap \beta(J) = \{0\}$ of J in Theorem 4.4.6 implies that arbitrary maps $k_1, k_2 \colon B \to J$ of any C^*-algebra B must be automatically C^*-morphisms if $h := k_1 + \beta k_2$ is a C^*-morphism and that $k_1 + \beta k_2$ is the unique decomposition of h as sum $h = k_1 + \beta k_2$ by maps $k_1, k_2 \colon B \to J$.

Thus, $h = k_1 + \beta k_2$ with $[k_1], [k_2] \in S(h_0\,;\, D, E)$ is the unique decomposition of $h := T^* H_0(\cdot) T$ with $h(D) \subseteq J + \beta J$.

The C^*-morphism H_0 dominates zero by Part (v). Hence Proposition 4.3.6(ii) applies and gives that $[h] \in S(H_0\,;\, D, E)$ for $h := x^* H_0(\cdot) x$ if $x \in E$ is a contraction with $(1 - xx^*) H_0(D) x = \{0\}$.

It happens in particular for all projections $x := P \in C$.

If, in addition, $x^* H_0(D) x \subset J + \beta J$, then above arguments show that $h = k_1 + \beta k_2$ for unique $k_1, k_2 \colon D \to J$, that satisfy $[k_1], [k_2] \in S(h_0\,;\, D, E)$.

(viii): Let $k, \ell \colon B \to J + \beta J$ C^*-morphisms with $[k] = [\ell]$ for a σ-unital C^*-algebra B, i.e., suppose that there exists $V \in \mathcal{U}(E)$ with $V k(\cdot) V^* = \ell(\cdot)$.

If we apply the before proven Part (ix) to $X := \{k(b_0)\}$ then we get $u \in \mathcal{U}_0(E)$ with $k(b)V^* = k(b)u$ for all $b \in B$, as in the proof of Part (i). Thus $u^* k(\cdot) u = V k(\cdot) V^* = \ell(\cdot)$.

This shows that if B is σ-unital and $k_1, k_2 \colon B \to J$ satisfy $[k_1] = [k_2]$ then there exists a (norm-) continuous path $\xi \in [0,1] \mapsto U_\xi \in \mathcal{U}(E)$ with $U_0 = 1$ and $U_1^* k_1(\cdot) U_1 = k_2$.

Let $b_0 \in B$ a strictly positive contraction in B, and let A_0 denote the *separable* C^*-subalgebra of $J + \beta J$ generated by $\beta h_0(D)$ and by the elements $k_1(b_0) U_\xi \in J$ for $n \in \mathbb{N}$ and $\xi \in [0,1]$. Notice that $A_0 = (A_0 \cap J) + \beta h_0(D)$, because $J \cap \beta J = \{0\}$. Let a_0 a strictly positive contraction of $A_0 \cap J$ and d_0 a strictly positive contraction of D. It follows that the C^*-subalgebra A of $J + \beta J$ generated by $\beta h_0(D)$ and $k_1(b) U_\xi \in J$ for $b \in B$ and $\xi \in [0,1]$ is σ-unital with strictly positive contraction $a_0 + \beta h_0(d_0)$, because again $A = (A \cap J) + \beta h_0(D)$, $A_0 \subset A$, and each $k_1(b) U_\xi$ is contained in the closed left-ideal generated by $k_1(b_0) U_\xi$, i.e., $a_0 \in A \cap J \subseteq J \cdot a_0$.

If we let $h \colon A \to J + \beta J$ denote the identity map on the σ-unital C^*-algebra A, then Part (i) applies to A and h. The Part (i) shows that there exists a unitary $W \in \mathcal{U}_0(E)$ with $aW^* = as^*$ for all $a \in A$, where we consider the isometries $s, t \in E$ from assumption (i) of Theorem 4.4.6. It implies $Wa = sa$ and $WaW^* = sas^*$ for $a \in A$, because $A = A \cdot A$ as set.

Equation (4.5) yields

$$(u_1W)(e+\beta h_0(d))(u_1W)^* = ses^* + t\beta h_0(d)t^* \quad \text{for all } d\in D, e\in A\cap J\,. \quad (4.7)$$

In particular, $t\beta h_0(\cdot)t^*(u_1W) = u_1W\beta h_0(\cdot)$. Equation (4.7) holds more generally for all $d\in D$ and all $b\in J$ with the property that $WbW^* = sbs^*$.

We define a norm-continuous group-morphism $\lambda_W\colon \mathcal{U}(E)\to\mathcal{U}(E)$ by

$$\lambda_W(U) := W^*u_1^*(U\oplus_{s,t}1)u_1W \quad \text{for } U\in\mathcal{U}(E)\,.$$

Clearly $\lambda_W(U^*) = \lambda_W(U)^*$. Straight calculation shows that $\lambda_W(U)\cdot\beta h_0(d) = \beta h_0(d)$ for $d\in D$ and all $U\in\mathcal{U}(E)$. The property $u_1\in J'\cap E$ implies that $\lambda_W(U^*)a\lambda_W(U) = U^*aU$ if $a\in J$ and $U\in\mathcal{U}(E)$ satisfy that $WaW^* = sas^*$ and $WU^*aUW^* = sU^*aUs^*$.

By our above construction of $A = (A\cap J) + \beta h_0(D)$ we get a norm-continuous path $\xi\in[0,1]\mapsto V_\xi\in\mathcal{U}(E)$ with $V_0 = 1$, $V_\xi^*k_1(\cdot)V_\xi = U_\xi^*k_1(\cdot)U_\xi$ and

$$V_\xi\cdot\beta h_0(\cdot) = h_0(\cdot) = \beta h_0(\cdot)\cdot V_\xi$$

if we define V_ξ by

$$V_\xi := \lambda_W(U_\xi)\,.$$

In particular, $V_\xi\in\beta h_0(D)'\cap E$ for each $\xi\in[0,1]$, because $\beta h_0(d)V_\xi = \beta h_0(d)$ for all $d\in D$. Thus, $u := V_1\in\mathcal{U}_0(\beta h_0(D)'\cap E)$ satisfies $u^*k_1(b)u = k_2(b)$ for $b\in B$ and $u\beta h_0(d) = h_0(d) = \beta h_0(d)u$.

If $k\colon B\to J$ is a C^*-morphism, B is separable and $T\in E$ an isometry then there exists a unitary $v\in\mathcal{U}_0(E)$ with $v^*k(\cdot)v = Tk(\cdot)T^*$ by Part (i) (proven via Part (ix)).

If we apply above observations to $k_1 := k$ and $k_2 := Tk(\cdot)T^*$ with $k_2 = v^*k_1(\cdot)v$, then we obtain the existence of a unitary $u\in\mathcal{U}_0(\beta(h_0(D))'\cap E)$ that satisfies $\beta h_0(\cdot)u = \beta h_0$ and $u^*k(\cdot)u = Tk(\cdot)T^*$.

(x): Recall that $s_0^*H_0(\cdot)s_0 = h_0 + \beta h_0$. By Part (ii), $[h_0+\beta h_0] = [h_0] + [\beta h_0]$. Thus, there is a unitary $V\in\mathcal{U}(E)$ such that

$$V(h_0+\beta h_0)V^* = sh_0s^* + t\beta h_0t^*\,,$$

where the isometries s,t come from Theorem 4.4.6(i,vi). If we apply the assumption (i) of Theorem 4.4.6 to $k := h_0+\beta h_0$ and $T := V$, we get some $u\in\mathcal{U}_0(E)$ with

$$u^*s_0^*H_0(\cdot)s_0u = sh_0s^* + t\beta h_0t^*\,.$$

Thus, $s_2 := s_0us$ and $s_3 := s_0ut$ are isometries in E that satisfy $s_2s_2^*+s_3s_3^* = p_0$, $s_2^*H_0(\cdot)s_2 = h_0$ and $s_3^*H_0(\cdot)s_3 = \beta h_0$.

By Lemma 4.3.4(i), the equation $s_2^*H_0(\cdot)s_2 = h_0$ implies that $p_2 := s_2s_2^*\in C$ and $p_0 - p_2 = s_3s_3^*\in C$ because $p_0\in F\subseteq C$ by Theorem 4.4.6(iv,iii).

Since $p_0, p_2\in C$, we get $p_2H_0(a) = s_2h_0(a)s_2^*\in J$, and $(p_0-p_2)H_0(a) = s_3\beta h_0(a)s_3^*\in\beta J$ for $a\in D$.

To compare e.g. $s_2h_0s_2^*$ with $s_0h_0s_0^*$, we define linear maps $L_1, L_2 \colon D \to E$ by

$$L_1(a) := (p_2H_0(a)) - s_0h_0(a)s_0^* \quad \text{and} \quad L_2(a) := ((p_0-p_2)H_0(a)) - s_0\beta h_0(a)s_0^*\,.$$

Then $L_1 + L_2 = 0$, i.e., $L_1(a) = -L_2(a)$ for $a \in D$, $L_1(D) \subseteq J$ and $L_2(D) \subseteq \beta J$.

The orthogonality $J \cdot \beta J = J \cap \beta J = \{0\}$ yields $L_1(a) = 0$ and $L_2(a) = 0$ for $a \in D$. Thus, $p_2H_0(\cdot) = s_0h_0s_0^*$ and $(p_0 - p_2)H_0(\cdot) = s_0(\beta h_0)s_0^*$, i.e., $s_2h_0s_2^* = s_0h_0s_0^*$ and $s_3(\beta h_0)s_3^* = s_0(\beta h_0)s_0^*$.

We let $G_1 := h_0(D)' \cap E$ and $G_2 := \beta G_1$, then $s_0G_js_0^* \subseteq C$ for $j = 1, 2$, $s_2G_1s_2^* \subseteq C$ and $s_3G_2s_3^* \subseteq C$ as the following calculations show:

The elements $p_0 = s_0s_0^*$, $p_2 := s_2s_2^*$ and $p_0 - p_2 := s_3s_3^*$ are in C, i.e., commute with the elements of $H_0(D)$. The definitions of s_0, s_2 and s_3 give us that

$$s_0^*H_0s_0 = h_0 + \beta h_0\,, \quad s_2^*H_0s_2 = h_0 \quad \text{and} \quad s_3^*H_0s_3 = \beta h_0\,.$$

Thus,

$$H_0s_0 = s_0(h_0 + \beta h_0)\,, \quad H_0s_2 = s_2h_0\,, \quad \text{and} \quad H_0s_3 = s_3\beta h_0\,.$$

It implies also

$$(h_0 + \beta h_0)s_0^* = s_0^*H_0\,, \quad h_0s_2^* = s_2^*H_0\,, \quad \text{and} \quad \beta h_0s_3^* = s_3^*H_0\,.$$

Let $x \in G_1$, $d \in D$, then

$$H_0(d)s_2xs_2^* = s_2h_0(d)xs_2^* = s_2xh_0(d)s_2^* = s_2xs_2^*H_0(d)\,.$$

The verifications of $s_3G_2s_3^* \subseteq C$ and $s_0G_js_0^* \subseteq C$ $(j = 1, 2)$ are similar.

The isometries $s, t \in E$ commute element-wise with $h_0(D) \cup H_0(D)$ and are β-invariant by Theorem 4.4.6(i), and since $t_0 \in C$ by Theorem 4.4.6(iv), the partial isometries $s_2ss_2^*$, $s_2ts_2^*$, $s_3ss_3^*$, $s_3ts_3^*$, $s_0ss_0^*$, $s_0ts_0^*$, $t_0st_0^*$ and $t_0tt_0^*$. are all in C.

It shows that the projections $p_2 = s_2s_2^* \leq p_0$, $p_0 - p_2$ and p_0 are properly infinite in p_2Cp_2, $(p_0 - p_2)C(p_0 - p_2)$ and p_0Cp_0 and there (!) $[p_2] = 0$, $[p_0 - p_2] = 0$ and $[p_0] = 0$, i.e., this hereditary C^*-subalgebras of C contain copies of $\mathcal{O}_2$ unital.

Since $1 - p_2 \geq 1 - p_0 = t_0t_0^*$ and $p_2, t_0 \in C$ it follows that $1 - p_2$, $1 - p_0$ and $1 = 1_C$ are full and properly infinite projections in C with same class in $\mathrm{K}_0(C)$. Thus, by Lemma 4.2.6(ii), there exists an isometry $t_2 \in C$ with $t_2t_2^* = 1 - p_2$.

There exist $u_2, v_2 \in \mathcal{U}(C)$ such that

$$u_2^*p_2u_2 = p_2 + t_2p_2t_2^* \qquad \text{and} \qquad v_2^*p_2v_2 = p_2 \oplus_{s,t} p_2\,.$$

This happens because p_2 is properly infinite with $[p_2] = 0$ in $\mathrm{K}_0(p_2Cp_2)$, because the projections $p_2 + t_2p_2t_2^*$ and $p_2 \oplus_{s,t} p_2$ are in the ideal of C generated by p_2, and the complementary projections $1 - p_2$, $1 - (p_2 + t_2p_2t_2^*)$ and $1 - (p_2 \oplus_{s,t} p_2)$ are respectively the ranges of the isometries t_2, t_2^2 and $t_2 \oplus t_2$.

The Murray–von-Neumann equivalences

$$q_0 := p_0 + t_0p_0t_0^* \sim p_0 \sim p_0 \oplus_{s,t} p_0 =: r_0$$

and

$$q_2 := p_2 + t_2 p_2 t_2^* \sim p_2 \sim p_2 \oplus_{s,t} p_2 =: r_2$$

can be given in C explicitly by the partial isometries $z_0 := s_0 s s_0^* + s_0 t s_0^* t_0^*$, and $z_2 := s_2 s s_2^* + s_2 t s_2^* t_2^*$, respectively by $y_0 := (s_0 \oplus_{s,t} s_0) s_0^*$ and $y_2 := (s_2 \oplus_{s,t} s_2) s_2^*$. Thus, below listed unitary elements $u_0, u_2, v_0, v_2 \in \mathcal{U}(C)$ satisfy $u_0 q_0 = p_0 u_0$, $u_2 q_2 = p_2 u_2$, $v_0 p_0 = r_0 v_0$ and $v_2 p_2 = r_2 v_2$.

$$u_0 := s_0 s s_0^* + s_0 t s_0^* t_0^* + t_0 (t_0^*)^2 \,, \tag{4.8}$$

$$u_2 := s_2 s s_2^* + s_2 t s_2^* t_2^* + t_2 (t_2^*)^2 \,, \tag{4.9}$$

$$v_0 := s s_0 s^* s_0^* + t s_0 t^* s_0^* + s t_0 s^* t_0^* + t t_0 t^* t_0^* \,, \tag{4.10}$$

$$v_2 := s s_2 s^* s_2^* + t s_2 t^* s_2^* + s t_2 s^* t_2^* + t t_2 t^* t_2^* \,. \tag{4.11}$$

The MvN-equivalences can be seen also from

$$\begin{aligned} p_0 &= (1 \oplus_{s,t} 1) \oplus_{s_0,t_0} 0 \,, \\ p_0 + t_0 p_0 t_0^* &= 1 \oplus_{s_0,t_0} (1 \oplus_{s_0,t_0} 0) \\ p_0 \oplus_{s,t} p_0 &= (1 \oplus_{s_0,t_0} 0) \oplus_{s,t} (1 \oplus_{s_0,t_0} 0) \,, \end{aligned}$$

because the below given Lemma 4.4.8 recognizes that u_0, u_2 v_0 and v_2 conjugate the used iterations of Cuntz addition.

The elements $S := s_0^* s_2$ and $T := s_0^* s_3$ are isometries in E with $S^* T = 0$, because s_2 and s_3 are isometries with $s_2 s_2^* + s_3 s_3^* = p_0 = s_0 s_0^*$.

If B is σ-unital and $h\colon B \to J$ is a C^*-morphism then there exists a unitary $U \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ with $U^* k(\cdot) U = S k(\cdot) S^*$ and $\beta h_0(d) U = h_0(d)$ for $d \in D$ by Part (viii). Take $u_h := U$.

(xi): By assumption (v) of Theorem 4.4.6, there exists a unitary $u \in 1 + F \subseteq C$ such that $u^* p_0 u = p$. Let $q := u^* p_2 u$ and $r := u^* (p_0 - p_2) u$ with $p_0 \geq p_2 \in C$ the projection from Part (x).

Since $F \subseteq C$ and $p_0, p_2 \in C$, also $q, r \in C$ and $q + r = p$.

The projections satisfy $q \in \mathcal{N}(H_0(D), J)$ and $r \in \mathcal{N}(H_0(D), \beta J)$, because $u \in C$ and $p_2 H_0(\cdot) = s_0 h_0 s_0^* \in J$ and $(p_0 - p_2) H_0(\cdot) = \beta(s_0 h_0 s_0^*) \in J$.

More generally, suppose that are given a non-zero projection $p \in F$ and projections $p, r \in C$ with $p = q + r$ with $q H_0(D) \subseteq J$ and $r H_0(D) \subseteq \beta J$.

By assumption 4.4.6(v), there exists $u \in \mathcal{U}(1 + F) \subseteq C$ with $u^* p_0 u = p$. Let $p_3 := u q u^* \in C$. Then $p_3 H_0(a) = u(q H_0(a)) u^* \in J$ and $(p_0 - p_3) H_0(a) \in \beta J$ for $a \in D$.

Thus, $[p_3 H_0(\cdot)] = [q H_0(\cdot)]$ and $[(p_0 - p_3) H_0(\cdot)] = [r H_0(\cdot)]$.

The same calculation as in proof of Part (x) with p_2 replaced by p_3 shows that $p_3 H_0(\cdot) = s_0 h_0 s_0^*$ and $(p_0 - p_3) H_0(\cdot) = s_0 \beta h_0 s_0$. Thus $[q H_0(\cdot)] = [h_0] + [0] = [h_0]$ and $[r H_0(\cdot)] = [\beta h_0] + [0] = [\beta h_0]$ by Part (i).

(xii): Let $p \in F$ a projection with $p \geq p_0$ and $p \neq p_0$, where $p_0 = s_0 s_0^*$ for the isometries s_0, t_0 in assumption (iv) of Theorem 4.4.6 with $p_0 \in F$, $t_0 \in C$ and $t_0 t_0^* = 1 - p_0$.

If we let $B := D$, $k := h_0 + \beta h_0$ and $T := s_0$ in Part (i), we get a unitary $W \in \mathcal{U}_0(E)$ with

$$W^*(h_0 + \beta h_0)W = s_0(h_0 + \beta h_0)s_0^* = p_0 H_0(\cdot)\,.$$

The elements t_0, p_0 and $p - p_0$ are in C. Thus, the proof of Part (xii) reduces to to the construction of $V \in \mathcal{U}(C)$ that satisfies

$$p_0 V = p_0 = V p_0 \quad \text{and} \quad V^*(p - p_0)V = t_0 p_0 t_0^* \tag{4.12}$$

Since p_0, $p - p_0$ and p are non-zero commuting projections in F, we find by Part (ix) unitaries

$$V_1, V_2 \in \mathcal{U}_0(F + \mathbb{C} \cdot 1) \subseteq \mathcal{U}_0(C)$$

with $V_1^* p_0 V_1 = p - p_0$ and $V_2^* p_0 V_2 = p$, and define $V \in E$ by

$$V := p_0 + V_1^* p_0 t_0^* + V_2^*(1 - p_0)t_0^*\,.$$

Straight calculation shows that V is unitary and fulfills Equations (4.12). The unitary V is in C because p_0, t_0, V_1 and V_2 are in C. □

Proof of Theorem 4.4.6 (Part 1: φ is surjective).
The C^*-morphism H_0 dominates h_0 by the equation $[h_0] + [H_0] = [H_0]$ in Lemma 4.4.7(iv). Thus

$$S(h_0\,;\, D, E) \subseteq S(H_0\,;\, D, E)\,,$$

and $[k] + [h_0] = [h] + [h_0]$ implies $[k] + [H_0] = [h] + [H_0]$. Since $[h_0] = [h_0] + [h_0]$ and $[H_0] = [H_0] + [H_0]$, the mapping

$$\varphi \colon [k] + [h_0] \mapsto [k] + [H_0]$$

is well-defined and additive, i.e., is a group homomorphism from $G(h_0\,;\, D, E) = S(h_0\,;\, D, E) + [h_0]$ into $G(H_0\,;\, D, E) = S(H_0\,;\, D, E) + [H_0]$. It is useful to consider $G(h_0\,;\, D, E)$ and $G(H_0\,;\, D, E)$ as subgroups of the big commutative semigroup $[\mathrm{Hom}(D, E)]$ of all unitary equivalence classes of C^*-morphisms from D into E with Cuntz addition, as introduced below Definition 4.3.1. This kind of subgroups of $[\mathrm{Hom}(D, E)]$ have natural morphism that are considered in parts (iii) and (iv) of the Lemma 4.2.3 in more generality on semi-groups.

We show that the group homomorphism φ is *surjective*:
Let $[g] \in S(H_0\,;\, D, E)$. By assumption (ii) of Theorem 4.4.6, there exists a unitary $u \in E$ with

$$T(a) := u^*(g(a) \oplus_{s_0, t_0} H_0(a))u - H_0(a) \in J \qquad \text{for all } \; a \in D\,.$$

We use here the isometries s_0, t_0 from assumption (iv) of Theorem 4.4.6 to realize the Cuntz sum $\oplus := \oplus_{s_0,t_0}$, simply to make calculations more transparent. Compare Proposition 4.3.2(ii) for the change of u if one takes another realization of $\oplus$ by other copies of $\mathcal{O}_2$ in E.

The image $T(D)$ of the linear map T generates a separable C^*-subalgebra $B \subseteq J$, because D is separable. Let $f \in J$ be a strictly positive element of B.

By assumption 4.4.6(iii), there exists a non-zero projection $p \in F \subseteq C$ with $pf = fp = f$. It follows that $(1-p)c = c(1-p) = 0$ for every $c \in B$, and that p commutes element-wise with $H_0(D)$. In particular, $(1-p)T(\cdot) = 0 = T(\cdot)(1-p)$, i.e., $pT(a) = T(a) = T(a)p$ for all $a \in D$.

Since $F \cdot H_0(D) \subseteq J + \beta J$ and $F \subseteq C$ by assumption (iii) of Theorem 4.4.6, we get for $a \in D$ that

$$pu^*(g(a) \oplus_{s_0,t_0} H_0(a))u = pT(a) + pH_0(a) \in J + \beta J\,,$$

$$pu^*(g(a) \oplus_{s_0,t_0} H_0(a))u = u^*(g(a) \oplus_{s_0,t_0} H_0(a))up$$

and

$$(1-p)u^*(g \oplus_{s_0,t_0} H_0)u = (1-p)H_0\,.$$

Let s_0, t_0 the isometries from Theorem 4.4.6(iv) with $t_0 \in C := H_0(D)' \cap E$ and $s_0 s_0^* = p_0 = 1 - t_0 t_0^*$ with $p_0 \in F \subseteq C$. By assumption (v) of Theorem 4.4.6, there is a unitary $v \in 1 + F \subseteq C$ with $v^* p v = p_0$.

We get $pv = vp_0$ and that $W := uvs_0$ is an isometry in E such that the projection $WW^* = uvp_0v^*u^* = upu^*$ commutes element-wise with $(g \oplus_{s_0,t_0} H_0)(D)$.

It implies that $p_0 = s_0 s_0^*$ commutes element-wise with the image of $(uv)^*\,(g \oplus_{s_0,t_0} H_0)\,uv$.

We define a C^*-morphism $h \colon D \to J + \beta J$ by

$$h := W^*(g \oplus H_0)W = s_0^*(uv)^*\,(g \oplus_{s_0,t_0} H_0)\,uvs_0\,.$$

The commutation properties of $p_0 = s_0 s_0^*$ implies that

$$s_0 h(\cdot) s_0^* = p_0(uv)^*\,(g \oplus_{s_0,t_0} H_0)\,uv = v^*(pV(\cdot) + pH(\cdot))v^*\,.$$

The equations

$$(1-p_0)(uv)^*\,(g \oplus_{s_0,t_0} H_0)\,uv = v^*(1-p)(T(\cdot) + H_0(\cdot))v = (1-p_0)H_0(\cdot) = t_0 H_0(\cdot) t_0^*$$

and

$$s_0 h(\cdot) s_0^* = p_0(uv)^*\,(g \oplus_{s_0,t_0} H_0)\,uv\,,$$

together lead to

$$v(h \oplus_{s_0,t_0} H_0)v^* = u^*(g \oplus_{s_0,t_0} H_0)u\,.$$

If we use that $s_0 s_0^* = p_0 \in F \subseteq C$, $t_0 \in C$ and $s_0 s_0^* + t_0 t_0^* = 1$ by assumptions (iii) and (iv) of Theorem 4.4.6, we get

$$t_0 H_0(\cdot) t_0^* = (1-p_0)H_0(\cdot) == (1-p_0)(uv)^*(g \oplus H_0)uv$$

and

$$h \oplus_{s_0,t_0} H_0 = s_0 h(\cdot) s_0^* + (1-p_0) H_0(\cdot) = (uv)^* (g \oplus_{s_0,t_0} H_0) uv \,.$$

Thus,

$$[h] + [H_0] = [g] + [H_0] \in S(H_0\,;\, D, E)\,.$$

It says that H_0 dominates $h \oplus_{s_0,t_0} H_0$ by Definition 4.4.1. (Compare also Definition 4.3.3, Lemma 4.3.4(iv) and Proposition 4.3.6.)

The equation $h = s_0^*(h \oplus_{s_0,t_0} H_0)s_0$ shows that H_0 dominates also h, i.e., $[h] \in S(H_0\,;\, D, E)$.

Since $v \in F + 1 \subseteq C$ is unitary, we get that $v^* H_0(\cdot) v = H_0$, and, by definition of T and v, that $p_0 v^* T(a) v = v^* T(a) v \in J$ for $a \in D$. It implies

$$s_0 h(a) s_0^* - p_0 H_0(a) = p_0\big((vu^*(g(a) \oplus_{s_1,t_1} H_0(a))uv) - H_0(a)\big) = p_0 v^* T(a) v\,.$$

By assumption 4.4.6(iv), $s_0^* H_0(a) s_0 = h_0(a) + \beta h_0(a)$ and $s_0^* s_0 = 1$. Thus,

$$h(a) - h_0(a) - \beta(h_0(a)) = s_0^* v^* T(a) v s_0 \in J \qquad \text{for } a \in D\,.$$

Since $h_0(D) \subseteq J$ it follows that $h(D) \subseteq J + \beta(J)$ and $h = k + \beta h_0$ for a unique C^*-morphism $k\colon D \to J$, because $h_0(D) \subseteq J$ and $J \cap \beta(J) = \{0\}$.

Now $[h] = [k + \beta h_0] \in S(H_0\,;\, D, E)$ and $h(D) \subseteq J + \beta J$ implies that the homomorphism h_0 dominates k by Lemma 4.4.7(vii), i.e., $[k] \in S(h_0\,;\, D, E)$.

By Lemma 4.4.7(iii,i,iv),

$$[k] + [H_0] = [k] + [h_0 + \beta h_0] + [H_0] = [h] + [h_0] + [H_0] = [h] + [H_0]\,.$$

It follows $[k] + [H_0] = [h] + [H_0] = [g] + [H_0]$. Thus, the group morphism

$$\varphi\colon [k] + [h_0] \in G(h_0\,;\, D, E) \mapsto [k] + [H_0] \in G(H_0\,;\, D, E)$$

is *surjective*. □

We start here the collection of observations that will be used in the proof of the necessity and sufficiency of the Condition (DC) for the *injectivity* of the natural epimorphism from $G(h_0\,;\, D, E)$ onto $G(H_0\,;\, D, E)$. It is essentially a detailed study of its possibly existing kernel.

LEMMA 4.4.8. *Let* (s,t) *and* (s_0, t_0) *the isometries from assumptions* (i) *respectively* (iv) *of Theorem* 4.4.6 *and* (s_2, t_2) *the isometries in Lemma* 4.4.7(x)*.*

The elements u_0*,* u_2*,* v_0 *and* v_2 *– as defined by equations* (4.8)*,* (4.9)*,* (4.10) *and* (4.11) *in the proof of Lemma* 4.4.7(x) *– are in* $\mathcal{U}(C)$ *and have the properties that, for* $a, b, c, d \in E$ *and* $k \in \{0, 1\}$*,*

$$u_k^*\big((a \oplus_{s,t} b) \oplus_{s_k,t_k} c\big)u_k = a \oplus_{s_k,t_k} (b \oplus_{s_k,t_k} c)\,,$$

and

$$v_k^*\big((a \oplus_{s_k,t_k} c) \oplus_{s,t} (b \oplus_{s_k,t_k} d)\big)v_k = (a \oplus_{s,t} b) \oplus_{s_k,t_k} (c \oplus_{s,t} d)\,.$$

In particular, for all C^**-morphisms* $k, \ell\colon D \to J$ *and* $h := k + \beta\ell$*,*

$$u_0^*\big((h \oplus_{s,t} (h_0 + \beta h_0)) \oplus_{s_0,t_0} H_0\big)u_0 = h \oplus_{s_0,t_0} H_0\,,$$

$$u_2^*\big((k \oplus_{s,t} h_0) \oplus_{s_2,t_2} H_0\big)u_2 = k \oplus_{s_2,t_2} H_0\,,$$

and

$$v_0^*\big(((k+\beta h_0) \oplus_{s_0,t_0} H_0) \oplus_{s,t} ((\ell+\beta h_0) \oplus_{s_0,t_0} H_0)\big)v_0 = ((k \oplus_{s,t} \ell)+\beta h_0) \oplus_{s_0,t_0} H_0\,.$$

Proof. The elements u_0, u_2, v_0 and v_1 of E are in C because $s,t \in C$ by assumption (i) of Theorem 4.4.6, t_0, t_2, $s_0 s s_0^*$, $s_0 t s_0^*$, $s_2 s s_2^*$ and $s_2 t s_2^*$ are in C by assumption (iv) of Theorem 4.4.6 and Lemma 4.4.7(x).

Straight calculation shows that u_k and v_k are unitary for $k \in \{0,2\}$ and have the quoted transformation properties.

Use $(h_0+\beta h_0) \oplus_{s_0,t_0} H_0 = H_0$ and $h_0 \oplus_{s_2,t_2} H_0 = H_0$ in the last equations. □

Lemma 4.4.9. *Let $A := H_0(D)$, $U \in \mathcal{U}\big(\mathcal{N}(A, J+\beta J)+\mathbb{C}\cdot 1\big)$ and $q^*q = q \in F$. There exists a projection $p \in F$ with $p \geq q$, $p \neq q$ and*

$$U^*(H_0(\cdot)p)U = U^*H_0(\cdot)Up\,, \tag{4.13}$$

$$U^*(H_0(\cdot)(1-p))U = H_0(\cdot)(1-p)\,. \tag{4.14}$$

More generally we find for each separable subset $X \subseteq \mathcal{N}(A, J+\beta J)$ and each projection $q \in F$ a projection $p \in F$ with $p \geq q$ and $p \neq q$ and $H_0(\cdot)x = H_0(\cdot)pxp$ for all $x \in X$.

Proof. Multiplying U by suitable $\xi \in \mathbb{C}$ we may suppose that $U = 1+y$ with $y \in \mathcal{N}(A, J+\beta J))$.

More generally, let $\mathcal{Y}$ a separable C^*-subalgebra of $\mathcal{N}(A, J+\beta J))$. Then the separable subset $X := (A\cdot\mathcal{Y}) \cup (\mathcal{Y}\cdot A)$ of $J+\beta J$, generates a separable C^*-subalgebra B of $J+\beta J$.

Let $f \in B_+ \in \beta(J)+J$ a strictly positive contraction for B. By assumption (iii,c) of Theorem 4.4.6 there exists a projection $p \in F$ with $f = pfp$, $p \geq q$ and $p \neq q$. Thus $pb = b = bp$ for all $b \in B$. In particular, $(1-p)B = \{0\} = B(1-p)$.

Since $A = H_0(D)$ and $p \in F \subseteq C$, it follows all $y \in \mathcal{Y}$ that $H_0(\cdot)py = H_0(\cdot)y = H_0(\cdot)yp$, $pyH_0(\cdot) = yH_0(\cdot) = ypH_0(\cdot)$ and $(1-p)H_0(\cdot)y = 0 = H_0(\cdot)y(1-p)$. $H_0(\cdot)y = H_0(\cdot)pyp$

If we let $U := 1+y$ then we get $H_0(\cdot)pU = H_0(\cdot)Up$ and

$$H_0(\cdot)(1-p)U = (1-p)H_0(\cdot)U = (1-p)H_0(\cdot)\,.$$

In particular, $U^*pH_0(\cdot)U = U^*H_0(\cdot)Up$.

Notice that $(1-p)^2 = 1-p \in C$. It implies for all $c,b \in D$ that

$$U^*H_0(c^*b)(1-p)U = ((1-p)H_0(c)U)^*(1-p)H_0(b)U = H_0(c^*b)(1-p)\,.$$

□

Lemma 4.4.10. *Let $A := H_0(D)$. The maps*

$$\lambda_0\colon \mathcal{U}(\beta h_0(D)' \cap E) \ni u \mapsto s_0 u s_0^* + (1-p_0) \in \mathcal{U}(E)$$

and

$$\lambda_2 \colon \mathcal{U}(E) \ni u \mapsto \; s_2 u s_2^* + (1-p_2) \in \mathcal{U}(E)$$

are norm-continuous group morphisms into $\mathcal{U}(\mathrm{Der}(A,J)) \cap (p_0 E p_0 + (1-p_0))$.

If moreover $\beta h_0(\cdot) u = \beta h_0$, *then* $1 - \lambda_0(u) \in \mathcal{N}(A,J) \cap p_0 E p_0$.

For each $u \in \mathcal{U}(E)$, $1 - \lambda_2(u) \in p_2 \mathcal{N}(A,J) p_2 = p_2 E p_2 \subseteq \mathcal{N}(A,J)$.

For every σ*-unital hereditary* C^**-subalgebra* B *of* J *there exists a unitary* $w_B \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ *that satisfies* $\beta h_0(\cdot) w_B = \beta h_0$ *and*

$$b w_B = b s_2^* s_0 \quad \text{for all } b \in B .$$

In particular, for each C^**-morphism* $k \colon D \to J$ *with* $k(D) \subseteq B$ *and the unitary* $W := w_B \oplus_{s_0,t_0} 1 \in \mathcal{U}_0(\mathrm{Der}(A,J))$ *holds* $(1-W) \in \mathcal{N}(A,J) \cap p_0 E p_0$ *and*

$$\big((k + \beta h_0) \oplus_{s_0,t_0} H_0\big) W = W \big(k \oplus_{s_2,t_2} H_0\big) .$$

Proof. Recall that $s_0(\beta h_0)s_0^* = (p_0 - p_2)H_0 = s_3(\beta h_0)s_3^*$ and $(1-p_2)H_0 = t_2 H_0 t_2^*$ by Lemma 4.4.7(x). By assumption (iv) of Theorem 4.4.6, $t_0 \in C$ and $t_0 t_0^* = 1 - p_0$. We obtain

$$s_0(\beta h_0)s_0^* + t_0 H_0 t_0^* = t_2 H_0 t_2^* = (1-p_2)H_0 .$$

It follows that

$$\lambda_0(u) := s_0 u s_0^* + (1-p_0) \in p_0 E p_0 + (1-p_0)$$

is a unitary in E.

The $\lambda_0(u)$ commute elementwise with $s_0 \beta h_0(D) s_0^* \cup (1-p_0)H_0(D)$ if $u \in \mathcal{U}(\beta h_0(D)' \cap E)$.

In particular, $\lambda_0(u)$ commutes with the elements of $(1-p_2)H_0(D)$ if $u \in \mathcal{U}(\beta h_0(D)' \cap E)$. It follows that $\lambda_0(u) - 1 \in \mathrm{Der}(A,J) \cap p_0 E p_0$

If u satisfies moreover that $\beta h_0(\cdot) \cdot u = \beta h_0$ then $(1-\lambda_0(u))H_0(\cdot)(1-p_2) = 0$ and $(1-\lambda_0(u)^*)H_0(\cdot)(1-p_2) = 0$. It implies

$$1 - \lambda_0(u) \in \mathcal{N}(A,J) ,$$

because $H_0(D)p_2 = s_2 h_0(D) s_2^* \subseteq p_2 J p_2$.

The element $1 - \lambda_2(u)$ is in $p_2 \mathcal{N}(A,J) p_2$ for *every* $u \in \mathcal{U}(E)$ because $p_2 = s_2 s_2^* \in \mathcal{N}(A,J)$ implies $s_2 E s_2^* = p_2 \mathcal{N}(A,J) p_2$.

If only $u \in \mathcal{U}(\beta h_0(D)' \cap E)$ we see at least that the unitaries $1 - \lambda_0(u)$ and $1 - \lambda_2(u)$ are in $\mathrm{Der}(A,J) \cap p_0 E p_0$, because they commute with the elements in $(1-p_2)H_0(D)$ and $p_2 H_0 = s_2 h_0 s_2^*$ takes values in J.

The map λ_0 satisfies $\lambda_0(1) = 1$ and is a norm-continuous group homomorphism from $\mathcal{U}(\beta h_0(D)' \cap E)$ into the intersection of $\mathcal{U}(\mathrm{Der}(A,J))$ with $p_0 E p_0 + (1-p_0)$. In particular, λ_0 maps $\mathcal{U}_0(\beta h_0(D)' \cap E)$ into the intersection of $\mathcal{U}_0(\mathrm{Der}(A,J))$ and $p_0 \mathcal{U}_0(E) p_0 + (1-p_0)$.

Since $p_2 H_0 = s_2 h_0 s_2^*$ has image in J we get that $p_2 E p_2 = s_2 E s_2^* \subset \mathcal{N}(A,J) \subseteq \mathrm{Der}(A,J)$, and that λ_2 maps $\mathcal{U}(E)$ into the unitaries of $p_2 E p_2 + (1-p_2)$. Thus, $\lambda_2(\mathcal{U}_0(E))$ is contained in $\mathcal{U}_0(\mathcal{N}(A,J) + \mathbb{C}\cdot 1) \subseteq \mathcal{U}_0(\mathrm{Der}(A,J))$.

Notice that $p_2 E p_2 + (p_0 - p_2) \subset p_0 E p_0$.

Let B a σ-unital hereditary C^*-subalgebra of J, and $h\colon B \hookrightarrow J$ the inclusion map $h(b) = b$.

By Lemma 4.4.7(x), there exists a unitary $u_h \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ with $u_h \beta h_0(\cdot) = \beta h_0 = \beta h_0(\cdot) u_h$ and $b u_h = h(b) u_h = h(b) s_2^* s_0 = b s_2^* s_0$ for $b \in B$. Thus $u_h^* b u_h = s_0^* s_2 b s_2^* s_0$ for $b \in B$, and we get that $w_B := u_h$ has the claimed additional properties, i.e., $b w_B = b s_2^* s_0$ for $b \in B$, $w_B \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ and $\beta h_0(\cdot) w_B = \beta h_0$. Then $W := \lambda_0(w_B) \in \mathcal{U}_0(\mathrm{Der}(A,J))$, $(1-W) \in \mathcal{N}(A,J) \cap p_0 E p_0$ and $(k \oplus_{s_2,t_2} H_0) W = W (k \oplus_{s_0,t_0} H_0)$ for each $k\colon D \to J$ with $k(D) \subseteq B$:

By definition of w_B, we have $b w_B = b s_2^* s_0$ for $b \in B$ and $w_B \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ and $\beta h_0(\cdot) w_B = \beta h_0$.

Let $W := \lambda_0(w_B) = w_B \oplus_{s_0,t_0} 1$. Then $W \in \mathcal{U}_0(\mathrm{Der}(A,J))$, $(1-W) \in \mathcal{N}(A,J) \cap p_0 E p_0$ and

$$s_0^*(b \oplus_{s_0,t_0} 0) W = b s_0^* s_0 w_B s_0^* = b w_B s_0^* = b s_2^* s_0 s_0^* = b s_2^*$$

for $b \in B$, because $s_0 s_0^* = p_0$ and $s_2 s_2^* \leq p_0$ by Lemma 4.4.7(x).

We combine this with

$$p_0 \cdot (b_j \oplus_{s_0,t_0} 0) = b_j \oplus_{s_0,t_0} 0\,,$$

for $j \in \{1,2\}$, and obtain for $b := b_2^* b_1$ that

$$W^*(b \oplus_{s_0,t_0} 0) W = b \oplus_{s_2,t_2} 0\,.$$

Since $\lambda_0(w_B)(D)$ commutes element-wise with elements in the image of the map $s_0 \beta h_0 s_0^* + t_0 H_0 t_0^* = t_2 H_0 t_2^*$, we get for C^*-morphisms $k\colon D \to E$ with the property $k(D) \subseteq B \subseteq J$ that $W(k \oplus_{s_2,t_2} H_0) = \big((k + \beta h_0) \oplus_{s_0,t_0} H_0\big) W$. □

LEMMA 4.4.11. *Let $A := H_0(D)$ and $u \in \mathcal{U}(\mathrm{Der}(A, J + \beta J))$.*

There exist $v \in \mathcal{U}_0(F + \mathbb{C}\cdot 1) \subseteq \mathcal{U}_0(C)$ and C^*-morphisms $k_1, k_2\colon D \to J$ with $(1-v) \in F \subset C \cap \mathcal{N}(A, J+\beta J)$, $[k_1], [k_2] \in S(h_0\,;\, D, J)$ and

$$(k_1 + \beta k_2) \oplus_{s_0,t_0} H_0 = v u^* H_0(\cdot) u v^*\,.$$

If moreover $u \in \mathcal{U}(\mathrm{Der}(A,J))$, then necessarily $k_2 = h_0$, and there exists $W \in \mathcal{U}_0(\mathrm{Der}(A,J))$ such that $(1-W) \in \mathcal{N}(A,J) \cap p_0 E p_0$ and

$$k_1 \oplus_{s_2,t_2} H_0 = W^*\big((k_1 + \beta h_0) \oplus_{s_0,t_0} H_0\big) W = W^* v u^* H_0(\cdot) u v^* W\,.$$

PROOF. Let $u \in \mathcal{U}(\mathrm{Der}(A, J + \beta J))$. The image of the corresponding H_0-derivation

$$\delta_u \colon d \in D \mapsto u H_0(d) - H_0(d) u$$

is contained in a separable C^*-subalgebra $B \subset J + \beta J$. Let $f \in B$ a strictly positive contraction in B.

By assumptions (iii) of Theorem 4.4.6, there exists a projection $p \in F \subset C$ with $p \geq p_0$, $pf = f$ and $p \neq p_0$. It follows that $pb = b = bp$ for all $b \in B$.

In particular, $u^*\delta_u(d) = u^*\delta_u(d)p$ for $d \in D$. It implies for $d \in D$ that

$$u^*H_0(d)up = H_0(d)p - u^*(uH_0(d) - H_0(d)u) \in J + \beta J$$

because $u \in \mathrm{Der}(A, J + \beta J)$, $F \cdot H_0(D) \subset J + \beta J$ and $F \subseteq C$ by assumptions (iii) of Theorem 4.4.6.

Moreover, $(1-p)\delta_u(d) = 0$ and $\delta_u(d)(1-p) = 0$ imply $(1-p)uH_0(d) = (1-p)H_0(d)u$, and $uH_0(d)(1-p) = H_0(d)u(1-p)$.

Thus, $H_0(d)(1-p) = u^*H_0(d)u(1-p)$ and $u^*H_0(d)up \in J + \beta J$ for all $d \in D$. It follows that $1-p$ and p commute with the elements of $u^*H_0(D)u$ and that

$$k\colon D \ni d \mapsto u^*H_0(d)up$$

is a C^*-morphism from D into $J + \beta J$.

Part (ix) of Lemma 4.4.7 provides a unitary $v \in (1 + F) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ with $vpv^* = p_0$.

Since $F \subset C$ by assumption (iii) of Theorem 4.4.6, we get that

$$t_0H_0(d)t_0^* = H_0(d)(1-p_0) = (uv^*)^*H_0(d)uv^*(1-p_0)\,.$$

In particular, $1-p_0$ and p_0 commute element-wise with $(uv^*)^*H_0(D)uv^*$. Recall that $pv^* = v^*p_0$. We use that $p_0 = s_0s_0^*$ by assumption (iv) of Theorem 4.4.6 and define a C^*-morphism $h\colon D \to E$ by

$$h(d) := s_0^*(uv^*)^*H_0(d)uv^*s_0 = (v^*s_0)^*k(d)(v^*s_0)\,.$$

Then $h(D) \subseteq J + \beta J$ and

$$h \oplus_{s_0,t_0} H_0 = (uv^*)^*H_0(\cdot)uv^*\,.$$

Hence, $h = T^*H_0(\cdot)T$ for the isometry $T := uv^*s_0$ and $[h] \in S(H_0\,; D, E)$.

Moreover, $h(D) \subseteq J + \beta J$ because $k(D) \subseteq J + \beta J$. Thus Lemma 4.4.7(vii) applies to h and shows that there are unique C^*-morphisms $k_1, k_2\colon D \to J$ with $[k_1], [k_2] \in S(h_0\,; D, E)$ and $h = k_1 + \beta k_2$.

Finally, we obtain that

$$(k_1 + \beta k_2) \oplus_{s_0,t_0} H_0 = v(u^*H_0(\cdot)u)v^*\,.$$

Suppose now that $u \in \mathcal{U}(\mathrm{Der}(A, J))$. Then, $uv^* \in \mathcal{U}(\mathrm{Der}(A, J))$ for $v \in \mathcal{U}(C)$, because $C \subseteq \mathrm{Der}(A, J)$, and, for $d \in D$ holds

$$s_0\big(k_1(d) - h_0(d) + \beta(k_2(d) - h_0(d))\big)s_0^* = v^*u^*H_0(d)uv - H_0(d)\,, \tag{4.15}$$

because $0 \oplus_{s_0,t_0} H_0 = (1-p_0)H_0(\cdot)$ and $p_0H_0 = s_0(h_0 + \beta h_0)s_0^*$.

The element $s_0(k_1(d) - h_0(d))s_0^*$ and the right side of Equation (4.15) are in J, but the element $s_0\beta(k_2(d) - h_0(d))s_0^*$ is in βJ for all $d \in D$. Thus, $k_2 = h_0$ by the assumption $J \cap \beta J = \{0\}$ in Theorem 4.4.6.

Suppose now that $k_2 = h_0$. By Lemma 4.4.10 there exists $W \in \mathcal{U}_0(\mathrm{Der}(A, J))$ such that $W \in p_0 E p_0 + (1 - p_0)$ and

$$W(k_1 \oplus_{s_2,t_2} H_0) = ((k_1 + \beta h_0) \oplus_{s_0,t_0} H_0)W\,.$$

Hence, $k_1 \oplus_{s_2,t_2} H_0 = W^* v u^* H_0(\cdot) u v^* W$. □

We do not suppose that $k_1, k_2 \in S(h_0\,;\, D, E)$ in the following lemma.

Lemma 4.4.12. *Let $A := H_0(D)$ and $k_1, k_2 \colon D \to J$ C^*-morphisms – not necessarily in $S(H_0\,;\, D, E)$. The following are equivalent:*

(i) $[k_1] + [h_0] = [k_2] + [h_0]$ *in the semi-group* $[\mathrm{Hom}(D, J)]$.

(ii) *There exists* $U_0 \in \mathcal{U}\big(\mathcal{N}(A, J + \beta J) + \mathbb{C} \cdot 1\big)$ *such that*

$$\big((k_1 + \beta h_0) \oplus_{s_0,t_0} H_0\big)U_0 = U_0\big((k_2 + \beta h_0) \oplus_{s_0,t_0} H_0\big)\,. \tag{4.16}$$

(iii) *There exists* $U_1 \in \mathcal{U}\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$ *such that* $U_1 \in \mathcal{U}_0\big(\mathrm{Der}(A, J)\big)$ *and*

$$\big((k_1 + \beta h_0) \oplus_{s_0,t_0} H_0\big)U_1 = U_1\big((k_2 + \beta h_0) \oplus_{s_0,t_0} H_0\big)\,. \tag{4.17}$$

(iv) *There exists* $U_2 \in \mathcal{U}_0\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$ *such that*

$$(k_1 \oplus_{s_2,t_2} H_0)U_2 = U_2(k_2 \oplus_{s_2,t_2} H_0)\,. \tag{4.18}$$

In particular, a C^*-morphism $k \colon D \to E$ satisfies $[k] + [h_0] = [h_0]$ in $[\mathrm{Hom}(D, E)]$, if and only if, $k(D) \subseteq J$ and there exists $u \in \mathcal{U}\big(\mathcal{N}(A, J + \beta J) + \mathbb{C} \cdot 1\big)$ and $v \in \mathcal{U}(C)$ such that

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = v u^* H_0(\cdot) u v^*\,.$$

If $[k_0] + [h_0] = [h_0]$ then there exists $U_3 \in \mathcal{U}_0\big(\mathrm{Der}(A, J)\big)$ with $(1 - U_3) \in \mathcal{N}(A, J)$ and

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = U_3^* H_0(\cdot) U_3\,.$$

Proof. Let $A := H_0(D)$. Part (iii) implies Part (ii) with $U_0 := U_1$ because $\mathcal{N}(A, J) \subseteq \mathcal{N}(A, J + \beta J)$.

(iv)⇒(iii): If we apply Lemma 4.4.10 to the separable C^*-subalgebra B of J that is generated by $k_1(D) \cup k_2(D)$, then we get a unitary $W \in \mathcal{U}_0(\mathrm{Der}(A, J))$ such that for $i \in \{0, 1\}$ holds

$$\big((k_i + \beta h_0) \oplus_{s_0,t_0} H_0\big)W = W\big(k_i \oplus_{s_2,t_2} H_0\big)\,.$$

Let $U_1 := W U_2 W^*$. The C^*-algebra $\mathcal{N}(A, J)$ is a closed ideal of the C^*-subalgebra $\mathrm{Der}(A, J) \subset E$ with $1_E \in \mathrm{Der}(A, J)$. It implies that $\mathcal{U}_0\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$ is a normal subgroup of $\mathcal{U}\big(\mathrm{Der}(A, J)\big)$. Thus $U_2 \in \mathcal{U}_0\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$. This implies together with $W \in \mathcal{U}\big(\mathrm{Der}(A, J)\big)$ that $U_1 \in \mathcal{U}_0\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$.

Straight calculation shows that U_1 fulfills the equation (4.17).

(ii)⇒(i): Suppose that $U_0 \in \mathcal{U}\big(\mathcal{N}(H_0(D), J + \beta J) + \mathbb{C} \cdot 1\big)$ satisfies Equation (4.16).

By Lemma 4.4.9 with $U := U_0$ and $q := p_0$, there exists a projection $p \in F$ with $p \geq p_0$ and $p \neq p_0$ such that $U_0^*\big(H_0(\cdot)(1-p)\big)U_0 = H_0(\cdot)(1-p)$. If we use that $t_0 H_0 t_0^* = (1-p_0)H_0$ we get:

$$U_0^*\big(s_0(k_1+\beta h_0)s_0^* + (p-p_0)H_0(\cdot)\big)U_0 = s_0(k_2+\beta h_0)s_0^* + (p-p_0)H_0(\cdot)$$

By Part (xii) of Lemma 4.4.7 there exists unitaries $V \in \mathcal{U}(C)$ and $W \in \mathcal{U}_0(E)$ such that $W^*(h_0+\beta h_0)W = p_0 H_0$, $Vp_0 = p_0 = p_0 V$ and

$$V^*(p-p_0)H_0 V = t_0 W^*(h_0+\beta h_0)W t_0^*\,.$$

We define a unitaries

$$V_0 := s_0 s^* + t_0 W^* t^* \quad \text{and} \quad V_1 := V_0^* V^* U_0 V V_0\,.$$

Straight calculation, using assumptions (i) of Theorem 4.4.6, shows that

$$V_1^*\big((k_1 \oplus_{s,t} h_0) + \beta(h_0)\big)V_1 = (k_2 \oplus_{s,t} h_0) + \beta(h_0)\,.$$

If we use that $J \cdot \beta J = \{0\}$ then we get finally

$$V_1^*(k_1 \oplus_{s,t} h_0)V_1 = (k_2 \oplus_{s,t} h_0)\,,$$

that yields $[k_1]+[h_0] = [k_2]+[h_0]$.

(i)⇒(iv): Suppose that $[k_1]+[h_0] = [k_2]+[h_0]$. Let $h_i := k_i \oplus_{s,t} h_0$ for $i \in \{1,2\}$. Then $[h_1] = [h_2]$ in $[\mathrm{Hom}(D,J)]$ and there exists $v \in \mathcal{U}(E)$ with $v^* h_1 v = h_2$.

Since $h_i(D) \subseteq J$, we can find even a unitary $u \in \mathcal{U}_0(E)$ with $u^* h_1 u = h_2$ by Lemma 4.4.7(viii). We define a unitary $W := u \oplus_{s_2,t_2} 1$ in $s_2\mathcal{U}_0(E)s_2^* + (1-p_2)$. Then $1-W \in \mathcal{N}(A,J)$ and W is contained in $\mathcal{U}_0(\mathcal{N}(A,J)+\mathbb{C}\cdot 1) \subset \mathcal{U}_0(\mathrm{Der}(A,J))$, because $p_2 H_0(\cdot) = s_2 h_0 s_2^*$ by Lemma 4.4.7(x) and because $x^* H_0(D) \subset J$ implies $xEx^* \subseteq \mathcal{N}(A,J)$. Obviously, $W^*(h_1 \oplus_{s_2,t_2} H_0)W = h_2 \oplus_{s_2,t_2} H_0$.

By Lemma 4.4.8, the element $u_2 := s_2 s s_2^* + s_2 t s_2^* t_2^* + t_2(t_2^2)^*$ from equation (4.9) is in $\mathcal{U}(C)$ and satisfies

$$\big((a \oplus_{s,t} b) \oplus_{s_2,t_2} c)\big)u_2 = u_2\big(a \oplus_{s_2,t_2} (b \oplus_{s_2,t_2} c)\big) \quad \text{for} \quad a,b,c \in E\,.$$

Part (x) of Lemma 4.4.7 shows that $h_0 \oplus_{s_2,t_2} H_0 = p_2 H_0 + (1-p_2)H_0 = H_0$. Thus, $k_i \oplus_{s_2,t_2} H_0 = u_2^*(h_i \oplus_{s_2,t_2} H_0)u_2$ for $i \in \{1,2\}$, and with $U_2 := u_2^* W u_2$ we get

$$U_2^*(k_1 \oplus_{s_2,t_2} H_0)U_2 = k_2 \oplus_{s_2,t_2} H_0\,.$$

The algebra $\mathcal{N}(A,J)$ is an ideal of $\mathrm{Der}(A,J)$. This implies that the group $\mathcal{U}_0(\mathcal{N}(A,J)+\mathbb{C}\cdot 1)$ is a normal subgroup of $\mathcal{U}(\mathrm{Der}(A,J))$ that is contained in $\mathcal{U}_0(\mathrm{Der}(A,J))$.

If we use that W is contained in $\mathcal{U}_0(\mathcal{N}(A,J)+\mathbb{C}\cdot 1)$ and $C \subseteq \mathrm{Der}(A,J)$ then we can see that the above defined U_2 is an element of $\mathcal{U}_0(\mathcal{N}(A,J)+\mathbb{C}\cdot 1)$.

Moreover $(1-W)H_0(\cdot)(1-p_2) = 0$ implies that also $(1-W^*)H_0(\cdot)(1-p_2) = 0$, and thus $(1-W) \in \mathcal{N}(A,J)$. It follows $(1-U_2) \in \mathcal{N}(A,J)$.

Finally we consider the special case $k_1 := k$ and $k_2 := h_0$:

If a C^*-morphism $k\colon D \to E$ satisfies $[k] + [h_0] = [h_0]$ in $[\mathrm{Hom}(D, E)]$, then there exists $V \in \mathcal{U}(E)$ with $k \oplus_{s,t} h_0 = V^* h_0(\cdot)V$. Thus $k = T^* h_0(\cdot)T$ for the isometry $T := Vs$, i.e., $[k] \in S(h_0\,;\, D, E)$ and $k(D) \subseteq J$.

Recall that $[h_0] = [h_0] + [h_0]$ and $H_0 = (h_0 + \beta h_0) \oplus_{s_0,t_0} H_0$ by assumptions (i,iv) of Theorem 4.4.6.

Therefore, the equivalence of Parts (i), (ii) and (iii) shows that $[k] + [h_0] = [h_0]$ is equivalent to $k(D) \subseteq J$ and the existence of $U_0 \in \mathcal{U}\big(\mathcal{N}(A, J + \beta J) + \mathbb{C} \cdot 1\big)$, respectively of $U_1 \in \mathcal{U}\big(\mathcal{N}(A, J) + \mathbb{C} \cdot 1\big)$ with $U_1 \in \mathcal{U}_0\big(\mathrm{Der}(A, J)\big)$, such that, for $j \in \{0, 1\}$:

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = U_j^* H_0 U_j\,.$$

(Notice that necessarily $U_0^* U_1 \in C$.)

We find $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that $1 - \xi U_1 \mathcal{N}(A, J)$ and can replace U_1 by ξU_1 in Part (iii).

If $u \in \mathcal{U}\big(\mathcal{N}(A, J + \beta J) + \mathbb{C} \cdot 1\big)$ and $v \in \mathcal{U}(C)$ are given such that

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = vu^* H_0(\cdot)uv^*\,,$$

then we can take $U_0 := vuv^*$ to satisfy Formula (4.16) in Part (ii). □

Next Lemma 4.4.13 true? To be checked: ??

But it could work with kind of "stabilization"?

Lemma 4.4.13. *Let $A := H_0(D) \subseteq E$, $C := A' \cap E$ and $I := J + \beta J$ as in Theorem 4.4.6.*

For each separable C^*-subalgebra $B \subseteq \mathcal{N}(A, I) \subset \mathrm{Der}(A, I)$ there exist $u \in (F + 1) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1) \subseteq \mathcal{U}_0(C)$ and an isometry $T \in E$ such that

$$u^* bu - s_0 T^* bT s_0^* \in \mathrm{Ann}(A, E) \quad \textit{for all } b \in B\,.$$

In particular, for each contraction $e \in 1 + \mathcal{N}(A, I)$ with $1 - e^* e \in \mathrm{Ann}(A)$ and $1 - ee^* \in \mathrm{Ann}(A)$ there exists a unitary in $u \in \mathcal{N}(A, I) + 1$ with $u - e \in \mathrm{Ann}(A)$.

We do not know if this also holds with $I := J \triangleleft E$ (replacing $J + \beta J$) and $u^* bu - s_2 T^* bT s_2^* \in \mathrm{Ann}(A, E)$ for $b \in B$ (in place of the known weak conclusion $u^* bu - s_0 T^* bT s_0^* \in \mathrm{Ann}(A, E)$).

Is it used or needed somewhere?

It needs certainly a more elaborate proof if it is true.

Proof. In other words, we have to show that for each separable C^*-subalgebra $B \subseteq E$ with the property $B \cdot H_0(D) \subseteq I := J + \beta J$ there exists an isometry $T \in E$ and a unitary $u \in 1 + F$ with $u \in \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ such that $(u^* bu - s_0 T^* bT s_0^*) \cdot H_0(D) = \{0\}$ for every $b \in B$.

Let $B \subseteq \mathcal{N}(A, I)$ a separable C^*-subalgebra and $b_0 \in B$ a strictly positive contraction in B.

Then let $e \in G_+ \subseteq I_+$ a strictly positive contraction for the separable C^*-subalgebra G of I generated by $B \cdot A$ (recall $A := H_0(D)$).

Notice that the linear span of all elements in $(BA)^n$, $(AB)^n$, $A(BA)^n = (ABBA)^n$ and $B(AB)^n = (BAAB)^n$ for $n = 1, 2, \ldots$ is dense in G. It follows that $BG \cup GB \subseteq G$ and $AG \cup GA \subseteq G$, i.e., that $A \cup B \subset \mathcal{N}(G, E)$.

The elements $x := b_1 a_1 + a_2 b_2$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$ build a set of generators of G, that is invariant under involution.

`Check next:`

$b_0 x, a_0 x \in G$ for $b_0 \in B$ and the linear span

$AB \cup BA \cup AB^2A \cup BA^2B \cup (AB)^n \cup (BA)^n \cup A(BA)^n$ generates $C^*(A \cup B)$????

Let $f_n \in I_+$ a sequence of positive contractions such that $f_n f_{n+1} = f_n$, $\lim_n f_n e = e = \lim_n e f_n$, $\lim_n \|(1 - f_n)b(1 - f_n)a\| = 0$, $\lim_n \|[b, f_n]\| = 0$ and $\lim_n \|[a, f_n]\| = 0$ for $b \in B$ and $a \in A$

????

Let $g := \sum_n 2^{-n} f_n$. There exists a projection $p \in F \subseteq C$ with $p \geq p_0$ and $p \neq p_0$ such that $pg = g = gp$ by assumption (iii,c) of Theorem 4.4.6.

(And there exist projections $q, r \in E$, but not in F, with $p = q + r$ and $q, \beta(r) \in \mathcal{N}(A, J)$ by Lemma 4.4.7(xi).)

`Write below new proof`
`fitting to above!!`

If $BA \subseteq J$, then $e, f_n, g \in J$ and

??? $rg, gr \in \operatorname{Ann}(A, E)$,

because $Ar = 0 = rA$ and $rgA = 0$??? Using an with respect to $C^*(A, r)$ approximately central approximate unit $(c_\tau) \in J_+$ of J, we get $c_\tau g \to g$ ...

Thus, $(1 - p)BA = \{0\}$ and $BA(1 - p) = B(1 - p)A = \{0\}$. It follows also that $AB(1 - p) = \{0\}$ and $A(1 - p)B = \{0\}$. Thus $(b - pbp) \in \operatorname{Ann}(A)$ for all $b \in B$, because $b - pbp = b(1 - p) + (1 - p)bp$.

Moreover $pb^*(1 - p)cp \in \operatorname{Ann}(A)$ for all $b, c \in B$, because $(pb^*(1 - p)cp)^* = (pc^*(1 - p)bp)$ and $(pb^*(1 - p)cp)^*(pb^*(1 - p)cp)A = \{0\}$. We get $pBp \subseteq \mathcal{N}(A, I)$ because $p \in C \subseteq \operatorname{Der}(A, I)$ and $\mathcal{N}(A, I)$ is an ideal of $\operatorname{Der}(A, I)$.

There exists $u \in (1 + F) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ with $u^* p u = p_0$.

Let $\gamma_0(b) := p_0 u^* b u p_0$. Then $u^* \gamma_0(b) u = pbp$.

Is $b - qbq = (b - up_2 u^* b u p_2 u^*) H_0(d) = 0$ for all $b \in B$? Likely not! More general open:

`Is there in case` $I = J$ `a decomposition`
$u^* b u - s_2 \gamma(b) s_2^* \in \operatorname{Ann}(A, E)$???

Let $T := us_0$ and $\gamma(b) := T^*bT$. Then $pbp = us_0\gamma(b)s_0^*u^*$. Thus $u^*bu - s_0\gamma(b)s_0^* \in \mathrm{Ann}(A)$ for all $b \in B$.

$H_0(d)T = us_0s_0^*H_0(d)s_0 = us_0(h_0(d) + \beta h_0(d))$.

There exist $u \in (F+1) \cap \mathcal{U}_0(F + \mathbb{C}\cdot 1) \subseteq \mathcal{U}_0(C)$ and an isometry $T \in E$ such that, for all $b \in B$, $u^*bu - s_2T^*bTs_2^* \in \mathrm{Ann}(A)$ in case $I = J$ (respectively $u^*bu - s_0T^*bTs_0^* \in \mathrm{Ann}(A)$ in case $I = J + \beta J$):

To be filled in

There exists an isometry $T \in E$ with $T^*H_0(\cdot)T = 0$ and $TT^* + p_0 \leq 1$:

By Lemma 4.4.7(vi), there exists an isometry $s_1 \in E$ with $s_1s_1^* \leq 1 - p_0$ and $s_1s_1^*H_0(\cdot) = 0$. Thus $T := s_1$ is as desired. (One could also use directly $T := R$ for R in the proof of Lemma 4.4.7(vi).)

To be filled in. Check. ?? The "naive" definition should be the following definition:

Let $U(e)$ the Halmos unitary of e defined in Remark 4.2.4 and $s_1, t_1 \in E$ the isometries from Lemma 4.4.7(vi), and define

$$u := [t_1, s_1]U(e)[t_1, s_1]^* = t_1et_1^* + s_1e^*s_1^* - t_1(1 - ee^*)^{1/2}s_1^* + s_1(1 - e^*e)^{1/2}t_1^*$$

Then $s_1e^*s_1^*, t_1(1 - ee^*)^{1/2}s_1^*, s_1(1 - e^*e)^{1/2}t_1^* \in \mathrm{Ann}(A)$ and $t_1a = a = at_1 = t_1^*a$ for all $a \in A$.

Thus $ua = t_1et_1^*a = t_1ea$ and $au = aet_1^*$ for $a \in A$. Need modification of s_1, t_1 such that $t_1ea = ea$ for $a \in A$ and $aet_1^* = ae$ for $a \in A$.

Notice that $ea - ae = g$ is contained in a separable C^*-subalgebra G of I, $t_1ea = t_1(g + ae) = t_1g + ae = ea + (t_1 - 1)g$. $aet_1^* = (t_1ea)^* = ((t_1 - 1)g)^* + ae$ for suitable $g \in G$

Thus we have to find a unitary V in $C := A' \cap E$ such that $s_1^*Vg = 0$ for all $g \in G$. Then still $V^*s_1s_1^*V \leq 1 - p_0$ and $V^*s_1s_1^*Va = V^*s_1s_1^*aV = 0$ for $a \in A$. We can rename V^*s_1 by R and use it to construct a suitable correction of s_1, t_1:

To find $V \in F$, we take $q \in F$ with $q \geq p_0$, $q \neq p_0$ and $qg = g = gq$ for all $g \in G$ (is possible by Parts (iii,iv) of Theorem 4.4.6). Then s_1V has the property $V^*s_1s_1^*V \leq (1 - q) = V^*(1 - p_0)V \leq (1 - p_0)$.

Let $R := V^*s_1$. Then $R^*A = \{0\}$ and $R^*G = \{0\}$ for $V \in \mathcal{U}_0(F)$ with $V^*p_0V = q$ as defined above. Since E is properly infinite there exists isometries S, T with $SS^* \leq RR^*$, $SS^* + TT^* = 1$ and $Ta = a = aT = aT^*$, $Tg = g = gT = gT^*$ for $a \in A$ and $g \in G$.

Now define $u := [T, S]U(e)[T, S]^*$. Then we get that $(e - u)A = \{0\} = A(e - u)$.

A construction of u in the special case that $e := s_2fs_2^* + 1 - p_2$ in case $I = J$ (respectively $e := s_0fs_0^* + 1 - p_0$ in case $I = J + \beta J$) where $f \in E$ is a contraction such that $2p_2 - s_2f^*fs_2^* - s_2ff^*s_2^* \in \mathrm{Ann}(A, E)$ (respectively $2p_0 - s_0f^*fs_0^* - s_0ff^*s_0^* \in \mathrm{Ann}(A, E)$).

Let $Z := [s_2, T, 1 - p_2 - TT^*] \in M_{1,3}(E)$ if $I = J$ (respectively let $Z := [s_0, T, 1 - p_0 - TT^*]$ if $I = J + \beta J$) and $U(f) \in M_2(E)$ the Halmos unitary of f as defined in Remark 4.2.4.

Then $ZZ^* = 1$ and $u := Z(U(f) \oplus 1)Z^*$ is a unitary that satisfies $u \in \mathcal{N}(A, I) + 1$ and $u - e \in \mathrm{Ann}(A, E)$. □

REMARK 4.4.14. Let $A := H_0(D)$, $I := J + \beta J$ and $u \in \mathcal{U}(\mathrm{Der}(A, I))$. By Lemma 4.4.11 there exist $v \in \mathcal{U}(C)$ and a C^*-morphism $k \colon D \to I$, i.e., $k = k_1 + \beta k_2$ with C^*-morphisms $k_1, k_2 \colon D \to J$, that satisfy

$$k \oplus_{s_0,t_0} H_0 = v^* u^* H_0(\cdot) u v \, .$$

The following are equivalent:

(i) $[k] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$.
(ii) There are $w_1, w_2 \in \mathcal{U}(C)$ such that $1 - w_1 u w_2 \in \mathcal{N}(A, I)$.
(iii) There exist $w \in \mathcal{U}(C)$ such that $wuv \in \mathcal{U}_0(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$.

PROOF. The proof of the equivalences in Parts (i)–(iii) of Remark 4.4.14 is similar to the proof of Lemma 4.4.12 with $\oplus_{s_2,t_2}$ and $V := u_2$ replaced by $\oplus_{s_0,t_0}$ and $V := u_0$ in Lemma 4.4.8 and in proof of Lemma 4.4.7(x). □

Let C, $\mathrm{Ann}(A, E)$, $\mathrm{Der}(A, I)$, $\mathcal{N}(A, I)$, $\mu \colon \mathrm{K}_1(C) \to \mathrm{K}_1(\mathrm{Der}(A, I))$ and $\Gamma\big(A, I, E\big)$ as introduced in Definitions 4.4.4 and 4.4.5.

Notice that $\mathrm{Der}(A, I_0) \subseteq \mathrm{Der}(A, I)$ if $I_0 \subseteq I$ for an ideal $I_0 \lhd E$. In particular, $C := A' \cap E = \mathrm{Der}(A, \{0\}) \subseteq \mathrm{Der}(A, I)$.

Recall that $\mathcal{N}(A, I)$ is an ideal of $\mathrm{Der}(A, I)$ and that $\mathcal{N}(A, I_0) \subseteq \mathcal{N}(A, I)$ if $I_0 \subseteq I$. In particular, $\mathrm{Ann}(A, E) := \mathcal{N}(A, \{0\}) \subseteq C \cap \mathcal{N}(A, I)$.

We use below the group morphisms

$$\mu \colon \mathrm{K}_1(C) \to \mathrm{K}_1(\mathrm{Der}(A, I)) \quad \text{and} \quad \mu' \colon \mathrm{K}_1(\mathcal{N}(A, I)) \to \mathrm{K}_1(\mathrm{Der}(A, I))$$

given by the inclusion maps $C \hookrightarrow \mathrm{Der}(A, I)$ and $\mathcal{N}(A, I) \hookrightarrow \mathrm{Der}(A, I)$. Recall that

$$\Gamma(A, J, E) := \mathrm{K}_1(\mathrm{Der}(A, J)) / \mu(\mathrm{K}_1(C)) \, .$$

LEMMA 4.4.15. *Let $A := H_0(D)$, $C := A' \cap E$ and $I := J + \beta J$ (respectively $I := J$). Let $\mathrm{Ann}(A, E) := \mathcal{N}(A, \{0\})$, $\mathrm{Der}(A, I)$ and $\mathcal{N}(A, I)$ as in Definition 4.4.4.*

(i) *$\mathcal{N}(A, I)$ is a closed ideal of $\mathrm{Der}(A, I)$ and a full hereditary C^*-subalgebra of E. It holds*

$$\mathrm{Der}(A, I) = C + \mathcal{N}(A, I) \, .$$

In particular there is a natural isomorphism

$$C / (C \cap \mathcal{N}(A, I)) \;\cong\; \mathrm{Der}(A, I) / \mathcal{N}(A, I) \, ,$$

(ii) *The annihilator* $\mathrm{Ann}(A,E)$ *of* A *in* E *is a closed ideal of* C *and is a full hereditary* C^**-subalgebra of* E.

It satisfies

$$\mathrm{Ann}(A,E) = \mathcal{N}(A,J)\cap\beta(\mathcal{N}(A,J)) \subseteq \mathcal{N}(A,I)\,.$$

In particular, the quotient maps $\pi\colon C \to C/\mathrm{Ann}(A,E)$ *and* $\mathrm{Der}(A,I)\to C/(C\cap\mathcal{N}(A,I))$ *induce surjective maps on the* K_**-groups, e.g.* $[\pi]_1(\mathrm{K}_1(C)) = \mathrm{K}_1(C/\mathrm{Ann}(A,E))$ *:*

For every contraction $c\in C$ *with* $1-c^*c, 1-cc^*\in\mathrm{Ann}(A,E)$ *there exists a unitary* $u\in C$ *such that*

$$1-u(c\oplus_{s,t}1)\in\mathrm{Ann}(A,E)\,.$$

(iii) *The* C^**-algebras* $\mathrm{Der}(A,I)$, C, $\mathcal{N}(A,I)+\mathbb{C}\cdot 1$ *and all its non-zero quotients and unifications of closed non-zero ideals are* K_1*-surjective* C^**-algebras.*

(iv) $\mathcal{U}(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$ *is a closed* normal *subgroup of* $\mathcal{U}(\mathrm{Der}(A,I))$ *and* $\mathcal{U}(C)$ *is a closed subgroup of* $\mathcal{U}(\mathrm{Der}(A,I))$ *.*

(v) *Let* $\mathcal{G}(I)$ *denote the set* $\mathcal{U}(C)\cdot\mathcal{U}(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$ *of products* $u\cdot v$ *with* $u\in\mathcal{U}(C)$ *and* $v\in\mathcal{U}(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$ *.*

The set $\mathcal{G}(I)$ *is an* open subgroup *of* $\mathcal{U}(\mathrm{Der}(A,I))$ *that satisfies*

$$\mathcal{G}(I)\oplus_{s,t}1\subseteq\mathcal{G}(I)\,.$$

In particular, $\mathcal{U}_0(\mathrm{Der}(A,I))\subseteq\mathcal{G}(I)$ *.*

(vi) *For each* $u\in\mathcal{U}(E)$ *there exists a unitary* $v\in 1+\mathrm{Ann}(A,E)\subset C$ *such that* $[v\cdot u]=0$ *in* $\mathrm{K}_1(E)$.

For each unitary in $u\in\mathcal{U}(\mathcal{N}(A,J+\beta J)+\mathbb{C}\cdot 1)$ *with the* additional *property* $u\in\mathrm{Der}(A,J)$ *there exists unitaries* $v,w\in\mathcal{U}(C)$ *such that*

$$v\,u\,w\in\mathcal{U}_0(\mathcal{N}(A,J)+\mathbb{C}\cdot 1)\subseteq\mathcal{U}_0(\mathrm{Der}(A,J))\,.$$

In particular, the image $\mu'(\mathrm{K}_1(\mathcal{N}(A,J)))\subseteq\mathrm{K}_1(\mathrm{Der}(A,J))$ *of* $\mathrm{K}_1(\mathcal{N}(A,J))$ *is contained in the image* $\mu(\mathrm{K}_1(C))$ *in* $\mathrm{K}_1(\mathrm{Der}(A,J))$, *and the mapping*

$$\mathcal{G}(J)\ni u\mapsto[u]\in\mathrm{K}_1(\mathrm{Der}(A,J))$$

maps $\mathcal{G}(J)$ onto $\mu(\mathrm{K}_1(C))\subseteq\mathrm{K}_1(\mathrm{Der}(A,J))$.

(vii) *The natural group morphisms* $\mathrm{K}_1(C)\to\mathrm{K}_1(C/\mathrm{Ann}(A,E))$ *and*

$$\mathrm{K}_1\big(\mathrm{Der}(A,I)\big)\to\mathrm{K}_1\big(\mathrm{Der}(A,I)/\mathcal{N}(A,I)\big)$$

are surjective.

(viii) *The natural group epimorphism* $\varphi\colon[k]+[h_0]\mapsto[k]+[H_0]$ *from* $G(h_0\,;\,D,E)$ *onto* $G(H_0\,;\,D,E)$ *is injective, if and only if, the natural group morphism*

$$\mathrm{K}_1(C/\mathrm{Ann}(A,E))\to\mathrm{K}_1\big(C/(C\cap\mathcal{N}(A,J))\big)$$

is surjective.

(ix) *This map* $\varphi\colon[k]+[h_0]\mapsto[k]+[H_0]$ *is injective, if and only if,*

$$u\oplus_{s,t}1\in\mathcal{G}(J)\quad\textit{for all}\;\; u\in\mathcal{U}(\mathrm{Der}(A,I))\cap\mathcal{U}_0(E)\,. \tag{4.19}$$

(x) *A sufficient condition for the injectivity of* φ *is that the natural group morphism*

$$\mathcal{U}\big(C/\operatorname{Ann}(A,E)\big) \to \mathcal{U}\big(C/(C\cap\mathcal{N}(A,I))\big)$$

contains $1\oplus\mathcal{U}\big(C/(C\cap\mathcal{N}(A,I))\big)$ *in its image.*

PROOF. Recall that $A := H_0(D)$, and that we let alternatively $I := J$ or $I := J+\beta J$ to avoid repeats. We need sometimes additional considerations for the case $I := J$.

(i): Clearly, $\operatorname{Der}(A,I) = \pi_I^{-1}\big(\pi_I(A)'\cap(E/I)\big)$. Thus, $\operatorname{Der}(A,I)$ is a C^*-subalgebra of E and

$$C := A'\cap E = \operatorname{Der}(A,\{0\}) \subseteq \operatorname{Der}(A,I)\,.$$

In the same way $\mathcal{N}(A,I) = \pi_I^{-1}(\operatorname{Ann}(\pi_I(A),E/I))$, where $\operatorname{Ann}(\pi_I(A),E/I)$ means the two-sided annihilators in E/I of the elements of $\pi_I(A)$.

Since $\operatorname{Ann}(\pi_I(A),E/I)$ is a closed ideal of $\pi_I(A)'\cap(E/I)$, the C^*-algebra $\mathcal{N}(A,I)$ is a closed ideal of $\operatorname{Der}(A,I)$.

In particular, $C+\mathcal{N}(A,I)$ is a C^*-subalgebra of $\operatorname{Der}(A,I)$ in general for all closed ideals I of E.

Let $e\in E_+$ with $eH_0(d)-H_0(d)e \in I := J+\beta J$ for all $d\in D$, i.e., $e\in \operatorname{Der}(A,I)$. Since D is separable, the image $T(D)$ of $T(d) := eH_0(d)-H_0(d)e$ is contained in a separable C^*-subalgebra B of $J+\beta J$. Let $f\in B_+$ a strictly positive contraction for B. By assumption (iii) of Theorem 4.4.6, there is a projection $p\in F$ with $pf = f = fp$, $p_0\le p$ and $p_0\ne p$.

Since $F\subseteq C$ by assumption (iii) of Theorem 4.4.6, it follows for $d\in D$ that

$$0 = (1-p)T(d) = (1-p)eH_0(d) - H_0(d)(1-p)e \quad\text{and}\quad 0 = T(d)(1-p)\,.$$

Thus, $(1-p)e, pe(1-p)\in C$ and $e-pep = (1-p)e+pe(1-p)\in C$. By assumptions (iii,v) of Theorem 4.4.6, $pH_0(D) = H_0(D)p\subset J+\beta J$. Thus, $pep\in\mathcal{N}(H_0(D),J+\beta J)$.

It also shows that $\operatorname{Der}(A,I) = C+\mathcal{N}(A,I)$ in the case where $I = J+\beta J$:

We can use that $\operatorname{Der}(A,J)\subset\operatorname{Der}(A,J+\beta J)$ in the case where $I = J$:

If $e\in\operatorname{Der}(A,J)_+$, then $e\in\operatorname{Der}(A,J+\beta J)$ and there exists a projection $p\in F$ with $e-pep\in C$ by our above considerations. Since $C = \operatorname{Der}(A,\{0\})\subset\operatorname{Der}(A,J)$ and $F\subseteq\mathcal{N}(A,J+\beta J)$ it follows that

$$pep = e-(e-pep)\in\operatorname{Der}(A,J)\cap\mathcal{N}(A,J+\beta J)\,.$$

It turns out that the latter implies that pep is in $\mathcal{N}(A,J)$:

By Part (xi) of Lemma 4.4.7 there exist projections $q,r\in C$ with the properties $p = q+r$, $[qH_0(\cdot)] = [h_0]$ and $[rH_0(\cdot)] = [\beta h_0]$. In particular $q\in C\cap\mathcal{N}(A,J)$ and $r\in C\cap\mathcal{N}(A,\beta J)$.

For $d \in D$ holds $rH_0(d) = H_0(d)r \in \beta J$. Thus,

$$\beta(J) \ni rerH_0(d) - H_0(d)rer = r(eH_0(d) - H_0(d)e)r \in J\,.$$

Since $J \cap \beta J = \{0\}$ it follows that $rer \in C$. Hence, $pep - rer \in \mathrm{Der}(A, J)$ and

$$e - (pep - rer) = (e - pep) + rer \in C\,.$$

We show that the element $pep - rer = qer + req + qeq$ is in $\mathcal{N}(A, J)$, and get that $e \in C + \mathcal{N}(A, J)$:

Since $q, r \in C$, we get e.g. from $qerH_0(d) - H_0(d)qer = q(eH_0(d) - H_0(d)e)r$ and $e \in \mathrm{Der}(A, J)_+$ that $qer, req, rer \in \mathrm{Der}(A, J)$. It follows from $J \cap \beta J = \{0\}$, $H_0(d)r \in \beta J$, $qH_0(d) \in J$ and $reqH_0(d) - H_0(d)req \in J$ that necessarily $H_0(d)req = 0$ for all $d \in D$. Since $(qer)^* = req \in \mathrm{Der}(A, J)$ it follows $qerH_0(d) = 0$ for all $d \in D$.

$(re + qe)(qH_0(d)) \in J$ and $qer\, H_0(d) = 0$ imply $(qer + req + qeq)H_0(d) \in J$.

Since $pep - rer$ is self-adjoint it follows that $pep - rer \in \mathcal{N}(A, J)$.

Summing up we get $e = x + y$ with $x = (e + rer) - (pep) \in C$ and $y = pep - rer \in \mathcal{N}(A, J)$.

Thus, $\mathrm{Der}(A, I)_+ \subseteq C + \mathcal{N}(A, I)$. Since $C + \mathcal{N}(A, I)$ is a C^*-subalgebra of the C^*-algebra $\mathrm{Der}(A, I)$ it follows $\mathrm{Der}(A, I) = C + \mathcal{N}(A, I)$ for $A := H_0(D)$ and $I := J + \beta J$ (respectively $A := H_0(D)$ and $I := J$).

The ideal $C \cap \mathcal{N}(A, I)$ of C is the kernel of the restriction to C of the epimorphism $\mathrm{Der}(A, I) \to \mathrm{Der}(A, I)/\mathcal{N}(A, I)$. Since $\mathrm{Der}(A, I) = C + \mathcal{N}(A, I)$, this quotient map defines an isomorphism

$$\mathrm{Der}(A, I)/\mathcal{N}(A, I) \cong C/(C \cap \mathcal{N}(A, I))\,.$$

(ii): The (two-sided) annihilator $\mathrm{Ann}(A, E)$ is a closed ideal of C, because

$$c_1ec_2a = c_1eac_2 = 0 \quad \text{and} \quad ac_1ec_2 = c_1aec_2 = 0$$

if $e \in \mathrm{Ann}(A, E)$, $a \in A$ and $c_1, c_2 \in C$.

The C^*-subalgebra $\mathrm{Ann}(A, E)$ is *hereditary* in E, because $e_1Ee_2 \subseteq \mathrm{Ann}(A, E)$ if $e_1, e_2 \in \mathrm{Ann}(A, E)$.

If $e \in \mathrm{Ann}(A, E)$, i.e., if $ea = 0 = ae$ for all $a \in A \subseteq E$, then $e \in \mathcal{N}(A, J) \subseteq \mathcal{N}(A, I)$ and $e \in C = A' \cap E$. Thus $\mathrm{Ann}(A, E)$

Since $\beta H_0 = H_0$, it follows $\beta A = A$ and $\beta(\mathrm{Ann}(A, E)) = \mathrm{Ann}(A, E)$.

Thus, $\mathrm{Ann}(A, E) \subseteq \mathcal{N}(A, J) \cap \beta\mathcal{N}(A, J)$.

We get $\beta\mathcal{N}(A, J) = \mathcal{N}(A, \beta J)$ from $\beta(A) = A$.

If $e \in E$ satisfies $eH_0(D) \subseteq J$ and $eH_0(D) \subseteq \beta J$, then $eH_0(D) \subseteq J \cap \beta J = \{0\}$.

Thus, $\mathcal{N}(A, J) \cap \beta\mathcal{N}(A, J) = \mathrm{Ann}(A, E)$.

The *-monomorphism H_0 dominates zero by Lemma 4.4.7(vi), i.e., there exists an isometry $s_1 \in E$ with $s_1^* H_0(\cdot) s_1 = \{0\}$. It implies that $s_1 s_1^* \in \mathrm{Ann}(A, E)$ if $A := H_0(D)$, i.e., $\mathrm{Ann}(A, E)$ is full in E.

(iii): The unital algebra $C := A' \cap E \subseteq \mathrm{Der}(A, I)$ contains the isometries s, t with orthogonal ranges by assumption (i) of Theorem 4.4.6, and $\mathcal{N}(A, I)$ is a non-zero ideal of $\mathrm{Der}(A, I)$ by Part (i).

All non-zero quotients of C and of $\mathrm{Der}(A, I)$ and all (unitization of) non-zero ideals of this quotients are K_1-surjective by Remark 4.2.17.

(iv): The groups $\mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ and $\mathcal{U}(C)$ are norm-closed subgroups of $\mathcal{U}(\mathrm{Der}(A, I))$, because unitary groups of C^*-algebras are norm-closed.

The group $\mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ is normal in $\mathcal{U}(\mathrm{Der}(A, I))$ because $\mathcal{N}(A, I)$ is an ideal of $\mathrm{Der}(A, I) = C + \mathcal{N}(A, I)$.

(v): The set $\mathcal{G}(I)$ of products $\mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ is a subgroup of $\mathcal{U}(\mathrm{Der}(A, I))$ because $\mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ is a normal subgroup of $\mathcal{U}(\mathrm{Der}(A, I))$ by Part (iv).

If $u \in \mathcal{U}_0(\mathrm{Der}(A, I))$ then u is a product of exponentials, i.e., there exist $t_m \in \mathrm{Der}(A, I)$ $(m = 1, \cdots, n)$ such that $t_m^* = -t_m$ and $u = \exp(t_1) \cdot \ldots \cdot \exp(t_n)$. We find $c_m \in C$ $(m = 1, \ldots, n)$ with $c_m^* = -c_m$ and $t_m - c_m \in \mathcal{N}(A, I)$, because $\mathrm{Der}(A, I) = C + \mathcal{N}(A, I)$ by Part (i). If we define $v \in \mathcal{U}_0(C)$ by $v := \exp(c_1) \cdot \ldots \cdot \exp(c_n)$, then $v^* u \in 1 + \mathcal{N}(A, I)$, because $\mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ is a normal subgroup of $\mathcal{U}(\mathrm{Der}(A, I))$ and $\exp(-c_m) \exp(t_m) \in 1 + \mathcal{N}(A, I)$ for $m = 1, \ldots, n$.

It follows that $\mathcal{G}(I)$ is an *open* subgroup of $\mathcal{U}(\mathrm{Der}(A, I))$.

Recall that every open subgroup G_2 of a topological group G_1 is also closed in G_1, because G_2 is the complement in G_1 of the union of open sets $G_2 \cdot g$ with $g \in G_1 \setminus G_2$.

If $u, v \in \mathcal{U}(C)$ then $u \oplus_{s,t} v = sus^* + tvt^* \in \mathcal{U}(C)$ because $s, t \in C$.

If $w \in \mathcal{U}(\mathcal{N}(A, I) + \mathbb{C} \cdot 1)$ then $w = (\xi 1) \cdot (e + 1)$ for some $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $e \in \mathcal{N}(A, I)$.

The unitary $s^* w s + tt^* = (\xi \cdot ss^* + tt^*)(ses^* + 1)$ is in $\mathcal{G}(I)$, because $\xi \cdot ss^* + tt^* \in \mathcal{U}_0(C)$ and $ses^* \in \mathcal{N}(A, I)$.

Thus $\mathcal{G}(I) \oplus_{s,t} 1 \subseteq \mathcal{G}(I)$.

(vi): By Lemma 4.4.7(vi) there exist isometries $s_1, t_1 \in E$ with $s_1 s_1^* + t_1 t_1^* = 1$ and s_1 satisfies $s_1^* A = \{0\} = A s_1$. It implies that $s_1 \, E \, s_1^* \subseteq \mathrm{Ann}(A, E)$. If $u \in \mathcal{U}(E)$ then $[u] = [u \oplus' 1] = [(1 - s_1 s_1^*) + s_1 u s_1^*]$ in $\mathrm{K}_1(E)$ by Lemma 4.2.6(v,2).

Thus $v := (1 - s_1 s_1^*) + s_1 u^* s_1^*$ is a unitary in $1 + \mathrm{Ann}(A, E)$ with $[vu] = 0$ in $\mathrm{K}_1(E)$.

Remaining TEXT:

For each unitary in $u \in \mathcal{U}(\mathcal{N}(A,I) + \mathbb{C}\cdot 1)$ with the additional property $u \in \mathrm{Der}(A,J)$ there exists unitaries $v, w \in \mathcal{U}(C)$ such that

$$v\,u\,w \in \mathcal{U}_0(\mathcal{N}(A,J) + \mathbb{C}\cdot 1) \subseteq \mathcal{U}_0(\mathrm{Der}(A,J))\,.$$

In particular, the image $\mu'(\mathrm{K}_1(\mathcal{N}(A,J))) \subseteq \mathrm{K}_1(\mathrm{Der}(A,J))$ of $\mathrm{K}_1(\mathcal{N}(A,J))$ is contained in the image $\mu(\mathrm{K}_1(C))$ in $\mathrm{K}_1(\mathrm{Der}(A,J))$, and the mapping

$$\mathcal{G}(J) \ni u \mapsto [u] \in \mathrm{K}_1(\mathrm{Der}(A,J))$$

maps $\mathcal{G}(J)$ *onto* $\mu(\mathrm{K}_1(C)) \subseteq \mathrm{K}_1(\mathrm{Der}(A,J))$.

??

(old ?) TEXT:

`This ref part is only granted modulo` $\mathrm{Ann}(A,I)$ `or only in` $\mathrm{K}_1(\mathcal{N}(A,I))$`?`

`Next text is different to ORIGINAL.`

For each unitary in $u \in \mathcal{U}(\mathcal{N}(A,I) + \mathbb{C}\cdot 1)$ (respectively $u \in \mathcal{U}(E)$) there exists a unitary $v \in \mathcal{U}(C) \cap (1 + \mathcal{N}(A,I))$ (respectively only $v \in \mathcal{U}(C)$)

OR ?: $v \in 1 + \mathrm{Ann}(A,E)$???

such that $v\,u \in \mathcal{U}_0(\mathcal{N}(A,I) + \mathbb{C}\cdot 1)$

Respectively only (minimal possible): $vu \in \mathcal{U}_0(\mathrm{Der}(A,I)) \cdot \mathcal{U}(C)$.

(respectively $v \in \mathcal{U}(C)$ such that $vu \in \mathcal{U}_0(E)$ and a unitary $w \in 1 + \mathrm{Ann}(A,E) \subseteq C$ with $[wu] = 0$ in $\mathrm{K}_1(E)$).

In particular, the image $\mu'(\mathrm{K}_1(\mathcal{N}(A,I)))$ in $\mathrm{K}_1(\mathrm{Der}(A,I))$ is contained in the image $\mu(\mathrm{K}_1(C))$ of $\mathrm{K}_1(C)$ in $\mathrm{K}_1(\mathrm{Der}(A,I))$, and the image of the map

$$\mathcal{G}(I) \ni u \mapsto [u] \in \mathrm{K}_1(\mathrm{Der}(A,I))$$

coincides with $\mu(\mathrm{K}_1(C)) \subseteq \mathrm{K}_1(\mathrm{Der}(A,I))$.

END TEXT

`Use in next proof lem. 4.4.7(vi)? in place of 4.4.7(iv)??`

By Lemma 4.4.7(vi), there exist isometries $s_1, t_1 \in E$ with $s_1 s_1^* + t_1 t_1^* = 1$ and $s_1^* H_0(D) = \{0\}$, i.e., $s_1 E s_1^* \subseteq \mathrm{Ann}(A,E)$.

`CASE` $u \in \mathcal{U}(E)$ `:`

Let $u \in \mathcal{U}(E)$. Define $v := (u \oplus_{R,T} 1)^* \in \mathrm{Ann}(A,E)$. Then $v \in C$ and $[v] = -[u]$ in $\mathrm{K}_1(E)$, i.e., $[vu] = 0$ in $\mathrm{K}_1(E)$ and $(vu)^* H_0(\cdot) vu = u^* H_0(\cdot) u$.

NEXT ??????

There exists a unitary $v \in \mathcal{U}(C) \cap (1 + \mathrm{Ann}(A,I))$ such that $vu \in \mathcal{U}_0(\mathcal{N}(A,I) + \mathbb{C}\cdot 1)$.

????

(vi): `New approach:`

Next to PROOF of a Lemma ??:

Let $u \in \mathcal{U}(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$. Then $u \in \mathrm{Der}(A,I)$ and by Lemma 4.4.11 there exist $k\colon D \to I$ and $v \in \mathcal{U}(C)$ such that

$$k \oplus_{s_0,t_0} H_0 = v^*u^*H_0(\cdot)uv\,,$$

where $k = k_1 + \beta k_2$ for C^*-morphisms $k_1, k_2\colon D \to J$. Moreover, $k_2 = h_0$ in case $I = J$.

Lemma 4.4.8 (in case $I = J$) and Remark 4.4.14 (in case $I = J + \beta J$) show that $v^*uv \in \mathcal{U}(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$ implies that $[k]+[h_0+\beta h_0] = [h_0+\beta h_0]$ (which is equivalent to $[k_1]+[h_0] = [h_0]$ in case $I = J$), that this is equivalent to the existence of a unitary $U \in \mathcal{U}_0(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$ with

$$k \oplus_{s_0,t_0} H_0 = U^*H_0(\cdot)U\,.$$

It follows that $w := uv \cdot U^* \in \mathcal{U}(C)$, and $w^*uv = U^* \in \mathcal{U}_0(\mathcal{N}(A,I)+\mathbb{C}\cdot 1)$.

(vii): Clearly $\mathrm{Ann}(A,E)$ is a closed ideal of C and is a hereditary C^*-subalgebra $\mathrm{Ann}(A,E)$ of E. It is full in E because there exist an isometry $s_1 \in E$ with $s_1^*H_0(\cdot)s_1 = 0$ by Lemma 4.4.7(vi).

Since $\mathcal{N}(A,I) = \pi_J^{-1}(\mathrm{Ann}(\pi_J(A), E/J))$, it is a hereditary C^*-subalgebra of E that contains $\mathrm{Ann}(A,E)$. Thus $\mathcal{N}(A,I)$ is full in E. By Part (i), $\mathcal{N}(A,I)$ is a closed ideal of $\mathrm{Der}(A,I)$ that is hereditary and full in E.

Since $\mathrm{Ann}(A,E)$ is a closed ideal of C and $\mathcal{N}(A,E)$ is a closed ideal of $\mathrm{Der}(A,E)$, and both are full and hereditary in E, Part (vii) follows from Lemma 4.2.20(o).

END Proofs (i)-(vii).

TEXT OF GENERAL CONCLUSIONS:

Finally, the K_*-theory 6-term exact sequence

$$0 \to (C \cap \mathcal{N}(A,I)) \to C \to C/(C\cap\mathcal{N}(A,I)) \to 0\,.$$

shows that the group morhism $\mathrm{K}_1(C) \to \mathrm{K}_1(C/(C\cap\mathcal{N}(A,I)))$ is surjective, if and only if, $\mathrm{K}_0(C\cap\mathcal{N}(A,I))) \to \mathrm{K}_0(C)$ is injective.

The latter is related to the injectivity of $\varphi\colon [k]+[h_0] \mapsto [k]+[H_0]$.

???????

END TEXT

BEGIN TEXT OF GENERAL Conclusion:

The group morphism $\varphi\colon [k]+[h_0] \mapsto [k]+[H_0]$ is injective, if and only if, the natural group morphism

$$\mathrm{K}_1\big(C/\mathrm{Ann}(A,E)\big) \to \mathrm{K}_1\big(C/(C\cap\mathcal{N}(A,I))\big)$$

is surjective.

This is the case if and only if $u \oplus_{s,t} 1 \in \mathcal{G}$ for all $u \in \mathcal{U}(\mathrm{Der}(A,I))$.

A *sufficient* condition for the latter is that the natural group morphism

$$\mathcal{U}(C/\operatorname{Ann}(A,E)) \to \mathcal{U}(C/(C\cap\mathcal{N}(A,I)))$$

is surjective.

Notice that $\operatorname{Ann}(A,E) \subseteq C\cap\mathcal{N}(A,I)$ are ideals of C and that $\operatorname{Ann}(A,I)$ is a full hereditary C^*-subalgebra of E. By Lemma 4.2.20, $\mathrm{K}_*(C/\operatorname{Ann}(A,I))$ is natural isomorphic to the kernel of $\mathrm{K}_*(C) \to \mathrm{K}_*(E)$, and the natural sequences

$$\mathrm{K}_*(\operatorname{Ann}(A,I)) \to \mathrm{K}_*(C) \to \mathrm{K}_*(C/\operatorname{Ann}(A,I))$$

are split-exact sequences.

Recall that $G(H_0\,;\,D,E)$ (respectively $G(h_0\,;\,D,E)$) is isomorphic to the kernel of $\mathrm{K}_0(C)\to\mathrm{K}_0(E)$ (respectively of $\mathrm{K}_0(h_0(D)'\cap E)\to\mathrm{K}_0(E)$).

□

PROOF OF THEOREM 4.4.6 (PART 2: KERNEL OF φ IN $G(h_0,D,E)$).

Let $\mathcal{A}$ (temporary) denote the set of $k\in\operatorname{Hom}(D,E)$ with $k(D)\subseteq J$ and the property that $k\oplus H_0$ is unitary equivalent to H_0 by a unitary in E, i.e., with $[k]+[H_0]=[H_0]$ in $[\operatorname{Hom}(D,E)]_u$. Recall that $[k]+[H_0]$ is the unitary equivalence class of $k\oplus H_0$ in the elements in $\operatorname{Hom}(D,E)$, which is independent from the chosen isometries $S,T\in E$ with $SS^*+TT^*=1$ for the sum $\oplus := \oplus_{S,T}$.

If $k\in\operatorname{Hom}(D,E)$ is in $\mathcal{A}$, i.e., $k(D)\subseteq J$ and and $[k]+[H_0]=[H_0]$, then k satisfies

- (1a) $[k]\in S(h_0\,;\,D,E)$, (by Lemma 4.4.7(vii)), and
- (1b) $[k]+[h_0]$ is in the kernel of $\varphi\colon G(h_0\,;\,D,E)\to G(H_0\,;\,D,E)$. (Because $([k]+[h_0])+[H_0]=[k]+([h_0]+[H_0])=[k]+[H_0]$ by Lemma 4.4.7(iv).)
- (2) $[k+\beta h_0]+[H_0]=[H_0]$. (Because $[k+\beta h_0]=[k]+[\beta h_0]$ for $k\in\operatorname{Hom}(D,E)$ with $k(D)\subseteq J$ and $[\beta h_0]+[H_0]=[H_0]$ by Lemma 4.4.7(i, iv).)

Conversely properties (1a) and (1b) or properties (1a) and (2) imply for $k\in\operatorname{Hom}(D,E)$ that $k\in\mathcal{A}$.

Indeed: (1a) implies $k(D)=T^*h_0(D)T\in J$ for some isometry $T\in E$ and $\varphi([k]+[h_0])=[H_0]$ says by definition of φ that $([k]+[h_0])+[H_0]=[H_0]$, i.e., that $[k]+[H_0]=[k]+([h_0]+[H_0])=([k]+[h_0])+[H_0]=[H_0]$ by Lemma 4.4.7(iv).

Thus, the set of unitary equivalence classes $[k]\in[\operatorname{Hom}(D,E)]$ of the elements $k\in\mathcal{A}$ is identical with the set of elements $[k]$ in the semi-group $S(h_0\,;\,D,E)$ that satisfy $\varphi([k]+[h_0])=[H_0]$.

The set $\mathcal{A}$ contains h_0 by Lemma 4.4.7(iv). If $k\in\mathcal{A}$ and $[h]=[k]$ then $h\in\mathcal{A}$ (here $[\cdot]$ denotes the unitary equivalence classes of elements of $\operatorname{Hom}(D,E)$). If $k_1,k_2\in\mathcal{A}$ then $k_1\oplus k_2\in\mathcal{A}$.

This is because [??????] =?.

Let $k \in \mathrm{Hom}(D,J) \subset \mathrm{Hom}(D,E)$, then

$$[k] + [H_0] = [H_0] \quad \Leftrightarrow \quad [k + \beta h_0] + [H_0] = [H_0]\,,$$

because, by Lemma 4.4.7(iv,i,iii),

$$[k + \beta h_0] + [H_0] = [k + \beta h_0] + [0] + [H_0] = [k] + [\beta h_0] + [H_0] = [k] + [H_0]\,.$$

Thus, if we take s_0 and t_0 as in assumption (iv) of Theorem 4.4.6, then $[H_0] = [k] + [H_0]$ becomes equivalent to $[H_0] = [(k + \beta h_0) \oplus_{s_0,t_0} H_0]$, i.e., is equivalent to the existence $u \in \mathcal{U}(E)$ that satisfies following *equation*:

$$s_0(k(\cdot) + \beta h_0(\cdot))s_0^* + (1 - p_0)H_0(\cdot) = (k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot)u\,. \tag{4.20}$$

The equation shows that $u \in \mathrm{Der}(H_0(D), J)$ and that the unitary u is determined by k in the right side of the equation up to multiplications Vu by unitaries $V \in \mathcal{U}(C)$.

We can also use the isometries s_2, t_2 of Lemma 4.4.7(x) and get directly $v \in \mathcal{U}(E)$ with

$$k \oplus_{s_2,t_2} H_0 = v^* H_0(\cdot)v \tag{4.21}$$

This unitary v is again in $\mathrm{Der}(H_0(D), J)$ and is determined by the left side, i.e., by k, only up to multiplications Uv with $U \in \mathcal{U}(C)$.

By Lemma 4.4.10 there exists a unitary $W \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J)) \cap (p_0Ep_0 + (1 - p_0))$ with $((k + \beta h_0) \oplus_{s_0,t_0} H_0)W = W(k \oplus_{s_2,t_2} H_0)$.

It follows that $W^*u^*H_0(\cdot)uW = v^*H_0(\cdot)v$, i.e., $\mathcal{U}(C)uW = \mathcal{U}(C)v$. In particular, $\mu(\mathrm{K}_1(C)) + [u] = \mu(\mathrm{K}_1(C)) + [v]$ in $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$, because $[W] = 0$ in $\mathrm{K}_1\big(\mathrm{Der}(H_0(D), J)\big)$.

Recall here, that we have defined $\mu\colon \mathrm{K}_1(C) \to \mathrm{K}_1\big(\mathrm{Der}(H_0(D), J)\big)$ by the natural *-monomorphism $C \hookrightarrow \mathrm{Der}(H_0(D), J)$.

The above considerations show that the class of the unitaries u and v in Equations (4.20) and (4.21) define for $k \in \mathcal{A}$ the same class

$$\gamma_0(k) := [u] + \mu(\mathrm{K}_1(C)) = [v] + \mu(\mathrm{K}_1(C)) \tag{4.22}$$

in $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))/\mu(\mathrm{K}_1(C))$.

We get that that $\gamma_0(h_0) = \mu(\mathrm{K}_1(C))$, because $s_0(h_0 + \beta h_0)s_0^* = p_0 H_0$ and $t_0 H_0 t_0^* = (1 - p_0)H_0$ by Assumptions (iii) and (iv) of Theorem 4.4.6.

If we use the definition of the unitary v in Equation (4.21) then we obtain that $\gamma_0(k) = \gamma_0(U^*kU)$ for all unitaries $U \in E$:
Indeed, the property $k(D) \subseteq J$ implies that there exists a unitary $V \in \mathcal{U}_0(E)$ with $U^*k(\cdot)U = V^*k(\cdot)V$, cf. Lemma **??**(????).

Let $v \in \mathcal{U}(\mathrm{Der}(H_0(D), J)$ as in Equation (4.21) and $W := v(V \oplus_{s_2,t_2} 1)$ then

$$(U^*k(\cdot)U) \oplus_{s_2,t_2} H_0 = W^* H_0(\cdot) W\,.$$

It implies that $\mathcal{U}(C)W = \mathcal{U}(C)v(V \oplus_{s_2,t_2} 1)$. The unitary $V \oplus_{s_2,t_2} 1$ is in $s_2\mathcal{U}_0(E)s_2^* + t_2t_2^*$. The group $s_2\mathcal{U}_0(E)s_2^* + t_2t_2^*$ is contained in the connected component of $\mathcal{U}(\mathrm{Ann}(H_0(D), J) + \mathbb{C}1) \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J))$.

It implies $[W] = [v] \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ and $\gamma_0(U^*k(\cdot)U) = \gamma_0(k)$, by the defining equation (4.22) of γ_0.

It follows that there is a unique mapping γ from the set $[\mathcal{A}]$ of elements $[k] \in S(h_0\,;\, D, E)$ with the property that $[k] + [h_0]$ is in the kernel of

$$S(h_0\,;\, D, E) \to G(h_0\,;\, D, E) \to G(H_0\,;\, D, E)$$

to $\mathrm{K}_1\big(\mathrm{Der}(H_0(D), J)\big)/\mu\big(\mathrm{K}_1(C)\big)$, such that γ satisfies $\gamma_0(k) = \gamma([k])$ for $k \in \mathcal{A}$ and $\gamma([h_0]) = \mu(\mathrm{K}_1(C))$.

We show now that γ **is additive:**
Let $k_1, k_2 \in \mathcal{A}$ and take $u_1, u_2 \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ with $k_j \oplus_{s_2,t_2} H_0 = u_j^* H_0 u_j$ for $j \in \{1, 2\}$.

By Lemma 4.4.8, if (s_2, t_2) are the isometries in Lemma 4.4.7(x) and the unitary $v_2 \in C$ is defined by Equation (4.11) given in proof of Lemma 4.4.7(x), then for $a, b, c, d \in E$ holds:

$$v_2^*\big((a \oplus_{s_2,t_2} c) \oplus_{s,t} (b \oplus_{s_2,t_2} d)\big)v_2 = (a \oplus_{s,t} b) \oplus_{s_2,t_2} (c \oplus_{s,t} d)\,.$$

If we let $a := k_1(d)$, $b := k_2(d)$, $c = d := H_0(d)$ then we get for $k_j \oplus_{s_2,t_2} H_0 = u_j^* H_0 u_j$ and the unitary $U := (u_1 \oplus_{s,t} u_2) \cdot v_2$ the equation

$$U^* H_0 U = v_2^*\big((u_1^* H_0 u_1) \oplus_{s,t} (u_2^* H_0 u_2)\big)v_2 = (k_1 \oplus_{s,t} k_2) \oplus_{s_2,t_2} H_0\,.$$

It follows that $\gamma_0(k_1 \oplus_{s,t} k_2) = \mu(\mathrm{K}_1(C)) + [U]$ Since $[U] = [u_1] + [u_2] + [v_2]$ and $v_2 \in C$, it follows that

$$\gamma_0(k_1 \oplus_{s,t} k_2) = \gamma_0(k_1) + \gamma_0(k_2)$$

in $\mathrm{K}_1(\mathrm{Der}(H_0(D)), J)/\mu(\mathrm{K}_1(C)) =: \Gamma\big(H_0(D), J, E\big)$.

Thus, γ_0 is additive with respect to Cuntz addition on $\mathcal{A}$ with $\gamma_0(h_0) = \mu(\mathrm{K}_1(C))$ and defines an additive map γ_1 from the sub-semigroup of those elements $[k]$ of $S(h_0\,;\, D, E)$ with $[k] + [h_0]$ in the kernel of φ.

`NEXT steps:`

Step (next 1):

`TEXT:`

`Let` $k \in \mathcal{A}$ `then` $\gamma_0(k) = \mu(\mathrm{K}_1(C))$
`if and only if` $[k] + [h_0] = [h_0]$:

It shows that $\gamma_1([k]) = \mu(\mathrm{K}_1(C))$ and $k \oplus_{s_2,t_2} H_0 = u^* H_0 u$ if and only if $[u] \in \mu(\mathrm{K}_1(C))$.

Notice that $[\mathcal{A}] + [h_0]$ is a subgroup of $G(h_0\,;\, D, E)$ and is identical with the kernel of φ.

It will be shown below that $k \in \mathcal{A}$ and $\gamma_0(k) = 0$ together imply that $[k]+[h_0] = [h_0]$. It follows that $\gamma_1|[\mathcal{A}] + [h_0]$ "restricted" to the kernel of φ in $G(h_0\,;\, D, E)$ is faithful on the kernel $\varphi^{-1}([H_0])$ of φ.

The neutral element "0" of $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))/\mu(\mathrm{K}_1(C))$ is $\gamma_0(h_0) = \mu(\mathrm{K}_1(C))$ and the invariance under unitary equivalence that $[k] + [h_0] = [h_0]$ implies

$$\gamma_0(k) = \gamma_1([k]) = \mu(\mathrm{K}_1(C)) \,.$$

Suppose that $k \in \mathcal{A}$ and $\gamma_0(k) = \mu(\mathrm{K}_1(C))$. It follows that there exists $v \in \mathcal{U}(C)$ and $w \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$ with $(v \oplus s, t1)w = (u \oplus_{s,t} 1)$ (cf. Lemma 4.2.6(v,2)).

Each $w \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$ decomposes by Lemma 4.4.15(v) into a product $w = w_1 w_2$ with $w_1 \in \mathcal{U}(C)$ and $w_2 \in \mathcal{U}(\mathbb{C}\cdot 1 + \mathcal{N}(H_0(D), J))$.

`Is it not` $\mathcal{U}(\mathbb{C}1 + \mathcal{N}(H_0(D), J))$ `???`

It follows that $(k \oplus_{s,t} h_0) \oplus_{s_2,t_2} H_0 = w_2^* H_0 w_2$.

This causes by Lemma **??** that $[k \oplus h_0] = [h_0]$ provided that in addition ???????

Step (next 2): γ_1 is surjective:

(a) $\mathrm{Der}(H_0(D), J)$ and C are K_1-surjective.

i.e., For $x \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ exists $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ with $x = [u]$.

compare next with places above!

We define for $k \in \mathcal{A}$ a class $\mathcal{U}(k) = \mathcal{U}(C)u \subset \mathcal{U}(\mathrm{Der}(H_0(D), J))$ of $\mathcal{U}(C)\backslash\mathcal{U}(\mathrm{Der}(H_0(D), J))$ by $u \in \mathcal{U}(k)$ if and only if u satisfies Equation (4.20).

If $k \in \mathcal{A}$, then the set $\mathcal{U}(k) \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J))$ is an element of the right homogenous space

$$\mathcal{U}(C)\backslash\mathcal{U}(\mathrm{Der}(H_0(D), J))$$

given by right multiplication of elements of $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ by elements of $\mathcal{U}(C) \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J))$.

Clearly, the unitary u in Equation (**??**) is determined by k only up to multiplications vu with $v \in \mathcal{U}(C)$. We denote this subset of $\mathcal{U}(E)$ by

$$\mathcal{U}(k) := \mathcal{U}(C)u$$

if a C^*-morphism $k\colon D \to J$ satisfies $[k] + [H_0] = [H_0]$. Thus, $\mathcal{U}(k) = \mathcal{U}(C) \cdot \mathcal{U}(k)$.

$\mathcal{U}(k) \subseteq \mathcal{U}(\mathrm{Der}(H(D), J))$:

The elements of $\mathcal{U}(k)$ are contained in $\mathrm{Der}(H(D), J)$, because

$$p_0 H(\cdot) = s_0(h_0 + \beta h_0)s_0^* \,,$$

by assumption 4.4.6(iv), and this together with Equation (**??**) implies

$$H_0(a)u - uH_0(a) = us_0(k(a) - h_0(a))s_0^* \in J \quad \text{for all } \ a \in D \,.$$

Thus $u \in \mathrm{Der}(H_0(D), J)$. Since $C \subseteq \mathrm{Der}(H_0(D), J)$ it follows that

$$\mathcal{U}(C)u =: \mathcal{U}(k) \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J))$$

The class

$$\mathcal{U}(k) \in \mathcal{U}(C) \setminus \mathcal{U}(\mathrm{Der}(H_0(D), J))$$

defines a class

$$\gamma_0(k) := [u] + \mu(\mathrm{K}_0(C)) \in \Gamma := \Gamma(H_0(D), J, E) := \mathrm{K}_0(\mathrm{Der}(H_0(D), J))/\mu(\mathrm{K}_0(C))\,.$$

`Next steps:`

(next 1)

$\gamma_0(k_1) = \gamma_0(k_2)$ if $[k_1] = [k_2]$, i.e., $\gamma_0(k)$ is invariant under unitary equivalence.

Let $w \in \mathcal{U}(E)$ with $w^* k_2(\cdot) w = k_1$.

If $[k_1] = [k_2]$ then there exists by Lemma 4.4.7(viii) a unitary $w \in \mathcal{U}_0(\beta h_0(D)' \cap E)$ such $k_1 + \beta h_0 = w^*(k_2 + \beta h_0) w$.

Let $\xi \in [0,1] \mapsto w(\xi) \in \mathcal{U}(\beta h_0(D)' \cap E)$ a continuous path with $w(0) = 1$ and $w(1) = w$. The continuous path

$$v(\xi) := s_0 w(\xi) s_0^* + t_0 t_0^* \in \mathcal{U}(E)$$

satisfies $v(\xi) = 1$ and for $d \in D$ that

$$v(\xi) H_0(d) - H_0(d) v(\xi) = s_0(w(\xi) h_0(d) - h_0(d) w(\xi)) s_0^* \in J\,,$$

i.e., $v(\xi) \in \mathrm{Der}(H_0(D), J)$. It follows that $v(1) \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$. Thus, $[v(1)] = 0$ in $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ and

$$\gamma_0(k_2) = \gamma_0(k_1) + (\mu(\mathrm{K}_1(C)) + [v(1)]) = \gamma_0(k_1)\,.$$

Notice that $w \in \mathcal{U}_0(E)$ also implies that $v \in \mathcal{U}_0\big(\mathcal{N}(H_0(D), J + \beta J) + \mathbb{C} \cdot 1\big)$.

An *alternative proof* of the equivalence of γ_0 with respect to unitary equivalence can be given directly by Lemma 4.4.7(i):

The equality $[k_1] = [k_2]$ yields by ???????????

Consequence of the invariance:

We can define

$$\gamma_1([k]) := \gamma_0(k)$$

for a representative $k \in \mathrm{Hom}(D, J)$ of the class $[k] \in S(h_0\,;\, D, E)$.

(next 2)

$\gamma_1([k_1] + [k_2]) = \gamma_1([k_1]) + \gamma_1([k_2])$, i.e., the mapping

$\gamma_1 \colon S(h_0\,;\, D, E) \to \Gamma := \Gamma(H_0(D), J, E) := \mathrm{K}_1(\mathrm{Der}(H_0(D), J))/\mu(\mathrm{K}_1(C))$

is an additive map from the semi-group $S(h_0\,;\, D, E)$ into the group Γ.

Use Lemma 4.4.8 and that $s = \beta(s), t = \beta(t) \in (h_0(D) \cup H_0(D))' \cap E$ by assumptions of Theorem 4.4.6, both cases of definitions with $\oplus_{s_0,t_0}$ or with $\oplus_{s_2,t_2}$.

(next 3)

$\mathcal{U}(h_0) = \mathcal{U}(C)$, i.e., $\gamma_1([h_0]) = \gamma_0(h_0) = 0$.

In particular, $\gamma_1([k]+[h_0]) = \gamma_1([k])$ for all $k\colon D\to J$ with $[k]+[H_0]=[H_0]$.

Thus, $\gamma([k]) := \gamma_1([k])$ defines a group homomorphism from the kernel $\varphi^{-1}([H_0]) \subseteq G(h_0\,;D,E)$ of $\varphi\colon G(h_0\,;D,E) \to G(H_0\,;D,E)$ into $\Gamma := \Gamma(H_0(D),J,E)$.

(next 4)

The homomorphism γ is surjective:

For every $x\in \mathrm{K}_1(\mathrm{Der}(H_0(D),J))$ there exists $u\in \mathcal{U}(\mathrm{Der}(H_0(D),J))$ with $[u]=x$, because $C\subseteq \mathrm{Der}(H_0(D),J)$ is properly infinite and this implies that $\mathrm{Der}(H_0(D),J)$ (respectively C) is K_1-surjective by Lemma 4.2.6(v).

It suffices to find $k\colon D\to J$, $v\in\mathcal{U}(C)$ and $w\in\mathcal{U}_0(\mathrm{Der}(H_0(D),J))$ with

$$(k+\beta h_0)\oplus_{s_0,t_0} H_0 = w^*v^*u^*H_0(\cdot)uvw\,.$$

Give Ref's for next blue!!

In fact, there exists $v\in\mathcal{U}_0(F+\mathbb{C}\cdot 1)\cap(1+F)$ such that $uv\in\mathcal{U}(k)$ for some C^*-morphism $k\colon D\to J$ with $[k]+[H_0]=[H_0]$.

Thus, $[uv]=[u]+[v]=[u]$ in $\mathrm{K}_1(\mathrm{Der}(H_0(D),J))$, and $\gamma([k]+[h_0]) = \mu(\mathrm{K}_1(C))+[u]$ in Γ. It shows that γ is surjective.

(next 5)

If $k\colon D\to J$ is a C^*-morphism, $y\in\mathcal{N}(J+\beta J)$, $1+y\in\mathcal{U}(E)$ and

$$s_0(k+\beta h_0)s_0^* + (1-p_0)H_0(\cdot) = (1+y^*)H_0(\cdot)(1+y)\,,$$

then $[k]+[h_0]=[h_0]$:

Indeed, by Lemma 4.4.9 there exists a projection $p\in F$ with $p\geq p_0$, $p\neq p_0$ and

$$(1+y^*)(1-p)H_0(\cdot)(1+y) = (1-p)H_0(\cdot)\,,$$

because it implies that

$$(1+y^*)pH_0(\cdot)(1+y) = s_0(k+\beta h_0)s_0^* + (p-p_0)H_0(\cdot)\,.$$

By Part (xii) of Lemma 4.4.7, there exists a unitary $V\in C$ and $u_2\in\mathcal{U}_0(E)$ with $up_0=p_0=p_0u$ and

$$V^*(p-p_0)H_0(\cdot)V = t_0u_2^*(h_0+\beta h_0)u_2t_0^*\,.$$

It follows for $W:=V(p_0+t_0u_2^*t_0^*)$ that

$$(k+\beta h_0)\oplus_{s_0,t_0}(h_0+\beta h_0) = W^*(1+y^*)pH_0(\cdot)(1+y)W\,.$$

By part (ix) of Lemma 4.4.7 there exists a unitary $u_3\in\mathcal{U}_0(F+\mathbb{C}\cdot 1)\subset C$ with $u_3^*pu_3=p_0=s_0s_0^*$ and, by assumption (iv) of Theorem 4.4.6, $p_0H_0(\cdot)=s_0(h_0+\beta h_0)s_0^*$. We obtain with the unitary $U:=u_3(1+y)W\in\mathcal{U}(E)$ that

$$(k+\beta h_0)\oplus_{s_0,t_0}(h_0+\beta h_0) = U^*\big((h_0+\beta h_0)\oplus_{s_0,t_0}0\big)U\,.$$

If we use now that $J\cap\beta J=0$ and $h_0(D)\cup k(D)\subset J$ then we get

$$k \oplus_{s_0,t_0} h_0 = U^*\big((h_0 \oplus_{s_0,t_0} 0)\big)U\,,$$

which is up to multiplication of U by unitaries in $(s_0 h_0(D) s_0^*)' \cap E$ equivalent to $[k] + [h_0] = [h_0] + [0]$. But we know from Part (ii) of Lemma 4.4.7 that $[h_0] = [h_0] + [0]$.

(next 6)

The kernel of γ is $\{[h_0]\}$:

If $\gamma([k]) = \mu(\mathrm{K}_1(C))$, and $u \in \mathcal{U}(k) \subseteq \mathrm{Der}(H_0(D), J)$, then there exist $V \in \mathcal{U}(C)$ and $W \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$ such that $u \oplus_{S,T} 1 = VW$.

`Definition of` S, T `??`

It follows that

$$((k \oplus_{s,t} h_0) + \beta h_0) \oplus_{s_0,t_0} H_0 = (u \oplus_{S,T} 1)^* H_0(\cdot)(u \oplus_{S,T} 1) = W^* H_0(\cdot) W\,.$$

By Lemma 4.4.15(iv ????) ????, $W = V_1 W_1$ with $V_1 \in C$ and $W_1 = y + 1 \in \mathcal{U}(\mathcal{N}(H_0(D), J) + \mathbb{C}\cdot 1)$ with $y \in \mathcal{N}(H_0(D), J)$. Thus, for $W_1 := y + 1$,

$$s_0((k \oplus_{s,t} h_0) + \beta h_0) s_0^* + (1 - p_0) H_0(\cdot) = W_1^* H_0(\cdot) W_1\,.$$

Since

$$C = H_0(D)' \cap E = \mathrm{Der}(H_0(D), \{0\}) \subseteq \mathrm{Der}(H_0(D), J)\,,$$

we can define for $k \in \mathrm{Hom}(D, J)$ with $[k] \in S(h_0\,; D, E)$ in the kernel of φ an element $\mathcal{U}(k)$ of the (right-sided) homogenous space $\mathcal{U}(C)\backslash\mathcal{U}(\mathrm{Der}(H_0(D), J))$ by

$$\mathcal{U}(k) := \{vu\,;\ v \in \mathcal{U}(C)\,\}$$

where is u some unitary satisfying Equation (**??**).

Obvious examples are $\mathcal{U}(h_0) = \mathcal{U}(C)$ and $\mathcal{U}(0) = \mathcal{U}(C)u_\aleph$, where $u_\aleph := s_1 s_0^* + t_1 t_0^*$ with the isometries s_1, t_1 of Lemma 4.4.7(vi).

If $[k] = [k']$, then $[k + \beta h_0] = [k' + \beta h_0]$ by Lemma 4.4.7(viii), and the unitary equivalence can be realized by a unitary $w \in \mathcal{U}_0(E)$ with $w^*(k + \beta h_0)w = k' + \beta h_0$.

It implies that

$$\mathcal{U}(k') = \mathcal{U}(k) \cdot (w \oplus_{s_0,t_0} 1)\,.$$

Notice that $w \oplus_{s_0,t_0} 1$ is a unitary in $1 + \mathcal{N}(H_0(D), J)$ and is in $\mathcal{U}_0(\mathcal{N}(H_0(D), J) + \mathbb{C}\cdot 1)$. In particular, $\mathcal{U}(k)$ and $\mathcal{U}(k')$ are contained in the same two-sided class of

$$\mathcal{U}(C)\backslash\mathcal{U}(\mathrm{Der}(H_0(D), J))/\mathcal{U}_0(\mathcal{N}(H_0(D), J) + \mathbb{C}\cdot 1)\,.$$

It follows that $\mathcal{U}(k) \subseteq \mathcal{U}(C)\cdot\mathcal{U}(\mathcal{N}(H_0(D), J))$ if $[k] = [h_0]$.

The isometries s_1, t_1 of Lemma 4.4.7(vi) satisfy $p_0 t_1 = t_1 p_0 = p_0$, $t_1 \in C$, $H_0(\cdot)s_1 = 0$, $s_1^* H_0(\cdot) = 0$, $H_0(\cdot)t_1 = H_0$, and $t_1^* H_0(\cdot) = H_0$. It yields $t_1 a t_1^* = a$ for all $a \in s_0 E s_0^*$, and $t_1(1 - p_0)H_0(\cdot)t_1^* = (1 - p_0)H_0(\cdot)$.

Let $u \in \mathcal{U}(E)$, then

$$u_1 := u^* \oplus_{s_1,t_1} u := s_1 u^* s_1^* + t_1 u t_1^* \tag{4.23}$$

has the properties

$$u_1^* H_0(\cdot) u_1 = t_1 u^* H_0(\cdot) u t_1^* \qquad \text{and} \quad u_1 \in \mathcal{U}_0(E)\,.$$

The above equations together imply for $u \in \mathcal{U}(k) \subseteq \mathcal{U}(E)$ in Equation (**??**), that the unitary $u_1 \in \mathcal{U}_0(E)$ defined by (4.23) again satisfies Equation (**??**):

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u_1^* H_0(\cdot) u_1\,.$$

It follows that $u_1 \in \mathcal{U}(k) \cap \mathcal{U}_0(E) \neq \emptyset$ if $[k] \in S(h_0\,;\, D, E)$ and $[k] + [H_0] = [H_0]$.

We seek for an estimate of the set $\mathcal{U}(k_1 \oplus_{s,t} k_2) \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J))$ for $k_j \colon D \to E$ with $[k_j] \in S(h_0\,;\, D, E)$ and $[k_j] + [H_0] = [H_0]$, $j = 1, 2$.

New generators S, T of $\mathcal{O}_2$ come from the generators (s, t) and (s_0, t_0) in Theorem 4.4.6(i,iv) by

$$S := s_0 s s_0^* + t_0 s t_0^* \quad \text{and} \quad T := s_0 t s_0^* + t_0 t t_0^* \tag{4.24}$$

Then $S, T \in C$: use $\beta(s) = s$, $\beta(t) = t$, $s, t \in h_0(D)' \cap H_0(D)' \cap E$,

$SH_0 = S s_0 (h_0 + \beta h_0) s_0^* + S t_0 H_0 t_0^* = s_0 (s h_0 + s \beta h_0) s_0^* + t_0 s H_0 t_0^* = s_0 (h_0 s + (\beta h_0) s) s_0^* + t_0 H_0 s t_0^* = H_0 S$. Similarly, $TH_0 = H_0 T$.

Since $S s_0 = s_0 s$, $T s_0 = s_0 t$, $S p_0 = p_0 S$, $T p_0 = p_0 T$ and

$$(\beta h_0) \oplus_{s,t} (\beta h_0) = \beta(h_0 \oplus_{s,t} h_0) = \beta h_0\,,$$

we get for $k_1, k_2 \colon D \to J$ that

$$\big((k_1 + \beta h_0) \oplus_{s_0,t_0} H_0\big) \oplus_{S,T} \big((k_2 + \beta h_0) \oplus_{s_0,t_0} H_0\big) \;=\; \big((k_1 \oplus_{s,t} k_2) + \beta h_0\big) \oplus_{s_0,t_0} H_0\,.$$

If $k_1, k_2 \in S(h_0\,;\, D, E)$ with $[k] + [H_0] = [H_0]$ and $[k'] + [H_0] = [H_0]$, then for $k_3 := k_1 \oplus_{s,t} k_2$ holds $[k_3] = [k_1] + [k_2]$ and $[k_3] + [H_0] = [H_0]$.

The class $\mathcal{U}(k_3)$ is given by

$$\mathcal{U}(k_1 \oplus_{s,t} k_2) = \mathcal{U}(C) \cdot (\mathcal{U}(k_1) \oplus_{S,T} \mathcal{U}(k_2))\,,$$

because, for $u_1 \in \mathcal{U}(k_1)$ and $u_2 \in \mathcal{U}(k_2)$

$$\big(k_1 \oplus_{s,t} k_2) + \beta h_0\big) \oplus_{s_0,t_0} H_0 = (u_1 \oplus_{S,T} u_2)^* H_0(\cdot)(u_1 \oplus_{S,T} u_2)\,.$$

It follows that

$$\mathcal{U}(k \oplus_{s,t} h_0) = \mathcal{U}(C) \cdot (\mathcal{U}(k) \oplus_{S,T} 1)\,,$$

and, if $[k] + [h_0] = [k'] + [h_0]$, that there is a unitary $u_0 \in \mathcal{U}_0(E)$ with

$$u_0^*(k \oplus_{s,t} h_0) + \beta h_0) u_0 = k' \oplus_{s,t} h_0) + \beta h_0,$$

i.e., then

$$\mathcal{U}(k' \oplus_{s,t} h_0) = \mathcal{U}(C) \cdot (\mathcal{U}(k) \oplus_{S,T} 1) \cdot (u_0 \oplus_{s_0,t_0} 1)\,.$$

Since

$$u_0 \oplus_{s_0,t_0} 1 \in \mathcal{U}_0(E) \oplus_{s_0,t_0} 1 \subseteq \mathcal{U}_0(\mathcal{N}(H_0(D), J) + \mathbb{C} \cdot 1)\,,$$

this implies in particular

$$\mathcal{U}(k' \oplus_{s,t} h_0) \subseteq \mathcal{U}(C) \cdot (\mathcal{U}(k) \oplus_{S,T} 1) \cdot \mathcal{U}_0(\mathcal{N}(H_0(D), J) + \mathbb{C} \cdot 1)\,.$$

We show below that the weaker condition

$$\mathcal{U}(k') \subseteq \mathcal{U}(C) \cdot (\mathcal{U}(k) \oplus_{S,T} 1) \cdot \mathcal{U}_0(\mathcal{N}(H_0(D), J + \beta J) + \mathbb{C} \cdot 1)\,,$$

implies conversely that

$$[k] + [h_0] = [k'] + [h_0]\,.$$

It follows the equivalence of

$$[k] + [h_0] = [h_0] = [h_0] + [h_0]$$

and

$$\mathcal{U}(k) \subseteq \mathcal{U}(C) \cdot \mathcal{U}_0(\mathcal{N}(H_0(D), J + \beta J) + \mathbb{C} \cdot 1)\,.$$

If we let ????????, then we can find a unitary $u \in E$ that satisfies Equation (**??**) with the additional condition

$$1 \oplus_{S,T} u \in \mathcal{U}_0(E)\,.$$

`Next also mentioned above ?`

Let $(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot) u$. Since $p_0 H_0(\cdot) = s_0(h_0(\cdot) + \beta h_0(\cdot)) s_0^*$ by assumption (iv) of Theorem 4.4.6, we get that

$$u^* H_0(a) u - H_0(a) = s_0(k(a) - h_0(a)) s_0^* \in J \quad \text{for all} \quad a \in D\,.$$

It follows that $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$, $u(1-p_0) \in C$ and $up_0u^* H_0(D) \subseteq J + \beta J$.

Since $1 - up_0u^* = u(1-p_0)^2 u^* \in C$, the map

$$a \in D \mapsto up_0u^* H_0(a)$$

is a C^*-morphism from D into $J + \beta(J)$, because $p_0 u^* H_0(a) u - p_0 H_0(a) \in J$ and $p_0 H_0(a) \in J + \beta(J)$ for all $a \in D$.

Let $(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot) u$. It implies $u \in \mathrm{Der}(H_0(D), J)$.

We get (see above)

$$((k \oplus_{s,t} h_0)?????\beta h_0) \oplus_{s_0,t_0} H_0 = U^* H_0(\cdot) U$$

with $U := u \oplus_{S,T} 1$.

Suppose now more generally that $u \in \mathrm{Der}(H_0(D), J + \beta J)$ is given with the additional property and $u \oplus_{s,t} 1 = wv$ with unitaries $w \in C$ and $v = 1 + x$ with $x \in \mathcal{N}(H_0(D), J + \beta J)$.

Let $W := sS^* + tT^*$. Since $s, t, S, T \in C$ it follows that $W \in C$ and $u \oplus_{S,T} 1 = W^*(u \oplus_{s,t} 1) W$.

We get $u \oplus_{S,T} 1 = W^* w W \cdot W^* v W$, and $W^* v W = 1 + y$ with $W^* w W \in C$ and $y = W^* x W \in \mathcal{N}(H_0(D), J + \beta J)$.

Thus, $U^* H_0(a) U = (1 + y^*) H_0(a) (1 + y)$. Since $H_0(D) y \subseteq J + \beta J$ and D is separable, there is a separable C^*-subalgebra $B \subseteq J + \beta J$ that contains $H_0(D) y$. Let $f \in B_+$ is a strictly positive contraction B. By assumptions (iii), (iv) and (v) of

Theorem 4.4.6 there exists a projection $q \in F \subseteq C$ with $qf = f$, $q \geq p_0$ and $q \neq p_0$. It follows that $qH_0(a)y = H_0(a)y = H_0(a)yq$. Thus $qH_0(a)(1+y) = H_0(a)(1+y)q$ and $qU^*H_0(a)U = (W^*vW)^*qH_0(a)W^*vW$.

Since $q = p_0 + r$ with $r = q - p_0 = q(1-p_0) \leq t_0t_0^*$, $r \in E$, it follows for

???????

that

$$s_0((k \oplus_{s,t} h_0) + \beta h_0)s_0^* + rH_0(\cdot) = (W^*vW)^*qH_0(a)W^*vW\,.$$

There are unitaries $u_1, u_2 \in 1 + F$ such that $u_1ru_1^* = p_0$ and $u_2qu_2^* = p_0$. We get $rH_0(\cdot) = u_1^*p_0H_0(\cdot)u_1$ and $qH_0(\cdot) = u_2^*p_0H_0(\cdot)u_2$. Since $p_0H_0(\cdot) = s_0(h_0+\beta h_0)s_0^* = (h_0 + \beta h_0) \oplus_{s_0,t_0} 0$ and $J \cdot \beta J = 0 = J \cap \beta J$, we obtain for the summands with values in J that

$$s_0(k \oplus_{s,t} h_0)s_0^* + u_1^*s_0h_0(\cdot)s_0^*u_1 = (W^*vW)^*u_2^*s_0h_0(a)s_0^*u_2W^*vW\,,$$

with $p_0u_1^*s_0 = 0$. Can define $V_1 \in \mathcal{U}(E)$ such that

$$t_1V_1^*s_0h_0(\cdot)s_0^*V_1t_1^* = u_1^*s_0h_0(\cdot)s_0^*u_1 = rH_0(\cdot)$$

It follows that

$$[k \oplus_{s,t} h_0] + [h_0] + [0] = [h_0] + [0]\ .$$

`From` $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ `to` (k, uw):

Recall that $F \subseteq C \cap \mathcal{N}(H_0(D), J)$ in the following calculations.

SURJECTIVITY:

Let $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$, then there exists an element $g \in F$ and a C^*-morphism $k\colon D \to J$ with $[k] \in S(h_0\,;\, D, E)$, such that $w := 1 + g \in \mathcal{U}_0(F + \mathbb{C}\cdot 1)$ and

$$(uw)^*H_0(\cdot)uw = (k + \beta h_0) \oplus_{s_0,t_0} H_0\,.$$

Indeed: Let $\partial_u(x) := ux - xu$ for $x \in E$. Then $u^*\partial_u(H_0(D)) \subseteq J$. and generates a separable C^*-subalgebra B of J. Let $b_0 \in B_+$ a strictly positive contraction of B and $f := b_0 + \beta(b_0)$. By assumptions 4.4.6(iii,iv,v) there are projections $p, q \in F$ with $p_0 \leq q \leq p$ with $qf = f$, $p_0 \neq q$, $q \neq p$. By Lemma 4.4.7(ix) there exists $w \in \mathcal{U}_0(F + \mathbb{C}1)$ such that $q = wp_0w^*$. Since $w \in \mathcal{U}_0(C) \subseteq \mathrm{Der}(H_0(D), J)$, we get $uw \in \mathrm{Der}(H_0(D), J)$ and $[uw] = [u] \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$.

Then $(1-q)u^*H_0(a)u = (1-q)H_0(a)$ and $w \in \mathcal{U}_0(C)$. Thus $(1-p_0)(uw)^*H_0(a)(uw) = (1-p_0)H_0(a)$ and

$$h(a) := s_0^*p_0(uw)^*H_0(a)(uw)s_0$$

is a C^*-morphism. Since uws_0 is an isometry we get $[h] \in S(H_0\,;\, D, E)$. The C^*-morphism h satisfies

$$h \oplus_{s_0,t_0} H_0 = (uw)^*H_0(\cdot)(uw)\,,$$

and $h(a)-s_0^*H_0(a)s_0 \in J$ for $a \in D$. Since $s_0^*H_0(a)s_0 = h_0(a)+\beta h_0(a)$ by Theorem 4.4.6(iv), $h = k + \beta h_0$ with $[k] \in S(H_0\,;\, D, E)$ and $k(D) \subseteq J$, and by Lemma 4.4.7(vii) that $[k] \in S(h_0\,;\, D, E)$, $uw \in \mathcal{U}(k)$, and $[uw] = [u] \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$. In particular,

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = (uw)^* H_0(\cdot)(uw)\,.$$

`Where is next needed??`

The C^*-morphism $h'\colon a \in D \mapsto h'(a) := upu^*H_0(a) = us_0h(\cdot)s_0^*u^*$ maps D into J, therefore it satisfies $[h'] = [h] + [0] = [h]$, $[h'] = [k + \beta h_0]$ and $h' \oplus_{s_0,t_0} H_0 = u_1^* H_0(\cdot)u_1$ for some unitary $u_1 \in \mathrm{Der}(H_0(D), J + \beta(J))$ such that $u_1 := v_1uw_1$ for some $v_1 \in C$ and $w_1 \in \mathcal{N}(H_0(D), J + \beta(J))$.

`Next: From (DC) to (wDC)?`

We consider now the special case, where $u \in \mathcal{U}_0(E)$ and $k \in S(h_0; D, E)$ satisfy

$$(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot)u\,,$$

and suppose in addition that $u \in \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J + \beta J))$.

Let B_0 the separable C^*-algebra of J that is generated by $k(D) \cup h_0(D)$, $g \in B_0$ a strictly positive contraction of B_0. By assumption (iii) of Theorem 4.4.6 there exists $q_1 \in F$, $q_1 \geq p_0$ with $q_1(g + \beta g) = g + \beta g$ and $q_1 \neq p_0$.

Above we have seen that $(k + \beta h_0) \oplus_{s_0,t_0} H_0 = u^* H_0(\cdot)u\,.$ implies $u \in \mathrm{Der}(H_0(D), J)$.

We suppose now that a given $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ satisfies the assumption of the decomposition condition (DC), i.e., that there is $v \in C$ such that $w := v(u \oplus_{s,t} 1) \in 1 + \mathcal{N}(H_0(D), J + \beta(J))$.

This implies $(u \oplus_{s,t} 1) \in \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J + \beta J))$.

We show that the latter implies that $[k] + [h_0] = [h_0]$:

`Recall:`

(DC) If $u \in \mathcal{U}_0(E)$ satisfies $uH_0(a) - H_0(a)u \in J$ for all $a \in D$, then there exists a unitary $v \in C := H_0(D)' \cap E$ such that

$$v(u \oplus_{s,t} 1) \in \mathcal{N}(H_0(D), J + \beta(J)) + \mathbb{C} \cdot 1\,.$$

THIS IMPLIES:

If $u \in \mathcal{U}_0(E)$ and $u \in \mathrm{Der}(H_0(D), J)$ then

$[u] = [u \oplus 1] = [w] - [v]$ with $v \in \mathcal{U}(C)$ and $w \in \mathcal{U}(\mathcal{N}(H_0(D), J))$.

Thus, the kernel of $\mathrm{K}_1(\mathrm{Der}(H_0(D), J)) \to \mathrm{K}_1(E)$ is contained in $\gamma(\mathrm{K}_1(C))$.

We know that $\mathrm{K}_1(C) \to \mathrm{K}_1(E)$ is surjective, cf. . ?????????, because H_0 dominates (absorbs) zero.

Let $q_1 \geq p_0$ as above. Then $w = 1 + y$ for some $y \in \mathcal{N}(H_0(D), J + \beta(J))$. There is a separable C^*-algebra $B \subseteq J$ such that $B + \beta(B)$ contains $H_0(D)y \cup yH_0(D)$. If

$b_0 \in B_+$ is a strictly positive contraction for B, then there exists a projection $q_2 \in F \subseteq C$ with $q_2 \geq q_1$, $q_2 \neq q_1$ and $q_2(b_0 + \beta(b_0)) = b_0 + \beta(b_0)$ by Theorem 4.4.6(iii). It follows $q_2 H_0(a) y = H_0(a) y = H_0(a) y q_2$, $q_2 H_0(a) w = H_0(a) q_2 + q_2 H_0(a) y = H_0(a) w q_2$ and $q_2 w H_0(a) = w H_0(a) q_2$ in the same way. Moreover $q_2 - q_1 \in F$ is a non-zero projection and there is a unitary $z_3 \in (1 + F) \cap \mathcal{U}_0(F + \mathbb{C} \cdot 1)$ with $z_3^*(q_2 - q_1) z_3 = p_0$.

Let $z_1, z_2 \in 1 + F$ unitaries with $z_1^* q_1 z_1 = p_0$ and $z_2^* q_2 z_2 = p_0$, cf. assumption (v) of Theorem 4.4.6.

We get:

??? Then $p_0 H_0(\cdot) z^* w z = H_0(\cdot) z^* w z p_0$.

`New approach:`

The crucial points are:

$$q_2 u^* H_0 u = q_2 w^* H_0 w = w^* q_2 H_0(\cdot) w$$

and

$$q_2((k + \beta h_0) \oplus_{s_0, t_0} H_0) = s_0(k + \beta h_0) s_0^* + (q_2 - p_0) H_0$$

and that $q_2 H_0$ and $(q_2 - p_0) H_0$ are unitarily equivalent to $p_0 H_0 = s_0(h_0 + \beta h_0) s_0^*$ by unitaries $v_1, v_2 \in C$: Say $q_2 H_0 = v_1^* p_0 H v_1$ and $(q_2 - p_0) H_0 = v_2^* p_0 H v_2$.

Since h_0 dominates zero, we get that $[q_2 u^* H_0 u] = [h_0 + \beta h_0]$. Let

$$h_1(a) := t_0^*(q_2 - p_0) H_0(a) t_0 = t_0^* v_1 p_0 H_0(a) v_1^* t_0 \,.$$

Then $t_0 h_1(a) t_0^* = (q_2 - p_0) H_0(a)$. Thus $[k + \beta h_0] + [h_1] = [h_0 + \beta h_0]$.

We show that h_1 dominates zero:

By Theorem 4.4.6(iii) – with $f := 0$ – there is a projection $q_3 \in F$ with $q_2 \leq q_3$ and $0 \neq q_4 = q_3 - q_2 \leq 1 - p_0 = t_0 t_0^*$. It follows that the projection $q_5 := t_0^* q_4 t_0$ is Murray–von-Neumann equivalent to $q_4 = t_0 q_5 t_0^*$ and satisfies $h_1(\cdot) q_5 = 0$. The assumption (v) of Theorem 4.4.6 allows to find a unitary $u_4 \in 1 + F \subseteq C$ with $u^* q_4 u = q_0 = s_0 s_0^*$. The element $T := s_0^* u^* t_0 q_5 \in E$ satisfies $T^* T = 1$ and $T T^* = q_5$. Thus, $T^* h_1(\cdot) T = 0$, and h_1 dominates zero.

It follows $[0] + [h_1] = [h_1]$, i.e.,

$$t_0 h_1(\cdot) t_0^* = (q_2 - p_0) H_0(\cdot) = v_2^* p_0 H_0(\cdot) v_2$$

is unitarily equivalent to h_1 by Proposition 4.3.5(i). We obtain

$$[h_1] = [p_0 H_0(\cdot)] = [0] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$$

by Lemma 4.4.7(ii). It yields $[k + \beta h_0] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$. Summing up, we get the existence of a unitary $U \in E$ with

$$s_0(k + \beta h_0) s_0^* + t_0(h_0 + \beta h_0) t_0^* = U^*(h_0 + \beta h_0) U \,.$$

Since $k(D) \cup h_0(D) \subseteq J$ and $J \cap \beta(J) = \{0\}$, it follows that $s_0 k(a) s_0^* + t_0 h_0(a) t_0^* = U^* h_0(a) U$ for all $a \in D$, i.e., $[k] + [h_0] = [h_0]$.

`Next Part of proof not complete !!! ??`

The plan (!!!) is to apply assumption 4.4.6(iii) to suitable separable C^*-subalgebras $M \subseteq J + \beta J$ to get projection $p, q \in F$ with $p_0 \leq p$, $p_0 \leq q$ and $p_0 \neq p$, such that there exists a unitary $V \in E$ with the (needed) property

$$V^* H_0(\cdot) q V = s_0(k(\cdot) + \beta h_0(\cdot)) s_0^* + (p - p_0) H_0(\cdot)\,.$$

By Part (v), there are unitaries U_1, U_2 and U_3 in $1 + F \subseteq C$ with $U_1^* p_0 U_1 = q$, $U_2^* p_0 U_2 = p - p_0$ and $U_3^* p_0 U_3 = p$. We get that

$$U_4^*(H_0(\cdot)p_0)U_4 = (k + \beta h_0) \oplus_{s_0,R} ((U_2^*(H_0(\cdot)p_0)U_2 \oplus_{R,s_0} 0)$$

where $U_4 := U_1 V$, $R \in C$ is an isometry with with $RR^* = 1 - p_0$ and $R(p - p_0) = (p - p_0)R = (p - p_0)$, e.g. $R := U_3^*(1 - p_0)U_2 + (p - p_0)$. It implies

$$[h_0 + \beta h_0] + [0] = [k + \beta h_0] + [h_0 + \beta h_0] + [0]\,.$$

Since $h_0 + \beta h_0$ absorbs zero, we get $[h_0 + \beta h_0] = [k + \beta h_0] + [h_0 + \beta h_0]$ on unitaries equivalence classes (with Cuntz addition outside the brackets $[.]$ and ordinary addition of linear maps inside $[.]$). The unitary equivalence of $h_0 + \beta h_0$ and $(k + \beta h_0) \oplus (h_0 + \beta h_0) = (k \oplus h_0) + (\beta h_0 \oplus \beta h_0)$ induces a unitary equivalence of h_0 and $k \oplus h_0$ because $J \cap \beta(J) = 0$. This means $[k] + [h_0] = [h_0]$.

□

Lemma 4.4.16. *The assumptions* (i)–(vi) *in Theorem 4.4.6 imply the following properties* (i)-(iv)*.*

(i) *There exist isometries $s_2, t_2 \in E$ with $s_2 s_2^* + t_2 t_2^* = 1$, $t_2 \in C$, $s_2^* H_0(\cdot) s_2 = h_0$, $p_2 := s_2 s_2^* \leq p_0$.*

Moreover, p_2 is a properly infinite projection in the ideal $C \cap \mathcal{N}(H_0(D), J)$ of C with $0 = [p_2]$ in $\mathrm{K}_0(C \cap \mathcal{N}(H_0(D), J))$, i.e., there exists a unital C^*-morphism from $\mathcal{O}_2$ into $p_2 C p_2$.

The isometry s_2 satisfies also $\beta(s_2)^* H_0(\cdot) \beta(s_2) = \beta h_0$.

(ii) *The projection p_0 is the sum $p_0 = q + r$ of properly infinite projections $q, r \in C$ with $q \in \mathcal{N}(H_0(D), J)$ and $r \in \mathcal{N}(H_0(D), \beta J)$. For each decomposition $p_0 = q + r$ of this kind holds $[qH_0(\cdot)] = [h_0]$ and $[rH_0(\cdot)] = [\beta h_0]$.*

(iii) *If $e \in E$ satisfies $e^* H_0(D) e \subseteq J + \beta J$ then there exist elements $x, y, z \in E$ with $\max\{\|x\|, \|y\|, \|z\|\} \leq \|e\|$ and a unitary $w \in 1 + F$ such that $e = wqx + wry + z$, $H_0(D)z = \{0\}$, $z^*(wqx) = 0$ and $z^*(wqy) = 0$, where $q, r \in C$ are projections with $p_0 = q + r$ and $q, \beta(r) \in \mathcal{N}(H_0(D), J)$.*

(iv) *Let $T_1, \ldots, T_n \in E$ such that $k(a) := \sum_{j=1}^n T_j^* H_0(a) T_j \in J + \beta J$ for all $a \in D$. Then there exist $S_1, S_2 \in E$, such that $k = k_1 + \beta k_2$ with $k_\ell(a) = S_\ell^* h_0(a) S_\ell$ for $a \in D$, $\ell \in \{1, 2\}$.*

In particular, if $[k] \in S(H_0\,;\, D, E)$ and $k(D) \subseteq J + \beta J$, then $k = k_1 + \beta k_2$ with unique $k_1, k_2 \in S(h_0\,;\, D, E)$.

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PROOF. (i): By Lemma 4.4.7(ii), there exists a unitary $w \in E$ with

$$w^*(h_0 + \beta h_0)w = sh_0s^* + t\beta h_0t^* .$$

The isometry $T := ws \in E$ satisfies $T^*(h_0 + \beta h_0)T = h_0$. Let $s_2 := s_0T$. Then $p_2 := s_2s_2^* \leq p_0 = s_0s_0^*$ and $s_2^*H_0(\cdot)s_2 = h_0$. Thus $p_2H_0(D) = s_2h_0(D)s_2^* \subseteq J$, $p_2 \in C$, $p_2Cp_2 = s_2(h_0(D)' \cap E)s_2^*$ and $p_2 \in \mathcal{N}(H_0(D), J)$.

We get $p_2Cp_2 \subseteq C \cap \mathcal{N}(H_0(D), J)$, because $p_2cp_2 \leq \|c\|p_2$ for $c \in C_+$, $p_2 \in C \cap \mathcal{N}(H_0(D), J)$ and because $\mathcal{N}(H_0(D), J) = (\pi_J)^{-1}(\mathrm{Ann}(\pi_J(H_0(D)), J))$ is a hereditary C^*-subalgebra of E.

In particular, $s_2ss_2^*$ and $s_2ts_2^*$ generate a unital copy $s_2C^*(s,t)s_2^*$ of $\mathcal{O}_2$ that is unitally contained in the corner p_2Cp_2 of the closed ideal $C \cap \mathcal{N}(H_0(D), J)$ of C.

Since there is a unital C^*-morphism from $\mathcal{O}_2$ into p_2Cp_2 – with unit p_2 –, p_2 is properly infinite in $C \cap \mathcal{N}(H_0(D), J)$ and $[p_2] = 0$ in $\mathrm{K}_0(C \cap \mathcal{N}(H_0(D), J))$.

It follows that $[1 - p_2] = [1] = 0$ in $\mathrm{K}_0(C)$.

Since $t_0t_0^* \leq 1 - p_2$ and $t_0 \in C$, we get that $1 - p_2$ is full and properly infinite in C. By Lemma 4.2.6(ii), 1 and $1 - p_2$ are MvN-equivalent in C, i.e., there exists an isometry $t_2 \in C$ with $t_2t_2^* = 1 - s_2s_2^*$.

The equation $\beta(s_2)^*H_0(\cdot)\beta(s_2) = \beta h_0$ holds, because $\beta H_0 = H_0$. The corner $\beta(p_2)C\beta(p_2)$ of C is a corner of the ideal $C \cap \mathcal{N}(H_0(D), \beta(J))$ of C.

(ii): Let s_2 as in Part (i). Since

$$[h_0] + [\beta h_0] = [h_0 \oplus_{s,t} \beta h_0] = [h_0 + \beta h_0]$$

by Lemma 4.4.7(ii), and since $s_0^*H_0(\cdot)s_0 = h_0 + \beta h_0$ by assumption (iv) of Theorem 4.4.6, there exists a unitary $w \in E$ such that $z = (s_2 \oplus_{s,t} \beta(s_2))ws_0^*$ is a partial isometry in C with $z^*z = p_0$ and $zz^* = sp_2s^* + t\beta(p_2)t^*$ – compare Lemma 4.3.4(ii) and proof of Proposition 4.4.3(i).

The inequality $ss^* + t\beta(p_2)t^* \leq 1$, implies $s(1 - p_2)s^* \leq 1 - zz^*$. Using that $p_2 \leq p_0$, and that $1 - p_0 = t_0t_0^*$ and $s(1 - p_0)s^* = st_0t_0^*s^*$ are properly infinite projections in C, we obtain that $1 - zz^*$ is properly infinite in C. Since $1 - zz^*$ is properly infinite and $[1 - zz^*] = [1 - p_0] = [t_0t_0^*] = [1]$ in $\mathrm{K}_0(C)$, Lemma 4.2.6(ii) applies to $1 - zz^*, 1 \in C$, and there exists an isometry $T \in C$ with $TT^* = 1 - zz^*$.

The element $W := z + Tt_0^*$ is a unitary in C with

$$p_0 = (z^*z)^2 = W^*zz^*W = W^*(sp_2s^* + t\beta(p_2)t^*)W .$$

Hence $q := W^*sp_2s^*W = (W^*ss_2)(W^*ss_2)^*$ and $r := W^*t\beta(p_2)t^*W$ are properly infinite projections in C that are MvN-equivalent in C to p_2 and $\beta(p_2)$, and satisfy $p_0 = q + r$.

Since $\beta H_0 = H_0$ and $s^*W, t^*W \in C$, we get $q \in \mathcal{N}(H_0(D), J)$ and $r \in \mathcal{N}(H_0(D), \beta J)$, because $p_2 \in \mathcal{N}(H_0(D), J)$.

If $p_0 = q + r$ with $q \in \mathcal{N}(H_0(D), J)$ and $r \in C \cap \mathcal{N}(H_0(D), \beta J)$ are given, then $k_1(a) := qH_0(a)$ and $k_2(a) := rH_0(a)$ satisfy, by assumption (iv) of Theorem 4.4.6, that

$$k_1(a) + k_2(a) = p_0 H_0(a) = s_0(h_0(a) + \beta h_0(a)) s_0^* \,.$$

Since $J \cap \beta J = \{0\}$ and $\beta(s_0) = s_0$ it follows that $k_1 = s_0 h_0(\cdot) s_0^* = \beta k_2$, i.e., $[k_1] = [\beta k_2] = [h_0] + [0] = [h_0]$.

(iii): Let $e \in E$ and $e^* H_0(a^* a) e \in J + \beta J$ for all $a \in D$. Then $H_0(D) e \subseteq J + \beta J$. Let $f \in J + \beta J$ a strictly positive element of the separable C^*-subalgebra A of $J + \beta J$ that is generated by $H_0(D) e$.

There exists a projection $p \in F$ with $p_0 \leq p$ and $pf = f$ by assumption (iii) of Theorem 4.4.6. In particular $H_0(D)(1 - p)e = \{0\}$.

By assumption (v) of Theorem 4.4.6, there is a unitary $w \in 1 + F$ with $w p_0 w^* = p$. Let $z := (1-p)e$, $x := q w^* e$, and $y := r w^* e$ for projections $q, r \in C$ with $p_0 = q + r$, $q, \beta(r) \in \mathcal{N}(H_0(D), J)$. Then $\max\{ \|x\|, \|y\|, \|z\| \} \leq \|e\|$ and $wqx + wry + z = e$, and $z^* w q = 0 = z^* w r$ follows from $(1-p) w p_0 = 0$.

(iv): Let $k \colon D \to E$ with $[k] \in S(H_0\,; D, E)$ and $k(D) \subseteq J + \beta J$. By definition of $S(H_0\,; D, E)$, there exists an isometry $T \in E$ with $T^* H_0(\cdot) T = k$.

More generally, if $T_1, \ldots, T_n \in E$ such that $k(a) := \sum_{j=1}^n T_j^* H_0(a) T_j \in J + \beta J$ for all $a \in D$, then $k(a) = T^* H_0(a) T$ for $T := \sum_{j=1}^n S_j T_j$ with isometries $S_1, \ldots, S_n \in C^*(s,t) \subseteq H_0(D)' \cap E$. In particular, $T^* H_0(D) T \subseteq J + \beta J$. By Part (ix), there exists a unitary $w \in 1 + F \subseteq C$, and $x, y, z \in E$ with norms $\leq \|T\|$ such that $T = wqx + wry + z$ and $H_0(D) z = \{0\}$, where $q, r \in C$ are projections with $q, \beta(r) \in \mathcal{N}(H(D), J)$. It follows $T^* H_0(\cdot) T = x^* q H_0(\cdot) x + y^* r H_0(\cdot) y$.

The c.p. maps $k_1 := x^* q H_0(\cdot) x$ and $k_0 := y^* r H_0(\cdot) y$ satisfy $k = k_0 + k_1$. Let $k_2 := \beta k_0$, then $k = k_1 + \beta k_2$. Since $J \cap \beta(J) = \{0\}$ and $\beta^2 = \mathrm{id}$, the maps k_1 and k_2 are uniquely determined by the map $k = k_1 + \beta k_2$.

Now use that $[q H_0(\cdot)] = [h_0]$, $[r H_0(\cdot)] = [\beta h_0]$ by Lemma 4.4.16(ii). It means that there are unitaries $u_1, u_2 \in E$ with $q H_0(\cdot) = u_1^* h_0(\cdot) u_1$ and $r H_0(\cdot) = u_2^* \beta h_0(\cdot) u_2$.

It follows that $k_1 = S_1^* h_0(\cdot) S_1$ with $S_1 := u_1 x$ and $k_2 = S_2^* h_0(\cdot) S_2$ with $S_2 = \beta(u_2 y)$.

If $\|T\| \leq 1$ then $\|S_j\| \leq 1$ for $j = 1, 2$, and k is a c.p. contraction. If k is moreover multiplicative then k_1 and k_2 must be multiplicative, because $J \cdot \beta J = J \cap \beta J = \{0\}$. If k is multiplicative, then we can use that h_0 dominates zero by Lemma 4.4.7(v), and find, by Proposition 4.3.6(ii), isometries $T_1, T_2 \in E$ with $T_1^* h_0(\cdot) T_1 = k_1$ and $T_2^* h_0(\cdot) T_2 = k_2$. This means that $k_1, k_2 \in S(h_0\,; D, E)$, if, moreover, $\|T\| \leq 1$ and k is multiplicative. □

LEMMA 4.4.17. *The following properties of* $\mathrm{Der}(H_0(D), J)$*,* $C = H_0(D)' \cap E$ *and* $\mathcal{N}(H_0(D), J)$ *follow from assumptions* (i)-(vi) *of Theorem 4.4.6.*

(o) *If* $Q \in C$ *is a projection and* $k(a) := Q \cdot H_0(a)$ *for* $a \in D$*, then* $[k] \in S(H_0\,; D, E)$.

If, in addition, $k(D) \subseteq J + \beta(J)$*, then* $k = k_1 + \beta k_2$ *for unique* C^**-morphisms* $k_j : D \to E$ *with* $[k_j] \in S(h_0\,; D, E)$*.*

(i) *The two-sided normalizer algebra* $\mathcal{N}(H_0(D), J)$ *is a full hereditary* C^**-subalgebra of* E *and is an ideal of the* C^**-algebra* $\mathrm{Der}(H_0(D), J)$ *of derivations of* $H_0(D)$ *into* J*. It holds* $\mathrm{Der}(H_0(D), J) = C + \mathcal{N}(H_0(D), J)$ *.*

All this is also true for the ideal $J + \beta(J)$ *in place of* J*.*

(ii) *If* $k : D \to J$ *is given with* $[k] + [H_0] = [H_0]$*, then there exists* $u \in \mathcal{U}(E)$ *with*

$$u^* H_0(\cdot)u \;=\; s_2 k(\cdot)s_2^* + (1 - p_2)H_0(\cdot)\,,$$

$u - p_2 u p_2 \in C$ *and* $u \in \mathrm{Der}(H_0(D), J)$*.*

With $v := u(s_2 s_0^* + t_2 t_0^*)$ *holds* $v^* H_0(a)v = s_0 k(a)s_0^* + (1 - p_0)H_0(a)$ *for all* $a \in D$*, and* $v - p_0 v p_0 \in C$ *and* $v \in \mathrm{Der}(H_0(D), J + \beta J)$*, i.e.,*

$$v^* H_0(a)v - H_0(a) \;\in J + \beta(J) \quad \forall a \in D\,.$$

The above described u *and* v *in* $\mathcal{U}(E)$ *are uniquely defined by* $k : D \to J$ *up to left-multiplication by unitaries in* C*.*

Compare with (xv) ???

If $[k] + [h_0] = [h_0]$ *in* $S(h_0\,; D, E)$*, then* $u = w_1 u_1$ *with* $u_1 \in 1 + \mathcal{N}(H_0(D), J)$ *and* $w_1 \in C$*.*

(iii) *If* $G \subseteq \mathcal{U}(\mathrm{Der}(H_0(D), J + \beta J))$ *is a separable subgroup of the unitaries in* $\mathrm{Der}(H_0(D), J + \beta J)$*, then there exist a projection* $p \in F \subseteq C$ *and a unitary* $w \in 1 + F$*, such that* $p \geq p_0$*,* $p \neq p_0$*,* $w^* p w = p_0$ *and* $(1 - p)u \in C$ *for all* $u \in G$*.*

For each $u \in G$ *there are* C^**-morphisms* $k_1^{(u)}, k_2^{(u)} \in S(h_1\,; D, E)$ *such that*

$$w^* u^* H_0(\cdot)uw = (k_1^{(u)} + \beta k_2^{(u)}) \oplus_{s_0, t_0} H_0\,.$$

If $u \in \mathrm{Der}(H_0(D), J)$ *then* $k_2^{(u)} = h_0$*.*

(iv) *If* G *is a separable subgroup of* $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ *then there exist isometries* $S, T \in E$ *with* $TT^* + SS^* = 1$*,* $S^* u^* H_0(\cdot)uS = 0$ *and* $Tu^* H_0(\cdot)u = u^* H_0(\cdot)u = u^* H_0(\cdot)uT$ *for all* $u \in G$*.*

(v) *If* $u \in \mathrm{Der}(H_0(D), J)$ *is unitary, and* $S, T \in E$ *are the isometries from Part (iv) for a separable subgroup* G *of* $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ *that contains* u*, then the unitary* $W := Su^* S^* + TuT^* \in \mathcal{U}_0(E)$ *satisfies* $(W^*)^k H_0(\cdot)W^k = (u^*)^k H_0(\cdot)u^k$ *for all* $k \in \mathbb{Z}$*.*

In particular, $W^k (u^*)^k \in C$ *for all* $k \in \mathbb{Z}$ *and* $[W]_E = 0$ *in* $\mathrm{K}_1(E)$*.*

(vi) *Next calculation needed?*

For $u \in \mathrm{Der}(H_0(D), J)$ *and* S, T *from (iii) – depending on a chosen closed separable subgroup* G *of* $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ *that contains* u *– holds* $v[u]^* H_0(\cdot)v[u] = u^* H_0(\cdot)u$*, where*

$$v[u] := (TuT^* \,+\, SS^*)\,.$$

Two unitaries u_1 *and* u_2 *in* $G \subseteq \mathrm{Der}(H_0(D), J)$ *have same class* $[u_1] = [u_2]$ *in* $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$*, if and only if,* $v[u_1]$ *and* $v[u_2]$ *are homotopic in* $\mathcal{U}(\mathrm{Der}(H_0(D), J))$

????

Is it at least true modulo the closed ideal of $\mathrm{Der}(H_0(D), J)$ *generated by the annihilators of* $H_0(D)$*?*

Homotopy not in $\mathcal{U}(\mathrm{Der}(H_0(D), J))$??

Because the S is not in $\mathrm{Der}(H_0(D), J)$.

I.e., the (multiple-valued) map $u \mapsto v[u] \in \mathcal{U}(E)$ *has the property that* $v[u_1]^* v[u_2] \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$*, if and only if,* $u_1, u_2 \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ *have the same* K_1*-class in* $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$*.*

(vii) *If* $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$*, then there exists a unitary* $w \in 1 + F \subseteq C$ *and* $k \in S(h_0\,;\, D, E)$*, such that* $(uw)^* H_0(\cdot) uw = k \oplus_{s_0,t_0} H_0$*.*

Why not s_2, t_2 in place of s_0, t_0?

So far it is only shown that

$$(uw)^* H_0(\cdot) uv = (k + \beta h_0) \oplus_{s_0,t_0} H_0$$

for suitable $v \in \mathcal{U}(C)$ *and* $k \in S(h_0\,; D, E)$*.*

Notice that then one can replace uv *by* $v^* uv$*. Is* $u^* v^* uv \in \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J))$*.*

Is $1 \oplus w \in \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J))$ *if* $[w] = 0 \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$*?*

What about next? Typo?

(viii) *For every* $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ *there exist* $w \in \mathcal{U}(C)$ *and* $k \in S(h_0; D, E)$ *with* $w^*(u^* H_0(\cdot) u) w = k \oplus_{s_0,t_0} H_0$*, such that* $1 - uw \in \mathcal{N}(H_0(D), J) u$

?????

?? What is required above? Typos?

(ix) *The unitary group of* $\mathrm{Der}(H_0(D), J)$ *is invariant under forming* $u_1 \oplus_{s,t} u_2$ *(modulo multiplication with unitaries in* $\mathcal{N}(H_0(D), J) + 1$ *or in* C*).*

The sum $u_1 \oplus_{s,t} u_2$ *corresponds to*

$$(k_1 \oplus_{s,t} k_2) + \beta h_0$$

up to multiplication by a unitary $v \in \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J))$ *if* $k_1, k_2 \in S(h_0\,;\, D, E)$ *are morphisms with*

$$u_j^* H_0(\cdot) u_j = (k_j + \beta h_0) \oplus_{s_0,t_0} H_0\,.$$

(x) *If* $[v_1] = [v_2] \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$*, then* $[k_1 + \beta h_0] + [h_0] = [k_2 + \beta h_0] + [h_0]$*.*

(xi) *A* C^**-morphism* $k \colon D \to J$ *satisfies* $[k + \beta h_0] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$ *if and only if there exists a unitary* v *in* $1 + \mathcal{N}(H_0(D), J)$ *such that*

$$s_2 k(\cdot) s_2^* + (1 - p_2) H_0(\cdot) \;=\; v^* H_0(\cdot) v\,.$$

It implies $[k] + [H_0] = [H_0]$*.*

(xii) $\pi_J(u) \in \mathcal{U}_0\big(\pi_J(H_0(D))' \cap (E/J)\big)$*, if and only if, there is path* $v(t)$ *in* $\mathcal{U}_0(\mathrm{Der}(H_0(D), J))$ *with* $v(0) \in 1 + J \subseteq 1 + \mathcal{N}(H_0(D), J)$ *and* $v(1) = u$*.*

(xiii) *If* $u = u_1 u_2$ *with* u_1 *a unitary in* C *and* u_2 *a unitary in the unitization of* $\mathcal{N}(H_0(D), J)$ *then* $u^* H_0(\cdot) u = s_2 k(\cdot) s_2^* + (1 - p_2) H_0(\cdot)$ *for* $k \colon D \to J$ *implies that* $[k + \beta h_0] + [h_0] = [h_0 + \beta h_0]$*.*

(xiv) *If* $k \colon D \to J$ *and* $[k + \beta h_0] + [h_0] = [h_0 + \beta h_0]$*, then* $[k] + [h_0] = [h_0]$*.*

(xv) *If $k\colon D \to J$ and $u \in \mathcal{U}(E)$ are given with $u^* H_0(\cdot) u = k(\cdot) \oplus_{s_2,t_2} H_0(\cdot)$, then $u \in \mathrm{Der}(H_0(D), J)$.*

What about the unitary $W_0 := s_2^* s_0 + t_2^* t_0$ that transforms

$$k(\cdot) \oplus_{s_2,t_2} H_0(\cdot)$$

into $k(\cdot) \oplus_{s_0,t_0} H_0(\cdot)$??

Is NOT useful as transformation

If $u \in \mathrm{Der}(H_0(D), J)$ then there exists a unitary $w \in C := H_0(D)' \cap E$ with $0 = [wu] \in \mathrm{K}_1(E)$ ([8]).

Then $[k] + [h_0] = [h_0]$ if and only if there exist $v \in C$ with $vu \in \mathcal{N}(H_0(D), J) + 1$.

Part (xv) is shown in the below ????? given proof of Theorem 4.4.6.

What about the general relations between $u, u' \in \mathcal{U}(E)$ for k and k' with $[k'] + [h_0] = [k] + [h_0]$?

What about the unitary for $[k_1 + \beta h_0] + [k_2 + \beta h_0]$?

(xvi) *The set $P := \mathcal{U}(C) \cdot \mathcal{U}(\mathcal{N}(H_0(D), J) + \mathbb{C} \cdot 1)$ is an open subgroup of $\mathcal{U}(\mathrm{Der}(H_0(D), J))$. $\mathcal{U}(\mathcal{N}(H_0(D), J) + \mathbb{C} \cdot 1)$ is a closed normal subgroup of $\mathcal{U}(\mathrm{Der}(H_0(D), J))$.*

The set of $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ with the property that $u \oplus_{s,t} 1 \in P$ is an open subgroup of $\mathcal{U}(\mathrm{Der}(H_0(D), J))$.

Each element of $\mathrm{Der}(H_0(D), J)$, $\mathcal{N}(H_0(D), J) + \mathbb{C} \cdot 1$ and C are K_1-surjective and satisfy that $[u_1] = [u_2]$ if and only if $(u_1^* u_2) \oplus_{s,t} 1$ is in $\mathcal{U}_0(\mathrm{Der}(H_0(D), J))$ (respectively in $\mathcal{U}_0(\mathcal{N}(H_0(D), J))$, or $\mathcal{U}_0(C)$).

In particular:

If $x = [u] \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ for some $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ and if $x = \mu_1(y) + \mu_2(z)$ for some $y = [v_1] \in \mathrm{K}_1(C)$ and $z = [v_2] \in \mathrm{K}_1(\mathcal{N}(H_0(D), J))$, where $\mu_1\colon \mathrm{K}_1(C) \to \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ and $\mu_2\colon \mathrm{K}_1(\mathcal{N}(H_0(D), J)) \to \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ are induced by the canonical inclusion. Then $(u^* v_1 v_2) \oplus_{s,t} 1 \in \mathcal{U}_0(\mathrm{Der}(H_0(D), J))$.

more details? others? ??

Since $\mathcal{N}(H_0(D), J)$ is an ideal of $\mathrm{Der}(H_0(D), J)$ and is a full hereditary C^*-subalgebra of E and since $C := H_0(D)' \cap E \subseteq \mathrm{Der}(H_0(D), J)$ contains the copy of $C^*(s,t) \subseteq C$ of $\mathcal{O}_2$, the natural maps from $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ and $\mathcal{U}(E)$ into $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ and $\mathrm{K}_1(E)$ are surjective. We can use Lemma 4.2.20(ii) to obtain that $[u_1] = [u_2] \in \mathrm{K}_1(\mathrm{Der}(H_0(D), J))$ if $[u_1] = [u_2] \in \mathrm{K}_1(E)$ and $u_2^* u_1 \in 1 + \mathcal{N}(H_0(D), J)$.

Proof. (o): Let $T := s_1(1 - Q) + t_1 Q$ for the isometries s_1, t_1 as in Lemma 4.4.7(vi). Then $T^* T = 1$ and $T^* H_0(\cdot) T = Q H_0(\cdot) = k$. Thus, $[k] \in S(H_0\,;\, D, E)$.

If moreover $k(D) \subseteq J + \beta J$, then Lemma 4.4.16(iv) applies and $k = k_1 + \beta k_2$ with unique $k_1, k_2 \in S(h_0\,;\, D, E)$.

[8] Notice here that $(wu)^* H_0(\cdot) wu = u^* H_0(\cdot) u$.

(i): Let $A := H_0(D) \subseteq E$. Clearly $\mathrm{Der}(A, J) = \pi_J^{-1}\big(\pi_J(A)' \cap (E/J)\big)$. Thus, it is a C^*-subalgebra of E and $C := A' \cap E \supset \mathrm{Der}(A, J)$. In the same way $\mathcal{N}(A, J) = \pi_J^{-1}(\mathrm{Ann}(\pi_J(A), E/J))$, where $\mathrm{Ann}(\pi_J(A), E/J)$ means the two-sided annihilators of the elements of $\pi_J(A)$. Since $\mathrm{Ann}(\pi_J(A), E/J)$ is a closed ideal of $\pi_J(A)' \cap (E/J)$, the C^*-algebra $\mathcal{N}(A, J)$ is a closed ideal of $\mathrm{Der}(A, J)$.

Similar arguments work with $J + \beta(J)$ in place of J. Thus, $\mathcal{N}(A, J + \beta(J))$ is a closed ideal of $\mathrm{Der}(A, J + \beta(J))$.

Let $e \in E$ with $ea - ae \in J$ for all $a \in A$, i.e., $e \in \mathrm{Der}(A, J)$. Since $A := H_0(D)$ is separable, the C^*-algebra $M \subseteq J$ generated by the image of $a \in A \mapsto ea - ae$ is separable, and M contains a strictly positive element f. Assumption (iii) of Theorem 4.4.6, gives a projection $p \in F \subseteq C$ with $p_0 \leq p$ and $pf = f = fp$. Thus $px = x = xp$ for all $x \in M$, and it follows that $e(1-p), (1-p)e \in C$. Since $p \in C$ we get that $e - pep \in C$, and $pepa - apep \in J$ for all $a \in A$.

Let $u \in 1 + F \subseteq C$ with $u^*pu = p_0$. Then $pep = ugu^*$ for $g := p_0u^*eup_0$ and g satisfies $\partial_g(a) = p_0u^*(ea - ae)up_0 \in J$ for $\partial_g(a) := ga - ag$ and $a \in A$, i.e., $g \in \mathrm{Der}(A, J)$.

By Lemma 4.4.16(ii), p_0 is the sum $p_0 = q + r$ of properly infinite projections $q, r \in C$ with $q \in \mathcal{N}(A, J)$ and $r \in \mathcal{N}(A, \beta J)$.

Since $\mathcal{N}(A, J)$ is an ideal of $\mathrm{Der}(A, J) \supset C$ and $q \in \mathcal{N}(A, J)$, we get that

$$gq = g(1-r),\ qg = (1-r)g,\ (1-r)gr = qgr \ \in \mathcal{N}(A, J)\,.$$

It follows $g - rgr \in \mathcal{N}(A, J)$ and $rgr = g - (g - rgr) \in \mathrm{Der}(A, J)$.

The C^*-algebra $\mathcal{N}(A, \beta(J))$ is a hereditary C^*-subalgebra of E, and $r \in \mathcal{N}(A, \beta(J))$. Thus, $rgr \in \mathcal{N}(A, \beta(J))$. Since $rgra - argr \in J \cap \beta J = \{0\}$ for all $a \in A$, we get $rgr \in C$. Since $u \in 1 + F \subseteq C$, it follows $urgru^* \in C$.

Let $e_1 := (e - pep) + urgru^*$ and $e_2 := u(g - rgr)u^*$. Since $pep = ugu^*$, we get $e = e_1 + e_2$. Above we have seen that $e_1 \in C$ and $e_2 \in \mathcal{N}(A, J)$.

We consider $\mathrm{Der}(A, J + \beta J)$ and – more generally – the case where X is a countable subset of $\mathrm{Der}(A, J+\beta J)$: Every $e \in X$ satisfies $\partial_e(a) := ea - ae \in J + \beta J$ for all $a \in A$, and the linear map ∂_e is the sum $\partial_e = T_e + \beta S_e$, where $T_e, S_e \colon A \to J$ are bounded linear maps, because $J \cap \beta J = \{0\}$. Let G denote a separable C^*-subalgebra of $J \subseteq \beta J$ that contains the separable subsets $T_e(A) \cup S_e(A)$ for all $e \in X$ and satisfies $\beta(G) = G$. Let $f \in G_+$ a strictly positive contraction of G. There is a projection $p \in F \subseteq C$ with $p \geq p_0$ and $pf = f$ by 4.4.6(iii). It follows that $ea - ae = pepa - apep$ and $e - pep \in C$ for all $e \in X$. By 4.4.6(v), there is a unitary $w \in 1 + F \subseteq C = A' \cap E$ with $w^*pw = p_0$.

It follows that $p\partial_e(a) = \partial_e(a) = \partial_e(a)p = p\partial_e(a)p = \partial_{pep}$ for $a \in A$ and $e \in X$.

In particular, $(1-p)\partial_e(A) = \{0\} = \partial_e(A)(1-p)$ for all $e \in X$.

Recall that $F \subseteq \mathcal{N}(A, J + \beta J) \cap C$ and that $\mathcal{N}(A, J + \beta J)$ is a hereditary C^*-subalgebra of E. Thus $pep \in \mathcal{N}(A, J + \beta J)$ for each orthogonal projection

$p \in \mathcal{N}(A, J + \beta J)$ and $e \in E$. Since $p \in F \subseteq \mathcal{N}(A, J + \beta J)$ and $w \in 1 + F \subseteq C$, we get that $e - pep \in C$ and $pep \in \mathcal{N}(A, J + \beta J)$ for all $e \in X$. In particular, $\mathrm{Der}(A, J + \beta J) = C + \mathcal{N}(A, J + \beta J)$.

(ii): If $[k] + [H_0] = [H_0]$ then $[k \oplus_{s_2,t_2} H_0] = [H_0]$. There exists $u \in \mathcal{U}(E)$ with

$$u^* H_0(\cdot) u \;=\; k \oplus_{s_2,t_2} H_0 \;=\; s_2 k(\cdot) s_2^* + (1 - p_2) H_0(\cdot)\,,$$

where p_2, s_2 and t_2 are as in Lemma 4.4.16(i). It follows $ut_2 \in C$, because $t_2 \in C$, $(1 - p_2) = t_2 t_2^*$ and $H_0(\cdot) u (1 - p_2) = u(1 - p_2) H(\cdot)$. Furthermore, up_2u^* commutes with $H_0(D)$ and, for $a \in D$, $up_2u^* H_0(a) = us_2 k(a) s_2^* u^* \subseteq J$. In particular, $k(a) = s_2^* u^* H_0(a) u s_2$. Recall that $p_2 H_0(\cdot) = s_2 h_0(\cdot) s_2^*$.

We get $u^* H_0(a) u - H_0(a) = s_2(k(a) - h_0(a)) s_2^* \in J$ for all $a \in D$, because $k(D) \subseteq J$. Thus, $u \in \mathrm{Der}(H_0(D), J)$.

Clearly $v^* H_0(\cdot) v = s_0 k(\cdot) s_0^* + (1 - p_0) H_0(\cdot)$ with $v := u(s_2 s_0^* + t_2 t_0^*)$. Then $vt_0 \in C$ and $v \in \mathrm{Der}(H_0(D), J + \beta(J))$.

Since $p_2, p_0 \in \mathcal{N}(H_0(D), J + \beta J)$ and $s_2 s_0^* = p_2 (s_2 s_0^*) p_0$, the partial isometry $s_2 s_0^*$ is in $\mathrm{Der}(H_0(D), J + \beta(J))$ but is not in $\mathrm{Der}(H_0(D), J)$, because $p_0 = s_0 s_0^* = (s_2 s_0^*)^* s_2 s_0^*$ is not in $\mathrm{Der}(H_0(D), J)$.

`Next also (xv):`

If $[k] + [h_0] = [h_0]$ in $S(h_0\,;\, D, E)$, then $u = w_1 u_1$ with $u_1 \in 1 + \mathcal{N}(H_0(D), J)$ and $w_1 \in C$.

(iii): Let $Y \subseteq \mathcal{U}(E)$ an at most countable subset of the unitaries in E, and suppose that $u \in \mathrm{Der}(H_0(D), J + \beta J)$ for all $u \in Y$. If G is the (norm-)closure of the subgroup of $\mathcal{U}(E)$ generated by Y, then $G \subseteq \mathrm{Der}(H_0(D), J + \beta J)$, because $\mathrm{Der}(H_0(D), J + \beta J)$ is a C^*-algebra by Part (i).

If we take for X in the proof of Part (i) the sub-group of G that is algebraical generated by Y, then we get a projection $p \in F \subseteq C$ and a unitary $w \in 1 + F$, such that $p \geq p_0$, $u - pup \in C$ for all $u \in G$, and $w^* p w = p_0$. Moreover $pup \in \mathcal{N}(H_0(D), J + \beta J)$ for all $u \in G$, because $w \in C$, $p_0 \in \mathcal{N}(H_0(D), J + \beta J)$ and $\mathcal{N}(H_0(D), J + \beta J)$ is an ideal of $\mathrm{Der}(H_0(D), J + \beta J)$.

It follows that $(1 - p) u^* H_0(\cdot) u = (1 - p) H_0(\cdot)$ for all $u \in G$. This implies that $(1 - p_0)(uw)^* H_0(\cdot) uw = (1 - p_0) H_0(\cdot)$, and $Q := uwp_0(uw)^*$ is a projection in C, i.e., p_0 commutes with $(uw)^* H_0(\cdot) uw$. Let $k(a) := s_0^* (uw)^* H_0(a) uws_0$, i.e., $s_0 k(a) s_0^* = p_0 (uw)^* H_0(a) uw$, then $k \in S(H_0\,;\, D, E)$ by Part (o) and

$$(uw)^* H_0(\cdot) uw = k \oplus_{s_0,t_0} H_0\,.$$

Since

$$(1 - p_0) \cdot \big((uw)^* H_0(\cdot) uw - H_0(\cdot)\big) = (1 - p_0)(wu)^* \partial_{wu} H_0(\cdot) = 0\,,$$

$p_0 H_0(\cdot) = s_0\big(h_0(\cdot) + \beta h_0(\cdot)\big) s_0^*$ and $uw \in \mathrm{Der}(H_0(D), J + \beta J)$ it follows that, for $a \in D$,

$$s_0\big(k(a) - h_0(a) - \beta h_0(a)\big) s_0^* \;=\; (uw)^* \partial_{uw}(H_0(a)) \in J + \beta J\,.$$

Thus, $k(a) \in J + \beta J$ for $a \in D$. Since $k \in S(H_0\,;\, D, E)$ it implies $k = k_1 + \beta k_2$ for unique $k_1, k_2 \in S(h_0\,;\, D, E)$ by Lemma 4.4.7(x).

If moreover $u \in \mathrm{Der}(H_0(D), J)$, then the above arguments show that $k(a) - h_0(a) - \beta h_0(a) \in J$ for each $a \in D$. This implies $k_2 = h_0$ because $J \cap \beta J = \{0\}$.

(iv): Let G a separable (norm-) closed subgroup of $\mathcal{U}(\mathrm{Der}(H_0(D), J + \beta J))$. By the proof of Part (iii) there exist a projection $p \in F \subseteq C$, and a unitary $w \in 1 + F$ such that $p \geq p_0$, $u - pup \in C$ for all $u \in G$, and $w^* p w = p_0$. Moreover $pup \in \mathcal{N}(A, J + \beta J)$ for all $u \in G$. It follows that $(1-p)u^* H_0(\cdot)u = (1-p)H_0(\cdot)$ for all $u \in G$. It follows that there exists an isometry R with $R^*(1-p)R = 1$, i.e., $RR^* \leq 1 - p$, and with $H_0(D)RR^* = \{0\}$:

Indeed, by Lemma 4.4.7(vi), there exist isometries $s_1, t_1 \in E$ with $t_1 \in C$, $s_1 s_1^* + t_1 t_1^* = 1$, $p_0 t_1 = p_0 = t_1 p_0$, $s_1 s_1^* H_0(\cdot) = 0$ and $t_1 H_0(\cdot) = H_0 = H_0(\cdot) t_1$. It implies $s_1 s_1^* = 1 - t_1 t_1^* \leq 1 - p_0$. Since $w \in C$ and $w p_0 w^* = p$, the isometry $R := w s_1$ satisfies $RR^* \leq 1 - p = w(1-p_0)w^*$, and $R^* H_0(\cdot) R = s_1^* H_0(\cdot) s_1 = 0$. Then $RER^* \perp H_0(D)$, and for all $u \in G$,

$$R^* u^* H_0(\cdot) u R = R^*(1-p) u^* H_0(\cdot) u R = R^*(1-p) H_0(\cdot) R = 0\,.$$

Let $S := Rs$ and $T := (1 - RR^*) + RtR^*$. Then S and T are isometries in E, $TT^* = (1 - RR^*) + Rtt^*R^*$, and $SS^* \leq RR^* \perp u^* H_0(D) u$, $Tu^* H_0(\cdot) = (1 - RR^*) u^* H_0(\cdot) = u^* H_0(\cdot)$, $H_0(\cdot) u T = H_0(\cdot) u$ for all $u \in G$, and $SS^* + TT^* = (1 - RR^*) + Rss^*R^* + Rtt^*R^* = 1$.

(v): Let $S, T \in E$ the isometries from Part (iv). It is well-known that $W_u := Su^*S^* + TuT^*$ is in $\mathcal{U}_0(E)$, and straight calculation shows $W_u^* H_0(\cdot) W_u = u^* H_0(\cdot) u$ for all $u \in G$. It follows $uW_u^* \in C$ and $[W_u]_E = 0$ in $\mathrm{K}_1(E)$.

(vi): For any $u \in E$ holds $v^* H_0(a) v = u^* H_0(a) u$ for $v := (SuS^* + TT^*)$. Then $w := vu^* = SuS^*u^* + TT^*u^*$ is in C.

TYPO ????????

Unitaries u_1 and u_2 in $\mathrm{Der}(H_0(D), J)$ have same class in $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$, if and only if, the corresponding – in this way defined – v_1 and v_2 are homotopic in $\mathcal{U}(\mathrm{Der}(H_0(D), J))$.

(I.e., the new sort of $v \in \mathcal{U}_0(E)$ has the property that, any two of them are homotopic in $\mathcal{U}(\mathrm{Der}(H_0(D), J))$ if they have the same K_1-class in $\mathrm{K}_1(\mathrm{Der}(H_0(D), J))$.)

This implies ????

(vii): To be shown: If $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$, then there exists a unitary $v \in 1 + F \subseteq C$ and $k \in S(h_0\,;\, D, E)$, such that with $w = uv$ holds

$$w^* H_0(\cdot) w = (k + \beta h_0) \oplus_{s_0, t_0} H_0\,.$$

`Compare with` $k \oplus_{s_2, t_2} H_0$ `??`

(viii): To be shown: ??????

For every $u \in \mathcal{U}(\mathrm{Der}(H_0(D), J))$ there exist $w \in \mathcal{U}(C)$ and $k \in S(h_0; D, E)$ with $w^*(u^* H_0(\cdot) u) w = k \oplus_{s_0,t_0} H_0$, such that $1 - uw \in \mathcal{N}(H_0(D), J)u$?????.

`It would imply:` $u^* - w \in u^* \mathcal{N}(H_0(D), J)u = \mathcal{N}(H_0(D), J)$. It is desirable, but still not prove-able. ????

(xiv): $[k + \beta h_0] + [h_0] = [h_0 + \beta h_0]$, is the same as $[k] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$ by Lemma 4.4.7(i). Since $[k] + [h_0 + \beta h_0] = [k \oplus_{s,t} (h_0 + \beta h_0)]$, there exists a unitary $u \in E$ with

$$(u^* sk(a) s^* u + u^* th_0(a) t^* u - h_0(a)) + (u^* t\beta h_0(a) tu - \beta h_0(a)) = 0$$

for all $a \in D$. Since $J \cap \beta(J) = J\beta(J) = \{0\}$ it follows $u^*\big(sk(\cdot)s^* + th_0(\cdot)t^*\big)u = h_0$, i.e., $[k] + [h_0] = [h_0]$.

(xv): Let $k\colon D \to J$ and $u \in \mathcal{U}(E)$ with $u^* H_0(\cdot) u = k(\cdot) \oplus_{s_2,t_2} H_0(\cdot)$.

Then $k = s_2^* u^* H_0(\cdot) u s_2 \in S(H_0\,; D, E)$ and $k \in S(h_0\,; D, E)$ by Lemma 4.4.7(x).

Suppose $[k] + [h_0] = [h_0]$, ...?????

Since $s, t \in h_0(D)' \cap E$ for the isometries s, t in Theorem 4.4.6(i), we get $W_0(h_0 \oplus_{s_2,t_2} h_0) W_0^* = h_0$ for the unitary $W_0 := s s_2^* + t t_2^*$.

If $[k] + [h_0] = [h_0]$, then there is a unitary $V_0 \in E$ with $V_0^*(k \oplus_{s_2,t_2} h_0) V_0 = h_0 = p_2 H_0(\cdot)$. The unitary $V_1 := V_0 W_0$ satisfies

$$V_1^*(k \oplus_{s_2,t_2} h_0) V_1 = h_0 \oplus_{s_2,t_2} h_0\,.$$

The unitary $T := (s_2)^2 s_2^* + s_2 t_2 s_2^* t_2^* + t_2 (t_2^*)^2$ satisfies for all $X, Y, Z \in E$ that

$$T\big(X \oplus_{s_2,t_2} (Y \oplus_{s_2,t_2} Z)\big) = \big((X \oplus_{s_2,t_2} Y) \oplus_{s_2,t_2} Z\big)T\,.$$

Notice that $H_0 = h_0 \oplus_{s_2,t_2} H_0$. It implies

$$H_0 = h_0 \oplus (h_0 \oplus H_0) = T^*((h_0 \oplus h_0) \oplus H_0)T$$

Let $W_1 := V_1 \oplus 1$ and $v := T^* W_1^* T = T^*(V_1^* \oplus 1)T$. Then $rv = r = vr$ for $r := T^* t_2 t_2^* T = t_2^2 (t_2^2)^* = t_2(1 - p_2) t_2^*$, $1 - r = p_2 + t_2 p_2 t_2^*$ is in $C \cap \mathcal{N}(H_0(D), J)$, because the latter is an ideal of C and $t_2 \in C$. It implies $t_2 H_0(D) p_2 t_2^* \subseteq J$. Thus $v \in 1 + (1 - r)E(1 - r) \subseteq 1 + \mathcal{N}(H_0(D), J)$, and satisfies:

$$v^* H_0 v = v^* T^*((h_0 \oplus h_0) \oplus H_0) T v = T^* W_1((h_0 \oplus h_0) \oplus H_0) W_1^* T =$$
$$T^*\big((k \oplus h_0) \oplus H_0\big)T = \big(k \oplus (h_0 \oplus H_0)\big) = k \oplus_{s_2,t_2} H_0\,.$$

It follows that $uv^* = w \in C$, i.e., that u is the product wv of a unitary $v \in 1 + \mathcal{N}(H_0(D), J)$ and a unitary $w \in C$.

`Check again:`

Now we consider the opposite direction: Suppose that $u = wv$ with v a unitary in $1 + \mathcal{N}(H_0(D), J)$ and $w \in \mathcal{U}(C)$. Then

$$v^* H_0(\cdot) v = k(\cdot) \oplus_{s_2,t_2} H_0(\cdot)\,.$$

and there is normal $g \in \mathcal{N}(H_0(D), J)$ with $v = g + 1$. The sets $H_0(D)g$ and $\partial_v(H_0(D)) = \partial_g(H_0(D))$ together generate a separable C^*-subalgebra of J. By

Theorem 4.4.6(ii) there exists $p \in F$ with $p \geq p_0$, $p \neq p_0$ such that $p(H_0(a)g - gH_0(a))p = H_0(a)g - gH_0(a)$ and $H_0(\cdot)gp = H_0(\cdot)g = H_0(\cdot)pg$. If we add to the last equation $H_0(\cdot)p$, then we get $H_0(\cdot)vp = H_0(\cdot)pv$. It follows $v^*H_0(\cdot)vp = v^*H_0(\cdot)pv$, $H_0(\cdot)pvp - pvpH_0(\cdot) = H_0(\cdot)v - vH_0(\cdot)$ and $v^*H_0(\cdot)v(1-p) = H_0(\cdot)(1-p)$. Since $p = p_0 + r$ for some $r \neq 0$, it follows $r = rt_0t_0^*$, $v^*H_0(\cdot)pv = v^*H_0(\cdot)vp = s_2k(\cdot)s_2^* + rH_0(\cdot)$. By Theorem 4.4.6(v), there exist unitaries $W_1, W_2 \in 1 + F$ with $W_1^*pW_1 = p_0$ and $W_2^*rW_2 = p_0$. Then $H_0(\cdot)p = W_1H_0(\cdot)p_0W_1^*$ and $H_0(\cdot)r = W_2(H_0(\cdot)p_0)W_2^*$. Recall that $H_0(\cdot)p_0 = s_0(h_0 + \beta h_0)s_0^*$. It gives

$$[h_0 + \beta h_0] + [0] = [k] + [h_0 + \beta h_0] + [0]\,.W_2H_0(\cdot)p_0W_2^*t_2)t_2^*$$

By Lemma 4.4.7(ii,i), this is equivalent to $[h_0 + \beta h_0] = [k + \beta h_0] + [h_0]$ It follows $[h_0] = [k] + [h_0]$ by Part (xiv).

`??` `to be filled in` □

We study at first the case, where the **OLD !!!! assumptions (I) and (II)** are satisfied (in addition to (i)-(vii)):

If $u \in \mathrm{Der}(H_0(D), J)$ is unitary, then, by Lemma 4.4.17(v), there exists $w \in C$ such that $u^*H_0(a)u = (wu)^*H_0(a)(wu)$ for all $a \in A$ and $wu \in \mathcal{U}_0(E)$.

Then by assumption (I) there exists a unitary $v \in C \cap \mathcal{U}_0(E)$ such that for $u' = wuv \in \mathcal{U}_0(E)$

$$(u' - 1)\beta(J) = \{0\}\,.$$

`to be filled in ??` ??????????????????????

`Start:`
`towards isomorphism of kernel and` Γ

Now we take a more K-theoretic approach for the study of the **general situation**:

Let $C_2 := h_0(D)' \cap E$. Then $G(h_0\,;\,D, E) \cong \ker(\mathrm{K}_0(C_2) \to \mathrm{K}_0(E))$ by ??????????? **??** from Lemma 4.2.20 ???

It holds $C_2 \cong p_2Cp_2$ via the *-isomorphism $b \in C_2 \mapsto s_2bs_2^*$.

The subalgebras $\mathrm{Ann}(H_0(D), E)$ and $C \cap \mathcal{N}(H_0(D), J)$ are ideals of C, and $\mathrm{Ann}(H_0(D), E) \subseteq C \cap \mathcal{N}(H_0(D), J)$.

The projection $p_2 + \mathrm{Ann}(H_0(D), E)$ is properly infinite in $C/\mathrm{Ann}(H_0(D), E)$, which is a full C^*-subalgebra of

$$J_2 := (C \cap \mathcal{N}(H_0(D), J))/\mathrm{Ann}(H_0(D), E)\,.$$

Since H_0 dominates zero, we get that $\mathrm{Ann}(H_0(D), E)$ is a full hereditary C^*-subalgebra of E. Now Lemma 4.2.20 shows that

$$G(H_0\,;\,D, E) \cong \mathrm{K}_0(C/\mathrm{Ann}(H_0(D))) \cong \ker(\mathrm{K}_0(C) \to \mathrm{K}_0(E))\,.$$

The same happens with $\mathrm{Ann}(h_0(D),E) \subseteq C_2$ in E. Moreover, $s_2 b s_2^* \in \mathrm{Ann}(H_0(D),E)$, if and only if, $b \in \mathrm{Ann}(h_0(D),E)$. Let

$$A := C/\,\mathrm{Ann}(H_0(D),E)\,.$$

The algebra J_2 is a closed ideal of A with $A/J_2 \cong C/(C \cap \mathcal{N}(H_0(D),J))$ and the natural map $G(h_0\,;\,D,E) \to G(H_0\,;\,D,E)$ transforms to

$$\mathrm{K}_0(C_2/\,\mathrm{Ann}(h_0(D),E)) \cong \mathrm{K}_0(J_2) \to \mathrm{K}_0(A)\,.$$

The surjectivity of $G(h_0\,;\,D,E) \to G(H_0\,;\,D,E)$ is equivalent to the exactness of the sequence

$$\mathrm{K}_1(A) \to \mathrm{K}_1(A/J_2) \to_\partial \mathrm{K}_0(J_2) \to \mathrm{K}_0(A) \to 0\,.$$

One can use the identity $\mathrm{Der}(H_0(D),J) = C + \mathcal{N}(H_0(D),J)$, i.e., that

$$A/J_2 \cong \mathrm{Der}(H_0(D),J)/\,\mathcal{N}(H_0(D),J)\,,$$

the natural map $\mathrm{K}_1(C) \to \mathrm{K}_1(\mathrm{Der}(H_0(D),J))$ and the splitting (by Lemma 4.2.20) of the exact sequence

$$0 \to \mathrm{K}_1(\mathcal{N}(H_0(D),J)) \to \mathrm{K}_1(\mathrm{Der}(H_0(D),J)) \to \mathrm{K}_1(A/J_2) \to 0\,,$$

to verify that the quotient $\mathrm{K}_1(A/J_2)/\partial(\mathrm{K}_1(A))$ of $\mathrm{K}_1(A/J_2)$ by the image of $\partial\colon \mathrm{K}_1(A) \to \mathrm{K}_1(A/J_2)$ is isomorphic to $\Gamma(H_0(D),J,E)$ as defined before Theorem 4.4.6.

It seems likely that the following argument gives that the natural map $\mathrm{K}_1(A) \to \mathrm{K}_1(A/J_2)$ is zero, i.e., that $\mathrm{K}_1(J_2) \to \mathrm{K}_1(A)$ is surjective too. The only difficult point in such an attempt is the verification of assumption (ii) of Theorem 4.4.6 for the “new” system.

Let $\mathcal{P}_\infty$ denote the unique pi-sun algebra in the UCT-class with $\mathrm{K}_0(\mathcal{P}_\infty) \cong \{0\}$ and $\mathrm{K}_1(\mathcal{P}_\infty) \cong Z$ If we repeat all arguments above with E, J, β, F, H_0, h_0, s, t, s_0, t_0, and u_1, but *replaced* by $E \otimes \mathcal{P}_\infty$, $J \otimes \mathcal{P}_\infty$, $F \otimes 1$, $\beta \otimes \mathrm{id}$, $H_0(\cdot) \otimes 1$, $h_0(\cdot) \otimes 1$, $s \otimes 1$, ..., then one gets also that $\mathrm{K}_1(J_2) \to \mathrm{K}_1(A)$ is surjective, provided assumption (ii) of Theorem 4.4.6 can be verified by the new system, at least in a suitable approximate or asymptotic sense.

Remark 4.4.18. One can show that the following conditions (I) and (II) together imply that $[k \oplus h_0] \mapsto [k \oplus H_0]$ is injective.

(I) If $u \in \mathcal{U}_0(E)$ is in $C + \mathcal{N}(H_0(D),J)$ then there exists a unitary $w \in C \cap \mathcal{U}_0(E)$ such that $(wu-1)\beta(J) = \{0\}$.

`Alternatively(?): ... such that` $vu \in 1 + \mathcal{N}(H_0(D), J + \beta(J))$.

(II) If $v \in \mathcal{U}_0(E)$ is in $C + \mathcal{N}(H_0(D),J)$, $vp_0v^* \in C \cap \mathcal{N}(H_0(D),J)$ and $(v-1)\beta(J) = \{0\}$, then $k(\cdot) := (vp_0v^*)H_0(\cdot)$ satisfies

$$[k] + [h_0 + \beta h_0] = [h_0 + \beta h_0]\,.$$

5. Generalizations and limitations

The following remarks and corollaries discuss some limitations of the arguments in above given proofs and generalize them partly for "soft" asymptotic considerations.

We need in later chapters local estimates for an approximate version of Proposition 4.3.5(i). The following remark and corollary give estimates in very special situations, well applicable e.g. to singly generated C^*-algebras D.

REMARK 4.5.1. Let $h_1\colon D \to E$ a C^*-morphism and $s_1, s_2, t \in E$ isometries with $s_1 s_1^* + s_2 s_2^* = 1$.

Define a c.p. contraction $h_2\colon D \to E$ by $h_2(a) := t^* h_1(a) t$ for $a \in D$ and isometries $t_1 := (1 - tt^*) + ts_1 t^*$, $t_2 := ts_2$ in E.

Let $C(s_1, s_2; y) := \| [s_1, y] \| + \| [s_2, y] \|$ denote the sum of norms of the commutators of $y \in E$ with s_1 and s_2, and define by

$$\operatorname{curv}(h_2, a) := \max\big(\| h_2(a^*a) - h_2(a^*)h_2(a) \| ,\ \| h_2(aa^*) - h_2(a)h_2(a^*) \| \big)^{1/2}$$

the **local curvature** of the completely positive contraction h_2 at $a \in D$, i.e., the failure of h_2 to be multiplicative at the elements a and a^*.

Then the isometries t_1, t_2 satisfy $t_1 t_1^* + t_2 t_2^* = 1$ and for the Cuntz sum holds

$$\|h_1(a) - (h_1 \oplus_{t_1,t_2} h_2)(a)\| \le C\big(s_1, s_2; h_2(a)\big) + \operatorname{curv}(h_2, a)\,.$$

If $h_3\colon D \to E$ is another C^*-morphism, then we denote by

$$\operatorname{var}(h_2, h_3; a) := \max\{ \|h_2(x) - h_3(x)\| \,;\ x = a,\ a^*a,\ aa^* \}\,,$$

the **local variation** of h_2 and h_3 at $a \in D$, i.e., the distance of h_2 and h_3 on $\{a, a^*, a^*a, aa^*\}$.

The C^*-morphism $h_1 \oplus_{t_1,t_2} h_3\colon D \to E$ satisfies, for a *contraction* $a \in E$, that

$$\begin{aligned} &\|h_1(a) - (h_1 \oplus_{t_1,t_2} h_3)(a)\| \\ \le\ & C\big(s_1, s_2; h_3(a)\big) + C\big(s_1, s_2; h_3(a) - h_2(a)\big) + 4 \cdot \operatorname{var}(h_2, h_3; a)^{1/2} \\ \le\ & C\big(s_1, s_2; h_3(a)\big) + 10 \cdot \operatorname{var}(h_2, h_3; a)^{1/2}\,. \end{aligned}$$

PROOF. Obviously, the elements $t_1 := (1 - tt^*) + ts_1 t^*$ and $t_2 := ts_2$ of E are isometries with $t_1 t_1^* + t_2 t_2^* = 1$. Let $p := tt^*$,

$$X(a) := h_1(a) - \big(h_1(a) \oplus_{t_1,t_2} h_2(a)\big)$$

and recall that $h_2 := t^* h_1(\cdot) t$.

Thus, $ph_1(a)p = th_2(a)t^*$, $pt_1 h_1(a) t_1^* p = ts_1 h_2(a) s_1^* t^*$ and

$$pX(a)p = th_2(a)t^* - ts_1 h_2(a) s_1^* t^* - ts_2 h_2(a) s_2^* t = t\Delta(a)t^*$$

for $\Delta(a) := h_2(a) - (h_2 \oplus_{s_1,s_2} h_2)(a)$. Clearly, for $y \in E$ and $q_k := s_k s_k^*$,

$$\|y - (y \oplus_{s_1,s_2} y)\| \le \|yq_1 - s_1 y s_1^*\| + \|yq_2 - s_1 y s_1^*\| = \|ys_1 - s_1 y\| + \|ys_2 - s_2 y\| \le 4\|y\|\,,$$

because $q_1 + q_2 = 1$, $s_j^* q_k = \delta_{j,k} s_j$ and $\|y s_j^*\| = \|y\|$. It follows that

$$\|pX(a)p\| \le \|\Delta(a)\| \le C\big(s_1, s_2; h_2(a)\big)\,.$$

The equations $(1-p)t_1 = (1-p) = t_1^*(1-p)$ and $pt_1 = ts_1$, $pt_2 = t_2 = ts_2$ show that

$$p \cdot (h_1(a) - X(a)) = t(h_2(a) - \Delta(a))t^* = (h_1(a) - X(a)) \cdot p\,,$$

$(1-p)X(a)(1-p) = (1-p)(h_1(a) - t_1 h_1(a) t_1^*)(1-p) = 0$, and that

$$pX(a)(1-p) + (1-p)X(a)p = ph_1(a)(1-p) + (1-p)h_1(a)p\,.$$

If we use that $\|x \pm y\| = \max(\|x\|, \|y\|)$ if $x^* y = 0 = x y^*$, then we get for $z \in E$ that

$$\begin{aligned} \|[z,p]\| &= \|pz(1-p) - (1-p)zp\| = \\ \|pz(1-p) + (1-p)zp\| &= \max(\,\|pz(1-p)\|\,,\ \|(1-p)zp\|\,)\,. \end{aligned}$$

It follows that above estimates sum up to

$$\|X(a)\| \,\le\, C\big(s_1, s_2; h_2(a)\big) \,+\, \max\big(\,\|ph_1(a)(1-p)\|,\, \|(1-p)h_1(a)p\|\,\big)\,.$$

Use now that for $p := tt^*$ and $h_2 := t^* h_1(\cdot) t$ holds

$$\|h_2(a^*a) - h_2(a)^* h_2(a)\| \,=\, \|(1-p)h_1(a)p\|^2\,.$$

We obtain

$$\operatorname{curv}(h_2, a) = \max\big(\,\|\, ph_1(a)(1-p)\,\|\,,\, \|\,(1-p)h_1(a)p\,\|\,\big) \,=\, \|[h_1(a), tt^*]\|\,.$$

This implies the proposed estimate $C\big(s_1, s_2; h_2(a)\big) + \operatorname{curv}(h_2, a)$ of $\|h_1(a) - (h_1 \oplus_{t_1,t_2} h_2)(a)\|$.

To verify the estimate of the norm of $h_1(a) - (h_1 \oplus_{t_1,t_2} h_3)(a)$ consider following inequalities:

$$\begin{aligned} \|h_1(a) - (h_1 \oplus_{t_1,t_2} h_3)(a)\| &\le \|h_3(a) - h_2(a)\| + \|h_1(a) - (h_1 \oplus_{t_1,t_2} h_2)(a)\|\,, \\ C\big(s_1, s_2; h_2(a)\big) &\le C\big(s_1, s_2; h_3(a)\big) + C\big(s_1, s_2; h_2(a) - h_3(a)\big) \\ \text{and}\quad C\big(s_1, s_2; h_2(a) - h_3(a)\big) &\le 4\|h_2(a) - h_3(a)\|\,. \end{aligned}$$

A rough estimate of $\operatorname{curv}(h_2, a)^2$ can be seen from

$$\begin{aligned} \|h_2(a^*a) - h_2(a)^* h_2(a)\| &\le \|h_3(a^*a) - h_2(a^*a)\| + \|h_3(a)^* h_3(a) - h_2(a)^* h_2(a)\| \\ &\le (2\|a\| + 1) \cdot \operatorname{var}(h_2, h_3; a)\,. \end{aligned}$$

Since $\operatorname{var}(h_2, h_3; a^*) = \operatorname{var}(h_2, h_3; a)$, we get $\operatorname{curv}(h_2, a)^2 \le 3 \cdot \operatorname{var}(h_2, h_3; a)$ for $\|a\| \le 1$.

Notice that

$$\|h_3(a) - h_2(a)\| \le \sqrt{2} \cdot \|h_3(a) - h_2(a)\|^{1/2} \le \sqrt{2} \cdot \operatorname{var}(h_2, h_3; a)^{1/2}\,,$$

for all *contractions* $a \in E$, in particular, $4\|h_3(a) - h_2(a)\| \le 6 \cdot \operatorname{var}(h_2, h_3; a)^{1/2}$.

It leads in case $\|a\| \le 1$ to the proposed estimates

$$C\big(s_1, s_2; h_3(a)\big) + C\big(s_1, s_2; h_3(a) - h_2(a)\big) + 4 \cdot \operatorname{var}(h_2, h_3; a)^{1/2}$$

and the bigger upper estimate

$$C\big(s_1, s_2; h_3(a)\big) + 10 \cdot \mathrm{var}(h_2, h_3; a)^{1/2}$$

for the norm of $h_1(a) - (h_1 \oplus_{t_1,t_2} h_3)(a)$. □

The proof of the following corollary is based on the estimates in Remark 4.5.1.

COROLLARY 4.5.2. *If $T, S \in E$ are isometries and $h, k \colon D \to E$ are C^*-morphisms, then there exists a unitary $U \in E \otimes \mathcal{O}_2$ such that*

$$\|U^*(h(a) \otimes 1)U - k(a) \otimes 1\| \ \leq\ 8 \cdot \mu(S, T;\, h, k;\, a)^{1/2}\,, \quad \forall\, a \in D\,, \tag{5.1}$$

where $\mu(S, T;\, h, k;\, a)$ denotes – for fixed $a \in D$ – the maximum of

$$\{\|S^* h(x) S - k(x)\|\,,\ \|T^* k(x) T - h(x)\|\,;\ x = a, a^*a, aa^*\}\,.$$

The Inequality (5.1) holds even if the unit element of E is not necessarily properly infinite, i.e., no orthogonality of SS^* and TT^* is required, i.e., it is not necessary that $C^*(S, T)$ is an image of $\mathcal{E}_2$.

PROOF. Given h, k, S, T, consider the C^*-morphisms

$$h_1 := h(\cdot) \otimes 1 \quad \text{and} \quad k_1 := k(\cdot) \otimes 1\,,$$

and define the “linking” completely positive contractions

$$h_2 := (S^* h(\cdot) S) \otimes 1 \quad \text{and} \quad k_2 := (T^* k(\cdot) T) \otimes 1\,.$$

We can take $s_1 := 1 \otimes r_1 \in E \otimes \mathcal{O}_2$ and $s_2 := 1 \otimes r_2$ in Remark 4.5.1 for the canonical generators r_1, r_2 of $\mathcal{O}_2 = C^*(r_1, r_2)$. Then s_1 and s_2 commute with the elements of $h_i(D) \cup k_j(D)$, $i, j \in \{1, 2\}$, and the estimates in Remark 4.5.1 apply to the C^*-morphisms h_1, $h_3 := k_1$ and the c.p. contraction $h_2 = t^* h_1(\cdot) t$ for $t := S \otimes 1$, respectively to k_1, $k_2 = t^* k_1(\cdot) t$ for $t := T \otimes 1$ and $k_3 := h_1$ (in place of h_1, t, h_2, h_3 in Remark 4.5.1).

Notice that $C\big(s_1, s_2; h_i(a)\big)$, $C\big(s_1, s_2; k_j(a)\big)$ and $C\big(s_1, s_2; h_i(a) - k_j(a)\big)$ are zero for $i, j = 1, 2$, and that the estimates in Remark 4.5.1 changes therefore to simpler ones, as e.g. to

$$\|h_1(a) - (h_1 \oplus_{t_1,t_2} k_1)(a)\| \ \leq\ 4 \cdot \mathrm{var}(h_2, k_1; a)^{1/2}\,,$$

and – with interchanged roles –

$$\|k_1(a) - (k_1 \oplus_{t_3,t_4} h_1)(a)\| \ \leq\ 4 \cdot \mathrm{var}(k_2, h_1; a)^{1/2}\,,$$

where we let $t_1 := (1 - SS^*) \otimes 1 + SS^* \otimes r_1$, and $t_2 := S \otimes r_2$ (build by $t := S \otimes 1$ from $s_k := 1 \otimes r_k$), and $t_3 := (1 - TT^*) \otimes 1 + TT^* \otimes r_1$ and $t_4 := T \otimes r_2$ (build by $t := T \otimes 1$ from $s_k := 1 \otimes r_k$).

If $U := t_1t_4^* + t_2t_3^*$ then $U(k_1 \oplus_{t_3,t_4} h_1) = (h_1 \oplus_{t_1,t_2} k_1)U$, and we obtain the estimate

$$\|U^*h_1(a)U - k_1(a)\| \le$$
$$\|h_1(a) - (h_1 \oplus_{t_1,t_2} k_1)(a)\| + \|k_1(a) - (k_1 \oplus_{t_3,t_4} h_1)(a)\| \le$$
$$4 \cdot \mathrm{var}(h_2, k_1; a)^{1/2} + 4 \cdot \mathrm{var}(k_2, h_1; a)^{1/2} \le$$
$$8 \cdot \mu(S,T;\, h,k;\, a)^{1/2} .$$

Here we have used that

$$\mu(S,T;\, h,k;\, a) = \max(\|k_2(x) - h_1(x)\|, \|h_2(x) - k_1(x)\| ;\; x = a, a^*a, aa^*) ,$$

i.e., that

$$\mu(S,T;\, h,k;\, a) = \max\bigl(\mathrm{var}(h_2, k_1; a), \mathrm{var}(k_2, h_1; a)\bigr) .$$

Notice finally that $\|U^*(h(a) \otimes 1)U - k(a) \otimes 1\| = \|U^*h_1(a)U - k_1(a)\|$. □

REMARK 4.5.3. The methods of estimates in the proofs of Remark 4.5.1 and Corollary 4.5.2 can be generalized to prove the following more flexible and general result (with different $U \in E$):

If $s_1, s_2, s_1', s_2' \in E$ are isometries, $s_1s_1^* + s_2s_2^* = 1 = s_1'(s_1')^* + s_2'(s_2')^*$, $h, k \colon D \to E$ are C^*-morphisms, and $S, T \in E$ are isometries, then there exists a unitary $U \in E$ that satisfies for all contractions $a \in D$ the inequality

$$\|U^*h(a)U - k(a)\| \le C\bigl(s_1, s_2; h(a)\bigr) + C\bigl(s_1', s_2'; k(a)\bigr) + 20 \cdot \mu(S,T;\, h,k;\, a)^{1/2} ,$$

where $\mu(S,T;\, h,k;\, a)$ is the maximum of

$$\{\|S^*h(x)S - k(x)\| ,\; \|T^*k(x)T - h(x)\| ;\; x = a, a^*a, aa^*\} ,$$

as defined in Corollary 4.5.2.

PROOF. Let $h_1 := h$, $h_2 := S^*h(\cdot)S$, and $h_3 := k$ and use the Cuntz addition $\oplus' := \oplus_{t_1,t_2}$ with isometries t_1, t_2 build by (s_1', s_2', S) in place of (s_1, s_2, t) in Remark 4.5.1, then we get that

$$\|h(a) - (h(a) \oplus' k(a))\| \le C\bigl(s_1', s_2'; k(a)\bigr) + 10 \cdot \mathrm{var}(S^*hS, k; a)^{1/2} .$$

If we take for Remark 4.5.1 $h_1 := k$, $h_2 := T^*k(\cdot)T$, $h_3 := h$ and use $\oplus := \oplus_{t_1,t_2}$ with t_1, t_2 build by (s_1, s_2, T) in place of (s_1, s_2, t) in Remark 4.5.1, then we obtain the estimate:

$$\|k(a) - (k(a) \oplus h(a))\| \le C\bigl(s_1, s_2; h(a)\bigr) + 10 \cdot \mathrm{var}(T^*kT, h; a)^{1/2} .$$

Now we use that

$$\mu(S,T;\, h,k;\, a) = \max\bigl(\mathrm{var}(S^*hS, k; a), \mathrm{var}(T^*kT, h; a)\bigr)$$

and take a unitary $U \in E$ with $U^*(x \oplus' y)U = y \oplus x$ for $x, y \in E$, then we get the upper estimate

$$C\bigl(s_1', s_2'; k(a)\bigr) + C\bigl(s_1, s_2; h(a)\bigr) + 20 \cdot \mu(S,T;\, h,k;\, a)^{1/2}$$

of $\|U^*h(a)U - k(a)\|$. □

REMARK 4.5.4. Let $A := H_0(D)$ and $C := A' \cap E$. The algebras $\mathcal{N}(A, J)$, $\beta(\mathcal{N}(A, J)) = \mathcal{N}(A, \beta J)$ and $\mathcal{N}(A, J + \beta J)$ are hereditary C^*-subalgebras of E, $\mathcal{N}(A, J) \cap \mathcal{N}(A, \beta J) = \mathrm{Ann}(A, E)$ (because $J \cap \beta J = 0$), and $\mathcal{N}(A, J) \cup \mathcal{N}(A, \beta J) \subseteq \mathcal{N}(A, J + \beta J)$.

The elements $s_2, s_3 \in E$ defined `check ref!!` Part (x) of Lemma 4.4.7 have the property that the partial isometry $Z := s_2 s_3^*$ satisfies $Z^*Z \in \mathcal{N}(A, \beta J)$, $ZZ^* \in \mathcal{N}(A, J)$, and $Z^*Z, ZZ^* \notin \mathrm{Ann}(A, J)$. Thus, $Z \in \mathcal{N}(A, J + \beta J) \setminus \mathrm{Ann}(A, J)$.

Conjecture: The partial isometry Z is not contained in the smallest β-invariant C^*-subalgebra of $\mathcal{N}(A, J + \beta J)$ generated by $\mathcal{N}(A, J)$, i.e.,

$$Z \notin C^*(\mathcal{N}(A, J) \cup \mathcal{N}(A, \beta J)) .$$

A positive answer seems likely, because otherwise there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{N}(A, J)$ such that $Z = b_1^*\beta(a_1) + \ldots + b_n^*\beta(a_n)$.

REMARK 4.5.5. The observations in Part (i) of Lemma 4.4.7 and the assumptions of Theorem 4.4.6 together *are not enough* to to give a general proof that $[k_1] = [k_2]$ and $k_1(B) \subseteq J$ imply $[k_1 + \beta k_3] = [k_2 + \beta k_3]$ for *all* C^*-morphisms $k_j \colon B \to J$, $j \in \{1, 2, 3\}$, of an arbitrary separable C^*-algebra B.

But, since $[k_1] = [k_2]$ and $k_1(B) \subseteq J$ together imply that there exists $u \in \mathcal{U}_0(E)$ with $u^*k_1(\cdot)u = k_2$ by Lemma 4.4.7(i), one can construct, with help of suitable paths of positive contraction in J_+, a norm-continuous path of unitaries $U(\xi) \in \mathcal{U}_0(J + \mathbb{C}1) \subset \mathcal{U}_0(E)$ for $\xi \in [0, \infty)$ such that $U(0) = 1$ and

$$\lim_{\xi \to \infty} U(\xi)^* k_1(b) U(\xi) = k_2(b) \quad \text{for all } b \in B .$$

Then $U(\xi)c = c = cU(\xi)$ for all $c \in \beta J \subset \mathrm{Ann}(J, E)$ and

$$\lim_{\xi \to \infty} U(\xi)^*(k_1(b) + \beta k_3(b))U(\xi) = k_2(b) + \beta k_3(b) \quad \text{for all } b \in B ,$$

i.e., $k_1 + \beta k_3$ and $k_2 + \beta k_3$ are unitarily homotopic in E in the sense of Definition 5.0.1, which is sometimes weaker than k_1 and k_2 being homotopic or unitary equivalent.

By symmetry – given by β – this shows also that for C^*-morphisms $k_j \colon B \to J$, $j \in \{1, 2, 3, 4\}$, the equations $[k_1] = [k_2]$ and $[k_3] = [k_4]$ imply that $k_1 + \beta k_3$ and $k_2 + \beta k_4$ are unitarily homotopic in E.

We apply Theorem 4.4.6 mainly to special cases where the ideal J is moreover sub-Stonean, i.e., has the property that, for each $a, b \in J_+$ with $ab = 0$, there exist contractions $c, d \in J_+$ with $ac = a$, $bd = b$ and $cd = 0$.

One can show that $[k_1] = [k_2]$ and $k_j(B) \subset J$ for $j \in \{1, 2\}$ imply $[k_1 + \beta k_3] = [k_2 + \beta k_3]$ for *all* C^*-morphisms $k_3 \colon B \to J$ if J is sub-Stonean in addition to the other assumptions of Theorem 4.4.6.

It is not clear if the assumptions of Theorem 4.4.6 allow to deduce from $[k_j] + [h_0] = [h_0]$ ($j \in \{1, 2\}$) that $[k_1 + \beta k_2] + [h_0 + \beta h_0] = [h_0 + \beta h_0]$ in the case where J is *not* sub-Stonean.

PROOF. If $[k_1] = [k_2]$ and $k_1(B) \subseteq J$ then there exists $u \in \mathcal{U}_0(E)$ with $u^* k_1(\cdot) u = k_2$ by Lemma 4.4.7(i).

There are $T_1, \ldots, T_n \in E$ with $T_m^* = -T_m$ for $m = 1, \ldots, n$, i.e., with $(iT_m)^* = iT_m$ in case of complex E, such that $u = \exp(T_1) \cdot \ldots \cdot \exp(T_n)$.

We can define C^*-morphisms $h_m \colon B \to J$ $(m = 1, \ldots, n-1)$ by $h_1 := k_1$, $h_{m+1}(\cdot) := \exp(-T_m) h_m(\cdot) \exp(T_m)$. Notice that $k_2 = h_{n+1}$.

Something unclear here!

Unitary homotopy, as defined in Definition 5.0.1 is an equivalence relation (in particular is *transitive*) and it suffices to show that of

????? of Theorem 4.4.6 ????

i.e., $k_1 + \beta k_3$ and $k_2 + \beta k_3$ are unitarily homotopic in E in the sense of Definition 5.0.1, which is sometimes weaker than being homotopic or unitary equivalent.

$h_m + \beta k_3$ and $h_{m+1} + \beta k_3$ are unitarily homotopic for $m = 1, \ldots, n$ (respectively are unitary equivalent by a unitary in $\mathcal{U}_0(J + \mathbb{C} \cdot 1)$ if J is sub-Stonean).

Thus, we may suppose that $n = 1$ and that $k_2 = \exp(-T) k_1(\cdot) \exp(T)$ with $T^* = -T$.

Let $T \in E$ with $T^* = -T$, and suppose first that J is a sub-Stonean C^*-algebra. In fact, we need only the weaker property of J that for every $g \in J_+$ there exists $f \in J_+$ with $\|f\| \leq 1$ and $gf = g$. Then we find a positive contraction $f \in J_+$ with $xf = x$ for all $x \in k_1(B)$ and a positive contraction $e \in J_+$ with $fe = e$ and $fT^\ell = f(eTe)^\ell$ for all $\ell \in \mathbb{N}$.

Indeed:
$k_1(B)$ contains a strictly positive contraction g_1, and there exists a contraction $f \in J_+$ with $fg_1 = g_1$. The C^*-subalgebra of J, generated by the countable subset $Y := \{ f, fT^n \,;\ n \in \mathbb{N} \}$ of J, contains a strictly positive contraction g_2. We find a positive contraction $e \in J_+$ with $g_2 e = g_2$. It satisfies $ye = y$ for all $y \in Y$, which implies $fT = fTe = (fe)(Te) = f(eTe)$, and that

$$fT^{n+1} = fT^{n+1}e = (fT^n)Te = (fT^n e)Te = fT^n(eTe)\,.$$

It follows by induction that $fT^n = f(eTe)^n$. Thus, $f\exp(T) = f\exp(eTe)$ and

$$k_1(b)\exp(T) = k_1(b) f \exp(T) = k_1(b) f \exp(eTe) = k_1(b)\exp(eTe) \quad \text{for all } b \in B\,.$$

If we use that $(eTe)^* = eT^*e = -eTe$ and $B^* \cdot B = B$, then we obtain that $\exp(eTe) \in \mathcal{U}_0(J + \mathbb{C} \cdot 1) \subset \mathcal{U}_0(E)$ and, for all $b \in B$,

$$\exp(-eTe)(k_1(b) + \beta k_3(b))\exp(eTe) = (\exp(-T) k_1(b) \exp(T)) + \beta k_3(b)\,.$$

We can reduce the proof of the general *unitary homotopy* to the sub-Stonean case if we consider k_1 and $\exp(-T)k_1(\cdot)\exp(T)$ as maps into suitable sub-Stonean ideals, e.g. if we use that

(1) $I := \mathrm{C_b}([0,\infty),J)/\,\mathrm{C_0}([0,\infty),J)$ is naturally isomorphic to a closed ideal of

$$\mathrm{C_b}([0,\infty),E)/\,\mathrm{C_0}([0,\infty),E) \supset E\,,$$

(2) $J = E \cap I$ is a C^*-subalgebra of I, and
(3) I is a sub-Stonean C^*-algebra.

Property (3) is easy to see. The above considerations show that, for given $T \in E$ with $T^* = -T$, there is a positive contraction $e \in I_+$ such that

$$\exp(-T)k_1(\cdot)\exp(T) \;=\; \exp(-eTe)k_1(\cdot)\exp(eTe)\,.$$

It means equivalently that $\exp(-T)k_1(b)\exp(T) = \lim_{\xi\to\infty} U(\xi)^*k_1(b)U(\xi)$ for all $b \in B$, where we define

$$U(\xi) :=\; \exp(g(\xi)Tg(\xi)) \in \mathcal{U}_0(J + \mathbb{C}\cdot 1) \subset \mathcal{U}_0(E)$$

with $g \in \mathrm{C_b}([0,\infty),J)$ a positive contraction that satisfies $g + \mathrm{C_0}([0,\infty),J) = e$. In particular, $\xi \mapsto U(\xi)$ is norm-continuous and $U(\xi) \in \mathcal{U}_0(\beta J' \cap E)$. It follows that $k_1 + \beta k_3$ and $\exp(-T)k_1(\cdot)\exp(T) + \beta k_3$ are unitarily homotopic via $\xi \mapsto U(\xi)$ for $T \in E$ with $T^* = -T$.

This proves finally that $[k_1] = [k_2]$ and $k_j(B) \subseteq J$ for $j = 1, 2, 3$ imply the unitary homotopy of $k_1 + \beta k_3$ and $k_2 + \beta k_3$. □

REMARK 4.5.6. There are also isometries s_1 and t_1 with $s_1s_1^* + t_1t_1^* = 1$ and $t_1H_0(\cdot) = H_0 = H_0(\cdot)t_1$ as proposed in Part (vi) of Lemma 4.4.7, but with interchanged role of p_0 and $1 - p_0$, i.e., we find those isometries $s_1, t_1 \in E$ with the additional properties $s_1^*p_0s_1 = 1$, $(1 - p_0)t_1 = (1 - p_0) = t_1(1 - p_0)$ instead of $s_1^*(1 - p_0)s_1 = 1$, $p_0t_1 = p_0 = t_1p_0$.

We do not know if in some applications (!) for given $g \in (J + \beta J)_+$ there exists isometries s_1, t_1 with above properties and the additional property that $t_1g = g = gt_1$, i.e., that in addition to $p_0s_1 = s_1$ holds $gs_1 = 0$. It means that there exists a full properly infinite projection $q \in p_0Ep_0$ with $qg = 0$.

PROOF. By Proposition 4.3.6(v), it suffices to find an isometry $R \in E$ with $(1 - p_0)R = 0$ and $R^*H_0(\cdot)R = 0$, i.e., with $R^*p_0H_0(\cdot)R = 0$.

There exists $u \in \mathcal{U}(E)$ with $s_0(h_0 + \beta h_0)s_0^* = u^*(h_0 + \beta h_0)u$ by Lemma 4.4.7(i). Let $R := s_0ut_0$. Then $p_0 = s_0s_0^*$, $s_0^*H_0s_0 = h_0 + \beta h_0$ and $s_0^*t_0 = 0$ imply $p_0R = R$, and $R^*H_0(\cdot)R = 0$. □

Here are some additional remarks concerning the C^*-subalgebras $C^*(p,q)$ and $C^*(p,q,1)$ generated by projections $p, q \in A$.

REMARK 4.5.7. Let A a unital C^*-algebra that is $\mathbb{Z}_2$-graded by $\beta_A \in \mathrm{Aut}(A)$ and let $F_1, F_2 \in A$ symmetries (i.e., $F_k^* = F_k$ and $F_k^2 = 1$) of degree one (i.e., $\beta_A(F_k) = -F_k$).

Then the unital C^*-subalgebra $B := C^*(F_1, F_2)$ of A satisfies $\beta_A(B) = B$.

It follows from Parts (iii) and (v) of Lemma 4.1.3 that there exists a grading preserving unital C^*-epimorphism φ from the universal (real or complex) C^*-algebra $C^*(P,Q,1) \subset \mathrm{C}([0,\pi/2],M_2)$ onto B with $\varphi(1-2P) = F_1$ and $\varphi(1-2Q) = F_2$.

Here we take the *odd* $\mathbb{Z}_2$-grading on $C^*(P,Q,1)$ given as the restriction to $C^*(P,Q,1)$ of the $\mathbb{Z}_2$-grading β on $\mathrm{C}([0,\pi/2],M_2)$ defined by $\beta(f)(\varphi) := Z^* f(\varphi) Z$ for $f \in \mathrm{C}([0,\pi/2],M_2)$ with $Z := [\zeta_{jk}] \in M_2 \subset \mathrm{C}([0,1],M_2)$ with $\zeta_{jk} := j-k$, $j,k \in \{1,2\}$.

In particular, if $\|F_1 - F_2\| < 2$ then there exists $h \in C^*(F_1,F_2)$ of degree zero (i.e., $\beta_A(h) = h$), with $h^* = -h$, and $\|h\| < \pi/2$ such that $F_2 = \exp(-h)F_1\exp(h)$.

The $\mathbb{Z}_2$-grading β of $\mathrm{C}([0,\pi/2],M_2)$ is odd in case $M_2 := M_2(\mathbb{R})$ and is even if $M_2 := M_2(\mathbb{C})$, because there is no symmetry S in $M_2(\mathbb{R})$ and $\mu \in \mathbb{R}$ with $Z = \mu S$, but $(iZ)^* = iZ =: S$ is a symmetry in $M_2(\mathbb{C})$.

Remark 4.5.8. A continuous path $\xi \in [0,1] \mapsto p_\xi \in A$ of projections p_ξ with $p_0 = p$ and $p_1 = q$ for projections $p,q \in A$ with $\|p-q\| < 1$, can be defined by $p_\xi := (U(\xi)+1)/2$ with symmetries $U(\xi) := |T(\xi)|^{-1}T(\xi)$ obtained from the path

$$T(\xi) := \sin(\xi\pi/2)(2q-1) + \cos(\xi\pi/2)(2p-1)$$

of self-adjoint invertible operators $T(\xi)$ with $T(\xi)^2 \geq 2(1-\|p-q\|^2)$ in $A + \mathbb{C}\cdot 1$, cf. [**73**, prop. 4.6.6] and proof of [**816**, prop. 5.2.6].

Parts (v) and (vi) of Lemma 4.1.3 show that the minimal length of a continuous path $\xi \mapsto u(\xi) \in \mathcal{U}(A)$ of unitary operators with $u(0) = 1$ and $u(1)^*pu(1) = q$ is exactly $\arcsin\|p-q\|$, because the *lower* bound for this length can be seen in M_2 itself.

Remark 4.5.9. There is an alternative way in the *complex case* to get the parameters for the C^*-morphisms of the universal unital C^*-algebra $C^*(P,Q,1)$ generated by projections P,Q into M_2 by considering the isomorphism of $C^*(P,Q,1)$ with universal C^*-algebras

$$C^*(P,Q,1) = C^*(s,t) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong C^*(\mathbb{Z} \rtimes \mathbb{Z}_2) \cong \mathrm{C}(S^1) \rtimes \mathbb{Z}_2$$

with self-adjoint isometries $s := 1-2P$, $t := 1-2Q$, $\mathbb{Z} \subset \mathbb{Z}_2 * \mathbb{Z}_2$ given by $n \in \mathbb{Z} \mapsto (st)^n \in \mathbb{Z}_2 * \mathbb{Z}_2$ and the $\mathbb{Z}_2$-action $n \mapsto -n$ on $\mathbb{Z}$ realized by $(st)^n \mapsto s(st)^n s = (st)^{-n}$. Finally, $C^*(\mathbb{Z})$ will be naturally identified with $\mathrm{C}(\widehat{\mathbb{Z}}) = \mathrm{C}(S^1)$. We get a parametrization by $\varphi \in (0,\pi/2)$ of the simple quotients of $\mathrm{C}(S^1) \rtimes \mathbb{Z}_2$ with the 4 characters attached at 0 and $\pi/2$ in the same way as the parametrization given by the embedding of $C^*(P,Q,1)$ into $\mathrm{C}([0,\pi/2],M_2)$ with the 4 characters attached at 0 and $\pi/2$.

Let $F := [f_{jk}] = \mathrm{diag}(-1,1)Z \in M_2$ the flip-symmetry with $f_{jk} := |1-\delta_{jk}|$. Then the inner automorphism $U^*(\cdot)U$ of $M_2(\mathbb{C})$ given e.g. by the unitary $U := [u_{jk}] \in M_2(\mathbb{C})$ with entries $u_{11} := -i/\sqrt{2} =: -u_{21}$ and $u_{12} := u_{22} := 1/\sqrt{2}$ transforms all pairs of unitary elements $\mathrm{diag}(e^{2i\varphi},e^{-2i\varphi}), F \in M_2(\mathbb{C})$ into the pairs $t(\varphi), s$ defined in our case as

$$(1-2P)(1-2Q(\varphi)),\ 1-2p \in \mathrm{O}(2) \subset M_2(\mathbb{R}).$$

Notice here that

$$st(\varphi) = (1-2P)(1-2Q(\varphi)) = (1-2P)\exp(-\varphi Z)(1-2P)\exp(\varphi Z) = \exp(2\varphi Z)$$

with P, Z, Q as in Lemma 4.1.3(iii). Then it suffices to check that $U^*U = 1_2$, $FU = U\,\mathrm{diag}(-1,1)$ and $\mathrm{diag}(i,-i)U = UZ$, to obtain that the parameter $\varphi \in [0, \pi/2]$ defines the same C^*-morphism from $C^*(P, Q, 1)$ into M_2.

6. On K-theory of corona C^*-algebras

The homotopy of id and δ_2 and that $\mathcal{O}_2$ is a pi-sun algebra can be used to derive that they are unitary equivalent in

$$\ell_\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)\,.$$

It turns out that this property implies that $\mathcal{O}_2$ has a central sequence of unital copies of $\mathcal{O}_2$...

Some almost $\mathcal{E}$-theory results:

Suppose that B is a σ-unital C^*-algebra and that $\mathcal{M}(B)$ is properly infinite, i.e., that $\mathcal{O}_\infty$ is unitally contained in $\mathcal{M}(B)$, let A a separable C^*-subalgebra of B, and suppose that there is a sequence of point-norm continuous paths of c.p. contractions $V_{n,\xi} \colon A \to B$ ($n \in \mathbb{N}$, $\xi \in [0,1]$) given, where $\mathcal{M}(B)$ is properly infinite, i.e., there are isometries $S, T \in \mathcal{M}(B)$ with $S^*T = 0$.

Is this the Def. of "properly infinite" C*-algebras?

We can manage that $SS^* + TT^* = 1$ if $[1_{\mathcal{M}(B)}] = 0$ in $\mathrm{K}_0(\mathcal{M}(B))$ (e.g. this is the case if B is stable) and other-wise we find the isometries $S, T \in \mathcal{M}(B)$ such that $1 - SS^* + TT^*$ is a properly infinite full projection, e.g. we can replace S, T by TS, ST and have $T^2(T^2)^* \leq 1 - (TSS^*T^* + STT^*S^*)$.

Notice that $[1 - SS^* + TT^*] = -[1]$ in $\mathrm{K}_0(\mathcal{M}(B))$.

We define

(i) a unital C^*-morphism $\lambda \colon \mathcal{O}_\infty \to \mathcal{M}(B)$ by $\lambda(T_n) := S_n := T^{n-1}S$ (with $T^0 := 1$) and

(ii) $V_{\infty,\alpha} \colon A \to B_\infty := \ell_\infty(B)/c_0(B)$ for $\alpha \in \{0,1\}$ by

$$V_{\infty,\alpha}(a) := (V_{1,\alpha}(a), V_{2,\alpha}(a), \ldots) + c_0(B)\,.$$

Now fix a linear filtration $X_1 \subseteq X_2 \subseteq$ of A by finite-dimensional subsets with $\bigcup_n X_n$ dense in A.

Then there exists "Cuntz-averages"

$$W_n := \sum_{k=1}^{k_n} S_k V_{n,\xi_k} S_k^*$$

of the $V_{n,\xi}$ ($\xi \in [0,1]$) with suitable $0 = \xi_1 < \xi_2 < \ldots < \xi_{k_n} = 1$ and unitaries $U_n \in \mathcal{M}(B)$ with

(iii) $\|V_{n,\tau_k}(a) - V_{n,\tau_{k+1}}(a)\| < 2^{-n}$ for $a \in X_n$, and

(iv) $S^*U_n = S_1^*S^*$, $S_k^*T^*U_n = S_{k+1}^*S^*$ for $k = 1, \ldots, k_n - 1$ and $S_{k_n}^*T^*U_n = T^*$, i.e., with suitable partial isometry $Z_n \in \mathcal{M}(B)$ the unitary U_n is given by

$$U_n := SS_1^*S^* + TS_1S_2^*S^* + \ldots + TS_kS_{k+1}^*S^* + TS_{k_n-1}S_{k_n}^*S^* + TS_{k_n}T^* + Z_n .$$

The partial isometry $Z_n \in \mathcal{M}(B)$ used in the definition of the unitary U_n exists by our assumptions on the isometries S, T, because $S_k = T^{k-1}S$.

$$Y := SS_1^*S^* + TXS^* + TS_{k_n}T^* ,$$

with

$$X := \sum_{k=1}^{k_n-1} S_kS_{k+1}^* .$$

Then $X^*S_{k_n} = 0 = S_{k_n}^*X$, $X^*X = \sum_{k=1}^{k_n-1} S_{k+1}S_{k+1}^*$ and

$$XX^* =???? = \sum k = 1^{k_n-1} S_kS_k^* .$$

$S_kS_{k+1}^*S_{\ell+1}S_\ell^* = \delta_{k,\ell}S_kS_k^*$

$$\sum_{k,\ell=1}^{k_n-1} S_kS_{k+1}^*\, S_{\ell+1}S_\ell^* = \sum_{k=1}^{k_n-1} S_kS_k^*$$

It follows

$$Y^*Y = (\sum_{j=1}^{k_n} S_jS_j^*) \oplus_{S,T} 1$$

and

$$YY^* = 1 \oplus_{S,T} (\sum_{j=1}^{k_n} S_jS_j^*)$$

The projection $P_{k_n} := 1 - \sum_{j=1}^{k_n} S_jS_j^*$ is full and properly infinite by construction of the S_k, because $P_{k_n}S_\ell = S_\ell$ for $\ell > k_n$.

Then $W_\infty(a) := (W_1(a), W_2(a), \ldots) + c_0(B)$ is a c.p.c. map from A into B_∞ and $U_\infty = (U_1, U_2, \ldots) + c_0(B)$ is a unitary in $\mathcal{M}(B_\infty)$ that satisfies

$$U_\infty^*(V_{\infty,0} \oplus_{S,T} W_\infty)U_\infty = W_\infty \oplus_{S,T} V_{\infty,1}$$

in $B_\infty := \ell_\infty(B)/c_0(B)$.

The same happens for B_ω in place of B_∞, simply by passing to the quotient B_ω of B_∞ and by observing that this epimorphism maps $\mathcal{M}(B_\infty)$ unitally into $\mathcal{M}(B_\omega)$.

A bit more technical work is needed for the construction of ?????

The key trick of N.Ch. Phillips in [**627**] is the construction of suitable interpolations between the above defined “Cuntz averages” for point norm continuous maps $V_{t,\xi} \in \mathrm{CP}(A, B)$ ($t \in [0, \infty)$, $\xi \in [0, 1]$), i.e., construct fitting strictly continuous paths $A \ni a \mapsto W_t(a) \in B$ in the approximately inner c.p.c. maps $W_t \in \mathrm{CP}(A, B)$

that satisfy that $V_{t,0} \oplus W_t$ and $W_t \oplus V_{t,0}$ are asymptotic unitarily equivalent by a norm-continuous path $t \to U(t)$ of unitaries in $\mathcal{M}(B)$.

We do this in case of special classes of $V_{t,\xi} \in \mathcal{C} \subseteq \mathrm{CP}(A, B)$ that cover the cases considered by N.Ch. Phillips.

Further plan for topics:
Applications to unital endomorphisms of $\mathcal{O}_\infty$
(all are unitarily homotopic),
permutation $C*$-morphisms of $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$, $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$
(are all unitarily homotopic)
and to the n-repeat $\delta_n \colon \mathcal{O}_{\mathrm{n}} \to \mathcal{O}_{\mathrm{n}}$
(δ_n unitarily homotopic to $\mathrm{id}_{\mathcal{O}_{\mathrm{n}}}$,
unital endomorphisms are all unitarily homotopic to id
existence of asymptotic central path of copy of $\mathcal{O}_2$ in $\mathcal{O}_2$).

Following should be a remark in Chp.4?
But with target $\mathcal{O}_2$ replaced by $Q(\mathbb{R}_+, \mathcal{O}_2)$? Partly some appears even in ?? Chp.1.!. But at beginning of Chp.5 there are some calculations in this direction

All unital C^*-morphisms $h \colon \mathcal{O}_2 \to A$ are unitarily homotopic to each other, if and only if, they are homotopic, – and in “good cases” of A (e.g. if A is K_1-injective)– if and only if, the related unitaries are in $\mathcal{U}_0(A)$?

$h_1, h_2 \colon \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, A)$ both are in “general position” and represent the zero element of $[\mathrm{Hom}_u(\mathcal{O}_2, \mathrm{Q}(\mathbb{R}_+, A))]$, because they dominate each other. In general the unitary equivalence can not be given by a unitary in $\mathcal{U}_0(\mathrm{Q}(\mathbb{R}_+, A))$. Example: The unital free product $\mathcal{O}_2 * \mathrm{C}(S^1) \cong \mathcal{O}_2 * \mathcal{O}_2$.

The latter could be deduced from a very special case of Theorem A: There is a unital $g \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$, and $a \mapsto h(a) := g(a \otimes 1)$ is unitarily homotopic inside $\mathcal{O}_2$ to $\mathrm{id}_{\mathcal{O}_2}$. Then $h_1(g((\cdot) \otimes 1))$ and $h_2(g((\cdot) \otimes 1))$ dominate each other asymptotically by

Corollary ?? (on the asymptotic domination of residually nuclear C^*-morphisms $h_k \colon A \to B$ of stable separable C^*-algebras A and B, where A is exact and separable and B is s.p.i.).

Suppose that E is strongly purely infinite, and that A is separable and exact, (or if E is not s.p.i., that A is separable, s.p.i. and exact). Then every nuclear C^*-morphism $\varphi \colon A \to E$ extends to a C^*-morphism $\psi \colon A \otimes \mathcal{O}_\infty \to \mathrm{Q}(\mathbb{R}_+, E)$, by Proposition 3.0.4.

Let $g_k \colon A \otimes \mathcal{O}_\infty \to E$ $(k = 1, 2)$ C^*-morphisms such that the C^*-morphisms $h_k(a) := g_k(a \otimes 1)$ generate the same m.o.c. cone in $\mathrm{CP}(A, E)$, then h_1 and h_2 1-step-dominate each other in $\mathrm{Hom}(A, E^\infty)$ or $\mathrm{Hom}(A, \mathrm{Q}(\mathbb{R}_+, E))$.

If the h_k are injective and nuclear, then they generate the same m.o.c. cone, if and only if, for each $a \in A_+$, the elements $h_1(a)$ and $h_2(a)$ generate the same ideal of E.

In particular, for any two injective unital nuclear C^*-morphisms $g_1, g_2 \colon A \otimes \mathcal{O}_2 \to E$ that induce the same action of $\mathrm{Prim}(E)$ on A the C^*-morphisms h_1 and h_2 are unitarily homotopic.

Because one finds isometries $S, T \in \mathrm{Q}(\mathbb{R}_+, E)$ with $S^* h_1(\cdot) S = h_2$ and $T^* h_2(\cdot) T = h_1$ by simplicity and nuclearity of the h_k and the pure algebraic fact that there commutant algebras contain (possibly different) copies of $\mathcal{O}_2$ unitally.

Then we get from Chapter 4 that h_1 and h_2 both are unitarily equivalent in $\mathrm{Q}(\mathbb{R}_+, E)$ to $h_1 \oplus h_2$.

The situation of unital C^*-morphisms form $\mathcal{O}_2$ into the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ is different:

The (usual) *homotopy classes* of the unital $h \colon \mathcal{O}_2 \to \mathrm{Q}^{\mathrm{s}}(B)$ are in bijective correspondence to $\mathrm{K}_0(B) \cong \mathrm{K}_1(\mathrm{Q}^{\mathrm{s}}(B))$ by the K_1-bijectivity of stable coronas by Proposition **??**, but there is *only one* (sic !) class of *unitary homotopy*.

More generally, for every unital C^*-algebra E with properly infinite unit that satisfies $0 = [1_E] \in \mathrm{K}_0(E)$ there is up to unitary homotopy only one unital C^*-morphism $h \colon \mathcal{O}_2 \to E$, cf. Section **??**, but the homotopy classes of unital $h \in \mathrm{Hom}(\mathcal{O}_2, E)$ is identical with $\mathcal{U}(E)/\mathcal{U}_0(E)$, which is isomorphic to $\mathrm{K}_1(E)$ if and only if E is K_1-injective in addition (As is e.g. the case where E stable corona $E = \mathrm{Q}^{\mathrm{s}}(A)$ of a σ-unital C^*-algebra A or where E is strongly purely infinite.)

An example is the Calkin algebra $\mathcal{C} := \mathcal{C}(\ell_2) := \mathcal{L}(\ell_2)/\mathbb{K}(\ell_2)$. Then each unital C^*-morphism $\psi \colon \mathcal{O}_2 \to \mathcal{C}$ is up to unitary equivalence given by multiplying the $\pi_{\mathbb{K}} \circ \rho(s_j)$ with a unitary in $\mathcal{C}$ from the left. The multiplication with images of Toeplitz operators T^n or $(T^*)^m$ exhausts all cases (up to unitary equivalence in $\mathcal{C}$?).

Corollary 4.6.1. *There is a copy of $\mathcal{O}_{\mathrm{n}}$ unitally contained in ${\mathcal{O}_{\mathrm{n}}}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_{\mathrm{n}})$, i.e., there exists a point-norm continuous path of u.c.p. maps $V_t \colon \mathcal{O}_{\mathrm{n}} \otimes \mathcal{O}_{\mathrm{n}} \to \mathcal{O}_{\mathrm{n}}$ such that $\lim_{t\to\infty} \|V_t(a \otimes 1) - a\| = 0$ for all $a \in \mathcal{O}_{\mathrm{n}}$, $\lim_{t\to\infty} \|[a, V_t(1 \otimes b)]\| = 0$ for all $a, b \in \mathcal{O}_{\mathrm{n}}$ and*

$$\lim_{t\to\infty} \|V_t(1 \otimes b^* b) - V_t(1 \otimes b)^* V_t(1 \otimes b)\| = 0 \quad \textit{for all } b \in \mathcal{O}_{\mathrm{n}}\,.$$

The proof should follow from a "unsuspended" E-theory argument concerning asymptotic homotopy of id and δ_n considered in $\mathrm{Q}(\mathbb{R}_+ \times [0,1], \mathcal{O}_{\mathrm{n}})$, respectively the then unitary equivalence $\delta_n(a) = U^* a U$ in $\mathrm{Q}(\mathbb{R}_+, \mathcal{O}_{\mathrm{n}})$ for $a \in \mathcal{O}_{\mathrm{n}} \subseteq \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_{\mathrm{n}})$. The unital embedding $\mathcal{O}_{\mathrm{n}} \to {\mathcal{O}_{\mathrm{n}}}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_{\mathrm{n}})$ is defined by the elements $U s_j$ $(j = 1, \ldots, n)$.

Corollary 4.6.2. *For every strongly purely infinite, unital, separable and nuclear C^*-algebra A there is a copy of $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$ unitally contained in $A' \cap \mathrm{Q}(\mathbb{R}_+, A)$.*

It uses that $\mathcal{O}_\infty \subseteq \mathcal{M}(A)$ (unitally) and the nuclear map $a \to s_1 a s_1^* + s_2 a s_2^*$ is asymptotical 1-step dominated by id_A, because every nuclear ideal system preserving c.p. map is approximately 1-step inner. (See study of nuclear c.p. maps in s.p.i. algebras).

Find contraction $T \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(A))$ with $s_1 a s_1^* + s_2 a s_2^* = T^* a T$ it implies that $\delta_{jk} \cdot a = s_j^* T^* a T s_k = t_j^* a t_k$ for the contractions $t_k := T s_k$.

More ???? Tensor by $\mathbb{K}$ first or not? ...

We shall see in Chapter 10 that this implies that $A \otimes \mathcal{O}_\infty \cong A$ for every purely infinite separable nuclear A, using that there exists a unital C^*-morphism $\eta \colon \mathcal{O}_\infty \to \mathrm{C}([0,1], \mathcal{O}_\infty \otimes \mathcal{O}_\infty)$ with $\pi_0(\eta(b)) = b \otimes 1$ and $\pi_1(\eta(b)) = 1 \otimes b$ by $\eta(s_n)(t) :=$????? ?????????????? `exact formulations needed!!!` And where is the proof? of .

COROLLARY 4.6.3. *If B is separable, nuclear and strongly purely infinite then there exists a norm-continuous path of *-monomorphisms $h_t \colon B \otimes \mathcal{O}_\infty \to B$ such that* $\lim_{t\to\infty} \|h_t(b \otimes 1) - b\| = 0$.

Next corollary will be used in Chapter 5.

COROLLARY 4.6.4. *Suppose that A is a unital C^*-algebra and $h_1, h_2 \in \mathrm{Hom}(\mathcal{O}_2, A)$ are unital C^*-morphisms, then there exists a norm-continuous path $t \in [0, \infty) \mapsto u(t) \in \mathcal{U}(A)$ such that $h_2(x) = \lim_{t\to\infty} u(t)^* h_1(x) u(t)$ for all $x \in \mathcal{O}_2$.*

PROOF. There exist a unital C^*-morphism $k_0 \colon \mathcal{O}_2 \to {\mathcal{O}_2}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$, by Corollary 4.6.1.

`Give exact and precise reference!`

We write $\mathcal{O}_2 := C^*(s_1, s_2)$, $s_1^* s_1 = 1 = s_2^* s_2$, $s_1 s_1^* + s_2 s_2^* = 1$.

The unital C^*-morphisms $h_1, h_2 \colon \mathcal{O}_2 \to A$ extend to unital C^*-morphisms H_1 and H_2 from $\mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ to $\mathrm{Q}(\mathbb{R}_+, A)$ with $H_\ell | \mathcal{O}_2 = h_\ell$ ($\ell \in \{1, 2\}$).

Then $k_\ell := H_\ell \circ k_0$ are unital C^*-morphisms from $\mathcal{O}_2$ into $h_\ell(\mathcal{O}_2)' \cap \mathrm{Q}(\mathbb{R}_+, A)$ ($\ell \in \{1, 2\}$).

It suffices to show that there exists a unitary $U \in \mathrm{Q}(\mathbb{R}_+, A)$ with $U^* h_1(s_j) U = h_2(s_j)$, $j \in \{1, 2\}$.

`To be filled in` ?? □

`?? Is next Lemma 4.6.5 in a good position? Move it to Chapter 5?`

LEMMA 4.6.5. *Suppose that B is a stable C^*-algebra and that $\psi \colon D \to \mathcal{M}(B)$ is a C^*-morphism from separable D to $\mathcal{M}(B)$.*

The relative commutant $\delta_\infty(\psi(D))' \cap \mathcal{M}(B)$ of the image of the infinite repeat $\delta_\infty \circ \psi$ has trivial K_*-groups and is closed in the strict topology.

The infinite repeat $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ is given by $\delta_\infty(b) := \sum_n s_n b s_n^*$ for $b \in \mathcal{M}(B)$ where $s_1, s_2, \ldots$ is a sequence of isometries in $\mathcal{M}(B)$ with $\sum_n s_n s_n^* = 1$ strictly convergent, cf. Remark 5.1.1(8) and Lemma 5.1.2(i,ii).

Proof. Let $F := \delta_\infty(\psi(D))' \cap \mathcal{M}(B)$. It is closed under strict convergence. Certainly, $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B) \subseteq F$. By Lemma 5.1.2(i),

early citations ? ??

there are isometries $t_1, t_2, \ldots$ in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ such that $\sum t_n t_n^*$ strictly converges to 1. By Lemma 5.1.2(ii), there are isometries r_1 and r_2 in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ with $r_1 r_1^* + r_2 r_2^* = 1$.

Let $h\colon \mathcal{M}(B) \to \mathcal{M}(B)$ denote the infinite repeat of id which is defined by $t_1, t_2, \ldots$, cf. Remark 5.1.1(8).The element $h(b) = \sum$ is in F if $b \in F$, because F is strictly closed. Thus, $h|F$ is a unital *-endomorphism of F. Let $u := r_1 t_1^* + \sum r_2 t_n t_{n+1}^*$. The right side converges strictly to a unitary u in $\mathcal{M}(B)$ by proof of Lemma 5.1.2(i) (see e.g. Remark 5.1.1(2) for the strict convergence). Since its summands are in F, we get $u \in F$. It holds $u^*(b \oplus_{r_1,r_2} h(b))u = h(b)$ for $b \in F$ by the convergence argument in Remark 5.1.1(2) and in the proof of Lemma 5.1.2(ii). Thus $\mathrm{K}_*(F) = 0$ ([9]). □

The following Lemma 4.6.6 is a K_1-counterpart to Proposition 4.4.3(iii). 4.4.3(iii) correct? ??

Lemma 4.6.6. *Suppose that B is stable and σ-unital, D is stable and separable, and that $H\colon D \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism such that $\delta_\infty \circ H$ is unitarily equivalent to H (by an unitary in $\mathcal{M}(B)$).*

Let $E := \mathrm{Q}^{\mathrm{s}}(B) = \mathcal{M}(B)/B$, $H_0 := \pi_B \circ H$, $C := H_0(D)' \cap E$, and let $\mathcal{U}(H(D), B)$ denote the set of unitaries in $\mathcal{M}(B)$ that commute modulo B element-wise with $H(D)$.

(i) *$\mathcal{U}(H(D), B)$ is closed under Cuntz addition and $U \mapsto [U + B]$ is an additive epimorphism from $\mathcal{U}(H(D), B)$ onto the kernel of $i_1\colon \mathrm{K}_1(C) \to \mathrm{K}_1(E)$, where $i_1 = \mathrm{K}_1(i)$ for the inclusion map $i\colon C \hookrightarrow E$.*

(ii) *Let $U_1, U_2 \in (H(D), B)$ and let $U := U_1 \oplus 1$, $U' := U_2 \oplus 1$. Then $[U_1 + B] = [U_2 + B]$ in $\mathrm{K}_1(H_0(D)' \cap E)$,*
if and only if,
there exist $U'' \in \mathcal{U}(H(D), B)$ such that $U'' - U \in B$ and U' is homotopic to U'' in $\mathcal{U}(H(D), B)$,
if and only if,
there exist $U'', V \in \mathcal{U}(H(D), B)$ such that $V^* U' V$ is homotopic to U'' in $\mathcal{U}(H(D), B)$ and $(U'' - U)H(D) \subseteq B$.

[9] More generally, the existence of a unital endomorphism $h\colon F \to F$ with $[h \oplus \mathrm{id}_F] = [h]$ and $h \circ \mathcal{C} \subseteq \mathcal{C}$ implies $\mathrm{KK}(\mathcal{C}; A, F) = 0$ for all our generalized KK-theories in Chapter 8.

PROOF. There exists a unital C^*-morphism $\mathcal{O}_2 \to \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$. The latter is contained in $H(D)' \cap \mathcal{M}(B)$. It defines the needed isometries $s_1, s_2 \in H(D)' \cap \mathcal{M}(B)$, cf. e.g. Lemma 5.1.2(ii).

?? Is it good to ref Chp 5 here?

(i): It is easy to see that $\mathcal{U}(H(D), B)$ is closed under Cuntz addition, because $\mathcal{O}_2$ commutes with $H(D)$. The image of $\mathcal{U}(H(D), B)$ under $U \to U + B \in E$ goes into the unitary group of C. $\mathrm{K}_1(\mathcal{M}(B)) = 0$ because B is stable. Thus $[U+B]_E = 0$ in $\mathrm{K}_1(E)$ because $[U]_{\mathcal{M}(B)} = 0$ in $\mathrm{K}_1(\mathcal{M}(B))$. The Cuntz addition in $\mathcal{M}(B)$ induces the Cuntz addition in E: $(U_1 \oplus U_2) + B = (U_1 + B) \oplus (U_2 + B)$. By Lemma 4.2.6(v),the natural map from the unitaries of C into $\mathrm{K}_1(C)$ is an additive epimorphism with respect to Cuntz addition on C.

It remains to show that $U \to [U + B]$ is an epimorphism onto the kernel of $\mathrm{K}_1(C) \to \mathrm{K}_1(E)$. Let $x \in \mathrm{K}_1(C)$ be in the kernel of $i_1 \colon \mathrm{K}_1(C) \to \mathrm{K}_1(E)$. Then there is a unitary in $u \in C$ with $[u]_C = x$, $[u]_E = 0$. Since $1_E \subseteq \mathcal{O}_2 \subseteq C \subseteq E$, we can use Lemma 4.2.6(v,2) and get $[u \oplus 1]_C = [u]_C = x$ and that $u \oplus 1$ is homotopic to 1 in the unitaries of E. Thus, there is a unitary $U \in \mathcal{M}(B)$ with $U + B = u \oplus 1$. U commutes element-wise with $H(D)$ modulo B, because $U + B = u \oplus 1 \in C = ((H(D) + B)/B)' \cap E$.

(ii): Let $U' := U_1 \oplus 1$, $U'' := U_2 \oplus 1$ and $v_1 := U_1 + B$ and $v_2 := U_2 + B$ the corresponding unitaries in C. By Lemma 4.2.6(v,2), $[v_1]_C = [v_2]_C$ if and only if $v_1 \oplus 1 = U' + B$ is homotopic to $v_2 \oplus 1 = U'' + B$ in the unitaries of C. This homotopy can be lifted to a continuous arc $t \in [0, 1] \mapsto U(t)$ in the unitaries of $\mathcal{M}(B)$ starting from $U(0) = U'$ and ending at $U'' := U(1)$. Then $U'' - U \in B$. But if V is a unitary in $\mathcal{M}(B)$ such that $V + B \in C$ then V commutes element-wise with $H(D)$ modulo B. Thus $U(t) \in \mathcal{U}(H(D), B)$ for $t \in [0, 1]$.

Conversely, let $U := U_1 \oplus 1$, $U' := U_2 \oplus 1$ and suppose that there exist unitaries $V, U'' \in \mathcal{U}(H(D), B)$ and a continuous arc $t \in [0, 1] \mapsto U(t)$ into $\mathcal{U}(H(D), B)$ with $U(0) = V^* U' V$, $U(1) = U''$ and $(U'' - U)H(D) \subseteq B$.

Let $v_1 := U_1 + B$, $v_2 := U_2 + B$, $v_3 := V + B$ and $v_4 := U'' + B$. Then $[v_2]_C = [v^3(v_2 \oplus 1)v_3]_C = [v_4]_C$ in $\mathrm{K}_1(C)$, because $[1]_C = 0$ and $t \mapsto U(t) + B$ connects $v^3(v_2 \oplus 1)v_3$ and v_4 in the unitaries of C.

On the other hand, $v_1 \oplus 1, v_4 \in C$, $(v_1 \oplus 1) - v_4$ is in the annihilator $\mathrm{Ann}(H_0(D))$ of $H_0(D)$ in E, and $[v_1 \oplus 1]_E = [v_4]_E = 0$ in $\mathrm{K}_1(E)$. By Proposition 5.5.12(ii), $H_0 \colon D \to E$ dominates zero, because $(\pi_B)^{-1}(H_0(D)) = B + H(D)$ contains the stable and non-degenerate C^*-subalgebra $H(D)$. Therefore, the two-sided annihilator $\mathrm{Ann}(H_0(D))$ of $H_0(D) \subseteq E$ is a *full* hereditary C^*-subalgebra of E. The annihilator $\mathrm{Ann}(H_0(D))$ is an ideal of $C = H_0(D)' \cap E$. Thus, Lemma 4.2.20(ii) applies to $N := \mathrm{Ann}(H_0(D)) \subseteq C \subseteq E$ and $v_1 \oplus 1$, $v_4 \in C$. We get $[v_1] = [v_1 \oplus 1] = [v_4] = [v_1]$ in $\mathrm{K}_1(C)$. □

COROLLARY 4.6.7. *Suppose that $J \subseteq E \subseteq F$ are C^*-subalgebras of F and satisfy the following conditions* (i)–(v)*:*

(i) $1_F \in E$.
(ii) J *and is an ideal of* F.
(iii) *For every* $b \in J_+$ *there is* $c \in J_+$ *with* $\|c\| = 1$ *and* $cb = b$.
(iv) *For every* $c \in J_+$ *there is an isometry* $t \in E$ *with* $ct = 0$.
(v) E *contains a copy of* $\mathcal{O}_2$ *unitally.*

If A *is a* σ*-unital* C^**–algebra, and* $h_1, h_2 \colon A \to J$ *are* C^**-morphisms, then the following* (a)–(c) *are equivalent:*

(a) $h_2 = U^* h_1(\cdot) U$ *for a unitary* U *in the* connected component $\mathcal{U}_0(E)$ of 1 *in the group* $\mathcal{U}(E)$ *of unitaries of* E.
(b) h_1 *and* h_2 *are unitarily equivalent in* F.
(c) h_1 *and* h_2 *are unitarily equivalent in* $\mathcal{M}(J)$.

Proof. (a)⇒(b)⇒(c): There are natural unital C^*-morphisms

$$E \to F \to \mathcal{M}(J)$$

that fix the elements of J by assumptions (i) and (ii).

(c)⇒(a): Suppose that $U \in \mathcal{M}(J)$ is a unitary with $h_2 = U^* h_1(\cdot) U$. Let $e \in A_+$ a strictly positive element of A, and let $b := h_1(e) + h_2(e)$. By (iii) and (iv), there are $c \in J_+$ and $t \in E$ with $\|c\| \leq 1$, $cb = b$, $ct = 0$ and $t^* t = 1$. It follows $ch_k(\cdot) = h_k = h_k(\cdot)c$ $(k = 1, 2)$, and $t^*(h_1(a) + h_2(a))t = 0$ for $a \in A$. Let $x := cUc$. Then x is a contraction in $J \subseteq E$ that satisfies $x^* h_1(\cdot) x = h_2$ and $x h_2(\cdot) x^* = h_1$. Since $t \in E$ and E contains a copy of $\mathcal{O}_2$ unitally by (v), the Parts (iii) and (iv,c) of Proposition 4.3.6 apply: There is $V \in \mathcal{U}_0(E)$ with $V^* h_1(\cdot) V = h_2$. □

Remark 4.6.8. We consider in Chapters 7 and 9, the case, where $F := \mathrm{Q}(\mathrm{C}_0(X, B)) := \mathcal{M}(\mathrm{C}_0(X, B)) / \mathrm{C}_0(X, B)$, $E := \mathrm{C}_\mathrm{b}(X, \mathcal{M}(B)) / \mathrm{C}_0(X, B)$ and $J := \mathrm{Q}(X, B) := \mathrm{C}_\mathrm{b}(X, B) / \mathrm{C}_0(X, B)$ for stable σ-unital B and a σ-compact locally compact Hausdorff space X.
Note that in this particular case, J is the (two-sided) annihilator $\mathrm{Ann}(C, F)$ of the C^*-subalgebra $C := \pi_{\mathrm{C}_0(X,B)}(\mathrm{C}_0(X) \cdot 1)$ of the center of F.

Finally, we say some words about the groups $G_\approx(h_0\,;\, D, E)$ in case where D and h_0 are unital and E is K_1-injective, because this is important for the comparison of stable extensions with unital extension in case where $E = \mathrm{Q}^\mathrm{s}(B) = \mathcal{M}(B)/B$ for some σ-unital stable B.

Remark 4.6.9. Suppose that D is *unital*, that $h_0 \colon D \to E$ is *unital* and that isometries $t_1, t_2 \in E$ are canonical generators of a copy of $\mathcal{O}_2$ in E. Let $h \in \mathrm{Hom}(D, E)$.

Recall that $[h]_\approx \in [\mathrm{Hom}(D, E)]_\approx$ denotes the class of $k \in \mathrm{Hom}(D, E)$ with $k = u^* h(\cdot) u$ for a unitary in the *connected component* $\mathcal{U}_0(E)$ of 1 in in the unitary group $\mathcal{U}(E)$, and that $[\mathrm{Hom}(D, E)]_\approx$ becomes a semigroup with Cuntz addition $[h_1]_\approx + [h_2]_\approx := [h_1 \oplus_{s_1, s_2} h_2]_\approx$ (see Proposition 4.3.2).

Suppose now – in addition – that $[h_0]_\approx + [h_0]_\approx = [h_0]_\approx$ in $([\mathrm{Hom}(D,E)]_\approx, +)$. The proof of Proposition 4.3.5(iii) shows that this equivalently means that there is $u \in \mathcal{U}_0(E)$ such that $s_1 := ut_1$, $s_2 := ut_2 \in h_0(D)' \cap E$.

Notice that $s_1^* s_1 = s_2^* s_2 = 1$ and $s_1 s_1^* + s_2 s_2^* = 1$, and that $[h_1 \oplus_{t_1,t_2} h_2]_\approx = [h_1 \oplus_{s_1,s_2} h_2]_\approx$ for all $h_1, h_2 \in \mathrm{Hom}(D,E)$. Therefore we can write $\oplus$ for $\oplus_{s_1,s_2}$.

We can define a subset $S_\approx(h_0\,;\, D,E) \subseteq [\mathrm{Hom}(D,E)]_\approx$:

$$S_\approx(h_0\,;\, D,E) := \{[h]_\approx \,;\, \exists (h' \in \mathrm{Hom}(D,E)) : [h]_\approx + [h']_\approx = [h_0]_\approx\}.$$

Since $[h_0]_\approx + [h_0]_\approx = [h_0]_\approx$, the set $S_\approx(h_0\,;\, D,E)$ is a sub-semigroup of $[\mathrm{Hom}(D,E)]_\approx$. The set $G_\approx(h_0\,;\, D,E) := [h_0]_\approx + S_\approx(h_0\,;\, D,E)$ is a subgroup of $S_\approx(h_0\,;\, D,E)$ that is naturally isomorphic to the Grothendieck group $\mathrm{Gr}(S_\approx(h_0\,;\, D,E))$ of $S_\approx(h_0\,;\, D,E)$ by Lemma 4.2.3.

The definition of $S_\approx(h_0\,;\, D,E)$ shows that $[h]_\approx \in S_\approx(h_0\,;\, D,E)$ implies that there is $u \in \mathcal{U}_0(E)$ with $h = s_1^* u^* h_0(\cdot) u s_1$. In particular, $h\colon D \to E$ is unital.

Conversely, if $u \in \mathcal{U}_0(E)$ and if $h := s_1^* u^* h_0(\cdot) u s_1$ is multiplicative, then $us_2(s_2u)^* = 1 - us_1(s_1u)^*$ commutes with $h_0(D)$ and $s_1 s_1^* = 1 - s_2 s_2^*$ commutes with $u^* h_0(D) u$ (by Lemma 4.3.4(i)), and $h' := s_2^* u^* h_0(\cdot) u s_2$ is a unital C^*-morphism with $[h]_\approx + [h']_\approx = [h_0]_\approx$. $[h]_\approx$ is in $G_\approx(h_0\,;\, D,E)$, if and only if, there are $u, v \in \mathcal{U}_0(E)$ with $h = s_1^* u^* h_0(\cdot) u s_1$ and $h_0 = s_1^* v^* h(\cdot) v s_1$. Thus, *there is a natural semigroup morphism* $[h]_\approx \in S_\approx(h_0\,;\, D,E) \to [h] \in S(h_0\,;\, D,E)$. *It defines a natural group morphism of Grothendieck groups:* $G_\approx(h_0\,;\, D,E) \to G(h_0\,;\, D,E)$.

PROPOSITION 4.6.10. *Suppose that D and E are unital, that $h_0\colon D \to E$ is unital, and that isometries $t_1, t_2 \in E$ are canonical generators of a copy of $\mathcal{O}_2$ in E with $[h_0]_\approx = [h_0 \oplus_{t_1,t_2} h_0]_\approx$.*

If E is K_1-injective, then the group-morphisms of Remark 4.6.9 define natural exact sequences:

(i) $\mathrm{K}_1(h_0(D)' \cap E) \to \mathrm{K}_1(E) \to G_\approx(h_0\,;\, D,E) \to G(h_0\,;\, D,E) \to 0$. *Here* $\mathrm{K}_1(h_0(D)' \cap E) \to \mathrm{K}_1(E)$ *is induced by* $h_0(D)' \cap E \hookrightarrow E$.

(ii) $0 \to G(h_0\,;\, D,E) \to G(h_0 \oplus 0; D,E) \to \mathrm{K}_0(E)$, *where* $[h] \mapsto [h] + [0]$ *for* $[h] \in G(h_0\,;\, D,E)$ *and* $[k] \mapsto [k(1)] \in \mathrm{K}_0(E)$ *for* $[k] \in G(h_0 \oplus 0; D,E)$.

PROOF. We use the $s_1, s_2 \in h_0(D)' \cap E$ defined in Remark 4.6.9.

(i): An element $h \in \mathrm{Hom}(D,E)$ defines an element $[h]$ of $S(h_0\,;\, D,E)$, if and only if, there is $h' \in \mathrm{Hom}(D,E)$ with $[h] + [h'] = [h_0]$. It follows that the last observation of Remark 4.6.9 applies almost verbatim to $S(h_0\,;\, D,E)$ and $G(h_0\,;\, D,E)$, in particular:
$[h] \in G(h_0\,;\, D,E)$, if and only if, there are $u, v \in \mathcal{U}(E)$ with $h = s_1^* u^* h_0(\cdot) u s_1$ and $h_0 = s_1^* v^* h(\cdot) v s_1$.
Since $u(1 \oplus u^*) s_1 = u s_1$ and since $[u(1 \oplus u^*)] = 0$ in $\mathrm{K}_1(E)$ we may suppose that, in addition, $[u] = 0 = [v]$ in $\mathrm{K}_1(E)$.

Now we use the assumption that E is K_1-injective. Then for $h \in \mathrm{Hom}(D, E)$ holds: $[h] \in G(h_0\,;\, D, E)$ if and only if $[h]_\approx \in G_\approx(h_0\,;\, D, E)$. Thus:
If E is K_1-injective, then the natural morphism $G_\approx(h_0\,;\, D, E) \to G(h_0\,;\, D, E)$ is *surjective*.

If $[h]_\approx$ is in $G_\approx(h_0\,;\, D, E)$ then $[h] \in G(h_0\,;\, D, E)$. Then $[h] = [h_0]$ if and only if $h = u^* h_0(\cdot) u$ for some $u \in \mathcal{U}(E)$. Thus, the kernel of $G_\approx(h_0\,;\, D, E) \to G(h_0\,;\, D, E)$ is the set of equivalence classes $[u^* h_0(\cdot) u]_\approx$ with $u \in \mathcal{U}(E)$. If $u_1^* h_0(\cdot) u_1 = v^* u_2^* h_0(\cdot) u_2 v$ for $v \in \mathcal{U}_0(E)$, then $(u_2 v u_2^*) u_2 u_1^* \in h_0(D) \cap E$, i.e., $u_2 u_1^* \in \mathcal{U}_0(E) \cdot \mathcal{U}(h_0(D)' \cap E)$.
Conversely, $[u_1^* h_0(\cdot) u_2]_\approx = [u_2^* h_0(\cdot) u_2]_\approx$ if $u_2 u_1^* \in \mathcal{U}_0(E) \cdot \mathcal{U}(h_0(D)' \cap E)$. In particular, $\varsigma[u] \in \mathrm{K}_1(E) \mapsto [u^* h_0(\cdot) u]_\approx$ is well-defined, because $\mathrm{K}_1(E) \cong \mathcal{U}(E)/\mathcal{U}_0(E)$ by Lemma 4.2.6(iv) and Lemma 4.2.10.
Since $(u^* h_0(\cdot) u) \oplus v^* h_0(\cdot) v = (u \oplus v)^* h_0(\cdot)(u \oplus v)$ the map $\varsigma \colon \mathrm{K}_1(E) \to G_\approx(h_0\,;\, D, E)$ is additive by Lemma 4.2.6(iv). It follows: $\ker(\varsigma) = \{[u]\,;\, u \in \mathcal{U}_0(E) \cdot \mathcal{U}(h_0(D)' \cap E)\}$. In particular, $\ker(\varsigma)$ is contained in the image of the natural map $\eta_1 \colon \mathrm{K}_1(h_0(D)' \cap E) \to \mathrm{K}_1(E)$, for the inclusion $\eta \colon h_0(D)' \cap E \hookrightarrow E$.

The natural map $\mathcal{U}(h_0(D)' \cap E)/\mathcal{U}_0(h_0(D)' \cap E) \to \mathrm{K}_1(h_0(D)' \cap E)$ is surjective by Lemma 4.2.6(v). Therefore, if $[u] \in \mathrm{K}_1(E)$ is in $\eta_1(\mathrm{K}_1(h_0(D)' \cap E))$, there is $v \in \mathcal{U}(h_0(D)' \cap E)$ with $[v] = [u]$. We get $u \in \mathcal{U}_0(E) \cdot \mathcal{U}(h_0(D)' \cap E)$ by K_1-injectivity of E.

(ii): The map $[h] \to [h] + [h_0] + [0]$ defines (obviously) a semigroup morphism from $S(h_0\,;\, D, E)$ into $G(h_0 \oplus 0\,;\, D, E)$.

The map is injective on $G(h_0\,;\, D, E)$:
Let $[h] \in G(h_0\,;\, D, E)$ with $[h_0 \oplus 0] = [h \oplus h_0 \oplus 0]$, then $[h_0] = [h \oplus h_0]$: By assumption, there is a unitary $u \in E$ with $u^* s_1 h_0(\cdot) s_1^* u = s_1^2 h(\cdot)(s_1^*)^2 + s_1 s_2 h_0(\cdot) s_2^* s_1^*$. In particular, $u^* s_1 s_1^* u = s_1 s_1^*$. Thus $v := s_1^* u s_1$ is unitary, and $v^* h_0(\cdot) v = h \oplus h_0$.

The map $\tau \colon [h] \in [\mathrm{Hom}(D, E)] \mapsto [h(1)] \in \mathrm{K}_0(E)$ is a semigroup morphism by Lemma 4.2.6(i). Since $[1] = 0$ we get $[h(1) \oplus h_0(1) \oplus 0] = 0$ in $\mathrm{K}_0(E)$ if $h(1) = 1$. Hence, the image of $G(h_0\,;\, D, E) \to G(h_0 \oplus 0\,;\, D, E)$ is in the kernel of $G(h_0 \oplus 0\,;\, D, E) \to \mathrm{K}_0(E)$.

Now let $[k] \in S(h_0 \oplus 0\,;\, D, E)$ and $[k(1)] = 0$ in $\mathrm{K}_0(E)$. We are going to show that there is $[h] \in G(h_0\,;\, D, E)$ with $[k] + [h_0] + [0] = [h] + [h_0] + [0]$ in $G(h_0 \oplus 0\,;\, D, E)$:

There is an isometry $t \in E$ with $p = k(1) \oplus 1 \oplus 0 \geq tt^*$ and $[p] = 0$ in $\mathrm{K}_1(E)$, thus, there is an isometry $s \in E$ with $ss^* = p$, cf. Lemma 4.2.6(ii).

By assumption, there is an isometry $r \in E$ with $r^* s_1 h_0(\cdot) s_1^* r = k$. Let $h_1 := s^*(k \oplus h_0 \oplus 0)s$, then $h_1(1) = 1$ and $h_1 := z_1^* h_0(\cdot) z_1 + z_2^* h_0(\cdot) z_2 = w^* h_0(\cdot) w$ for $z_1 := s_1^* r (s_1^2)^*$, $z_2 := s_2^* s_1^*$ and $w := s_1 z_1 + s_2 z_2$. Then $w^* w = h(1) = 1$ and $h_1 \in \mathrm{Hom}(D, E)$, hence $h_1 \in S(h_0\,;\, D, E)$.

There is a unitary $u \in E$ with $u^*(h_1 \oplus h_0 \oplus 0) u = k \oplus h_0 \oplus 0$:

We have $s h_1(\cdot) s^* = k \oplus h_0 \oplus 0$.

Since $(ss_2)^* ss_1 = 0$, there is an isometry $q \in E$ with $qq^* = 1 - (ss_1)(ss_1)^*$. Let $v := s_1 s_1^* s^* + s_2 q^* \in \mathcal{U}(E)$ with $v^* s_1 = s_1 s$ and $v^* s_2 = q$. We get:

$$v^*(h_1 \oplus 0)v = s_1(k \oplus h_0 \oplus 0)s_1^* = (k \oplus h_0 \oplus 0) \oplus 0\,.$$

Since $[0]+[0] = [0]$ and $[h_0]+[h_0] = [h_0]$ in $([\mathrm{Hom}(D,E)],+)$, we get $[h]+[h_0]+[0] = [k] + [h_0] + [0]$ for $h := h_1 \oplus h_0$. Note that $[h] \in G(h_0\,;\,D,E)$. □

The proof of the isomorphism $\mathrm{KK}(\mathcal{C};\,A, B_{(1)}) \cong \mathrm{Ext}(\mathcal{C};\,A,B)$ for trivially graded A and B in Chapter 8 uses a characterization of the defining relations of elements given in Chapter 5, and this characterization

Precise reference? To Chp.5 ?? ??

uses the following observation:

Lemma 4.6.11. *Let $h\colon D \to E$ a C^*-morphism such that $C := h(D)' \cap E$ contains a copy of $\mathcal{O}_2 = C^*(s_1, s_2)$ unitally, and that h dominates zero.*

If $p, q \in C$ are projections and $V \in \mathcal{U}(E)$ a unitary that satisfy $V^* h(\cdot)pV = h(\cdot)q$, $[p] = [q]$ in $\mathrm{K}_0(E)$ and $[V] = 0$ in $\mathrm{K}_1(E)$, then there exist projections $r_1, r_2 \in \mathrm{Ann}(h(D),E) \subseteq C$ such that $[p \oplus r_1] = [q \oplus r_2]$ in $\mathrm{K}_0(C)$.

Moreover, there exists a unitary $U \in \mathcal{U}_0(h(D)' \cap E)$ with $U^*(p \oplus r_1 \oplus 1 \oplus 0)U = q \oplus r_2 \oplus 1 \oplus 0$, where $\oplus = \oplus_{s_1,s_2}$.

Proof. The proof can be reduced to the splitting of the 6-term exact into the split-exact sequence

$$0 \to \mathrm{K}_*(\mathrm{Ann}(h(D),E)) \to \mathrm{K}_*(C) \to \mathrm{K}_*(C/\,\mathrm{Ann}(h(D),E)) \to 0$$

with splitting maps $\mathrm{K}_*(C) \to \mathrm{K}_*(\mathrm{Ann}(h(D),E)) \cong \mathrm{K}_*(E)$ given by the inclusion map $C \hookrightarrow E$, see Lemma 4.2.20.

Here we give an alternative elementary approach, that gives explicit elements that are useful for the study of generalized extension groups $\mathrm{Ext}(\mathcal{C}\,;\,A,B)$ considered in Chapter 5. We write $\oplus$ for the specified Cuntz sum $\oplus_{s_1,s_2}$.

Since $\mathrm{K}_0(C)$ and $\mathrm{K}_0(E)$ are groups, it suffices to consider the case where p is the range of an isometry in C and q is properly infinite in C. We can do this simply by passing from (p,q,V) to the projections $p_1 := p \oplus (1-p)$ and $q_1 := q \oplus (1-p)$, and the unitary $V_1 \in \mathcal{U}(C)$ given by $V_1 := V \oplus_{s_1,s_2} 1$. Then $p_1 = TT^*$ for the isometry $T := s_1 p + s_2(1-p) \in C$. Thus, p_1 is range of an isometry. and $[p_1]_C = [p]_C + [1-p]_C = 0$ in $\mathrm{K}_0(C)$, $[q_1]_C = [q]_C - [p]_C$ in $\mathrm{K}_0(C)$ and $[q_1]_E = [q]_E - [p]_E = 0$ in $\mathrm{K}_0(E)$. Since $h(\cdot) \oplus h(\cdot) = h$, the unitary V_1 satisfies $V_1^* h(\cdot) p_1 V_1 = h(\cdot) q_1$.

To keep notation simple we suppose in the following that p itself is a range of an isometry. Let $t_1 \in C$, $t_2 \in E$ the isometries defined in Proposition 4.3.6(i) with $t_1 t_1^* + t_2 t_2^* = 1$ and $h(D)t_2 = \{0\}$.

The projections $p' := t_1 p t_1^* + t_2 t_2^*$ and $q' := t_1 q t_1^* + t_2 t_2^*$ are in C, satisfy $ph(\cdot) = p'h(\cdot)$ and $qh(\cdot) = q'h(\cdot)$ and have same classes $0 = [p]_C = [p']$ and $[q]_C = [q']_C$ in $\mathrm{K}_0(C)$ and $[p']_E = [p]_E = [q]_E = [q']_E$ in $\mathrm{K}_0(E)$. This is because

$t_2t_2^* = 1 - t_1t_1^* \in C$ and $t_1^*t_1 = 1$ imply $[t_2t_2^*] = 0$ in $\mathrm{K}_0(C)$ and $\mathrm{K}_0(E)$. Moreover, $t_2\mathcal{O}_2t_2^*$ is unitally contained in $F := t_2Et_2^* = (1-t_1t_1^*)E(1-t_1t_1^*)$ and $fh(D) = \{0\}$ for all $f \in F$. Notice that F is contained in the closed ideal $\mathrm{Ann}(h(D), E)$ of C.

The projections 1, $t_2t_2^*$, $1-t_1^*pt_1$ and $1-t_1^*qt_1$ are in C and majorize the properly infinite projection $t_2t_2^*$ of C. The have all the same class $[1] = 0$ in $\mathrm{K}_0(E)$ and $t_2t_2^*$ is full in E. Thus, by Lemma 4.2.6(ii), there are partial isometries $v, w \in E$ with $v^*v = 1 - t_1pt_1^*$, $vv^* = t_2t_2^*$, $w^*w = 1 - t_1qt_1^*$ and $ww^* = t_2t_2^*$. Let $S := t_1pt_1^* + v$ and $T := t_1qt_1^* + w$, then S and T are isometries in E with $SS^* = p'$, $TT^* = q'$, $St_1pt_1^* = t_1pt_1^*$ and $Tt_1qt_1^* = t_1qt_1^*$. Thus, $S^*h(\cdot)S = t_1pt_1^*h(\cdot)t_1pt_1^* = t_1h(\cdot)pt_1^*$ and $T^*h(\cdot)T = t_1h(\cdot)qt_1^*$.

It follows, that $W^*S^*h(\cdot)SW = T^*h(\cdot)T$ for the unitary $W = t_1Vt_1^* + t_2t_2^*$. Hence, $SWT^* \in C$ by Lemma 4.3.4(ii), and defines a Murray–von-Neumann equivalence between p' and q'. Thus $[p] = [p'] = [q'] = [q]$ in $\mathrm{K}_0(C)$. By Lemma 4.2.6(iv,b), there is $U \in \mathcal{U}_0(C)$ with $U^*(p \oplus 1 \oplus 0)U = q \oplus 1 \oplus 0$. □

7. Additional remarks and further plans for Chp.4

`plan: ??`

`Some of following is contained in Lem.4.2.6 and around.`

(0.1) $V(A)$ `common name?`

Recall that $V(A)$ denotes the set of all Murray–von-Neumann equivalence classes $[p]$ of projections $p \in A$. It has *partially* defined commutative "addition" $[p] + [q] := [r + s]$, if there exist projections $r, s \in A$ with $sr = 0$ and $[r] = [p]$ and $[s] = [q]$.

If the addition of MvN-classes – $[p]$ with $[q]$, $[q]$ with $[r]$, $[p] + [q]$ with $[r]$ and $[p]$ with $[q] + [r]$ – are all defined, then the associative law holds: $[p] + ([q] + [r]) = ([p] + [q]) + [r]$.

The addition is always defined if the unit element $1_{\mathcal{M}(A)}$ of the multiplier algebra $\mathcal{M}(A)$ of A is properly infinite, and then $V(A)$ is a commutative semigroup. If, in addition, A contains a full projection p, then $[q] \in V(A) \to [e_{11} \otimes q] \in V(\mathbb{K} \otimes A)$ is an isomorphism of semigroups, and $\mathrm{K}_1(A)$ is naturally isomorphic to the Grothendieck group of the semigroup $V(A)$.

Lemma 4.7.1. *Suppose that p is a full and properly infinite projection in a C^*-algebra A.*

For any projection $q \in A$ there exists a projection $r \in pAp$ such that $p - r$ is full and properly infinite in A and $q \sim r$.

For any two projections $r, s \in A$ the sum $[r] + [s]$ exists in $V(A)$. Its representative is full and properly infinite if one of the projections is full and properly infinite.

$[q] \in [p] + V(A)$ if and only if q is full and properly infinite in A.

If $p \in A$ is a full and properly infinite projection, then for any projection $p \in A$ there exists a full and properly infinite projection $r \in A$ with $[q] + [r] = [p]$.

In particular, the Murray–von-Neumann equivalence classes of full and properly infinite projections in A build a subgroup $[p] + V(A)$ of $V(A)$ with (unique) neutral element $[r]$ that satisfies $[p] + [r] = [p]$.

If $p \in A$ is full and properly infinite, then $[p] + [q] = [p] + [r]$ in $V(A)$, if and only if, $[q] = [r]$ in $\mathrm{K}_0(A)$.

$\mathrm{K}_0(A)$ is naturally isomorphic to the sub-semigroup $[p] + V(A)$ of $V(A)$.

If $p \in A$ is a properly infinite projection, i.e., if there exist partial isometries $u, v \in A$ with $u^*u = v^*v = p$, $uu^* \leq p$, $vv^* \leq p$ and $u^*v = 0$, then $p_0 := p - vv^*$ is a properly infinite projection in A that generates the same ideal as p, and p_0Ap_0 contains a copy of $\mathcal{O}_2$ unitally, in particular $[p_0] = 2[p_0]$ in $V(A)$.

LEMMA 4.7.2. *If $p \in A$ is full and properly infinite, then $u \in \mathcal{U}(pAp) \mapsto [(1-p) + u] \in \mathrm{K}_1(A)$ is a surjective map.*

If A is unital, $u \in \mathcal{U}(A)$, $p \in A$ projection with $0 = [u] \in \mathrm{K}_1(A)$, $\|pu - up\| < 1$ and p and $1 - p$ both full and properly infinite in A. Then u is in the connected component $\mathcal{U}_0(A)$ of 1_A in $\mathcal{U}(A)$.

`Above lem contained in` K_1`-injectivity/surjectivity stuff?`

(1.0) Suppose that A contains a full and properly infinite projection $p \in A$. Then:

The map $u \in \mathcal{U}(pAp) \mapsto [u + (1-p)] \in \mathrm{K}_1(A)$ defines a group epimorphism from $\mathcal{U}(pAp)$ *onto* $\mathrm{K}_1(A)$.

The algebra A contains also a full projection p_0 with $[p_0] = 2[p_0]$ in $V(A)$, i.e., with the property that p_0Ap_0 contains a copy of $\mathcal{O}_2$ unitally.

The natural semigroup morphism $[q] \in V(A) \mapsto [q] \in \mathrm{K}_0(A)$ is surjective.

The set $V(A) + [p_0]$ is a subgroup of $V(A)$ that is identical with the set of the MvN–equivalence classes of all full properly infinite projections in A.

The restriction of the semigroup epimorphism $V(A) \to \mathrm{K}_0(A)$ to $V(A) + [p_0]$ is a group *isomorphism* from $V(A) + [p_0]$ onto $\mathrm{K}_0(A)$.

Suppose that A is unital and 1_A is properly infinite.

It allows to define "additions" in $\mathcal{U}(A)$ and in $\mathrm{Proj}(A)$ by $U \oplus V := s_1Us_1^* + s_2Vs_2^* + (1 - s_1s_1^* - s_2s_2^*)$ and $p \oplus q := s_1ps_1^* + s_2qs_2^*$.

It defines an addition in $\mathcal{U}(A)/\mathcal{U}_0(A)$ and in $V(A)$ that is compatible with the multiplication in $\mathcal{U}(A)/\mathcal{U}_0(A)$, – respectively with the addition in $V(A \otimes \mathbb{K})$.

It turns out that $q \to [q \otimes e_{1,1}] \in V(A \otimes \mathbb{K})$ induces an isomorphism from $(V(A), +)$ onto $V(A \otimes \mathbb{K})$ with the usual bloc-wise addition. (In fact: $(p+q) \otimes e_{1,1}$ is MvN-equivalent to $p \otimes e_{1,1} + q \otimes e_{2,2}$ if $pq = 0$ and for each projection $p \in A \otimes \mathbb{K}$ there exists $q \in A$ with $[q \otimes e_{1,1}] = [p]$ in $V(A \otimes \mathbb{K})$.)

Furthermore, $U \oplus V$ is homotopic to $(UV) \oplus 1$ in $\mathcal{U}(A)$. $[U \oplus 1] = [U]$ in $\mathrm{K}_1(A)$ for all $U \in \mathcal{U}(A)$. It holds $U \oplus 1 \in \mathcal{U}_0(A)$ iff $[U] = 0$ in $\mathrm{K}_1(A)$. Moreover, if $T \in A$ is an isometry, such that $1 - TT^*$ is a full properly infinite projection then $[TUT^* + (1 - TT^*)] = [U]$ in $\mathrm{K}_1(A)$ for all $U \in \mathcal{U}(A)$, and $[U] = 0$ implies that $TUT^* + (1 - TT^*) \in \mathcal{U}_0$.

(1.1) In particular, every C^*-algebra A with a properly infinite unit 1_A is K_*-surjective, i.e., the natural semigroup morphism $V(A) \to \mathrm{K}_0(A)$ and the natural group morphism $\mathcal{U}(A) \to \mathrm{K}_1(A)$ are surjective.

(1.2) Every unital s.p.i. algebras is K_*-injective in the sense that: This is now shown in Chp. 4 in case of unital p.i. C^*-algebras, with help of the squeezing property. Refere to it!

(0) $V(A) + [1_A]$ is the subgroup of $V(A)$ of all full properly infinite projections. The natural morphism from $V(A)$ to $\mathrm{K}_0(A)$ defines an isomorphism from $V(A) + [1_A]$ onto $\mathrm{K}_0(A)$.
The element $[1_A]$ is the *neutral* element of $V(A) + [1_A]$, if and only if, A contains a copy of $\mathcal{O}_2$ unitally.

(1) The natural group morphism from $\mathcal{U}(A)$ to $\mathrm{K}_1(A)$ is surjective and has kernel $\mathcal{U}_0(A)$.

(1.3) If A is a unital s.p.i. algebra, then 1_A is properly infinite and the group $V(A) + [1_A]$ consists of all MvN-equivalence classes of *full* projections in A (because every element of A is properly infinite).

There is a copy of $\mathcal{O}_\infty = C^*(s_1, s_2, \ldots)$ unitally contained in A, because A is, in particular,

purely infinite, i.e., each non-zero element of A is properly infinite.

Exact reference! ??

We have seen in Chapter 3 that every s.p.i. algebra B satisfies the following property:
For every finitely generated Abelian C^*-subalgebra $A \subseteq B_\omega$ with at most 1-dimensional maximal ideal space $\mathrm{Prim}(A)$ *there exists a C^*-morphism*

$$h \colon A \otimes \mathcal{O}_\infty \to B_\omega \quad \text{with} \quad h(a \otimes 1) = a \ \forall\, a \in B\,.$$

(Notice that $U = h(U \otimes (tt^*)) + h(U \otimes (1 - tt^*))$ is homotopic to the unitary $h(U \otimes (tt^*)) + h(1 \otimes (1 - tt^*)) = TUT^* + (1 - TT^*)$ with $T := h(1 \otimes t)$ if $1_B, U \in A$ and U is unitary, because the unitary group of $\mathrm{C}(S^1, \mathcal{O}_2)$ is connected.)

Now use that $\mathrm{K}_*(B_\omega) = (\mathrm{K}_*(B))_\omega$ (as $\mathbb{Z}_2$-graded groups), and get: A unitary $U \in B$ (respectively a projection $p \in B$) is homotopic to 1 in $\mathcal{U}(B)$ (respectively to $q \in B$ in $\mathrm{Proj}(B)$), if and only if, it is homotopic to 1 (respectively to q) in the unitaries (respectively projections) of B_ω.

Thus, it suffices to proof the K_1-injectivity of B_ω. This follows from the homotopy of $U \sim_h TUT^* + (1 - TT^*)$ in $\mathcal{U}(h(C^*(U) \otimes \mathcal{O}_\infty))$, where $T := h(1 \otimes t)$, for a non-unitary isometry $t \in \mathcal{O}_\infty$.

More generally, if for every commuting pair (b_1, b_2) of self-adjoint elements in a given unital C^*-algebra B and $\varepsilon > 0$ there exists isometries $s, t \in B$ with $s^*t = 0$ and $\|sb_j - b_js\| < \varepsilon$ $(j = 1, 2)$ then B is K_1-injective.

This is, because then 1_B is properly infinite, and – with $\varepsilon := 1/4$ – the following criteria holds for $u = b_1 + ib_2$ with $[u] = 0$ in $\mathrm{K}_1(B)$ and $p := ss^*$.

which criterium ? from Kt_1-injectivity proposition?????, squeezing property?? $\|s^*ut\| < 2\varepsilon$.

(2) **Transfer** to here, and generalize, the unitary homotopy for $(\mathrm{id} \oplus h_0)^k$ considered in Chapter 11. **??**.

(3) Let $J \subseteq E$ a closed ideal, and let $H_0 \colon D \to E$ a *-monomorphism such that $H_0(D)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally, and that there is an isometry $T \in E$ with $TT^*H_0(D) = \{0\}$.

Recall that $S(H_0\,; D, E)$ is the set of unitary equivalence classes $[h]$ where $h \in \mathrm{Hom}(D, E)$ is dominated by H_0. Then, obviously, $[h] \in S(H_0\,; D, E)$ with $h(D) \subseteq J$ is a sub-semigroup $S(H_0\,; D, E; J)$ of $S(H_0\,; D, E)$.

Let B a C^*-subalgebra of E. We denote by $V(B)$ the Murray–von-Neumann equivalence classes of projections in B with (partial) addition given by sums of orthogonal representatives. If $\mathcal{M}(B)$ has a properly infinite unit, then $V(B)$ is a commutative semigroup such that $[p] \in V(B) \to [e_{11} \otimes p] \in V(\mathbb{K} \otimes B)$ is an isomorphism from $V(B)$ onto $V(\mathbb{K} \otimes B)$.

We define a map $\theta_0 \colon S(H_0\,; D, E) \to V(H_0(D)' \cap E)$ by $\theta_0(h) := [tt^*]$ for some isometry $t \in E$ with $h := t^*H_0(\cdot)t$.

Let $B := H_0(D)' \cap \mathcal{N}(H_0(D), J)$, where $\mathcal{N}(H_0(D), J) \subseteq E$ denotes the hereditary C^*-subalgebra of E of the elements $x \in E$ with $xH_0(D) \cup H_0(D)x \subseteq J$, i.e.,

$$\mathcal{N}(H_0(D), J) = (\pi_J)^{-1}(\mathrm{Ann}(\pi_J(H_0(D)), E/J)) .$$

There is a natural semigroup morphism $[p]_B \in V(B) \mapsto [p]_C \in V(C)$ where $C := H_0(D)' \cap E$.

The following lemma is used to compare the semigroups $\mathrm{SR}(\mathcal{C}\,; A, B)$ of Chapter 7 with WvN-equivalence classes of projections in $H_0(D)' \cap \mathcal{N}(H_0(D), J)$.

Lemma 4.7.3. *Let $(H_0\,; D, E; J)$, B and θ_0 as above (with $[H_0] + [0] = [H_0]$ and $[H_0] + [H_0] = [H_0]$). Let $C := H_0(D)' \cap E$. Then:*

(o) *B is a closed ideal of C and $V(B)$ is a hereditary sub-semigroup of $V(C)$.*

(i) *The map θ_0 is well-defined and is a semigroup isomorphism from $S(H_0\,; D, E)$ onto the sub-semigroup of classes $[p] \in V(C)$ with the property that there is an isometry $t \in E$ with $[tt^*] = [p]$ in $V(C)$.*

(ii,a) *Let $V_0(C)$ the kernel of the natural morphism $V(C) \to \mathrm{K}_0(C)$.*
If $T \in E$ is an isometry with $TT^*H_0(D) = \{0\}$, then $TT^* \in C$,

$$\theta_0(S(H_0\,; D, E)) = [TT^*] + V_0(C) .$$

(ii,b) *The Grothendieck group of* $[TT^*] + V_0(C)$ *is naturally isomorphic to the kernel of* $\mathrm{K}_0(C) \to \mathrm{K}_0(E)$.

If $F := \mathrm{Ann}(H_0(D), E)$ *denotes the two-sided annihilator of* $H_0(D)$ *in* E*, then* F *is an ideal of* C*, the Grothendieck group* $G(H_0\,;\, D, E) := H_0 + S(H_0\,;\, D, E)$ *of* $S(H_0\,;\, D, E)$*, and it holds:*

$$G(H_0\,;\, D, E) \cong \mathrm{K}_0(C/F) \cong \ker(\mathrm{K}_0(C) \to \mathrm{K}_0(E))\,.$$

(iii) *Let* $V_0(B)$ *denote the kernel of* $V(B) \to \mathrm{K}_0(E)$. $[TT^*] \in V_0(B)$ *and* $\theta_0(S(H_0\,;\, D, E;\, J)) = \theta_0(S(H_0\,;\, D, E)) \cap V(B)$.

If $T \in E$ *is an isometry with* $TT^* H_0(D) = \{0\}$ *then* $TT^* \in V_0(B)$, $V_0(B) = V(B) \cap V_0(C)$ *and*

$$\theta_0(S(H_0\,;\, D, E; J)) = [TT^*] + V_0(B)\,.$$

PROOF. Let $C := H_0(D)' \cap E$.

(o): If $b, c, e \in C$ and $bH_0(D) \subseteq J$, then $H_0(D)cbe = cbeH_0(D) = cbH_0(D)e \subseteq J$. Thus, B is an ideal of C. The same argument shows that $[p]_C = [q]_C$ in $V(C)$ and $p \in B$ implies $q \in B$ and $[p]_B = [q]_B$ in $V(B)$. It follows that $V(B)$ is a hereditary sub-semigroup of $V(C)$, in the sense that $[p] + [q] \in V(B) \subseteq V(C)$ implies $[p], [q] \in V(C)$.

(i): See the arguments in the proof of Part (i) of Proposition 4.4.3.

(ii,a): Since $TT^* H_0(a) = 0 = H_0(a)TT^*$ for all $a \in D$, the projection TT^* is in $B \subseteq C$. Let $r, s \in C$ canonical generators of a unital copy of $\mathcal{O}_2$, and let p a projection in C with $[p] = 0$ in $\mathrm{K}_0(E)$. Then $q := rTT^*r^* + sps^*$ is a projection in C with $[q] = [TT^*] + [p] = [1_C] + [p]$ in $V(C)$, and there exists an isometry $t \in E$ with $tt^* \in C$ and $[q] = [tt^*]$ in $V(C)$.

To see this, we can use the isomorphism $V(E) + [1] \cong \mathrm{K}_0(E)$ and the fact that $V(E) + [1]$ is identical with the MvN-classes of all properly infinite elements: Since $[q] = [p] + [TT^*] = [p] + [1]$ in $V(E)$, the projection q is properly infinite and full in E. Since $[1] = 0$ in $\mathrm{K}_0(E)$ it follows that $[q] = [p] = 0$ in $\mathrm{K}_0(E)$. Thus $[q] = [1]$ in $V(E)$.

(ii,b): $F \to C \to E$ defines an isomorphism $\mathrm{K}_0(G) \cong \mathrm{K}_0(E)$ that makes that the sequence $0 \to \mathrm{K}_*(F) \to \mathrm{K}_*(C) \to \mathrm{K}_*(C/F) \to 0$ is split exact with $\mathrm{K}_*(C) \to \mathrm{K}_*(E) \cong \mathrm{K}_*(F)$ as splitting map.

(iii): Let $[h] \in S(H_0\,;\, D, E; J)$ and $h = t^* H_0(\cdot)t$. Then $tt^* \in C$ by (i) and $tt^* H_0(a^*a)tt^* \in J$ for all $a \in D$. Thus, $H_0(D)tt^* \subseteq J$ and $tt^* \in B$. It follows that $\theta_0(S(H_0\,;\, D, E; J)) \subseteq V(B) \subseteq V(C)$.

If $\theta_0(h) \in V(B)$ and $h = t^* H_0(\cdot)t$ for some isometry $t \in E$, then $tt^* \in C$ and $[tt^*]_C = [q]_C$ in $V(C)$ for some $q \in B$. It follows $tt^* \in B$ and $h = t^*(tt^*)H_0(\cdot)t$ maps D into J, i.e., $h \in S(H_0\,;\, D, E; J)$.

Hence, $\theta_0(S(H_0\,;\, D, E; J)) = \theta_0(S(H_0\,;\, D, E)) \cap V(B)$.

Now notice that $([TT^*]+V(C))\cap V(B) = [TT^*]+V(B)$, because $[TT^*] \in V(B)$ and $V(B)$ is hereditary in $V(C)$. □

Can the Grothendieck group of $S(H_0\,;\, D, E; J)$ be calculated with help of K_*-groups of B, C and J ?

REMARK 4.7.4. The following observation yields alternative proofs of some parts of Lemma 4.2.6, e.g. of Part (v,5).

A unitary $u \in \mathcal{U}(E)$ satisfies $u\,tt^* = tt^*$ for some isometry $t \in E$, if and only if, u is conjugate to $(u \oplus 1) + p_{[-1]}$, i.e., if there is a unitary $V \in \mathcal{U}(E)$ with $V^*uV = (u \oplus 1) + p_{[-1]}$. The unitary V can be chosen such that $[V] = 0$ in $\mathrm{K}_1(E)$.

PROOF. If $V \in \mathcal{U}(E)$ satisfies $V^*uV = (u \oplus 1) + p_{[-1]} = s_1us_1^* + (1 - s_1s_1^*)$, then $uVs_2 = Vs_2$, thus $utt^* = tt^*$ for the isometry $t := Vs_2$.

If $t \in E$ is an isometry with $utt^* = tt^*$, then let $P := t(1 - s_1s_1^*)t^*$. The projection P is splitting, because $P \geq (ts_2)(ts_2)^*$ and $P(ts_1)(ts_1)^* = 0$. Since P is MvN-equivalent to $1 - s_1s_1^*$, there exists a unitary V_1 with $V_1^*PV_1 = 1 - s_1s_1^*$. It follows $V_1^*uV_1(1 - s_1s_1^*) = (1 - s_1s_1^*)$ and $u_1 := s_1^*V_1^*uV_1s_1$ is a unitary in E such that $u = s_1u_1s_1^* + (1 - s_1s_1^*)$, i.e., $V_1^*uV_1 = (u_1 \oplus 1) + p_{[-1]}$.

We use now the "corrected" direct sum formula $U \oplus' V := (U \oplus V) + p_{[-1]}$ on the unitary elements $U, V \in \mathcal{U}(E)$, and get $V_1^*uV_1 = u_1 \oplus' 1$. It follows that $V_2^*(u \oplus' 1)V_2 = (u_1 \oplus' 1) \oplus' 1$ for $V_2 := V_1 \oplus' 1$. The corrected direct sum is still commutative and associative up to unitary equivalence by the unitaries U_c and U_d defined in Equations (2.3) and (2.4), i.e., $(U \oplus' V) \oplus' W$ and $U \oplus' (V \oplus' W)$ are unitarily equivalent by unitaries in $\mathcal{U}_0(E)$. We get that there is a unitary V in the classes $\mathcal{U}_0(E)V_1$ such that $V^*uV = u \oplus' 1$.

We can replace V by $V' := V(1 \oplus' V^*)$ that satisfies $0 = [V'] \in \mathrm{K}_1(E)$. □

CHAPTER 5

Generalized Weyl–von Neumann Theorems

?? Revise section structure !!

Remove two-fold text !! very urgent!!!

We generalize in this chapter the Weyl–von-Neumann theorem along lines indicated by Voiculescu and Kasparov ([**798**], [**404**], see [**42**] for an elementary account and use Lemma 2.1.22 to see that they are special cases of Theorem A). The definition, viewpoints and results are different from the generalization of some aspects of the classical Weyl–von Neumann theorem by S. Zhang in [**845**].

The generalized Weyl–von Neumann Theorem 5.6.2 will be an ingredient of the proofs of Theorem A and of the important special case of Theorem M, – the Theorem 6.3.1 –, considered in Chapter 6 under the additional assumption that the C^*-algebra B in Theorem M has *residual nuclear separation* (cf. Definition 1.2.3), i.e., that the m.o.c. cone $\mathcal{C}_{rn} \subseteq \mathrm{CP}_{\mathrm{nuc}}(B, B)$ is separating for B. This additional assumption will be removed finally in Chapter 12, by showing that in ultrapowers and asymptotic algebras those additional conditions are in the considered cases for sufficiently big separable C^*-subalgebras are satisfied, all this only by using combinations of results obtained in all Chapters 7-11 together.

We *assume* in Chapter 6, in addition to the assumptions of Theorem M, that the related universal weakly residually nuclear C^*-morphism $H_{rn} \colon B \to \mathcal{M}(B)$ from B into its multiplier algebra $\mathcal{M}(B)$ is non-degenerate, faithful and that the induced ideal-system action of $\prime(B)$ on B is simply the identity map ????

([1])

???? HERE and a bit below ???????

Explain H_{rn} by use of action and by use of suitable $\mathcal{C}$.

In case of trivially graded stable separable A, B and non-degenerate m.o.c. cone $\mathcal{C}$ we should have exact sequences

$$0 \to \mathrm{Ext}(\mathcal{C}; A, B) \to \mathrm{K}_0(\pi_B(H_{\mathcal{C}}(A))' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(\mathrm{Q}(B)) \cong \mathrm{K}_1(B) \to 0\,,$$

and

$$0 \to \mathrm{KK}(\mathcal{C}; A, B) \to \mathrm{K}_1(\pi_B(H_{\mathcal{C}}(A))' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B) \to 0\,.$$

[1] The author got in between, by a self-contained study of relations between coherent and non-coherent Dini-spaces, refining and applying basic results from lattice theory, that each separable stable C^*-algebra has this property anyway. But here we use an other way of proof that considers properties of ultra-powers of strongly purely infinite C^*-algebras.

Since $H_{\mathcal{C}}(A)' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally we get K_*-surjectivity.

What about K_*-injectivity here??

For $\mathrm{K}_0(H_{\mathcal{C}}(A)' \cap \mathcal{M}(B))$ only the full properly infinite projections in $H_{\mathcal{C}}(A)' \cap \mathcal{M}(B)$ are uniquely defined by its value in $\mathrm{K}_1(H_{\mathcal{C}}(A)' \cap \mathcal{M}(B))$.

We must check if $H_{\mathcal{C}}(A)' \cap \mathcal{M}(B)$ is K_1-bijective! ?

... for the considered stable separable B and that there exists a C^*-morphism $h\colon A \to \mathcal{M}(B)$ that defines the action of $\mathrm{Prim}(B)$ on A.

Relation to more general $\mathcal{C}$ and induced actions have to be explained / refer to Chapter 3.

This ??? additional assumptions will be removed in Chapter 12 completely using results of Chapters 3, and 7 -11 .

Here in Chapter 5 we allow weaker/stronger ??? assumptions.

Perhaps Chapter 3 should present the relations between actions and cones ...

In particular, $C' \cap \mathrm{Q^s}(B)$ should be σ-sub-Stonean ...???

Should here also obtain finally that $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ is (unitally) contained in $\mathcal{O}_2$, and that this allows to prove that $\mathcal{O}_2$ can be written as its own infinite tensor product

??? if each unital ind-lim of $\mathcal{O}_2$ is $\mathcal{O}_2$.

Need the approximate unitary equivalence of all unital endomorphisms of $\mathcal{O}_2$ and more generally of $\mathcal{O}_{\mathrm{n}}$, $\mathcal{O}_\infty$.

¡ ¡ ¡ Given at end of Chapter 4 ??

Its generalization to weakly residually nuclear maps – and, more generally and systematic, to maps in a given m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ – will be used in Chapters 6 and 12 for the final proof of Theorem K.

`Compare above and below text`

First we prove an almost "tautological" universal version of a generalized Weyl–von-Neumann theorem (Theorem 5.6.2) and show some of its consequences:

We apply it to give an alternative proof for the stability criteria of M. Rørdam and J. Hjelmborg for σ-unital C^*-algebras and its extensions (cf. [**373**], [**688**]). We specialize it to the cases of extensions with splitting c.p. maps that are weakly nuclear, weakly residually nuclear, or are in suitable general m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$. It gives the base to define general cone-related Ext-groups $\mathrm{Ext}(\mathcal{C};\, A, B)$ and we display explicit defining relations that will be used in Chapters 8 and 9 to establish the functorial isomorphism of $\mathrm{Ext}(\mathcal{C};\, A, B)$ with $\mathrm{KK}(\mathcal{C};\, A, B_{(1)})$ – alone the lines of early work of G. Kasparov – and its isomorphism with the a generalized version of "Rørdam groups" $\mathrm{R}(\mathcal{C};\, A, B)$ defined in Chapter 7, that generalize groups of asymptotic morphisms appearing (implicitly) in early work of G. Elliott and M. Rørdam.

Later applications of our general WvN-theorems (in the spirit of Voiculescu and Kasparov) for the proofs of isomorphisms between the in Chapters 1, 4 and 7 – 9 considered types of operator class groups use the key results from Chapters 3 and 4.

We need some basic definitions, e.g. our version of unitarily homotopic morphisms in the following Definition 5.0.1 will be used frequently.

Recall that the C^*-algebra $\mathcal{M}(A) \subseteq \mathcal{L}(A, A)$ denotes the multiplier algebra of a C^*-algebra A.

DEFINITION 5.0.1. Let $h_j \colon A \to \mathcal{M}(B)$, $j = 1, 2$, C^*-morphisms, and let V a completely positive contraction from A into the multiplier algebra $\mathcal{M}(B)$ of B.

We call h_1 and h_2 **unitarily homotopic** if there is a *norm-continuous* map $t \mapsto U(t)$ from the non-negative real numbers $\mathbb{R}_+$ into the *unitaries* in $\mathcal{M}(B)$, such that,

(i) $h_2(a) - U(t)^* h_1(a) U(t) \in B$ for all $t \in \mathbb{R}_+$ and $a \in A$, and
(ii) $\lim_{t\to\infty} \| h_2(a) - U(t)^* h_1(a) U(t) \| = 0$ for all $a \in A$.

Let B σ-unital and stable. Question:
What happens if we require only that
$t \mapsto U(t)$ satisfies (i, ii) and is only *strictly* continuous?

Comments on this question to be checked:

It says then only that $h_1 \colon A \to \mathcal{M}(B)$ and $h_2 \colon A \to \mathcal{M}(B)$ are unitarily equivalent in $C := \mathrm{C}_{\mathrm{b,st}}\big([0, \infty), \mathcal{M}(B)\big) / \mathrm{C}_{\mathrm{b}}([0, \infty), B)$, where we take the natural injective "constant" C^*-morphism $\mathcal{M}(B) \subset \mathrm{C}_{\mathrm{b,st}}\big([0, \infty), \mathcal{M}(B)\big)$.

In case of our Definition 5.0.1 we get that $U \in \mathrm{C}_{\mathrm{b}}([0, \infty), \mathcal{M}(B))$.

Is the strict topology – restricted to $\mathrm{C}_{\mathrm{b}}([0, \infty), B)$ – is the same as the norm topology on $\mathrm{C}_{\mathrm{b}}([0, \infty), B)$?

We have often to do with the case where B is stable and σ-unital. Then $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}_0(\mathcal{M}(B))$, and there exist $y_1, \ldots, y_n \in \mathcal{M}(B)$ with $y_k^* = -y_k$ such that $U(1) = \exp(y_1) \cdot \ldots \cdot \exp(y_1)$. Then $h_2(a) - U(1)^* h_1(a) U(1) \in B$ for all $a \in A$. Let $V := \pi_B(U(1))$ Thus, $\pi_B \circ h_2 = V^* \pi_B \circ h_1(\cdot) V \in \mathrm{Q}(B)$. and $V \in \mathcal{U}_0(\mathrm{Q}(B))$. If, in addition, h_2 is unitarily equivalent to $\delta_\infty \circ h_2$ in $\mathcal{M}(B)$, then we can replace here $\pi_B \circ h_2$ by $\pi_B \circ \delta_\infty \circ h_2$.

The path $t \mapsto U(t) \in \mathcal{U}(\mathcal{M}(B))$ defines an element U in $\mathcal{U}(\mathcal{M}(\mathrm{C}_0(\mathbb{R}_+, B))) = \mathcal{U}_0(\mathcal{M}(\mathrm{C}_0(\mathbb{R}_+, B)))$ with the property $h_2(a)U - Uh_1(a) \in \mathrm{C}_0(\mathbb{R}_+, B)$, where we use the natural unital inclusions $\mathcal{M}(B) \subset \mathcal{M}(\mathrm{C}(\mathbb{R}_+, B)) \subseteq \mathcal{M}(\mathrm{C}_0(\mathbb{R}_+, B))$

Notice here that $\mathrm{C}_0(\mathbb{R}_+, B)$ is again stable and σ-unital if B has this properties.

If $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, then we can define Cuntz addition $h_1 \oplus h_2$ on $\mathrm{Hom}(A, \mathcal{M}(B))$ (cf. Chapter 4). We say that "h_1 **asymptotically absorbs** h_2" if h_1 and $h_1 \oplus h_2$ are unitarily homotopic.

The C^*-morphism h_1 **asymptotically dominates** a completely positive contraction $V\colon A \to \mathcal{M}(B)$ if there is a *norm-continuous* map $t \mapsto S(t)$ from the non-negative real numbers $\mathbb{R}_+$ into the *isometries* in $\mathcal{M}(B)$, such that,

(i) $V(a) - S(t)^* h_1(a) S(t) \in B$ for $t \in \mathbb{R}_+$ and $a \in A$, and
(ii) $\lim_{t\to\infty} \| V(a) - S(t)^* h_1(a) S(t) \| = 0$ for all $a \in A$.

Notice that h_1 and h_2 are unitarily homotopic, (respectively h_1 asymptotically dominates V), if and only if, h_1 and h_2 are unitarily equivalent in (respectively h_1 dominates V in) $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))/\,\mathrm{C_0}(\mathbb{R}_+, B) \supset \mathcal{M}(B)$. If B is unital, then this algebra is the same as $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B)$.

This "algebraic" reformulation allows to apply the observations on Cuntz additions from Sections 3 and 4 of Chapter 4, e.g. it implies immediately that h_1 asymptotically absorbs h_2, i.e., h_1 is unitarily homotopic to $h_1 \oplus h_2$, if and only if, h_1 asymptotically dominates $h_1 \oplus h_2$ and $h_2 \oplus h_2$ is unitarily homotopic to h_2, cf. Proposition 4.3.5(i,iii).

`The asymptotic domination of all unital` $h_1, h_2\colon \mathcal{O}_2 \to E$
`(of each other) should be clear ??`
`Compare all arguments with observations in Chapter 4!!!`

From nuclearity of $\mathcal{O}_2$ (and $\mathcal{D}_2 := {\mathcal{O}_2}^{\otimes\infty} \cong \mathcal{O}_2$?) one gets:

$\mathcal{O}_\infty \in {\mathcal{O}_2}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ unitally, and that there exist paths of approximate inner u.c.p. maps $V_t\colon \mathcal{O}_2 \to \mathcal{O}_2$ with $V_\infty\colon \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ is a unital C^*-morphism with $V_\infty(\mathcal{O}_2) \subseteq {\mathcal{O}_2}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ implemented by some isometry in $T \in \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ with $T^* a T = V_\infty(a)$ for $a \in \mathcal{O}_2$. Thus, the images of h_1 and h_2 commute with unital copies of $\mathcal{O}_2$ and 1-step dominate each other in $\mathrm{Q}(\mathbb{R}_+, E)$ by nuclearity of $\mathcal{O}_2$. It implies that h_1 and h_2 are unitary equivalent by a unitary in $\mathrm{Q}(\mathbb{R}_+, E)$. It means that unital C^*-morphisms $h_1, h_2\colon \mathcal{O}_2 \to E$ are unitary homotopic (with $B := E$ and $A := \mathcal{O}_2$).

The unitary homotopy of id with δ_2 on $\mathcal{O}_2$, implies that id is unitary homotopic to an endomorphism $k\colon \mathcal{O}_2 \to \mathcal{O}_2$ with $k(\mathcal{O}_2)' \cap \mathcal{O}_2 \cong \mathcal{O}_2$, and moreover there exists unital C^*-morphisms $k, k'\colon \mathcal{O}_2 \to \mathcal{O}_2$ with $k'(\mathcal{O}_2) \subseteq k(\mathcal{O}_2)' \cap \mathcal{O}_2$ and linear span of $k(\mathcal{O}_2) \cdot k'(\mathcal{O}_2)$ dense in $\mathcal{O}_2$.

Thus, all unital $h_1, h_2\colon \mathcal{O}_2 \to E$ are unitarily homotopic inside $h_k(\mathcal{O}_2) \subseteq E$ to unital homomorphisms $g_k\colon \mathcal{O}_2 \to h_k(\mathcal{O}_2) \subseteq E$ such that $g_k(\mathcal{O}_2)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally. This implies that both $h_k(\mathcal{O}_2)' \cap \mathrm{Q}(\mathbb{R}_+, E)$ contain (possibly different) unital copies of $\mathcal{O}_2$.

Since they 1-step dominate each other and commute with (possibly different) copies of $\mathcal{O}_2$ in $\mathrm{Q}(\mathbb{R}_+, E)$, they are unitarily equivalent in $\mathrm{Q}(R_+, E)$.

It means that unital $h_1, h_2\colon \mathcal{O}_2 \to E$ are unitarily homotopic with respect to $A := \mathcal{O}_2$ and $B := E$. In particular h_1 is homotopic to some unitary equivalent of h_2, because near generators of copies of $\mathcal{O}_2$ in E are homotopic.

This applies to all unital C^*-morphisms $h_1, h_2 \colon \mathcal{O}_2 \to E$ of $\mathcal{O}_2$ into a unital C^*-algebra E and shows that they are unitarily homotopic (with respect to $A := \mathcal{O}_2$ and $B := E$):

For every unital C^*-algebra E with properly infinite unit element that satisfies $0 = [1_E] \in \mathrm{K}_0(E)$ there is up to unitary homotopy exactly one unital C^*-morphism $h \colon \mathcal{O}_2 \to E$.

`Does the latter not require that` E `is` K_1`-injective?`
`Answer: No, as above/below considerations show!`

It shows that *unitary equivalence* implies unitary homotopy, but in general even not point-norm homotopy of h_1 and h_2 in $\mathrm{Hom}(A, B)$.

A unitary homotopy with norm-continuous path of unitaries $t \mapsto U(t) \in \mathcal{U}(\mathcal{M}(B))$ implies homotopy – in $\mathrm{Hom}(A, \mathcal{M}(B))$ with point-norm topology – if A is σ-unital and $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}_0(\mathcal{M}(B))$. In particular, $h_1 \otimes \mathrm{id}_{\mathbb{K}}$ and $h_2 \otimes \mathrm{id}_{\mathbb{K}}$ are homotopic in $\mathrm{Hom}(A \otimes \mathbb{K}, J \otimes \mathbb{K})$, if h_1 and h_2 are unitarily homotopic in $\mathcal{M}(B \otimes \mathbb{K})$ and if J is an *ideal* of $\mathcal{M}(B)$ that contains $h_1(A)$.

Unitary equivalent C^*-morphisms $h_1, h_2 \colon A \to \mathcal{M}(B)$ are both non-degenerate if one of its is non-degenerate (i.e., $h_k(A)B = B$).

But this is not necessarily the case for unitarily homotopic C^*-morphisms:

For example let $B := A := \mathbb{K}$, $h_1(a) := a$, $h_2(a) := TaT^*$ with $T \in \mathcal{L}(\ell_2)$ an isometry with $(1-TT^*)\ell_2$ of infinite dimension (i.e., related to a unital copy of $\mathcal{O}_2$). One can find here a norm-continuous path $t \in [0, \infty) \colon U(t) \in \mathcal{U}(\mathcal{M}(\mathbb{K}))$ of unitaries in $\mathcal{L}(\ell_2)$ with $U(0) = 1$ and $\|U(t)^* a U(t) - TaT^*\| \to 0$ for $t \to \infty$. This particular sort of *unitary* homotopy can be given explicit by using that $\mathcal{U}(\mathcal{M}(\mathbb{K})) = \mathcal{U}_0(\mathcal{M}(\mathbb{K}))$ and the fact that any trace-preserving *-endomorphism h_2 of the algebra $\mathbb{K}$ of compact operators of a separable (complex) Hilbert space is unitary homotopic to $h_1 := \mathrm{id}_{\mathbb{K}}$.

(One can use here that the set of all isometries $T \in \mathcal{L}(\ell_2(\mathbb{N})) \cong \mathcal{M}(\mathbb{K})$ with $(1-TT^*)\ell_2$ of infinite dimension is path-connected in operator norm to the isometry $T(\delta_n) := \delta_{2n}$, $n = 1, 2, \ldots$.)

Notice that it is in general not possible to start with the additional condition $U(0) = 1$ in the definition of unitary homotopy as the following counterexample shows:

Let $\mathcal{H} := \ell_2(\mathbb{N})$, $B := \mathbb{K}(\mathcal{H})$, $a_0 \in \ell_\infty(\mathbb{N}) \subseteq \mathcal{L}(\mathcal{H})$ the projection given by $(0, 1, 0, 1, \ldots)$, and $A := C^*(1, a_0) \subseteq \ell_\infty(\mathbb{N})$.

There exists $T \in \mathcal{M}(B) = \mathcal{L}(\ell_2)$ with $Ta_0 - a_0T \notin \mathbb{K}$, $T^* = -T$, $\|T\| < 1$, e.g. $T := 3^{-1}(S^* - S)$ for the 1-step forward shift S of $\ell_2(\mathbb{N})$ (Toeplitz operator with index $= -1$), $S(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots)$. Then $\exp(-T)a_0\exp(T) - a_0 \notin \mathbb{K} =: B$.

Let $h_1 := \mathrm{id}_A$ and $h_2(a) := \exp(-T)h_1(a)\exp(T)$, i.e., $\exp(T)h_2(a) - h_1(a)\exp(T) = 0$, $U(t) := \exp(T)$ for all $t \in [0, \infty)$ (constant). In particular $h_2(a) - U(t)^* h_1(a) U(t) = 0 \in \mathbb{K} =: B$ for all $t \in [0, \infty)$.

It does not exists $t \mapsto V(t) \in \mathcal{U}(\mathcal{M}(\mathbb{K}))$ norm-continuous, unitary, with $V(0) = 1$, $V(t)^* h_1(a) V(t) - h_2(a) \in \mathbb{K}$ for all $t \in [0, \infty)$ and $a \in A$, because it would imply $[\exp(T), h_1(a_0)] \in \mathbb{K}$.

A similar example is on some other place. **??**

????????

because there are natural isomorphisms $\mathrm{KK}(A, B) \cong \mathrm{Ext}(A, SB)$ for separable stable σ-unital A and B, and because

$$\mathrm{KK}(A, B) \cong \ker(\mathrm{K}_1(H_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B)),$$

cf. Chapter 8.

We shall see below and in Chapter 7, that it does *not* matter if we replace in the above definitions of asymptotic domination and of asymptotic absorption the norm-continuity of $t \mapsto S(t)$, respectively of $t \mapsto U(t)$ by the – much weaker – strict continuity if A is *stable*, h_1 is *non-degenerate* and $V(A) \subseteq B$ (respectively $h_2(A) \subseteq B$, respectively h_2 is also non-degenerate and $h_1(A) \cup h_2(A) \subseteq B$). It means, that we can *replace* a weakly continuous path $t \mapsto S(t)$ (respectively $t \mapsto U(t)$) in this cases by *norm-continuous* paths that do the same job, but are rather different from the given weakly continuous paths and have a different isometry $S(t_0)$ (respectively unitary $U(t_0)$) at the starting point $t_0 \geq 0$.

`Give Ref's to places of proofs`
`of ‘‘strictly ..." to ‘‘norm- ...’’ !!!`
`Give example of different start points of paths !!!`

The situation of unital C^*-morphisms form $\mathcal{O}_2$ into the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ of B (see Definition 5.4.7) is different:

The homotopy classes of the unital C^*-morphisms $h\colon \mathcal{O}_2 \to \mathrm{Q}^{\mathrm{s}}(B)$ are equivalent to $\mathcal{U}(\mathrm{Q}^{\mathrm{s}}(B))/\mathcal{U}_0(\mathrm{Q}^{\mathrm{s}}(B))$, thus are in bijective correspondence to $\mathrm{K}_1(\mathrm{Q}^{\mathrm{s}}(B))$ by the K_1-bijectivity of stable coronas, cf. Proposition 4.2.15. Thus, they are classified by $\mathrm{K}_0(B) \cong \mathrm{K}_1(\mathrm{Q}^{\mathrm{s}}(B))$.

But there is *only one* full non-unital C^*-morphism $h\colon \mathcal{O}_2 \to \mathrm{Q}^{\mathrm{s}}(B)$ up to *unitary homotopy*. `<-- Example? Why?`

`Sort above/below text (red and blue)!!!`

?????? Give Ref's!!!

The unital C^*-morphisms $h\colon \mathcal{O}_2 \to \mathcal{M}(B)$ for σ-unital stable B are *all* homotopic, because of $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}_0(\mathcal{M}(B))$ by the Cuntz-Higson Theorem, cf. Proposition 4.2.15, *and* unital C^*-morphisms $h_0, h_1\colon \mathcal{O}_2 \to A$ are unitarily homotopic to each other, if and only if, the unitary $u := h_0(s_1)h_1(s_1^*) + h_0(s_2)h_1(s_2^*)$ is in $\mathcal{U}_0(A)$, by Corollary ?? cf. Section ?? of Chapter 4. Clearly they are all *homotopic* if and only if $\mathcal{U}(A) = \mathcal{U}_0(A)$.

But the homotopy classes of unital $h \in \mathrm{Hom}(\mathcal{O}_2, E)$ is identical with $\mathcal{U}(E)/\mathcal{U}_0(E)$, which is isomorphic to $\mathrm{K}_1(E)$ if and only if E is K_1-injective (in addition).

All *non-unital* C^*-morphisms $h \in \mathrm{Hom}(\mathcal{O}_2, E)$ with the property that the projections $h(1)$ and $1 - h(1)$ are full in E and that $1 - h(1)$ is properly infinite are homotopic in $\mathrm{Hom}(\mathcal{O}_2, E)$ and are unitary homotopic. Clearly, the latter happens for all non-unital $h \in \mathrm{Hom}(\mathcal{O}_2, E)$ if E is simple and purely infinite.

Compare for the non-purely infinite case the elements of $\mathrm{Hom}(\mathcal{O}_2, E)$ for the stable corona $E := \mathrm{Q}^{\mathrm{s}}(M_{2^\infty})$ of M_{2^∞} in Section 8.

1. Strict convergence in Multiplier algebras

Here we give some later used unconditional strict convergence results for completely bounded maps defined with help of certain strictly "unconditional square-summable" sequences. We introduce some natural C^*-subalgebras in $\mathcal{M}(B)$ of stable (not necessarily σ-unital) B, and describe a certain "expanding" path of projections in $\mathcal{O}_2 \otimes \mathbb{K} \subset \mathcal{M}(B)$ for σ-unital stable C^*-algebras B.

A very important point for application of extension theory is the "stability" of the σ-unital C^*-algebras under consideration. We select for reference some elementary equivalent descriptions in Part (8). There are others and other proofs for the here given equivalences and descriptions. We use (or misuse) our proof of the following Remarks 5.1.1 – in particular of Parts (1) and (8) – to give also a survey on multiplier algebras and their strict topology from our viewpoints and with our terminology ([2]).

Remarks 5.1.1. The **strict topology** on $\mathcal{M}(B) \subseteq \mathcal{L}(B)$ is given by the semi-norms $d \in \mathcal{M}(B) \mapsto \|L_d(a)\| + \|R_d(b)\|$, where $a, b \in B$ ([3]), this topology is different to the **strong topology** on $\mathcal{M}(B) \subseteq \mathcal{L}(B)$, induced by the *strong operator topology* on $\mathcal{L}(B)$. The latter is defined by the semi-norms $T \mapsto \|T(a)\|$ with $a \in B$ and $T \in \mathcal{L}(B)$ only.

(Since $\mathcal{M}(B)$ has a natural C^*-algebra imbedding as a C^*-subalgebra of B^{**}, we can write $R_d(a)$ as ad and $L_d(a)$ as da for $a \in B$ and $d \in \mathcal{M}(B) \subseteq \mathcal{L}(B)$.)

The strong and the strict topology coincide on the set of normal elements $d \in \mathcal{M}(B)$, because $\|ad\| = \|da^*\|$ for $d \in \mathcal{M}(B)$ with $d^*d = dd^*$ and $a \in B$. In particular this topologies coincide on the unitary group of $\mathcal{M}(B)$ and on bounded sets of self-adjoint elements in $\mathcal{M}(B)$.

A *bounded* net in $\{b_\tau\} \subset \mathcal{M}(B)$ converges strongly (resp. strictly = *strongly) to an element of $b \in \mathcal{M}(B)$ if $\lim \|b_\tau c - bc\| = 0$ (resp. if $\lim \|b_\tau c - bc\| = 0$ and

[2] Keep always in mind the simple observation in Part (2) of the the following Remarks.

[3] In some cases, e.g. $B = \mathbb{K}$, it is on bounded parts the same as the *-*strong topology.*

$\lim \|db_\tau - db\| = 0$) for all c in a subset $X \subseteq \mathcal{M}(B)$ (resp. for all d in a subset $Y \subseteq \mathcal{M}(B)$ such that $\operatorname{span}(XB)$ and $\operatorname{span}(BY)$ are dense in B.

If B is σ-unital and $e \in B_+$ is a strictly positive element, the strict topology is metrizable with metric $\rho(b,c) := \|(b-c)e\| + \|e(b-c)\|$ on *bounded parts* of $\mathcal{M}(B)$.

A *series* $\sum_n d_n$ means the sequence of its partial sums $e_k := \sum_{n=1}^k d_n$ with $d_n \in \mathcal{M}(B)$. It converges **unconditional** strictly to $T \in \mathcal{M}(B)$ if $\sum_n d_{\gamma(n)}$ converges also strictly in $\mathcal{M}(B)$ for each permutation γ of $\mathbb{N}$. The series $\sum_n d_n$ denotes also the sum (if it exists).

If $\sum_n d_n$ converges unconditional in strict topology, then for its sum holds $\sum_n d_{\gamma(n)} = \sum_n d_n$ for each permutation γ of $\mathbb{N}$, because this is true for the series $\sum_n f(d_n)$ and each linear functional $f \in B^*$, where B^* is considered as a (complemented) subspace of $\mathcal{M}(B)^*$ by $B \triangleleft \mathcal{M}(B)$.

(1) A C^*-morphism $h\colon A \to \mathcal{M}(B)$ extends *uniquely* to a unital and *strictly continuous* C^*-morphism $\mathcal{M}(h)$ from $\mathcal{M}(A)$ into $\mathcal{M}(B)$, if and only if, the linear span of the set $h(A) \cdot B$ is dense in B. (In fact the closed linear span of $h(A) \cdot B$ is identical with the *set* of elements $h(a) \cdot b$, $a \in A$ and $b \in B$, because the proof of G.K. Pedersen shows that the $y \in X$ in the factorization $e \cdot y = x$ can be taken in the closed convex cone $\overline{B_+ \cdot x}$ of X, cf. G.K. Pedersen's proof a C^*-version of the Cohen Factorization Theorem outlined in Section 11 of Appendix A.

Def. of ‘‘non-degenerate’’ is used first in Part (7), but also
in Chapters 1, 2 and 3
(and perhaps Chapter 4 ?)

We call the C^*-morphism h **non-degenerate** if $h(A)B$ is dense in B. A C^*-subalgebra $C \subseteq \mathcal{M}(B)$ is a **non-degenerate subalgebra** if CB is dense in B.

(2) Suppose that $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are sequences in $\mathcal{M}(B)$ such that the sums $\sum x_n x_n^*$ and $\sum y_n y_n^*$ converge strongly to elements X and Y of $\mathcal{M}(B)_+$. We identify $\ell_\infty(\mathcal{M}(B))$ naturally with the multiplier algebra of $c_0(B) \cong B \otimes c_0$.

Original d_n have been changed to b_n
for later notational reasons.
Check changes in proofs (!), other references, applications!!

Then, for every bounded sequence $b = (b_1, b_2, \ldots) \in \ell_\infty(\mathcal{M}(B))$, the series $\sum y_n b_n x_n^*$ converges strictly and unconditionally to an element $\Gamma(b_1, b_2, \ldots)$ in $\mathcal{M}(B)$. It defines a strictly continuous and completely bounded linear map

$$\Gamma : \quad (b_1, b_2, \ldots) \in \ell_\infty(\mathcal{M}(B)) = \mathcal{M}(c_0(B)) \;\mapsto\; \sum y_n b_n x_n^* \in \mathcal{M}(B)\,.$$

The mapping Γ maps $c_0(B)$ into B and has cb-norm

$$\|\,\Gamma\,\|_{\mathrm{cb}} \;\leq\; \big(\|X\| \cdot \|Y\|\big)^{1/2}\,.$$

This convergence observation turns the proofs of Lemma 5.1.2, Proposition 5.4.1 and some others into straight-forward calculations – sometimes left to the reader, as e.g. in the proof of Corollary 2.2.11(i) where this observation is used implicitly.

(3) The applications of the unconditional strict convergence of the expressions Γ – observed in Remark (2) – use sometimes quasi-central commutative approximate units of σ-unital C^*-algebras B to construct suitable sequences (x_n) and (y_n) that satisfy the assumptions in Part (2):

Let $e \in B_+$ any strictly positive element in B of norm $\|e\| = 1$, $A \subseteq \mathcal{M}(B)$ a separable C^*-subalgebra, and take a linear filtration of A by vector sub-spaces $X_n \subseteq A$ of finite dimension with the properties that $X_n = X_n^* := \{x^* ;\ x \in X_n\} \subseteq X_{n+1}$, $\dim(X_n) < \infty$ and $\bigcup_n X_n$ is dense in A.

Separation arguments for convex combinations of $f(e)$ with suitable non-decreasing functions f show that there exists a strictly decreasing zero-sequence $\alpha_1 > \alpha_2 > \cdots$ in $(0,1)$ and non-decreasing functions $f_n \in \mathrm{C}_0((0,1])_+$, $n := 0, 1, 2, \ldots$, with properties $f_0(t) := t$, $f_n(t) = 0$ for $t \in [0, \alpha_n]$,

$$f_n f_{n+1} = f_n,\ \ \|f_n\| = 1,\ \text{ and }\ \|f_0 - f_n f_0\| < 4^{-n}\,,$$

and the property that, for each $b \in X_n$ and $m > n$, the commutators have estimates

$$\|\,[f_n(e), b]\,\| \ +\ \|[(f_m(e) - f_n(e))^{1/2}, b]\,\| \ <\ 4^{-n}\|b\|\,.$$

The properties imply $f_n|[\varepsilon, 1] = 1$ and $\|\, e - f_n(e)e\,\| < \varepsilon$ for $n \geq 1 - \log(\varepsilon)$.

(4) Especially let $x_n := y_n := g_n := (f_n(e) - f_{n-1}(e))^{1/2}$ for $n > 1$ and $x_1 := y_1 := g_1 := f_1^{1/2}(e)$ in Remark (2) with f_n selected for $X_n \subseteq A \subseteq \mathcal{M}(B)$ as in Remark (3).

The sum $\sum_n g_n^* g_n$ converges strictly to 1 and the in Remark (2) defined map $\Gamma \colon \ell_\infty(\mathcal{M}(B)) \to \mathcal{M}(B)$ with $x_n := y_n := g_n^*$ is a *unital* completely positive map with $a - \Gamma(a, a, \ldots) \in B$ for all $a \in A$.

(5) If $g_1, g_2, \ldots$ are as in Remarks (3) and (4), constructed for a given filtration X_n of $A \subseteq \mathcal{M}(B)$ and a given strictly positive contraction $e \in B_+$, and if $d_1, d_2, \ldots$ is a bounded sequence in $\mathcal{M}(B)$ with $\sum_n \|d_n g_n - g_n d_n\| < \infty$, then the series $\sum_n d_n g_n$ converges strictly to an element $d \in \mathcal{M}(B)$.

In particular, let $P \subseteq \mathbb{N}$ and (d_n) defined by $d_n := 1$ if $n \in P$ and $d_n := 0$ if $n \in \mathbb{N}\backslash P$. The corresponding positive contraction $S := S_P := \sum_{n\in P} g_n \in \mathcal{M}(B)_+$ is not necessarily in B.

But we get from Remark (4) that

$$Sa - aS\,,\ SaS \,+\, (1 - S^2)^{1/2} a (1 - S^2)^{1/2} \,-\, a \ \in B \quad \forall\, a \in A\,.$$

The operator S satisfies $S^2 = \sum_{n\in P} g_n^2$ if $|p - q| > 1$ for all $p, q \in P$ with $p \neq q$.

(6) Suppose that a sequence of contractions $d_1, d_2, \ldots \in \mathcal{M}(B)$, a self-adjoint contraction $a^* = a \in \mathcal{M}(B)$ and the sequence $g_1, g_2, \ldots$ defined in Remark (4) satisfy

(i) $\sum_\ell \nu_\ell < \infty$ for $\nu_\ell := \sup_{n\geq \ell}\bigl(\max_{1\leq k\leq n} \|d_k g_n - g_n d_k\|\,\bigr)$,

(ii) $\sum_\ell \mu_\ell < \infty$ for $\mu_\ell := \sup_{n\geq \ell} \|g_n a - a g_n\|$, and
(iii) $\lim_{n\to\infty} \|g_n d_n^* a d_{n+1} g_{n+1}\| = 0$.

Then $d_1, d_2, \ldots, a$ and $d := \sum_n d_n g_n$ satisfy

$$d^* a d \,-\, \Gamma(d_1^* a d_1\,,\; d_2^* a d_2\,,\; \ldots) \,\in\, B\,.$$

(7) D is a **corner** of B, if D is a hereditary C^*-subalgebra of B and $D+\mathrm{Ann}(D)$ is a *non-degenerate* C^*-subalgebra of B.

It says equivalently that there is a projection $p \in \mathcal{M}(B)$ with $D = pBp$.

A subset $X \subseteq B$ **generates a corner** of B, if the closure of the linear span of $C^*(X) \cdot B \cdot C^*(X)$ is a corner of B.

(In fact the set of products $c \cdot b \cdot c$ with $b \in B$ and contractions $c \in C^*(X)_+$ in an approximate unit of $C^*(X)$ is identical with the hereditary C^*-subalgebra of B generated by $C^*(X)$ as the Cohen factorization theorem [**621**, Thm. 4.1] shows, cf. also Theorem A.11.1.)

(8) Recall that B is a **stable C^*-algebra** if there exists an isomorphism ψ from $B \otimes \mathbb{K}$ onto B, where $\mathbb{K} := \mathbb{K}(\ell_2)$ denotes the compact operators $\mathbb{K} := \mathbb{K}(\ell_2)$ on $\ell_2(\mathbb{N})$.

This is equivalent to the existence of a C^*-algebra C and a C^*-algebra isomorphism from $C \otimes \mathbb{K}$ onto B.

The following criterium is necessary and sufficient for the stability of a (not necessarily σ-unital) C^*-algebra B:
There exists a sequence $s_1, s_2, \ldots$ of isometries in $\mathcal{M}(B)$ such that $\sum_n s_n s_n^*$ converges strictly to 1 *in $\mathcal{M}(B)$.*

Given a sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$ of isometries with $\sum_n s_n s_n^*$ strictly convergent to 1, then $\delta_\infty(b) := \sum_n s_n b s_n^*$ for $b \in \mathcal{M}(B)$ defines a faithful strictly continuous unital *-endomorphism of $\mathcal{M}(B)$.

It holds $u^* \delta_\infty(b) u = \sum_n t_n b t_n^*$ by a unitary $u \in \mathcal{M}(B)$ with $t_n = u s_n$ for $n = 1, 2, \ldots$ if $t_1, t_2, \ldots$ is any other sequence of isometries in $\mathcal{M}(B)$ with $\sum_n t_n t_n^* = 1$ (strictly convergent).

The unitary equivalence class $[\delta_\infty]$ has representatives $\delta_\infty \colon \mathcal{M}(B) \to \mathcal{M}(B)$ determined by $s_1, s_2, \ldots$. We call this class (and sometimes the endomorphisms $\sum_n s_n(\cdot)s_n$ in this class) the **infinite repeat** (on $\mathcal{M}(B)$) of the identity mapping $\mathrm{id}_{\mathcal{M}(B)}$ of $\mathcal{M}(B)$. Then $\delta_\infty(b)$ (or its unitary equivalence class $[\delta_\infty(b)]$) is the *infinite repeat of the element $b \in \mathcal{M}(B)$.*

There exists a C^*-morphism $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ with following properties (a,b,c):

(a) $\mu(\mathbb{K}) \subset \delta_\infty(B)' \cap \mathcal{M}(B)$,
(b) $\delta_\infty(b)\mu(k) \in B$ for $b \in B$ and $k \in \mathbb{K}$, and

(c) the (unique) C^*-morphism $\varphi\colon B \otimes \mathbb{K} \to B$ with $\varphi(b \otimes k) = \delta_\infty(b)\mu(k)$ is an isomorphism from $B \otimes \mathbb{K}$ *onto* B that maps $J \otimes \mathbb{K}$ onto J for each ideal J of B.

If a C^*-morphism $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ satisfies conditions (a,b,c) then $\mu(\mathbb{K}) \cdot B = B$.

The C^*-morphisms $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with properties (a,b,c) are uniquely defined by the given sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$ of isometries with $\sum s_n s_n^* = 1$ – *up to unitary equivalence by unitaries in* $\delta_\infty(B)' \cap \mathcal{M}(B)$.

One of the C^*-morphisms $\mu\colon \mathbb{K} \to \delta_\infty(B)' \cap \mathcal{M}(B)$ with properties (a,b,c) can be defined by $\mu(p_{j,k}) := s_j s_k^*$ for $j, k \in \mathbb{N}$, if the unitary equivalence class $[\delta_\infty]$ is realized by given isometries $s_1, s_2, \ldots$, i.e., if $\delta_\infty(\cdot) := \sum_n s_n(\cdot)s_n^*$.

The isomorphism φ from $B \otimes \mathbb{K}$ onto B defined by $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with properties (a,b,c) has the property that the corresponding C^*-morphism $b \mapsto \varphi(b \otimes p_{1,1})$ is a *-endomorphism of B, that is approximately 1-step inner as a c.p. contraction.

The C^*-morphism $\varphi((\cdot) \otimes p_{1,1})$ is unitarily homotopic to id_B if B is σ-unital.

If $b \in B_+$, then $\varphi(b \otimes e_{1,1})$ is Murray–von-Neumann equivalent to b. In particular, $\varphi(J \otimes \mathbb{K}) = J$ for each $J \in \mathcal{I}(B)$.

This property of the "natural" φ is different from the properties of very random and possibly not well-behaved *-isomorphism ψ from $B \otimes \mathbb{K}$ onto B – as allowed in the *definition* of stability –, because in general the isomorphism $\varphi \circ \psi^{-1}$ of B does not fix the ideal-system $\mathcal{I}(B)$ of B and is not approximately unitarily equivalent to id_B by unitaries in $\mathcal{M}(B)$.

The non-degenerate *-monomorphism $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with properties (a,b,c) has the "dual" property that

$$\mu(\mathbb{K})' \cap \mathcal{M}(B) = \delta_\infty(\mathcal{M}(B))\,.$$

It implies that $\delta_\infty(\mathcal{M}(B))$ is closed in $\mathcal{M}(B)$ with respect to the strict topology on $\mathcal{M}(B)$.

The strictly continuous extension $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2) \to \mathcal{M}(B)$ of μ is a unital *-monomorphism, has image in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ and the norm-closed unit-ball of $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ is also *closed in the strict topology* of $\mathcal{M}(B)$.

The strictly closed C^*-subalgebra $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ of $\mathcal{M}(B)$ can be larger than $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$, e.g. if the center of $\mathcal{M}(B)$ is not trivial.

For each non-degenerate C^*-morphism $\psi\colon \mathbb{K}(\ell_2) \to \mathcal{M}(B)$, the topology induced via $\mathcal{M}(\psi)^{-1}$ on $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$ by the strict topology on $\mathcal{M}(B)$ coincides on bounded parts with the *-strong operator topology on $\mathcal{L}(\ell_2)$.

(9) The $(s_1, s_2, \ldots)$-existence criteria for stability in Remark (8) shows that

B and $B + h(A)$ are stable if A is stable and $h\colon A \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism.

In particular, B is stable if B contains a non-degenerate stable C^*-subalgebra.

`To be worked out in detail:`

(10) Let $C_1, C_2, \ldots$ a contable sequence of separable unital C^*-algebras, $\mathbb{K}\otimes D$ a σ-unital stable C^*-algebra, and let $B := \mathbb{K} \otimes D \otimes C_1 \otimes C_2 \otimes \cdots$ the infinite tensor product. We define unital C^*-subalgebras $G_1 \subset G_2 \subset \cdots$ of $\mathcal{M}(B)$ by the following conditions:

`To be defined. Sort of infinite` $\mathcal{E}(C_n, C_{n+1}) \subseteq \mathcal{M}(B)$ `build with help an approximately central approximate unit of` B `that is approximately central for` $h(A)$ `of a given` `C*-morphism`

$$h \colon A \to \mathcal{M}(B).$$

?????

Then for every C^*-morphism $h \colon A \to \mathcal{M}(B)$ of a separable C^*-algebra A there exists $n_0 := n_0(h(A)) \in \mathbb{N}$ such that $[h(A), G_n] \subseteq B$ for all $n \geq n_0$ and, for each $a \in A$, $\|[h(a), g_n]\| \to 0$ for any sequence of contractions $g_n \in G_n$.

(With n_0 depending from an approximate unit in $\mathbb{K} \otimes D$ that is quasi-central for $h(A)$.)

TO BE FILLED IN – for almost central sequences...

Lemma, ???? Idea ???, corona property ...

Case $B := \mathbb{K} \otimes D \otimes C_1 \otimes C_2 \otimes \cdots$ $A \subseteq \mathcal{M}(B)$ separable C^*-subalgebra. Then one can find *tensorial residuum* if $1 \otimes 1_1 \otimes \cdots \otimes 1_{n_k} \otimes C_{n_{k+1}} \otimes \cdots$ commuting modulo B with A... provided that the C_n are unital simple and separable ... MORE conditions ???

Proofs of Remarks (1)–(10):

We say here something about the different definitions of multipliers, multiplier algebras, its relation to factorization properties of C^*-algebra modules and the interrelation between the (by them defined) strict, strong and weak* topologies on bounded subsets of C^*-subalgebrasof multiplier algebras. And we explain the often required "non-degeneracy" of modular morphisms for the often needed verification of unconditional strict convergence of series.

(0.1) **General multiplier algebras**:

If B is a Banach algebra and $\rho \colon B \to \mathcal{L}(X)$ is an algebra homomorphism, then the "action" ρ of B on X, respectively the (left) B-module X defined by $b \cdot x := \rho(b)(x)$ is called **non-degenerate** if $\text{span}(\rho(B)X)$ is dense in X.

To simplify some arguments we use later the special case for C^*-algebras B of the Cohen factorization theorem ([4]) given by G.K. Pedersen [**621**, Thm. 4.1] (with 13 lines of proof):

[4] P. Cohen [**?**] proved 1959 that Banach algebras B with bounded approximate left-unit have the factorization property. In 1964, E. Hewitt [**363**] and P. Koosis [**491**] showed that this factorization property also holds for non-degenerate Banach B-modules.

If a C^*-algebra B acts on a Banach space X from left with $\|b\cdot x\| \leq \|b\|\|x\|$ and if $\mathrm{span}(B \cdot X)$ *is dense in X, then for each $\varepsilon > 0$ and $x \in X$ there exists $e \in B_+$ and $y \in X$ with $e \cdot y = x$, $\|e\| \leq 1$, $\|y - x\| < \varepsilon$.*

In fact, the proof of G.K. Pedersen, shows that the $y \in X$ in the factorization $e \cdot y = x$ can be chosen in the closed convex cone $\overline{B_+ \cdot x}$ of X, cf. Section 11 of Appendix A.

Since one can here replace X likewise by $\overline{B \cdot x}$ or by $c_0(X)$ with left action $b \cdot (x_1, x_2, \ldots) = (b \cdot x_1, b \cdot x_2, \ldots)$, we can find a positive contraction $a \in B$ and $y \in \overline{B \cdot x}$ (respectively $(y_1, y_2, \ldots) \in c_0(X)$) with $b \cdot y = x$ and $\|x - y\| < \varepsilon$ (respectively with $b \cdot y_n = x_n$ and $\|x_n - y_n\| < \varepsilon$ for $n = 1, 2, \ldots$). The latter implies in particular for $X := B$ that for $a, b \in B$ there are $e \in B_+$, $c \in \overline{B_+ a}$ and $d \in \overline{B_+ b}$ with $ec = a$, $ed = b$ and $\|e\| \leq 1$.

If B is a σ-unital C^*-algebra and X a non-degenerate B-module with $\|b\cdot x\| \leq \|b\|\,\|x\|$, then one can find, for each given strictly positive contraction $b_0 \in B_+$ and $\varepsilon > 0$ and $x \in X$, a factorization $e \cdot y = x$ with $\|y - x\| < \varepsilon$ where $y \in X$ and $e := \varphi(b_0)$, φ an increasing continuous function with $f(0) = 0$ and $f(1) \leq 1$.

Except the latter, all this is also true for Banach algebras B with an approximate unit consisting of contractions. (But one has to use for such algebras B the estimates from the generalization of the Cohen factorization given by E. Hewitt or P. Koosis.)

In particular, the following definition and arguments work more generally for Banach algebras B that contains an approximate unit $\{e_\tau\}$ with $\| e_\tau \| \leq 1$.

Notice that the non-degeneracy of the action of B on X implies that $x \in \overline{B \cdot x}$ and and thus, that $B \cdot x = \{0\}$ is equivalent to $x = 0$.

This allows to extend the action of B on X in a unique way to a left module action of the **left-multiplier algebra** of B , i.e., of the operator-norm closed subalgebra $\mathcal{M}_\ell(B)$ of $\mathcal{L}(B)$ of bounded **left multipliers** $L \in \mathcal{L}(B)$ of B, that are defined by its property

$$L(ab) = L(a)b \quad \text{for all } a, b \in B\,.$$

(The elements $L \in \mathcal{M}_\ell$ are also called "left centralizers" of B, where we have to consider B as subalgebra of $\mathcal{L}(B)$.)

The extension of the B-modul X to an $\mathcal{M}_\ell(B)$-modul X with $(L_b) \cdot' x = b \cdot x := \rho(b)(x)$ is uniquely defined for $L \in \mathcal{M}_\ell(B)$ and $y \in X$ via Cohen factorization $y := b \cdot x$ by

$$L \cdot' (b \cdot x) := L(b) \cdot x \quad \text{for } b\,.$$

The definition of $L \cdot' y$ is justified, because the right side is equal to $\lim_\tau L(e_\tau b) \cdot x$, because L is bounded and $e_\tau b \to b$. Since $L(ea) \cdot x = L(e) \cdot (a \cdot x) = L(eb) \cdot y$ for $a \cdot x = b \cdot y$, it follows that the action of $\mathcal{M}_\ell(B)$ is well-defined. It is then not difficult to check that $(L, x) \mapsto L \cdot' x$ is bi-linear in L and $x \in X$ and $\|L \cdot' x\| \leq \|L\|\|x\|$. See [**621**, thm. 5.9] and proof of [**621**, thm. 5.10] for the case where B is a C^*-algebra.

Obviously an operator $T \in \mathcal{L}(B)$ is in $\mathcal{M}_\ell(B)$ if and only $TR_b = R_bT$ for all $b \in B$, where $R_b(a) := ab$ for $a \in B$. This shows that $\mathcal{M}_\ell(B)$ is closed in $\mathcal{L}(B)$ with respect to the strong operator topology.

If a Banach algebra B contains (two-sided) bounded approximate unit, then the Cohen factorization theorem, applied to B as a left module over B, shows that the subalgebra $\mathcal{M}_\ell(B) \subset \mathcal{L}(B)$ is the closure of the (then to B isomorphic) subalgebra $\{L_b\,;\ b \in B\} \subset \mathcal{L}(B)$ with respect to the strong operator topology on $\mathcal{L}(B)$.

We use now that for every *Banach space* X, the space $\mathcal{L}(X)$ is *complete* as locally convex vector space with respect to the *strong* operator topology (an immediate consequence of the Banach-Steinhaus uniform boundedness theorem).

It implies that $\mathcal{M}_\ell(B)$ equipped with the strong operator topology is the *completion* of B with respect to the uniform structure defined on $B \times B$ by the system of semi-metrics $\rho_c(a, b) := \|(b - a)c\|$ $(c \in B)$, if B has a *bounded two-sided approximate unit*.

Let $\eta_B \colon B \hookrightarrow B^{**}$ the natural isometry from B into its second conjugate Banach *space* B^{**}. We identify the elements $b \in B$ later with its image $\eta_B(b) \in B^{**}$.

If B has a (left) approximate unit consisting of *contractions* in B then there exists an element $e \in B^{**}$ with the properties $\|e\| \leq 1$ and $(R_b)^{**}(e) = \eta_B(b)$ for all $b \in B$.

It follows that the map $T \in \mathcal{L}(B) \mapsto T^{**}(e) \in B^{**}$ defined on $\mathcal{L}(B)$ has an *isometric* restriction λ to $\mathcal{M}_\ell(B)$ that satisfies

$$(R_b)^{**}\lambda(T) = \eta_B(Tb) \quad \text{and} \quad \|\lambda(T)\| = \|T\| \quad \text{for } b \in B,\ T \in \mathcal{M}_\ell(B)\,. \tag{1.1}$$

If $X \in B^{**}$ satisfies $(R_b)^{**}T \in \eta_B(B)$ for all $b \in B$ then $T(b) := \eta_B^{-1}((R_b)^{**}X)$ defines an element $T \in \mathcal{M}_\ell(B)$ with $\lambda(T) = X$.

If, moreover, the linear span of the elements $(R_b)^*\rho$ for $b \in B$ and $\rho \in B^*$ is norm-dense in B^* then the element $e \in B^{**}$ with the property $(R_b)^{**}(e) = b$ for all $b \in B$ and the map $\lambda\colon \mathcal{M}_\ell(B) \to B^{**}$ with the quoted property is unique.

The latter is the case for C^*-algebras B, by the Kaplansky density theorem for $B \subseteq B^{**}$.

Notice that that $(R_b)^{**}(X) = X \cdot \eta_B(b)$ in the W*-algebra B^{**} for $b \in B$ and $X \in B^{**}$ if B is a C^*-algebra B, i.e., the equation (1.1) becomes

$$\gamma(T)\eta_B(a) = \eta_B(Ta)\,.$$

It implies that γ is an isometric algebra isomorphism from $\mathcal{M}_\ell$ into B^{**} with $\gamma(L_b) = \eta_B(b)$.

If $X \in B^{**}$ satisfies $X\cdot \in B \subset B^{**}$ then there exists $T \in \mathcal{M}_\ell(B)$ with $X = \gamma(T)$ for $T(b) := \eta_B^{-1}(X\eta_B(b))$.

It defines a multiplicative isometric algebra representation of $\mathcal{M}_\ell(B)$ into the W*-algebra B^{**}. The image of λ is the algebra of left-multipliers of $\eta_B(B) \subseteq B^{**}$ inside B^{**}.

Recall that each C^*-algebras B has an approximate unit consisting of contractions and the linear span of the elements $(R_b)^*(\rho)$ with $b \in B$ and $\rho \in B^*$ (where $R_b(a) := ba$) is norm-dense in B^*, cf. [**616**, thm. 4.1.2] and use that each ρ is in the norm closure of $(R_b)^* \cdot \rho$ ($b \in B$, $\|b\| \leq 1$), because B is *-ultra-strong dense in second conjugate W*-algebra B^{**} by Kaplansky density theorem cf. [**616**, thm. 2.3.3]. In particular, in this case $e = 1 := 1_{B^{**}}$ and $\gamma(T) := T^{**}(1)$ for $T \in \mathcal{M}_\ell(B)$.

Thus a natural isometric linear map $\lambda\colon \mathcal{M}_\ell(\eta_B)\colon \mathcal{M}_\ell(B) \hookrightarrow B^{**}$ is given by the restriction of the natural linear map

$$T \in \mathcal{L}(B) \mapsto T^{**}(e) \in B^{**}$$

to $\mathcal{M}_\ell(B)$, where T^{**} here is the bi-adjoint of the operator $T \in \mathcal{L}(B)$ and e is the unique cluster point of an approximate unit consisting of contractions in B.

The restriction of the map $T \mapsto T^{**}(e)$ to the elements in the algebra of left multipliers $\mathcal{M}_\ell(B) \subseteq \mathcal{L}(B)$ of the map $T \to T^{**}(1)$ from $\mathcal{L}(B)$ to B^{**} is isometric and multiplicative if the *Banach* algebra B contains a two-sided approximate unit consisting of contractions and the elements $(R_b)^*(\rho)(\cdot) := \rho((\cdot)b)$ for $\rho \in B^*$ and $b \in B$ are norm-dense in the dual Banach space B^* of B. Here e denotes the (then) unique element in B^{**} with $(R_b)^{**}(e) = b$ for all $b \in B$. In particular the map $\gamma\colon \mathcal{M}_\ell(B) \to B^{**}$ is uniquely defined for all C^*-algebras B.

The general (two-sided) **multiplier algebra** of a Banach algebras B (not necessarily with involution operation on B) is defined as the set of pairs (L, R) of operators $L, R \in \mathcal{L}(B)$ that satisfy $(R(b)) \cdot c = b \cdot (L(c))$, $L(ab) = L(a)b$ and $R(ab) = aR(b)$ for $a, b, c \in B$. Notice that left and right multiplication (L_a, R_a) by $a \in B$ defines a two-sided multiplier for each $a \in B$.

The property $(R(b)) \cdot c = b \cdot (L(c))$ for all $b, c \in B$ of the pair $(L, R) \in \mathcal{L}(B) \times \mathcal{L}(B)$ alone implies that $a(L(bc) - L(b)c) = 0 = (R(ab) - aR(b))c$ for all $a, b, c \in B$. Thus implies automatically that L and R are left and right multipliers if e.g. B contains a (two-sided) approximate unit. Moreover, then the pair (L, R) is determined uniquely by L alone and the property that to this (special sort of) left multipliers L there exists a right multiplier such that (L, R) is a two-sided multiplier.

In a Banach *-algebra B one can transform each right multiplier R into a left multiplier $\widehat{R} \in \mathcal{M}_\ell(B)$ by the anti-linear map on $\mathcal{L}(B)$ (of order 2) given by $\widehat{R}(a) := R(a^*)^*$. If a Banach *-algebra B contains an approximate unit then the two-sided multipliers are the pairs of left multipliers $(L, \widehat{R})$ that satisfy $(\widehat{R}(a))^* b = a^* L(b)$, where $\widehat{R}$ is uniquely determined by L (if R exists for L). We rename them and describe them as pairs $(S, T) \in \mathcal{M}_\ell(B) \times \mathcal{M}_\ell(B)$ and the defining property is the relation

$$a^* S(b) = T(a)^* b \quad \text{for all } \; a, b \in B\,, \tag{1.2}$$

that is equivalent to $\widehat{S}(b)a = bT(a)$ for $a, b \in B$.

If B contains an approximate unit (e_τ) with $\|e_\tau\| \leq 1$ then Equation (1.2) implies that $\|S\| = \|T\|$. In this way we can define on the two-sided multiplies (S,T) a norm $\|(S,T)\| := \max\{\|S\|, \|T\|\} = \|S\|$.

(0.2) **Two-sided multipliers of C^*-algebras**: We use the *equivalent* definition for the two-sided multiplier algebra $\mathcal{M}(B)$ in case of C^*-algebras B that better suites for our applications:

We define $\mathcal{M}(B)$ as the norm-closed sub-algebra of $\mathcal{M}_\ell(B) \subseteq \mathcal{L}(B)$ consisting of all operators $T \in \mathcal{M}_\ell(B)$ with the property that there exists $S \in \mathcal{L}(B)$ that satisfies Equation (1.2).

This equations yield automatically that the operator-norm inherited from $\mathcal{L}(B)$ is the same as the C^*-norm of the C^*-algebra $\mathcal{M}(B)$.

The reader should notice the we have in fact a stronger observation (going back to an old remark of J. von Neumann):
Alone the property that S and T are abstract maps from B into B (without supposing linearity as precondition) that satisfy $a^*S(b) = T(a)^*b$ for all $b, c \in B$ allows to see, using the property $\|d^*d\| = \|d\|^2 = \|dd^*\|$ of C^*-norms and the closed graph theorem, that S and T must be *linear* operators that are bounded and have the properties $\|T\| = \|S\|$, $T(bc) = (Tb)c$, $S(bc) = (Sb)c$ and that S is uniquely determined by T. Then $S := T^* \in \mathcal{M}_\ell(B)$ is *well-defined* for T in this closed subalgebra $\mathcal{M}(B)$ of $\mathcal{M}_\ell(B)$ by $T^* := S$.

The natural *-monomorphism of B onto an essential ideal of $\mathcal{M}(B) \subseteq \mathcal{L}(B)$ is given by $b \in B \mapsto L_b \in \mathcal{L}(B)$, where $L_b(c) := bc$ for $b, c \in B$. We *identify* B with its image in $\mathcal{M}(B)$ via $b \leftrightarrow L_b$.

The so-called *strong topology* and *strict topology* on $\mathcal{M}(B) \subseteq \mathcal{L}(B)$ is given by the restriction to $\mathcal{M}(B)$ of the usual strong operator topology on $\mathcal{L}(B)$, that is given by the family of semi-norms $T \mapsto \|Tb\|$ with $b \in B$, and the strict topology is given by the family of semi-norms $T \mapsto \|Tb\|$ and $T \mapsto \|T^*b\|$, where $T^* \in \mathcal{L}(B)$ is defined as above by $(T^*a)^*b = a^*(Tb)$ for $a, b \in B$.

Obviously this implies that in case of σ-unital B we can take a strictly positive contraction $b_0 \in B_+$ and get with $T \mapsto \|Tb_0\| + \|T^*b_0\|$ a *defining norm* on $\mathcal{M}(B)$ that defines the strict topology on *bounded parts* of $\mathcal{M}(B)$.

Let $\eta_B \colon B \hookrightarrow B^{**}$ the natural *-monomorphism from B into its second conjugate W*-algebra B^{**}. Sometimes we identify the elements $b \in B$ with its image $\eta_B(b) \in B^{**}$.

A natural and unital C^*-monomorphism $\mathcal{M}(\eta_B) \colon \mathcal{M}(B) \hookrightarrow B^{**}$ is given by the restriction of the natural linear map

$$\gamma \colon T \in \mathcal{L}(B) \mapsto T^{**}(1) \in B^{**}$$

to $\mathcal{M}(B)$, where T^{**} here is defined as the bi-adjoint of the operator $T \in \mathcal{L}(B)$ and $1 := 1_{B^{**}}$ is the unit element of the W*-algebra B^{**}.

Look at next. Is it mentioned above?

The restriction of the map $T \mapsto T^{**}(1)$ (with $1 = 1_{B^{**}}$) to the elements in the algebra of left multipliers $\mathcal{M}_\ell(B) \subseteq \mathcal{L}(B)$ of the map $T \to T^{**}(1)$ from $\mathcal{L}(B)$ to B^{**} is isometric and multiplicative if the Banach algebra B contains a two-sided approximate unit consisting of contractions and the elements $(R_b)^*(\rho)(\cdot) := \rho((\cdot)b)$ for $\rho \in B^*$ and $b \in B$ are norm-dense in the dual Banach space B^* of B. Here $1 := 1_{B^{**}}$ denotes the (then) unique element in B^{**} with $(R_b)^{**}(1) = b$ for all $b \in B$.

Thus, if B is a C^*-algebra, then it is a C^*-algebra monomorphism if restricted to $\mathcal{M}(B) \subseteq \mathcal{L}(B)$ because, $B_+ \ni e_\tau \to 1$ in B^{**} with $\sigma(B^{**}, B^*)$ topology, e.g.

$$T(b) = T^{**}(1 \cdot b) = \lim_\tau T(e_\tau \cdot b) = T^{**}(e_\tau)b = T^{**}(1)b$$

for all $b \in B^{**}$. To see this, use that T^{**} is $\sigma(B^{**}, B^*)$-continuous and that the unit ball of B is $\sigma(B^{**}, B^*)$-dense in the unit ball of B^{**}.

The image of $T \in \mathcal{M}(B) \mapsto T^{**}(1_{B^{**}})$ is a C^*-algebra isomorphism from $\mathcal{M}(B)$ onto the C^*-subalgebra of the W*-algebra B^{**} of elements $d \in B^{**}$ with the property $dB \subseteq B$ and $Bd \subseteq B$, where we have the elements $b \in B$ naturally identified with its images $\eta_B(b) \in B^{**}$ in the second conjugate W*-algebra B^{**}.

We say that a C^*-morphism $h\colon B \to M$ from a C^*-algebra B into a W*-algebra M that is **weakly non-degenerate** if 1_M is contained in the $\sigma(M, M_*)$-closure of $h(B)$.

If h is weakly non-degenerate then there exists a unique C^*-morphism $\mathcal{M}(h)\colon \mathcal{M}(B) \to M$ with the property $\mathcal{M}(h)|B = h$. The C^*-morphism $\mathcal{M}(h)$ is unital and maps $\mathcal{M}(B)$ into the C^*-subalgebra of M given by the two-sided multipliers $m \in M$ of $h(B)$ in M, i.e., $mh(B) \cup h(B)m \in h(B)$ for $m \in \mathcal{M}(h)(\mathcal{M}(B))$.

If $h\colon B \to M$ is *faithful* and *weakly non-degenerate* then $\mathcal{M}(h)$ is faithful and each two-sided multiplier m of $h(B)$ is in the image $\mathcal{M}(h)(\mathcal{M}(B))$, because then $L\colon b \mapsto h^{-1}(mh(b))$ and $R\colon b \mapsto h^{-1}(h(b)m)$ define a pair $(L, R) \in \mathcal{L}(B) \times \mathcal{L}(B)$ that defines an element $T \in \mathcal{M}(B)$ with $\mathcal{M}(h)(T)h(b) = mh(b)$. Now use that 1_M is in the $\sigma(M, M_*)$-closure of $h(B)$.

The latter applies to $B := \mathbb{K}(\mathcal{H})$, $M := \mathcal{L}(\mathcal{H})$ and to $h\colon b \in \mathbb{K}(\mathcal{H}) \to b \in \mathbb{K}(\mathcal{H})$ and yields that $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ is naturally isomorphic to $\mathcal{L}(\mathcal{H})$ by an isomorphism that fixes the compact operators.

The strict closure of any convex subset $Z \subseteq \mathcal{M}(B)$ coincides with the intersection of $\mathcal{M}(B) \subseteq B^{**}$ with its closure $\overline{Z}$ in the $\sigma(B^{**}, B^*)$-topology, because all strictly continuous linear functionals $f\colon \mathcal{M}(B) \to \mathbb{C}$ are of the form $f = g \circ \mathcal{M}(\eta_B)$ with $g \in B^* \subseteq B^{**}$.

This can be seen from the existence of $b_1, \ldots, b_n \in B$ for a given *strictly* continuous linear functionals $f \in \mathcal{M}(B)^*$ that satisfy $|\, f(T) \,| \leq \sum_{k=1}^n \|Tb_k\| + \|T^*b_k\|$ for all $T \in \mathcal{M}(B)$, and from the fact that the unit-ball of B is strictly dense in the unit ball of $\mathcal{M}(B)$ because each C^*-algebra has an approximate unit $\{e_\tau\}$ consisting of positive contractions:

The net $e_{\tau'}(Te_\tau) \in B$ – with possibly different τ', τ - converges strictly to T.

The strict density of the unit ball of B in the unit ball of $\mathcal{M}(B)$ implies also directly that a strictly continuos linear functional f on $\mathcal{M}(B)$ satisfies $\|f\| = \|f|B\|$. Hence, f is *determined* by its restriction to B, i.e., is the same as $g|\,\mathcal{M}(B)$ of the natural extension of $g := f|B \in B^*$ to an $\sigma(B^*, B^{**})$-continuous extension to an element of $\mathcal{M}(B) \subseteq B^{**}$.

If $\rho\colon B \to \mathcal{L}(\mathcal{H})$ is a non-degenerate *-representation of B on a Hilbert space $\mathcal{H}$, i.e., if $\mathrm{span}(\rho(B)\mathcal{H})$ is dense in $\mathcal{H}$, then ρ is also a non-degenerate C^*-morphism from B into $\mathcal{M}(\mathbb{K}(\mathcal{H}))$.

This is because the closure of the linear span $\mathrm{span}(\rho(B)\mathbb{K}(\mathcal{H}))$ is equal to $\rho(B) \cdot \mathbb{K}(\mathcal{H})$ by Cohen factorization and is a closed right ideal R of $\mathbb{K}(\mathcal{H})$.

Indeed: It is easy to see that $\mathbb{K}(\mathcal{H})\mathcal{H} = \mathcal{H}$ and that $R\mathcal{H} = P\mathcal{H}$ and $P\mathbb{K}(\mathcal{H}) = R$ with a unique projection $P \in \mathcal{L}(\mathcal{H})$ for all closed right ideals R of $\mathbb{K}(\mathcal{H})$.

Since $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ is naturally isomorphic to $\mathcal{L}(\mathcal{H})$, a *-representation $\rho\colon B \to \mathcal{L}(\mathcal{H})$ is non-degenerate if and only if ρ is non-degenerate as a C^*-morphism from B into $\mathcal{M}(\mathbb{K}(\mathcal{H}))$.

(This says equivalently that $\rho(B)\mathcal{H} = \mathcal{H}$ if and only if $\rho(B)\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$.)

If the latter is the case we get the unique unital strictly continuous extension $\mathcal{M}(\rho)\colon \mathcal{M}(B) \to \mathcal{M}(\mathbb{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$ with the property $\mathcal{M}(\rho)(b) = \rho(b)$ for $b \in B$.

`Next has also overlaps with (0.3):`

`MOVE TO to` $\mathcal{M}(\mathbb{K})$ `study !?!` CROSS REF to????

(uniqueness extensions $\mathcal{M}(h)$ of non-degenerate h are explained in ???????? where??? "uniqueness of $\mathcal{M}(h)$ for non-degenerate h"

Should be in Chp. 3 – or in Chp.2, or even in Chp.1)

The *-representation $\rho\colon B \to \mathcal{L}(\mathcal{H}) \cong \mathcal{M}(\mathbb{K}(\mathcal{H}))$ is non-degenerate as C^*-morphism from B to $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ if and only if ρ is non-degenerate as a representation of B over $\mathcal{H}$, i.e., the linear span of $\rho(B)\mathbb{K}(\mathcal{H})$ dense in $\mathbb{K}(\mathcal{H})$ if and only if the linear span of $\rho(B)\mathcal{H}$ is dense in $\mathcal{H}$.

`See also further above/below for next:`

The latter holds because $\mathbb{K}(\mathcal{H})\mathcal{H} = \mathcal{H}$. (Alternative argument: A closed right ideal R of $\mathbb{K}$ is equal to $\mathbb{K}$ if and only if the linear span of $R\mathcal{H}$ is dense in $\mathcal{H}$.)

The proof of the uniqueness is similar to the below given proof in proof-part (1a) of the uniqueness of $\mathcal{M}(h)\colon A \to \mathcal{M}(B)$ for non-degenerate $h\colon A \to \mathcal{M}(B)$.

The existence of $\mathcal{M}(\rho)$ comes by restriction to $\mathcal{M}(B) = \mathcal{M}(\eta_B)(\mathcal{M}(B)) \subseteq B^{**}$ of the unital and normal extension $\overline{\rho}\colon B^{**} \to \mathcal{L}(\mathcal{H})$ of ρ to B^{**}, of ρ – using that $\mathcal{L}(\mathcal{H}) \cong (\overline{\mathcal{H}}\widehat{\otimes}\mathcal{H})^*$, with $\widehat{\otimes}$ the maximal uniform Grothendieck tensor product, i.e., $\overline{\mathcal{H}}\widehat{\otimes}\mathcal{H}$ is the ideal of nuclear operators X with the norm $\|X\|_{nuc}$, and that $\overline{\mathcal{H}}\widehat{\otimes}\mathcal{H}$ is a complemented subspace of $\mathcal{L}(\mathcal{H})^*$ – as adjoint $\overline{\rho}$ of $\rho^*|\overline{\mathcal{H}}\widehat{\otimes}\mathcal{H}$. The proofs are straight-forward by calculations (cf. functional analysis textbooks).

Moreover, $\mathcal{M}(\rho)(\mathcal{M}(B))$ is contained in the bi-commutant $\rho(B)''$, which is the smallest von-Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $\rho(B)$.

It is the *-strong closure of $\rho(B)$ if ρ is non-degenerate, by using the bi-commutation theorem of J. von Neumann.

Indeed, if $U \in \mathcal{L}(\mathcal{H})$ is unitary and $U\rho(b) = \rho(b)U$ for all $b \in B$, then $(U\mathcal{M}(\rho)(T) - \mathcal{M}(\rho)(T)U)\rho(b) = 0$ for all $b \in B$. Since $\rho(B)\mathcal{H}$ is dense in $\mathcal{H}$ it follows $U\mathcal{M}(\rho)(T) = \mathcal{M}(\rho)(T)U$, i.e., $\mathcal{M}(\rho)(T) \in \rho(B)''$.

If, in addition, ρ is faithful on B then $\mathcal{M}(\rho)$ is faithful on $\mathcal{M}(B)$, because $\mathcal{M}(\rho)(T) = 0$ implies $\rho(TB) = \{0\}$, i.e., $TB = \{0\}$ and $T = 0$.

If the non-degenerate ρ is moreover faithful, then the image $\mathcal{M}(\rho)(\mathcal{M}(B))$ of $\mathcal{M}(\rho)$ is identical with the set of operators $Y \in \mathcal{L}(\mathcal{H})$ with $Y\rho(b), Y^*\rho(b) \in \rho(B)$ for all $b \in B$. Indeed: Such Y defines bounded linear maps $T, S \in \mathcal{L}(B)$ by $Tb := \rho^{-1}(Y\rho(b))$ and $Sb := \rho^{-1}(Y^*\rho(b))$ that satisfy $T(bc) = (Tb)c$ and $(Sb)^*c = b^*(Tc)$ for $b, c \in B$. Thus, $T \in \mathcal{M}(B)$ and $\mathcal{M}(\rho)(T)\rho(b) = \rho(Tb) = Y\rho(b)$ for $b \in B$. Since $\rho(B)\mathcal{H}$ has dense span in $\mathcal{H}$ it follows $\mathcal{M}(\rho)(T) = Y$.

In particular, $\mathcal{M}(B)$ can be identified with the C^*-subalgebra of the bi-dual W*-algebra $B^{**} \supseteq B$ given by the elements $T \in B^{**}$ with $Tb, bT \in B$ for all $b \in B$. This is because B^{**} is isomorphic to a von-Neumann algebra on a suitable Hilbert space $\mathcal{H}$ and the corresponding representation of B is *-strongly dense in B^{**}.

We notice that any non-degenerate *-representation $\rho\colon B \to \mathcal{L}(\mathcal{H}) \cong \mathcal{M}(\mathbb{K}(\mathcal{H}))$ is also a non-degenerate C^*-morphism into $\mathcal{M}(\mathbb{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$, despite of different definitions.

The norm-closed linear span $R \subseteq \mathbb{K}(\mathcal{H})$ of $\rho(B) \cdot \mathbb{K}(\mathcal{H})$ is a non-degenerate left B-module via ρ. Application of Cohen factorization shows that $\rho(B) \cdot \mathbb{K}(\mathcal{H}) = R$ is a closed right-ideal of $\mathbb{K}(\mathcal{H})$. The closed right ideals R of $\mathbb{K}(\mathcal{H})$ are all of the form $R = P \cdot \mathbb{K}(\mathcal{H})$ for some orthogonal projection $P \in \mathcal{L}(\mathcal{H})$ given by the closed subspace $R \cdot \mathcal{H}$ of $\mathcal{H}$.

Since $\rho(B)\mathcal{H} = \mathcal{H}$ by non-degeneracy of ρ and – obviously – $\mathbb{K}(\mathcal{H}) \cdot \mathcal{H} = \mathcal{H}$ we get $P = 1$. This shows that $\rho\colon B \to \mathcal{M}(\mathbb{K}(\mathcal{H}))$ is non-degenerate, i.e., $\rho(B) \cdot \mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$, if we apply Cohen factorization for the non-degenerate left B-module $\mathbb{K}(\mathcal{H})$.

We obtain that $\mathcal{M}(\rho)\colon \mathcal{M}(B) \to \mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$ is strictly continuous.

`NEW:`

(0.3) **The strict topology on** $\mathcal{M}(B)$:

Recall that the strict topology on the two-sided multiplier algebra $\mathcal{M}(B) \subseteq \mathcal{L}(B) \oplus \mathcal{L}(B)$ is given on the pairs $(L, R) \in \mathcal{M}(B)$ by semi-norms $(L, R) \mapsto \|L(b)\| + \|R(b)\|$ with $b \in B$. Clearly the family defines the same locally convex topology as the family of semi-norms given by $(L, R) \mapsto \| L(b) \| + \| R(c) \|$ with $b, c \in B$.

In case of C^*-algebras B (or Banach *-algebras with bounded approximate unit) this strict topology is given by the semi-norms $T \in \mathcal{M}(B) \mapsto \|Tb\| + \|T^*b\|$ for $b \in B_+$.

The norm topology on B is the same as the strict topology on $B \subseteq \mathcal{M}(B)$, if and only if, B is unital, because B is strictly dense in $\mathcal{M}(B)$, an approximate unit $\{ e_\tau \}$ in the positive contractions converges strictly to 1 in $\mathcal{M}(B)$.

If B is non-unital and σ-unital, then B contains a sequence $b_1, b_2, \ldots B_+$ with $\|b_n\| = 1$ such that $b_n \to 0$ strictly in $\mathcal{M}(B)$.

`Older stuff below in blue:`

The finite rank operators $a \in \mathbb{K}(\mathcal{H})$ are norm-dense in $\mathbb{K}(\mathcal{H})$. Thus, is suffices to find, for each finite rank projection $p \in \mathbb{K}(\mathcal{H})$ and $\varepsilon > 0$, a positive contraction $e = e(p, \varepsilon) \in B$ with $\| \rho(e)x_k - x_k \| < \varepsilon/\sqrt{n}$ for $k = 1, \ldots n$ where $n :=$ rank of p and $x_1, \ldots, x_n \in \mathcal{H}$ is an orthonormal basis of $p\mathcal{H}$, because then $\|p - \rho(e)p\| < \varepsilon$. Since the positive elements $e \in B_+$ with $\|e\| < 1$ build an approximate unit for B and $\mathrm{span}(\rho(B)\mathcal{H})$ is dense in $\mathcal{H}$ we can find such $e \in B_+$. In fact, the Cohen factorization theorem for non-degenerate left B-modules gives an $e \in B_+$ with $\rho(e)x_k = x_k$ and $\|e\| = 1$.

`TO DO still here`

(0.4) **The strict topology on** $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$:

`HERE WE NEED FIRST that` $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$.

Has been shown with several methods above(?).

Recall that we have seen above that $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ is naturally isomorphic to $\mathcal{L}(\mathcal{H})$ by the isomorphism from $\mathcal{L}(\mathcal{H})$ to $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ that fixes $\mathbb{K}(\mathcal{H})$.

Let $\mathcal{H}$ a Hilbert space (of arbitrary dimension) and $\mathbb{K} := \mathbb{K}(\mathcal{H})$ the essential closed ideal of $\mathcal{L}(\mathcal{H})$ of compact operators on $\mathcal{H}$.

There is natural isomorphism from $\mathcal{M}_\ell(\mathbb{K})$ onto $\mathcal{L}(\mathcal{H})$ that fixes the elements of $\mathbb{K}$, because $\mathbb{K}$ is an ideal of $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ is in a natural way the second conjugate of $\mathbb{K}$.

In particular, $\mathcal{M}_\ell(\mathbb{K}) = \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\mathcal{H})$ and the natural isomorphism from $\mathcal{M}(\mathbb{K})$ onto $\mathcal{L}(\mathcal{H})$ is continuous with respect to the strict topology on $\mathcal{M}(\mathbb{K})$ and *-ultra-strong topology on $\mathcal{L}(\mathcal{H})$.

In the same way $\mathcal{M}_\ell(\mathbb{K}) \to \mathcal{L}(\mathcal{H})$ is continuous with respect to the strong operator topology on $\mathcal{M}_\ell(\mathbb{K}) \subset \mathcal{L}(\mathbb{K})$ induced from $\mathbb{K}$ and the strong operator topology on $\mathcal{L}(\mathcal{H})$ induced from $\mathcal{H}$.

It turns out that the strict topology on $\mathcal{M}(\mathbb{K})$ and the strong* topology on $\mathcal{L}(\mathcal{H})$ coincide on bounded parts. (Similar: Both strong topologies on bounded parts of $\mathcal{M}_\ell(\mathbb{K}) = \mathcal{L}(\mathcal{H})$ coincide.)

It follows that for any (non-zero) non-degenerate representation $\psi \colon \mathbb{K} \to \mathcal{M}(B)$ (or any *weakly* non-degenerate $\psi \colon \mathbb{K} \to M$ into a W*-algebra M) induce via $\mathcal{M}(\psi)^{-1}$ on bounded parts of $\mathcal{M}(\mathbb{K})$ exactly the strict topology.

In particular, on bounded parts of $\mathcal{M}(\mathbb{K})$ the strict, *-strong and *-ultrastrong topology coincide.

Alternatively:

Since $\mathbb{K}$ is essential in $\mathcal{L}(\mathcal{H})$ there is a natural unital *-monomorphism (say η) from $\mathcal{L}(\mathcal{H})$ into $\mathcal{M}(\mathbb{K})$.

Conversely the natural *-representation $T \in \mathbb{K} \mapsto T \in \mathcal{L}(\mathcal{H})$ of $\mathbb{K}$ on $\mathcal{H}$ extends to a faithful unital C^*-morphism (say λ) from $\mathcal{M}(\mathbb{K})$ into the W*-algebra $\mathbb{K}^{**} \cong \mathcal{L}(\mathcal{H})$.

The unital C^*-morphisms $\lambda \circ \eta$ form $\mathcal{M}(\mathbb{K})$ to $\mathcal{M}(\mathbb{K})$ fixes the elements of $\mathbb{K} \subseteq \mathcal{M}(\mathbb{K})$. The uniqueness of an extension of the non-degenerate C^*-morphism $\mathrm{id}_{\mathbb{K}}$ to a C^*-morphism from $\mathcal{M}(\mathbb{K})$ into $\mathcal{M}(\mathbb{K})$ follows from above discussed Cohen factorization factorization theorem and implies that $\lambda \circ \eta = \mathrm{id}_{\mathcal{M}(\mathbb{K})}$. Since η and λ are faithful it follows that η must be surjective, i.e., η is an isomorphism from the *abstract* C^*-algebra $\mathcal{M}(\mathbb{K}) \subseteq \mathcal{L}(\mathbb{K})$ onto $\mathcal{L}(\mathcal{H})$ that fixes $\mathbb{K}(\mathcal{H})$.

We compare the strict topology on $\mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$ with the *-ultra-strong topology on $\mathcal{L}(\mathcal{H})$ and consider the topologies induced on $\mathcal{L}(\ell_2(\mathbb{N}))$ by $\mathcal{M}(\rho)^{-1}$ from *any* non-degenerate *-representation $\rho\colon \mathbb{K}(\ell_2(\mathbb{N})) \to \mathcal{L}(\mathcal{H})$:

We get

(α)

Bounded norm-closed convex parts of $\mathcal{M}(\rho)(\mathbb{K}(\ell_2(\mathbb{N})))$ are strictly closed ??? in $\mathcal{M}(\mathbb{K}(\mathcal{H}))$???,

and that the by $\mathcal{M}(\rho)^{-1}$ induced topology on $\mathcal{M}(\mathbb{K}(\ell_2(\mathbb{N})))$ is the same as the strict topology with respect to $\mathbb{K}(\ell_2(\mathbb{N}))$.

(β) Strict topology in the multiplier algebra $\mathcal{M}(\mathbb{K})$

(respectively $\mathcal{M}(B) \subseteq B^{**}$ is finer or equal to *-ultra-strong topology, and they coincide on bounded parts.

(But it is not unlikely that they are globally different, because its restrictions to $\ell_\infty(\mathbb{N}) \subseteq \mathcal{L}(\ell_2(\mathbb{N})) \cong \mathcal{M}(\mathbb{K}(\ell_2(\mathbb{N})))$ coincides with the strong operator topologies given by the multiplier actions of $\ell_\infty(\mathbb{N})$ on $c_0(\mathbb{N})$ respectively on $\ell_2(\mathbb{N})$.)

Strict topology in the multiplier algebra $\mathcal{M}(\mathbb{K})$

(respectively $\mathcal{M}(B) \subseteq B^{**}$ is finer or equal to *-ultra-strong topology, and they coincide on bounded parts.

`Older stuff below in blue:`

The finite rank operators $a \in \mathbb{K}(\mathcal{H})$ are norm dense in $\mathbb{K}(\mathcal{H})$. Thus, is suffices to find, for each finite rank projection $p \in \mathbb{K}(\mathcal{H})$ and $\varepsilon > 0$, a positive contraction $e = e(p, \varepsilon) \in B$ with $\| \rho(e)x_k - x_k \| < \varepsilon/\sqrt{n}$ for $k = 1, \ldots n$ where $n :=$ rank of p and $x_1, \ldots, x_n \in \mathcal{H}$ is an orthonormal basis of $p\mathcal{H}$, because then $\|p - \rho(e)p\| < \varepsilon$.

Since the positive elements $e \in B_+$ with $\|e\| < 1$ build an approximate unit for B and $\mathrm{span}(\rho(B)\mathcal{H})$ is dense in $\mathcal{H}$ we can find such $e \in B_+$. In fact, the Cohen factorization theorem for non-degenerate left B-modules gives an $e \in B_+$ with $\rho(e)x_k = x_k$ and $\|e\| = 1$.

The same argument shows that the *-ultra-strong topology on $\mathcal{M}(\mathbb{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$ is coarser than the strict topology on $\mathcal{M}(\mathbb{K}(\mathcal{H}))$, because the *-ultra-strong topology is the *-strong topology for the left multiplication $\mathcal{L}(\mathcal{H})$ on the Hilbert-Schmidt class operators ($HS(\mathcal{H}) \cong \overline{\mathcal{H}} \otimes_2 \mathcal{H}$):

There the same argument as for given above for $B := \mathbb{K}(\mathcal{H})$ works for the left multiplication on (new) $B := HS(\mathcal{H})$, i.e., holds for the action of $\mathcal{L}(\mathcal{H}) = \mathcal{M}(\mathbb{K})$ on the Hilbert space tensor product $\overline{\mathcal{H}} \otimes_2 \mathcal{H} \cong HS(\mathcal{H})$.

`Alternatively (done above):`

The Cohen factorization theorem applies to left action of $\mathbb{K}(\mathcal{H})$ on the Hilbert-Schmidt class operators and gives that the strict topology on $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ is finer than the *-ultra-strong topology on all subsets of $\mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$.

It follows that they coincide on bounded parts of $\mathcal{M}(\mathbb{K})$.

But it says nothing about the behavior on unbounded parts.

(1.) **Existence and uniqueness of $\mathcal{M}(h)$:**

(1.0) **Uniqueness and strict continuity of $\mathcal{M}(h)$:**

Let A and B C^*-algebras, $h\colon A \to \mathcal{M}(B)$ a C^*-morphism and $\psi\colon \mathcal{M}(A) \to \mathcal{L}(B)$ any algebra homomorphism with the property $\psi|A = h$. Then ψ and h have the property $\psi(X)(h(a)b) = \psi(X)(\psi(a)b) = \psi(Xa)b = h(Xa)b$ for $X \in \mathcal{M}(A)$, $a \in A$ and $b \in B$.

Thus, ψ is uniquely determined by its restriction $h := \psi|A$ on the closed linear span $L \subseteq B$ of $h(A)B$ and $\psi(1)b = b$ for $b \in L$. In particular ψ is unital and is *uniquely determined* if the linear span of $h(A)B$ is dense in B.

If $\mathrm{span}(h(A)B)$ is dense in B, then ψ is also a C^*-morphism from $\mathcal{M}(A)$ into $\mathcal{M}(B) \subseteq \mathcal{L}(B) := \mathcal{L}(B, B)$, because $\psi(X)(h(a)b) = h(Xa)b$ implies

$$(\psi(X)(h(a_1)b_1))^*(h(a_2)b_2) = b_1^* h(a_1^* X^* a_2)b_2 = (h(a_1)b_1)^*(\psi(X^*)(h(a_2)b_2)),$$

i.e., $(\psi(X)b_1)^* b_2 = b_1^*(\psi(X^*)b_2)$ if we pass to the closure of $\mathrm{span}(h(A)B)$. We get that $\psi(X)$ is in $\mathcal{M}(B) \subseteq \mathcal{L}(B)$ and $\psi(X^*) = \psi(X)^*$ if the linear span of $h(A)B$ is dense in B.

Such extensions ψ of h are automatically strictly continuous if $h(A)B$ is dense in $B\,zz$:

Then the B is also a non-degenerate left Banach module over A with left-action $a \cdot b := h(a)b$ for $a \in A$ and $b \in B$, and the Cohen factorization theorem applies to it:

For each $b_1, b_2 \in B$ there exist contractions $a_1, a_2 \in A$ and $c_1, c_2 \in B$ with $\|c_j - b_j\| < 1/2$ such that $b_j = h(a_j)c_j$. It follows that the semi-norm

$$X \mapsto \|\psi(X)b_1\| + \|\psi(X^*)b_2\|$$

on $\mathcal{M}(A)$ is less or equal to the semi-norm by

$$X \mapsto (1 + \|b_1\| + \|b_2\|)(\|Xa_1\| + \|X^*a_2\|).$$

This latter semi-norms are continuous with respect to the strict topology on $\mathcal{M}(A)$. It implies that ψ is strictly continuous.

If conversely a *unital and strictly continuous* $\psi\colon \mathcal{M}(A) \to \mathcal{M}(B)$ with $\psi(a) = h(a)$ exist, then an approximate unit $(e_\tau) \subset A_+$ of A converges strictly to $1 \in \mathcal{M}(A)$, and $h(e_\tau)b = \psi(e_\tau)b$ converges to $b = \psi(1)b$ for each $b \in B$. Thus $h(A)B$ is dense in B i.e., h is non-degenerate.

(1.1) **Existence of** $\mathcal{M}(h)$:

Suppose that $h\colon A \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism, i.e., the linear span of $h(A)B$ is dense in B.

The Cohen factorization theorem applies to the Banach left A-module B with product $a \cdot b := h(a)b$.

Then B is also in a unique way a Banach module over $\mathcal{M}_\ell(A)$ with a product $L \cdot' b \in B$ defined by $L \cdot' b := L(a) \cdot c = h(L(a))c$ for $b = h(a)c$ and $L \in \mathcal{M}_\ell(A)$. It satisfies $L(a) \cdot b = L \cdot' (h(a)b)$, $\|L \cdot' b\| \leq \|L\| \cdot \|b\|$. This defines a contractive Banach algebra homomorphism from $\mathcal{M}_\ell(A)$ into $\mathcal{L}(B)$.

But in this special case holds the additional property $L \cdot' (bc) = (L \cdot' b)c$ for $a \in A$, $b, c \in B$ and $L \in \mathcal{M}_\ell(A)$. Indeed, let $b = h(a)d$, then $bc = h(a)dc$ and $L \cdot' bc = L(a) \cdot dc = h(L(a))dc = (h(L(a))d)c == (L \cdot' b)c$.

Hence $\mathcal{M}_\ell(L)(b) := L \cdot' b$ defines a contractive Banach algebra homomorphism $\mathcal{M}_\ell(h)$ from $\mathcal{M}_\ell(A)$ into $\mathcal{M}_\ell(B) \subseteq \mathcal{L}(B)$ with $\mathcal{M}_\ell(L_a)(b) = h(a)b$ for $a \in A$ and $b \in B$.

Since – in our picture – $\mathcal{M}(A) \subseteq \mathcal{M}_\ell(A) \subseteq \mathcal{L}(A)$ and $\mathcal{M}(B) \subseteq \mathcal{M}_\ell(B)$, we can define $\mathcal{M}(h)\colon \mathcal{M}(A) \to \mathcal{M}_\ell(B)$ as the restriction of $\mathcal{M}_\ell(h)$ to $\mathcal{M}(A)$, i.e., by $\mathcal{M}(h)(X)b := X \cdot' b$ for $X \in \mathcal{M}(A)$ and $b \in B$. This becomes automatically a C^*-morphism from $\mathcal{M}(A)$ into $\mathcal{M}(B)$ as we have seen in proof-part (1.0).

`Older version for the existence proof:`

The following is a special case of the above considered *weakly non-degenerate* C^*-morphism from B into a W*-algebra.

Let $\rho\colon B \to \mathcal{L}(\mathcal{H})$ a faithful and non-degenerate *-representation on a Hilbert space $\mathcal{H}$. The representation ρ extends to a faithful unital *-representation $\mathcal{M}(\rho)\colon \mathcal{M}(B) \to \mathcal{L}(\mathcal{H})$ with $\mathcal{M}(\rho)(b) = \rho(b)$ for $b \in B$.

Let $d\colon A \to \mathcal{L}(\mathcal{H})$ the *-representation defined by $d(a) = \mathcal{M}(\rho)(h(a))$. In particular, $d(a)\rho(b) = \rho(h(a)b)$.

Since ρ and h are non-degenerate, the linear spans of $h(A)B$ and $\rho(B)\mathcal{H}$ are dense in B respectively in $\mathcal{H}$. Thus, the linear span of $d(A)\rho(B)\mathcal{H} = \rho(h(A)B)\mathcal{H}$ is dense in $\mathcal{H}$, and $d\colon A \to \mathcal{L}(H)$ is non-degenerate. We know from Proof-part (1.1) that a non-degenerate *-representation $d\colon A \to \mathcal{L}(\mathcal{H})$ extend uniquely to a non-degenerate *-representation $\mathcal{M}(d)\colon \mathcal{M}(A) \to \mathcal{L}(\mathcal{H})$ with $\mathcal{M}(d)(X)d(a) = d(Xa)$.

We get

$$\mathcal{M}(d)(X)\rho(h(a)b) = \mathcal{M}(d)(X)d(a)\rho(b) = d(Xa)\rho(b) = \rho(h(Xa)b)$$

for all $X \in \mathcal{M}(A)$, $a \in A$ and $b \in B$. Since the linear span of $h(A)B$ is dense in B, it follows that $\mathcal{M}(d)(X)\rho(b) \in \rho(B)$ and $\mathcal{M}(d)(X)^*\rho(b) = \mathcal{M}(d)(X^*)\rho(b) \in \rho(B)$ for all $b \in B$. Thus, $\mathcal{M}(d)(\mathcal{M}(A)) \subseteq \mathcal{M}(\rho)(\mathcal{M}(B))$ by the characterization of the image $\mathcal{M}(\rho)(\mathcal{M}(B))$ of $\mathcal{M}(B)$ in $\mathcal{L}(\mathcal{H})$.

We let $\mathcal{M}(h) := \mathcal{M}(\rho)^{-1}(\mathcal{M}(d)(X))$. This is a unital C^*-morphism from $\mathcal{M}(A)$ into $\mathcal{M}(B)$ that satisfies $\mathcal{M}(h)(X)h(a)b = h(Xa)b$ and is the desired extension to $\mathcal{M}(A)$, because

$$\mathcal{M}(\rho)(\mathcal{M}(h)(X)h(a)b) = \mathcal{M}(d)(X)\rho(h(a)b) = d(Xa)\rho(b) = \mathcal{M}(\rho)(h(Xa)b)\,.$$

(2) **Proofs for Part(2)**: Let $p_{j,k}$ denote the matrix units of $\mathbb{K}$. Then the series $c := \sum_n x_n \otimes p_{1,n}$ and $d := \sum_n y_n \otimes p_{1,n}$ are unconditional strictly convergent series in $\mathcal{M}(B \otimes \mathbb{K})$ ([5]).

Indeed, partial sums, i.e., $c(S) := \sum_{n\in S} x_n \otimes p_{1,n}$ over a finite set $S \subset \mathbb{N}$ of indices, are norm-bounded by $\|X\|^{1/2}$ respectively by $\|Y\|^{1/2}$.

More precisely: Let $p_m := p_{1,1} + \cdots + p_{m,m}$, where $p_{j,k} \in \mathbb{K}$ are partial isometries that define canonical matrix units $\mathbb{K}$ for the standard basis of ℓ_2. Then

$$gXh^* \otimes p_{1,1} - (g \otimes 1)c(S)((h \otimes 1)c(S))^* = g(X - \sum_{n\in S} x_n x_n^*)h^* \otimes p_{1,1}\,,$$

for all $g, h \in \mathcal{M}(B)$, $\sum_{n\in S} x_n x_n^* \leq X$, $c(S)(e \otimes p_m) = \sum_{n\in S, n\leq m}(x_n e) \otimes p_{1,n}$ and $(e \otimes p_m)c(S) = \sum_{n\in S, n\leq m}(ex_n) \otimes p_{1,n}$ for each $e \in \mathcal{M}(B)$.

Since the partial sums $c(S)$ are bounded in $\mathcal{M}(B \otimes \mathbb{K})$ by $\|X\|^{1/2}$, and since the semi-norms

$$\rho_m(s,t) := \max(\|(s-t)(e \otimes p_m)\|, \|(e \otimes p_m)(s-t)\|)$$

for $m \in \mathbb{N}$ and positive contractions $e \in B_+$ define the strict topology on bounded parts of $\mathcal{M}(B \otimes \mathbb{K})$, it follows that $\sum_n x_n \otimes p_{1,n}$ converges strictly in $\mathcal{M}(B \otimes \mathbb{K})$.

That the convergence is also unconditional can be seen as follows:

For each contraction $e \in B_+$, $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n := n(e, m, \varepsilon) \in \mathbb{N}$ such that $n \geq m$, $c(S)(e \otimes p_m) = c(S')(e \otimes p_m)$ and $\|(e \otimes p_m)(c(S) - c(S'))\| < \varepsilon$ for all finite subsets $S, S' \subset \mathbb{N}$ with $\{1, \ldots, n\} \subseteq S' \cap S$. The latter property implies the unconditional strict convergence of $c := \sum_n x_n \otimes p_{1,n}$ in $\mathcal{M}(B \otimes \mathbb{K})$, i.e., that for each permutation γ of the positive integers $\mathbb{N}$ the series $\sum_n x_{\gamma(n)} \otimes p_{1,\gamma(n)}$ converges strictly to the same element of $\mathcal{M}(B \otimes \mathbb{K})$ as $c := \sum_n x_n \otimes p_{1,n}$ does.

Thus, the series c and d are unconditional strictly convergent and its sums are *elements* of $\mathcal{M}(B \otimes \mathbb{K})$. We use again c and d as notation for the sums of the series,

[5] The convergence is usually not absolute!

because in this particular case the entries of the series can be restored from the sum, e.g.

$$x_n \otimes p_{1,n} = c(1 \otimes p_{n,n}) .$$

The c and d are row matrices if one considers $\mathcal{M}(B \otimes \mathbb{K})$ naturally as a subset of the set of all infinite matrices $[T_{i,j}] \in M_\infty(\mathcal{M}(B))$ with entries $T_{i,j}$ from $\mathcal{M}(B)$:

If we use a fixed non-degenerate inclusion of $c_0(B) = B \otimes c_0$ into $B \otimes \mathbb{K}$ then we get "natural" unital C^*-monomorphisms

$$\mathcal{M}(B) \otimes 1 \subset \ell_\infty(\mathcal{M}(B)) \cong \mathcal{M}(c_0(B)) \subset \mathcal{M}(B \otimes \mathbb{K}) \subset M_\infty(\mathcal{M}(B)) ,$$

where the first two strict inclusion are strictly continuous. This inclusions are naturally defined: If we choose a system of matrix units $p_{j,k}$ for $\mathbb{K}$, we get natural inclusions $c_0 \subset \mathbb{K}$, $c_0(B) = B \otimes c_0 \subset B \otimes \mathbb{K}$ and can use the isomorphism $\ell_\infty(\mathcal{M}(B)) \cong \mathcal{M}(c_0(B))$. The map $b \in \mathcal{M}(B \otimes \mathbb{K}) \mapsto dbc^*$ is obviously a strictly continuous completely bounded map with cb-norm equal to $\|c\| \cdot \|d\| \leq (\|S\|\|T\|)^{1/2}$. The in Remark (2) defined map Γ satisfies $\Gamma(b) \otimes p_{1,1} = dbc^*$ for $b \in \ell_\infty(\mathcal{M}(B)) \subset \mathcal{M}(B \otimes \mathbb{K})$.

The proposed unconditional strict convergence of the series $\Gamma(b_1, b_2, \ldots)$ follows now from the unconditional strict convergence of the sum $\sum_n (\delta_{n,k} b_k)$ to $(b_1, b_2, \ldots)$ in $\mathcal{M}(c_0(B)) \cong \ell_\infty(\mathcal{M}(B))$.

Since $d^* bc^* \in B \otimes p_{1,1}$ for $d \in B \otimes \mathbb{K}$, we get that $\Gamma(b_1, b_2, \ldots) \in B$ for $(b_1, b_2, \ldots) \in c_0(B)$.

(3): Let $\varphi_0(t) := t$ for $t \in [0, 1]$ and

$$\varphi_k(t) := \min(1, \max(2^k t - 1, 0)) = (2^k t - 1)_+ - (2^k t - 2)_+$$

for $k = 1, 2, \ldots$. Then there exists $1 =: k_0 < k_1 < k_2 < \cdots$ and $f_n(t)$ in the convex span of $\varphi_{k_n}, \ldots, \varphi_{k_{n+1}-1}$ such that the functions f_n have the properties quoted in Part (3). Here we use the method of Akemann and Pedersen as described in proof of [**43**, thm. 2] or in [**616**, thm. 3.12.14, cor. 3.12.15]. See more details and discussion in Section 3. The proposed estimates for norms of $[(f_{n+1}(e) - f_n(e))^{1/2}, x]$ follow also from the estimates in Lemma 5.3.2.

(4): Let $T(a) := a - \Gamma(a, a, \ldots)$ the linear map from A into $\mathcal{M}(B)$ with norm ≤ 2, where Γ is defined as in Part (2) with $x_n := y_n := g_n := (f_n(e) - f_{n-1}(e))^{1/2}$ for $n > 1$ and $x_1 := y_1 := g_1 := f_1^{1/2}(e)$ in Remark (2) with f_n selected for $X_n \subset A \subseteq \mathcal{M}(B)$ as in Remark (3).

$x_n := y_n := g_n := (f_n(e) - f_{n-1}(e))^{1/2}$.

Check Notations and estimates !?? **??**

If $a^* = a \in X_n$ with $\|a\| \leq 1$ then $T(a) = \sum_n (ag_n - g_n a) g_n$ and $\|ag_n - g_n a\| < 4^{-n}$ by part (3). Since $ag_n \in B$, it follows $T(a) \in B$ for a in a dense linear subspace of A, and that $T(A) \subseteq B$ by continuity of T.

(5): Let $g_1, g_2, \ldots$ as in (4) and $d_1, d_2, \ldots \in \mathcal{M}(B)$ as in (5), then $g_n h_n = g_n = h_n g_n$ for $h_1 := f_2(e)^{1/2}$ and $h_n := (f_{n+1}(e) - f_{n-2}(e))^{1/2}$ for $n > 1$ where $f_0 := 0$,

$$\sum d_n g_n = \sum h_n (d_n g_n) h_n + \sum (1 - h_n) d_n g_n \,,$$

$\|(1 - h_n) d_n g_n\| < \|d_n g_n - g_n d_n\|$, and $\sum h_n^2$ converges strictly to 2 in $\mathcal{M}(B)$.

Since $g_p g_q = 0$ if $|p - q| > 1$, it follows $S_P^2 = \sum_{n \in P} g_n^2$ if $|p - q| > 1$ for all $p \neq q \in P$.

It holds $Sa - aS \in B$ for the contractions $a^* = a \in X_n$, because $\|g_n a - a g_n\| < 4^{-n}$ by part (3). Thus, the continuous derivation $a \mapsto Sa - aS$ from A into $\mathcal{M}(B)$ maps A into B, i.e., $\pi_B(S) \in \pi_B(A)' \cap (\mathcal{M}(B)/B)$.

It follow that $\pi_B(SaS + (1 - S^2)^{1/2} a (1 - S^2)^{1/2}) = \pi_B(a)$ for all $a \in A$.

(6): Since $\sum \|d_n g_n - g_n d_n\| \leq \sum_n \nu_n < \infty$, the series $\sum_n d_n g_n$ converges (unconditional) strictly in $\mathcal{M}(B)$ by part (5). Let $d := \sum_n d_n g_n$ denote its sum.

Denote by $Z_{k,j}$ the sum of the series $\sum_n g_{2n-j}(d_{2n-j}^* a d_{2n-j+k}) g_{2n-j+k}$ for $j = 0, 1$ and $k \in \mathbb{N}$, that is strictly convergent in $\mathcal{M}(B)$ by part (2), because $\|d_{2n-j}^* a d_{2n-j+k}\| \leq \|a\|$ and the sums $\sum_n g_{2n-j+k}^2$ and $\sum_n g_{2n-j}^2$ are strictly convergent in $\mathcal{M}(B)$ by part (5).

Let $h_n := (g_{n-1}^2 + g_n^2 + g_{n+1}^2 + g_{n+2}^2)^{1/2}$ ($g_0 := 0$). The series $\sum_n h_n^2$ converges strictly to some positive element in $\mathbb{R}_+ - B_+$. By condition (iii) and part (3),

$$Z_{1,0} + Z_{1,1} = \sum_n h_n (g_n d_n^* a d_{n+1} g_{n+1}) h_n \in B \,,$$

because $g_n \in B$ and the sequence $(\, g_n d_n^* a d_{n+1} g_{n+1} \,)_n$ is in $c_0(\mathcal{M}(B))$ by condition (iii). It follows that $S_1 := (Z_{1,0} + Z_{1,1})^* + (Z_{1,0} + Z_{1,1}) \in B$.

If $k > 1$ then the elements $z_n := g_{2n-j}(d_{2n-j}^* a d_{2n-j+k}) g_{2n-j+k}$ satisfy

$$\|z_n\| \leq \|a g_{2n+k-j} - g_{2n+k-j} a\| + 2 \max_{1 \leq k \leq 2n-j+k} \|d_k g_{2n+k-j} - g_{2n+k-j} d_k\|$$

for $k > 1$ and $j \in \{0, 1\}$, by conditions (i) and (ii). Moreover, $z_n^* z_m = 0 = z_m z_n^*$ for $n \neq m$. Thus $\sum_n z_n$ converges *absolute* to $Z_{k,j}$ for $k > 1$, $j = 0, 1$, and its norm can be estimated by

$$\|Z_{k,j}\| \leq \mu_{2+k-j} + 2\nu_{2+k-j} \,.$$

Since $(\sum_\ell \mu_\ell) + 2(\sum_\ell \nu_\ell) < \infty$, it follows that

$$\sum_k \|Z_{k,j}\| \leq \sum_k \mu_{2+k-j} + 2 \sum_k \nu_{2+k-j} < \infty$$

for $j = 0, 1$. It implies that the series $S_2 := \sum_{k>1} (Z_{k,0} + Z_{k,1})^* + (Z_{k,0} + Z_{k,1})$ with entries in B converges to an element of B.

It is not difficult to see from the partial sums that the series corresponding to $d^* a d - \Gamma(d_1^* a d_1, d_2^* a d_2, \ldots)$ converges *strictly* to the sum $S_1 + S_2 \in B$.)

(7): The isomorphism $D + \mathrm{Ann}(D) \cong D \oplus \mathrm{Ann}(D)$ implies

$$\mathcal{M}(D + \mathrm{Ann}(D)) \;\cong\; \mathcal{M}(D) \oplus \mathcal{M}(\mathrm{Ann}(D))$$

and that the natural *-monomorphism $\iota\colon D + \mathrm{Ann}(D) \to B$ has a unital strictly continuous extension $\mathcal{M}(\iota)\colon \mathcal{M}(D + \mathrm{Ann}(D)) \to \mathcal{M}(B)$ if ι is non-degenerate.

Define $p := \mathcal{M}(\iota)(1_{\mathcal{M}(D)})$. Then $pBp = D$. Conversely, if $D = pBp$ then $(1-p)B(1-p) = \mathrm{Ann}(D)$ and $pep + (1-p)e(1-p)$ becomes an approximate unit of B if e runs through an approximately central (for p) approximate unit of B.

(8): `final part of TEXT in (8):`

Recall that B is a **stable C^*-algebra** if there exists an isomorphism ψ from $B \otimes \mathbb{K}$ onto B for the compact operators $\mathbb{K} = \mathbb{K}(\ell_2)$ on $\ell_2(\mathbb{N})$.

This is equivalent to the existence of a C^*-algebra C and a C^*-algebra isomorphism from $C \otimes \mathbb{K}$ onto B.

The following criterium is necessary and sufficient for the stability of a (not necessarily σ-unital) C^*-algebra B:
There exists a sequence $s_1, s_2, \ldots$ of isometries in $\mathcal{M}(B)$ such that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(B)$.

It holds $u^* \delta_\infty(b) u = \sum_n t_n b t_n^*$ by a unitary $u \in \mathcal{M}(B)$ with $t_n = u s_n$ for $n = 1, 2, \ldots$ if $t_1, t_2, \ldots$ is any other (countable) sequence of isometries in $\mathcal{M}(B)$ with $\sum_n t_n t_n^* = 1$.

In this sense the unitary equivalence class $[\delta_\infty]$ has "representatives" $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ determined by $s_1, s_2, \ldots$. We call this class (and sometimes its members $\sum_n s_n(\cdot)s_n$) the **infinite repeat** (on $\mathcal{M}(B)$) of the identity mapping $\mathrm{id}_{\mathcal{M}(B)}$ of $\mathcal{M}(B)$.

Then $\delta_\infty(b)$ (or its unitary equivalence class $[\delta_\infty(b)]$) is the *infinite repeat of the element* $b \in \mathcal{M}(B)$.

Moreover, there exists C^*-morphism $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with the properties

(a) $\mu(\mathbb{K}) \subset \delta_\infty(B)' \cap \mathcal{M}(B)$,
(b) $\delta_\infty(b)\mu(k) \in B$ for $b \in B$ and $k \in \mathbb{K}$, and
(c) the (unique) C^*-morphism $\varphi\colon B \otimes \mathbb{K} \to B$ with $\varphi(b \otimes k) = \delta_\infty(b)\mu(k)$ is an *isomorphism* from $B \otimes \mathbb{K}$ *onto* B.

The C^*-morphism $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with properties (a,b,c) is uniquely defined by the given sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$ of isometries with $\sum s_n s_n^* = 1$ – up to unitary equivalence by unitaries in $\delta_\infty(B)' \cap \mathcal{M}(B)$.

One of the corresponding C^*-morphisms $\mu\colon \mathbb{K} \to \delta_\infty(B)' \cap \mathcal{M}(B)$ can be defined by $\mu(p_{j,k}) := s_j s_k^*$ for $j, k \in \mathbb{N}$, if the unitary equivalence class $[\delta_\infty]$ is realized by given isometries $s_1, s_2, \ldots$, i.e., $\delta_\infty(\cdot) := \sum_n s_n(\cdot)s_n^*$.

The isomorphism φ from $B \otimes \mathbb{K}$ onto B defined by $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ – with properties (a,b,c) – has the property that the corresponding C^*-morphism $b \mapsto \varphi(b \otimes e_{11})$ is a *-endomorphism of B, that is approximately 1-step inner if considered as a c.p. contraction.

The C^*-morphism $\varphi((\cdot)\otimes p_{1,1}) = s_1(\cdot)s_1^*$ is unitarily homotopic to id_B if B is σ-unital:

??????????????

Moreover, if $b \in B_+$, then $\varphi(b\otimes p_{1,1})$ is Murray–von-Neumann equivalent to b. In particular,

$$\varphi(J\otimes \mathbb{K}) = J \quad \forall\ J \lhd B\,.$$

This property of the "natural" φ is different from the properties of very random and possibly not well-behaved *-isomorphism ψ from $B\otimes\mathbb{K}$ onto B – as allowed in the *definition* of stability –, because in general the isomorphism $\varphi\circ\psi^{-1}$ of B does not fix the ideal-system $\mathcal{I}(B)$ of B and is not approximately unitarily equivalent to id_B by unitaries in $\mathcal{M}(B)$.

The non-degenerate *-monomorphism $\mu\colon \mathbb{K}\to\mathcal{M}(B)$ has the important "dual" property that

$$\mu(\mathbb{K})'\cap\mathcal{M}(B) \;=\; \delta_\infty(\mathcal{M}(B))\,.$$

It shows that $\delta_\infty(\mathcal{M}(B))$ is closed in $\mathcal{M}(B)$ with respect to the strict topology of $\mathcal{M}(B)$.

The strictly continuous extension $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K})\cong\mathcal{L}(\ell_2)\to\mathcal{M}(B)$ of μ is a unital *-monomorphism, has image in $\delta_\infty(\mathcal{M}(B))'\cap\mathcal{M}(B)$ and the norm-closed unit-ball of $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ is also *closed in the strict topology* of $\mathcal{M}(B)$.

The strictly closed C^*-subalgebra $\delta_\infty(\mathcal{M}(B))'\cap\mathcal{M}(B)$ of $\mathcal{M}(B)$ can be larger than $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$, e.g. if the the center of $\mathcal{M}(B)$ is not trivial, because the center of $\mathcal{M}(B)$ is identical with the commutant of $\delta_\infty(B)\cdot\mu(\mathbb{K})$ in $\mathcal{M}(B)$, and $\delta_\infty(B)\cdot\mu(\mathbb{K})$ generates B if the center of $\mathcal{M}(B)$ is not trivial.

For each non-zero C^*-morphism $\rho\colon \mathbb{K}(\ell_2)\to\mathcal{M}(B)$, the topology induced by $\mathcal{M}(\rho)^{-1}$ on $\mathcal{M}(\mathbb{K})=\mathcal{L}(\ell_2)$ from the strict topology on $\mathcal{M}(B)$ coincides on bounded parts with the *-strong operator topology on $\mathcal{L}(\ell_2)$. Moreover, $\mathcal{M}(\rho)(\mathbb{K})$ is strictly closed if ρ is non-degenerate.

HERE: BEGIN NEW PROOF of (8) 27.10.2015 19:30

If B is stable then there exists a sequence of isometries $s_1, s_2, \ldots$ in $\mathcal{M}(B)$ such that $\sum s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(B)$:

Is φ good notation?

If B is stable then there exist (by definition of stability) a C^*-algebra C and a C^*-algebra isomorphism $\varphi\colon C\otimes\mathbb{K}\to B$ from $C\otimes\mathbb{K}$ onto B, where $\mathbb{K}:=\mathbb{K}(\ell_2(\mathbb{N}))$.

If C is unital, then $\rho(k) := \varphi(1\otimes k)$ defines a non-degenerate C^*-morphism from $\mathbb{K}$ into B.

If C is not unital, then the strictly continuous extension $\mathcal{M}(\varphi)\colon \mathcal{M}(C\otimes\mathbb{K})\to\mathcal{M}(B)$ is a strictly continuous isomorphism from $\mathcal{M}(C\otimes\mathbb{K})$ onto $\mathcal{M}(B)$.

The algebra $C\otimes\mathbb{K}$ is an essential ideal of $\mathcal{M}(C)\otimes\mathbb{K}$. Thus, there is a unique *-monomorphism γ from $\mathcal{M}(C)\otimes\mathbb{K}$ into $\mathcal{M}(C\otimes\mathbb{K})$ with $\gamma(T\otimes k_1)(c\otimes k_2) =$

$(Tc) \otimes (k_1 k_2)$ for $T \in \mathcal{M}(C)$, $c \in C$ and $k_1, k_2 \in \mathbb{K}$. Then $\gamma|(1 \otimes \mathbb{K})$ is non-degenerate. It implies that

$$\rho(k) := \mathcal{M}(\varphi)(\gamma((1 \otimes k)))$$

is a non-degenerate C^*-morphism from $\mathbb{K}$ into $\mathcal{M}(B)$. Recall that here "non-degenerate" means that $\rho(\mathbb{K})B = B$ (by using e.g. Cohen factorization).

The *-monomorphism ρ extends uniquely to a faithful strictly continuous unital C^*-morphism $\mathcal{M}(\rho)$ from $\mathcal{M}(\mathbb{K})$ into $\mathcal{M}(B)$.

Any bijective map η from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ defines isometries $t_n \in \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$ with $\sum_n t_n t_n^*$ converging *-strongly to 1 if we let $t_n(e_k) := e_{\eta(n,k)}$.

Since, as discussed above, on bounded parts of $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ the *-strong, *-ultra-strong and strict topology coincide, and since μ is strictly continuous, we get that the isometries $s_n := \mathcal{M}(\gamma)(t_n)$ define a sequence of isometries in $\mathcal{M}(B)$ with the property that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(B)$.

Suppose now that B is a C^*-algebra such that its multiplier algebra $\mathcal{M}(B)$ contains a sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$. We define a rather canonical C^*-isomorphism φ from $B \otimes \mathbb{K}$ onto B:

The isometries $s_1, s_2, \ldots$ define (the unitary equivalence class of) the *infinite repeat*

$$\delta_\infty(b) := \sum_n s_n b s_n^*$$

for $b \in \mathcal{M}(B)$, because it follows from Remark (2) – with $x_n := y_n := s_n^*$ and $b_n := b$ – that $\delta_\infty(b) := \sum_n s_n b s_n^*$ is strictly unconditional convergent. The unconditional convergence allows to check that δ_∞ is a unital *-endomorphism of $\mathcal{M}(B)$.

It holds $u^* \delta_\infty(b) u = \sum_n t_n b t_n^*$ by a unitary $u \in \mathcal{M}(B)$ with $t_n = u s_n$ for $n = 1, 2, \ldots$ if $t_1, t_2, \ldots$ is any other sequence of isometries in $\mathcal{M}(B)$ with $\sum_n t_n t_n^* = 1$ (strictly convergent), because Part (2) shows that the sums $\sum_n t_n s_n^*$ and $\sum_n s_n t_n^*$ are unconditionally strictly convergent to contractions $u, v \in \mathcal{M}(B)$. Then the unconditionality of the strict convergence of all four series allows to verify that $vu = uv = 1$ by calculation.

The Remark (2) also implies that δ_∞ is strictly continuous and is determined up to unitary equivalence if it is build from different choices of sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ with $\sum_n s_n s_n^* = 1$ (strictly), compare Lemma 5.1.2(i,ii) and its proof.

There is a natural C^*-morphism $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ that is uniquely defined on the elementary matrices $p_{j,k}$ by $\mu(p_{j,k}) := s_j s_k^*$, as calculation shows.

Notice that μ can be different from the formerly defined $\rho \colon \mathbb{K} \to \mathcal{M}(B)$, e.g. if $B = C \otimes \mathbb{K}$ with unital C.

We show that above defined $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ has the in Part (8) listed properties (a), (b) and (c):

Ad(a): $\mu(\mathbb{K}) \subset \delta_\infty(B)' \cap \mathcal{M}(B)$ because for $b \in B$ and $j, k \in \mathbb{N}$

$$s_j s_k^* (\sum_n s_n b s_n^*) = s_j b s_k^* = (\sum_n s_n b s_n^*) s_j s_k^* .$$

Ad(b): $\delta_\infty(b)\mu(k) \in B$ for $b \in B$ and $k \in \mathbb{K}$, because the calculation in Part(a) implies that $(b \otimes k) \mapsto \delta_\infty(b)\mu(k)$ extends to a C^*-morphism $\varphi \colon B \otimes \mathbb{K} \to \mathcal{M}(B)$ with $\varphi(b \otimes p_{j,k}) \in B$. Since the linear span of matrix units $p_{j,k}$ is dense in $\mathbb{K}$ it follows that $\varphi(B \otimes \mathbb{K}) \subseteq B$.

Ad(c): The C^*-morphism $\varphi \colon B \otimes \mathbb{K} \to B$ is with $\varphi(b \otimes k) = \delta_\infty(b)\mu(k)$ is an *isomorphism* from $B \otimes \mathbb{K}$ *onto* B.

Indeed: By Lemma 2.2.3 the kernel of φ is $= \{0\}$, because from $\varphi(b \otimes k) = 0$ it would follow that $s_j b s_k^* = \delta_\infty(b) s_j s_k^* = 0$ for a matrix unit $p_{j,k} \in \mathbb{K}$ with $p_{j,j} k p_{k,k} = \xi p_{j,k} \neq 0$.

The sequence $s_1, s_2, \ldots$ defines a partial isometry $T \in \mathcal{M}(B \otimes \mathbb{K})$ by $T := \sum_n s_n \otimes p_{1,n}$ with $T^*T = 1$ and $TT^* = 1 \otimes p_{1,1}$ by $s_n s_m^* \otimes p_{1,n} p_{m,1} = \delta_{m,n} 1 \otimes p_{1,1}$.

$T(b \otimes p_{j,k})T^* = T \sum_n b s_n^* \otimes p_{j,k} p_{n,1} = T((b s_k^*) \otimes p_{j,1})$ =?????

$\sum_n s_n b s_k^* \otimes p_{1,n} p_{j,1} = s_j b s_k^* \otimes p_{11} = (\delta_\infty(b) \cdot s_j s_k^*) \otimes p_{1,1}$. This means $T(b \otimes p_{j,k})T^* = \varphi(b \otimes p_{j,k}) \otimes p_{1,1}$. Thus, $TcT^* = \varphi(c) \otimes p_{1,1}$ for all $c \in B \otimes \mathbb{K}$. Moreover, $T^*T = 1$ and $T(B \otimes \mathbb{K})T^* = B \otimes p_{1,1}$.

`Until HERE new 2015 Oct 30`

??? by ??? the is a ???? natural C^*-morphism $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ with the in Part (8) quoted properties (a,b,c) uniquely defined by the given sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$ of isometries with $\sum s_n s_n^* = 1$ – up to unitary equivalence by unitaries in $\delta_\infty(B)' \cap \mathcal{M}(B)$.

`The below given arguments do not systematically prove the statements in (8). Write new proofs!`

??

(only for parts of 8?);

Suppose $s_1, s_2, \ldots \in \mathcal{M}(B)$ are isometries with $\sum s_n s_n^* = 1$ (strictly convergent) are given.

FROM “STABLE” TO $s_1, s_2, \ldots$

If, conversely, isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ are given with $\sum_n s_n s_n^*$ is (unconditional) strictly convergent to $1_{\mathcal{M}(B)}$ then we can define a strictly continuous unital positive map $\delta_\infty \colon \mathcal{M}(B) \to \mathcal{M}(B)$. The strict continuity of δ_∞ and the (easy to check) multiplicative property of $\delta_\infty | B$ imply that δ_∞ is a strictly continuous $*$-endomorphism of $\mathcal{M}(B)$.

??????

It turns out that there exists C^*-morphism $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ with the properties $\mu(\mathbb{K}) \subset \delta_\infty(B)' \cap \mathcal{M}(B)$ such that $\delta_\infty(b)\mu(k) \in B$ for $b \in B$ and $k \in \mathbb{K}$ *and* such that the (unique) C^*-morphism $\varphi\colon B \otimes \mathbb{K} \to B$ with $\varphi(b \otimes k) = \delta_\infty(b)\mu(k)$ is a *-isomorphism from $B \otimes \mathbb{K}$ onto B.

`Indicate proof of non/uniqueness:`

The C^*-morphism μ with the described properties is determined up to unitary equivalence by unitaries in $\delta_\infty(B)' \cap \mathcal{M}(B)$.

The isomorphism φ from $B \otimes \mathbb{K}$ onto B defined by $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ with properties (a,b,c) has the property that the corresponding C^*-morphism $b \mapsto \varphi(b \otimes e_{11})$ is a *-endomorphism of B, that is approximately 1-step inner if considered as a c.p. contraction. Moreover, if $b \in B_+$, then $\varphi(b \otimes e_{11})$ is Murray–von-Neumann equivalent to b. In particular,

$$\varphi(J \otimes \mathbb{K}) = J \quad \forall\, J \triangleleft B\,.$$

This property of the "natural" φ is different from the properties of very random and possibly not well-behaved *-isomorphisms ψ from $B \otimes \mathbb{K}$ onto B – as allowed in the *definition* of stability –, because in general the isomorphism $\varphi \circ \psi^{-1}$ of B does not fix the ideal-system $\mathcal{I}(B)$ of $B$and is not approximately unitarily equivalent to id_B by unitaries in $\mathcal{M}(B)$.

The non-degenerate *-monomorphism $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ has the important property that

$$\mu(\mathbb{K})' \cap \mathcal{M}(B) = \delta_\infty(\mathcal{M}(B))\,.$$

In particular:
$\delta_\infty(\mathcal{M}(B))$ is closed in $\mathcal{M}(B)$ with respect to the strict topology of $\mathcal{M}(B)$.

The strictly continuous extension $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2) \to \mathcal{M}(B)$ of μ is a unital *-monomorphism, has image in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ and the norm-closed unit-ball of $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ is also closed in the strict topology of $\mathcal{M}(B)$.

The strictly closed C^*-subalgebra $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ of $\mathcal{M}(B)$ can be larger than $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$, e.g. if the center of $\mathcal{M}(B)$ is not trivial, because the center of $\mathcal{M}(B)$ is identical with the commutant of $\delta_\infty(B)\cdot\mu(\mathbb{K})$ in $\mathcal{M}(B)$, and $\delta_\infty(B)\cdot\mu(\mathbb{K})$ generates B.

`Next should be partly in the proof`

Since μ is non-degenerate and $B \neq \{0\}$, μ is faithful and extends to a faithful and strictly continuous C^*-morphism $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ by the considerations (0) given in the proof.

Let $d\colon B \to \mathcal{L}(\mathcal{H})$ any non-degenerate *-representation on a Hilbert space $\mathcal{H}$ (i.e., $d(B)\mathcal{H} = \mathcal{H}$, using factorization).

compare ?????????

Then $d\colon B \to \mathcal{M}(\mathbb{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$ is non-degenerate in the sense of Part (1), and $\mathcal{M}(d)\colon \mathcal{M}(B) \to \mathcal{M}(\mathbb{K}(\mathcal{H}))$ is strictly continuous.

Thus $\mathcal{M}(d) \circ \mathcal{M}(\mu)$ is strictly continuous on $\mathcal{M}(\mathbb{K}(\ell_2))$ and the topology induced on $\mathcal{M}(\mathbb{K})$ by $\mathcal{M}(\mu)^{-1}$ from the strict topology of $\mathcal{M}(B)$ is between the strict topology of $\mathcal{M}(\mathbb{K})$ and the topology on $\mathcal{M}(\mathbb{K})$ that is induced by $(\mathcal{M}(d) \circ \mathcal{M}(\mu))^{-1}$ on $\mathcal{M}(\mathbb{K})$ from the strict topology of $\mathcal{M}(\mathbb{K}(\mathcal{H}))$.

The strict topology on $\mathcal{M}(\mathbb{K}(\mathcal{H}))$ is on bounded parts the same as the *-strong topology.

Every non-degenerate representation of $\mathbb{K}(\mathcal{H})$ on some other Hilbert space $\mathcal{H}_2$ induces on $\mathcal{M}(\mathbb{K}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$ the same strict topology as its "native" coming from the identical representation over $\mathcal{H}$.

The topology induced by $\mathcal{M}(\mu)^{-1}$ on $\mathcal{M}(\mathbb{K}(\ell_2)) = \mathcal{L}(\ell_2(\mathbb{N}))$ from the strict topology on $\mathcal{M}(B)$ is the same for each non-degenerate C^*-morphism $\mu\colon \mathbb{K}(\ell_2) \to \mathcal{M}(B)$.

This "universal" strict topology on $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$ coincides on bounded parts of $\mathcal{L}(\ell_2)$ with the *-strong operator topology on $\mathcal{L}(\ell_2)$. In particular, $\mathcal{M}(\mu)(\mathcal{M}(\mathbb{K}))$ *is strictly closed in* $\mathcal{M}(B)$ *if* $\mu(\mathbb{K})$ *is non-degenerate.*

The strict closure $\overline{h(A)}^{\text{str}} \subseteq \mathcal{M}(B)$ of the image $h(A) \subseteq \mathcal{M}(B)$ of any non-degenerate C^*-morphism $h\colon A \to \mathcal{M}(B)$ of a stable C^*-algebra A contains the image $\mathcal{M}(h)(\mathcal{M}(A))$ of the strictly continuous unital extension $\mathcal{M}(h)\colon \mathcal{M}(A) \to \mathcal{M}(B)$, just by strict continuity of $\mathcal{M}(h)$, but *in general* $\mathcal{M}(h)(\mathcal{M}(A))$ is *not* strictly closed for non-degenerate h.

The closure of $\mathcal{M}(h)(\mathcal{M}(A))$ in $\mathcal{M}(B)$ with respect to the strict topology of $\mathcal{M}(B)$ coincides with the intersection of $\mathcal{M}(B) \subseteq B^{**}$ with the $\sigma(B^{**}, B^*)$ closure of $\mathcal{M}(h)(\mathcal{M}(A))$ in B^{**}.

(a) *There exists a sequence* $s_1, s_2, \ldots$ *of isometries in* $\mathcal{M}(B)$ *such that* $\sum_n s_n s_n^*$ *converges strictly to* 1 *in* $\mathcal{M}(B)$*, if and only if,* B *is stable:*

There exists a sequence of isometries $t_1, t_2, \ldots \in \mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$ with $\sum t_n t_n^*$ strictly convergent to 1. The isometries t_n can be given explicitly in $\mathcal{L}(\ell_2)$ as $t_n := I \circ J_n$ by a fixed isometry I from $\ell_2 \otimes \ell_2$ onto ℓ_2 composed with the isometries $J_n(x) := e_n \otimes x$ for $x \in \ell_2(\mathbb{N})$ and for the canonical basis $\{e_1, e_2, \ldots\}$ of $\ell_2(\mathbb{N})$.

The strict convergence $\sum t_n t_n^* = 1$ in $\mathcal{M}(\mathbb{K})$ follows from the fact that the *-strong topology on $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$ coincides with the strict topology *on bounded parts* of $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$.

The here used coincidence of the *-strong topology and the strict topology *on bounded parts* of $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ follows from the norm-density of the finite rank operators in $\mathbb{K}$.

Since the *-ultra-strong topology coincides with the *-strong topology on bounded parts of $\mathcal{L}(\ell_2)$, we get that strict (given by $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K}(\ell_2))$ and *-ultra-strong topology coincide on bounded parts.

`Do the strict topology and *-ultra-strong topology coincide also on unbounded parts of` $\mathcal{L}(\mathcal{H})$`?`

It is – perhaps – the case if and only if, strict and *-ultra-strong topology coincide on $\mathcal{M}(\mathbb{K}(\ell_2(\mathbb{N})))$. The latter seems not to be the case (?!).

Are the two the strong operator topologies on $\ell_\infty(\mathbb{C}) = \mathcal{M}(c_0\mathbb{C})$ for its natural actions on c_0 or ℓ_2 different on *unbounded* parts?

Does there exist an unbounded *net* in ℓ_∞ that that converges in the strong operator topology on $\mathcal{L}(\ell_2)$ to zero but does not converge in the strong operator topology in $\mathcal{M}(c_0) = \ell_\infty \subset \mathcal{L}(c_0)$?

It is the same problem with the "global" difference of the strong operator topology for the left multiplication of $\mathcal{L}(\ell_2)$ on $\mathbb{K}$, Hilbert-Schmidt class, ore trace class.

Semi-norms for *-ultra-strong topology on $\mathcal{L}(\mathcal{H})$:

$T \mapsto (\sum_n \|T^*x_n\|^2 + \|Tx_n\|^2)^{1/2}$ where $x_n \in \ell_2$, $\sum_n \|x_n\|^2 < \infty$. It is the same as $\mathrm{tr}((T^*T + TT^*)S)^{1/2} = (\|TS^{1/2}\|_2^2 + \|T^*S^{1/2}\|_2^2)^{1/2}$ for some positive trace class operator S. Then $R := S^{1/2}$ is Hilbert-Schmidt, and

$$\max(\|TR\|_2, \|T^*R\|_2) \leq (\|TR\|_2^2 + \|T^*R\|_2^2)^{1/2} \leq \|TR\|_2 + \|T^*R\|_2$$

Notice here that $\|X\| = \|X\|_\infty \leq \|X\|_p$ for all $p \geq 1$ and $X \in \mathcal{L}(\ell_2)$ with $(\|X\|_p)^p = \mathrm{tr}((X^*X)^{p/2}) < \infty$.

Alternatively:

The Hilbert-Schmidt class operator R can be Cohen factorized into $aR_1 = R$ with $a \in \mathbb{K}$ and R_1 in Hilbert-Schmidt class.

Since $\|TR\|_2 = \|TaR_1\|_2 \leq \|Ta\| \cdot \|R_1\|_2$ for $T \in \mathcal{L}(\ell_2) = \mathcal{M}(\mathbb{K})$ it follows that *-ultra-strong topology is coarser than strict topology or is equal to it:

Result:
If an (possibly unbounded) net converges to an operator T in strict topology, then it converges also in *-ultra-strong topology.

Recall that strict topology on $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$ is defined by the semi-norms $T \mapsto \|Ta\| + \|T^*b\|$ where $a, b \in \mathbb{K}$.

If we use Cohen factorization, then the strict topology on $\mathcal{L}(\ell_2)$ is also defined by the semi-norms $T \mapsto \|Tc\| + \|T^*c\|$ with $c \in c_0(\mathbb{N})_+ \subset \mathbb{K}(\ell_2(\mathbb{N}))$.

Restricted to $T \in \ell_\infty(\mathbb{N})$ it is the strong operator norm on $\ell_\infty(\mathbb{N})$ defined by the multiplication $c \mapsto Tc$ of elements $c \in c_0(\mathbb{N})$ with T.

If one wants to compare this with the right multiplication action of ℓ_∞ on the Hilbert-Schmidt class HS $\cong \ell_2(\mathbb{N} \times \mathbb{N})$, then one has to use the restriction of the *-ultra-strong topology on $\mathcal{L}(\ell_2)$ to $\ell_\infty \subset \mathcal{L}(\ell_2)$.

The (left-)actions of $\mathcal{M}(\mathbb{K})$ on $\mathbb{K}$ and HS are isometric with respect to the operator norm on $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$.

The induced strong topologies have same (countable) zero *sequences*:

If X is a Banach space and $T_1, T_2, \ldots \in \mathcal{L}(X)$ converges to zero in the strong (or the "ultra-strong") topology to zero, then $\|T_1\|, \|T_2\|, \ldots$ is a bounded sequence by the uniform boundedness theorem.

Now one can use that the norms $\|T\|$ of $T \in \mathcal{M}(\mathbb{K}) \subset \mathcal{L}(\mathbb{K})$ and $T \in \mathcal{L}(\ell_2) = \mathcal{M}(\mathbb{K})$ coincide and that strict and *-ultra-strong topologies on bounded parts of $\mathcal{M}(\mathbb{K})$ coincide.

It follows that differences in the convergence structures for *-ultra-strong or strict topology on $\mathcal{M}(\mathbb{K})$ can be only found if one considers unbounded nets of high cardinality that converge in *-ultra-strong topology but not in strict topology.

Suppose that there exists an isomorphism ψ from a C^*-algebra $C \otimes \mathbb{K}$ onto B. Let $\mathcal{M}(\psi)$ denote its strictly continuous extension to an isomorphism from $\mathcal{M}(C \otimes \mathbb{K})$ onto $\mathcal{M}(B)$. Its inverse is given by $\mathcal{M}(\psi)^{-1} = \mathcal{M}(\psi^{-1})$ and, therefore, is strictly continuous. Notice that there are natural strictly continuous unital *-monomorphisms $\mathcal{M}(\varphi_1)\colon \mathcal{M}(C) \to \mathcal{M}(C \otimes \mathbb{K})$ and $\mathcal{M}(\varphi_2)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(C \otimes \mathbb{K})$ that are the strictly continuous extensions of the non-degenerate C^*-morphisms $\varphi_1\colon C \to \mathcal{M}(C \otimes \mathbb{K})$ and $\varphi_2\colon \mathbb{K} \to \mathcal{M}(C \otimes \mathbb{K})$ that are determined by the properties $\varphi_1(c)(c_1 \otimes k) = (c \cdot c_1) \otimes k$ and $\varphi_2(k)(c \otimes k_1) = c \otimes (k \cdot k_1)$. We write $1 \otimes t$ for $\mathcal{M}(\varphi_2)(t)$ if $t \in \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$.

There is a non-degenerate C^*-morphism $\lambda\colon \mathbb{K} \to \mathcal{M}(B)$ that is given by $\lambda(k) := \mathcal{M}(\psi)(1 \otimes k)$ for $k \in \mathbb{K}$, i.e., $\lambda = \mathcal{M}(\psi) \circ \varphi_2$. It extends to a unital *-monomorphism $\mathcal{M}(\lambda) = \mathcal{M}(\psi) \circ \mathcal{M}(\varphi_2)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$, that is strictly continuous on bounded parts. The isometries $s_n := \mathcal{M}(\lambda)(t_n) = \mathcal{M}(\psi)(1 \otimes t_n)$ have the property that $\sum s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(B)$, because $\sum t_n t_n^*$ converges strictly to 1 and $\mathcal{M}(\lambda)$ is strictly continuous on bounded parts.

Conversely suppose that $\mathcal{M}(B)$ contains a sequence $s_1, s_2, \ldots$ of isometries in $\mathcal{M}(B)$ such that $\sum_n s_n s_n^*$ converges strictly to 1.

Then $\delta_\infty(b) := \sum_n s_n b s_n^*$ converges strictly to an element $\delta_\infty(b)$ of $\mathcal{M}(B)$ and the map $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ is strictly continuous and unital on $\mathcal{M}(B)$.

Indeed: If $b \in B$ and $\varepsilon > 0$, then there is $m \in \mathbb{N}$ with $\|b - (s_1 s_1^* + \ldots + s_m s_m^*)b\| < \varepsilon/2$ and there exists a contraction $e \in B_+$ with $\| s_k^* b - e s_k^* b \| < \varepsilon/2m$. It gives $\|b - \delta_\infty(e)b\| < \varepsilon$.

Thus $\delta_\infty(B)B$ is dense in B, i.e., $\delta_\infty|B$ is non-degenerate. It follows that $\delta_\infty = \mathcal{M}(\delta_\infty|B)$ is the unique strictly continuous unital extension of $\delta_\infty|B$ to $\mathcal{M}(B)$.

Then there is a unique C^*-morphism $\mu\colon \mathbb{K} \to \delta_\infty(B)' \cap \mathcal{M}(B)$ that satisfies $\mu(e_{k,\ell}) := s_k s_\ell^*$.

The range projections $p_n := s_n s_n^*$ satisfy $p_m \leq 1 - p_n$ because $p_n + p_m \leq 1$ for $m \neq n$. It implies $p_m p_n = 0$. Thus $\sum_n s_n s_n^* \leq 1$ implies $s_i^* s_j = s_i^* p_i p_j s_j = 0$ for $i \neq j$. In particular, B is the closure of its *-subalgebra $B_0 := \bigcup_n P_n B P_n$ for $P_n := p_1 + \cdots + p_n \in \mathcal{M}(B)$.

Let $e_{i,j}$ denote the canonical matrix units of $\mathbb{K} \subset M_\infty(\mathbb{C})$. We define an isometry $T \in \mathcal{M}(B \otimes \mathbb{K})$ with $TT^* = 1_{\mathcal{M}(B)} \otimes e_{1,1}$ by the "row matrix"

$$T := s_1 \otimes e_{1,1} + s_2 \otimes e_{1,2} + \ldots . \tag{1.3}$$

We use the natural embedding

$$\mathcal{M}(B \otimes \mathbb{K}) \subset M_\infty(\mathcal{M}(B)) \subseteq B^{**}\overline{\otimes}\mathcal{L}(\ell_2(\mathbb{N}))$$

that describes the elements of $\mathcal{M}(B \otimes \mathbb{K})$ as matrices over $\mathcal{M}(B)$ then above T becomes the row matrix

$$T \cong [s_1, s_2, \ldots] \in M_{1,\infty}(\mathcal{M}(B)) \subset M_\infty(\mathcal{M}(B))\,.$$

The sum $\sum_n s_n \otimes e_{1,n}$ is unconditional strictly convergent in $\mathcal{M}(B\otimes\mathbb{K})$ by Remark (2) if we take there $y_n := s_n \otimes 1$, $d_n := 1 \otimes 1$ and $x_n := 1 \otimes e_{n,1}$. Since

$$(s_n^* \otimes e_{m,1})T = (s_n^* \otimes e_{m,1})(s_n \otimes e_{1,n}) = 1 \otimes e_{m,n}\,,$$

we get $T^*(s_n \otimes e_{1,n}) = 1 \otimes e_{n,n}$ and $T^*T = 1\otimes 1$. The identities $(s_k \otimes e_{1,k})(s_n^* \otimes e_{n,1}) = \delta_{k,n} s_k s_n^* \otimes e_{11}$, imply that $TT^* = 1 \otimes e_{11}$. It follows that the map

$$H\colon b \in \mathcal{M}(B) \mapsto H(b) := T^*(b \otimes e_{11})T$$

defines a unital *-monomorphism H from $\mathcal{M}(B)$ into $\mathcal{M}(B \otimes \mathbb{K})$ that is strictly continuous (on bounded parts) and maps B into $B \otimes \mathbb{K}$, because $T \in \mathcal{M}(B \otimes \mathbb{K})$.

The *-monomorphism H maps $\mathcal{M}(B)$ *onto* $\mathcal{M}(B\otimes\mathbb{K})$, i.e., is an isomorphism, H maps B *onto* $B \otimes \mathbb{K}$, and the inverse map $H^{-1}\colon \mathcal{M}(B \otimes \mathbb{K}) \to \mathcal{M}(B)$ is given on $\mathcal{M}(B) \odot \mathbb{K}$ by $H^{-1}(b \otimes e_{jk}) = \delta_\infty(b) s_j s_k^*$:

Indeed, if $a \in \mathcal{M}(B\otimes\mathbb{K})$, then $a = T^*(TaT^*)T = H(TaT^*)$ and $TaT^* = b\otimes e_{11}$ for some $b \in \mathcal{M}(B)$.

Since H is verified to be an isomorphism from $\mathcal{M}(B)$ onto $\mathcal{M}(B \otimes \mathbb{K})$, $H(B)$ must be a closed ideal of $\mathcal{M}(B\otimes\mathbb{K})$. The definition $H := T^*((\cdot)\otimes e_{11})T$ of H with $T \in \mathcal{M}(B \otimes \mathbb{K})$ implies that $H(B) \subseteq B \otimes \mathbb{K}$.

Since B is an essential ideal of $\mathcal{M}(B)$ that is dense in $\mathcal{M}(B)$ with respect to the strict topology, $H(B)$ is a closed ideal of $B\otimes\mathbb{K}$ that is strictly dense in $B\otimes\mathbb{K}$. Thus $H(B) = B \otimes \mathbb{K}$.

(Alternatively one can prove that $b \otimes e_{n,m} \in T^*(B \otimes e_{1,1})T$ by showing that $B \otimes 1 = T^*(\delta_\infty(B) \otimes e_{1,1})T$, $T^*(s_n s_m^* \otimes e_{1,1})T = 1 \otimes e_{n,m}$ and $\delta_\infty(B) s_n s_m^* \in B$. See more below.)

Let $\psi := H|B$ and $\varphi\colon B \otimes \mathbb{K} \to B$ the inverse of $\psi = H|B$. Then $\mathcal{M}(\varphi)$ and $\mathcal{M}(\psi)$ are strictly continuous and $H = \mathcal{M}(\varphi)^{-1}$.

We can use the above defined isomorphism $\varphi\colon B \otimes \mathbb{K} \to B$ to define the other in part (8) proposed C^*-morphisms:

We have that $(s_\ell^* \otimes e_{11})T = 1 \otimes e_{1,\ell}$. It implies $T^*(s_k b s_\ell^*) \otimes e_{11}T = b \otimes e_{k,\ell}$ and $T^*(\delta_\infty(b) \otimes e_{11})T = b \otimes 1$ for all $b \in \mathcal{M}(B)$.

`Next blue NEW / BETTER APPROACH?:`

Let $\delta_\infty(b) := \sum_n s_n b s_n^*$ where $b \in \mathcal{M}(B)$, $s_1, s_2, \ldots \in \mathcal{M}(B)$ with $\sum_n s_n s_n^* = 1_{\mathcal{M}(B)}$ and $s_m^* s_n = \delta_{m,n} 1_{\mathcal{M}(B)}$.

Let $T \in \mathcal{M}(B \otimes \mathbb{K})$ the isometry defined by Equation (1.3).

There is a non-degenerate C^*-morphism $\mu\colon \mathbb{K} \to \delta_\infty(B)' \cap \mathcal{M}(B)$ with $T^*(\mu(e_{i,j}) \otimes e_{11})T = 1 \otimes e_{i,j}$, e.g. $\mu(c) \otimes e_{11} := T(1 \otimes c)T^*$.

We can μ define $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ explicit by $\mu(e_{k,\ell}) := s_k s_\ell^*$.

The equations $T(1 \otimes e_{m,1}) = s_m \otimes e_{1,1}$ imply $(1 \otimes e_{1,n})T^* = s_n^* \otimes e_{1,1}$ and

$$T(1 \otimes e_{m,n})T^* = T(1 \otimes e_{m,1})(1 \otimes e_{1,n})T^* = s_m s_n^* \otimes e_{11}\,.$$

Since

$$\delta_\infty(b) s_m s_n^* = s_m b s_n^* = s_m s_n^* \delta_\infty(b)\,,$$

the C^*-morphism $[\alpha_{jk}] \to \sum_{jk} \alpha_{jk} s_j s_k^*$ extends to a C^*-morphism $\mu\colon \mathbb{K} \to \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ and is uniquely defined by $\mu(k) \otimes e_{11} = T(1 \otimes k)T^*$.

`Better: To be shown directly with above` H:

By Remark (2), $\delta_\infty(b) := \sum_n s_n b s_n^*$ is unconditional strictly convergent for each $b \in \mathcal{M}(B)$, and δ_∞ defines a strictly continuous unital *-monomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(B)$, cf. Proof of Lemma 5.1.2(i,ii), or compare the proof of the equation $\delta_\infty(b) \otimes e_{1,1} = T(b \otimes 1)T^*$ where $T \in \mathcal{M}(B \otimes \mathbb{K})$ is the isometry defined by Equation (1.3).

(b) *There exists a non-degenerate *-monomorphism* $\mu\colon \mathbb{K} \to \mathcal{M}(B)$, *where* $\mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))$, *with the properties* $\mu(\mathbb{K}) \subseteq \delta_\infty(B)' \cap \mathcal{M}(B)$ *such that* $\delta_\infty(b)\mu(k) \in B$ *for* $b \in B$ *and* $k \in \mathbb{K}$ and *such that the unique C^*-morphism* $\psi\colon B \otimes \mathbb{K} \to B$ *with* $\varphi(b \otimes k) = \delta_\infty(b)\mu(k)$ *is a *-isomorphism from* $B \otimes \mathbb{K}$ *onto* B.

The above defined isomorphism $\varphi\colon B \otimes \mathbb{K} \to B$, with $H = \mathcal{M}(\varphi)^{-1}$, $\psi := \varphi^{-1} = H|B$, $T(b \otimes k)T^* = (\delta_\infty(b)\mu(k)) \otimes e_{11}$ and $H(b) = T^*(b \otimes e_{1,1})T$, does the job if μ is defined by $\mu(k) \otimes e_{1,1} := T(1 \otimes k)T^*$.

(c) *The non-degenerate *-monomorphism* $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ *has the property* $\mu(\mathbb{K})' \cap \mathcal{M}(B) = \delta_\infty(\mathcal{M}(B))$. *In particular,* $\delta_\infty(\mathcal{M}(B))$ *is closed in* $\mathcal{M}(B)$ *with respect to the strict topology of* $\mathcal{M}(B)$.

This can be seen by using the isomorphism $\mathcal{M}(\varphi)\colon \mathcal{M}(B \otimes \mathbb{K}) \to \mathcal{M}(B)$ from $\mathcal{M}(B \otimes \mathbb{K})$ onto $\mathcal{M}(B)$.

(d) *The strictly continuous extension* $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2) \to \mathcal{M}(B)$ *of* μ *is a strictly continuous unital *-monomorphism, has image in* $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ *and* $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ *is closed in the strict topology of* $\mathcal{M}(B)$.

`Next new version of proof ?`

The natural extension $\theta\colon \mathcal{M}(B) \to B^{**}$ of the unital *-monomorphism $\eta_B\colon B \to B^{**}$ from $\mathcal{M}(B)$ into the W*-algebra B^{**} is continuous with respect to the strict topology on $\mathcal{M}(B)$ and the *-ultra-strong topology on B^{**}.

If $h\colon A \to \mathcal{M}(B)$ is any non-degenerate *-morphism, then μ extends uniquely to a unital and strictly continuous map $\mathcal{M}(h)\colon \mathcal{M}(A) \to \mathcal{M}(B)$. Then $\mathcal{M}(h)(1) = 1$ and $\mathcal{M}(h)(\mathcal{M}(A))$ is in $\mathcal{M}(B)$ the strict closure of $h(A)$. Since $\theta\colon \mathcal{M}(B) \to B^{**}$ is continuous with respect to the strict topology on $\mathcal{M}(B)$ and the *-ultra-strong topology on B^{**}, it follows that the *-ultra-strong closure of $\theta(h(A))$ in B^{**} contains

$\theta(\mathcal{M}(h)(A))$. In particular, the *-ultra-weak closure of $h(A)$ contains the unit $1 := 1_{B^{**}}$ of B^{**}.

If M is a von-Neumann algebra and $\Phi\colon \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K}) \to M$ is a C^*-morphism such that the unit 1_M of M is contained in the *-ultra-weak closure of $\Phi(\mathbb{K})$ then Φ is normal on $\mathcal{L}(\ell_2)$, i.e., is a unital W*-algebra isomorphism from $\mathcal{L}(\ell_2)$ onto the *-ultra-weak closure of $\Phi(\mathbb{K})$.

In particular, $\Phi(\mathcal{M}(\mathbb{K}))$ is closed in the *-ultra-strong topology of B^{**}, and the topology induced from this topology on $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ is the same as the *-ultra-strong topology on $\mathcal{L}(\ell_2)$.

Now take $\mathbb{K}$ and $\mu\colon \mathbb{K} \to \mathcal{M}(B)$ in place of A and h above. Let $\Psi := \theta \circ \mathcal{M}(\mu)$, then we get that the restriction of the strict topology on $\mathcal{M}(B)$ to $\mathcal{M}(\mu)(\mathcal{M}(\mathbb{K}))$ is finer than the topology induced from the *-ultra-strong topology on $\mathcal{M}(\mathbb{K})$. Since $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ is strictly continuous with respect to the strict topology on $\mathcal{M}(\mathbb{K})$ and on $\mathcal{M}(B)$, it follows that the topology induced on $\mathcal{M}(\mathbb{K})$ by $\mathcal{M}(\mu)^{-1}$ is equal to or coarser than the strict topology on $\mathcal{M}(\mathbb{K})$. So it is between the *-ultra-strong topology and the strict topology on $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$. It is easy to see - by the definitions and by the remark in Part (i) – that the latter both coincides on bounded parts of $\mathcal{M}(\mathbb{K})$.

Thus, on bounded parts of $\mathcal{M}(\mathbb{K})$ the topology induced from the strict topology on $\mathcal{M}(B)$ via $\mathcal{M}(\mu)^{-1}$ coincide.

The image $\mathcal{M}(\mu)(\mathcal{M}(\mathbb{K}))$ is strictly closed, because θ is continuous with respect to the strict and *-ultra-strong topology, and $\theta(\mathcal{M}(\mu)(\mathcal{M}(\mathbb{K})))$ is *-ultra-strongly closed in B^{**}.

If one uses the isomorphism $\mathcal{M}(\varphi)$ from $\mathcal{M}(B \otimes \mathbb{K})$ onto $\mathcal{M}(B)$, then one can see that $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ is the isomorphic image of $1 \otimes \mathcal{L}(\ell_2) \subseteq \mathcal{M}(B \otimes \mathbb{K})$.

It is not difficult to see that $1 \otimes \mathcal{L}(\ell_2)$ is strictly closed in $\mathcal{M}(B \otimes \mathbb{K})$ and that the induced topology on $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ coincides on bounded parts of $\mathcal{L}(\ell_2)$ with the *-strong topology.

Indeed, if $e \in B_+$ is a strictly positive positive contraction, then $C^*(e) \otimes \mathbb{K}$ is a non-degenerate C^*-subalgebra of $B \otimes \mathbb{K}$ and $1 \otimes \mathcal{M}(\mathbb{K})$ is naturally contained in $\mathcal{M}(C^*(e) \otimes \mathbb{K}) \subseteq \mathcal{M}(B \otimes \mathbb{K})$ and the representation $T \in \mathcal{M}(\mathbb{K}) \mapsto 1 \otimes T \in$ is strictly continuous, because it extends a non-degenerate representation of $\mathbb{K}$. Thus, the topology on $\mathcal{M}(\mathbb{K})$ induced from $T \mapsto 1 \otimes T$ is coarser than the strict topology on $\mathcal{M}(\mathbb{K})$. Let $\chi\colon C^*(e) \to \mathbb{C}$ a character. It defines a *-epimorphism λ from $C^*(e) \otimes \mathbb{K}$ onto $\mathbb{K}$. Then $\mathcal{M}(\lambda)\colon \mathcal{M}(C^*(e) \otimes \mathbb{K}) \to \mathcal{M}(\mathbb{K})$ is strictly continuous, and its restriction to $1 \otimes \mathcal{M}(\mathbb{K})$ is continuous with respect to the strict topology on $\mathcal{M}(C^*(e) \otimes \mathbb{K})$ defined by $C^*(e) \otimes \mathbb{K}$.

The strict topology on $\mathcal{M}(C^*(e) \otimes \mathbb{K})$ coming from $C^*(e) \otimes \mathbb{K}$ coincides on bounded parts with the restriction to $\mathcal{M}(C^*(e) \otimes \mathbb{K})$ of the strict topology on $\mathcal{M}(B \otimes \mathbb{K})$, because on bounded parts the strict topology can be defined by the semi-norms $d \in \mathcal{M}(B \otimes \mathbb{K}) \mapsto \|d(e \otimes k_1)\| + \|(e \otimes k_2)d\|$ with $k_1, k_2 \in \mathbb{K}_+$ of finite rank. In particular this topologies coincide on bounded parts of $1 \otimes \mathcal{M}(\mathbb{K})$.

Summing up, we get strictly continuous C^*-morphisms $\mathcal{M}(\mu) \colon \mathcal{M}(\mathbb{K}) \to 1 \otimes \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(C^*(e) \otimes \mathbb{K}) \subseteq \mathcal{M}(B \otimes \mathbb{K})$ and $\mathcal{M}(\lambda) \colon \mathcal{M}(C^*(e) \otimes \mathbb{K}) \to \mathcal{M}(\mathbb{K})$ such that the strict topology on bounded parts of $\mathcal{M}(C^*(e) \otimes \mathbb{K})$ coincides with the topology induced from the strict topology of $\mathcal{M}(B \otimes \mathbb{K})$. Moreover $\mathcal{M}(\lambda) \circ \mathcal{M}(\mu) \colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$ is a strictly continuous isomorphism of $\mathcal{M}(\mathbb{K})$ Thus it is given by an inner automorphism of $\mathcal{M}(\mathbb{K})$. It follows that all the induced strict topologies are the same on bounded parts of $\mathcal{M}(\mathbb{K})$.

The topology on $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$ induced from the strict topology on $\mathcal{M}(\mathbb{K})$ is on bounded parts the same as the *-strong topology on $\mathcal{L}(\ell_2)$.

Thus, the topology induced by $\mathcal{M}(\mu)^{-1}$ on $\mathcal{L}(\ell_2)$ is on bounded parts the same as the *-strong topology on $\mathcal{L}(\ell_2)$.

Since the unit-ball of $\mathcal{L}(\ell_2)$ is complete in the *-strong topology, it follows that $\mathcal{M}(\mu)(\mathcal{M}(\mathbb{K}))$ is closed in $\mathcal{M}(B)$ with respect to the *strict* topology on $\mathcal{M}(B)$.

By Lemma 5.1.2(i,ii), all infinite repeats δ_∞ are unitarily equivalent in $\mathcal{M}(B)$. Therefore we need only to verify the observations by choosing case by case a suitable "fitting" infinite repeat δ_∞.

The isomorphism ψ from $B \otimes \mathbb{K}$ onto B extends uniquely to a strictly continuous *-isomorphism $\mathcal{M}(\psi)$ from $\mathcal{M}(B \otimes \mathbb{K})$ onto $\mathcal{M}(B)$. The natural *-monomorphism $\lambda \colon \mathbb{K} \to \mathcal{M}(B \otimes \mathbb{K})$ with $\lambda(k_1)(b \otimes k_2) = b \otimes k_1 k_2$ is non-degenerate and extends to a strictly continuous unital *-monomorphism $\mathcal{M}(\lambda)$ from $\mathcal{M}(\mathbb{K})$ into $\mathcal{M}(B \otimes \mathbb{K})$.

Since there is an isomorphism ι from $\mathbb{K} \otimes \mathbb{K}$ onto $\mathbb{K}$, given by an isomorphism $\ell_2 \cong \ell_2 \otimes_2 \ell_2$, there exists isometries $t_1, t_2, \ldots \in \mathcal{M}(\mathbb{K})$ such that $\sum_n t_n t_n^*$ strictly convergent to 1, and that the commutant $\delta_\infty(\mathbb{K})' \cap \mathcal{M}(\mathbb{K})$ of $\delta_\infty(\mathbb{K})$ contains a non-degenerate copy $H(\mathbb{K})$ of $\mathbb{K}$, that satisfies $\delta_\infty(k_1)H(k_2) = \iota(k_1 \otimes k_2)$ where $\delta_\infty(k) := \sum_n t_n k t_n^*$.

Let $s_n := \mathcal{M}(\psi) \circ \mathcal{M}(\lambda)(t_n)$. Then $s_1, s_2, \ldots$ is a sequence of isometries in $\mathcal{M}(B)$ such that $\sum_n s_n s_n^*$ converges strictly to 1.

And $\delta_\infty(b) := \sum_n s_n b s_n^*$ is a non-degenerate C^*-morphism, such that $\mu \colon \mathbb{K} \to \delta_\infty(B)' \cap \mathcal{M}(B)$ (given by $\mu(k) := \mathcal{M}(\psi)(H(k))$) satisfies that $\delta_\infty(b)\mu(k) \in B$. It follows that there is a (new) C^*-morphism φ from $B \otimes \mathbb{K}$ into B that satisfies $\varphi(J \otimes \mathbb{K}) = J$ for all closed ideals of B and defines an isomorphism from $B \otimes \mathbb{K}$ onto B that is considerably different from the above defined by an arbitrary isomorphism of B and some $C \otimes \mathbb{K}$.

In particular, the unique strictly continuous extension $\mathcal{M}(\mu) = \mathcal{M}(\psi) \circ \mathcal{M}(H) \colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ of $\mu \colon \mathbb{K} \to \mathcal{M}(B)$ is a unital *-monomorphism from $\mathcal{L}(\ell_2)$ into $\mathcal{M}(B)$, and has the property that $\delta_\infty(\mathcal{M}(B))$ and $\mathcal{M}(\mu)(\mathcal{L}(\ell_2))$ commute and the C^*-algebra generated by $\delta_\infty(B) \cdot \mu(\mathbb{K})$ is strictly dense in $\mathcal{M}(B)$.

NEXT old or new ???

There exists isometries $t_1, t_2, \ldots$ in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ such that $\sum_n t_n t_n^*$ converges strictly to 1. This is because $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ is strictly closed and there is a non-degenerate *-monomorphism $\mu\colon \mathbb{K}(\ell_2) \to \mathcal{M}(B)$ with image in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$. Then $\mathcal{M}(\mu)\colon \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2) \to \mathcal{M}(B)$ is strictly continuous, unital and has image in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$.

(9): Recall that a C^*-morphism $h\colon A \to \mathcal{M}(B)$ is *non-degenerate* if and only if the linear span of $h(A)B$ is dense in B. The Cohen factorization theorem shows that this is equivalent to $B = h(A) \cdot B$ (as set of products $h(a)b$).

It follows that non-degenerate h satisfies $h(A)(h(A)+B) \subseteq h(A)+B$ and that $h(A)(h(A)+B)$ is norm-dense in the C^*-subalgebra $h(A)+B$ of $\mathcal{M}(B)$.

Indeed, if $a \in A$, $b \in B$ and $\varepsilon > 0$ are given, then let $\gamma := \varepsilon/(3 + \|b\|)$ We find $f \in A$ with $\|b - h(f)b\| < \gamma$. There exists a positive contraction $e \in A_+$ with $\|f - ef\| < \gamma$ and $\|a - ea\| < \gamma$. Straight calculation shows that $\|(h(a) + b) - h(e)(h(a) + b)\| < \varepsilon$.

Let $H\colon A \to \mathcal{M}(h(A)+B)$ the C^*-morphism with $H(e)(h(a)+b) := h(e)(h(a)+b)$ for $a, e \in A$ and $b \in B$. The elements $t_n := \mathcal{M}(h)(s_n) \in \mathcal{M}(B)$ (respectively $t_n := \mathcal{M}(H)(s_n) \in \mathcal{M}(B)$ are isometries in $\mathcal{M}(h)(\mathcal{M}(A)) \subseteq \mathcal{M}(B)$ (respectively in $\mathcal{M}(H)(\mathcal{M}(A)) \subseteq \mathcal{M}(h(A) + B)$), and $\sum t_n t_n^*$ converges strictly to 1 in $\mathcal{M}(B)$ (respectively in $\mathcal{M}(h(A)+B)$), because $\mathcal{M}(h)$ (respectively $\mathcal{M}(H)$) is strictly continuous and unital by Part (1).

It follows that B and $h(A)+B$ are stable by Part (8). □

LEMMA 5.1.2. *Suppose that B is a stable and σ-unital C^*-algebra, and let $s_1, s_2, \ldots \in \mathcal{M}(B)$ a sequence of isometries such that $\sum_n s_n s_n^*$ converges strictly to 1 in $\mathcal{M}(B)$, – see Remark 5.1.1(8) for the existence of $\{s_1, s_2, \ldots\}$.*

(i) *The map $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ is a strictly continuous unital *-monomorphism. If $\delta'_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ comes from another sequence $t_1, t_2, \ldots$ of isometries in $\mathcal{M}(B)$ such that $\sum t_n t_n^*$ converges strongly to 1, then δ'_∞ is unitarily equivalent to δ_∞. In particular, δ_∞^2 is unitarily equivalent to δ_∞.*

The algebra $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ is strictly closed and contains a sequence of isometries $q_1, q_2, \ldots$ such that $\sum q_n q_n^*$ converges strictly to 1.

(ii) *$\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ (with canonical generators r_1, r_2) unitally, and $\delta_\infty \oplus_{r_1,r_2} \mathrm{id}_{\mathcal{M}(B)}$ is unitarily equivalent to δ_∞.*

(iii) *Let $D := \mathcal{O}_2 \otimes \mathbb{K}$ denote the stabilized Cuntz algebra $\mathcal{O}_2$, $q_1 \le q_2 \le \ldots$ an increasing sequence of projections in D and p_0 is a nonzero projection in D.*

Then there exists norm-continuous maps $t \mapsto U(t)$ from $\mathbb{R}_+$ into the set of unitaries in $1 + D \subseteq \mathcal{M}(D)$ such that $U(0) = 1$ and $q_n \le U(t)^* p_0 U(t)$ and $q_n \ne U(t)^* p_0 U(t)$ for $t \ge n$.

If, in addition, $q_n \neq q_{n+1}$ *for* $n = 1, 2, \ldots$, *then one can find the path* $t \mapsto U(t) \in \mathcal{M}(D)$ *such that* $q_n \leq U(t)^* p_0 U(t) \leq q_{n+3}$ *and* $U(t)^* p_0 U(t) \notin \{q_n, q_{n+3}\}$ *for* $t \in [n+1, n+2]$.

(iv) *There are isometries* $S, T \in \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ *with* $SS^* + TT^* = 1$ *and a norm-continuous map* $t \in [0, \infty) \mapsto U(t) \in \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ *into the unitaries of* $\mathcal{M}(B)$, *such that the norm-continuous maps*

$$S_0, T_0 \colon t \in \mathbb{R}_+ \mapsto S_0(t), T_0(t) \in \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$$

given by $S_0(t) := U(t)^* S U(t)$ *and* $T_0(t) := U(t)^* T U(t)$ *for* $t \in \mathbb{R}_+$ *satisfy* $\lim_{t\to\infty} S_0(t)^* b = 0$ *and* $\lim_{t\to\infty} T_0(t)^* b = b$ *for every* $b \in B$.

(v) *Let* C *a separable* C^**-subalgebra of* $\mathcal{M}(B)$, $T \colon C \to \mathcal{M}(B)$ *a completely positive contraction, and suppose that there exists a contraction* $g \in \mathcal{M}(B)$ *with* $g^* c g - \delta_\infty(T(c)) \in B$ *for every* $c \in C$.

Then there exists a norm-continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ *into the contractions in* $\mathcal{M}(B)$ *such that* $S(t)^* c S(t) - \delta_\infty(T(c)) \in B$ *for every* $t \in \mathbb{R}_+$, *and* $\lim_{n\to\infty} \|\delta_\infty(T(c)) - S(t)^* c S(t)\| = 0$ *for all* $c \in C$.

If, moreover, $1_{\mathcal{M}(B)} \in C$ *and* T *is unital then* $t \mapsto S(t)$ *can be found such that, moreover,* $S(t)^* S(t) = 1$ *for* $t \in \mathbb{R}_+$.

(vi) *Suppose that* $1_{\mathcal{M}(B)} \in C \subseteq \mathcal{M}(B)$, C *is separable, and that* T *is a unital* C^**-morphism such that there exists a contraction* $g \in \mathcal{M}(B)$ *with* $g^* c g - \delta_\infty(T(c)) \in B$ *for every* $c \in C$. *Then* id_C *asymptotically absorbs* T, *i.e.,* $\mathrm{id}_C \oplus T$ *and* id_C *are unitarily homotopic in the sense of Definition 5.0.1.*

(vii) *If* $f \in \mathcal{M}(B)$ *and if* $e \in B_+$ *is a strictly positive element, then the norm closures* B_0 *of* $f^* B f$ *and* B_1 *of* $\delta_\infty(f^*) B \delta_\infty(f)$ *are* σ*-unital with strictly positive elements* $f^* e f$, *respectively with* $\delta_\infty(f^*) e \delta_\infty(f)$.

There is a natural isomorphism ψ *from* B_1 *onto* $B_0 \otimes \mathbb{K}$, *that is given by*

$$\psi(b) := [s_j^* b s_k]_{jk} \in B_0 \otimes \mathbb{K} \subseteq M_\infty(B_0)\,.$$

PROOF. (i): The strict convergence of the n-th partial sums of $\sum s_n d s_n^*$ for $d \in \mathcal{M}(B)$ and then of $\delta_\infty(d_\tau)$ to $\delta_\infty(d)$ for a strictly convergent bounded net in $\{d_\tau\} \subseteq \mathcal{M}(B)$ with strict limit $d \in \mathcal{M}(B)$ can be seen from the argument in Remark 5.1.1(2). Since $\sum_n s_n s_n^* = 1$ (strictly), the map δ_∞ is unital. It is clear that δ_∞ is an isometric and completely positive map. It is multiplicative because

$$\delta_\infty(d)\delta_\infty(e) s_n s_n^* = s_n d s_n^* s_n e s_n^* = \delta_\infty(de) s_n s_n^*\,.$$

The sum $\sum s_n t_n^* =: U$ is strictly convergent to a unitary because the strong convergence of U and that of U^* can be seen by the argument in Remark 5.1.1(2). This also shows that $U \in \mathcal{M}(B)$ is unitary and $U^* \delta_\infty(d) U = \delta'_\infty(d)$ for $d \in \mathcal{M}(B)$.

The sufficient criterium in Remark 5.1.1(2) shows that $\sum_{n,m} s_n s_m (s_n s_m)^*$ converges unconditional strictly to $1_{\mathcal{M}(B)}$. Thus, δ_∞^2 is an infinite repeat of $\mathrm{id}_{\mathcal{M}(B)}$.

Compare Remark 5.1.1(8)
and use $\rho(e_{jk}) := s_j s_k^*$????

''It is clear that'' ???????

The map $T \in \mathcal{M}(B) \mapsto ST - TS \in \mathcal{M}(B)$ is strictly continuous for each $S \in \mathcal{M}(B)$, thus with $S := \delta_\infty(d)$ we get that $\{\delta_\infty(d)\}' \cap \mathcal{M}(B)$ is strictly closed for every $d \in \mathcal{M}(B)_+$. It implies that $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ is strictly closed as intersection of strictly closed C^*-subalgebras. It contains a non-degenerate copy of $\mathbb{K}(\ell_2)$ by Remark 5.1.1(8). Therefore, there is a strictly continuous unital *-monomorphism $\rho \colon \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2) \to \mathcal{M}(B)$ such that the image of ρ commutes with $\delta_\infty(\mathcal{M}(B))$.

If $t_1, t_2, \ldots$ is a sequence of isometries in $\mathcal{M}(B)$ (with $\sum t_n t_n^* = 1$ strictly) then $\rho(t_1), \rho(t_2), \ldots$ are isometries in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ (with $\sum \rho(t_n)\rho(t_n)^* = 1$ strictly).

(ii): Let the isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ the generators of a copy of $\mathcal{O}_\infty$ as established in Remark 5.1.1(8). By part (i), there exists a unital *-monomorphism Λ from $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$ into $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$. Take isometries $R_1, R_2 \in \mathcal{L}(\ell_2)$ with $R_1R_1^* + R_2R_2^* = 1$ and let $r_k := \Lambda(R_k) \in \mathcal{M}(B)$ for $k = 1, 2$. Then $C^*(r_1, r_2)$ is a copy of $\mathcal{O}_2$ in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ that contains $1_{\mathcal{M}(B)}$.

The map $\delta_\infty \oplus_{r_1,r_2} \mathrm{id}_{\mathcal{M}(B)}$ is an infinite repeat $\delta'_\infty(b) := \sum_n t_n b t_n^*$ of the identity map $\mathrm{id}_{\mathcal{M}(B)}$ of $\mathcal{M}(B)$ that is given by the sequence of isometries $t_1 := r_2$, $t_{n+1} := r_1 s_n$ for $n = 1, 2, \ldots$, because the series $\sum t_n t_n^* = r_2 r_2^* + r_1(\sum s_n s_n^*) r_1^*$ converges strictly to 1. It follows that $\delta_\infty \oplus_{r_1,r_2} \mathrm{id}_{\mathcal{M}(B)}$ is unitarily equivalent to δ_∞ by (i).

(iii): We use here (and only here !) the notation $p < q$ for projections if $p \leq q$ and $p \neq q$.

Where next is really used? For what?

Let $p_0 := 1 \otimes e_{11} \in D$ and a $p \in D := \mathcal{O}_2 \otimes \mathbb{K}$ non-zero projection. Since D has an approximate unit consisting of projections we find projections $r, s \in D$ and a unitary $v_1 \in \mathcal{U}_0(D + \mathbb{C} \cdot 1)$ in the connected component of 1 inside the unitaries in $D + 1$ such that $p_0 < r < s$, $p < r < s$, $v_1(1 - s) = (1 - s)$ and $v_1^* p_0 v_1 = p$.

Is r really necessary for something above?

Let $p, q, r, s \in D$ with $p < q < s$ and $p < r < s$. Then there exists a continuous map $t \in [0, 1] \mapsto v(t) \in D$ with $v(t)^*v(t) = v(t)v(t)^* = v(0) = s - p$ and $v(1)^* q v(1) = r$.

Use below that $\mathcal{O}_2$ has property (sq), because $\mathcal{O}_2 = \mathcal{E}_2/\mathbb{K}$.
It implies: $\mathcal{O}_2$ is K_1-bijective.

Every projection in $\mathcal{O}_2$ is properly infinite,
and
δ_2 is homotopic to id.
(Implies $\mathrm{K}_*(\mathcal{O}_2) = 0$.

This happens because, by [**172**], any two projections $P, Q \in \mathcal{O}_2 \setminus \{0,1\}$ are unitarily equivalent in $\mathcal{O}_2 \cong (s-p)D(s-p)$, and $\mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$.

The unitary equivalence of non-trivial projections in $\mathcal{O}_2$ implies that all non-zero projections in $D = \mathcal{O}_2 \otimes \mathbb{K}$ are MvN-equivalent, thus $(s-p)D(s-p)$ is isomorphic to $\mathcal{O}_2$, and we can pass from $\mathcal{O}_2$ to $(s-p)D(s-p)$.

If there is $n_0 \in \mathbb{N}$ such that $q_n = q_{n_0}$ for all $n \geq n_0$, then we find projections $r, s \in D$ and a continuous map $t \mapsto w(t) \in D + 1$ into the unitaries of $D + 1$ such that $w(0) = 1$, $q_{n_0} < r < s$ and $q := w(1)^* p_0 w(1) < s$.

Thus $0 < r < s$ and $0 < q < s$, and there is a continuous map $t \mapsto v(t) \in D$ with $v(t)^* v(t) = v(t)v(t)^* = v(0) = s$ and $v(1)^* q v(1) = r$. Let $U(t) := w(t)(v(t) + (1-s))$ for $t \in [0,1]$ and $U(t) = U(1)$ for $t > 1$. The map $t \mapsto U(t)$ is a continuous path in the $\mathcal{U}_0(D + \mathbb{C}1) \cap (1 + D)$ of the desired type.

If the sequence $q_1 \leq q_2 \leq \ldots$ becomes not stationary, than we may assume that $0 < q_n < q_{n+1}$ for all $n \in \mathbb{N}$, – simply by re-indexing the non-equal q_n and changing the parameter $t \in \mathbb{R}$. Therefore we find continuous maps $t \in [0,1] \mapsto v_n(t) \in D$ into the partial unitaries of D such that $v_n(t)^* v_n(t) = v_n(t) v_n(t)^* = v_n(0) = q_{n+3} - q_n$ and $v_n(1)^* q_{n+1} v_n(1) = q_{n+2}$.

We find a projection $s \in D$ with $0 < q_3 < s$ and a continuous map $t \mapsto w(t)$ from $[0,1]$ into the unitaries of $D + 1$, such that $q := w(1)^* p_0 w(1) < s$, $w(0) = 1$. Then there is a continuous map $t \mapsto v_0(t) \in D$ with $v_0(t)^* v_0(t) = v_0(t) v_0(t)^* = v_0(0) = s$ and $v_0(1)^* q v_0(1) = q_3$.

Let $U(t) := w(t)(v_0(t) + (1-s))$ for $t \in [0,1]$ and, by induction, $U(n+t) := U(n)(q_{n+2} + v_{n+1}(t) + (1 - q_{n+4}))$ for $t \in [0,1]$ and $n = 1, 2, \ldots$. Then $t \mapsto U(t)$ is a continuous map from $\mathbb{R}_+$ to the unitaries in $D + 1$, with $U(0) = 1$, $U(n)^* p_0 U(n) = q_{n+2}$ and $q_n \leq U(t)^* p_0 U(t)$ and $U(t)^* p_0 U(t) \neq q_n$ for $t \in [n, n+1]$.

(iv): As noticed in Remark 5.1.1(8) (combined with part (i)), there is a *non-degenerate* *-monomorphism $\psi\colon \mathbb{K} \to \mathcal{M}(B)$ such that $\mathcal{M}(\psi)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ is a unital strictly continuous *-monomorphism with image in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$. We identify $\mathcal{M}(\mathbb{K})$ with such a copy $\mathcal{M}(\psi)(\mathcal{M}(\mathbb{K}))$ of $\mathcal{M}(\mathbb{K})$ in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$.

Since $\ell_2 \otimes_2 \ell_2 \cong \ell_2$ there is a non-degenerate copy D of $\mathcal{O}_2 \otimes \mathbb{K}$ contained in $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$, i.e., $\overline{D\mathbb{K}} = \mathbb{K}$ and the inclusion $\mathcal{M}(D) \subseteq \mathcal{M}(\mathbb{K})$ is unital. $p_0 := 1_{\mathcal{O}_2} \otimes p_{11}$ is in D, but is not in $\mathbb{K}$. There exists an increasing sequence of projections $q_0 < q_1 < q_2 < \cdots$ in $1_{\mathcal{O}_2} \otimes \mathbb{K} \subseteq D$ with $q_0 \leq q_1$, $q_1 - q_0 \neq 0$, $q_1 - q_2 \neq 0$, and $\lim_{n\to\infty} \|q_n k - k\| = 0$ for every $k \in \mathbb{K}$. It follows that (q_n) converges strictly to $1 \in \mathcal{M}(B)$.

By (iii) there is a norm-continuous map $t \mapsto U(t)$ into the unitaries of $D + 1 \subseteq \mathcal{M}(\mathbb{K})$ such that $q_n \leq U(t)^* p_0 U(t)$ for $t \geq n$.

In $\mathcal{M}(\mathbb{K}) = \mathcal{L}(\ell_2)$ there are isometries T and S with $p_0 T = p_0$, $TT^* = q_1$, $SS^* = 1 - q_1$, because $1 - p_0$, $q_1 - p_0$ and $1 - q_1$ are not in $\mathbb{K}$ (recall: $D \cap \mathbb{K} = \{0\}$). Let $S_0(t) := U(t)^* S U(t)$ and $T_0(t) := U(t)^* T U(t)$. Then $q_n S(t) = q_n (U(t)^* p_0 U(t)) U(t)^* S U(t) = q_n U(t)^* p_0 \, S U(t) = 0$ and $q_n T(t) =$

$q_n(U(t)^*p_0\,U(t))U(t)^*TU(t) = q_n(U(t)^*p_0\,U(t)) = q_n$ for $t > n$, and the maps $t \to S_0(t), T_0(t) \in \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B)$ are norm-continuous.

Thus, $\lim_{t\to\infty} S_0(t)^*q_nb = 0$ and $\lim_{t\to\infty} T_0(t)^*q_nb = q_nb$ for every $b \in B$ and $n \in \mathbb{N}$.

The desired result follows, because $\bigcup_n(q_nB)$ is dense in B.

(v): Let $S(t) := gS_0(t)$ where $S_0(t) \in \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ is as in (iv). Then $t \mapsto S(t)$ is a norm-continuous map from $[0,\infty)$ into the contractions in $\mathcal{M}(B)$,

$$(\delta_\infty \circ T)(c) - S(t)^*cS(t) = S_0(t)^*(\delta_\infty(T(c)) - g^*cg)S_0(t) \in B$$

for $t \in \mathbb{R}_+$, and $\lim_{t\to\infty} \|S(t)^*cS(t) - \delta_\infty(T(c))\| = 0$.

If $1 \in C$ and $T(1) = 1$, then $1 - S(t)^*S(t) = S_0(t)^*(\delta_\infty(T(1)) - g^*1g)S_0(t)$ is in B and tends to zero in norm. Since the norm-limit of $S(t)^*S(t)$ is 1, there is $t_0 \geq 0$ such that $S(t)^*S(t)$ is invertible for every $t \geq t_0$. Then $t \to (S(t_0+t)^*S(t_0+t))^{-1/2} \in B$ is norm-continuous, $S'(t) := S(t_0+t)(S(t_0+t)^*S(t_0+t))^{-1/2}$ is norm-continuous and satisfies $S'(t)^*S'(t) = 1$, $S'(t)^*cS'(t) - \delta_\infty(T(c)) \in B$ and $\lim S'(t)^*cS'(t) = \delta_\infty(T(c))$ for $c \in C$.

(vi): The statement is independent of the chosen copy and generators $\{s_1, s_2\}$, $\{t_1, t_2\}$ of $\mathcal{O}_2$ in $\mathcal{M}(B)$, because the unital completely positive map $\mathrm{id}_C \oplus_{s_1,s_2} T$ is unitarily equivalent to $\mathrm{id}_C \oplus_{t_1,t_2} T$ by the unitary $w = t_1s_1^* + t_2s_2^*$. In the following $\oplus$ is formed with help of the same generators of $\mathcal{O}_2 \subseteq \mathcal{M}(B)$. Let $V := \delta_\infty \circ T$. By (ii), there is a unitary $U_0 \in \mathcal{M}(B)$ with $U_0^*V(\cdot)\,U_0 = T \oplus V$. If we have found a norm-continuous map $t \mapsto U_1(t)$ into the unitaries of $\mathcal{M}(B)$ with $V(c) \oplus c - U_1(t)^*c\,U_1(t) \in B$ and $\lim_{n\to\infty} \|V(c) \oplus c - U_1(t)^*c\,U_1(t)\| = 0$ for all $c \in C$, then

$$U(t) := U_1(t)(U_0 \oplus 1)U_d(1 \oplus U_1(t)^*)$$

is as desired, where U_d is a unitary in $\mathcal{U}_0(\mathcal{O}_2)$ with $U_d^*(a \oplus b) \oplus c)U_d = a \oplus (b \oplus c)$ as in Proposition 4.3.2(iv).

Thus, it is enough to prove the existence of $U(t)$ for $V = \delta_\infty \circ T$ (in place of T). By (iv) there exists a norm continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ into the isometries of $\mathcal{M}(B)$ such that $S(t)^*c\,S(t) - V(c) \in B$ for every $t \in \mathbb{R}_+$, and $\lim_{n\to\infty} \|V(c) - S(t)^*c\,S(t)\| = 0$ for all $c \in C$. The relative commutant of $V(C)$ in $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, by Part (ii).

Now we prove a **more general result** than stated in Part (vi):
Suppose that C is a C^*-subalgebra $C \subseteq \mathcal{M}(B)$ with $1_{\mathcal{M}(B)} \in C$ and that V is a unital C^*-morphism $V\colon C \to \mathcal{M}(B)$ such that there are a unital copy of $\mathcal{O}_2$ in $V(C)' \cap \mathcal{M}(B)$ and a norm-continuous map $t \mapsto S(t)$ from $\mathbb{R}_+$ into the contractions of $\mathcal{M}(B)$ with $S(t)^*c\,S(t) - V(c) \in B$ and $V(c) = \lim S(t)^*c\,S(t)$. Then there is a norm-continuous map $t \mapsto U(t)$ from $\mathbb{R}_+$ into the unitaries of $\mathcal{M}(B)$ such that $c \oplus V(c) - U(t)^*c\,U(t) \in B$ for every $t \in \mathbb{R}_+$ and

$$\lim_{t\to\infty} \|\, c \oplus V(c) \;-\; U(t)^*c\,U(t)\,\| = 0 \qquad \text{for } c \in C\,.$$

By $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))$ we denote the bounded continuous maps from $\mathbb{R}_+$ into $\mathcal{M}(B)$. We let $E := \mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))/\mathrm{C_0}(\mathbb{R}_+, B)$ and consider $\mathcal{M}(B)$ as a subalgebra of E. Then the assumptions imply that $C \hookrightarrow E$ dominates the C^*-morphism $V: C \to E$ and that $V(C)' \cap E$ contains a copy of $\mathcal{O}_2$ unitally. Thus, by Proposition 4.3.5(i) there is a unitary $u \in E$ such that $u^*cu = c \oplus V(c)$ for $c \in C$.

Let $t \mapsto W(t)$ a contraction in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))$ which represents u. Then $1 - W(t)^*W(t)$ and $1 - W(t)W(t)^*$ are in B, $\lim \|1 - W(t)W(t)^*\| = 0$, $\lim \|1 - W(t)^*W(t)\| = 0$. Thus there exists $t_0 \geq 0$ with $\|1 - W(t)^*W(t)\| \leq 1/2$ and $\|1 - W(t)W(t)^*\| \leq 1/2$ for $t \geq t_0$. Let $U(t) := W(t+t_0)(W(t+t_0)^*W(t+t_0))^{-1/2}$ for $t \geq 0$. Then $t \mapsto U(t)$ is as desired.

(vii): Let $g := \delta_\infty(f)$, $p_k := s_k s_k^*$ and $q_n := \sum_{k \leq n} p_k$. Recall $B_0 := \overline{f^*Bf}$ and $B_1 := \overline{g^*Bg}$. We have $p_k g = g p_k = s_k f s_k^*$, $p_k B_1 \subseteq B_1$ $B = s_j^*(s_j B s_k^*) s_k \subseteq s_j^* B s_k \subseteq B$, and $p_j g B g p_k = s_j f^* s_j^* B s_k f s_k^* = s_j f^* B f s_k^*$. Thus, $p_j B_1 p_k = s_j B_0 s_k^*$ and $s_j^* B_1 s_k = B_0$. Since $q_n g = g q_n$, $\|q_n g^* b g - g^* b g\| \leq \|g\|^2 \|(q_n b - b) g\| \to 0$ for $n \to \infty$ and $b \in B$, we get $q_n B_1 + B_1 q_n \subseteq B_1$ and $\lim_{n\to\infty} \|q_n b q_n - b\| = 0$ for all $b \in B_1$.

Let $\psi(b) := [s_j^* b s_k]_{j,k} \in \mathcal{M}_\infty(B_0)$. The restriction of ψ to $q_n B_1 q_n$ defines a *-isomorphism from $q_n B_1 q_n$ onto $M_n(B_0)$. It follows that ψ is a *-isomorphism from B_1 onto $B_0 \otimes \mathbb{K} \subseteq M_\infty(B_0)$ because $\lim_{n\to\infty} \|q_n b q_n - b\| = 0$ for all $b \in B_1$.

Let $\rho \colon B \to \mathbb{C}$ a positive state with $\rho(f^*ef) = 0$. Since $b \mapsto \rho(f^*bf)$ is a positive functional on B and e is strictly positive, we get $\rho(f^*Bf) = \{0\}$ and $\rho(B_0) = \{0\}$, i.e., f^*ef is a strictly positive element of B_0. In the same way one gets that g^*eg is strictly positive in B_1. □

2. Some properties of corona C^*-algebras

HERE STARTS DISCUSSION ON σ-sub-Stonean C^*-algebras:

DEFINITION 5.2.1. A C^*-algebra E is **sub-Stonean** (also called **SAW*-algebra**) if, for each positive contractions $a, b \in E_+$ with $ab = 0$, there exists a positive contraction $c \in E_+$ with $ca = a$ and $bc = 0$ ([6]).

???? Wrong Definition ?:

The C^*-algebra E is a **σ-sub-Stonean** C^*-algebra if $A' \cap E$ is sub-Stonean for every *separable* C^*-subalgebra A of E.

????

My comment:

This (!) is an other property than given in my original definition in [**448**, def. 1.4].

This original definition seems to be at least formally “stronger”? It is unclear if they are equivalent.

[6] In particular, for each $e \in E_+$ there exist a positive contraction $f \in E_+$ with $fe = e = ef$.

"Recall" that a C^*-algebra C is an SAW*-algebra if any two σ-unital C^*-subalgebras A and B of C are "orthogonal" (i.e., $A \cdot B = \{0\}$) if and only if they are "separated" (i.e., there exists positive contractions $e, f \in C$ with $ef = 0$, $A = eAe$ and $B = fBf$. $\Leftarrow$ is perhaps not the original definition of SAW*-algebras?

This applies to $E := C^*(e)$ and $F := C^*(f)$. It causes that there exists contractions $g, h \in C_+$ with $gh = 0$, $F = gFg$ and $G = hGh$. It implies that $\rho_\varphi \cdot \varphi(f) \leq g$ for each $\varphi \in \mathrm{C}_0(0,1]_+$ with suitable $\rho_\varphi \in (0,1]$. Does it imply that there exists a positive contraction $T \in \overline{gCg}$ with $Tf = f = fT$?

What is it the correct Definition of sub-Stonean C^*-algebras? (Look to Pedersen and others?)

???? Definition ???? A C^*-algebra C is an SAW*-algebra if any two σ-unital subalgebras A and B of C are "orthogonal", if and only if, they are "separated".

Original ?: positive f, g?: [**292**, def. 2.9]

My Definition [**448**, def. 1.4] in "green":

We call a C^*- algebra C σ-sub-Stonean if for every separable C^*-subalgebra $A \subseteq C$ and every $b, c \in C_+$ with $bc = 0$ and $bAc = \{0\}$ there are positive contractions $f, g \in A' \cap C$ with $fg = 0$, $fb = b$ and $gc = c$.

Definition [**292**, def. 2.9]:

A C^*-algebra C is σ-sub-Stonean if for every separable subalgebra A of C and all positive b and c in C such that $bAc = \{0\}$ there are contractions f and g in $A' \cap C$ such that $fg = 0$, $fb = b$ and $gc = c$.

Gives $fb = b$ and $fc = 0$, $f^*fg = 0$, $f^*fc = 0$ and $fP_b = P_b$.

Missing:

$bc = 0$ (becomes stronger, because of weaker assumptions)

and

Positivity of f, g (becomes weaker conclusion).

Therefore difficult to compare!

Is it really equivalent to my Def?

(Implies: $fc = 0$, $f = v(f^*f)^{1/2}$, $(f^*f)^{1/2}g = 0$, $v(f^*f)^{1/2}P_b = P_b$???? . f^*f has properties: $f^*fg = 0$, $fP_b = P_b$. Is $g^*c = c$? $gP_c = P_c$ with P_c support of c.

Can here replace A by $C^*(A, b)$ then it covers partly our Definition ... but I can not see that one can deduce from [**292**, def. 2.9] that the commutant $A' \cap C$ is again σ-sub-Stonean as it is in our Definition ...

The definition [**292**, def. 2.9] is not identical to my Definition:

A C^*-algebra C is σ-sub-Stonean if, for every separable subalgebra A of C and all positive b and c in C such that $bAc = \{0\}$, there are contractions f and g in $A' \cap C$ such that $fg = 0$, $fb = b$ and $gc = c$.

Definitions [**292**, def. 2.5] of Farah and Hart:

Two subalgebras A, B of an algebra C are **orthogonal** if $ab = 0$ for all $a \in A$ and $b \in B$.

They are **separated** if there is a positive element $c \in C$ such that $cac = a$ for all $a \in A$ and $cb = 0$ for all $b \in B$.

My definition [**448**, def. 1.4] is stated here again ([7]):

DEFINITION 5.2.2. We call a C^*-algebra C **σ-sub-Stonean** if for every separable C^*-subalgebra $A \subset C$ and every $b, c \in C_+$ with $bc = 0$ and $bAc = \{0\}$ there are positive contractions $f, g \in A' \cap C$ with $fg = 0$, $fb = b$ and $gc = c$.

My comments:

Obviously, if C is σ-sub-Stonean, then C is sub-Stonean (case $A := \{0\}$), and $B' \cap C$ is σ-sub-Stonean for every separable C^*-subalgebra B of C (consider $C^*(B, A)$ in place of A in the definition).

It is easy to see, that if D is a hereditary C^*-subalgebra of C, then D is σ-sub-Stonean, if and only if, for every $d \in D_+$ there is a positive contraction $e \in D$ with $ed = d$.

Proof of last:
There is a positive contraction e in D with $eb = b$, $ec = c$ and $ea = a$ for all $a \in A$. Then $bC^*(A, e)c = \{0\}$. If f', g' are positive contractions in C that commute with $C^*(A, e)$ and satisfy $f'g - 0$, $fb = b$ and $g'c = c$, then $f := f'e$ and $g = g'e$ in I_+ are as desired.

In particular, $\mathrm{Ann}(d, C)$ is σ-sub-Stonean for every $d \in C_+$ if C is σ-sub-Stonean.

Further, if C is σ-sub-Stonean and $I \triangleleft C$ is a σ-sub-Stonean closed ideal of C, then C/I is σ-sub-Stonean. (An exercise.)

Proof of last:
Let $A \subset C/I$ separable, $b, c \in (C/I)_+$ with $bc = 0$ and $bAc = \{0\}$, $A_1 \subset C$ separable with $\pi_I(A_1) = A$, $f \in C$ a self-adjoint contraction with $f = b^{1/2} - c^{1/2}$, d a strictly positive element of $C^*(f_+ A_1 f_-) \subset I$. Let $e \in I$ a positive contraction with $ed = d$, and let $b' := f_+(1-e)f_+, c' := f_-(1-e)f_- \in C_+$. Then $b'A_1c' = \{0\}$, $b'c' = 0$ and $\pi_I(b') = b$, $\pi_I(c') = c$. Since C is σ-sub-Stonean there are $f', g' \in A_1' \cap C$ with $f'b' - b'$ and $g'c' = c'$. $f := \pi_I(f')$ and $g := \pi_I(g')$ are as desired.

If C is sub-Stonean, then C satisfies almost a sort σ-variant for this for given:

There are always contractions $g, h \in C_+$ with $gh = 0$ that are "quasi-invariant" under the unitaries in $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(C)$, – in the sense that e.g. $u^*gu \in \overline{gCg}$ for all $u \in \mathcal{U}(A + \mathbb{C} \cdot 1)$ and satisfy $b \in \overline{gCg}$ and $c \in \overline{hCh}$.

Definition of Farah and Hart ([**292**, def. 2.9]):

[7] It was modified or wrongly displayed in [**292**, def. 2.9] to something that ?????? seems to be even not equivalent to [**448**, def. 1.4].

A C^*-algebra C is σ-sub-Stonean if for every separable subalgebra A of C and all positive b and c in C such that $bAc = \{0\}$ there are contractions f and g in $A' \cap C$ such that $fg = 0$, $fb = b$ and $gc = c$.

My question: Each SAW*-algebra (Def.???) has this property?

Let B the separable C^*-algebra generated by AcA and $e \in B_+$ a strictly positive contraction in B. Then $be = 0$ and there are positive contractions $f, g \in C_+$ with $fg = 0$, $bf = b$ and $ge = e$. It follows $gbc = bc$ for all $b \in B$

Definition 5.2.3. The algebra E is σ_2**-sub-Stonean** if for every separable C^*-subalgebra A of E, every positive contraction $e \in A' \cap E$ and every sequence $g_1, g_2, \ldots$ of positive contractions with $eg_n = 0$, $g_{n+1}g_n = g_n$ and $\lim_n \|g_n a - a g_n\| = 0$ for all $a \in A_+$ there exists a positive contraction $f \in A' \cap E$ with $fg_n = g_n$ for all $n \in \mathbb{N}$ and $fe = 0$.

`Next blue argument not clear !`

(If one can take a subsequence $h_k = g_{n_k}$ and then takes $g := \sum 2^{-n} g_n$ or $G := \sum_k 2^{-n_k}(h_{k+1} - h_k)$, then $ge = 0$ and $Ge = 0$, the proposed positive contraction $f \in A' \cap E$ satisfies $fg = g$ – which is equivalent to $fg_n = g_n$ for each $n = 1, 2, \ldots$ –, and each $a \in A$ derives $D = \overline{gEg}$ into D:

$$ageg - gega = (ag - ga)eg + (gaeg - geag) + (ge(ag - ga)) \ \in D$$

for each $a \in A$ and $e \in E$, because for all $a \in A$ $ga = \sum_n 2^{-n} g_n a \in D$ by $\lim_k \|(g_n a - a g_n) g_k\|$.)

A C^*-algebra C has **AA-CRISP** (**asymptotically abelian, countable Riesz separation property**) if the following holds:

Assume a_n, b_n, for $n \in \mathbb{N}$, are positive elements of C such that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

for all n. Furthermore assume that D is a separable *subset* of C such that for every $d \in D$ we have

$$\lim_n \|[a_n, d]\| = 0 .$$

Then there exists a positive $c \in C$ such that $a_n \leq c \leq b_n$ for all n and $[c, d] = 0$ for all $d \in D$.

The algebra E satisfies **Kasparov's technical property** (**KTP**) if E has the following property:

If B_1 and B_2 are σ-unital C^*-subalgebras of E with $B_1 \cdot B_2 = \{0\}$ and $\Delta \subset E$ is a separable subspace with $xb - bx \in B_1$ for all $x \in \Delta$ and $b \in B_1$, then there exists a positive contraction $f \in E_+$ such that $f \cdot B_1 = \{0\}$, $(1 - f) \cdot B_2 = \{0\}$ and $fx = xf$ for all $x \in \Delta$.

We say that a C^*-algebra M has **KTP** if the following holds:
Assume A, B, and C are subalgebras of M such that $A \perp B$ and C derives B. Furthermore assume A and B are σ-unital and C is separable. Then there is a

positive element $d \in M$ such that $d \in C' \cap M$, the map $x \mapsto xd$ is the identity on B, and the map $x \mapsto dx$ annihilates A.

Every derivation of a separable subalgebra of M is of the form δ_b for some $b \in M$.

(Latter is Part of Definition? Is it ∂_b?)

Kasparov's Technical Theorem: [**73**, thm.12.4.2]

Let J be a σ-unital C^*-algebra. Let A_1 and A_2 be σ-unital C^*-subalgebras of $\mathcal{M}(J)$, and Δ a separable subspace of $\mathcal{M}(J)$. Suppose $A_1 \cdot A_2 \subseteq J$, and that Δ derives A_1. Then there are $M, N \in \mathcal{M}(J)$ such that

$$0 \leq M \leq 1,\ \ N = 1 - M,\ \ M \cdot A_1 \subseteq J,\ \ N \cdot A_2 \subseteq J,\ \ \text{and}\ \ [M, \Delta] \subseteq J\,.$$

The result can fail if A_1 and A_2 are not σ-unital, even if B_1? and B_2? are commutative and $\Delta = 0$ [Choi and Christensen 1983]:

Kasparov's Technical Theorem [**73**, thm.12.4.2] says – in an equivalent formulation – that $E := \mathrm{Q}(A) := \mathcal{M}(A)/A$ has "Kasparov's Technical Property" if A is σ-unital.

If B_1 and B_2 are orthogonal σ-unital C^*-subalgebras of E with $B_1 \cdot B_2 = \{0\}$ and Δ a separable subspace with $[x, b] \in B_2$ for all $b \in B_2$ and $x \in \Delta$. Then we can find strictly positive contractions $e_1 \in B_1$ and $e_2 \in B_2$ and we get $[x, b] \in \overline{e_1 E e_1}$ for all $x \in \Delta$ and $b \in B_1$.

The algebras B_1 and B_2 can not be completely re-discovered from (e_1, e_2, Δ). But if we find a positive contraction $T \in E_+$ with $Te_2 = e_2$ and $Te_1 = 0$, and $[T, d] = 0$ for all $d \in \Delta$ then $f := T$ satisfies Kasparov's technical property for (B_1, B_2, Δ).

We can conversely consider two positive $e_1, e_2 \in E_+$ with $e_1 e_2 = 0$ and a countable subset X of E with the property that $[x, e_1] \in \overline{e_1 E e_1}$ for all $x \in X$. (E.g. by taking strictly positive contractions $e_1 \in A_1$ and $e_2 \in A_2$).

If we find a contraction $T \in E_+$ with $Tx = xT$ for all $x \in X$, $Te_2 = e_2$ and $Te_1 = 0$. Then T has the required properties.

In fact it suffices to find $T \in C^*(X)' \cap E$, $\|T\| \leq 1$, $T \geq 0$ and $Te_1 = 0$, $Te_2 = e_2$.

If $e_1, e_2 \in E_+$ with $e_1 e_2 = 0$ and a countable subset X of E with the property that $[x, e_1] \in \overline{e_1 E e_1}$ for all $x \in X$ are given, then one can consider the C^*-algebras $B_2 := C^*(e_2)$ and define Δ as the linear span of X. A C^*-algebra B_1 can be defined as closure of $\bigcup_n B_{1,n}$, where the $B_{1,n}$ are inductively defined by $B_{1,1} := C^*(e_1)$, $B_{1,n+1} := C^*(B_{1,n}, [y, x], x \in X, y \in ???)$.

Then e_1 is a strictly positive element of B_1 and $[B_1, y] \subseteq B_1$ for all y in the linear span of X.

Assuming that E satisfies the Kasparov technical property, we get a contraction $f \in E_+$ with $f \cdot B_1 = \{0\}$, $f \in C^*(X)' \cap E$ and $fe_1 = e_1$.

A C^*-algebra C has AA-CRISP (asymptotically abelian, countable Riesz separation property) if the following holds:

Assume a_n, b_n, for $n \in \mathbb{N}$, are positive elements of C such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Furthermore assume D is a separable subset of C such that for every $d \in D$ we have

$$\lim_n \|[a_n, d]\| = 0\,.$$

Then there exists a positive $c \in C$ such that $a_n \leq c \leq b_n$ for all n and $[c, d] = 0$ for all $d \in D$.

So far we have not seen examples that separate the stronger definitions from the sub-Stonean C^*-algebras. For abelian C^*-algebras the definitions are equivalent.

(But compare the study [**292**] of I. Farah and B. Hart in the case of corona algebras.:

" We present unified proofs of several properties of the corona of σ-unital C^*-algebras such as $AA-CRISP$, SAW^*, being sub-σ-Stonean in the sense of Kirchberg, and the conclusion of Kasparov's Technical Theorem.

Although our results were obtained by considering C^*-algebras as models of the logic for metric structures, the reader is not required to have any knowledge of model theory of metric structures (or model theory, or logic in general). The proofs involve analysis of the extent of model-theoretic saturation of corona algebras.

Countable saturation of corona algebras Ilijas Farah, Bradd Hart,
arXiv:1112.3898v2 (revised 17 Oct 2012)

Related properties of coronas of σ-unital stable C^*-algebras A is given in the following proposition.

PROPOSITION 5.2.4. *Let A a stable σ-unital C^*-algebra, $D \subset A$ a σ-unital hereditary C^*-subalgebra.*

(i) *D is unitary homotopic to a corner of A.*
(ii) *The corona $\mathrm{Q}(A) := \mathcal{M}(A)/A \cong \mathrm{Q}^{\mathrm{s}}(A)$ is is a σ_2-sub-Stonean C^*-algebra (in sense of Definition 5.2.3).*

PROOF. (i): `To be filled in` **??** (ii): It is essentially Kasparov's lemma. □

QUESTION 5.2.5. Let B a sub-Stonean C^*-algebra (in the sense that for each self-adjoint contraction $c \in B$ there exists a self-adjoint contraction $d \in B$ such that $dc = |c|$ (equivalently expressed: for $c_1, c_2 \in B_+$ with $c_1c_2 = 0$ exist contractions $d_1, d_2 \in B_+$ with $d_1d_2 = 0$ and $d_kc_k = c_k$ for $k \in \{1, 2\}$.)

Let $e \in B_+$ a positive contraction, $D := \overline{eBe}$ and define the (two-sided) **normalizer** of D in B by

$$\mathcal{N}(D) := \{b \in B\,;\; bD \cup Db \subseteq D\}\,.$$

It is easy to see that $\mathcal{N}(D)$ is a C^*-subalgebra of B and that $\psi_D(c)d := cd$ (for $d \in D$ and $c \in \mathcal{N}(D)$) defines a natural C^*-morphism $\psi_D \colon \mathcal{N}(D) \to \mathcal{M}(D)$ from $\mathcal{N}(D)$ into the (abstract two-sided) multiplier algebra $\mathcal{M}(D)$ of D. It is unital, because there exists a contraction $f \in B_+$ with $fe = e$.

Is $\psi_D(\mathcal{N}(D))$ sub-Stonean?

Clearly, the kernel of ψ_D is equal to the two-sided annihilator

$$\mathrm{Ann}(D) := \mathrm{Ann}(B, D) := \{b \in B\,;\; bD = Db = \{0\}\,\}$$

of D in B, which is a sub-Stonean hereditary C^*-subalgebra of B.

The interesting *question* says:
Is $\psi_D \colon \mathcal{N}(D) \to \mathcal{M}(D)$ surjective?

I.e., $\mathcal{M}(D)/D \cong \mathcal{N}(D)/\,\mathrm{Ann}(D)$ in a natural way.

It would give a nice new characterization of sub-Stonean C^*-algebras.

QUESTION 5.2.6. Let B a sub-Stonean C^*-algebra. A special case of Question 5.2.5 is the following question:

Let $b_1, b_2, \ldots \in B_+$ a sequence of mutually orthogonal contractions.

Does B_+ contain an contraction $c \in B_+$ such that c is contained in the in $\mathcal{N}(D)$ for $D := \overline{dBd}$ with $d := \sum_n 2^{-n} b_n$ and $c(b_n - (b_n - 1/2)_+) = (b_n - 1/2)_+$ for all $n \in \mathbb{N}$?

It seems (!) that Question 5.2.5 is equivalent to the following question:

QUESTION 5.2.7. Let B a sub-Stonean C^*-algebra and $a \in B_+$. Let $D := \overline{aBa}$. Suppose that $d_1, d_2, \ldots \in D_+$ are mutually orthogonal and such that $d_1 + d_2 + \cdots$ converges strictly to an element in $T \in \mathcal{M}(D)_+$.

Something more elaborate is needed ?

Does there exist an element $b \in B_+$ that satisfies $bD \cup Db \subseteq D$ and

$$\lim_n \|b(a - (1 - 1/n))_+ a - \sum_{k \le n} d_n (a - (1 - 1/n))_+ a\| = 0\,?$$

3. Functions that respect quasi-central approximate units

If we want to verify in applications the estimates that appear in the assumptions of the Parts (2)-(6) of Remark 5.1.1, then the below given Proposition 5.3.1 and its consequence, the Lemma 5.3.2, are useful, despite the fact that also the arguments in the proof of [**43**, thm. 2] can be used without controlling bound. This argument of Arveson in [**43**] is in essence the following:

Let K a convex subset of the positive contractions in A that contains an approximate unit of A, and $\varphi_1, \varphi_2, \ldots \in \mathrm{C}_0(0, 1]_+$ a given sequence of continuous function.

We need an element $e \in K$ with $\|y - ey\| \leq \varepsilon\|y\|$ for y in a given finite-dimensional subspace $Y \subseteq A$ that satisfies at the same time the commutation properties $\|[e,x]\| \leq \varepsilon\|x\|$ and $\|[\varphi_k(e),x]\| \leq \varepsilon\|x\|$ for $x \in X$ and $k = 1,\ldots,n$, where $X \subseteq \mathcal{M}(A)$ is a finite-dimensional linear subspace. This can be done by solving following inequalities by an inductive selection procedure with n^{th} step as follows:

First choose for each given $\varepsilon > 0$, n_0 and the considered continuous function $\varphi_k(\lambda)$ with $\varphi_k(0) = 0$, $k \leq n$, a polynomial $P_k(\lambda)$ with $P_k(0) = 0$ and $|\varphi_k(\lambda) - P_k(\lambda)| < \varepsilon/2$ for $\lambda \in [0,1]$ and then select from before constructed approximate units e_n with $e_n e_m = e_{\min m,n}$ a finite convex combination e of elements $e_{n_1},\ldots,e_{n_k}$ ($n_0 < n_1 < n_2 < \ldots < n_k$) such that $\|[e,x]\| \leq \eta\|x\|$ for x in a given finite-dimensional subspace $X_k \subseteq \mathcal{M}(A)$ in a (before chosen) filtration of A or of some separable subspace, where η can be taken e.g. as $\eta := \varepsilon/(2\max_{1\leq k\leq n}\{1 + Q_k'(1)\})$ with polynomials $Q_k(\lambda)$ build from $P_k(\lambda)$ by replacing all coefficients of P_k by its absolute values and $Q_k' := d/d\lambda\, Q_k$ is the derivative of Q_k. Those estimates are not perfect.

See [**11**, lem. 10.3] for the case of the polynomials $P(\lambda) := \lambda^n$.

Notice also that by [**11**, lem. 10.4], $T^* = T$, $\|T\| < 1$ and bounded X:

$$\|(1-T^2)^{1/2}X - X(1-T^2)^{1/2}\| \leq (\|T\| \cdot \|XT - TX\|)/(1 - \|T\|^2)^{1/2}\,.$$

Unfortunately we have to consider the case where $\|T\|$ is near to 1.

In our application we need a construction that produces a sufficiently fast commutation property such that sub-selections satisfy automatically better commutation and approximate-unit properties, i.e., we need to check estimates of commutator-norms for commutators of elements in given different convex subsets of an algebra. This must be used later to verify the quadratic unconditional strict convergence of some series build by selections from given sequences of elements in the multiplier algebra.

A control of such general estimate is possible by the above described inductive selection. But we need a general control of the commutators for all needed functions f itself, e.g. by suitable bounds for the – with respect to $t \in [0,1]$ non-decreasing – function

$$G(f,\cdot)\colon\ [0,1] \ni t \mapsto G(f,t) \in [0,\infty]\,,$$

defined by

$$G(f,t) := \sup\{\, \|[x,f(a)]\|\,;\ (a,x) \in P(t)\,\}$$

for the set of pairs

$$P(t) := \{\,(x,a)\,;\ x,a \in \mathcal{L}(\ell_2),\ \|x\| \leq 1,\, 0 \leq a \leq 1,\, \|xa - ax\| \leq t\,\}\,.$$

Here f is a given non-decreasing continuous real function on $[0,1]$. The definition shows that $t \mapsto G(f,t)$ is the minimal function that satisfies for given f the inequalities

$$\|[x,f(a)]\| \leq G(f,\|[x,a]\|)$$

for all contractions $x, a \in \mathcal{L}(H)$ with $a \geq 0$. Below we observe that it suffices to consider only self-adjoint contractions $x = x^*$ to define $G(f,t)$.

`Where` $G_u(f,t)$ `is used?`

`Where` $G_u(f,t)$ `is used?` To get applicable estimates of $G(f,t)$ it could be useful to consider also the functions $G_u(f,t)$ defined by

$$G_u(f,t) := \sup\{ \|u^* f(a) u - f(a)\| \,;\ (a,u) \in PU(t) \}$$

where $PU(t)$ is defined as the set of pairs

$$PU(t) := \{ (u,a) \,;\ u, a \in \mathcal{L}(H) \ \text{with}\ u^*u = 1 = uu^*, 0 \leq a \leq 1, \|[a,u]\| \leq t \},$$

i.e., it considers only differences $\|f(b) - f(a)\|$ of functions of unitary transformations $b := u^* a u$. Those have often easier estimates if the function f is operator monotone, e.g. for $f(t) = t^{1/2}$ it is easy to see that $\|f(b) - f(a)\| \leq f(\|b - a\|)$.

Always $\|[b,x]\| \leq \|[b_-, x]\| + \|[b_+, x]\| \leq 2\|x\|(\|b_-\| + \|b_+\|)$ for all $x \in \mathcal{L}(H)$

Is always $\|[b,x]\| \leq \|x\|\|b\|$ and each positive $b \in \mathcal{L}(H)$?

It implies that for $b \in \mathcal{L}(H)_+$ and $x \in \mathcal{L}(H)$ holds that

$$\|[b,x]\| \leq \inf_{t \in \mathbb{R}_+} \|[(b-t)_-, x]\| + \|[(b-t)_+, x]\| \,.$$

Is $\|[b,x]\| \leq \|x\|\|b\|$ for $b \in \mathcal{L}(H)_+$ and any $x \in \mathcal{L}(H)$ by the following estimates ??:

For each $\delta > 0$ there are vectors $\theta, \xi \in H$ with norm $= 1$ such that $\langle [b,x]\theta, \xi \rangle \geq 0$ and $\|[b,x]\| \leq \delta + \langle [b,x]\theta, \xi \rangle$. Notice that $\langle [b,x]\theta, \xi \rangle = \langle b\theta, x^*\xi \rangle - \langle x\theta, b\xi \rangle \geq 0$

It is a non-negative difference of two complex numbers of norm $\leq \|x\|\|b\|$. It follows that it is also a non-negative difference of two real numbers of norm $\leq \|x\|\|b\|$. (Still it could be $= 2\|x\|\|b\|$.)

If P is the orthogonal projection onto the linear span of $\{\theta, \xi, x\theta, x^*\xi\}$ then $\langle [b,x]\theta, \xi \rangle = \langle bP\theta, P^2 x^* P\xi \rangle - \langle P^2 x P\theta, bP\xi \rangle \geq 0$. The latter is equal to $\langle [PbP, PxP]\theta, \xi \rangle \leq \|[PbP, PxP]\|$.

Thus $\|[b,x]\| \leq \delta + \|[PbP, PxP]\|$ and we can reduce all to the case of $H = \mathbb{C}^4$.

Can suppose that $0 \in \operatorname{Spec}(PbP)$.

Let $P \in \mathcal{L}(H)$ an orthogonal projection that commutes with b and satisfies $Pb = b_+$ and $(1-P)b = b_-$. Then $[b,x] = bPx - xPb + b(1-P)x - x(1-P)b$ ...

??? says $b_+x - xb_+ + b_-x - xb_- = [b,x]$

and then ????? used where ???

`check above and next again:`

This ????????? can be seen by restriction to the case of self-adjoint contractions $X^* = X$ and $B \geq 0$, if we replace b by $B := \operatorname{diag}(\gamma + b, \gamma + b) \in M_2(\mathcal{L}(H)) \cong \mathcal{L}(H)$ with $\gamma := \|b_-\|$ and replace x by $X := \operatorname{diag}(x^*, x) \cdot Z \in M_2(\mathcal{L}(H))$, where $Z^* = Z := [\zeta_{jk}] \in M_2$ is defined by $\zeta_{jk} := 1 - \delta_{jk}$ and satisfies $Z^2 = 1$, $Z \operatorname{diag}(y,z) =$

$\mathrm{diag}(z,y)Z$. Thus, $ZBZ = B$, $ZX = \mathrm{diag}(x,x^*)$, $XZ = \mathrm{diag}(x^*,x)$. and $X^* = Z \cdot \mathrm{diag}(x,x^*) = X$.

It follows that $G(f,\cdot)$ restrict to the case of self-adjoint contractions $X = X^*$, because Then $\|[x,b]\| = \|[X,B]\|$ and $\|[x,f(b)]\| = \|[X,f(B)]\|$.

$B\,\mathrm{diag}(x^*,x) = \mathrm{diag}((\gamma+b)x^*, (\gamma+b)x)$,

$ZBX = \mathrm{diag}((\gamma+b)x, (\gamma+b)x^*)$,

$ZB = Z\,\mathrm{diag}((\gamma+b),(\gamma+b)) = \mathrm{diag}((\gamma+b),(\gamma+b))Z = BZ$,

$Z(XB - BX) = \mathrm{diag}(x,x^*)\cdot \mathrm{diag}(\gamma+b,\gamma+b) - \mathrm{diag}((\gamma+b)x,(\gamma+b)x^*)$ is the diagonal matrix with entries $x(\gamma+b)-(\gamma+b)x$ and $x^*(\gamma+b)-(\gamma+b)x^*$. Thus $\|XB - BX\| = \|x(\gamma+b)-(\gamma+b)x\| = \|[b,x]\|$ if $b^* = b$, $\gamma := \|b_-\|$ and $X := \mathrm{diag}(x^*,x)\cdot Z = X^*$.

Since one can X replace here again by $X + \|X_-\| \geq 0$ without changing the norms, it shows that all estimates can be done using positive operators in $\mathcal{L}(\ell_2)$.

In particular it suffices to consider self-adjoint contractions x and positive a if we want to find an estimate for $G(f,t)$.

`Check next arguments again`

Always $G(f,0) = 0$, because $f(a)$ commutes with x if $a = a^*$ commutes with x and $f(t)$ is a function on the spectrum of a. Moreover, if $t_1 > t_2 > \cdots$ we find (a_n,x_n) with $\|[x_n,a_n]\| \leq t_n$ and $G(f,t_n) - 2^{-n} < \|[x_n,f(a_n)]\| \leq G(f,t_n)$. The elements $a := (a_1,a_2,\ldots) + c_0(\mathcal{L}(H))$ and $x := (x_1,x_2,\ldots) + c_0(\mathcal{L}(H))$ of $\ell_\infty(\mathcal{L}(H))/c_0(\mathcal{L}(H))$ generate a C^*-subalgebra $C^*(a,x)$ that can be faithfully represented as a C^*-subalgebra of $\mathcal{L}(H)$, then $\|[x,a]\| \leq t_\infty := \lim_n t_n$ and $\|[x,f(a)]\| = \limsup_{n\to\infty} G(f,t_n) = \lim_n G(f,t_n)$. Since $t \mapsto G(f,t)$ is increasing, and $\|[x,f(a)]\| \leq G(f,t_\infty) \leq G(f,t_n)$, it follows that the monotone increasing function $t \mapsto G(f,t)$ is continuous from the right side.

(i.e., is upper semi-continuous ?) What about the left side?

If f is continuous, then $t \mapsto G(f,t)$ is continuous, i.e., is also continuous from the left side: Let $t_1 < t_2 < \ldots$ in $[0,1]$ and $t_\infty := \lim_n t_n$. Suppose $G(f,t_\infty) > \sup_n G(f,t_n)$ and let $0 < \varepsilon < G(f,t_\infty) - \sup_n G(f,t_n)$.

By definition of $G(f,t_\infty)$, there exists $(a,x) \in P(t_\infty)$ with $G(f,t_\infty) - \varepsilon < \|[x,f(a)]\| \leq G(f,t_\infty)$. If f is continuous on $[0,1]$, then $\lim_{\delta\to 0} \|f((1-2\delta)(a+\delta 1)) - f(a)\| = 0$.

Since $[x,(1-2\delta)(a+\delta 1)] = (1-2\delta)[x,a]$, and $G(f,(1-2\delta)t_\infty) \leq \sup_n G(f,t_n)$ and

$$\|[x,f((1-2\delta)(a+\delta 1))]\| \leq \sup_n G(f,t_n) \leq G(f,t_\infty) - \varepsilon \leq G(f,t_\infty)\,.$$

If we suppose that f is continuous, then $\|f((1-2\delta)(a+\delta 1)) - f(a)\| \to 0$ for $0 < \delta \to 0$. Thus, $\sup_n G(f,t_n) = G(f,t_\infty)$. This implies that $t \in [0,1] \mapsto G(f,t) \in [0,1]$ is continuous on $[0,1]$. The continuity implies – by the Lemma of Dini on automatic uniform convergence from below applied to $G(f,\cdot)$ on $[0,1]$ – that the value $G(f,t)$ is determined by the pairs (a,x) in $P(t)$ with a and b operators of finite rank.

Some words about lower bounds for $G(f,\cdot)$:

It is easy to check that $f(t) \leq G(f,t)$, e.g. $a(t) := ts_2s_2^* + s_3s_3^*$ and $x := s_1s_2^*$ of $\mathcal{O}_\infty = C^*(s_1,s_2,\ldots) \subseteq \mathcal{L}(\ell_2)$ satisfy $\|x\| = 1$, $\|a(t)\| = 1$, $a(t) \geq 0$ and $\|[x, f(a(t))]\| = f(t)$, $\|[x,a(t)]\| = t$. This shows that $f(t) \leq G(f,t)$ for all $t \in [0,1]$. Alternatively one can take elements $a := \mathrm{diag}(0,t,1) \in M_3$ and $x := [\xi_{jk}] \in M_3$ with $\xi_{13} := 1$ and $\xi_{jk} = 0$ for $(j,k) \neq (1,3)$.

If we allow in case of M_2 also the case with $\|a\| < 1$, then $a(t) := \mathrm{diag}(t,0)$ and the above defined self-adjoint orthogonal matrix $Z \in M_2$ again satisfy $\|[Z, f(a(t))]\| = f(t)$ and $\|[Z,a(t)]\| = t$ for $t \in [0,1]$.

The above reduction to the case of self-adjoint $x = x^*$ and $b \geq 0$ shows also that $\|xb - bx\| \leq \|x\|(\|b_+\| + \|b_-\|)$ for each $b^* = b = b_+ - b_-$, and any $x \in \mathcal{L}(\ell_2)$, because for $x^* = x$ and $b^* = b$ the element $T := i(bx - xb)$ is self-adjoint and its norm is given by the supremum of $|\langle Tv, v\rangle| = |\langle xv, bv\rangle - \langle bv, xv\rangle|$ for $v \in H := \ell_2$ with $\|v\| = 1$. To see that $|\langle Tv, v\rangle| \leq \|x\|\|b\|$, consider the orthogonal projection $P \in \mathbb{K}$ of rank ≤ 2 from H onto the linear span of v and xv. Then

$$|\langle Tv, v\rangle| = |\langle PxPv, bPv\rangle - \langle bPv, PxPv\rangle| = |\langle (PbPxP - PxPbP)v, v\rangle|\,.$$

This reduces the claim to the case where b and x are in M_2: If $\mathrm{rank}(P) = 1$ we get $\langle Tv, v\rangle = 0$, and otherwise PbP is positive and PxP is self-adjoint in $p(\mathbb{K})p \cong M_2$. It is easy to see that $\|bx - xb\| \leq 1$ for contractions $b, x \in M_2$ with $b \geq 0$, because one can reduce the general case to the case where b is diagonal and x has zero diagonal.

If we allow both of b and x to be non-positive but self-adjoint contractions, then $x := Z$ with $Z \in M_2$ as above defined and $b := \mathrm{diag}(1,-1)$ are self-adjoint contractions that satisfy $\|[x,b]\| = \|x\|(\|b_-\| + \|b_+\|) = 2$.

Since one can for positive $a \in M_2$ and $f \geq 0$ reduce the considerations to the case of diagonal a and $x = \mathrm{diag}(\alpha,\beta)\cdot Z$, one gets that $\|[x, f(a)]\| \leq f(\|[x,a]\|)$ for all contractions $x, a \in M_2(\mathbb{C})$ with $a \geq 0$ and any continuous non-decreasing f on $[0,1]$ with $f(0) \geq 0$.

Def. of P and ?

$\|[x, f(a)]\| = i\langle xv,\ f(a)v\rangle = i\langle PxPv, f(a)Pv\rangle \geq \|[PxP, Pf(a)P]\|$
for $P = P^*P$, $Pv = v$, $Pxv = xv$???
and $\|[PxP, Pf(a)P]\| \leq$??? if $Pf(a)P \geq f(PaP)$?

Some determination of $G(f,t)$ should be possible in other cases because one can restrict the definition to $a, x \in M_n(\mathbb{C})$ and get $G_n(f,t) \leq G(f,t)$ and $G(f,t) = \sup_n G_n(f,t)$ with uniform convergence by the Lemma of Dini. This reduction to M_n can be seen as follows:

The positive contractions in $c_0 \subseteq \mathbb{K}(\ell_2)$ contain a approximate unit (e_n) of $\mathbb{K}$ that is (in norm) quasi-central for $\mathcal{L}(\ell_2)$, i.e., for each separable subset of $X \subset$

$\mathcal{L}(\ell_2)$ there exists a suitable sub-sequence that approximately commutes with the elements of X and has the following properties: If $0 \leq e_n \leq 1$ satisfy $e_n e_m = e_m$ for $1 \leq m < n$ (that implies that each e_n is automatically of finite rank) and $e_n \to 1$ strictly in $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ and $\|[e_n, a]\| + \|[e_n, f(a)]\| \to 0$ for $n \to \infty$, then $\|[a, e_n x e_n]\| \to \|[a, x]\|$ and $\|[f(a), e_n x e_n]\| \to \|[f(a), x]\|$ for $n \to \infty$.

Thus, it suffices to consider self-adjoint contractions $x = x^* \in \mathbb{K}$ of finite rank to determine the value of $F(f, t)$. After this reduction we can modify the approximate unit (e_n) such that in addition $e_n x = x = x e_n$ for all n, and get $[e_n a e_n, x] = e_n [a, x] e_n$ and $\lim_n \|f(e_n a e_n) - f(a e_n^2)\| = 0$, $\lim_n \|f(e_n a e_n) - f(e_n^2 a)\| = 0$, $\lim_n \|f(e_n a e_n) - f(a e_n^2)\| = 0$, $\lim_n \|f(a e_n^2) x - f(a) x\| = 0$ and $\lim_n \|x f(a e_n^2) - x f(a)\| = 0$, but $[f(e_n a e_n), x] = [e_n^2 f(a) e_n^2, x]$ and $\|e_n^2 a^2 e_n^2 - (e_n a e_n)^2\| \to 0$ for $n \to \infty$. We obtain finally the convergences $\|[f(e_n a e_n), x]\| \to \|[f(a), x]\|$ and $\|[e_n a e_n, x]\| \to \|[a, x]\|$. Together with the continuity of $t \mapsto F(f, t)$ this shows that $F(f, t) = \lim_n F_n(f, t)$.

T.A. Loring and F. Vides study in [**541**] among others the special case of $f(t) = t^{1/2}$ and get good reason for the *conjecture* that $G(f, t) = t^{1/2}$ for $f(t) = t^{1/2}$, that was also suggested by G.K. Pedersen in [**619**]. Related results up to a (there not explicitly given) constant can be obtained from results of A.B. Aleksandrov and V.V. Peller, e.g. in [**11, 12**]. An estimate by Pedersen was given in [**619**, cor. 6.3]: If $0 \leq a \leq 1$ and $\|b\| = 1$, then $\|[a, b]\| \leq \varepsilon \leq 1/4$ implies $\|[a^{1/2}, b]\| \leq (1.25)\varepsilon^{1/2}$.

We give explicit estimates in the following Proposition 5.3.1, that are likely not the best possible, but its proof can be obtained easily by elementary observations.

PROPOSITION 5.3.1. *Let H a Hilbert space, and suppose that $\varphi(t)$ a continuous function on $[0, \infty)$ that is operator-monotone in $(0, \infty)$ and satisfies $\varphi(0) \geq 0$.*

For all positive $a, b \in \mathcal{L}(H)$,

$$\|\varphi(b) - \varphi(a)\| \leq \varphi(\|b - a\|) .$$

If $u \in \mathcal{L}(H)$ is unitary and $a \geq 0$, then

$$\| [u, \varphi(a)] \| \leq \varphi(\|[u, a]\|) . \tag{3.1}$$

If $a, x \in \mathcal{L}(H)$ are contractions with $a \geq 0$, then

$$\| [x, \varphi(a)] \| \leq 3\varphi(\| [x, a] \|) . \tag{3.2}$$

In particular, for all $a \geq 0$, every contraction x and each $\beta \in [1, \infty)$,

$$\|[x, a^{1/\beta}]\| \leq 3\|[x, a]\|^{1/\beta} .$$

PROOF. It suffices to consider $a_\varepsilon := (1 - 2\varepsilon)(\varepsilon + a)$ in place of a, because

$$x a_\varepsilon - a_\varepsilon x =: [x, a_\varepsilon] = (1 - 2\varepsilon)[x, a] ,$$

and $\varphi(\|[x, a]\|) = \lim_{\varepsilon \to 0} \varphi(\|[x, a_\varepsilon]\|)$ by continuity of φ on $[0, \infty)$, and it implies $[x, \varphi(a)] = \lim_{\varepsilon \to 0} [x, \varphi(a_\varepsilon)]$ by uniform continuity of φ on bounded subsets of $[0, \infty)$.

Notice that $\mathrm{Spec}(a_\varepsilon) \subseteq [(1-2\varepsilon)\varepsilon, \infty) \subset (0,\infty)$. Therefore we can use the variant of the Löwner theorem given by F. Hansen [**356**, thm. 4.9, rem. 5.3]:

For every positive operator monotone function φ, defined in the positive half-line $(0,\infty)$, there is a unique bounded positive measure μ on the closed interval $[0,1]$ such that for $t \in (0,\infty)$, $\lambda \in [0,1]$ and the functions

$$f_\lambda(t) := (\lambda + (1-\lambda)t)^{-1}t = \big(1 - \big(1 + ((1-\lambda)/\lambda)t\big)^{-1}\big)/\big(1-\lambda\big)$$

holds

$$\varphi(t) = \int_0^1 f_\lambda(t)\,\mathrm{d}\mu(\lambda) \quad \text{for } t > 0\,.$$

Any function given by such a measure is operator monotone, and the measure μ is a probability measure if and only if $\varphi(1) = 1$. The functions $f_\lambda \colon t \mapsto f_\lambda(t)$ are easily seen operator monotone, in particular the f_λ are monotone.

Let $\eta > 0$ and $a, b \in \mathcal{L}(\ell_2)_+$ such that $\eta \le a$ and $\eta \le b$ and let $\gamma := \|b-a\|$. Since f_λ is operator monotone and $b \le a + \gamma$, we get $f_\lambda(b) \le f_\lambda(a+\gamma)$. Moreover f_λ has the property $f_\lambda(t+\gamma) \le f_\lambda(t) + f_\lambda(\gamma)$ for $\gamma > 0$, because the function $x \to x/(1+x)$ is sub-additive for $x \in [0,\infty)$. Thus we get

$$f_\lambda(b) \le f_\lambda(a+\gamma) \le f_\lambda(a) + f_\lambda(\gamma)\,.$$

If we interchange here a and b it implies together that $\|f_\lambda(b) - f_\lambda(a)\| \le f_\lambda(\gamma)$. If we use the above integral representations of $\varphi(a)$ and $\varphi(b)$, we obtain the estimate

$$\|\varphi(b) - \varphi(a)\| \le \int_0^1 f_\lambda(\gamma)\,\mathrm{d}\mu(\lambda) = \varphi(\gamma) = \varphi(\|b-a\|)\,.$$

To get that $\|\varphi(b) - \varphi(a)\| \le \varphi(\|b-a\|)$ for arbitrary positive $a, b \in \mathcal{L}(\ell_2)$, we can consider $\delta + a$ and $\delta + b$ for $\delta > 0$ and then use that φ is uniformly continuous on bounded parts of $[0,\infty)$, e.g. that

$$\lim_{0<\delta\to 0} \|\varphi(a) - \varphi(\delta + a)\| = 0\,.$$

If u is unitary and $a \ge 0$, then $u\varphi(a)u^* = \varphi(uau^*)$ can be seen easily with help of uniform approximation of φ by polynomials on the interval $[0, \|a\|]$. For $b := u^*au$ holds $\|ua - au\| = \|b - a\|$ and $\|u\varphi(a) - \varphi(a)u\| = \|\varphi(b) - \varphi(a)\|$. This implies inequality (3.1).

To find an estimate of $\|[x,\varphi(a)]\|$ for a contraction x and $a \ge 0$ by $\|[x,a]\|$, we can replace a by the positive element $A := \mathrm{diag}(a,a) \in M_2(\mathcal{L}(\ell_2))$ and x by the self-adjoint contraction $X := \mathrm{diag}(x, x^*) \cdot Z \in M_2(\mathcal{L}(\ell_2))$, because then $\|[X,A]\| = \|[x,a]\|$ and $\|[X,\varphi(A)]\| = \|[x,\varphi(a)]\|$ ([8]).

Let $x^* = x$, $b \in \mathcal{L}(\ell_2)$ with $\|x\| < 1$ and $b \ge 0$. The **Cayley transformation** of $x \in \mathcal{L}(\ell_2)$ with $\|x\| < 1$ and $x^*x = xx^*$ is defined as the continuous operator-valued function

$$u(\lambda) := (\lambda + x)(1 + \lambda x^*)^{-1}$$

for $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$, cf. [**616**, p. 4]. Then $u(\lambda)$ is analytic for $|\lambda| < 1$, $u(0) = x$, $u(e^{is})$ is unitary for $s \in [0, 2\pi]$.

[8] Here $Z \in M_2$ has entries $z_{j,k} := 1 - \delta_{j,k}$.

Is this next blue general study useful here?

The general Cayley transformation is given for all $x \in \mathcal{L}(\ell_2)$ with $\|x\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ by

$$u(\lambda) := (1 - xx^*)^{-1/2}(\lambda 1 + x)(1 + \lambda x^*)^{-1}(1 - x^*x)^{1/2}.$$

Then $u(\lambda)$ is unitary for $|\lambda| = 1$ and $x = u(0)$. Moreover $\|u(\lambda)\| < 1$ for all $|\lambda| < 1$ (the latter by maximum modulus principle). See [**222**] or [**402**, ex. 10.5.5].

As in the considerations below, we get for $v_1, v_2 \in \mathcal{H} := \ell_2$ and $\rho(y) := \langle yv_1, v_2\rangle$ that

$$\rho([x,b]) = (2\pi)^{-1} \int_0^{2\pi} \rho([u(e^{is}), b]) \, \mathrm{d}s.$$

??????

Since $2\pi u(0) = \int_0^{2\pi} u(e^{is}) \, \mathrm{d}s$, we get

$$[x,b] = (2\pi)^{-1} \int_0^{2\pi} [u(e^{is}), b] \, \mathrm{d}s.$$

It implies $(2\pi) \cdot \|[x,b]\| \le \int_0^{2\pi} \|[u(e^{is}), b]\| \, \mathrm{d}s$. Hence, $\|[x,b]\| \le \sup_s \|[u(e^{is}), b]\|$.

If we let here $b := \varphi(a)$ then inequality (3.1) implies that

$$\|[x, \varphi(a)]\| \le \sup\{ \varphi(\|[u(e^{is}), a]\|) \,;\; s \in [0, 2\pi] \}.$$

We let $v(\lambda) := (\lambda + x)$ and $w(\lambda) := (1 + \lambda x^*)^{-1} = \sum_{n=0}^{\infty} (\lambda x^*)^n$. Then $u(\lambda) = v(\lambda)w(\lambda)$, and, for all $b \in \mathcal{L}(\ell_2)$, $[b, v(\lambda)] = [b, x]$, $[b, w(\lambda)] = -\lambda w(\lambda)[b, x^*]w(\lambda)$ and

$$[b, u(\lambda)] = [b, x]w(\lambda) + \lambda u(\lambda)[b, x^*]w(\lambda).$$

Since $u(\lambda)$ is unitary if $|\lambda| = 1$ we get for $x = x^*$ the estimate

$$\|[u(e^{is}), b]\| \le 2\|[x, b]\| \|w(e^{is})\|.$$

The inequality $\|e^{is}x^*\| = \|x\| < 1$ implies $\|w(e^{is})\| \le (1 - \|x\|)^{-1}$. Thus,

$$\varphi(\|[u(e^{is}), a]\|) \le \varphi((1 - \|x\|)^{-1} 2\|[x, a]\|),$$

and for all $x^* = x$ with $\|x\| < 1$,

$$\|[x, \varphi(a)]\| \le \varphi((1 - \|x\|)^{-1} 2\|[x, a]\|).$$

Since the latter formula holds for all $x = x^*$ with norm $\|x\| < 1$, we can replace x by $3^{-1}x$ and get by monotony of φ the estimate

$$\|[x, \varphi(a)]\| \le 3\varphi(\|[x, a]\|).$$

□

Compare next blue with above given similar remarks!!

A particular case is the operator monotone function $\varphi(t) := t^{1/2}$, where the estimate shows that $\|[x, a]\|^2 \le 9\|[x, a^2]\|$. It gives that the above defined and discussed continuous function $G(t) := G(\varphi, t) \le 3t^{1/2}$ for $\varphi(t) := t^{1/2}$ – defined as the "minimal" function with $G(\|[a,x]\|) \geq \|[a^{1/2},x]\|$ for contractions $x, a \in \mathcal{L}(\ell_2)$ with $a \geq 0$ – satisfies

$$t^{1/2} \leq G(t) \leq 3t^{1/2}\,.$$

Numerical methods indicate (but do not prove) that in this special case it could be possible to improve the constant 3 down to 1 by other methods, compare [**541**]. This is in accordance with a suggestion of G.K. Pedersen in [**619**].

The following lemma gives for explicit estimates of unconditional strict convergence at least some useful upper bounds (that are perhaps not minimal).

LEMMA 5.3.2. *Let $x, y \in A$ and $a, b \in A_+$ contractions and $\gamma > 0$. Suppose that*

$$\max(\,\|[x,a^2]\|\,,\,\|[x,b^2]\|\,) \leq \gamma \quad \textit{and} \quad \|y - b^2y\| + \|y^* - b^2y^*\| \leq \gamma\,.$$

Then we get the estimates

$$\|[x,(1-b)a]\| \leq 6\gamma^{1/2} \quad \textit{and} \quad \|y - by\| + \|y^* - by^*\| \leq \gamma\,.$$

PROOF. Let $a, b \in A_+$ and $x \in A$ contractions. We get from Proposition 5.3.1 that

$$\max(\,\|[x,a]\|\,,\,\|[x,b]\|\,)^2 \leq 9\max(\|[x,a^2]\|, \|[x,b^2]\|) \leq 9\gamma\,.$$

This implies by $[(1-b)a,x] = (1-b)[a,x] - [b,x]a$ that

$$\|[(1-b)a,x]\| \leq \|[a,x]\| + \|[b,x]\| \leq 3\|[a^2,x]\|^{1/2} + 3\|[b^2,x]\|^{1/2} \leq 6\gamma^{1/2}\,.$$

The estimates for y follows from $e := (1-b)^2 \leq (1-b^2)^2 =: f$ if $0 \leq b \leq 1$, because e.g. $\|y - by\|^2 = \|y^*ey\| \leq \|y^*fy\| = \|y - b^2y\|^2$. □

4. A "tautologic" Weyl–von Neumann type result

We prove a very general "tautologic" version of the classical Weyl–von Neumann theorem: the below stated Proposition 5.4.1.

Applications will be obtained later by verifying in some special cases the validity of assumptions (α) and (β) given in Proposition 5.4.1. The assumptions are sufficient but are not necessary for some derived properties e.g. as in Part (iii) of Proposition 5.4.1. Applications consider cases where the C^*-algebra C does not contain an element h with the property (α) required in Proposition 5.4.1 or is not unital. But often one can build a larger algebra C_1 and a suitable extension T_1 of T that contains an element h that satisfies the required Conditions (α) and (β) with C_1 and T_1 in place of C and T. But this extension, e.g. to an outer unitization is not always possible and needs some care. Here is a warning concerning application in the theory of extension groups $\mathrm{Ext}(\mathcal{C}\,;\,C, B)$: The nontrivial part in those applications – by trying to establish the assumptions of Proposition 5.4.1 – consists in showing that condition (β) still holds also for the *extended* T_1 on the bigger C^*-algebra C_1.

Adjoining a unit to C is only possible if the considered c.p. map T on C satisfies that $\pi_B \circ T$ is "zero-absorbing", i.e., if $\pi_B \cdot T$ "dominates zero" in the more precise sense, as discussed in Chapter 4.

`Does the Elliott-Kucerovsky result follow??` – from the special case where $C = C^*(E, d)$, with $d \in B$ strictly positive, and $\pi_B(E)$ separable, $T(d) = 0$ and $[T] \colon \pi_B(C) \to \mathcal{M}(B)$ has the property that, for each $a \in B_+$, the map $e \in E \mapsto a[T](\pi_B(e))a$ or the map $f \in \pi_B(E) \mapsto a[T](f)a$ is nuclear?

I.e.: When does it satisfy moreover the Condition (β)?? (That we can then apply Prop. 5.4.1 to it.)

PROPOSITION 5.4.1. *Let B a σ-unital C^*-algebra, C a separable C^*-subalgebra of $\mathcal{M}(B)$, and $T \colon C \to \mathcal{M}(B)$ a linear map with $\|T\| \leq 1$ that satisfies the following Conditions (α) and (β):*

- (α) *There exists $h \in C_+$ with $T(h) = 0$ and $h^{1/n}d \to d$ if $n \to \infty$ for every $d \in B$.*
- (β) *For every $a \in B_+$, every finite subset $X \subset C_+$ of contractions, and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^*cd - aT(c)a\| < \varepsilon$ for $c \in X$.*

Then the contraction $T \colon C \to \mathcal{M}(B)$ with this properties is completely positive and has following properties:

- (i) *There exists a sequence S_n of contractions in $\mathcal{M}(B)$ such that, for all $c \in C$ and $k \in \mathbb{N}$, and, for all $m, n \in \mathbb{N}$,*

$$S_n^* c S_m - \delta_{m,n} T(c) \in B \qquad \textit{and} \qquad \lim_{n \to \infty} \| \delta_{k,0} T(c) - S_n^* c S_{n+k} \| = 0 \,.$$

- (ii) *If B is* stable *then the completely positive contraction $V := \delta_\infty \circ T$ satisfies again Conditions (α) and (β) (with V there in place of T) and there is a norm-continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ into the contractions of $\mathcal{M}(B)$ such that, for $s, t \in \mathbb{R}_+$, $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $c \in C$ holds*

$$S(s)^* c\, S(t) - \delta_{s,t} T(c) \in B \quad \textit{and} \quad \lim_{t \to \infty} S(t+k)^* c\, S(t) = \delta_{k,0} \cdot T(c) \,. \qquad (4.1)$$

- (iii) *If in addition to the properties in (ii) T is a C^*-morphism and $T(C)B$ is dense in B, then the monomorphism $\eta_C \colon c \in C \ni c \mapsto c \in \mathcal{M}(B)$ asymptotically absorbs T, i.e., the C^*-morphisms $(\mathrm{id}_C \oplus T) \colon C \to \mathcal{M}(B)$ and η_C are unitarily homotopic in the sense of Definition 5.0.1 modulo B.*
- (iv) *If B is stable, $1_{\mathcal{M}(B)} \in C$ and $T(1) = 1$, or if $\pi_B \circ T$ "dominates zero" in the sense of Definition 4.3.3 ([9]), then the path $t \to S(t)$ in Part (ii) can be modified such that Properties (4.1) still hold, but that $S(t)$ has the additional property that $\pi_B(S(t))$ is an isometry for all $t \in \mathbb{R}_+$.*

  If the operators $\pi_B(S(t))$ are isometries, $\pi_B \circ T$ is a C^*-morphism and $(\pi_B \circ T)(C)' \cap (\mathcal{M}(B)/B)$ contains a copy $C^*(s_1, s_2)$ of $\mathcal{O}_2$ unitally,

[9] Compare also Lemma 4.3.4(i,iv).

then there exists a unitary $U \in \mathcal{M}(B)/B$ *with*

$$U^*\pi_B(c)U \;=\; \pi_B(c) \oplus_{s_1,s_2} \pi_B(T(c)) \quad \textit{for all } c \in C\,.$$

(v) *The contractions* S_n *in Part (i), respectively the* $S(t)$ *in Part (ii), can be chosen as isometries if* $1_{\mathcal{M}(B)} \in C$ *and* T *is unital.*

Collection of remarks:

General remarks:

Check and reorganize the proof !!!

Items have been changed !!!

Condition (α) is satisfied by some $h \in C_+$ with $T(h) = 0$ if $B = R$ for the closed right ideal $R := \{d \in B\,;\ \lim_n h^{1/n}d = d\}$. In particular it suffices to check this for d in a subset of B that generates B as a closed right ideal.

It suffices for Condition (β) to check the existence of $d_a \in \mathcal{M}(B)$ with

$$\|d_a^* c d_a - aT(c)a\| < \varepsilon \quad \text{for each} c \in X_\gamma$$

and for each $a \in B_+$ in an approximate unit for B and a family $\{X_\gamma\}$ of finite subsets $X_\gamma \subset B_+$ that have the property that the linear spans $L(X_\gamma)$ build an upward directed net of linear subspaces of B with $\bigcup_\gamma L(X_\gamma)$ dense in B. This implies then that T is a point-wise approximately inner completely positive map.

Are there weaker formulation of Condition (β) possible? e.g. :

(New β): For every $a \in B_+$, every pair of positive contractions $c_1, c_2 \in C_+$ and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$, – depending on $(a, c_1, c_2, \varepsilon)$ – with

$$\|d^* c_j d - aT(c_j)a\| < \varepsilon \quad \text{for } j \in \{1, 2\}$$

We can replace the element $d \in \mathcal{M}(B)$ in Condition (New β) or in Condition (β) by some $d \in B$, because $aT(c)a \in B$ for $a \in B_+$, $de_\lambda \in B$ and

$$\lim_\lambda \|aT(c)a \;-\; e_\lambda aT(c)ae_\lambda\| = 0$$

for any quasi-central approximate unit $\{e_\lambda\} \subseteq B_+$ of B ([10]).

Now we use Condition (α) and the positivity of T. They imply the following:

For any $\varphi, \psi \in \mathrm{C}_0(0,1]$ and each $c \in C$ we have that

$$T((1-\varphi(h))c(1-\psi(h))) = T(c),$$

i.e., $T(\varphi(h)c) = 0$ for all $c \in C$ and $\varphi \in \mathrm{C}_0(0,1]$.

For any finite subset $X \subseteq C_+$, $\varepsilon > 0$ and contraction $\varphi \in \mathrm{C}_0(0,1]_+$ there exists for each $x \in X$ an element $d_x \in B$ with $\|d_x^*(1-\varphi(h))x(1-\varphi(h))d_x - aT(x)a\| \,<\, \varepsilon\,.$

Since $\overline{hBh} = B$ by Condition (α), it implies that we can find a sequence of mutually orthogonal positive contractions $e_n \in C*(h)_+ \subseteq C \subset \mathcal{M}(B)$ such that,

[10] Quasi-central approximate units of B exist in B_+ for every C^*-algebra B.

for each $a \in B_+$ and $x \in C_+$, there exists a sequence of elements $d_k \in B$ and numbers $n_k \in \mathbb{N}$ such that

$$\|d_k^* e_{n_k} x e_{n_k} d_k - aT(x)a\| < 2^{-k} .$$

Unfortunately this seems only to work for each single element of finite subsets $X \subset C_+$.

Condition (α) implies that there exists a contraction $h \in C_+ \subseteq \mathcal{M}(B)$ with the property that for any given ??????

Question:

Is it enough to require in place of (β) for every $a \in B_+$, every pair contraction $c_1, c_2 \in C_+$ and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^* c_k d - aT(c_k)a\| < \varepsilon$?

Could it be that Condition (α) allows to use a kind of an approximate induction procedure:

Suppose we have verified (β) for all finite subsets $X \subset C_+$ with cardinality $\leq n$ (with fixed $n \geq 2$) and all $\varepsilon > 0$. Is it possible to derive then that (β) also holds for X with cardinality $\leq n+1$.

CONCERNING PASSAGE TO THE STABLE CASE:

We show first that the conditions (α) and (β) on a linear map $T : C \to \mathcal{M}(B)$ of norm $\|T\| \leq 1$ imply that T is completely positive (and $T = 0$ is possible).

Then each of the maps

$$T_1 : C \otimes 1 \ni c \otimes 1 \mapsto T(c) \otimes 1 \in \mathcal{M}(B \otimes \mathbb{K})$$

– where here 1 means the unit element $1_{\mathcal{M}(\mathbb{K})}$ of $\mathcal{M}(\mathbb{K})$ –, and

$$T_2 : C \otimes \mathbb{K} \ni c \mapsto (T \otimes \mathrm{id}_{\mathbb{K}})(c) \in \mathcal{M}(B) \otimes \mathbb{K} \subseteq \mathcal{M}(B \otimes \mathbb{K})$$

satisfy again the Condition (α) with $h \otimes 1 \in C \otimes 1$ for T_1 and for T_2 e.g. with $h \otimes q$ in place of h, where $q := \sum_n n^{-2} p_{nn}$.

Notice that T_1 is unitary equivalent to its infinite repeat, but T_2 is usually not, but the quotient C^*-morphism $\pi_{B\otimes\mathbb{K}} \circ T_2 : C \otimes \mathbb{K} \to \mathrm{Q}^{\mathrm{s}}(B)$ dominates zero (in sense of Definition 4.3.3).

The elements $a \in (B \otimes \mathbb{K})_+$ can be approximated in norm by elements in $(b_0 \otimes q)(B \otimes \mathbb{K})_+(b_0 \otimes q)$ with a strictly positive element $b_0 \in B_+$.

In case of $T_2 : C \otimes \mathbb{K} \to \mathcal{M}(B \otimes \mathbb{K})$ it is enough to consider finite sets $X \subset C \otimes \mathbb{K}$ containing only tensors $c \otimes p$ with $c \in C_+$ and $p \in \mathbb{K}$ a projection.

It follows that it suffices to check the Condition (β) for completely positive maps T_1 or T_2 (in place of T) only on those finite subsets $X \subseteq (C \otimes 1)_+$ that contain finitely many elements $c \otimes 1$ with $c \in C_+$ in case of T_1, and those finite subsets $X \subseteq (C \otimes \mathbb{K})_+$ that contain only elements $c \otimes p$ with $c \in C_+$ and $p \in \mathbb{K}$ in case of T_2 .

????? the elements $(b_0 \otimes q)^{1/n}$ in place of the required arbitrary elements $a \in (B \otimes \mathbb{K})_+$ in the test criterium for Condition (β).

The linear span of the elements $c \otimes p$ with $c \in C_+$ and $p \in \mathbb{K}$ a rank-one projection in $\mathbb{K}$ is dense in $X \subseteq (B \otimes \mathbb{K})_+$.

For every $a \in B_+$, every finite subset $X \subset C_+$ of contractions and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^*cd - aT(c)a\| < \varepsilon$ for $c \in X$.

$h \otimes (\sum_n n^{-2} p_{nn}) \in B \otimes \mathbb{K}$

in place of B,

??? map from $T \otimes \mathrm{id}_{\mathbb{K}}$ also the properties (α) and (β) for $B \otimes \mathbb{K}$ in place of B with $h \otimes (\sum_n n^{-2} p_{nn})$ in place of h and $d \otimes 1$ in place of d?

If we can shown that $\|T \otimes \mathrm{id}_{\mathbb{K}}\| \leq 1$, then it suffices to consider stable B.

We have anyway $\|T \otimes 1_{\mathcal{M}(\mathbb{K})}\| \leq 1$, because $\|c \otimes 1\| = \|c\|$.

Desire / hope more:

Ad(i): $S_k^* S_\ell = 0$?? Perhaps at least sufficient that only $S_k^* S_\ell \in B$ for $k \neq \ell$?

The strong hope is that (moreover) $S_1, S_2, \ldots$ can be found such that:

$S_j^* S_k = 0$ for $j \neq k$

Or at least: $S_j^* S_k \in B$ and $\lim_n S_{n+k}^* S_n = 0$ for $j \neq k$, $k \geq 1$. ??

Could be related to extendability to unital case $\tilde{T}$??

Reality check to (i):

Contractions $S_1, S_2, \ldots$.

$S_j^* c S_k \in B$ for all $c \in C$ (Suffices: $S_j^* c S_k - \delta_{jk} T(c) \in B$ For each $c \in L_m$ and $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$ and $\varepsilon > 0$ there exist $n := n(m, \varepsilon)$ such that $\|S_j^c S_k\| \leq \varepsilon \|c\|$ for all $j, k \geq n$ and $c \in L_m$.

and, for all $c \in C$ and $n, m \in \mathbb{N}$,

$$S_n^* c S_m - \delta_{n,m} T(c) \in B \qquad \text{and} \qquad \lim_{n \to \infty} \|\delta_{k,0} \cdot T(c) - S_{n+k}^* c S_n\| = 0\,.$$

In the special case where $S_n^* S_{n+1} = 0$, let $\alpha(t) := t - n + 1$ for $t \in [n-1, n]$ and

$$S(t) := \alpha(t) S_{n+1} + (1 - \alpha(t)^2)^{1/2} S_n\,.$$

Calc: $S(t)^* S(t) = \alpha^2 S_n^* S_n + (1 - \alpha^2) S_{n+1}^* S_{n+1} + 0$ if $S_n^* S_{n+1} = 0$.

If $S_{n+1}^* c S_n \in B$ and $\|S_{n+1}^* c S_n\| \leq 2^{-(n+1)} \|c\|$ and $\|S_n^* c S_n - T(c)\| \leq 2^{-n} \|c\|$ and $\|S_{n+1}^* c S_{n+1} - T(c)\| \leq 2^{-(n+1)} \|c\|$ for $c \in L_n$, then we get:

$S(t)^* c S(t) = \alpha^2 S_n^* c S_n + (1 - \alpha^2) S_{n+1}^* c S_{n+1} + \alpha(1 - \alpha^2)^{1/2} S_n^* c S_{n+1} + \alpha(1 - \alpha^2)^{1/2} S_{n+1}^* c S_n$

It has estimates $\|S(t)^* c S(t) - T(c)\| \leq \|c\| \cdot \big(\alpha^2 2^{-n} + (1 - \alpha^2) 2^{-(n+1)} + \alpha(1 - \alpha^2)^{1/2} 2^{-n}\big) \leq 2^{-n}$

The map $t \in \mathbb{R}_+ \mapsto S(t)$ is norm-continuous path in the contractions in $\mathcal{M}(B)$ and satisfies for $s, t \in \mathbb{R}_+$ and $c \in C$,

$$S(s)^* c S(t) - \delta_{s,t} T(c) \in B \qquad \text{and} \qquad \lim_{t\to\infty} \|\delta_{s,0} T(c) - S(t+s)^* c S(t)\| = 0.$$

Indeed: Let $s \in [m-1, m]$ and $t \in [n-1, n]$, $\beta := s - m + 1$, $\alpha := t - n + 1$, $S(s) = \beta S_m + (1 - \beta^{1/2})^{1/2} S_{m+1}$, $S(t) := \alpha S_n + (1-\alpha^2)^{1/2} S_{n+1}$. If $c \in \mathcal{M}(B)$, then $S(s)^* c S(t) = \alpha\beta S_m^* c S_n + \alpha(1-\beta^{1/2})^{1/2} S_{m+1}^* c S_n +$

$(1-\alpha^2)^{1/2} \beta S_m^* c S_{n+1} + (1-\alpha^2)^{1/2}(1-\beta^{1/2})^{1/2} S_{m+1}^* c S_n$.

To (ii): requires to distinguish the unital, non-unital, or stable case ??

It seems that second part of (ii) is not so different to the case of NON-STABLE B. Compare all with Part (iv) !!!

Ad (iii): ''$\pi_B \circ T$ is a C*-morphism'' enough?? The unital, non-unital, or stable can give different results??

Addition in (iii) requires that $\pi_B(S_n)$ resp. $\pi_B(S(t))$ is an isometry and that $\mathcal{O}_2$ is unitally contained in $\pi_B(T(C))' \cap \mathrm{Q}(B)$.

General remark to proof: Notice that for a verification of assumption (β) it suffices to consider only positive contractions $a := a_m$ in a given approximate unit $(a_m)_{m\in\mathbb{N}}$ of B.

Proof. The following are elementary consequences of assumptions (α) and (β): If $Y = \{y_1, \ldots, y_n\} \subset C$ is any finite subset of contractions in C, then we can pass to the finite set $X := \{x_{1,1}, \ldots, x_{1,4}, \ldots, x_{n,1}, \ldots, x_{n,4}\}$ of the at most $4n$ positive contractions $x_{k,1}$, $x_{k,2}$, $x_{k,3}$, $x_{k,4}$, given by the polar decompositions $x_{k,1} - x_{k,2} = 2^{-1}(y_k^* + y_k)$ and $x_{k,3} - x_{k,4} = i2^{-1}(y_k^* - y_k)$ of the real and imaginary parts of the $y_k \in Y$. An application of Condition (β) to X gives that there is $d \in \mathcal{M}(B)$ with $\|d^* y_k d - aT(y_k)a\| < 4\varepsilon$ for $k = 1, \ldots, n$.

If we apply this observation to the set $Y := \{c_{1,1}, c_{1,2}, \ldots, c_{nn}\}$ of entries $c_{j,k}$ of a positive contraction $[c_{j,k}] \in M_n(C)_+$ and take $a \in \{e_m\}$ for an approximate unit $e_m \in B_+$ of B, then we can see – with help of Condition (β) with $\varepsilon/(4n^2)$ in place of ε –, that there exists $d_m \in \mathcal{M}(B)$ for each $a := e_m$ and $\varepsilon := 1/m$ such that in $M_n(B)$ holds

$$\| (d_m \otimes 1_n)^* [c_{j,k}]_{n,n} (d_m \otimes 1_n) - (e_m \otimes 1_n)[T(c_{j,k})]_{n,n}(e_m \otimes 1) \| < m^{-1} .$$

In particular, the matrix $[T(c_{j,k})]$ is positive in $M_n(\mathcal{M}(B))$ for each positive matrix $[c_{j,k}] \in M_n(C)_+$, i.e., we get that *the map* $T\colon C \to \mathcal{M}(B)$ *is a completely positive contraction.*

Therefore, we can tensor it with $\mathrm{id}_{\mathbb{K}}$ and get a c.p. contraction $T_s := T \otimes \mathrm{id}_{\mathbb{K}}$ from $C \otimes \mathbb{K}$ into $\mathcal{M}(B) \otimes \mathbb{K} \subseteq \mathcal{M}(B \otimes \mathbb{K})$. Let $e := \sum 2^{-n} p_{nn} \in \mathbb{K}$.

The c.p. contraction $T_s \colon C \otimes \mathbb{K} \to \mathcal{M}(B \otimes \mathbb{K})$ satisfies again the conditions (α) and (β) with $C \otimes \mathbb{K}$, $B \otimes \mathbb{K}$ and $h \otimes e$ in places of C, B and h, but now with the estimate

$$\|T_s(c) - (d \otimes e^{1/n})^* c(d \otimes e^{1/n})\| < \varepsilon ,$$

where d comes from condition (β) for suitably chosen $n \in \mathbb{N}$.

It is clear that $T_s(h \otimes e) = 0$ and $\lim (h \otimes e)^{1/n} c = c$ for each $c \in B \otimes \mathbb{K}$.

We can replace the d in Condition (β) by $da^\gamma \in B$ for sufficiently small $\gamma > 0$. Thus, the element $d \in \mathcal{M}(B)$ in the inequality of Condition (β) for a given finite set $X \subset C_{s.a.}$ of selfadjoint contractions $x = x_+ - x_-$ in C and $a \in B_+$ can be always chosen such that $d \in B$.

It follows for fixed $a \in B_+$ that the c.p. map $c \in C \mapsto aT(c)a \in B$ is "approximately 1-step inner" in the strict topology of $\mathcal{M}(B)$ as a map from $C \subseteq \mathcal{M}(B)$ into B by assumption (β), and we can apply Lemma 3.1.8 by Remark 3.1.7 to this approximately 1-step inner map. It gives an element $d \in B$ that satisfies condition (β) with norm $\|d\| \leq \|a\|$, because the c.p. map $c \in C \mapsto aT(c)a \in B$ has norm $\|a\|^2 \|T\| \leq \|a\|^2$.

If $b \in B$ is not positive, then $a := (bb^*)^{1/4}$, $b_0 := a^{-1}b := \lim_n (bb^* + 1/n)^{-1/4} b$ with norms $\|a\| = \|b\|^{1/2} = \|b_0\|$ and there exists $d_0 \in B$, depending on a, X and $\varepsilon > 0$, with $\|d_0\| \leq \|a\|$ and $\|d_0^* x d_0 - aT(x)a\| < \gamma := \varepsilon/(1 + \|b\|^{1/2})$ for $x \in X$. Thus $d := d_0 b_0$ satisfies $\|d^* x d - b^* T(x) b\| \leq \|b\|^{1/2}\gamma \leq \varepsilon$ and $\|d\| \leq \|b\|$.

Hence, we can use in the following also the c.p. maps $b^* T(\cdot) b$ with non-selfadjoint $b \in B$ in place of $aT(\cdot)a$ in Condition (β), and find the element $d \in B$ with $\| d^* c\, d - b^* T(c) b \| < \varepsilon$ for $c \in X$ with $\|d\| \leq \|b\|$ and $d \in B \cdot b$.

Now we bring Condition (α) into play:
If $\varphi \in \mathrm{C}_0(0,1]_+$ is any non-negative continuous function with $\varphi(0) = 0$ and $\|\varphi\| \leq 1$, then we can find moreover an element $d \in B$ depending on X, $b \in B$, φ and ε, such that $\|d\| \leq \|b\|$ and

$$\| d^*(1 - \varphi(h)) x (1 - \varphi(h)) d - b^* T(x) b \| < \varepsilon \quad \text{for all } x \in X . \tag{4.2}$$

This holds because the assumption $T(h) = 0$ from Condition (α) and the positivity of T imply $T(c\varphi(h)) = 0$ for $c \in C$ and $\varphi \in \mathrm{C}_0(0,1]$, which gives that

$$T((1 - \varphi(h)) c (1 - \varphi(h))) = T(c) \quad \text{for all } c \in C . \tag{4.3}$$

To get the requested element d from Condition (β) – with a replaced by our non-selfadjoint b (as discussed above) –, we replace $X = \{x_1, \ldots, x_n\}$ by

$$gXg := \{ g x_1 g, \ldots, g x_n g \} ,$$

where we let $g := 1 - \varphi(h)$, and use Equation (4.3). We find some $d \in B$ depending on X, b and ε, with $\|d\| \leq \|b\|$ such that d solves the Inequality (β) for gXg and b in place of X and a, i.e., d solves the Inequality (4.2).

We can go one step further and consider the specific elements $h_n := \psi_n(h)$ for increasing pice-wise linear functions $\psi_n \in \mathrm{C}_c(0,1]$, as e.g. defined by

$$h_n := \psi_n(h) \quad \text{for} \quad \psi_n(t) := \min(1, \max(0, 2^n t - 1))\,. \tag{4.4}$$

Notice that the positive contractions $h_{n+1}h_n = h_n$ for all $n \in \mathbb{N}$. Above considerations show that we can use in place of the before considered general $\varphi(h)$ any of the h_n.

This allows to choose another solutions $d \in B$ of the inequality in assumption (β) in the following manner:
Let $n_0 \in \mathbb{N}$, and X a finite set of selfadjoint contractions $x^* = x \in C$, an element $b \in B$ and $\varepsilon > 0$. Moreover we suppose that there is given an element $f \in B$ and let $\delta \in (0, \varepsilon)$, e.g. let $\delta := \varepsilon/(2 + 2\|b\|)$. Condition ($\alpha$) implies that

$$\lim_n h_n b = \lim_n h^{1/n} b = b \quad \text{for all} \quad b \in B\,. \tag{4.5}$$

It implies that the decreasing sequence $\{(1 - h_m)^2\}$ converges strictly to zero in $\mathcal{M}(B)$. Thus, there is a number $m_1 := m(f, \delta) > n_0$ with $\|f - h_m f\| < \delta$ and $\|xf - h_m xf\| < \delta$ for each $m \geq m_1$ and $x \in X$.

We can replace for $m > m_1 > n_0$ the finite set X of selfadjoint contractions by the set $(1 - h_m)X(1 - h_m)$ and apply Equation $T((1 - h_m)c(1 - h_m)) = T(c)$ for $c \in C$ – coming from Equation (4.3) – and find $d \in B$ with $\|d\| \leq \|b\|$ such that for $x \in X$ and $\varepsilon > 0$ holds

$$\| \, d^*(1 - h_m)x(1 - h_m)d \, - \, b^*T(x)b \, \| \, < \, \varepsilon/2\,. \tag{4.6}$$

The new element $d \in B$ depends from X, b, $m \in \mathbb{N}$ and ε. Moreover our choice of $m \in \mathbb{N}$ ensures that $\|d^*(1 - h_m)xf\| < \delta\|b\|$ for $x \in X$ and $\|d^*(1 - h_m)f\| < \delta\|b\|$.

Let $\delta \in (0, \varepsilon/(4 + 4\|d\|))$ and let $f \in B$. The elements $(1 - h_m)d$ and xf are in B. Therefore we find by Equation (4.5) a number $n \in \mathbb{N}$ such that $n > m + 2$, $\| \, h_n(1 - h_m)d - (1 - h_m)d \, \| \, < \, \delta$ and $\| \, h_n xf - xf \, \| \, < \, \delta$ for $x \in X$.

Then element $d_0 := h_n(1 - h_m)d$ satisfies $\|d_0\| \leq \|d\| \leq \|b\|$, and for $x \in X$,

$$\| \, d_0^* x d_0 - d^*(1 - h_m)x(1 - h_m)d \, \| \leq \, 2\delta\|b\| < \varepsilon/2$$

and

$$\|d_0^* xf - d^*(1 - h_m)xf\| \leq \|b\| \cdot \|h_n(1 - h_m)xf - (1 - h_m)xf\| \leq \delta\|b\| < \varepsilon/2\,.$$

It implies that h_n, h_m, $d, b \in B$ and all $x \in X$ satisfy following inequalities

$$\| \, (h_n(1 - h_m)d)^* x (h_n(1 - h_m)d) \quad - \quad b^*T(x)b \, \| \, < \, \varepsilon\,, \tag{4.7}$$

$$\|(h_n(1 - h_m)d)^* xf\| \quad < \quad \varepsilon/2\,. \tag{4.8}$$

and $\|(h_n(1 - h_m)d)^* xf\| \, \leq \, \|d\| \cdot \|xf - h_m xf\| \, < \varepsilon/2$ for $x \in X$.

This describes the idea for an induction procedure if $b_1, b_2, \ldots \in B$, a zero-sequence $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ (or finite sequence) and an increasing sequence of finite subsets $X_1 \subseteq X_2 \subseteq \cdots$ of the selfadjoint contractions in C are given.

To explain in more detail the proposed induction procedure that is indicated by the above observations, we rename above considered $b, f \in B$, $X \subseteq C$, $\varepsilon > 0$, and the from them defined $d \in B$, $m < n \in \mathbb{N}$ by b_1, f_1, X_1, ε_1, d_1, $m_1 + 2 < n_1$.

The f_1, i.e., in the above side condition involving f, was an "additional" information, e.g. we can put $f_1 := f := 0$ for $n := 1$. In the induction step (from $k-1$ to k) it will be *defined* below more generally by

$$f_k := \big(\sum_{j=1}^{k-1}(h_{n_j}(1-h_{m_j})d_j)(h_{n_j}(1-h_{m_j})d_j)^*\big)^{1/2}\,. \tag{4.9}$$

Let $X_1 \subseteq X_2 \subset C$ finite sets of contractions, $b_2 \in B$ an arbitrary element and $\varepsilon_2 \in (0, \varepsilon_1)$ given. Take from the former step d_1 and $n_1 > m_1 + 2$ in $\mathbb{N}$. Define $f_2 := \big(h_{n_1}(1-h_{m_1})d_1 d_1^*(1-h_{m_1})h_{n_1}\big)^{1/2}$ and find $m_2 > n_1 + 2$ such that

$$\|h_{m_2} x f_2 - x f_2\| < \varepsilon_2/2$$

for all $x \in X_2$. Notice that automatically $h_{m_2} f_2 = f_2$ because $h_{m_2} h_{n_1} = h_{n_1}$.

We find $d_2 \in B$ with $\|d_2\| \leq \|b_2\|$ such that, for $x \in X_2$,

$$\|\, d_2^*(1-h_{m_2})x(1-h_{m_2})d_2 \;-\; b_2^* T(x) b_2 \,\| \;<\; \varepsilon_2\,.$$

Let $\delta_2 := \varepsilon_2/(2+2\|b_2\|)$. After we have selected d_2 we apply the above consideration and find a suitable number $n_2 > m_2 + 1$ such that, for $x \in X_2$,

$$\|\, h_{n_2}(1-h_{m_2})d_2 - (1-h_{m_2})d_2 \,\| \;<\; \delta_2 \quad\text{and}\quad \|\, h_{n_2} x f_2 - x f_2 \,\| \;<\; \delta_2\,.$$

From now on we use the new notation $h[m,n] := h_n(1-h_m) = h_n - h_m$ for $m < n$ that avoids iterated indices and seems to be more transparent ([11]). The $h[n,m]$ are positive contractions that satisfy the identities $h[\ell,p]^\alpha h[m,n] = h[m,n]$, $h[m,n]h[p,q] = 0$, and $h[m,n] + h[n,p] = h[m,p]$ for $\alpha \in (0,1]$ and $\ell, m, n, p \in \mathbb{N}$ with $\ell < m < n < p < q$. Then we get for $x \in X_2$ and $h[m_2,n_2] = h_{n_2}(1-h_{m_2})$ that

$$\|\, (h[m_2,n_2]d_2)^* x (h[m_2,n_2]d_2) \;-\; b_2^* T(x) b_2 \,\| \;<\; \varepsilon_2\,,$$

$(h[m_2,n_2]d_2)^*(h[m_1,n_1]d_1) = 0$ by $m_2 > n_1$, and

$$\|\, (h[m_2,n_2]d_2)^* x f_2 \,\| \;<\; \varepsilon_2\,.$$

If we repeat this arguments for given contractions $b_1, b_2, \ldots, b_k, \ldots \in B$, selfadjoint contractions $c_1 := 1, c_2 = h, c_3, \ldots, c_k, \ldots \in C_+$ and a given decreasing zero sequence $\varepsilon_1 > \varepsilon_2 > \cdots > 0$, then we obtain by the above described method for each $k \in \mathbb{N}$ and $X_k := \{c_1, \ldots, c_k\}$, suitable positive integers with $\ell_1 := 1$, $\ell_k < m_k < n_k < \ell_{k+1}$ (with $\ell_1 := 1$ and element $d_k \in B$ with $\|d_k\| \leq \|b_k\|$ that satisfy the general Inequalities:

$$\|\, (h[m_k,n_k]d_k)^* x (h[m_k,n_k]d_k) \;-\; b_k^* T(x) b_k \,\| \;<\; \varepsilon_k \quad \text{for all } x \in X_k\,. \tag{4.10}$$

and $\|(h[m_k,n_k]d_k)^* x (h[m_{j-1},n_{j-1}]d_{j-1})\| < \varepsilon_k$ for $x \in X_k$ and $1 \leq j < k$.

[11] Observe here and in some later proofs that the expressions $[m,n]$ denote pairs of indices and *not* any sort of commutators!

What happens with $j = k-1$??? Perhaps we have to replace in Equation (4.10) $(h[m_k, n_k]d_k)^* x(h[m_k, n_k]d_k) - b_k^* T(x) b_k$ simply by $(h[m_k, n_k]d_k)^* x(h[m_j, n_j]d_j) - \delta_{jk} \cdot b_k^* T(x) b_k$ for $1 \leq j \leq k$

The latter inequalities follow from $\| (h[m_k, n_k]d_k)^* \, x \, f_n \| < \varepsilon_k$ for f_n as defined in Equation (4.9). We have automatically $(h[m_k, n_k]d_k)^*(h[m_j, n_j]d_j) = 0$ for $j = 1, \ldots, k-1$ by $h[m_k, n_k]h[m_j, n_j] = 0$ for $j < k$. This can be sees also from the equations $h[m_k, n_k]h[\ell_k, \ell_{k+1}] = h[m_k, n_k]$, for the suitable chosen sequence $1 := \ell_0 < \ell_1 < \ell_2 < \ldots$ in $\mathbb{N}$ with $\ell_k < m_k < n_k < \ell_{k+1}$. Such a sequence exists because we have chosen $m_{k+1} > 1 + n_k$.

In particular, for a given finite set $X \subseteq C$ of selfadjoint contractions in C, and given elements $b_1, b_2, \ldots, b_k \in B$ and $\varepsilon > 0$, we find elements $d_1, \ldots, d_k \in B$ with $\|d_j\| \leq \|b_j\|$ that satisfy

$$\|d_i^* x d_j - \delta_{i,j} b_i^* T(x) b_j\| < \varepsilon$$

for $1 \leq i, j \leq k$ and $x \in X$, i.e., we get the following observation:

Observation (1):
Given contractions $b_1, \ldots, b_n$ and $X \subset C$ a finite subset of the selfadjoint contractions, then, for every $\varepsilon > 0$, there exists contractions $d_1, \ldots, d_n \in B$ such that $d_j^* d_k = 0$ for $j < k$ and

$$\| \, d_j^* x d_k \, - \, \delta_{j,k} b_k^* T(x) b_k \, \| \, < \, \varepsilon / n^2 \quad \text{for } k, j = 1, \ldots, n, \text{ and } x \in X \, . \tag{4.11}$$

The Inequality (4.11) implies the approximate 1-step innerness of the c.p. map $\sum_{k=1}^n b_k^* T(\cdot) b_k :=$ `NAME? S ????????` with respect to the strict topology on $\mathcal{M}(B)$, i.e., the element $d := d_1 + d_2 + \cdots + d_n \in B$ and all $x \in X$ satisfy the Inequality

$$\| \, d^* x d \, - \, \sum_{k=1}^{n} b_k^* \, T(x) \, b_k \, \| \, < \, \varepsilon \, . \tag{4.12}$$

Proof of Part(i):

We are going to choose the following objects μ, X_n, f_k and S_n to make the rest of the proof transparent and almost "constructive" – in a certain sense:

(A) A bijection μ form $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ such that

$$\mu((k,1)) \, = \, \min \mu(\{k\} \times \mathbb{N}) \, < \, \mu((n,1))$$

for $k < n$ in $\mathbb{N}$. It can be used to define a (disjoint) decomposition $\bigcup_k M_k$ of $\mathbb{N}$ given by bijective and *order preserving* maps $\nu_k \colon \mathbb{N} \to \mathbb{N}$ with $k \leq \nu_k(1) < \nu_n(1)$ for $k < n$ in $\mathbb{N}$. Here $M_k = \nu_k(\mathbb{N}) = \mu(\{k\} \times \mathbb{N})$. We have $\nu_1(n) = n$ for all $n < \nu_2(1) = \min M_2$, $\nu_1(n) > n$ for $n \geq \nu_2(1)$, and $\nu_k(1) > k$ for all $k > 1$. It follows that $\nu_k(n) \geq k + n$ for $k > 1$, $n \in \mathbb{N}$ and $\nu_1(n) \geq n + 1$ for $n \in \mathbb{N}$.

(B) Filtration X_n of C and quasi-central approximate unit (e_n) for $V(X_n)$ defined from strictly positive contraction $e \in B_+$.

(C) By induction:
Find solutions m_k, n_k, d_k, and the such defined elements $h[m_k, n_k]d_k =: g_k$ of (4.10) for $b_k := e_k^{1/2}$, with $\varepsilon_k := 4^{-n}$ and from them f_k defined by Equation (4.9).

(D) The S_n are then (suitable) Γ-sums for the in (A) defined maps $\nu_n \colon \mathbb{N} \to \mathbb{N}$,

$$S_n := \Gamma_n(g_{\nu_k(1)} \, , g_{\nu_k(2)} \, , \ldots) \, ,$$

with $\Gamma_n(\cdot)$ defined by the $x^{(n)}$ and $y^{(n)}$ as where x_n and y_n have the components

components ?????????????

We use the above considerations concerning the inductive solutions of the Inequalities (4.10) and Part (3) of Remarks 5.1.1 to define a filtration of C given by linear spans $L_n := \mathrm{span}(X_n)$ with $X_n := \{c_1, \ldots, c_{k_n}\}$, defined from a sequence of selfadjoint contractions $c_1, c_2, \ldots \in C$ that is dense in the unit ball of the self-adjoint part $C_{s.a.}$ of C.

Let $e \in B_+$ a strictly positive contraction. We find a sequence of functions $f_n \in \mathrm{C}_0((0,1])_+$ such that $e_n := f_n(e)$ build an approximate unit of B that is quasi-central with respect to finite subsets $X_n \cup T(X_n)$ of $\mathcal{M}(B)$. See Part (3) of Remarks 5.1.1 for more details. We take the there defined adjustment of the sequence e_n, – in particular in relation to $T(X_n)$ – such that, for each (contraction) $x \in X_n$ and $m > n$,

$$\| \, [e_n, T(x)] \, \| \; + \; \| [(e_m - e_n)^{1/2}, T(x)] \, \| \; < \; 4^{-n} \, .$$

It follows for infinite subsets $M_1, M_2 \subseteq \mathbb{N}$ with $\min M_1 < \min M_2$ and order preserving bijective maps λ_j from $\mathbb{N}$ onto M_j, $\lambda_j \colon \mathbb{N} \to M_j$, that

`NEXT give good estimate:`

$$\|\Gamma_{j,k}(T(x), T(x), \ldots) - \delta_{j,k} T(x)\| \leq ?????$$

and $\Gamma_{j,k}(T(x), T(x), \ldots) - \delta_{j,k} T(x) \in B$ for $x \in X_n$ and $j,k \in \{1,2\}$. Here $\Gamma_{j,k}(a_1, a_2, \ldots)$ for $j,k \in \{1,2\}$ is defined as in Remark 5.1.1(2) with $y_1 := e^{1/2}_{\lambda_j(1)}$, $y_{n+1} := (e_{\lambda_j}(n+1) - e_{\lambda_j}(n))^{1/2}$ $x_1 := e^{1/2}_{\lambda_k(1)}$, and $x_{n+1} := (e_{\lambda_k}(n+1) - e_{\lambda_k}(n))^{1/2}$.

We define by induction successive elements

$$g_1 := h[m_1, n_1]d_1 \, , \;\; g_2 := h[m_2, n_2]d_2 \, , \;\; \ldots \in B$$

and a sequence $p_1 < p_2 < \ldots$ in $\mathbb{N}$ with $p_k + 1 < m_k$, $m_k + 1 < n_k$, $n_k + 1 < p_k$ such that

$$\|g_k^* x g_\ell \; - \; \delta_{k,\ell} e_k^{1/2} T(x) e_k^{1/2}\| \leq ???$$

for $x \in X_k$, $\ell \leq k$, and that $p_k ???? g_k = g_k$.

Let $M \subseteq \mathbb{N}$ an infinite subset, and let $\lambda \colon \mathbb{N} \to M$ the unique bijective order-preserving map from $\mathbb{N}$ onto M, as described in Part (i,A).

allow to make the following observations (?0.1) - (?0.4):

????

such that corresponding $\varphi_{k_n}(h)(1 - \varphi_{\ell_n}(h))d_n$

with $n_k > 2 + m_k$, next $m_{k+1} > 2 + n + k$????

can be used in place of d_n for

$$\|d_m^* x d_n - \delta_{n,m} g_n(e) T(x) g_n(e)\| < 2^{-\max m,n} \|x\| .$$

NEXT:

The sequences (φ_n) and $h_1, h_2, \ldots$ can be defined by Equation (**??**).

(1.) $h_1, h_2, \ldots \in C^*(h)_+ \subseteq C$.

" suitably " (TO BE MADE PRECISE!!)

positive contractions with $h_{n+1} h_n = h_n$, $h_n \to 1_{\mathcal{M}(B)}$ strictly, using that hB is dense in B by condition (α).

`Define a nice sequence of` φ_n.
`Such that` $h[m_k, n_k] d_k$ `with` $n_k > 2 + m_k$, `next` $m_{k+1} > 2 + n + k$
`can be used in place of` d_n `for`
$\|d_n^* x d_n - g_n(e) T(x) g_n(e)\| < 2^{-n} \|x\|$

(2.) $f_n \in C^*(e)_+ \subseteq B$ for a strictly positive contraction $e \in B_+$ with $f_{n+1} f_n = f_n$ and $\lim_n \|e - f_n e\| = 0$.

Since $A := C^*(e, C, T(C))$ is a separable C^*-subalgebra of $\mathcal{M}(B)$ we find for every linear filtration of A by finite-dimensional linear subspaces $K_1 \subseteq K_2 \subseteq \ldots$ of A with $\bigcup_n K_n$ dense in the rational convex hull of $\{f_1, f_2, \ldots\}$ an approximate unit $e_1, e_2, \ldots$ with $e_{n+1} e_n = e_n$ and $\|[e_n, y]\| \leq 4^{-n} \|y\|$ for $y \in K_n$. Compare Remark 5.1.1(3,4).

We can manage that K_n contains $L_n \cup T(L_n)$ from a before defined linear filtration $L_1 \subseteq L_2 \subseteq \ldots$ of C, given by $L_n := \operatorname{span}(X_n)$, $X_n := \{ c_1, c_2, \ldots, c_n \}$.

Thus, can find an approximate unit $e_1, e_2, \ldots$, with $\| [(e_{n+1} - e_n)^{1/2}, c] \| \leq 3(2^{-n}) \sqrt{2} \, \|c\|$

`Compare next with above given definition of` $h[m,n]$
`without taking roots!`

We define

$$h[j,k] := (h_k - h_j) = (h_k(1 - h_j))$$

for $k > j$. Notice that $h[j+1, k-1](h_k - h_j) = h_k - h_j$ and $h[j_1, k_1] h[j_2, k_2] = 0$ if $k_1 < j_2$.

Find $d_1, d_2, \ldots \in B$ and $m_1 < n_1 < m_2 < n_2 < \cdots$ in $\mathbb{N}$ with the properties:

(1.) $m_{k+1} > n_k + 3$

(2.) $\|d_n\| \leq 1$,

$$\|d_n^* c d_n - e_{n+1} T(c) e_{n+1}\| \leq 8^{-n} \|c\|$$

for all $c \in L_n$, e_n, e_{n+1} well-commute with $c \in L_n$ and with $T(\, c\,)$ if $c \in L_n$.

(3.) $d_k = (h_{n_k} - h_{m_k}) d_k$, in particular $d_k^* d_\ell = 0$ for $\ell \neq k$.

(4.) $\|(h_{n_{k+1}} - h_{m_{k+1}}) c \, d_k\| \leq 8^{-n} \| c \|$ for all $c \in L_n$.

(5.) In particular ???, $\|c\, d_k - h_{m_{k+1}} c\, d_k\|$ <?????.

`That is not clear !!!`

(6.) The contractions S_n should be defined as some $S_n = S_{\mu,\nu}$, where

$$S_{\mu,\nu} := \sum_{\ell} d_{\nu(\ell)} (e_{\mu(\ell+1)} - e_{\mu(\ell)})^{1/2}$$

for suitable injective maps $\nu\colon \mathbb{N} \to \mathbb{N}$ and $\mu\colon \mathbb{N} \to \mathbb{N}$.

(6.0) We take a bijective map $\lambda\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that $\lambda^{-1}(n,1) < \lambda^{-1}(n+1,1)$ and $\lambda^{-1}(n,1) = \min \lambda^{-1}(\{n\} \times \mathbb{N})$ for all $n \in \mathbb{N}$. E.g. take λ defined by the order on $\mathbb{N} \times \mathbb{N}$ given by $(1,1) < (1,2) < (2,1) < (1,3) < (2,2) < (3,1) < (1,4) < \ldots$, which comes from $\gamma\colon (m,n) \to k := m+n-1 \in \mathbb{N}$, $\gamma^{-1}(k)$ lexicographic ordered as e.g. $\gamma^{-1}(1) = \{(1,1)\}$, $\gamma^{-1}(2) = \{(1,2) < (2,1)\}$, and so on. Then λ defines a decomposition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets $M_n := \lambda^{-1}(\{n\} \times \mathbb{N})$, such that $k_n := \min M_n$ satisfies $k_n < k_{n+1}$.

The natural maps order preserving bijections $\mu_n\colon \mathbb{N} \to M_n$ satisfy $\mu_1(k) = k$ for $k = 1, \ldots, \mu_2(1) - 1$ and $\mu_n(k) > k$ for all other pairs $(n,k) \in \mathbb{N} \times \mathbb{N}$.

`find more flexible formulation by working with finite subsets!`
??

For each $\ell, m, n \in \mathbb{N}$, each finite-dimensional linear subspace ??? $L \subseteq C$??? of C, contractions $b_k \in B$, $\gamma_k \in (0,1)$, $k = 1, \ldots, \ell$ there exist ???? $L \subseteq C$????

(i) numbers $m(j,k)$, and $n(j,k)$, $j = 1, \ldots, m$, with $n+2 < m(j,k)$, $m(j,k) + 2 < n(j,k)$, $n(j,k) + 2 < m(j+1,k)$, $n(m,k) + 2 < m(1,k+1)$.

(ii) Contractions $d_{j,k} \in B$, $j = 1, \ldots, m$; $k = 1, \ldots, \ell$, with

$$\|d_{i,k}^* x d_{j,k} - b_k^* x b_k\| < \gamma_k\,, \quad \text{and} \quad \|d_{j,k}^* x d_{i,k'}\| < \gamma_{\min(k',k)}\,.$$

(iii) a number $\nu \in \mathbb{N}$ depending on $d_{j,k} \in B$, $j = 1, \ldots, m$, $k = 1, \ldots, \ell$,
numbers $n < n_1 < n_2$ <???
Let $b_1, b_2, \ldots \in B$, $\gamma_1, \gamma_2, \ldots \in (0,1)$ finite or infinite sequence,
?????

Consider the functions

$$\varphi_n(\xi) := \min\big(1, \max(0, 2^n\xi - 1)\big) = (2^n\xi - 1)_+ - (2^n\xi - 2)_+\,,$$

and let $h_n := \varphi_n(h)$ for $n = 1, 2, \ldots$.

The elements $h_n \in \mathcal{M}(B)_+$ are positive contractions with $h_m h_n = h_m$ for $n > m$ and $\|h_n h - h\| \leq 2^{1-n}$. Since $h^{1/k}$ converges strictly to 1 in $\mathcal{M}(B)$ by (β), the sequence $h_1, h_2, \ldots$ tends to 1 unconditional strictly in $\mathcal{M}(B)$, because $\|(1-h_n)h^{1/k}\| \leq 2^{(1-n)/k}$.

We define rational numbers $\gamma_k > 0$, $k = 1, 2, \ldots, n$ and select by induction elements $f_1, f_2, \ldots, f_{n-1}$ and $g_1, g_2, \ldots, g_n$ from the above defined increasing sequence $\mathcal{S} := \{h_1, h_2, \ldots\} \subset C^*(h)_+ \subset \mathcal{M}(B)$, and choose simultaneously contractions $d_1, \ldots, d_n \in B$ such that they fulfill together following equations and inequalities, where we let $g_1 := h_1$ at the first step:

(1.1) Find contractions $d_k \in B$ that fulfill inequality (4.13) for the given b_k, $x \in X$ and given $g_k \in \mathcal{S} := \{h_1, h_2, \ldots\}$:

$$\| d_k^*(1-g_k)x(1-g_k)d_k - b_k^* T(x) b_k \| < \gamma_k . \tag{4.13}$$

(1.2) Find $f_k \in \mathcal{S}$ with $f_k g_k = g_k$, $\|(1-f_k)d_k\| < \gamma_k$ and $\|(1-f_k)xd_k\| < \gamma_k$ for $x \in X$, where g_k, d_k, X are given.

(1.3) Take $g_{k+1} \in \mathcal{S}$ with $g_{k+1}f_k = f_k$ for given $f_k \in \mathcal{S}$ (if $k < n$).

We have seen above the possibility to find the contraction $d_k \in B$ (for given X, γ_k, b_k and g_k) that satisfies the inequality (4.13). The existence of f_k and g_{k+1} follow immediately from the properties of the sequence $\mathcal{S}$.

We apply the conditions on the inductively selected b_k, g_k and f_k to find an estimate:

Conditions (1.3) and (1.2) on g_k and f_k imply that $g_{k+1} \geq f_k \geq g_k$, and by induction that $g_\ell \geq f_k$ and $(1-g_\ell)(1-f_k) = (1-g_\ell)$ for all $\ell > k$. The elements d_ℓ and $(1-g_k)$ are contractions by condition (1.2), and we get from condition (1.2) that

$$\|d_\ell^*(1-g_\ell)x(1-g_k)d_k\| \leq \|(1-f_k)x(1-g_k)d_k\| < \gamma\,.$$

If we combine this with Inequalities (4.13) and (4.18) we get that the element $D := (1-g_1)d_1 + \ldots + (1-g_n)d_n \in B$ satisfies

$$\|D^*xD - V(x)\| < \varepsilon \quad \text{for all } \; x \in X\,,$$

because

$$\|D^*xD - \sum_k b_k^* V(x) b_k\| < n^2\delta = \varepsilon/2$$

by $\|d_\ell^*(1-g_\ell)x(1-g_k)d_k\| < \delta$ for $k \neq \ell$, and by the Inequalities (4.13).

Hence, $V := \delta_\infty \circ T \colon C \to \mathcal{M}(B)$ satisfies also condition (β).

Notice that $\delta_\infty \circ V = \delta_\infty^2 \circ T$ is unitarily equivalent to V by a unitary in $u \in \mathcal{M}(B)$ by Lemma 5.1.2(i), and that $T = s_k^* V(\cdot) s_k$ for each $k \in \mathbb{N}$ if $\delta_\infty(b) = \sum_k s_k b s_k^*$.

`Next assumption necessary?`
`It becomes relevant in Part (ii), where` B `is stable,`
`And not here!!!`

We suppose from now on that T is unitary equivalent to $\delta_\infty \circ T$. ????

`Begin of a bit revised old text:`

It seems better to define first the needed universal sequences of $e_k \in B$ and $d_k \in B$ with the "good" properties with respect to a given filtration L_n of C. And then select the corresponding $h_{n(k)} - h_{m(k)}$.

Then define S and more general S_n only with help of this sequences!

We define below a contraction $S \in \mathcal{M}(B)$ with the property $S^*cS - T(c) \in B$ for all $c \in C$.

The element S is defined as the sum of an unconditional strictly convergent series by

$$S := \sum_n (f_{n+1} - f_n)^{1/2} d_n (e_{n+1} - e_n)^{1/2}$$

where the $f_n = h_{k_n} \in \mathcal{S} := \{h_1, h_2, \ldots\}$ have to be suitably chosen, and $\{e_1, e_2, \ldots\} \subset B_+$ is an approximate unit of B that is quasi-central with respect of the C^*-subalgebra $A := C^*(e, C, T(C)) \subseteq \mathcal{M}(B)$ build from a strictly positive contraction $e \in B_+$ of B and contractions $d_n \in B$. They are "suitably chosen" in the sense that the series S converges strictly and unconditional to a contraction in B, $\|S\| \leq 1$, and

??????

$$\sum_{m,n} \|d_n^*(f_{n+1} - f_n)^{1/2} c (f_{m+1} - f_m)^{1/2} d_m \; - \; \delta_{n,m} e_{n+2} f_{n+2} c f_{n+2} e_{n+2}\| \; < \; \infty \, .$$

By assumption, B is σ-unital, i.e., there exists a strictly positive element $e \in B_+$ with $\|e\| = 1$.

Steps:

Choose:

(0) strictly positive contraction $e \in B_+$, $\|e\| = 1$.

(1) linear filtration of $C^*(e, C, T(C))$ by taking a sequence in the unit ball of $C^*(e, C, T(C))_+$ that contains a sub-sequence $c_1, c_2, \ldots$ that is dense in the positive part of the unit ball of C (notation: $C_+^{\leq 1}$???)

the positive rational convex combinations of the $\varphi_n(e)$,

$\mathcal{S} := \{h_1, h_2, \ldots\}$, $\varphi_n(h_k)$,

and with ????

defining a filtration $Y_1 \subset Y_2 \subset \ldots$ of $A := C^*(e, C, T(C))$ such that

Y_n contains e, $X_n := \{h, h_n, c_1, \ldots, c_n\}$, $T(X_n)$ and $y_1, \ldots, y_n$ of a dense sequence in the positive parts $C_+^{\leq 1}$ and $A_+^{\leq 1}$ of the unit balls of C_+ and of A_+.

The strict convergence of $\mathcal{S} := \{h_1, h_2, \ldots\}$ to $1_{\mathcal{M}(B)}$ implies that we can find for each linear subspace $Y \subseteq B$ of finite dimension and $\varepsilon > 0$ a number $m := m(Y, \varepsilon) \in \mathbb{N}$ such that $\|(1 - h_m)y\| + \|y(1 - h_m)\| < \varepsilon \|y\|$ for all $y \in Y$.

A reformulation of conditions (α) and (β) together says:

Let $L \subset C$ a finite-dimensional subspace of C, $m \in \mathbb{N}$, $a \in B$ a positive contraction and $\varepsilon > 0$ then there exists $d := d(L, \varepsilon) \in B$ with $\|d\| \leq 1$ and

$$\|d^*(1 - h_m)c(1 - h_m)d - aT(c)a\| \leq \varepsilon \|c\| \quad \forall\, c \in L \, .$$

And, given any $d \in B$ and $m \in \mathbb{N}$, $\gamma > 0$ we find $n \in \mathbb{N}$ with $n > m + 1$ such that $\|(1 - h_n)d\| < \gamma$, $\|(1 - h_n)cd\| < \gamma$ and $\|(1 - h_n)c(1 - h_m)d\| < \gamma$ for all $c \in L$.

We take a dense sequence in the positive contractions $c_1, c_2, \ldots \in C_+$ and denote by L_n the linear span of $\{h, h_{n+2}, c_1, \ldots, c_n\}$.

Find positive $e_1, e_2, \ldots \in C^*(e)_+ \subset B_+$ with $e_n e_m = e_{\min(m,n)}$ and $\|[e_n, T(x)]\| < 4^{-(n+1)}$ for $x \in L_n$, $\|x\| \leq 1$.

Then $\|[e_n^{1/2}, T(x)]\| < 2^{-n}$ for $x \in L_n$, $\|x\| \leq 1$, $\|[(e_n - e_m), T(x)]\| < 4^{-(n+1)} + 4^{-(m+1)}$ and $\|[(e_n - e_m)^{-1/2}, T(x)]\| < 2^{-\min(n,m)}$ for $x \in L_n$, $\|x\| \leq 1$.

The positive contractions $e_n := \varphi_n(e) \in B$ with above ???

???? the above defined functions φ_n satisfy ???$e'_n e'_m = e'_m$ for $m < n$ build an approximate unit of B, i.e., the sequence (??e'_1, ??$e'_2, \ldots$) converges unconditional strictly to 1 in $\mathcal{M}(B)$.

If we build elements e_n $(n = 1, 2, \ldots)$ by convex combinations of $e'_{k_n}, \ldots, e'_{l_n}$ with $k_n < l_n < k_{n+1}$ then again $e_n e_m = e_m$ for $m < n$ and $\lim e_n b = b$ for all $b \in B$.

By Remark 5.1.1(3)

– applied

to the ??? of the separable C^*-subalgebra $A := C^*(T(C) \cup C \cup \{1\}) \supset C$ of $\mathcal{M}(B)$, and to a dense sub-sequence $y_1, y_2, \ldots$ of the positive contractions in A – Let $x_1, x_2, \ldots$ a dense sequence in the positive contractions of C.

Let L_n and K_n the linear span of $X_n := \{h, h_n, x_1, \ldots, x_n\} \subset C$ respectively of $X_n \cup T(X_n) \cup \{y_1, \ldots, y_n\}$. Thus $L_n \cup T(L_n) \subseteq K_n$.

By Remarks **??**(??,???) we can find positive contractions $e_1, \ldots, e_n, \ldots \in \mathrm{C}_0(e)_+$ that satisfy $e_n e_m = e_{\min(m,n)}$, $\lim_n e_n = 1_{\mathcal{M}(B)}$ and that the commutators $[e_n, x] = e_n x - x e_n$ for $x \in K_n$ satisfy $\|[e_n, x]\| \leq 8^{-n}\|x\|$?????? $[e_n, x]$ and $[e_n, (1 - x^2)^{1/2}]$ for each $x \in K_n$

have norms

$$\max\{\, \|[e_n, x]\| \,,\ \|[e_n, (1 - x^2)^{1/2}]\|\,\} < 8^{-2n} \quad \text{for all } x \in K_n\,,\ \|x\| \leq 1\,.$$

`Convergence proof needs explicit dependence !!`

By Lemma 5.3.2 we get the desired estimates of $[e_n^{1/2}, x]$, $[(1 - e_n)^{1/2}, x]$ and $[(e_{n+k} - e_n)^{1/2}, x]$.

If $1 \leq k_1 < k_2 < \ldots$ is any sequence $\Sigma \subseteq \mathbb{N}$, and $c \in C$, then, – with $e_{k_0} := 0$ –, the series

$$T_\Sigma(c) := \sum_{k=0}^{\infty} (e_{k_{n+1}} - e_{k_n})^{1/2} T(c) (e_{k_{n+1}} - e_{k_n})^{1/2}$$

is unconditional strictly convergent in $\mathcal{M}(B)$ for each $c \in C$, and its "sum" $c \mapsto T_\Sigma(c)$ defines a c.p. contraction $T_\Sigma \colon C \to \mathcal{M}(B)$ with the property $T_\Sigma(c) - T(c) \in B$ for all $c \in C$

...?

Moreover $\|T_\Sigma(c) - T(c)\| \leq$??????? for $c \in L_n$.

Indeed, let $y_n := x_n := (e_{k_{n+1}} - e_{k_n})^{1/2}$ in Remark 5.1.1(2), then $\sum_n x_n^2 = 1 = \sum y_n^2$, the there defined map Γ corresponding to

??? $Y := X := [x_1, x_2, \ldots]$??? `look this up in 5.1.1(2) ???`

is a strictly continuous unital c.p. map from $\ell_\infty(\mathcal{M}(B)) = \mathcal{M}(c_0(B))$ into $\mathcal{M}(B)$ and $T_\Sigma(c) = \Gamma(T(c), T(c), \ldots)$ for $c \in C$ defines a c.p. contraction from C into $\mathcal{M}(B)$.

If ?????? ???????

Again by functional calculus – this time applied to the element h in condition (α), we find a sequence of contractions $h'_1, h'_2, \ldots \in C^*(h)_+ \subseteq C_+$ such that $h'_n h'_m = h'_m$ for $m < n$ and $\|(1 - h'_n)e_n\| < 8^{-n}$. Thus $\|h'_n b - b\| \to 0$ if $n \to \infty$ for each $b \in B$.

`Why not` h_n `(or` g_n, f_n`) in place of` h'_n`?`

If we use that T is positive, $h \geq 0$ and $T(h) = 0$ by assumption (α), we get $T(h'_n) = 0$, $T(c\,h'_n) = 0$, $T(h'_n\,c) = 0$ and

$$T(c) = T((1 - h'_n)c(1 - h'_n)), \quad \forall\, c \in C\,,\ \forall\, n \in \mathbb{N}\,.$$

Combining this with condition (β) we get the existence of contractions $g_{n,k} \in B$ $(k, n \in \mathbb{N})$ that satisfy, for $c \in L_n$,

$$\|\,(g_{n,k})^*(1 - (h'_k)^2)^{1/2}c(1 - (h'_k)^2)^{1/2}g_{n,k} - e_{n+3}T(c)e_{n+3}\,\| \leq 8^{-n-k}\,\|c\|\,. \quad (4.14)$$

We define the desired contractions $S_{n,p}$ as the sum of strictly converging series of the kind

$$(h'_{\ell(n)} - h'_{m(n)})g_{2n,m(n)}e_n^{1/2} + \sum_{k>n}(h'_{\ell(k)} - h'_{m(k)})g_{2k,m(k)}(e_k - e_{k-1})^{1/2}\,,$$

where $m(k) < \ell(k) < m(k+1) < \ell(k+1)$ with choice depending from $p \in \mathbb{N}$. Huge gaps between $\ell(k)$ and $m(k+1)$ can be used to make the $S_{n,p}$ orthogonal modulo B, i.e., $S^*_{n,p}S_{n,q} \in B$ and even such that $S^*_{n,p}cS_{n,q} \subset B$ for $c \in C$ and

$$\lim_{n\to\infty} \|S^*_{n,p}cS_{n,q} - \delta_{p,q}T(c)\| = 0\,.$$

More precisely, we select inductively a subsequence $h_n := h'_{k(n)}$ and finite subsequences $h_{n,m} := h'_{\ell(n,m)}$, $m = 1, \ldots, n$, from the above selected sequence $h'_1, h'_2, \ldots$ with following properties :

(1) $k(0) = 1$,
(2) $k(n) < \ell(n,m) < \ell(n, m+1) \leq k(n+1)$ for $m = 1, 2, \ldots, n-1$.
(3) $\|(1 - h'_{\ell(n,m+1)})g_{n,\ell(n,m)}\| < 8^{-n}$,
(4) $\|(1 - h'_{k(n+1)})g_{n,k(n)}\| < 8^{-n}$,
(5) $\|(1 - h'_{k(n+1)})c(1 - h'_{k(n)})g_{n,k(n)}\| < 8^{-n}$ and $\|(1 - h_{n+1})c(h_{j+1} - h_j)g_{j,k(j)}\| < 8^{-n}$ for $c \in X_{n+1}$, $j = 1, \ldots, n-1$, $n = 1, 2, \ldots$, and
(6) $(h_{\ell(n,m+1)} - h_{\ell(n,m)})(h_{n+1} - h_n) = h_{\ell(n,m+1)} - h_{\ell(n,m)}$, and then $(h_{\ell(n_1,m+1)} - h_{\ell(n_1,m)})(h_{\ell(n_2,m+3)} - h_{\ell(n_2,m+2)}) = 0$

Notice that $(1-h_k)(1-h_n) = 1-h_n$ for $k < n$, and $(h_{k+1}-h_k)(h_{n+1}-h_n) = 0$ for $k+1 < n$. Let

$$t_{n,m} := (h_{n+2,m} - h_n)g_{n,k(n)}e_{n+2}$$

for all $n \in \mathbb{N}$ and $m \leq n$.

Then the $t_{n,m}$ are contractions with $t^*_{n,m}t_{n,m} \leq e^2_{n+2}$, $t^*_{n,m_1}t_{k,m_2} = 0$ and $\|t^*_{n,m(2)}ct_{k,m(1)}\| < 8^{-n+1}2$ for $k+1 < n$, $c \in X_k$. We get $(h_{n+2} - h_{n-1})^{1/2}t_{n,m} = t_{n,m}$ for $n = 1, 2, \ldots$, $m \leq n$.

It follows

$$\|t^*_{k,m(1)}ct_{n,m(2)} - \delta_{k,n}e_{n+2}T(c)e_{n+2}\| < 8^{-n}3\|c\|$$

for $c \in L_n$, because $\|t_{n,m} - (1-h_n)g_{n,k(n)}e_{n+2}\| < 8^{-n}$.

This follows from the definitions and the inequality $\|(1-h_{n+2,m})g_{n,k(n)}\| < 8^{-n}$, because $h_{n+2,m} = h'_{\ell(n+2,m)}$ and $\|(1-h'_{\ell(n+2,m)})g_{n,k(n)}\| < 8^{-n}$.

The latter inequality is a consequence of the following properties of h'_n and $g_{n,k(n)}$:
The inequality $\|(1-h'_{k(n+1)})g_{n,k(n)}\| < 8^{-n}$ is property (3) of $h'_{k(n)}$ and $g_{n,k(n)}$, and that $k(n+1) < k(n+2) < \ell(n+2,m)$ follows from property (2) of $k(n)$ and $\ell(n,m)$. Then $0 \leq h'_nh'_m = h'_m \leq 1$ for $m < n$ by construction of h'_n. It follows that

$$0 \leq h'_{k(n+1)} = h'_{k(n+1)}h'_{\ell(n+2,m)} \leq h'_{\ell(n+2,m)} \leq 1\,,$$

It implies $(1-h'_{\ell(n+2,m)})^2 \leq (1-h'_{k(n+1)})^2$ and, therefore

$$\|(1-h'_{\ell(n+2,m)})g_{n,k(n)}\| \leq \|(1-h'_{k(n+1)})g_{n,k(n)}\| < 8^{-n}\,.$$

The definition of $S_{n,p}$ is justified by the unconditional strict convergence of the sum at the right hand side. The strict convergence can be seen with the argument mentioned in Remark 5.1.1(2):

We define $S_{n,p} \in \mathcal{M}(B)$ for $n \in \mathbb{N}$ and $p \leq n$ by

$$S_{n,p} := t_{2n,\min(p,2n)}e_n^{1/2} + \sum_{k>n} t_{2k,\min(p,2k)}(e_k - e_{k-1})^{1/2}.$$

We must show now:

(I) The series

$$\sum_{k>n} t_{2k,\min(p,2k)}(e_k - e_{k-1})^{1/2}$$

is unconditional strictly convergent, ...????

(II) $\lim_{n\to\infty} \|S_{n,p}\| = 1$, and $S^*_{n,p}cS_{n,q} - \delta_{p,q}T(c) \in B$ for all $c \in C$ and $n, p, q \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|S^*_{n,p}cS_{n,q} - \delta_{p,q}T(c)\| = 0\,.$$

`Old version:`

because one can express $S_{n,p}$ as product RDC of the bounded row matrix

$$R := [h_{2n+2}^{1/2}, (h_{2(n+1)} - h_{2(n-1)})^{1/2}, (h_{2(n+2)} - h_{(n+1)-2})^{1/2}, \ldots]$$

the contractive diagonal matrix

$$D := \operatorname{diag}(\alpha_1, \alpha_2, \ldots)$$

with diagonal entries $\alpha_1 := t_{2n,m}$ and $\alpha_\ell := t_{2(\ell+n),m}$, and the bounded column matrix C with transposed row

$$C^\top := [e_n^{1/2}, (e_{n+1} - e_n)^{1/2}, (e_{n+2} - e_{n+1})^{1/2}, \ldots] .$$

Moreover, Remark 5.1.1(2) shows that the $S_{n,p}$ are contractions.

It is not difficult to see that $?S_{k,m} - ?S_{n,m} \in B$ and, therefore

$$?S^*_{k,m} c S_{k,m} - ?S^*_{n,m} c S_{n,m} \in B$$

for every $c \in C$.

Now let $c \in X_m$ and, for $n > m$ let

$$a_m := e_m^{1/2} t^*_{2m} c t_{2m} e_m^{1/2} - T(c) e_m, \tag{4.15}$$
$$b_n := (e_n - e_{n-1})^{1/2} t^*_{2n} c t_{2n} (e_n - e_{n-1})^{1/2} - T(c)(e_n - e_{n-1}), \tag{4.16}$$
$$d_{m,n} := (e_n - e_{n-1})^{1/2} t^*_{2n} c t_{2m} e_m^{1/2} + e_m^{1/2} t^*_{2m} c t_{2n} (e_n - e_{n-1})^{1/2} , \tag{4.17}$$

and, for $i < j$,

$$f_{i,j} := (e_j - e_{j-1})^{1/2} t^*_{2j} c t_{2i} (e_i - e_{i-1})^{1/2} + (e_i - e_{i-1})^{1/2} t^*_{2i} c t_{2j} (e_j - e_{j-1})^{1/2} .$$

Then from $d^{1/2} T(c) d^{1/2} - T(c) d = (d^{1/2} T(c) - T(c) d^{1/2}) d^{1/2}$ we get $\|a_k\| < 8^{-2m} 3 + 8^{-k}$, $\|b_n\| < 8^{-2n} 3 + 8^{-n}$, $\|d_{m,n}\| < 8^{-2n+1} 2$ and $\|f_{i,j}\| < 8^{-2j+1} 2$ for $m < n$, $m < i < j$.

Since $e_m + (e_{m+1} - e_m) + \ldots = 1$ strictly, for $c \in X_m$,

$$S^*_m c S_m - T(c) = a_m + \sum_{n=m+1}^{\infty} b_n + \sum_{n=m+1}^{\infty} d_{m,n} + \sum_{m<i<j} f_{i,j} .$$

The right hand side are absolutely convergent series in B. The sums of the norms can be estimated by by $8^{-m} 4$.

It follows that $\|S^*_{k,m} c S_{k,m} - T(c)\| < 2^{-k}$ and that $S^*_{k,m} c S_{k,m} - T(c) \in B$ for $c \in X_k$. Since $S^*_k c S_k - S^*_n c S_n \in B$, we get that $S^*_{n,m} c S_{n,m} - T(c) \in B$ for each $n = 1, 2, \ldots$ and every element c of our dense sequence in the unit ball of C.

Thus, $S^*_{n,m} c S_{n,m} - T(c) \in B$ and $\lim_n \|S^*_{n,m} c S_{n,m} - T(c)\| = 0$ for every $c \in C$.

`Our desire was to deepen the results by proving orthogonality modulo` B `for the` S_n `...`

`still to check:`

We have $S^*_{n,m_1} c S_{n,m_2} \in B$ for $m_1 \neq m_2$ and $\|S^*_{n,m_1} c S_{n,m_2}\| \to 0$ for $n \to \infty$ if $c \in C$ or $c = 1$.

(ii): Recall that B is stable if and only if $\mathcal{M}(B)$ contains a sequence of isometries $s_1, s_2, \ldots$ such that $\sum_k s_k s_k^* = 1$, cf. Remark 5.1.1(8). Then $\delta_\infty(b) := \sum_k s_k b s_k^*$ converges unconditional strictly for $b \in \mathcal{M}(B)$, is a strictly continuous unital *-endomorphism and $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B) = \delta_\infty(B)' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally by Remark 5.1.1(8).

The contractive c.p. map $V := \delta_\infty \circ T$, i.e., $V(\cdot) := \sum_k s_k T(\cdot) s_k^*$ again satisfies Conditions (α) and (β):

Clearly V satisfies condition (α), because $V(h) = \delta_\infty(T(h)) = 0$.

Let $X \subseteq C_+$ a finite subset of the contractions in C_+, $b \in B$ a contraction and $\varepsilon > 0$.

There exists $q \in \mathbb{N}$ with

$$\| (1 - P_q) b \| < \varepsilon/4$$

for the projection $P_q := \sum_{k=1}^q s_k s_k^*$, because $\sum_k s_k s_k^*$ converges strictly to $1 \in \mathcal{M}(B)$.

It implies for $x \in X$ and the contractions $b_k := s_k^* b \in B$ that

$$\| b^* V(x) b - \sum_{k=1}^{q} b_k^* T(x) b_k \| < \varepsilon/2 \,. \tag{4.18}$$

We have seen at the end of the general observation, cf. Inequality (4.12), that, for finite $X \subset C_+$, $b_1, \ldots, b_q \in B$ and $\varepsilon > 0$, there exists an element $d \in B$ with

$$\|d^* x d - \sum_{k=1}^{q} b_k^* T(x) b_k\| < \varepsilon/2$$

Thus $d \in B$ solves the inequalities $\|d^* x d - b^* V(x) b\| < \varepsilon$ $(x \in X)$ for given finite $X \subseteq C_+$, $b \in B$, $\varepsilon > 0$ and $V := \delta_\infty \circ T$.

This confirms that $\delta_\infty \circ T$ satisfies also the Condition (β).

NEXT TO BE SORTED WITH NEW INSIGHT.

By Part (i), there exists a contraction $g \in \mathcal{M}(B)$ with $V(c) - g^* c g \in B$ for all $c \in C$.

Apply ???? Part (i) to $V =: \delta_\infty \circ T$ and C. Let $g := S_1$ with S_1 as in Part (i).

We use Lemma 5.1.2(v). Let $t \mapsto S'(t)$ the map for $V = \delta_\infty \circ T$ in Part (v) of Lemma 5.1.2, and let $s_1, s_2, \ldots$ be the generators of a unital copy of $\mathcal{O}_\infty$ in $\mathcal{M}(B)$ with $\sum_n s_n s_n^* = 1$ (strictly convergent) such that $\delta_\infty = \sum_n s_n(\,\cdot\,) s_n^*$, cf. Remark 5.1.1(8). Then $t \mapsto S(t) := S'(t) s_1$ is as desired for T.

Now notice that ?????????? ...

When $S(t)$ is an isometry ? E.g. if $S'(t)$ is a contraction with $S'(t)^* S(t) \geq s_1 s_1^*$ and $s_1^* s_1 = 1$??

When absorption happens?

(iii): If $1_{\mathcal{M}(B)} \in C$, then T is unital, because $T(C)B$ is dense in B.

By Part (ii), $V := \delta_\infty \circ T$ satisfies (α) and (β). Moreover $V(C)' \cap \mathcal{M}(B) \subset \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally.

By Part (i), there exists a contraction $g \in \mathcal{M}(B)$ with $V(c) - g^*cg \in B$ for all $c \in C$. The existence of $U(t)$ now follows from Lemma 5.1.2(vi).

If $1_{\mathcal{M}(B)}$ is *not* in C, then let $\widetilde{C} := C + \mathbb{C}1_{\mathcal{M}(B)}$ and $\widetilde{T}(c + z1) := T(c) + z1$ for $c \in C$ and $z \in \mathbb{C}$.

Then $\widetilde{T}$ is a unital C^*-morphism, and $\widetilde{C}$ and $V := \delta_\infty \circ \widetilde{T}$ again satisfy (α) and (β) (in place of C and T there).

Indeed: $V(h) = \delta_\infty(T(h)) = 0$ gives (α).

We show (β) for the *unital* C^*-morphism $V : \widetilde{C} \to \mathcal{M}(B)$:
It suffices to check the condition (β) for V on sets of the form $X \cup \{1\}$ and $a \in B_+$, where $\|a\| \le 1$ and X is a finite set of contractions in C.

By assumption, $T(C)B$ is dense in B. The set $\delta_\infty(B)B$ is dense in B, because δ_∞ is a unital and strictly continuous endomorphism of $\mathcal{M}(B)$. It follows that the set $V(C)B$ is dense in B, because $V(C)\delta_\infty(B)B = \delta_\infty(T(C)B)B$.

Let X be a finite subset of the contractions in C, $a \in B_+$ a contraction and $\varepsilon > 0$.

There exists a contraction

`e was used as strictly positive in B`

$e \in C_+$ with $\|V(e)a - a\| < \varepsilon/4$ and $\|ece - c\| < \varepsilon/4$ for $c \in X$, because V is multiplicative and $V(C)B$ is dense in B.

`(β) of V was shown above from (β) of T`

By assumption, $V|C = \delta_\infty \circ T$ satisfies (β).

Thus, there exists a *contraction* $d_1 \in B$ such that $\|d_1^* x d_1 - aV(x)a\| < \varepsilon/2$ for $x \in eXe \cup \{e^2\}$, cf. proof of part (i).

Let $d := ed_1 \in B$. Then $\|d^*cd - aV(c)a\| < \varepsilon$ for $c \in X \cup \{1\}$.

(iv): See the argument at the end of the proof of Part (v) of Lemma 5.1.2. □

`What are (α) and (β) in case T := 0 or`
`if T(c) := ρ(c)·1 for a (pure) state ρ on C with ρ(C ∩ B) = {0}?`

REMARK 5.4.2. *Desirable !!! is the following version for countably generated Hilbert D-modules and with a finite group action:*

Let G a finite group, D a σ-unital C^*-algebra with a G-action $\tau_D : G \to \mathrm{Aut}(D)$, E a countably generated right Hilbert D-module with an isometric G-action $\tau_E : G \to \mathrm{Iso}(E)$ into the linear isometries on E that is compatible with τ_D in the sense that, for $e \in E$, $g \in G$, ??? and

$$\tau_E(g)(e) \cdot \tau_D(g)(d) = \tau_E(g)(e \cdot d)$$

Now let C a separable C^*-algebra with a G-action $\tau_C\colon G \to \mathrm{Aut}(C)$, and suppose that $H_k\colon C \to \mathcal{L}(E)$, $k \in \{1,2\}$, are G-equivariant C^*-morphisms.

If they are both in general position, and generate the same m.o.c cone, then they should be unitarily homotopic ??

COROLLARY 5.4.3. *Suppose that B is σ-unital and stable, A is separable, and that $H_1, H_2\colon A \to \mathcal{M}(B)$ are C^*-morphisms, such that $\delta_\infty \circ H_k$ are non-degenerate for $k \in \{1,2\}$.*

Then $\delta_\infty \circ H_1$ and $\delta_\infty \circ H_2$ are unitarily homotopic, if and only if, for every contraction $b \in B$, H_j dominates weakly approximately inner the c.p. map $b^* H_k(\cdot) b$ for $j,k \in \{1,2\}$ in sense of Definition 3.10.1.

I.e., $\delta_\infty \circ H_1$ and $\delta_\infty \circ H_2$ are unitarily homotopic if, and only if, the (point-norm closed) m.o.c. cones $\mathcal{C}(H_1)$ and $\mathcal{C}(H_2)$ in $\mathrm{CP}(A,B)$ corresponding to H_1 and H_2 are identical.

Recall here that, by definition, the m.o.c. cone $\mathcal{C}(H_k) \subseteq \mathrm{CP}(A,B)$ is generated by $\{b^* H_k(\cdot) b\,;\ b \in B\}$ and is point-norm closed.

We can weaken the requirement of non-degeneracy of $\delta_\infty \circ H_k$ by supposing only that $\delta_\infty \circ H_k$ is *unitarily homotopic* to a non-degenerate C^*-morphism from A into $\mathcal{M}(B)$. Notice that this is a property of $\delta_\infty \circ H_k$ that is *not* the same as as the property that $\delta_\infty \circ H_k$ non-degenerate.

PROOF. The condition implies that H_1 and H_2 have the same kernel. Thus, w.l.o.g. we can assume that H_1 and H_2 are faithful.

Let $e \in B_+$ a strictly positive element of B, $F_j := \delta_\infty(H_j(A))$ and $C_j := C^*(F_j, e)$ for $j = 1,2$. Since $F_j \cap B = \{0\}$, there is a *unique* C^*-morphism T_j from C_j onto F_i (for $i \neq j \in \{1,2\}$) with $T_j(e) = 0$ and $T_j(\delta_\infty(H_j(a))) = \delta_\infty(H_i(a))$ for $a \in A$.

The C^*-morphism $\delta_\infty \circ T_j$ is unitarily equivalent to T_j, because δ_∞^2 is unitarily equivalent to δ_∞ by Lemma 5.1.2(i) and H_j is uniquely defined up to unitary equivalence.

C_j and T_j satisfy assumption (α) of Proposition 5.4.1 with $h := e$.

We show that (C_j, T_j) satisfies also condition 5.4.1(β):
Let $s_1, s_2, \ldots$ a sequence of isometries in $\mathcal{M}(B)$ such that $\sum_n s_n (s_n)^*$ strictly converges to $1_{\mathcal{M}(B)}$, and let $a \in B_+$, X a finite subset of C_j and $\varepsilon > 0$ given. Then there are finite subsets $Y \subseteq A_+$ and $Z \subseteq B$ such that X is a subset of $\{\delta_\infty(H_j(y)) + z\colon y \in Y, z \in Z\}$. Our assumptions imply that there are $b_1, \ldots, b_n \in B$ such that

$$\|a(\delta_\infty)^2(H_i(y))a - \sum_{1\le k\le n} b_k^* H_j(y) b_k\| < \varepsilon/2\,.$$

Since $\sum_n s_n(s_n)^*$ strictly converges to $1_{\mathcal{M}(B)}$, there is $m \in \mathbb{N}$ such that $\|d^* z d\| < \varepsilon/2$ for $z \in Z$ and $d := s_{m+1} b_1 + \ldots + s_{m+n} b_n$. Thus $\|a V_j(x) a - d^* x d\| < \varepsilon$ for $x = \delta_\infty(H_j(y)) + z$ in X.

By assumption, $T_j(C_j)B = \delta_\infty(H_i(A))B$ is dense in B. Thus Proposition 5.4.1(iii) applies to (C_j, T_j) and yields that H_1, $H_1 \oplus H_2$ and H_2 are mutually unitarily homotopic. □

COROLLARY 5.4.4. *Suppose that B is σ-unital and stable, A is separable, and that $\mathcal{S}$ is a countably generated matrix operator-convex cone of completely positive maps V from A into B (cf. Definition 3.2.2), such that $\mathcal{S}$ is* non-degenerate *in the following sense: For every $b \in B_+$ and $\varepsilon > 0$, there exists $a \in A_+$ and $V \in \mathcal{S}$ such that $(b-\varepsilon)_+$ is in the closed ideal of B that is generated by $V(a)$.*

Then there is a non-degenerate *C^*-morphism $H_0 \colon A \to \mathcal{M}(B)$ with following properties (i)–(iii):*

(i) *H_0 is unitary equivalent to $\delta_\infty \circ H_0$ by a unitary in $\mathcal{M}(B)$.*
(ii) *For every $V \in \mathcal{S}$ there exists a sequence $b_1, b_2, \ldots \in B$ such that, for every $a \in A$,*
$$\lim_{n\to\infty} b_n^* H_0(a) b_n = V(a)\,.$$
(iii) *For every $b \in B$ there exists a sequence $V_1, V_2, \ldots \in \mathcal{S}$ such that, for every $a \in A$,*
$$\lim_{n\to\infty} V_n(a) = b^* H_0(a) b\,.$$

H_0 is determined by (i)–(iii) up to unitary homotopy (cf. Definition 5.0.1).

H_0 is injective, if and only if, $\mathcal{S}$ is separating *for A, i.e., for every non-zero positive $a \in A_+ \setminus \{0\}$, there is $V \in \mathcal{S}$ with $V(a) \neq 0$.*

The example with $B := A := \mathcal{O}_2 \otimes \mathbb{K}$, $\mathcal{S} = \mathrm{CP}(A, B)$, $H_0(a) := a$ shows that H_0 is *not* determined by (ii) and (iii) of Corollary 5.4.4 alone.

PROOF. The uniqueness up to unitary homotopy follows from Corollary 5.4.3.

Let $a \in A_+$. By (ii) and (iii), $H_0(a) = 0$ if and only if $V(a) = 0$ for every $V \in \mathcal{S}$. In particular, H_0 is a monomorphism if and only if $a \in A_+$ and $V(a) = 0$ for all $V \in \mathcal{S}$ imply $a = 0$.

By assumption there exists a countable sequence $V_1, V_2, \ldots \in \mathcal{S}$ such that $\mathcal{S}$ is contained in the point-norm closure of the matrix operator-convex cone generated by $\{V_1, V_2, \ldots\}$. We can suppose that every V_n appears infinitely times in this sequence.

Kasparov–Stinespring dilations lead to Hilbert B-modules $\mathcal{H}_n$ and and C^*-morphisms $D_n \colon A \to \mathcal{L}(\mathcal{H}_n)$ such that for $b \in B$ and $c \in A$ there is $x \in \mathcal{H}_n$ with $\langle x, D_n(a)x\rangle = b^* V_n(c^*ac)b$ for $a \in A$, and that $a \in A \mapsto \langle y, D_n(a)y\rangle \in B$ is in the point-norm closure of the matrix operator-convex cone generated by V_n. (Recall here that $\mathcal{H}_n$ is a completion of a quotient of the algebraic tensor product $A \odot B$, and that the B-valued Hermitian form is given there by
$$\langle x, y\rangle := \sum_{j,k} b_j^* V(a_j^* c_k) d_k$$

for representatives $\sum_j a_j \otimes b_j$ of x and $\sum_k c_k \otimes d_k$ of y. We take it anti-linear in the first variable.)

Let $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$ and $D\colon A \to \mathcal{L}(\mathcal{H})$ denote the Hilbert B-module sum, respectively of the corresponding ℓ_∞-sum of the C^*-morphisms D_n.

Since every completely positive map V_n appears infinitely often in our sequence $(V_1, V_2, \ldots)$, the maps $W_y\colon a \in A \mapsto \langle y, D(a)y\rangle \in B$ for $y \in \mathcal{H}$ build a *convex* cone $\mathcal{S}_0$ of completely positive maps from A into B, which is then automatically matrix operator-convex. The map W_y is in the point-norm closure of $\mathcal{S}$ for every $y \in \mathcal{H}$, and every V_n is in the point-norm closure of $\mathcal{S}_0$, i.e., $\mathcal{S}_0$ and $\mathcal{S}$ have the same point-norm closure in $\mathrm{CP}(A, B)$. (Here we use the existence of approximate units in A and B.)

Since $\mathcal{H}_n$ and (therefore) $\mathcal{H}$ are countably generated over B and since B is σ-unital and stable, the Kasparov trivialization theorem leads to the Hilbert B-module isomorphisms $\mathcal{H} \oplus \mathcal{H}_B \cong \mathcal{H}_B \cong B$.

Thus there is a (degenerate) C^*-morphism $h\colon A \to \mathcal{M}(B)$ that satisfies properties (ii) and (iii) (with H_0 replaced by h).

Let B_0 denote the closed span of $\delta_\infty(h(A))B\delta_\infty(h(A))$, and let $e \in B_+$, $f \in A_+$ strictly positive elements. B_0 is a stable hereditary C^*-subalgebra of B and $\delta_\infty(h(f))e\delta_\infty(h(f))$ is a strictly positive element of B_0, cf. Lemma 5.1.2(vii). B is the closure of the linear span of $Bh(A)B$, because h satisfies property (ii) (with H_0 replaced by h). $Bh(A)B$ is contained in the closed span of BB_0B.

Thus, by a variant of the stable isomorphism theorem of L.G. Brown [**107**] (cf. Corollary 5.5.6) there is an isomorphism ψ from B onto $B_0 \subseteq B$, which is approximately unitary equivalent to the id_B by unitaries in $\mathcal{M}(B)$. $H(a)b := \psi^{-1}(\delta_\infty(h(a))\psi(b))$ defines a non-degenerate C^*-morphism from A into $\mathcal{M}(B)$.

Since ψ is approximately unitary equivalent to id_B, we get that that H satisfies (ii) and (iii). Thus $H_0 := \delta_\infty \circ H$ satisfies (i)–(iii) by Lemma 5.1.2(i). □

Remark 5.4.5. A "converse" of Corollary 5.4.4 holds obviously if A is separable and B is σ-unital:

For every non-degenerate C^*-morphism $h\colon A \to \mathcal{M}(B)$, the set $\mathcal{S}_h$ of maps

$$V_{h,b}\colon a \in A \mapsto b^*\delta_\infty(h(a))b \in B$$

is a singly generated matrix operator-convex cone of completely positive maps, and $V_{h,e}$ is a generator of $\mathcal{S}_h$ if e is a strictly positive element of B. The corresponding H_0 is given by $H_0 := \delta_\infty \circ h$.

Corollary 5.4.6. *Suppose that B is σ-unital and stable, A is separable and unital, $\mathcal{C} \subset \mathrm{CP}(A, B)$ a countably generated non-degenerate and faithful point-norm closed operator-convex cone. Let $H_0\colon A \to \mathcal{M}(B)$ the non-degenerate *-monomorphism from $A \to \mathcal{M}(B)$ as defined in Corollary 5.4.4. Further let $W\colon A \to \mathcal{M}(B)$ be a unital completely positive map with $b^*W(\cdot)b \in \mathcal{C}$ for all $b \in B$.*

Then $H_0(1) = 1$*, and there are isometries* $S, T \in \mathcal{M}(B)$ *with* $SS^* + TT^* = 1$ *and* $W(a) - S^* H_0(a) S \in B$ *for all* $a \in A$*.*

PROOF. Since A is unital and H_0 is non-degenerate, we get $H_0(1) = 1$. Property (i) in Corollary 5.4.4 implies $H_0(A) \cap B = \{0\}$.

Let $e \in B_+$ a strictly positive contraction, and let $C := C^*(e, H_0(A)) \subset \mathcal{M}(B)$. Then $C/(C \cap B) \cong H_0(A)$. Thus, there is a unique *-morphism $\psi \colon C \to A$ with $\psi(e) = 0$ and $\psi \circ H_0 = \mathrm{id}_A$. Let $V(c) := W(\psi(c))$ for $c \in C$. Then V is unital and $(C, h := e, V)$ satisfy the condition (α) of Proposition 5.4.1.

We have $C = (C \cap B) + H_0(A)$. Let $c_j = b_j + H_0(a_j)$ $(j = 1, \ldots, n)$ and $\varepsilon > 0$, $\delta := \varepsilon/3$. Since $V \circ H_0 = W$ and since $b^* W(\cdot) b \in \mathcal{C}$, there is $d_1 \in B$ with $\|d_1^* H_0(a_j) d_1 - W(a_j)\| < \delta$ for $j = 1, \ldots, n$. By Lemma 5.1.2(iv) there exists

$$t \mapsto S_0(t) \in \delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B) \subset H_0(A)' \cap \mathcal{M}(B)$$

with $\lim_{t\to\infty} b S_0(t) = 0$ for $b \in B$. There is $t \in \mathbb{R}_+$ with $\|d_1^* S_0(t)^* b_j S_0(t)(t) d_1\| < \delta$ for $j = 1, \ldots, n$. Thus $\|d^* c_j d - W(c_j)\| < \varepsilon$ for $j = 1, \ldots, n$. Hence, for every $b \in B$, the map $c \mapsto b^* V(c) b \in B$ is approximately 1-step inner, i.e., condition (β) of Proposition 5.4.1 is satisfied.

Thus $C \subset \mathcal{M}(B)$, $h := e$ and $V \colon C \to \mathcal{M}(B)$ satisfy the assumptions of Proposition 5.4.1(i,iv). It gives an isometry $S_1 \in \mathcal{M}(B)$ with $S_1^* H_0(a) S_1 - W(a) \in B$ for all $a \in A$.

By Lemma 5.1.2(iv), there exist isometries $s, t \in H_0(A)' \cap \mathcal{M}(B)$ with $ss^* + tt^* = 1$. Let $S := sS_1$, then $S^* H_1(\cdot) S = S_1^* H_1(\cdot) S_1$, $tt^* \leq 1 - SS^*$ and $[1 - SS^*] = [1]$ in $\mathrm{K}_0(\mathcal{M}(B))$. By Lemma 4.2.6(ii), there is an isometry T with $TT^* = 1 - SS^*$. $\square$

DEFINITION 5.4.7. The quotient $\mathrm{Q^s}(B) := \mathcal{M}(B \otimes \mathbb{K})/(B \otimes \mathbb{K})$ is called **stable corona** of the C^*-algebra B.

If B itself is stable, i.e., $B \cong B \otimes \mathbb{K}$, we write instead $\mathrm{Q}(B)$ for $\mathcal{M}(B)/B \cong \mathrm{Q^s}(B)$. But the notation $\mathrm{Q}(B)$ for $\mathcal{M}(B)/B$ will also be used if B is non-unital and not stable, i.e., it does not indicate that B is stable.

In Section 9 we define extension groups (depending on operator-convex cones $\mathcal{C} \subset \mathrm{CP}(A, B)$) by

$$\mathrm{Ext}(\mathcal{C}\,;\, A, B) := G(\pi_B \circ H_0\,;\; A, \mathrm{Q}(B))$$

where $\mathcal{C}$ is separating and satisfies the requirements of Corollary 5.4.4 and $H_0 \colon A \to \mathcal{M}(B)$ is as in Corollary 5.4.4. The following Corollary 5.4.9 describes its elements.

LEMMA 5.4.8. *Let* $\mathcal{S} \subset \mathrm{CP}(A, J)$ *a point-norm closed operator-convex cone,* $T \colon A \to C/J$ *a c.p. contraction, and* $V \colon A \to C$ *a c.p. lift of* T*, i.e.,* $\pi_J \circ V = T$*, such that* $c^* V(\cdot) c \in \mathcal{S}$ *for every* $c \in J$*.*

If A *is separable, then there is a c.p. contraction* $W \colon A \to C$ *in the point-norm closure of the operator convex hull* $\mathcal{C}(V) \subset \mathrm{CP}(A, C)$ *of* V *in* $\mathrm{CP}(A, C)$ *with* $\pi_J \circ W = T$*. In particular,* $c^* V(\cdot) c \in \mathcal{S}$ *for every* $c \in J$*.*

If, in addition, A, C and T are unital, and if $\mathcal{S}$ is non-degenerate ([12]), *then there are a positive contraction $e \in C_+$ and $\chi \in \mathcal{S}$ such that such that $\chi(1) = 1 - eV(1)e \in J_+$. Then $W = \chi(\cdot) + eV(\cdot)e$ is unital, $\pi_J \circ W = T$ and $c^*W(\cdot)c \in \mathcal{S}$ for all $c \in C$.*

Proof. Suppose that A is separable and that $e \in A_+$ is a strictly positive contraction. Let $g_\delta(t) := \min(t^\delta, t^{-1})$ for $t \in (0, \infty)$ and $g_\delta(0) := 0$, where $\delta \in (0, 1]$. For $\delta \in (0, 1]$ and $a \in A + \mathbb{C} \cdot 1$ let

$$V_\delta(a) := g_\delta(V(e^{2\delta})V(e^\delta a e^\delta)g_\delta(V(e^{2\delta}))\,,$$

and

$$T_\delta(a) := (T(e^{2\delta})^\delta T(e^\delta a e^\delta)(T(e^{2\delta}))^\delta\,,$$

Then $\pi_J \circ V_\delta = T_\delta$, $V_\delta \in \mathcal{C}(V)$, and $\|V_\delta\| = \|V_\delta(1)\| \leq 1$, because $V_\delta(1) = g_\delta(c)^2 c$ has norm ≤ 1 for $c = V(e^{2\delta}) \in C_+$. Let D the hereditary C^*-subalgebra of C/J generated by $T(e^2)$, i.e., D is the closure of $T(e^2)DT(e^2)$. The span of eA_+e is dense in A and $0 \geq T(eae) \leq T(e^2)$. Thus $T(A) \subset D$, and $T(e^2)^\delta \leq T(e^{2\delta})^\delta \in D_+$, and $f_\delta := T(e^{2\delta})^\delta$ is an increasing approximate unit for D_+. Since $e^\delta a e^\delta \to a$ if $\delta \searrow 0$, it follows, that $T_\delta(a) \to T(a)$ for $a \in A$ if $\delta \searrow 0$. Hence, T is in the point-norm closure of the c.p. maps $\pi_J \circ V_\delta$ where the $V_\delta \in \mathcal{C}(V)$ are contractions.

The set $\mathcal{C}(V)_1$ of contractions in $\mathcal{C}(V)$ is an operator-convex set, in the sense that $a^*V_1(\cdot)a + b^*V_2(\cdot)b$ is a contraction in $\mathcal{C}(V)$ for $a, b \in C$ with $\|a^*a + b^*b\| \leq 1$ and contractions $V_1, V_2 \in \mathcal{C}(V)$). Therefore one can "perturb" ([13]) the elements of the sequence $\{V_{1/n}\}$ inside $\mathcal{C}(V)_1$ to a sequence $W_n \in \mathcal{C}(V)_1$ that converges in point-norm to an element $W \in \mathrm{CP}(A, C)$ such that $c^*W(\cdot)c \in \mathcal{S}$ for every $c \in C$.

Suppose now that A, C and T are unital. Then $1 - g_1(V(1)) \in J_+$ and $d_1 := 1 - V(1)g_1(V(1)) \in J_+$. Let $e_1 := g_1(V(1))^{1/2}$. There exists $\chi_0 \in \mathcal{S}$ with $\chi_0(1) \geq (d_1 - 1/4)_+$. There is a contraction $f \in J$ with $f^*\chi_0(1)f = (d_1 - 1/2)_+$, cf. Lemma 2.1.9. Let $\chi_1 := f^*\chi_0(\cdot)f$. Then $\chi_1 \in \mathcal{S}$, $\chi_1(1) + e_1V(1)e_1 = (d_1 - 1/2)_+ + 1 - d_1 =: h \geq 1/2$. Now let $e := e_1h^{-1/2} \in C^*(d_1, 1) \subset C$ and $\chi := h^{-1/2}\chi_1(\cdot)h^{-1/2} \in \mathcal{S}$. Then $\pi_J(e) = 1$ and $W := \chi + eV(\cdot)e$ is unital with $c^*W(\cdot)c \in \mathcal{S}$ for all $c \in J$. □

Corollary 5.4.9. *Suppose that A is separable, that B is stable and σ-unital, and that $\varphi\colon A \to \mathrm{Q}(B) = \mathcal{M}(B)/B$ is a C^*-morphism.*

Let $e \in B_+$ denote a strictly positive contraction in B and let $\mathcal{S} \subset \mathrm{CP}(A, B)$ countably generated non-degenerate operator-convex cone. Consider the non-degenerate $H_0\colon A \to \mathcal{M}(B)$ as defined in Corollary 5.4.4 for $\mathcal{S}$, and let $h_0 := \pi_B \circ (H_0 \oplus 0)$. Then:

(i) $[h_0] + [h_0] = [h_0]$,
(ii) *The unitary equivalence class $[\varphi]$ of φ is in $S(h_0;\, A, \mathrm{Q}(B))$ (= the semi-group introduced in Section 4 of Chapter 4), if and only if, there is a*

[12] I.e., for every $e \in J_+$ and $\varepsilon > 0$ there are $\chi \in \mathcal{S}$ and $a \in A_+$ with $(e - \varepsilon)_+ \leq \chi(a)$

[13] By "controlled" perturbations on a compact subset $\Omega \subset A$ with $\mathrm{span}(\Omega)$ dense in A, cf. e.g. the proof of the existence of nuclear lifts in [**43**].

completely positive map $W\colon A \to \mathcal{M}(B)$ *with* $\pi_B \circ W = \varphi$ *such that* $b^*W(\cdot)b$ *in the point-norm closure of* $\mathcal{S}$ *for every* $b \in B$ ([14]).

(iii) $[\varphi \oplus 0] = [\varphi] + [0]$ ([15]) *is in* $G(h_0;\, A, \mathrm{Q}(B)) = [h_0] + S(h_0;\, A, \mathrm{Q}(B))$, (*cf. Chapter 4, Section 4*), *if and only if, there is completely positive contraction* $W\colon A \to \mathcal{M}(B)$ *with* $\pi_B \circ W = \varphi$ *such that*

(a) $b^*W(\cdot)b$ *in the point-norm closure of* $\mathcal{S}$ *for every* $b \in B$, *and,*

(b) *for each* $V \in \mathcal{S}$, *there is a sequence of elements* $d_1, d_2, \ldots \in B$ *such that* $\lim_n d_n^* e d_n = 0$ *and* $\lim_n d_n^* W(a) d_n = V(a)$ *for all* $a \in A$.

Proof. (i): The commutant $h_0(A)' \cap \mathrm{Q}(B)$ contains a copy of $\mathcal{O}_2$ unitally, because H_0 is unitarily equivalent to $\delta_\infty \circ H_0$, $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{L}(\ell_2)$ unitally, and because $\mathcal{L}(\ell_2)$ contains a copy of $\mathcal{O}_2 \otimes \mathcal{O}_2$ unitally. Thus the semigroup $S(h_0\,; A, \mathrm{Q}(B))$ is well-defined and is the set of unitary equivalence classes (by unitaries in $\mathrm{Q}(B)$) of C^*-morphisms $\psi\colon A \to \mathrm{Q}(B)$ dominated by h_0 (cf. Definition 4.3.3).

(ii): Let $\mathcal{D}$ denote the point-norm closure of $\mathcal{S}$.

Suppose that $W\colon A \to \mathcal{M}(B)$ is a completely positive map with $\pi_B \circ W = \varphi$ such that $b^*W(\cdot)b \in \mathcal{D}$ for every $b \in B$. The c.p. lift W of φ can be modified that it becomes *contractive*, cf. Lemma 5.4.8. It can be taken unital if A and $\varphi\colon A \to \mathrm{Q}(B)$ unital, because $W'(a) := d^{1/2} H_0(a) d^{1/2} + cW(a)c$ is as desired, if $c \in \mathcal{M}(B)_+$ with $0 \le c \le 1$, $1 - c \in B$ and $d := 1 - cV(1)c \in B$.

Let $J \subset A$ denote the kernel of $H_0\colon A \to \mathcal{M}(B)$ and let $H_1 = \delta_\infty \circ H_0$. H_1 is unitarily equivalent to H_1 (by condition (i) of Corollary 5.4.4) and, obviously, $\pi_B \circ H_1$ has the same kernel as H_0. $W(J) = \{0\}$, because $b^*W(\cdot)b \in \mathcal{D}$ for every $b \in B$ and H_0 satisfies (ii). Thus, there is a completely positive contrcaction $[W]_J\colon A/J \to \mathcal{M}(B)$ with $W = [W]_J \circ \pi_J$. Let $C := C^*(e, H_1(A))$, and define $V\colon C \to \mathcal{M}(B)$ by $V(c) := [W]_J(\Gamma^{-1}(\pi_B(c)))$, where $\Gamma\colon A/J \to \pi_B(H_1(A))$ is the isomorphism from A/J onto $\pi_B(H_1(A))$ with $\Gamma(a + J) := \pi_B(H_1(a))$. Note that $C \cap B = \overline{eCe}$, $C = (C \cap B) + \delta_\infty(H_0(A))$ and that $V(H_1(y) + z) = V(H_1(y)) = W(y)$ for $y \in A$ and $z \in C \cap B$.

Then $V(e) = 0$ and $e^{1/n} d \to d$ if $n \to \infty$ for every $d \in B$. In particular, $V(C \cap B) = \{0\}$. If $b \in B_+$, $\varepsilon > 0$ and if X is a finite subset of C_+, then there are $f \in B$ and finite subsets $Y \subset A$, and $Z \subset C \cap B$ such that $X \subset \{H_1(y) + z :\ y \in Y, z \in Z\}$ and $\|f^* H_0(y) f - bW(y)b\| < \varepsilon/2$ for $y \in Y$. The latter holds by property (ii) of H_0 in Corollary 5.4.4, because $bW(\cdot)b \in \mathcal{D}$. Since $\delta_\infty(g) = \sum_n s_n g s_n^*$ for all $g \in \mathcal{M}(B)$ with some sequence $s_1, s_2, \ldots$ of isometries in $\mathcal{M}(B)$ with $\sum_n s_n s_n^*$ strictly convergent to 1 in $\mathcal{M}(B)$, there is $m \in \mathbb{N}$ such that $\|f^* s_m^* z s_m f\| < \varepsilon/2$ for $z \in Z$. It follows $\|d^* x d - bV(x)b\| < \varepsilon$ for $x \in X$ and $d := s_m f$. Thus Proposition 5.4.1(i) applies: There is a contraction $g \in \mathcal{M}(B)$ such that $g^* c g - V(c) \in B$ for every $c \in C$. In particular, $g^* H_1(a) g - W(a) \in B$

[14]This particular c.p. lifts W of φ can be taken contractive. W can be chosen unital if A and φ are unital.

[15]Note that, in general, $[\varphi] + [0] \neq [\varphi]$ and $[\varphi] + [0] \neq [\varphi] + [h_0]$.

and $\pi_B(g)^*\pi_B(H_1(a))\pi_B(g)^* = \varphi(a)$ for $a \in A$. Since H_0 is unitarily equivalent to H_1 by property (i) of Corollary 5.4.4, and since $h_0 := \pi_B \circ H_0 \oplus 0$ (trivially) dominates zero, it follows from Proposition 4.3.6(ii) that h_0 dominates φ. This means $[\varphi] \in S(h_0, A, \mathrm{Q}(B))$.

Conversely, if φ is dominated by $h_0 = \pi_B \circ H_0 \oplus 0$ in $\mathrm{Q}(B) = \mathcal{M}(B)/B$, then there is a contraction $g \in \mathcal{M}(B)$ with $\pi_B(g^*H_0(\cdot)g) = \varphi$. For every $b \in B$, the map $W\colon a \in A \mapsto b^*g^*H_0(a)gb$ is in $\mathcal{D}$ by property (iii) of Corollary 5.4.4, i.e., $W := g^*H_0(\cdot)g$ is as desired.

We have seen that (i) is equivalent to $[\varphi] \in S(h_0, A, \mathrm{Q}(B))$.

But $[\varphi] \in S(h_0, A, \mathrm{Q}(B))$ implies $[\varphi \oplus 0] = [\varphi] + [0] \in S(h_0, A, \mathrm{Q}(B))$, because (trivially) $[0] \in S(h_0, A, \mathrm{Q}(B))$ and $S(h_0, A, \mathrm{Q}(B))$ is a semigroup by Proposition 4.4.2(i). Since $\varphi \oplus 0$ trivially dominates φ, $[\varphi] + [0] \in S(h_0, A, \mathrm{Q}(B))$ if and only if $[\varphi] \in S(h_0, A, \mathrm{Q}(B))$.

By Proposition 4.3.5(i), $[\varphi]+[0] \in G(h_0, A, \mathrm{Q}(B)$, i.e., $[\varphi]+[0] = [\varphi]+[0]+[h_0]$, if and only if $\varphi \oplus 0$ dominates h_0.

Suppose that $[\varphi]+[0]$ is in $G(h_0, A, \mathrm{Q}(B))$. Then there are contractions $g_1, g_2 \in \mathcal{M}(B)$ such that $\pi_B(g_1^*H_0(a)g_2) = \varphi(a)$ and $g_2^*g_1^*H_0(a)g_1g_2 - H_1(a) \in B$ for all $a \in A$ and $H_1 := \delta_\infty \circ H_0$, because H_1 is unitarily equivalent to H_0. Let $W(a) := g_1^*H_0(a)g_1$, then W satisfies (i), as we have seen above.

Let $V \in \mathcal{D}$, there is a sequence of elements $b_1, b_2, \ldots \in B$ such that $\lim_n b_n^*H_0(a)b_n = V(a)$ for all $a \in A$ (by property (ii) of H_0 in Corollary 5.4.4). One can see, with help of an approximate unit of A, that one manage that $\|b_n\|^2 \leq \|V\|$. Then $d_n := g_2s_nb_n$ satisfies $\lim_n d_n^*W(a)d_n = V(a)$ for all $a \in A$ and $\lim_n \|d_n^*ed_n\| \leq \|V\| \lim \|s_n^*(g_2^*eg_2)s_n\| = 0$, i.e., W satisfies also (ii).

Conversely, let $W\colon A \to \mathcal{M}(B)$ a contractive and completely positive lift of φ with (i) and (ii).

Then $[\varphi \oplus 0] \in S(h_0, A, \mathrm{Q}(A))$ by (i), as we have seen above. Since h_0 and $\varphi \oplus 0$ both dominate zero and since H_0 is unitarily equivalent to $H_1 = \delta_\infty \circ H_0$, it suffices to find a contraction $g \in \mathcal{M}(B)$ with $g^*W(a)g - H_0(a) \in B$ for all $a \in A$, to get that $\varphi \oplus 0$ dominates h_0 (cf. Proposition 4.3.6(ii)).

There is an isomorphism ψ from $\varphi(A)$ onto $H_0(A)$ such that $\psi(\varphi(a)) = H_0(a)$ for $a \in A$, because $W(a) \in B$ if and only if $H_0(a) = 0$: Indeed, above we have seen that $H_0(a) = 0$ implies $H_0(a^*a) = 0$ and $0 \leq W(a)^*W(a) \leq W(a^*a) = 0$. $W(a) \in B$ implies $\varphi(a^*a) = 0$, i.e., $W(a^*a) \in B$, and, by (ii), that $V(a^*a) = 0$ for all $V \in \mathcal{D}$, i.e., $H_0(a) = 0$ by property (iii) of H_0 in Corollary 5.4.4.

Let $C := C^*(e, W(A))$ and $V(c) := \psi(\pi_B(c))$. Then C is separable, $C \cap B = \overline{eCe}$, $C = (C \cap B) + W(A)$, $V(e) = 0$, $\lim_n e^{1/n}b = b$ for $b \in B$, $V(C \cap B) = \{0\}$, $V(W(a)) = H_0(a)$ for $a \in A$, and $\delta_\infty \circ V$ is unitarily equivalent to V, because H_0 is unitarily equivalent to $H_1 := \delta_\infty \circ H_0$. Moreover, $V\colon C \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism.

If $b \in B_+$, $X \subset A$ and $Y \subset C \cap B$ are finite and $\varepsilon > 0$, then there is $d \in B$ with $\|d^* y d\| < \varepsilon/2$ and $\|d^* V(x) d - bH_0(x)b\| < \varepsilon/2$ by property (ii) of W, because $bH_0(\cdot)b \in \mathcal{D}$. Thus, Proposition 5.4.1 applies to C and V: There is a contraction $g \in \mathcal{M}(B)$ with $g^* c g - V(c) \in B$ for all $c \in C$, in particular $g^* W(a) g - H_0(a) \in B$ for all $a \in A$. □

Corollary 5.4.10. *Suppose that A is separable, B is σ-unital, $D \subset \mathcal{M}(B)$ is a*

`Is ``simple'' necessary?`

simple, stable and σ-unital C^*-subalgebra of $\mathcal{M}(B)$ such that DB is dense in B.

*Let $H\colon A \to \mathcal{M}(D) \subset \mathcal{M}(B)$ a non-degenerate *-monomorphism with $H(A) \cap D = \{0\}$ such that $H_d\colon A \ni a \mapsto d^* H(a) d \in D$ is nuclear for all $d \in D$.*

Then

(i) *$H_0 := \delta_\infty \circ H$ satisfies properties (i)–(iii) of Corollary 5.4.4 for the operator-convex cone $\mathcal{S} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$.*
(ii) *H and H_0 are unitarily homotopic if $D \cong \mathbb{K}$ or if D is purely infinite.*

Proof. (i): Let $s_1, s_2, \ldots$ a sequence of isometries in $\mathcal{M}(D)$ such that $\sum s_n (s_n)^*$ converges strictly to 1. Then H_0 is unitarily equivalent to H_1 given by $H_1(a) := \sum s_n H(a) s_n^*$, and H_1 is unitarily equivalent to $\delta_\infty \circ H_1$, cf. Lemma 5.1.2(i). Let $q_m := \sum_{n \le m} s_n s_n^*$, $e \in D_+$ strictly positive, $b \in B$. Then

$$a \in A \to b^* e^{1/n} q_m H_1(a) q_m e^{1/m} b = \sum_{1 \le k \le m} b^* e^{1/m} s_k H(a) s_k^* e^{1/m} b$$

is nuclear, because $H\colon A \to \mathcal{M}(D)$ is weakly nuclear. It implies that $V_b := b^* H_1(\cdot) b$ is a nuclear map from A into B, because $\lim_m \|q_m e^{1/m} b - b\| = 0$ for $b \in B$. Hence $H_0 = U^* H_1(\cdot) U$ satisfies condition (i) and (iii) of Corollary 5.4.4 for the m.o.c. cone $\mathcal{S} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$.

The proof of Corollary 5.4.4 shows: Since H_1 is unitarily equivalent to $\delta_\infty \circ H_1$ and is non-degenerate, we get that the maps $V_b\colon a \mapsto b^* H_1(a) b$ build an operator-convex cone $\mathcal{C} \subset \mathrm{CP}_{\mathrm{nuc}}(A, B)$ (that is generated by V_f for a strictly positive element $f \in B_+$).

Let $a \in A_+$ with $\|a\| = 1$, $b \in B_+$ and $\varepsilon > 0$. Then there is $m \in \mathbb{N}$ such that $\|e^{1/m} H(a) e^{1/m}\| > 1/2$, and $\|b(1 - e^{1/m}) b\| < \varepsilon/2$, because $\|H(a)\| = 1$ and because DB is dense in B. Let $c := e^{1/m} H(a) e^{1/m} \in D$. Since D is simple, there are $f_1, \ldots, f_k \in D$ such that $\|e^{1/m} - \sum f_j^* c f_j\| < \varepsilon/2$. Now let $d := \sum_{1 \le j \le k} s_j^* e^{1/m} f_j b \in B$. Then $\|d^* H_1(a) d - b^2\| < \varepsilon$. Thus, Lemma 3.2.7 applies, and every nuclear map $V\colon A \to B$ is in the point-norm closure of $\mathcal{C}$. Hence, H_0 satisfies condition (ii) of Corollary 5.4.4 for $\mathcal{S} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$.

(ii): Since $\delta_\infty(H(a)) = \sum s_n a s_n^*$ is (up to unitary equivalence) independent from the sequence of isometries $s_1, s_2, \ldots$ with $\sum s_n s_n^* = 1$ (strictly), and since $\mathcal{M}(D) \to \mathcal{M}(B)$ is unital and strictly continuous, it suffices to consider the case, where $B = D$. Let $e \in D_+$ a strictly positive contraction, and let $C := C^*(H(A), e)$, $e \in C \cap B =: J \triangleleft C$ and there is a (unique) *-morphism $\psi \colon C \to A$ with $\ker(\psi) = C \cap B$ and $\psi \circ H = \mathrm{id}_A$. Let $V := H_0 \circ \psi = \delta_\infty \circ T$ (for $T := H \circ \psi$). Then $V \colon C \to \mathcal{M}(D)$ is a non-degenerate *-morphism with $V(e) = 0$. The map V is weakly nuclear, because $H_0 \colon A \to \mathcal{M}(B)$ is weakly nuclear.

For $b \in D = B$ the map $V_b \colon c \in C \mapsto b^* V(c) b \in D$ is nuclear and $V_b(C \cap D) = \{0\}$. It follows from Proposition 3.2.13(i) that V_b is approximately 1-step inner if D is purely infinite.

If $D \cong \mathbb{K}$, then $[V_b] \colon C/(C \cap D) \to D$ is necessarily nuclear. Then the map V_b is approximately 1-step inner by Remark 3.2.14. It follows that $(C, V, e, B = D)$ satisfy also condition (β) of Proposition 5.4.1 in both cases. Then $H_0 = \delta_\infty \circ H = H \oplus \delta_\infty \circ H$ is unitarily homotopic to H by Proposition 5.4.1(iii). □

QUESTIONS:

Suppose A separable (unital or stable), B σ-unital and stable.

Let $H_0 \colon A \to \mathrm{Q}(B)$ a C^*-morphism, e.g. is defined by the universal bi-module of all nuclear maps form A to B.

Is it next?

If we use that $B \otimes \mathbb{K} \cong B$, then this is given by $H_0 := \pi_B \circ \delta_\infty \circ (1 \otimes h)$ the infinite repeat of any faithful representation $h \colon A \to \mathcal{M}(\mathbb{K})$.

Now describe all $H \colon A \to \mathrm{Q}(B)$ that absorb H_0. ...???

Define $\mathcal{S}'(H)$ as the set of "u.c.p. maps $V \colon A \to B$" (??) with the property that there exists a sequence of contractions in $T_n \in \mathrm{Q}(B)$ such that

$$\pi_B(\delta_\infty \circ V(a)) - T_n^* H_o(a) T_n \to 0\,.$$

Let $W \colon A \to \mathcal{M}(B)$ a c.p. contraction.

Does H dominate $\pi \circ W$ if $b^* W(\cdot) b \in \mathcal{C}(\mathcal{S}'(H))$ for all $b \in B$?

The point is:

What happens with infinite strictly convergent sums

$$T(\cdot) = \sum_n b_n^* V_n(\cdot) b_n$$

with $V_n \in \mathcal{C}(\mathcal{S}'(H))$?

Cases $A = \mathbb{C}$, $A = \mathrm{C}_0(0,1]$?

What happens in unital/non-unital cases?

Check next blue text again!!

REMARK 5.4.11. G. Elliott and D. Kucerovsky, [**264**], have used – together with their own proof – the following older version of Proposition 5.4.1:

Let C be a unital separable C^*-algebra and let B be an essential closed two-sided ideal of C, so that we may view C as a unital subalgebra of $\mathcal{M}(B)$:

$$B \subseteq C \subseteq \mathcal{M}(B); \quad 1 \in C\,.$$

Let $T\colon C \to \mathcal{M}(B)$ be a completely positive map which is zero on B, and suppose that, for every $b \in B$, the map

$$b^*T(\cdot)b\colon\ C \to B,$$

given by $c \mapsto b^*T(c)b$, can be approximated (on finite subsets of C) by the maps

$$c \mapsto d^*cd,\ \ d \in B\,.$$

It follows that there exists a sequence $S_n \in \mathcal{M}(B)$ such that

$$T(c) - S_n^*cS_n \in B,\ \text{ for all, } c \in C\,,$$

and $\lim_n S_n^*cS_n = T(c)$ for all $c \in C$.

This can be seen from Proposition 5.4.1, if one takes for the element $h \in C_+$ in Condition (α) of Proposition 5.4.1 a strictly positive contraction of B. The approximation condition implies Condition (β).

Comments to it: Find out the ingredients!!

Let B a σ-unital C^*-algebra, C a separable C^*-subalgebra of $\mathcal{M}(B)$, and $T\colon C \to \mathcal{M}(B)$ a linear map with $\|T\| \leq 1$ that satisfies the following Conditions *(α)* and *(β)*:

(α) There exists $h \in C_+$ with $T(h) = 0$ and $h^{1/n}d \to d$ if $n \to \infty$ for every $d \in B$.

(β) For every $a \in B_+$, every finite subset $X \subset C_+$ of contractions and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^*cd - aT(c)a\| < \varepsilon$ for $c \in X$.

Question 5.4.12. Let A separable, B stable and σ-unital, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a countably generated operator convex cone, e.g. $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ (which is singly generated).

Is there a characterization of those $H\colon A \to \mathrm{Q}(B)$ that absorbs each $\pi_B \circ \delta_\infty \circ h\colon A \to \mathrm{Q}(B)$ with $h\colon A \to \mathcal{M}(B)$ that satisfies $b^*h(\cdot)b \in \mathcal{C}$ for all $b \in B$?

In the case of [**264**] it is

In case of $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ a necessary and sufficient criteria for absorbing was given by G. Elliott and D. Kucerovsky [**264**], cf. [**310**] for the non-unital case.

Has to consider as $C \subset \mathcal{M}(B)$ some separable C^*-subalgebra of $\mathcal{M}(B)$ with the properties that $\pi_B(C) = H(A)$ and that $C \cap B$ contains $e \in C \cap B$ that is strictly positiv in B. Then use $B \cong B \otimes \mathbb{K}$ and take a faithful representation $\rho\colon A \to \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B \otimes \mathbb{K})$ in "general position", i.e., ρ unitary equivalent to its infinite repeat.

To get absorbed it is necessary that H is faithful.

Then let $T(c) := \rho(H^{-1}\pi_B(c))$. Let $h := e$ for (α) a strictly positive element of $B \otimes \mathbb{K}$.

Check (β) ... for C, T ...

5. Ψ-equivariant Stability of extensions

We deduce from Proposition 5.4.1 some results concerning stability and a Ψ-equivariant version (respectively m.o.c. cone equivariant version) of L.G. Brown's stable isomorphism theorem [**107**, thm.2.8].

Recall that a hereditary C^*-subalgebra D of a C^*-algebra B is **full** if the linear span of BDB is dense in B, and that D is called a **corner** if there exists a (unique) projection $p \in \mathcal{M}(B)$ with $D = pBp$.

The following Corollary of the tautological Weil–von-Neumann type result Proposition 5.4.1 gives an alternative proof of the stability criteria for σ-unital C^*-algebras of Hjelmborg and Rørdam [**373**, thm. 2.1, prop. 2.2(b)].

Corollary 5.5.1. *Suppose that B is a non-zero σ-unital and stable C^*-algebra, and that D is a* full corner *of B, i.e., there is a projection $p \in \mathcal{M}(B)$ with $pBp = D$ and* span BDB *is dense in B.*

(i) *If, for every $g \in D_+$ and $\delta > 0$, there exist $b \in D$ with $\|b^*b - g\| < \delta$, and $\|b^*gb\| < \delta$, then there exists an isometry $S \in \mathcal{M}(B)$ with $SS^* = p$.*
(ii) *The isometry S of Part (i) defines an isomorphism φ from B onto D by $\varphi(b) := SbS^*$ for $b \in B$, and φ satisfies*

$$\varphi(J) = D \cap J \quad \textit{for all } J \in \mathcal{I}(B)\,.$$

(iii) *If $S_1, S_2 \in \mathcal{M}(B)$ are isometries in $\mathcal{M}(B)$ such that the maps $\varphi_k := S_k(\,\cdot\,)S_k^*$, $k \in \{1,2\}$, from B onto $D \subseteq B$ coincide (i.e., $\varphi_1 = \varphi_2$), then there exists a unitary U in the center of $\mathcal{M}(B)$ with $S_1 \cdot U = S_2$.*
(iv) *The isometries $S \in \mathcal{M}(B)$ with $SBS^* = D$ can be obtained from the polar decompositions $d := S(d^*d)^{1/2} = (dd^*)^{1/2}S$, of any element $d \in B$ with the property that d^*d is strictly positive in B and dd^* is a strictly positive element of D.*

Proof. The idea of our proof is different from that in [**373**], because our proof is based on the fact that full properly infinite projections are MvN-equivalent if they have the same class in K_0 (an observation of J. Cuntz, cf. proof of Lemma 4.2.6(ii)). We combine this with the "tautological" Weyl–von-Neumann type result in Proposition 5.4.1 to get Part (i). The other parts follow then by straight calculation.

(In fact all this conclusions of Proposition 5.4.1 are almost trivial elementary applications, consisting only in notational problems.)

(i):

TO BE SHOWN:

Let $p \in \mathcal{M}(B)$ a projection, and let $D := pBp$. Suppose that span BDB is dense in B, and that for every $g \in D_+$ and $\delta > 0$, there exist $b \in D$ with $\|b^*b - g\| < \delta$, and $\|b^*gb\| < \delta$.

Then there exists an isometry $S \in \mathcal{M}(B)$ with $SS^* = p$.

Recall that $\mathrm{K}_*(\mathcal{M}(B)) = 0$ because δ_∞ is unitarily equivalent to $\delta_\infty \oplus \mathrm{id}$, cf. Lemma 5.1.2(ii).

By Lemma 4.2.6(ii), it suffices to show that there is an isometry I in $\mathcal{M}(B)$ with $II^* \leq p$, because then p and $q := 1$ are both projections that majorize orthogonal ranges of two isometries in $\mathcal{M}(B)$ and have same class $[p] = [q]$ in $\mathrm{K}_0(\mathcal{M}(B)) = 0$. By Lemma 4.2.6(ii), the projections p and $q = 1$ are MvN-equivalent in $\mathcal{M}(B)$, i.e., there is an isometry $S \in \mathcal{M}(B)$ with $SS^* = p$.

(In fact, the existence of an operator $S_0 \in \mathcal{M}(B)$ with $x := 1 - S_0^*pS_0 \in B$ would be enough, because the stability of B causes the existence of an isometry $S_1 \in \mathcal{M}(B)$ with $\|S_1^*xS_1\| \leq 1/2$. Then $I := pS_0S_1(S_1^*S_0^*pS_0S_1)^{-1/2}$ is an isometry with $II^* \leq p$.)

We apply Proposition 5.4.1(i) to prove the existence of needed isometry I:

Take a strictly positive contraction $e \in B_+$ and let C denote the unital commutative C^*-subalgebra of $\mathcal{M}(B)$, which is generated by the commuting elements 1, p, pep and $(1-p)e(1-p)$. The intersection $C \cap B$ contains the strictly positive contraction $h := pep + (1-p)e(1-p) = 2^{-1}((1-2p)e(1-2p) + e)$ of B.

The projection $p \in \mathcal{M}(B)$ is not contained in B, because otherwise $p \in pBp = D$ and the assumptions in part (i) propose the existence of $b \in D$ (i.e., $b \in pBp$) with $pb = b = bp$, $\|b^*pb\| < 1/4$ and $\|b^*b - p\| < 1/4$, which is impossible for a non-zero projection p.

Let χ denote the (unique) character of C with $\chi(C \cap B) = 0$ and $\chi(p) = 1$. It satisfies $\chi(y) = \chi(pyp)$ for all $y \in C$, in particular, $\chi((1-p) + pep) = 0$.

The map $T(a) := \chi(a)1$ is a C^*-morphism from C into $\mathcal{M}(B)$. Obviously, $T(p) = T(1) = 1$ and T has the same kernel $T^{-1}(0) = C \cap B + \mathbb{C} \cdot (1-p)$ as $\chi \colon C \to \mathbb{C}$ has. In particular $T(h) = 0$ for the strictly positive element h of B. The strict positivity implies that $b = \lim h^{1/n}b$ for all $b \in B$.

Thus, C, T and h satisfy condition (α) of Proposition 5.4.1.

(β): For every $a \in B_+$, every finite subset $X \subset C_+$ and every $\varepsilon > 0$ there exists $d \in \mathcal{M}(B)$ with $\|d^*cd - aT(c)a\| < \varepsilon$ for $c \in X$.

`Part to be checked:`

We show that T satisfies also condition (β) of Proposition 5.4.1:

Let $a \in B_+$, X a finite subset of C_+ for $C := C^*(1, p, pep, (1-p)e(1-p))$ and $\varepsilon > 0$. We define $\gamma := \max\{\|c\| \colon c \in X\}$ and $\delta := \varepsilon/(4 + 4\gamma)$.

Notice that $pCp = C^*(pep) + \mathbb{C} \cdot p$, because $c = pcp + (1-p)c(1-p)$ for all $c \in C$.

The kernel of T is identical with the kernel $(1-p)C + p(C \cap D)$ of χ.

It follows that $\chi(C \cap D) \subseteq \chi(C \cap B) = \{0\}$ and $\chi(1-p) = 0$

In particular $pcp - \chi(c)p$ is in $C \cap D$ for $c \in C_+$.

Since $c = pc+(1-p)c = pcp+(1-p)c(1-p)$ for all $c \in C$ and since $\chi(1-p) = 0$ and $\chi(C \cap B) = \{0\}$, we get that

$pcp - \chi(c)p$ is in $C \cap D$ for $c \in C_+$. Thus, if $b \in pB$, then, for $c \in C_+$,

$$b^*cb = \chi(c)b^*b + b^*(pcp - \chi(c)p)b\,.$$

Since $pcp - \chi(c)p$ is selfadjoint,

$$\|b^*(pcp - \chi(c)p)b\| \leq \|b^*|(pcp - \chi(c)p)|b\|\,.$$

We consider the sum f of the absolute values $|pcp - \chi(c)p|$:

$$f := \sum_{c\in X} |pcp - \chi(c)p| \in C \cap D_+\,. \tag{5.1}$$

Then, for $c \in X$ and $b \in pB$,

$$\|b^*cb - aV(c)a\| \leq |\chi(c)|\|b^*b - a^2\| + \|b^*fb\|\,.$$

Thus, to verify (β), it suffices to find $b \in pB$ with $\|b^*b - a^2\| < 2\delta$ and $\|b^*fb\| < 2\delta$ for the f defined in Equation (5.1).

Since D is full, there exists $b_1, \ldots, b_m \in pB$ with $\|\sum b_j^*b_j - a^2\| < \delta$. Let $\rho := \delta/m(1+\|a\|^2+\delta)$. There is a contraction $g \in D_+$, such that $\|b_j^*gb_j - b_j^*b_j\| < \rho$ for $j = 1, \ldots, m$. By induction, the assumption on D implies that, for every $f, g \in D_+$, $m \in \mathbb{N}$, and $\rho > 0$, there exist $d_1, \ldots, d_m \in D$, such that $\|d_i^*d_j - \delta_{ij}g\| < \rho$ and $\|d_i^*fd_j\| < \rho$ for $i, j = 1, \ldots, m$. Let $b := \sum d_jb_j$. Then $\|b^*fb\| < m\cdot(\|a\|^2+\delta)\cdot\rho$ and $\|b^*b - \sum b_j^*b_j\| < m\cdot(1 + \|a\|^2 + \delta)\cdot\rho$. Thus $\|b^*fb\| < \delta$, $\|b^*b - a^2\| < 2\delta$, and $V = \delta_\infty \circ T$ satisfies also condition (β) of Proposition 5.4.1.

We are now in position to apply Parts (i) and (iv) of Proposition 5.4.1, and find an element S_1 in $\mathcal{M}(B)$, such that $\|S_1^*pS_1 - 1\| < 1$.

The element $I := pS_1(S_1^*pS_1)^{-1/2}$ is an isometry in $\mathcal{M}(B)$ with $II^* \leq p$.

Thus p is full and properly infinite in $\mathcal{M}(B)$.

(ii): The isometry $S \in \mathcal{M}(B)$ with $SS^* = p$ defines obviously an injective C^*-morphism $\varphi(b) := SbS^*$ $(b \in B)$ from B into $D = pBp$. It is surjective because $\varphi(SdS^*) = d$ for $d \in D$.

The hereditary C^*-subalgebra D of B is full in B. Therefore, the map $J \in \mathcal{I}(B) \mapsto D \cap J \in \mathcal{I}(B)$ is a bijective map. Since $S \in \mathcal{M}(B)$ we get that $SJS^* \subseteq D \cap J$. If $d \in D \cap J$, then $S^*dS \in J$ and $\varphi(S^*dS) = d$. Thus, $\varphi(J) = D \cap J$.

(iii):

If $\varphi(b) = S_1bS_1^* = S_2bS_2^*$ for isometries $S_1, S_2 \in \mathcal{M}(B)$ and $b \in B$, then we obtain that $S_1S_1^* = S_2S_2^* =: p$ from $S_1eS_1^* = S_2eS_2^*$ for for $e \in B$ in an approximate unit of B. Thus, $U := T^*S \in \mathcal{M}(B)$ is a unitary with $Ub = bU$ for all $b \in B$. Now use that B is strictly dense in $\mathcal{M}(B)$ the unitary U is in the center of $\mathcal{M}(B)$. Clearly $TU = S = UT$.

(iv): If $d \in B$ has the property that d^*d is strictly positive in B and dd^* is a strictly positive element of D, then the partial isometry $v \in B^{**}$ in the polar decomposition with $d := v(d^*d)^{1/2} = (dd^*)^{1/2}v$ satisfies, – by definition of the unique polar-decomposition in the W*-algebra B^{**} –, that v^*v is the open support projection $= 1$ of d^*d in B^{**} and vv^* is the support projection p of dd^* in B^{**}. Thus, $vv^* = p$, $v^*v = 1$.

Above formulas imply $B = \overline{(d^*d)^{1/2}B}$, $\overline{dB} = pB$ and $\overline{B(dd^*)^{1/2}} = Bp$. Using this equations, then we get $vB = \overline{dB} = pB \subseteq B$ from $d = v(d^*d)^{1/2}$, and $Bv = Bpv = \overline{Bd} \subseteq B$ from $(dd^*)^{1/2}v = d$.

Thus, v is an isometry in $\mathcal{M}(B)$ and can deserve as one of the above considered $S \in \mathcal{M}(B)$. □

(StC) looks strange? Better notation?

Corollary 5.5.2 ([**373**]). *A non-zero σ-unital C^*-algebra A is stable, if and only if, satisfies the following "stability criterium"* (StC)*:*

(StC) *For every $a \in A_+$ and $\varepsilon > 0$ there exists $d \in A$ such that $\|a - d^*d\| < \varepsilon$ and $\|d^*ad\| < \varepsilon$.*

Proof. Let $a \in A_+$ and $\varepsilon > 0$. If A is stable and $t_1, t_2, \ldots$ is a sequence of isometries in $\mathcal{M}(A)$ with $\sum t_k t_k^*$ strictly convergent to $1_{\mathcal{M}(A)}$, then there exists $n \in \mathbb{N}$ with $\|t_n^* a t_n\| < \varepsilon/(1 + \|a\|)$. If we let $d := t_n a^{1/2}$, then $d^*d = a$ and $\|d^*ad\| < \varepsilon$.

Conversely suppose that each for $a \in A_+$ and $\varepsilon > 0$, there exists $d \in A$ with $\|d^*ad\| < \varepsilon$ and $\|d^*d - a\| < \varepsilon$. Let $B := A \otimes \mathbb{K}$ and $D := A \otimes e_{11}$.

Then the corner D of the σ-unital stable C^*-algebra B satisfies the assumptions of Corollary 5.5.1. Thus, A is isomorphic to the stable algebra B. □

If A is σ-unital, then A/J is σ-unital. If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ are σ-unital C^*-subalgebras with $\bigcup_n A_n$ dense in A, then A is σ-unital: $e = \sum_n 2^{-n} e_n$ is strictly positive in A if e_n is strictly positive in A_n. This together implies that $\mathrm{indlim}(h_n \colon A_n \to A_{n+1})$ is σ-unital if each A_n is σ-unital.

It is easy to see that A satisfies the criteria of Corollary 5.5.2 if A is the inductive limit of an (arbitrary upward directed) family of C^*-algebras A_τ that satisfy the criteria (STC) if the inductive limit is still σ-unital, e.g. if the directed diagram of morphisms h_n is countable. Thus we get:

Corollary 5.5.3 ([**373**]). *The inductive limit of a* sequence *of σ-unital stable C^*-algebras is σ-unital and stable.*

A generalization of the Zhang's dichotomy [**846**] for simple purely infinite σ-unital C^*-algebras, is the following, cf. [**462**, thm. 4.24].

COROLLARY 5.5.4 (Generalized Zhang dichotomy). *A σ-unital purely infinite C^*-algebra B is stable if and only if B has no unital quotient.*

PROOF. Quotients of stable C^*-algebras are zero or stable. Thus B can not have a unital quotient if B is stable.

For the *non-trivial* direction suppose that B is p.i. and has no unital quotient. Let $0 \neq a \in B_+$, $0 < \varepsilon < 1$ and $\delta = \varepsilon/(1 + \|a\|)$.

The closed ideal J of B which is generated by the non-zero (!) annihilator $D := \mathrm{Ann}((a-\delta)_+, B)$ of $(a-\delta)_+$ in B is equal to B, because – otherwise – B/J is unital, $\mathrm{Spec}(\pi_J(a)) \subset \{0\} \cup [\delta/2, \|a\|]$ and $\pi_J(a)$ is a strictly positive element of B/J, cf. Lemma 2.5.14(i,ii) for more details. Since, by assumption, B has no unital quotient, it follows that $D = \mathrm{Ann}((a-\delta)_+, B)$ is a full hereditary C^*-subalgebra of B. Thus, there are $c_1, \ldots, c_n \in D_+$ and $e_1, \ldots, e_n \in B$ with $\|a - \sum e_k^* c_k e_k\| < \delta/2$. By the definition of p.i. algebras there is $f \in B$ with $\|a - f^*cf\| < \delta$, where $c := c_1 + \ldots + c_n \in D_+$. Let $d := c^{1/2} f$. Then $\|d^*d - a\| < \delta < \varepsilon$, $d^*ad = d^*(a - (a-\delta)_+)d$ and $\|d^*ad\| \leq \delta(\|a\| + \delta) < \varepsilon$. Thus, B satisfies the criteria (STC) of Corollary 5.5.2 and is stable. □

REMARK 5.5.5.

(i) Corollary 5.5.4 implies Zhang's dichotomy [**846**] for simple purely infinite algebras:

A σ-unital simple purely infinite C^*-algebra is unital or is stable.

(ii) Proposition 2.2.1, Brown's stable isomorphism theorem and the dichotomy together immediately imply that every hereditary C^*-subalgebra D of a simple purely infinite C^*-algebra A has an approximate unit consisting of projections. By a theorem of Brown and Pedersen, [**113**], this property is equivalent to the property that the real rank of A is zero, i.e., every selfadjoint element can be approximated by selfadjoint elements of A with finite spectra.

Thus *every simple purely infinite C^*-algebra has real rank zero.* See also the proof of Proposition 2.2.1(x).

(iii) We don't use for our results on classification that simple purely infinite algebras A have zero real rank. ($rr(A) = 0$ can be seen also by the asymptotic methods in Chapter 7 or immediately from Theorem B.)

The following Corollary 5.5.6 is implicitly contained in the proofs of [**107**]. `more precise ref` We derive it from Corollary 5.5.1.

COROLLARY 5.5.6 (Ψ-equivariant Brown stable isomorphism). *Suppose that D is a stable σ-unital hereditary C^*-subalgebra of a stable σ-unital C^*-algebra B, such that BDB is dense in B.*

Then there exists $d \in B$, such that

(i) *d^*d is a strictly positive element of B,*
(ii) *dd^* is a strictly positive element of D,*

(iii) *the polar decomposition* $v(d^*d)^{1/2}$ *of* d *in* B^{**} *defines an isomorphism* $\varphi := v(\cdot)v^*$ *from* B *onto* $D \subset B$*, such that*

(iv) $\varphi\colon B \to B$ *is unitarily homotopic to* id_B*, in particular* $\varphi(J) = D \cap J$ *for* $J \in \mathcal{I}(B)$*.*

It follows: if D_1 *is a* σ*-unital hereditary* C^**-subalgebra of a* σ*-unital* C^**-algebra* B_1*, which is not contained in a closed ideal* $J \neq B_1$*, then there exists a* **-isomorphism* ψ *from* $D_1 \otimes \mathbb{K}$ *onto* $B_1 \otimes \mathbb{K}$ *such that, for* $J \in \mathcal{I}(B_1)$*,*

$$\psi((D_1 \cap J) \otimes \mathbb{K}) = J \otimes \mathbb{K}.$$

Proof. Let t and s isometries in $\mathcal{M}(B)$ with $t^*s = 0$, and let E denote the hereditary C^*-subalgebra of B which is generated by $F := sDs^* + tBt^*$. Then $\mathrm{id}\,|F$ is a non-degenerate *-monomorphism from F into E. It extends to a unital strictly continuous *-monomorphism $H := \mathcal{M}(\mathrm{id}\,|E)$ from $\mathcal{M}(F) \cong \mathcal{M}(sDs^*) \oplus \mathcal{M}(tBt^*)$ into E. It follows that sDs^* and tBt^* generate a corner of E. But both are hereditary C^*-subalgebras of B, which generate B as closed ideal.

Thus, sDs^* and tBt^* are corners of E that generate E as closed ideal.

F is stable, because sDs^* and tBt^* are stable, indeed: If $s_1, s_2, \ldots$ is a sequence of isometries in $\mathcal{M}(F)$, such that $\sum_n s_n s_n^*$ converges strictly to 1, then $H(s_1), H(s_2), \ldots$ has the same property for E. Thus, E is stable by Remark 5.1.1(8).

Since $sDs^* \cong D$, tBt^* and and E are stable, Corollary 5.5.2 implies that the pairs $sDs^* \subset E$ and $tBt^* \subset E$ satisfy the assumptions of Corollary 5.5.1 (in place of the pair $D \subset B$). It follows, that there are elements $d_1, d_2 \in E$, such that $d_1^*d_1$ and $d_2^*d_2$ are strictly positive elements of E, $d_1d_1^*$ is a strictly positive element of sDs^* and $d_2d_2^*$ is a strictly positive element of tBt^*. Then $d := s^*d_1d_2^*t$ satisfies that d^*d is a strictly positive element of B and dd^* is a strictly positive element of D. Let $v(d^*d)^{1/2}$ be the polar decomposition. By Remark 2.3.1, $\varphi(a) = vav^*$ is an isomorphism from B onto D, and $\varphi^{-1} : D \to B$ is given by $a \mapsto v^*av$.

By Lemma 5.1.2(iv) there exists a norm-continuous map $t \in [1, \infty) \mapsto S_0(t)$ into the isometries in $\mathcal{M}(B)$ with $\lim_{t\to\infty} S_0(t)b = 0$ for $b \in B$. Let $G := \mathrm{C_b}([1,\infty), \mathcal{M}(B))/\,\mathrm{C_0}([1,\infty), B) \supset B$. Then $\mathcal{M}(B) \subset G$ naturally and G contains a copy of $\mathcal{O}_2$ unitally. Let $h_1, h_2\colon B \to E$ given by $h_1(b) := b \in F$??? or G and $h_2(b) := h_1(\varphi(b))$ for $b \in B$. Now consider the isometry $s := \{S_0(t)\}_{1 \le t < \infty} + \mathrm{C_0}(\mathbb{R}_+, B)$ in G which corresponds to $S_0 \in \mathrm{C_b}([0,\infty), \mathcal{M}(B))$, and the contraction $y \in G$ with representative $Y(t) := d^*(dd^* + 1/t)^{-1/2}$ for $t \in [0, \infty)$.

Then $s^*h_j(\cdot)s = 0$ for $j = 1, 2$, $y^*h_1(\cdot)y = h_2$ and $yy^*h_1(\cdot)yy^* = h_1$ Thus Proposition 4.3.6(iii) applies: there is a unitary u in F with $u^*h_1(\cdot)u = h_2$.

The unitary u comes from a continuous map $U(t)$ from $[0, \infty)$ into the unitaries of $\mathcal{M}(B)$ such that $u = \{U(t)\}_{t\in[0,\infty)} + \mathrm{C_0}(\mathbb{R}_+, B)$, cf. proof of Lemma 5.1.2(vi).

Since the unitary group of $\mathcal{M}(B)$ is (norm-) connected ([**180**]), one can choose such that $t \mapsto U(t)$ such that $U(0) = 1$. Clearly, $u^*h_1(\cdot)u = h_2$ means that

$\lim_{t\to\infty} \|U(t)^* bU(t) - \varphi(b)\| = 0$ for $b \in B$, i.e., that id_B and φ are unitarily homotopic.

It follows $\varphi(J) \subset D \cap J$ and $\varphi^{-1}(D \cap J) \subset J$, which means $\varphi(J) = D \cap J$ for $J \in \mathcal{I}(B)$. □

next Lemma is the old Lemma 7.15,
which is now n o t cited in Chapter 7

LEMMA 5.5.7. *Suppose that A is a* full *hereditary C^*-subalgebra of B (i.e., $ABA = A$ and $\overline{\mathrm{span}(BAB)} = B$), and that A and B are both σ-unital and stable. Then there exist x in B and an isomorphism φ from B onto A such that x^*x is strictly positive in B, xx^* is a strictly positive element in A and $\varphi(b) = ubu^*$ where u can be chosen as any contraction in B^{**} that satisfies $u^*x = (x^*x)^{1/2}$ and $xu^* = (xx^*)^{1/2}$.*

Moreover, $\varphi \colon B \to A \subset B$ is unitarily homotopic to id_B.

PROOF. The proof is implicitly contained in the proofs of Corollaries 5.5.1 and 5.5.6:
It is enough to find an $x \in B$ with x^*x strictly positive in B and xx^* strictly positive in A. We proceed as in [**107**]. Let $D \subset M_2(B)$ be the hereditary C^*-subalgebra which is generated by $\mathrm{diag}(e, f)$ where e is a strictly positive element of B and f is a strictly positive element of A. Then $F = \mathrm{diag}(A, B) \cong B \oplus A$ is naturally a C^*-subalgebra of D such that D is the closure of FDF. It follows that $\mathcal{M}(F)$ is unitally and strictly continuous contained in $\mathcal{M}(D)$. F is stable because A and B are stable. Thus there exists a copy C of $\mathbb{K}$ in $\mathcal{M}(F)$ with $\overline{CF} = F = \overline{FC}$. Then $C \subset \mathcal{M}(D)$ and $\overline{CD} = \overline{CFDF} = \overline{FDF} = D$ and similarly $\overline{DC} = D$. Then $\mathcal{M}(C) \subset \mathcal{M}(D)$ by a strictly continuous inclusion map such that $1_{\mathcal{M}(C)} \mapsto 1_{\mathcal{M}(D)}$. It follows that D is isomorphic to $p_{11}Dp_{11} \otimes \mathbb{K}$ where p_{11} is a rank one projection in C. To see this, we take $b_i \in \mathcal{M}(D)$ with $b_i^* b_j = \delta_{ij} p_{11}$ and with C the strict closure of the span of the $b_j b_i$ $(i, j = 1, 2, 3, \ldots)$. Since $\{\sum_{i=1}^n b_i b_i^*\}$ converges strictly to $1_{\mathcal{M}(D)}$, $d \mapsto (b_j^* d b_i)_{ji}$ defines a *-isomorphism from D onto $(p_{11}Dp_{11}) \otimes \mathbb{K}$. Since F is isomorphic to $B \oplus A$, $1_{\mathcal{M}(B)} \oplus 0$ and $0 \oplus 1_{\mathcal{M}(A)}$ define projections p, $q \in \mathcal{M}(D)$ with $p + q = 1$ and p, $q \in C'$. By definition of F we have that p, $q \in M_2(B)^{**} = M_2(B^{**})$ are given by $p = \mathrm{diag}(1, 0)$ and $q = \mathrm{diag}(0, r)$ where r is the open support projection of A. Since p, q are in C' it follows that under the isomorphism $D \cong (p_{11}Dp_{11}) \otimes \mathbb{K}$ (defined by C) p, q define $\bar{p} = pp_{11}$, $\bar{q} = qp_{11}$ in $\mathcal{M}(p_{11}Dp_{11})$ with $\bar{p} + \bar{q} = 1$. Furthermore one gets $p \cong \bar{p} \otimes 1_{\mathcal{M}(\mathbb{K})}$ and $q \cong \bar{q} \otimes 1_{\mathcal{M}(\mathbb{K})}$. ($\mathcal{M}(D) \cong \mathcal{M}(p_{11}Dp_{11}) \otimes \mathbb{K}$ is described in $\mathrm{Mat}_\infty(p_{11}Dp_{11})$ again by $T \in \mathcal{M}(D) \mapsto (b_j^* T b_i)_{ij}$ and we get $p \mapsto (b_j^* p b_i) = (b_j^* b_i p) = (\delta_{ij} p_{11} p) \otimes 1_{\mathcal{M}(\mathbb{K})}$. Similarly q maps to $p_{11} p \otimes 1_{\mathcal{M}(\mathbb{K})}$. $\bar{p}(p_{11}Dp_{11}\bar{p}$ generates $p_{11}Dp_{11}$ as a closed ideal because $(\bar{p} \otimes 1)(p_{11}Dp_{11} \otimes \mathbb{K})(\bar{p} \otimes 1) \cong pDp$ and pDp generates D as a closed ideal. A similar argument shows that $\bar{q}p_{11}Dp_{11}\bar{q}$ generates $p_{11}Dp_{11}$ as a closed ideal, i.e., $\bar{p}$ and $\bar{q}$ are full projections and $\bar{p} + \bar{q} = 1$).

Since D contains a strictly positive element, $p_{11}Dp_{11}$ also contains one. Thus the conditions of [**107**, lem. 2.5] are fulfilled and we get isometries $\bar{v}$, $\bar{w}$ in $\mathcal{M}(p_{11}Dp_{11} \otimes \mathbb{K})$ with $\bar{v}^*\bar{v} = \bar{w}^*\bar{w} = 1$ and $\bar{v}\bar{v}^* = \bar{p} \otimes 1_{\mathcal{M}(\mathbb{K})}$, $\bar{w}\bar{w}^* = \bar{q} \otimes 1_{\mathcal{M}(\mathbb{K})}$. In particular $\bar{p} \otimes 1_{\mathcal{M}(\mathbb{K})}$ and $\bar{q} \otimes 1_{\mathcal{M}(\mathbb{K})}$ are Murray- von Neumann equivalent in $\mathcal{M}(p_{11}Dp_{11} \otimes \mathbb{K})$. Now we use the isomorphism to $\mathcal{M}(D)$ and get a partial isometry $v \in \mathcal{M}(D)$ with $v^*v = p$ and $vv^* = q$. If we consider v as an element in $M_2(B^{**}) \supset \mathcal{M}(D)$ we get $v^*v = \operatorname{diag}(1,0)$, $vv^* = \operatorname{diag}(0,r)$. Let $y = v\operatorname{diag}(e,0)$, then y satisfies $y \in D$, $y^*y = \operatorname{diag}(e^2,0)$ is strictly positive in $pDp \cong B \oplus 0$, $yy^* = v\operatorname{diag}(e^2,0)v^*$ is strictly positive in $qDq \cong 0 \oplus A$. Thus we get that $y = (a_{ij})_{ij} \in M_2(B)$ with only a_{12} nonzero. Let x be a_{12} and $\varphi(b) = ubu^*$ for any u in B^{**} with $u^*x = (x^*x)^{\frac{1}{2}}$, $xu^* = (xx^*)^{\frac{1}{2}}$ and $\|u\| \leq 1$ (e.g. coming from the polar decomposition of x). Since φ is a completely positive contraction, it is sufficient to check that φ is multiplicative on the dense subset x^*Bx of B and here we only need that $xx^* = xu^*ux^*$ holds.

Similarly one checks that φ maps all onto the closure A of $(xx^*)^{1/2}B(xx^*)^{1/2}$, and that $\psi(a) := u^*au$ for $a \in A$ is the inverse of φ, cf. Remark **??**. □

REMARK 5.5.8. *Extensions E (given by an exact sequence $0 \to B \to E \to A \to 0$) of stable separable algebras A by stable separable algebras B are in general not stable.* E.g. there is an extension E of $A := \mathbb{K}$ by $B := C(\mathcal{Z}, \mathbb{K})$ for the Tychonoff product $\mathcal{Z}$ of countably many copies of the 2-dimensional sphere S^2, cf. [**685**].

We show that every extension E of a separable stable C^*-algebra A by a purely infinite σ-unital stable B is stable.

This follows from some more general stability criteria that we discuss now.

Notice that it it suffices to consider the case $A := \mathrm{C}_0(\mathbb{R}_+, \mathbb{K})$ by Remark 5.1.1(9).

LEMMA 5.5.9. *Suppose that B is stable and σ-unital, and that D is σ-unital. Let $\mathrm{Q}(B) := \mathcal{M}(B)/B$ and suppose that $h_1, h_2 \colon D \to \mathrm{Q}(B)$ are unitarily equivalent C^*-morphisms. Then:*

- (o) *$\mathcal{M}(B)$ and $\mathrm{Q}(B)$ have property* (sq) *of Definition 4.2.14 (and are K_1-bijective by Lemma 4.2.6(v) and Proposition 4.2.15).*
- (i) *$\mathrm{K}_*(\mathcal{M}(B)) = 0$ and $\mathcal{U}_0(\mathcal{M}(B)) = \mathcal{U}(\mathcal{M}(B))$.*
- (ii) *$E := \mathrm{Q}(B)$ contains a copy of $\mathcal{O}_2$ unitally, and is K_1-bijective, i.e., the natural map $\mathcal{U}(E) \to \mathrm{K}_1(E)$ defines an isomorphism $\mathcal{U}(E)/\mathcal{U}_0(E) \cong \mathrm{K}_1(E) \cong \mathrm{K}_0(B)$.*
- (iii) *If h_1 dominates zero in the sense of Definition 4.3.3 then there is a unitary $U \in \mathcal{U}(\mathcal{M}(B))$ such that $\pi_B(U)^*h_1(\cdot)\pi_B(U) = h_2(\cdot)$.*

  In particular, h_1 and h_2 are unitary equivalent by a unitary in $\mathcal{U}_0(E)$.

PROOF. (i): The triviality $\mathrm{K}_*(\mathcal{M}(B)) = 0$ easily follows from the unitary equivalence of δ_∞ and $\delta_\infty \oplus \mathrm{id}$.
The equation $\mathcal{U}_0(\mathcal{M}(B)) = \mathcal{U}(\mathcal{M}(B))$ was found by J. Cuntz and N. Higson [**180**],

see also [**557**] in case that B contains a full projection. Modification of an observation in [**180**] leads to a slightly stronger observation in Lemma A.23.1, which implies that $\mathcal{M}(B)$ satisfies the "squeezing" property (sq) of Definition 4.2.14, cf. Proposition 4.2.15. Thus, $\mathcal{M}(B)$ and, for every C^*-algebra A, every C^*-algebra quotient of $\mathcal{M}(B) \otimes^{\max} A$ is K_1-bijective by Proposition 4.2.15.

(ii): $\mathcal{O}_2 \subset \mathcal{L}(\mathcal{H}) = \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B)$ unitally. Thus, $E := \mathcal{M}(B)/B$ contains a copy of $\mathcal{O}_2$ unitally. E is K_1-surjective by Lemma 4.2.6(v) and is K_1-injective by Part (i). This follows also from Proposition 4.2.15.

(iii): The maps $h_1, h_2 \colon D \to E$ satisfy the requirements of Proposition 4.3.6(iv,d) by Part (ii): There exists $W \in \mathcal{U}(E)$ with $W^* h_1(\cdot) W = h_2$. The C^*-morphism h_1 dominates zero in the sense of Definition 4.3.3 if and only if there exists an isometry $s \in E$ with $s^* h_1(D) s = \{0\}$. Then $V := \big(sW^*s^* + (1 - ss^*)\big)W$ is a unitary in $\mathcal{U}(E)$ with $V^* h_1(\cdot) V = h_2$ and $0 = [V] \in \mathrm{K}_1(E)$. Thus, $V \in \mathcal{U}_0(E)$ by the K_1-injectivity of $E = \mathrm{Q}(B) = \mathcal{M}(B)/B$. Since V is in $\mathcal{U}_0(\mathcal{M}(B)/B)$ it is a finite product of exponentials $\exp(t_k)$, of $-t_k^* = t_k \in \mathcal{M}(B)/B$, $k = 1, \ldots, n$. There are $T_k = -T_k^*$ in $\mathcal{M}(B)$ with $\pi_B(T_k) = t_k$. Thus, $V = \pi_B(U)$ for the unitary $U := \exp(T_1) \cdot \ldots \cdot \exp(T_n)$ in $\mathcal{M}(B)$. □

The following Lemma 5.5.10 summarizes consequences of Proposition 4.3.6 and of the generalized Tietze extension theorem of Akemann, Pedersen and Tomiyama in the σ-unital case (see [**616**, prop. 3.12.10] for the *separable case*).

LEMMA 5.5.10. *Suppose that A and B are σ-unital, and that $h \colon A \to \mathcal{M}(B)/B$ is a C^*-morphism. Let J denote the kernel of h, and define*

$$\mathcal{M}(A, J) := \{T \in \mathcal{M}(A)\,;\; TA \cup AT \subset J\}\,.$$

Further, let $C := h(A)$, and let $\mathcal{N}(C) \subset \mathcal{M}(B)/B$ the normalizer algebra of C in $\mathcal{M}(B)/B$, i.e., $\mathcal{N}(C) := \{b \in \mathcal{M}(B)/B\,;\; bC \cup Cb \subset C\}$, and let $\mathrm{Ann}(C) :=$ *$\{b \in \mathcal{M}(B)/B \;\; bC = 0 = Cb\}$ denote the (two-sided) annihilator of C in $\mathcal{M}(B)/B$. Define $L \colon \mathcal{N}(C) \to \mathcal{M}(C)$ by $L(b)(c) := bc$ for $c \in C$ and $b \in \mathcal{N}(C)$.*

Then L is a C^*-morphism, and the following properties are valid:

(i) *h extends naturally to an epimorphism $\mathcal{M}(h)$ from $\mathcal{M}(A)$ onto $\mathcal{M}(C)$ with kernel $\mathcal{M}(A, J)$ (generalized Tietze extension).*
(ii) *If, in addition, h has a non-degenerate lift to a C^*-morphism $h_1 \colon A \to \mathcal{M}(B)$, then $\pi_B(\mathcal{M}(h_1)(\mathcal{M}(A))) \subset \mathcal{N}(C)$, and $\mathcal{M}(h) = L \circ \pi_B \circ \mathcal{M}(h_1)$.*
(iii) *The natural C^*-morphism L from the normalizer $\mathcal{N}(C) \subset \mathcal{M}(B)/B$ of C into the multiplier algebra $\mathcal{M}(C)$ of C is an epimorphism. The kernel of L is* $\mathrm{Ann}(C)$.
(iv) *If h dominates zero, cf. Def. 4.3.3, and if S is any isometry in $\mathcal{M}(C)$, then there exists a unitary U in the connected component of* 1 *in the unitaries of $\mathcal{M}(B)$, such that $\pi_B(U)^* h(\cdot) \pi_B(U) = Sh(\cdot)S^*$.*

PROOF. (i): The morphism $h \colon A \to C$ is non-degenerate, because h is a *-epimorphism from A onto C. Thus, there is a unique unital strictly continuous

C^*-morphism $\mathcal{M}(h)$ from $\mathcal{M}(A)$ into $\mathcal{M}(C)$. By definition of $\mathcal{M}(h)$, $\mathcal{M}(h)(T) = 0$, if and only if, $h(TA \cup AT) = \{0\}$.

The morphism $\mathcal{M}(h)$ is an epimorphism, because A is σ-unital. Indeed: This is, — in the separable case —, the generalized Tietze extension theorem [**616**, prop. 3.12.10]. If A is not separable — but σ-unital —, then one can reduce the proof to the separable case ([16]), as follows:
Let $c \in \mathcal{M}(C)_+$, let $e \in A_+$ a strictly positive element of A and take $b \in A$ with $h(b) = c \cdot h(e)$, by using that $h\colon A \to C$ is surjective. Then the separable C^*-subalgebras $F := C^*(b, e) \subseteq A$, and $G := h(F) \subseteq C$ satisfy $\mathcal{M}(F) \subseteq \mathcal{M}(A)$ and $c \in \mathcal{M}(G) \subseteq \mathcal{M}(C)$, $\mathcal{M}(h)|\mathcal{M}(F) = \mathcal{M}(h|F)$ and $\mathcal{M}(h|F)(\mathcal{M}(F)) = \mathcal{M}(G)$.

(ii): Let $b \in \mathcal{M}(A)$, $c \in C$ and $a \in A$ with $\pi_B(h_1(a)) = h(a) = c$. Then $\mathcal{M}(h)(b)c = h(ba) = \pi_B(h_1(ba))$, and $\pi_B(\mathcal{M}(h_1)(b))c = \pi_B(\mathcal{M}(h_1)(b)h_1(a))$.

(iii): It is obvious that $\mathrm{Ann}(C) \subset \mathcal{N}(C)$, and that the natural C^*-morphism L from $\mathcal{N}(C)$ into $\mathcal{M}(C)$ has kernel $\mathrm{Ann}(C)$.

We show that $\mathcal{N}(C) \to \mathcal{M}(C)$ is surjective:

Consider $D := (\pi_B)^{-1}(C)$ and the *-epimorphism $k := \pi_B|D$. Since C has a strictly positive element $d = h(e)$, and B has a strictly positive element p, the element $p + q$ is a strictly positive element of D, if $q \in D_+$ is such that $k(q) = d$.

By part (i), $\mathcal{M}(k)$ is an epimorphism from $\mathcal{M}(D)$ onto $\mathcal{M}(C)$.

The lifted C^*-morphism $k_1 := \mathrm{id}\,|D$ of k is non-degenerate, because $B \subset D$. Thus, $\mathcal{M}(k_1)$ is a unital map from $\mathcal{M}(D)$ into $\mathcal{M}(B)$, and $\pi_B(\mathcal{M}(k_1)(\mathcal{M}(D))) \subset \mathcal{N}(C)$.

By part (ii), the image of the natural C^*-morphism $\mathcal{N}(C) \to \mathcal{M}(C)$ contains the image of $\mathcal{M}(k)$, i.e., $\mathcal{N}(C) \to \mathcal{M}(C)$ is an epimorphism.

(iv): Let $h_2 := Sh(\cdot)S^*$. Then h_2 is a C^*-morphism from A into $h(A) = C \subset \mathcal{M}(B)/B$. By assumption, h dominates zero, i.e., there exists an isometry $V \in \mathcal{M}(B)/B$ such that $V^*h(A)V = \{0\}$. Thus, $V^*h_2(\cdot)V = 0$.

There exists a contraction y in $\mathcal{N}(C)$ such that $L(y) = S$ by part (ii). It follows $y^*h_2(\cdot)y = h$ and $yh(\cdot)y^* = h_2$.

By Proposition 4.3.6(iii), h_2 and h are unitarily equivalent by a unitary $u_1 \in \mathcal{M}(B)/B$. By Proposition 4.3.6(iv,c) the unitary equivalence can be realized by

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[16] There is a direct argument: Since $\mathcal{M}(h)(\mathcal{M}(A))$ contains C, it suffices to show that for a given contraction $c \in \mathcal{M}(C)_+$ there exist $d \in \mathcal{M}(A)_+$ with $\mathcal{M}(h)(d) - c \in C$. Let $e \in A_+$ be a strictly positive element of A with $\|e\| = 1$, and let $d := h(e)$. By Remark 5.1.1(3), there exist continuous functions $0 \leq f_n \leq 1$, with $f_n(0) = 0$, $f_{n+1}f_n = f_n$, such that $\lim f_n(t) = 1$ for $t \in (0, 1]$ and that the commutator norms $\|[g_n(d), c]\|$ are less than 2^{-n}, where $g_1 = f_1^{1/2}$, and $g_n := (f_{n+1} - f_n)^{1/2}$ for $n > 1$. Let $b_n \in A_+$ contractions with $h(b_n) = f_{n+2}(d)cf_{n+2}(d)$. By Remark 5.1.1(4), $s := \sum g_n(a)b_ng_n(a)$ and $t := \sum g_n(d)cg_n(d)$ are strictly convergent series in $\mathcal{M}(A)$ and $\mathcal{M}(C)$, respectively. Then $\mathcal{M}(h)(s) = t$ and $c - t \in C$, because $h(g_n(e)b_ng_n(e)) = g_n(d)cg_n(d)$, $\mathcal{M}(h)$ is strictly continuous, and $c - t$ is the sum of the norm-convergent series $\sum x_n$, where with $x_n := g_n(d)[g_n(d), c] \in C$.

a unitary $u = \pi_B(U)$ in $\pi_B(\mathcal{U}_0(\mathcal{M}(B))) = \mathcal{U}_0(M(B)/B)$, because $V^*(h(a) + h_2(a))V = 0$ for $a \in A$. □

LEMMA 5.5.11. *Suppose that A and B are σ-unital and let $h\colon A \to \mathrm{Q^s}(B)$ a C^*-morphism.*

Choose an isomorphism $\psi\colon \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ from $\mathbb{K} \otimes \mathbb{K}$ onto $\mathbb{K}$.

Let γ denote the natural isomorphism from $(B \otimes \mathbb{K}) \otimes \mathbb{K}$ onto $B \otimes (\mathbb{K} \otimes \mathbb{K})$, i.e., with $\gamma((b \otimes k_1) \otimes k_2) = b \otimes (k_1 \otimes k_2)$ for $b \in B$, $k_1, k_2 \in \mathbb{K}$, and let

$$\Phi_1\colon \mathcal{M}(B \otimes \mathbb{K}) \otimes \mathbb{K} \to \mathcal{M}(B \otimes \mathbb{K})$$

the non-degenerate monomorphism, is defined by

$$\Phi_1 := \mathcal{M}((\mathrm{id}_B \otimes \psi) \circ \gamma) | \mathcal{M}(B \otimes \mathbb{K}) \otimes \mathbb{K}.$$

(i) *There are isometries $s_0, t_0 \in \mathcal{M}(\mathbb{K})$ and a norm-continuous map $\tau \in (0,1] \mapsto I_\tau \in \mathcal{M}(\mathbb{K})$ into the isometries in $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(H)$, such that, for $b \in \mathbb{K}$,*

$$\begin{aligned} s_0 s_0^* + t_0^* t_0 &= 1, \\ \mathcal{M}(\psi)(1_{\mathcal{M}(\mathbb{K})} \otimes p_{11}) &= s_0 s_0^*, \\ s_0^* \psi(k \otimes p_{11}) s_0 &= k, \\ (I_\tau)^* \mathcal{M}(\psi)(1_{\mathcal{M}(\mathbb{K})} \otimes b) I_\tau &\in \mathbb{K}, \quad \text{and} \\ \lim_{\tau \to 0} \|(I_\tau)^* \mathcal{M}(\psi)(1_{\mathcal{M}(\mathbb{K})} \otimes b) I_\tau\| &= 0\,. \end{aligned}$$

(ii) *Let $r_1 \in \mathcal{M}(\mathrm{C}_0((0,1], \mathbb{K}))$ be the isometry, which is defined by $(r_1 f)(\tau) := I_\tau f(\tau)$ and $(r_1 f)(0) = 0$ for $f \in \mathrm{C}_0((0,1], \mathbb{K})$, $\tau \in (0,1]$. Let e a strictly positive element of B with $\|e\| = 1$, and let μ the non-degenerate C^*-morphism from $\mathrm{C}_0((0,1])$ into B with $\mu(g_0) = e$, where $g_0(\tau) := \tau$. Now define $r = \mathcal{M}(\mu \otimes \mathrm{id}_{\mathbb{K}})(r_1)$.*

Then Φ_1 and the isometries $s := 1_{\mathcal{M}(B)} \otimes s_0$, $t := 1_{\mathcal{M}(B)} \otimes t_0$ and r in $\mathcal{M}(B \otimes \mathbb{K})$ satisfy, for $b \in \mathcal{M}(B \otimes \mathbb{K})$,

$$\begin{aligned} ss^* + tt^* &= 1, \\ ss^* &= \mathcal{M}(\mathrm{id}_B \otimes \psi)(1_{\mathcal{M}(B)} \otimes (1_{\mathcal{M}(\mathbb{K})} \otimes p_{11})), \\ \Phi_1(b \otimes p_{11}) &= sbs^*, \\ r^* \Phi_1(\mathcal{M}(B \otimes \mathbb{K}) \otimes \mathbb{K}) r &\subset B \otimes \mathbb{K}, \quad \text{and} \\ \Phi_1((B \otimes \mathbb{K}) \otimes \mathbb{K}) &= B \otimes \mathbb{K}. \end{aligned}$$

(iii) $\Phi := [\Phi_1]\colon \mathrm{Q^s}(B) \otimes \mathbb{K} \to \mathrm{Q^s}(B)$ *is a monomorphism, which dominates zero and satisfies $\Phi(b \otimes p_{11}) = S^* b S$ with $S := \pi_{B \otimes \mathbb{K}}(s)$. The range projection of S is $SS^* = \Phi(1 \otimes p_{11})$.*

(iv) *h is unitarily equivalent to $\Phi(h(\cdot) \otimes p_{11})$ if and only if h dominates zero in the sense of Definition 4.3.3. If this is the case, then the unitary equivalence can be given by a unitary in the image of the unitaries in $\mathcal{M}(B \otimes \mathbb{K})$.*

(v) $\Phi \circ (h \otimes \mathrm{id}_{\mathbb{K}})$ *is the Busby invariant of the stable extension*

$$0 \to (B \otimes \mathbb{K}) \otimes \mathbb{K} \to E \otimes \mathbb{K} \to A \otimes \mathbb{K} \to 0,$$

where we identify $B \otimes \mathbb{K}$ *and* $(B \otimes \mathbb{K}) \otimes \mathbb{K}$ *naturally by* $(\mathrm{id}_B \otimes \psi) \circ \gamma$, *and define*

$$0 \to B \otimes \mathbb{K} \to E \to A \to 0,$$

as the extension corresponding to $h\colon A \to \mathrm{Q^s}(B)$.

(vi) *If* A *is stable, then* $\Phi(h(\cdot) \otimes p_{11})$ *is the Busby invariant of a stable extension.*

Proof. (i): $p := \mathcal{M}(\psi)(1_{\mathcal{M}(\mathbb{K})} \otimes p_{11})$ and $1 - p$ are infinite projections in $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(H)$. Thus, there are isometries $s_1, t_0 \in \mathcal{M}(\mathbb{K})$ with $s_1 s_1^* = p$ and $t_0 t_0^* = 1 - p$. Then $b \in \mathcal{M}(\mathbb{K}) \mapsto s_1^* \mathcal{M}(\psi)(b \otimes p_{11}) s_1$ is a unital weakly continuous *-endomorphism on $\mathcal{M}(\mathbb{K})$. Its image contains the corner $\psi(\mathbb{K} \otimes p_{11})$ of $\mathbb{K}$. Therefore, it is an isomorphism of $\mathcal{M}(\mathbb{K})$, and there exist a unitary $u \in \mathcal{M}(\mathbb{K})$ with $ubu^* = s_1^* \mathcal{M}(\psi)(b \otimes p_{11}) s_1$ for $b \in \mathcal{M}(\mathbb{K})$. Let $s_0 := s_1 u$, then s_0 and t_0 have the desired properties.

The construction of I_τ goes as follows. We let

$$\begin{aligned} p_n &:= \sum_{j=1}^{n} p_{jj} \\ q_n &:= \mathcal{M}(\psi)(p_n \otimes (1 - p_n)) \\ e_n &:= \mathcal{M}(\psi)(p_{n+2} \otimes (1 - p_n)). \end{aligned}$$

Then, $q_n \leq e_n$, $q_{n+1} \leq e_n$, and q_n, $e_n - q_n$, $e_n - q_{n+1}$ are infinite projections. It follows that there exists a unitary v commuting with e_n, such that $vq_nv^* = q_{n+1}$. Since $\mathcal{M}(\mathbb{K})$ is isomorphic to the von-Neumann algebra $\mathcal{L}(H)$, there is a $T \in \mathcal{M}(\mathbb{K})_+$ such that T commutes with e_n and $v = e^{iT}$.

Let I an isometry in $\mathcal{M}(\mathbb{K})$ with $II^* = q_n$. Then $I_\tau := e^{iT(\tau)}v$, with $T(\tau) := ((n+1) - (n+1)n\tau)T$, defines a norm-continuous map from $[1/(n+1), 1/n]$. into the isometries of $\mathcal{M}(\mathbb{K})$, such that $I_{1/n} = I$, $I_{n+1} I_{n+1}^* = q_{n+1}$ and $I_\tau I_\tau^* \leq e_n$ for $\tau \in (1/(n+1), 1/n)$.

By induction we get the norm-continuous map

$$\tau \in (0, 1] \mapsto I_\tau \in \mathcal{M}(\mathbb{K})$$

into the isometries of $\mathcal{M}(\mathbb{K})$ with the desired properties.

(ii): The properties of Φ_1, s and t can be seen by simple calculations on the level of elementary tensors. Then use that all the maps in question are strictly continuous, to get the general result.

On the property of r: It suffices to show that $r^*(1 \otimes \mathcal{M}(\psi)(1 \otimes \mathbb{K}))r$ is contained in $C^*(e) \otimes \mathbb{K}$.

The properties of $\tau \mapsto I_\tau$ in part (i) mean that $(r_1)^*(1 \otimes \mathcal{M}(\psi)(1 \otimes \mathbb{K}))r_1$ is contained in $\mathrm{C}_0((0, 1], \mathbb{K})$. Now apply $\mathcal{M}(\mu \otimes \mathrm{id}_{\mathbb{K}})$.

(iii): With $R := \pi_{B\otimes\mathbb{K}}(r)$ we have $R^*R = 1$ and $R^*\Phi(\cdot)R = 0$ by (ii), i.e., Φ dominates zero. The rest are straight calculations.

(iv): Since Φ dominates zero by (ii), also $\Phi(h(\cdot)\otimes p_{11}) = Sh(\cdot)S^*$ dominates zero. Thus, if h is unitarily equivalent to $h_2 := \Phi(h(\cdot)\otimes p_{11})$, then h dominates zero.

Conversely, if $V^*h(\cdot)V = 0$ for some isometry $V \in \mathrm{Q^s}(B)$, and $h_2 = Sh(\cdot)S^*$, then $W^*h_2(\cdot)W = 0$ for the isometry $W := SVS^* + (1 - SS^*)$. Thus, by Proposition 4.3.6(iii), there is a unitary u with $u^*h(\cdot)u = h_2$, because $h_2 = Sh(\cdot)S^*$, $h = S^*h_2(\cdot)S$. By Lemma 5.5.9(iii) the unitary equivalence can be realized by a unitary $u = \pi_B(U)$ in $\pi_B(\mathcal{U}_0(\mathcal{M}(B)/B))$.

(v): Straightforward calculations.

(vi): Let D be a σ-unital C^*-algebra, and let φ an isomorphism from $D\otimes\mathbb{K}$ onto A. Let $h_1 := h\circ\varphi$.

Let $\psi\colon \mathbb{K}\otimes\mathbb{K}\to\mathbb{K}$ be the isomorphism from $\mathbb{K}\otimes\mathbb{K}$ onto $\mathbb{K}$, and let $s_0 \in \mathcal{M}(\mathbb{K})$ an isometry with $s_0bs_0^* = \psi(b\otimes p_{11})$ for $b\in\mathbb{K}$, see part (i).

Let γ denote the natural isomorphism from $(D\otimes\mathbb{K})\otimes\mathbb{K}$ onto $D\otimes(\mathbb{K}\otimes\mathbb{K})$.

We define an isomorphism λ from $D\otimes\mathbb{K}$ onto $(D\otimes\mathbb{K})\otimes\mathbb{K}$ by $\lambda := \gamma^{-1}\circ(\mathrm{id}_D\otimes\psi^{-1})$.

Then $\nu(b) := \lambda^{-1}(b\otimes p_{11})$, for $b\in D\otimes\mathbb{K}$, defines a *-endomorphism of $D\otimes\mathbb{K}$, such that $\nu(d\otimes b) := d\otimes\psi(b\otimes p_{11})$ for $d\in D$, $b\in\mathbb{K}$. It follows, that $s_2 := 1_{\mathcal{M}(D)}\otimes s_0$ is an isometry in $\mathcal{M}(D\otimes\mathbb{K})$, such that $s_2bs_2^* = \nu(b)$ for $b\in D\otimes\mathbb{K}$.

Let $h_2 := \Phi\circ(h_1\otimes\mathrm{id}_{\mathbb{K}})\circ\lambda$ and $h_3 := \Phi(h_1(\cdot)\otimes p_{11})$. Then $h_3 = h_1\circ\nu$. The h_2 and h_3 both dominate zero, because Φ dominates zero.

It suffices to show, that h_2 and h_3 are unitarily equivalent, by a unitary $v = \pi_B(U)$ for some unitary U of $\mathcal{M}(B\otimes\mathbb{K})$, because the extension with Busby invariant $\Phi\circ(h\otimes\mathrm{id}_{\mathbb{K}})$ is stable, by part (v).

Let $C := h_2(D\otimes\mathbb{K})$. $S := \mathcal{M}(h_2)(s_2)$ is an isometry in $\mathcal{M}(C)$, such that $h_3 = Sh_2(\cdot)S^*$. Since h_3 dominates zero, by Lemma 5.5.10(iv), there exists a unitary $U\in\mathcal{M}(B)$ with $v^*h_2(\cdot)v = h_3$ for $v := \pi_B(U)$. □

Proposition 5.5.12 (Stability of extensions). *Suppose that*

$$0\to B\to E\to A\to 0$$

is an extension of A by B, where A and B are σ-unital and stable C^*-algebras.

Let $\iota\colon B\to E$, $\eta\colon E\to A$ and $\varphi\colon A\to\mathcal{M}(B)/B\cong\mathrm{Q^s}(B)$ denote the the defining morphisms of the above short-exact sequence, i.e., the corresponding monomorphism, respectively epimorphism, and the Busby invariant of E.

Further let θ denote an (arbitrary) isomorphism θ from B onto $B\otimes\mathbb{K}$, and $\Theta\colon \mathrm{Q}(B)\to\mathrm{Q^s}(B) := \mathrm{Q}(B\otimes\mathbb{K})$ is the isomorphism that is induced by the isomorphism θ.

Then following properties (i)–(vi) are equivalent.

(i) *The extension E is a stable C^*-algebra.*
(ii) *The Busby invariant φ dominates zero.*
(iii) *For every $a \in \mathcal{M}(B)_+$ with $\pi_B(a) \in \varphi(A)$, $b \in B_+$ and $\varepsilon > 0$, there exists $d \in B$ with $\|d^*ad\| < \varepsilon$ and $\|d^*d - b\| < \varepsilon$.*
(iv) *$\Theta \circ \varphi$ and $\Phi(\Theta \circ \varphi(\cdot) \otimes p_{11})$ are unitarily equivalent, where Φ denotes the natural monomorphism from $\mathrm{Q^s}(B) \otimes \mathbb{K}$ into $\mathrm{Q^s}(B)$ defined in Lemma 5.5.11(iii).*
(v) *There is an isomorphism λ from $E \otimes \mathbb{K}$ onto E, such that $B = \lambda(B \otimes \mathbb{K})$, and that $b \in B \mapsto \lambda(b \otimes p_{11})$ and $a \in A \mapsto [\lambda](a \otimes p_{11})$ are approximately inner in B respectively in A, where $[\lambda](a \otimes p) := \eta(\lambda(f \otimes p))$ for $a \in A$, $f \in \eta^{-1}(a)$ and $p \in \mathbb{K}$.*
(vi) *$1_{\mathcal{M}(E)}$ is properly infinite in $\mathcal{M}(E)$, i.e., the multiplier algebra $\mathcal{M}(E)$ of E contains two isometries with orthogonal ranges.*

The criteria (ii) and (vi) are not obvious, and (iii) is useful for applications. Criterium (iii) was established by J. Hjelmborg and M. Rørdam in [**373**].

Compare also proof of [**499**, thm. 2.4].

Compare ?????

Proof. The implications (v)⇒(i) and (i)⇒(vi) are trivial. (i)⇒(iv) follows from Lemma 5.1.2(ii).

(vi)⇒(iii): The natural C^*-morphism $h_1 \colon E \to \mathcal{M}(B)$ with $h_1(f)b = \iota^{-1}(f\iota(b))$ (for $f \in E$, $b \in B$) satisfies $\varphi \circ \eta = \pi_B \circ h_1$ for the Busby invariant $\varphi \colon A \to \mathrm{Q}(B)$ of the extension. This can be seen by the natural Busby isomorphism:

$$E \cong E_\varphi := A \oplus_{\varphi, \pi_B} \mathcal{M}(B) \subset A \oplus \mathcal{M}(B)\,,$$

that is given by $f \mapsto (\eta(f), h_1(f))$.

Let $C := \varphi(A) = \pi_B(h_1(E))$, and let $a \in \mathcal{M}(B)_+$ with $\pi_B(a) \in C$, $b \in B_+$, $\varepsilon > 0$ and let $\delta := \varepsilon/(2(\|a\| + \|b\| + 1))$.

By Lemma 5.5.10(i) and (ii) (with E in place of A), the epimorphism $h := \pi_B \circ h_1 \colon E \to C$ extends uniquely to a strictly continuous epimorphism $\mathcal{M}(h) \colon \mathcal{M}(E) \to \mathcal{M}(C)$.

By assumption there are isometries $T_1, T_2 \in \mathcal{M}(E)$ with $(T_2)^*T_1 = 0$. Thus $r_j := \mathcal{M}(h)(T_j)$ for $j = 1, 2$ are isometries in $\mathcal{M}(C)$ with orthogonal ranges.

Since A is stable, its image $C = \varphi(A)$ by φ is again stable: $A \cong D \otimes \mathbb{K}$ implies that D has a closed ideal J such that $C \cong (D \otimes \mathbb{K})/(J \otimes \mathbb{K}) \cong (D/J) \otimes \mathbb{K}$.

Let $s_1, s_2, \ldots$ a sequence of isometries in $\mathcal{M}(C)$ such that $\sum_n s_n(s_n)^*$ strictly converges to 1. Then there is $m \in \mathbb{N}$ with $\|s_m^* \pi_B(a) s_m\| < \delta$. Since $\mathcal{M}(C)$ contains a copy of $\mathcal{O}_2$ unitally, there is a unitary v in $\mathcal{M}(C)$, such that $vr_1 = s_m$. The unitary group of $\mathcal{M}(C)$ is (norm-)contractible by [**180**]. Hence, there is a unitary $U \in \mathcal{M}(E)$ with $\mathcal{M}(h)(U) = v$.

The C^*-morphism $h_1 \colon E \to \mathcal{M}(B)$ is non-degenerate, because $h_1(\iota(B)) = B$. Thus h_1 uniquely extends to a strictly continuous unital C^*-morphism $\mathcal{M}(h_1) \colon \mathcal{M}(E) \to \mathcal{M}(B)$.

By Lemma 5.5.10(ii), $\pi_B(\mathcal{M}(h_1)(\mathcal{M}(E))) \subset \mathcal{N}(C)$ and $\mathcal{M}(h) = L \circ \pi_B \circ \mathcal{M}(h_1)$ for the natural epimorphism L from $\mathcal{N}(C)$ onto $\mathcal{M}(C)$ with kernel $\mathrm{Ann}(C)$. Thus the isometry $I := \mathcal{M}(h_1)(UT_1) \in \mathcal{M}(B)$ satisfies $\pi_B(I) \in \mathcal{N}(C)$ and $\mathrm{dist}(I^*aI, B) = \|\pi_B(I^*aI)\|$.

It follows $\pi_B(I^*aI) = (L \circ \pi_B)(I^*aI) == \mathcal{M}(h)(UT_1)^* \pi_B(a) \mathcal{M}(h)(UT_1) = s_m^* a s_m$, and, therefore, there is $g = g^* \in B$ with $\|I^*aI - g\| < \delta$.

Now let $t_1, t_2, \ldots$ a sequence of isometries in $\mathcal{M}(B)$ such that $\sum t_n(t_n)^*$ strictly converges to 1 (exists, because B is stable). There is $k \in \mathbb{N}$ such that $\|t_k^* g t_k\| < \delta$. Let $d := I t_k b^{1/2} \in B$, then $d^*d = b$ and $\|d^*ad\| < 2\delta\|b\| < \varepsilon$.

(iii)⇒(ii): Let $e \in A_+$ and $b \in B_+$ strictly positive elements of A and B respectively. We find $a \in \mathcal{M}(B)_+$ with $\pi_B(a) = \varphi(e)$. It is enough to find an isometry $S \in \mathcal{M}(B)$ with $S^*aS \in B$.

Let $C := C^*(a, b, 1)$ denote the unital C^*-subalgebra of $\mathcal{M}(B)$ generated by a, b and 1. Then C is the unitization $\widetilde{D}$ of the C^*-subalgebra D of C which is generated by b and a. Let $\chi \colon C \to \mathbb{C} \cong C/D$ the natural character.

The C^*-morphism $V \colon C \to \mathcal{M}(B)$, with $V(c) := \chi(c)1$ for $c \in C$, satisfies condition (α) of Proposition 5.4.1 with $h := b$.

$f := b + a$ is a strictly positive element of D, and, by assumption, for every $\delta > 0$, there is $d \in B$ with $\|d^* f^\delta d\| < \delta$ and $\|d^*d - b^\delta\| < \delta$.

Thus V and C satisfy also condition (β) of Proposition 5.4.1.

Since $\delta_\infty \circ V = V$ and $V(a) = 0$, by Proposition 5.4.1(ii) and (iv), there exists an isometry $S \in \mathcal{M}(B)$ with $S^*aS \in B$.

(ii)⇒(iv): $\Theta \circ \varphi$ dominates zero if φ dominates zero. But then $\Theta \circ \varphi$ and $\Phi((\Theta \circ \varphi)(\cdot) \otimes p_{11})$ are unitarily equivalent by Lemma 5.5.11(iv).

(iv)⇒(i): Since A is stable and σ-unital, $\Phi((\Theta \circ \varphi)(\cdot) \otimes p_{11})$ is the Busby invariant of a stable extension $0 \to B \otimes \mathbb{K} \to E_1 \to A \to 0$ by Lemma 5.5.11(vi). Since $\Phi((\Theta \circ \varphi)(\cdot) \otimes p_{11})$ is unitarily equivalent to $\Theta \circ \varphi$, it follows that this equivalence is given by a unitary in the image in $\mathcal{M}(B \otimes \mathbb{K})/(B \otimes \mathbb{K})$ of the unitaries of $\mathcal{M}(B \otimes \mathbb{K})$, cf. Lemma 5.5.11(iv). Thus, the extension E_1 is equivalent to an extension $0 \to B \otimes \mathbb{K} \to E_2 \to A \to 0$ with Busby invariant $\Theta \circ \varphi$ –

in particular, $E_2 \cong E_1$ as C^*-algebras, and

?? ????.

Since E_2 and $0 \to B \to E \to A \to 0$ are determined up to equivalence by the Busby invariants $\Theta \circ \varphi$ and φ, it follows that E_2 and E are naturally isomorphic (in particular, as C^*-algebras). Thus E is a stable C^*-algebra.

(i)⇒(v): Let $\mu\colon E\otimes\mathbb{K}\to E$ any isomorphism from $E\otimes\mathbb{K}$ onto E, and let $\iota\colon B\to E$, $\eta\colon E\to A$ the defining morphisms for the exact sequence $0\to B\to E\to A\to 0$. Then there is a closed ideal J of E such that $\mu(J\otimes\mathbb{K})=\iota(B)$ and $J\otimes\mathbb{K}$ is the kernel of $\eta\circ\mu\colon E\otimes\mathbb{K}\to A$.

We define a new isomorphism λ from $E\otimes\mathbb{K}$ onto E by

$$\lambda := \mu\circ(\mathrm{id}_B\otimes\psi)\circ\gamma\circ(\mu^{-1}\otimes\mathrm{id}_{\mathbb{K}})\,,$$

where ψ is an isomorphism from $\mathbb{K}\otimes\mathbb{K}$ onto $\mathbb{K}$, and γ is the natural isomorphism from $(E\otimes\mathbb{K})\otimes\mathbb{K}$ onto $E\otimes(\mathbb{K}\otimes\mathbb{K})$.

Then $\lambda(\iota(B)\otimes\mathbb{K})=\iota(B)$ and the *-monomorphisms

$$b\in B\mapsto \iota^{-1}(\lambda(\iota(b)\otimes p_{11}))$$

and $a\in A\mapsto[\eta\circ\lambda](a\otimes p_{11})$ are approximately inner, because even the *-endomorphisms $g\in E\otimes\mathbb{K}\mapsto(\mathrm{id}_E\otimes\psi)(\gamma(g\otimes p_{11}))$ and $f\in E\mapsto\lambda(f\otimes p_{11})$ are unitarily homotopic to $\mathrm{id}_{E\otimes\mathbb{K}}$ respectively id_E. □

Remark 5.5.13. One can the criteria 5.5.12(iii) replace by the (formally weaker) criteria:

(iii*) *There exists $k\in\mathbb{N}$, such that, for every $b,g\in B_+$, $\varepsilon>0$, $e_1,\dots,e_k\in\mathcal{M}(B)$, with $\pi_B(e_j)\in\varphi(A)$ and $e_i^*e_j=\delta_{ij}e_1^*e_1$ for $i,j=1,\dots,n$, there is $d\in B$ with $\|d^*(e_1^*e_1)d\|<\varepsilon$, $\|d^*gd\|<\varepsilon$ and $\|d^*d-b\|<\varepsilon$.*

Indeed:

(iii)⇒(iii*): Take $k:=1$, and let $a:=g+e_1^*e_1$ in (iii).

(iii*)⇒(iii): Let $C:=\varphi(A)$, $a\in\mathcal{M}(B)_+$ with $\pi_B(a)\in C$, $b\in B_+$, $\varepsilon>0$, and let $c:=\pi_B(a)$, $\delta:=\varepsilon/(2+\|b\|+\varepsilon)$.

Since A is stable, C is stable, and there are isometries $s_1,s_2,\dots$ in $\mathcal{M}(C)$, such that $\sum p_j$ strictly converges to 1, where $p_j:=s_js_j^*$. Thus there is a projection $q_1:=p_1+\ldots+p_n$ in $\mathcal{O}_\infty\subset\mathcal{M}(C)$ such that $\|q_1cq_1-c\|<\delta$. Let $q_j:=p_{(j-1)n+1}+\ldots+p_{jn}$. Then there are partial isometries v_j in $\mathcal{M}(C)$ with $v_j^*v_j=q_1$ and $v_jv_j^*=q_j$. Let $f_j:=v_j(q_1cq_1)^{1/2}$, $j=1,\dots,k$. Then $f_j\in C$ and $f_i^*f_j=\delta_{ij}q_1cq_1$. By Proposition A.8.4, there exist $e_1,\dots,e_k\in\mathcal{M}(B)$, with $\pi_B(e_j)=f_j$ and $e_i^*e_j=\delta_{ij}e_1^*e_1$ for $i,j=1,\dots,n$.

It follows, that $g:=(|a-e_1^*e_1|-\delta)_+$ is in B_+. By assumption, there is $d\in B$ with $\|d^*e_1^*e_1d\|<\delta$, $\|d^*gd\|<\delta$ and $\|d^*d-b\|<\delta$.

Since $\delta<\varepsilon$ and $a\le e_1^*e_1+|a-e_1^*e_1|$, we get $\|d^*d-b\|<\varepsilon$ and $\|d^*ad\|<\varepsilon$.

Corollary 5.5.14. *Suppose that D and B are σ-unital C*-algebras and that $\psi\colon D\to\mathcal{M}(B)/B$ is a *-monomorphism. Let $E:=\pi_B^{-1}(\psi(D))$.*

(I) *If ψ has a* non-degenerate *"split" C*-morphism $H\colon D\to\mathcal{M}(B)$ with $\psi=\pi_B\circ H$, and if D is stable, then B and E are stable.*

(II) *If E is stable, then*

(i) *B and D are stable and $\psi \oplus 0$ is unitarily equivalent to ψ by a unitary $u \in \mathcal{U}_0(\mathrm{Q}(B))$, and*

(ii) *there exists isometries $S, T \in \mathcal{M}(B)$ with $SS^* + TT^* = 1$, $S^*ES \subseteq B$ and $(1 - T)E \cup E(1 - T) \subset B$.*

Proof. Ad(I): Suppose that D is stable and that $H \colon D \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism with $\psi = \pi_B \circ H$.

Then the σ-unital C^*-algebra B is stable by Remark 5.1.1(9), because $H(D)$ is a stable non-degenerate C^*-subalgebra of $\mathcal{M}(B)$, i.e., $\overline{H(D)B} = B$.

If $e \in E_+$, then there exist $b \in B$ and $d \in D$ with $e = b + H(d)$. Thus, for $\varepsilon > 0$ there exists a contraction $f \in D_+$ with

$$\|e - H(f)e\| \leq \|b - H(f)b\| + \|d - fd\| < \varepsilon\,,$$

i.e., $\overline{H(D)E} = E$ and $H(D)$ is a non-degenerate C^*-subalgebra of $\mathcal{M}(E)$. It follows that E is stable by Remark 5.1.1(9).

(II): Suppose that E is stable.

(i): The stability of E implies the stability of its ideal B and of its quotient D.

By Proposition 5.5.12(ii), the Busby invariant $\psi \colon D \to \mathcal{M}(B)/B \cong \mathrm{Q}^{\mathrm{s}}(B)$ dominates zero, i.e., there is an isometry $t \in \mathrm{Q}(B)$ with $t^*\psi(D)t = \{0\}$.

Define $\oplus := \oplus_{s_1,s_2}$. Then ψ and $\psi \oplus 0$ both dominate zero: $t^*\psi(\cdot)t = 0$ and $(t \oplus 1)^*(\psi(\cdot) \oplus 0)(t \oplus 1) = 0$.

Chose one of next:

Since B is σ-unital and stable, the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ of B is K_1-injective by Lemma 4.2.10.

Stable coronas $\mathrm{Q}^{\mathrm{s}}(B)$ of σ-unital C^*-algebras B have Property *(sq)* and are therefore K_1-bijective by Proposition 4.2.15.

If B is stable, then $\mathrm{Q}(B)$ is K_1-surjective by Lemma 5.5.9.

It follows that ψ and $\psi(\cdot) \oplus 0$ are unitary equivalent with a unitary $u \in \mathcal{U}_0(\mathrm{Q}^{\mathrm{s}}(B))$ by last conclusion in Parts (iii) and (iv,b) of Proposition 4.3.6.

Prop./Lem./Rem. in Chp. 4

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????????????????? **??**

Since B is stable, there exist isometries $S_1, S_2 \in \mathcal{M}(B)$ with $S_1S_1^* + S_2S_2^* = 1$. Let $s_j = \pi_B(S_j)$. By Lemma 4.2.6(iii) there is a unitary u_1 in $\mathrm{Q}(B)$ with $u_1s_2 = ss_2$, because $t_1 := ss_1s_2$ and $t_2 := ss_2$ and $r_1 := s_1s_2$ and $r_2 := s_2$ are pairs of isometries (t_1, t_2) and (r_1, r_2) with orthogonal ranges and the property that $1 - (t_1t_1^* + t_2t_2^*) \geq ss_1^2(ss_1^2)$ and $1 - (r_1r_1^* + r_2r_2^*) = s_1^2(s_1^2)^*$ are full and properly infinite projections in $\mathrm{Q}(B)$.

What is the application?

Then $\psi = u_1^*\psi(\cdot)u_1$ and $\psi_2 := \psi(\cdot) \oplus_{s_1,s_2} 0$ have the property $\psi_j(D)s_2 = 0$ for $j = 1, 2$.

Since B is stable, $\mathrm{Q}(B)$ is K_1-injective by Lemma 5.5.9.

`Better refer to Lemma for property` *sq* `in 4 ?`

By Proposition 4.3.6(iv,a ???), there exists a a unitary $V \in \mathcal{U}_0(\mathrm{Q}(B))$ with

??? **??** Proceed proof ! ???????????????

(ii): By the proof of part (ii) of Lemma 5.5.9, there exist isometries $S_1, S_2, T_1 \in \mathcal{M}(B)$ with $\pi_B(S_1) = s\pi_B(S_2)$ and $S_1S_1^* + T_1T_1^* = 1$. Thus, $\pi_B(S_1^*ES_1) = \pi_B(S_2)^*s^*\psi(D)s\pi_B(S_2) = \{0\}$. It follows $\pi_B(S_1(1-T_1)S_1^*E) = \{0\} = \pi_B(ES_1(1-T_1)S_1^*)$.

Now let $P := T_1T_1^* = 1 - S_1S_1^*$, $T := P + S_1T_1S_1^*$ and $S := S_1^2$. Then $T^*T = 1 = S^*S$, $TT^* = P + S_1PS_1^* = 1 - SS^*$ and $1 - T = S_1(1-T_1)S_1^*$. Hence, $S^*ES \subset B$ and $(1-T)E \cup E(1-T) \subset B$. □

Corollary 5.5.15. *Suppose that Y is a σ-compact locally compact space, that B is a σ-unital stable C^*-algebra and that C is a σ-unital C^*-subalgebra of the ideal $\mathrm{Q}(Y,B) := \mathrm{C_b}(Y,B)/\,\mathrm{C_0}(Y,B)$ of $\mathrm{Q}(\mathrm{C_0}(Y,B)) = \mathcal{M}(\mathrm{C_0}(Y,B))/\,\mathrm{C_0}(Y,B)$.*

(i) *There are isometries S and T in $\mathrm{C_b}(Y,\mathcal{M}(B)) \subset \mathcal{M}(\mathrm{C_0}(Y,B)) = \mathcal{M}(\mathrm{C_b}(Y,B))$ such that $SS^* + TT^* = 1$, and that $s := S + \mathrm{C_0}(Y,B) \in \mathrm{Q}(\mathrm{C_0}(Y,B))$ and $t := T + \mathrm{C_0}(Y,B) \in \mathrm{Q}(\mathrm{C_0}(Y,B))$ satisfy $s^*Cs = \{0\}$ and $(1-t)C = \{0\} = C(1-t)$.*

(ii) *If D is a σ-unital stable C^*-algebra, and $0 \to \mathrm{C_0}(Y,B) \to E_\varphi \to D \to 0$ is an extension which has Busby invariant $\varphi\colon D \to \mathrm{Q}(\mathrm{C_0}(Y,B))$ with image $\varphi(D)$ in $\mathrm{C_b}(Y,B)/\,\mathrm{C_0}(Y,B) \subset \mathrm{Q}(\mathrm{C_0}(Y,B))$, then E_φ is* stable.

(iii) *If f is a contraction in $\mathrm{C_b}(Y,B)$, then there is an isometry T in $\mathrm{C_b}(Y,\mathcal{M}(B))$ such that $g^*ag = t^*at$ for all $a \in C$, where $g := f + \mathrm{C_0}(Y,B)$ and $t := T + \mathrm{C_0}(Y,B)$.*

(iv) *There is an *-monomorphism $\rho\colon C \otimes \mathbb{K} \hookrightarrow \mathrm{C_b}(Y,B)/\,\mathrm{C_0}(Y,B)$ with $\rho(c \otimes p_{11}) = c$ for $c \in C$.*

Proof. (i): We use that $B \cong B \otimes \mathbb{K}$, and replace B by $B \otimes \mathbb{K}$. Since Y is σ-compact, there is a strictly positive function $\gamma \in C_0(Y)$ with $0 \le \gamma(y) \le 1$. Since $\gamma \in C_0(Y)_+$ is strictly positive, the subsets $Y_n := \gamma^{-1}[1/n, 1] \subset Y$ are compact and $Y = \bigcup_n Y_n$. Let $g \in C_+$ a strictly positive contraction for C. There is $f \in \mathrm{C_b}(Y, B \otimes \mathbb{K})_+$ with $g = f + \mathrm{C_0}(Y, B \otimes \mathbb{K})$ and $\|f(y)\| \le 1$. There are projections $p_n \in \mathbb{K}$ with $\|f(y)(1 \otimes p_n) - f(y)\| < 1/(n+1)$ for all $y \in Y_{n+1}$ for $n = 0, 1, 2, \ldots$. This can be seen, because $f|Y_n$ is in $\mathrm{C}(Y_n, B \otimes \mathbb{K}) \cong \mathrm{C}(Y_n, B) \otimes \mathbb{K}$. If we take linear interpolations $p(t) := (n-t)p_{n-1} + (t+1-n)p_n$ for $t \in [n-1, n]$, then we get a norm-continuous path $t \in [0,\infty) \mapsto p(t) \in \mathbb{K}$ from $\mathbb{R}_+$ into the positive contractions of $\mathbb{K}$ such that $\|f(y)(1 \otimes p(t)) - f(y)\| < 1/n$ for all $y \in Y_n$ and $t \ge n-1$. Thus, $\|f(y)(1 \otimes p(\gamma(y)^{-1})) - f(y)\| \le \gamma(y)$ for all $y \in Y$. Note that

$p(t) \leq q_n := \bigvee_{1\leq k\leq n} p_k \in \mathbb{K}$, $p(t) = p(t)q_n$ and $p(t)r_n - p(t) = p(t)(q_n r_n - q_n)$ for $t \leq n$.

There is a non-degenerate *-representation $\lambda\colon D := \mathcal{O}_2 \otimes \mathbb{K} \to \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K})$. We identify $d \in D$ with $\lambda(d)$ and $\mathcal{M}(\mathbb{K})$ with $\mathcal{L}(\mathcal{H})$ (for simplicity of notations).

Then we find projections $r_n \in D$ with $0 \neq r_1 \leq r_2 \leq \cdots$, $r_n \neq r_{n+1}$ and $\|q_n r_n - q_n\| < 1/n$ for $t \leq n$ because $q_n \in \mathbb{K}$. By Lemma 5.1.2(iii), there exists a *norm*-continuous path $t \mapsto U(t)$ from $\mathbb{R}_+$ into the unitaries in $1 + D \subset \mathcal{M}(\mathbb{K})$ such that $U(t)r_1U(t)^* \geq r_{n+1}$ for $t \geq n$.

Since r_1 and $1 - r_1 \geq r_2 - r_1 \in D \setminus \{0\}$ are properly infinite projections in $\mathcal{M}(\mathbb{K})$, there are isometries $S_0, T_0 \in \mathcal{M}(\mathbb{K})$ with $S_0S_0^* = 1 - r_1$ and $T_0T_0^* = r_1$. It follows $\|p(t)U(t)S_0\| \leq \|p(t)-p(t)r_{n+1}\|+\|r_{n+1}U(t)S_0\| < 1/n + 1$ for $t \in [n, n+1]$, because $r_{n+1}U(t)(1 - r_1) = 0$ for $t \geq n$. Thus, $\|p(\gamma(y)^{-1})U(\gamma(y)^{-1})S_0\| \leq \gamma(y)$.

We define a norm-continuous map $y \mapsto V(y)$ from Y into the unitary elements of $1 \otimes \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B \otimes \mathbb{K})$ by $V(y) := 1 \otimes U(\gamma(y)^{-1})$ for $y \in Y$. Then $S_1(y) := V(y)(1\otimes S_0)$ and $T_1(y) := V(y)(1\otimes T_0)$ define isometries $S_1, T_1 \in \mathrm{C_b}(Y, \mathcal{M}(B\otimes\mathbb{K}))$ with $S_1S_1^* + T_1T_1^* = 1$ and, for $y \in Y$,

$$\|f(y)S_1(y)\| \leq \|f(y)(1 \otimes p(\gamma(y)^{-1})) - f(y)\| + \|p(\gamma(y)^{-1})U(\gamma(y)^{-1})S_0\| \leq 2\gamma(y)\,.$$

Thus, $fS_1 \in \mathrm{C_0}(Y, B\otimes\mathbb{K})$, i.e., $gs_1 = 0$ and $s_1^*Cs_1 = \{0\}$ for $s_1 := S_1+\mathrm{C_0}(Y, B\otimes\mathbb{K})$.

Now we proceed as in the proof of Proposition 5.5.14: The isometries $S := S_1^2$, $T := T_1T_1^* + S_1T_1S_1^*$ and $s := S + \mathrm{C_0}(Y, B \otimes \mathbb{K})$ are as stipulated.

(ii): By (i) with $C := \varphi(D)$, Proposition 5.5.12 applies to φ, because φ dominates zero.

(iii) follows from (i) by Proposition 4.3.6(ii), with $\mathrm{C_b}(Y, \mathcal{M}(B))/\,\mathrm{C_0}(Y, B)$ in place of E.

(iv): Let $S, T \in \mathrm{C_b}(Y, \mathcal{M}(B))$ the isometries from part (i), and let $s := S + \mathrm{C_0}(Y, B)$,

$t := T + \mathrm{C_0}(Y, B)$, $t_1 := t$ and $t_n := st^n$ for $n = 2, 3, \ldots$. Then s, t, t_n are isometries in $\mathrm{C_b}(Y, \mathcal{M}(B))/\,\mathrm{C_0}(Y, B)$ with $t^*s = 0$, $t_m^*t_n = 0$ for $m < n$, and $t_1ct_1^* = tct^* = c$ for all $c \in C$. There is a *-morphism $\rho\colon C \otimes \mathbb{K} \to \mathrm{C_b}(Y, B)/\,\mathrm{C_0}(Y, B)$ with $\rho(c\otimes p_{jk}) = t_jct_k^*$ for canonical matrix units p_{jk} in $\mathbb{K}$. We get $\rho(c\otimes p_{11}) = tct^* = c$ for $c \in C$. □

`Compare here also [499, thm. 2.4.]:` Let A and B be separable C^*-algebras such that $B\otimes\mathbb{K}$ has the CFP (= Corona Factorization Property). Suppose that there is an extension of C^*-algebras of the form $0 \to B \to E \to A \to 0$. Then E is stable if and only if A and B are stable.

(E.K.: It is clear that A and B must be stable if $E \cong E \otimes \mathbb{K}$. Does Prop. 5.5.12(iii) apply in the opposite direction?
Implies "B is s.p.i. and σ-unital" that $B \otimes \mathbb{K}$ has the CFP?)

COROLLARY 5.5.16. *Extensions $0 \to B \to E \to A \to 0$ of stable σ-unital C^*-algebras A by σ-unital purely infinite stable C^*-algebras B are stable.*

PROOF. Suppose that B is a stable, σ-unital and purely infinite C^*-algebra, that A is a stable σ-unital C^*-algebra, and that $\varphi\colon A \to \mathrm{Q}(B)$ is a C^*-morphism, i.e., is the Busby invariant of an extension of A by B.

We show that the criterion (iii) of Proposition 5.5.12 is satisfied:

Let $C := \varphi(A)$, $E := \pi_B^{-1}(C)$, $a \in E_+$, $b \in B_+$ a strictly positive contraction in B, let $\varepsilon \in (0,1)$, and put $\delta := \varepsilon/5$. The annihilator $D := \mathrm{Ann}((a-\delta)_+)$ of $(a-\delta)_+$ in E generates a closed ideal J of E, such that $J = E$ or E/J is unital and $\pi_J(a)$ is invertible. Then $E/(J+B) \cong C/\pi_B(J)$ is a quotient of C. It is isomorphic to $A/\varphi^{-1}(\pi_B(J))$ and is unital (if non-zero). Since A is stable it has no unit. Thus $\varphi(A) = \pi_B(J)$, i.e., $J + B = E$ and $B/(B \cap J) \cong E/J$ is unital or zero. Since B is stable and E/J is unital we get $B = B \cap J \subset J$.

The hereditary C^*-subalgebra $D \cap B$ generates $B \cap J = B$ as a closed ideal of B, because B is an ideal of E. Therefore, there exist $d_1, \ldots, d_n \in B$ with $\|b - \sum d_j^* d_j\| < \delta$ and $d_j d_j^* \in (D \cap B)$. Let $c := \sum d_j d_j^*$, $g := d_j^* d_j$. Since B is purely infinite, there exist $f \in B$ with $\|f^* c f - g\| < \delta$. It follows, that $d := c^{1/2} f$ satisfies $\|d^* d - b\| < 2\delta < \varepsilon$ and $\|d^* a d\| < \|d\|^2 \delta < \varepsilon$, i.e., the stability criterion (iii) of Proposition 5.5.12 applies. □

REMARK 5.5.17. Suppose that A and B are stable and σ-unital, and that $\varphi\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(B)$ is a *-monomorphism. The above results contain the following observations: (i) The extension $E := \pi_B^{-1}(\varphi(A \otimes \mathcal{O}_2))$ is stable if and only if $E_0 := \pi_B^{-1}(\varphi(A \otimes 1))$ is stable.

(This is because $\varphi(A \otimes 1)$ is orthogonal to the range of an isometry in $\mathcal{M}(B)/B$ if and only if $\varphi(A \otimes \mathcal{O}_2)$ is orthogonal to it. Recall that E is always stable if B is purely infinite by Corollary 5.5.16.)

(ii) If E is stable, then there is a copy of $\mathcal{O}_2$ unitally contained in $\mathcal{M}(B)$, such that, for $a \in A$ and $c \in \mathcal{O}_2$,

$$\varphi(a \otimes 1)\pi_B(c) = \varphi(a \otimes c) = \pi_B(c)\varphi(a \otimes 1)\,.$$

(Indeed, then $\mathcal{M}(E)$ contains a unital copy of $\mathcal{O}_2 = C^*(s_1, s_2)$, and the epimorphism $\lambda := \varphi^{-1}\pi_B|E\colon E \to A \otimes \mathcal{O}_2$ extends to an epimorphism $\mathcal{M}(\lambda)$ from $\mathcal{M}(E)$ *onto* $\mathcal{M}(A \otimes \mathcal{O}_2)$, because λ is surjective and E and $A \otimes \mathcal{O}_2$ are both σ-unital, i.e., one can apply Proposition **??**.

(find precise cite! But it is also easy to check...) Pedersen Thm.: $\mathcal{M}(A) \to \mathcal{M}(A/J)$ surjective if A is σ-unital.

There is a unitary $V \in \mathcal{M}(A \otimes \mathcal{O}_2)$ with $V(1 \otimes s_k) = \mathcal{M}(\lambda)(s_k)$. Since the unitary group of $\mathcal{M}(A \otimes \mathcal{O}_2)$ is connected by Remark **??**,

Cuntz-Higson Theorem:
$\mathcal{U}(\mathcal{M}(A)) = \mathcal{U}_0(M(A))$ if A stable and σ-unital

there is a unitary $U \in \mathcal{M}(E) \subset \mathcal{M}(B)$ with $\mathcal{M}(\lambda)(U) = V$. One can see, that $C^*(U^*s_1, U^*s_2) \cong \mathcal{O}_2$ is a copy of $\mathcal{O}_2$ in $\mathcal{M}(B)$ with the desired property.)

(iii) If E_0 is stable and $\varphi_0(a) := \varphi(a \otimes 1)$, then (ii) is equivalent to $[\varphi_0] = [\varphi_0] + [\varphi_0]$ in $[\mathrm{Hom}(A, \mathrm{Q}(B)]$, i.e., φ_0 is unitarily equivalent to $\varphi_0 \oplus \varphi_0$ by a unitary in $\mathrm{Q}(B)$.

QUESTION 5.5.18. Is every extension

$$0 \to B \to E \to \mathrm{C}_0((0,1], \mathcal{O}_2 \otimes \mathbb{K}) \to 0$$

stable if B is stable and separable? (Compare Remark 5.5.17(i).)

6. W-vN type results for weakly nuclear maps

We deduce from Propositions 5.4.1 and 4.3.5(i) a proof of Kasparov's more general version, cf. Corollary 5.6.1, of Voiculescu's generalized Weyl–von Neumann theorem. It is implicitly contained in [**404**], but is not explicitly stated there. Voiculescu's generalization of the Weyl-von Neumann theorem reads now as a special case of Kasparov's generalization. The proof of Corollary 5.6.1 shows that we could deduce Corollary 5.6.1 as a logical sum of Voiculescu's generalized Weyl–von Neumann theorem and of Theorem 5.6.2.

Let A a (non-zero) *separable* C^*-algebra and B a stable σ-unital C^*-algebra. We have seen in Chapter 3 that the universal *-monomorphism $H_{\mathcal{C}} \colon A \to \mathcal{M}(B)$ for the matrix operator-convex cone $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ of completely positive nuclear maps from A into B is given up to unitary homotopy by the following universal construction:

Let $d \colon A \to \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ any faithful *non-degenerate* *-representation of A, in sense of Remark 5.1.1(i).

We apply the infinite repeat δ_∞ of Remark 5.1.1(8) to d, i.e., replace d by $\delta_\infty \circ d$ to make sure that d is in general position, i.e., that $d(A) \cap \mathbb{K} = \{0\}$ and that $d(A)\mathbb{K} = \mathbb{K}$. A crucial property is that this "general position" implies that d and $\delta_\infty \circ d$ are unitarily homotopic.

The *-representation d is unital if A is unital. If A is not unital, then the extension given by the Busby invariant

$$\beta_0 := \pi_{\mathbb{K}} \circ d \colon A \to \mathcal{M}(\mathbb{K})/\mathbb{K}$$

dominates zero in sense of Definition 4.3.3. Since B is stable there is up to unitary equivalence (and up to unitary homotopy) a unique non-degenerate *-monomorphism $H_0 \colon \mathbb{K} \to \mathcal{M}(B)$

Hilbert B-module that the generating

COROLLARY 5.6.1 (Kasparov,Voiculescu,Weyl–von-Neumann). *Suppose that B is a σ-unital C^*-algebra, $I \colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ a strictly continuous unital *-monomorphism and $C \subset \mathcal{M}(B)$ a separable C^*-subalgebra such that $C \subset I(\mathcal{M}(\mathbb{K}))$.*

Let $T \colon C \to \mathcal{M}(B)$ be a completely positive contraction, such that

(α') $T(C \cap I(\mathbb{K})) = 0$,

(β') *the completely positive contraction* $T' \colon C/(C \cap I(\mathbb{K})) \to \mathcal{M}(B)$ *is* weakly nuclear*, and*

(γ) $T(c) = 1$ *for* $c \in C \cap (1 + I(\mathbb{K}))$ *(if* $1_{\mathcal{M}(B)} \in C + I(\mathbb{K})$*).*

Then:

(i) id_C *asymptotically dominates* T *in the sense of Definition 5.0.1.*

(ii) *If, moreover,* $T \colon C \to \mathcal{M}(B)$ *is a* C^**-morphism, then* id_C *asymptotically absorbs* T*, i.e.,* $\mathrm{id}_C \oplus T \colon C \to \mathcal{M}(B)$ *and* id_C *are unitarily homotopic in the sense of Definition 5.0.1.*

Proof. To keep notations simple, we identify the elements of $\mathcal{M}(\mathbb{K})$ and $I(\mathcal{M}(\mathbb{K}))$ with help of I. (In fact, there is a σ-unital C^*-algebra A and an isomorphism λ from $A \otimes \mathbb{K}$ onto B such that $I(k) = \mathcal{M}(\lambda)(1_{\mathcal{M}(A)} \otimes k)$ for $k \in \mathbb{K}$, and C lives in $1 \otimes \mathcal{M}(\mathbb{K})$.)

The strict continuity of the unital *-monomorphism I implies that $\mathbb{K}B$ is dense in B. $\mathcal{M}(\mathbb{K})$ contains a copy of $\mathcal{O}_\infty$ with generators given by isometries $t_1, t_2, \ldots$ with $t_j^* t_k = \delta_{jk} 1$ such that $\sum t_k t_k^*$ converges strictly to $1_{\mathcal{M}(\mathbb{K})} = 1_{\mathcal{M}(B)}$. By Remark 5.1.1(8), it follows that B is stable.

The conditions (β') and (γ) allow us to reduce the proof to the case where C is unital and contains $\mathbb{K}$ and that T is unital with $T(\mathbb{K}) = 0$: By assumption (β'), $T' \colon C/(C \cap \mathbb{K}) \to \mathcal{M}(B)$ is weakly nuclear. Replace C by $C + \mathbb{K}$ if $1_{\mathcal{M}(B)} \in C + \mathbb{K}$. Then T' is unital by assumption (γ). Thus we can replace T by $T'\pi_{\mathbb{K}}$, because $(C + \mathbb{K})/\mathbb{K} \cong C/(C \cap \mathbb{K})$.

If $1_{\mathcal{M}(B)}$ is not in $C + \mathbb{K}$, then $(C + \mathbb{K} + \mathbb{C}1)/\mathbb{K}$ is naturally isomorphic to the (outer) unitization of $C/(C \cap \mathbb{K})$ and the unital extension T_1 of T' to $(C + \mathbb{K} + \mathbb{C}1)/\mathbb{K}$ is again completely positive and weakly nuclear by Lemma B.7.7(i). Thus we can replace C by $C + \mathbb{K} + \mathbb{C}1$ and T by $T_1 \pi_{\mathbb{K}}$.

Thus C and $V := \delta_\infty \circ T$ satisfy the condition (α) of Proposition 5.4.1, where we take a strictly positive element h of $\mathbb{K} \subset C$. The new C and V are now unital. Moreover, V is a unital C^*-morphism if the original $T \colon C \to \mathcal{M}(B)$ is a C^*-morphism.

Next, we shall show that C and V satisfy condition (β) of Proposition 5.4.1. Then (i) and (ii) will follow from parts (ii), (iv) and (iii) of Proposition 5.4.1.

Let $\rho \colon C/\mathbb{K} \to \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(H)$ be given by a faithful unital *-representation of the separable unital C^*-algebra $C/\mathbb{K}$ over a separable Hilbert space H. We define $C_1 := \delta_\infty(\rho(C/\mathbb{K}))$ and a unital completely positive map $V_1 \colon C_1 \to \mathcal{M}(B)$ by $V_1(e) := \delta_\infty(T'(c))$ for $e \in C_1$, $c \in C/\mathbb{K}$, $e = \delta_\infty(\rho(c))$. Then, by Remark 3.1.2(iii), V_1 is weakly nuclear and $V = \delta_\infty \circ \circ T = V_1 \gamma$, where $\gamma := \delta_\infty \circ \rho \circ \pi_{\mathbb{K}}$ is a unital C^*-morphism from $C \subset \mathcal{M}(\mathbb{K})$ onto $C_1 \subset \mathcal{M}(\delta_\infty(\mathbb{K}))$ with kernel $\mathbb{K}$. Note that $\mathcal{M}(\delta_\infty(\mathbb{K})) = \delta_\infty(\mathcal{M}(\mathbb{K})) \subset \mathcal{M}(\mathbb{K})$ by our choice of $\mathcal{O}_\infty$.

By Lemma 5.1.2(ii), the commutant of $\delta_\infty(\mathcal{M}(\mathbb{K}))$ in $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally. $D := C^*(\mathcal{O}_2 \cdot \delta_\infty(\mathbb{K})) \cong \mathcal{O}_2 \otimes \mathbb{K}$ is a simple purely infinite C^*-subalgebra of $\mathcal{M}(B)$ such that $\delta_\infty(\mathcal{M}(\mathbb{K})) \subset \mathcal{M}(D)$ and $\delta_\infty(\mathbb{K})D$ is dense in D. Thus $C_1 D \subset D$. DB contains $\delta_\infty(\mathbb{K})B$ and is therefore dense in B.

Let $c_1, \ldots, c_n$ be contractions in C and $\varepsilon > 0$. Since C_1 is a separable subalgebra of $\mathcal{M}(D) \subset \mathcal{M}(B)$ and since, for $a \in B_+$, the map $e \in C_1 \mapsto aV_1(e)a$ is nuclear, by Proposition 3.2.15(i), there exists $b \in B$ with $\|b^*\gamma(c_j)b - aV_1(\gamma(c_j))a\| < \varepsilon/3$ for $j = 1, \ldots, n$.

We find a projection $p \in \mathbb{K}$ with $6\|b - pb\| < \varepsilon/(1 + \|b\|)$, because $\mathbb{K}B$ is dense in B.

By Lemma 2.1.22, we find a partial isometry $v \in \mathbb{K}$ such that $v^*v = p$ and $3\|S(c_j) - v^*c_j v\| < \varepsilon/(\|b\|^2 + 1)$ for $j = 1, \ldots, n$, where the unital completely positive map $S\colon C \to p\mathbb{K}p \cong M_k$ is given by $S(c) := p\gamma(c)p$, $k := \mathrm{Dim}(p\ell_2)$ and satisfies $S(\mathbb{K}) = 0$. Let $d := vb$. Since $V(c_j) = V_1(\gamma(c_j))$, the triangle inequality gives that $\|d^*c_j d - aV(c_j)a\| < \varepsilon$ for $j = 1, \ldots, n$. Thus C and $V := \delta_\infty \circ T$ satisfy also condition (β) of Proposition 5.4.1. □

We study now weakly nuclear liftable semi-split essential extensions of separable C^*-algebras A by simple purely infinite σ-unital stable algebras B. The corresponding generalized Weyl–von Neumann Theorem 5.6.2 implies that the extension is split if its class in $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ is zero. The pure infiniteness and the weak nuclearity of the semi-split map is necessary in this statement, cf. Corollaries 5.7.1 – 5.7.3.

The following generalized Weyl–von Neumann Theorem 5.6.2 is in view of Proposition 5.4.1 almost equivalent to the "local" result in Corollary 3.10.14.

If B nuclear then *every* unital completely positive contraction T with $T(C \cap I(\mathbb{K})) = \{0\}$ respectively $T(C \cap D) = \{0\}$ can be considered in Corollary 5.6.1 and Theorem 5.6.2.

Our Theorem 5.6.2 is simply a corollary of Proposition 3.2.13, Proposition 5.4.1, and Lemma B.7.7 together. (We call it a theorem, because of its importance.)

THEOREM 5.6.2 (Generalized Weyl–von Neumann Theorem). *Suppose that B is σ-unital and stable, and that $D \subseteq \mathcal{M}(B)$ a*

simple purely infinite

C^*-subalgebra of $\mathcal{M}(B)$ such that DB is dense in B.

Let C be a separable C^*-subalgebra of $\mathcal{M}(B)$ with $CD \subseteq D$ and $T\colon C \to \mathcal{M}(B)$ a completely positive contraction such that

(α') *$T(C \cap D) = 0$,*
(α'') *$C \cap D$ generates a corner of D,*
(β') *$T\colon C \to \mathcal{M}(B)$ is weakly nuclear, and*
(γ) *$T(C \cap (1 + D)) \subseteq \{1_{\mathcal{M}(B)}\}$.*

Then

(i) id_C *asymptotically dominates* T *(in the sense of Definition 5.0.1).*
(ii) *If, moreover,* $T\colon C \to \mathcal{M}(B)$ *is a* C^**-morphism then* id_C *asymptotically absorbs* T*, i.e.,* $\mathrm{id}_C \oplus T\colon C \to \mathcal{M}(B)$ *and* id_C *are unitarily homotopic in the sense of Definition 5.0.1.*

Proof. The natural extension $T_1\colon d + c \mapsto T(c)$ of T to $D + C$ is still weakly nuclear by Lemma B.7.7(ii) and by assumptions (α'), (α'') and (β').

It is unital by assumption (γ), if $1_{\mathcal{M}(B)} = 1_{\mathcal{M}(D)}$ is in $C+D$. If $1_{\mathcal{M}(B)}$ is not in $C+D$, then the unital extension $T_2\colon d+z1+c \mapsto T(c)+z1$ is again weakly nuclear by Lemma B.7.7(i). The extension T_1 (respectively T_2) is a unital C^*-morphism if T is a C^*-morphism from C into $\mathcal{M}(B)$.

Thus, we can assume that C and T are unital and that the natural extension of T to $C + D$ is weakly nuclear.

Let $b \in B_+$ a strictly positive element of B. We find a sequence $d_1, d_2, \ldots$ in D_+ such that $d_n b$ tends to b, because DB is dense in B. Let $C_1 \subset C + D$ be the separable unital C^*-subalgebra of $\mathcal{M}(D)$ which is generated by C, 1 and $\{d_1, d_2, \ldots\}$, and let T_1 be the natural extension of T to C_1. Then C_1, $V := \delta_\infty \circ T_1$ and a strictly positive element h of $C_1 \cap D$ fulfills condition (α) of Proposition 5.4.1.

We show that condition (β) of Proposition 5.4.1 is also satisfied:
Since C_1 is a separable subalgebra of $\mathcal{M}(D) \subset \mathcal{M}(B)$ and since, for $a \in B_+$, the map $c \in C_1 \mapsto aV(c)a$ is nuclear, by Proposition 3.2.15(i), for $c_1, \ldots, c_n \in C_1$ and $\varepsilon > 0$, there exists $d \in B$ with $\|d^* c_j d - aV(c_j)a\| < \varepsilon$.

Thus C_1 and $V := \delta_\infty \circ T_1$ are unital and satisfy the assumptions (α) and (β) of Proposition 5.4.1 and the assumptions of Propositions 5.4.1(ii) and 5.4.1(iv). Hence (i) follows from Proposition 5.4.1(ii) and (iv).

Moreover, V is a C^*-morphism if the original $T\colon C \to \mathcal{M}(B)$ is a C^*-morphism, and (ii) follows then from Proposition 5.4.1(iii). □

We explain now the important role of exactness in our consideration, and why we need exactness at certain places as assumptions for generalized Weyl–von-Neumann theorems.

Corollary 5.6.3. *A* C^**-algebra* A *is exact if and only if, for every* σ*-unital* C^**-algebra* B*, every weakly nuclear map* $V\colon A \to \mathcal{M}(B)$ *is nuclear.*

Proof. Suppose that each weakly nuclear $V\colon A \to \mathcal{M}(B)$ is nuclear if B is σ-unital. Let $A \subset \mathcal{L}(H)$ for some Hilbert space H and $C \subset A$ a separable C^*-subalgebra of A. Then there exists a faithful *-representation $d\colon C \to \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(H_1)$ of C on a separable Hilbert space H_1. Since $\mathcal{M}(\mathbb{K})$ is injective, by Arveson extension theorem, [**43**], d extends to a completely positive contraction $V\colon A \to \mathcal{M}(\mathbb{K})$. $B := \mathbb{K}$ is σ-unital and nuclear. Thus V is weakly nuclear and therefore nuclear by assumption. $d^{-1}\colon d(C) \to \mathcal{L}(H)$ extends to a completely positive contraction $T\colon \mathcal{M}(\mathbb{K}) \to \mathcal{L}(H)$ by injectivity of $\mathcal{L}(H)$. TV is nuclear and $TV(a) = a$ for $a \in C$. Thus the inclusion map $A \hookrightarrow \mathcal{L}(H)$ is nuclear. But this is equivalent to the exactness of A, cf. Remark 3.1.2(ii).

Now suppose conversely that A is exact and that B is σ-unital and $V\colon A \to \mathcal{M}(B)$ is a weakly nuclear contraction.

We can assume that A and V are unital: If A is not unital, then $V_1(a + z1) = V(a) + z1$ is a unital extension to the unitization of A. The u.c.p. mapV_1 is still weakly nuclear by Lemma B.7.7(i), and the unitization of A is again exact.

If A is unital, then we take a state ψ on A and consider $W(a) = V(a)+\psi(a)(1-V(1))$ for $a \in A$. W is unital and weakly nuclear. The tensor criteria in Remark 3.1.2(i) shows that V is nuclear if W is nuclear.

It suffices to approximate V on unital separable subalgebras C of A by unital nuclear maps. The subalgebras of A are again exact by Remark 3.1.2(ii) and by the Arveson extension theorem. Therefore, by Remark 3.1.2(ii), $H_0\colon C \to \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B \otimes \mathbb{K})$ is nuclear for a faithful unital *-monomorphism $H_0\colon C \to \mathcal{M}(\mathbb{K})$ with $H_0(C) \cap \mathbb{K} = \{0\}$, e.g. take $H_0 := \delta_\infty \circ \rho$ for some unital faithful *-representation $\rho\colon C \to \mathcal{L}(H)$.

$$c \in C \mapsto V(c) \otimes 1_{\mathcal{M}(\mathbb{K})} \subset \mathcal{M}(B \otimes \mathbb{K})$$

is still weakly nuclear. Therefore, by Corollary 5.6.1, there exists a sequence of isometries $S_n \in \mathcal{M}(B \otimes \mathbb{K})$ such that $\lim_n \|V(c) \otimes 1 - S_n^* H_0(c) S_n\| = 0$ for $c \in C$. The nuclearity of H_0 implies that $V|C$ is nuclear. Thus $V\colon A \to \mathcal{M}(B)$ is nuclear, because the nuclearity is a "local" property of V by Definition 3.1.1. □

Let A a separable C^*-algebra and $\rho\colon A \to \mathcal{L}(\mathcal{H}) \cong \mathcal{M}(\mathbb{K})$ a faithful non-degenerate *-representation in general position, i.e., $\rho(A) \cap \mathbb{K} = \{0\}$. If B is stable, the composition $H\colon A \to \mathcal{M}(B)$ of ρ with $\mathcal{M}(\mathbb{K}) \hookrightarrow \mathcal{M}(B)$ is a non-degenerate *-monomorphism with $H(A) \cap B = \{0\}$, and H is weakly nuclear. Let $H_0 := \pi_B \circ H$ the corresponding *-monomorphism from A into $\mathrm{Q}^{\mathrm{s}}(B) = \mathrm{Q}(B)$. We use the notation $[H_0]$ for the unitary equivalence class of $H_0\colon A \to \mathrm{Q}^{\mathrm{s}}(B)$ by unitaries in $\mathrm{Q}^{\mathrm{s}}(B)$.

Corollary 5.6.4 (Equivalence and Dominance in $\mathcal{M}(B)$ and $\mathrm{Q}^{\mathrm{s}}(B)$). *Suppose that B is σ-unital and stable, $D_i \subset \mathcal{M}(B)$ are σ-unital stable C^*-subalgebra with D_iB dense in B, for $i = 1, 2$, and that A is a separable C^*-algebra which is unital or is stable. Furthermore, assume that (for $i = 1, 2$) D_i is simple and purely infinite or $D_i \cong \mathbb{K}$.*

Further let H and H_0 be as above.

(i) *Let $h_1, h_2\colon A \to \mathrm{Q}^{\mathrm{s}}(B)$ be *-monomorphisms such that $h_1(A) \subset \pi_B(\mathcal{M}(D_1))$, $h_1(A) \cap \pi_B(D_1) = 0$, and that h_2 has a completely positive weakly nuclear lift $T_2\colon A \to \mathcal{M}(B)$. (Suppose in addition that that $h_i(1) = 1$ for $i = 1, 2$, if A is unital.)*

   Then h_1 dominates h_2 in $\mathrm{Q}^{\mathrm{s}}(B)$.

(ii) *Let $h_i\colon A \to \mathcal{M}(D_i) \subset \mathcal{M}(B)$ be weakly nuclear *-monomorphisms such that $h_i(A) \cap D_i = 0$ for $i = 1, 2$. (Suppose, in addition, that $h_i(1) = 1_{\mathcal{M}(B)}$ for $i = 1, 2$ if A is unital.)*

Then h_1 and h_2 are unitarily homotopic in the sense of Definition 5.0.1.

(iii) *$S(H_0, A, \mathrm{Q}^{\mathrm{s}}(B))$ is the $\oplus$-semigroup of the unitary equivalence classes of those C^*-morphisms from A into $\mathrm{Q}^{\mathrm{s}}(B)$ that have a weakly nuclear completely positive lift, and which are unital if A is unital.*

(iv) *If B itself is simple and purely infinite, then, for every *-monomorphism h from A into $\mathrm{Q}^{\mathrm{s}}(B)$, the Cuntz sum $h \oplus H_0$ is unitarily equivalent to h in $\mathrm{Q}^{\mathrm{s}}(B)$. (Here h is assumed to be unital, if A is unital.) In particular,*

$$S(H_0, A, \mathrm{Q}^{\mathrm{s}}(B)) \cap [\mathrm{Mon}(A, \mathrm{Q}^{\mathrm{s}}(B))] = G(H_0, A, \mathrm{Q}^{\mathrm{s}}(B))\,.$$

Proof. In the stable case the unitized monomorphisms are still monomorphisms. Therefore it suffices to consider the unital case.

(i): Note that $\pi_B T_2 = h_2$. Consider a unital separable $C \subset \mathcal{M}(D_1)$ such that C contains a strictly positive element of D_1 and $\pi_B(C) = h_1(A)$. Let $T := T_2 h_1^{-1} \pi_B | C$. T is weakly nuclear. Thus Theorem 5.6.2(i) or Corollary 5.6.1(i) applies. Let $s := \pi_B(S(1))$. Then s is an isometry in $\mathrm{Q}^{\mathrm{s}}(B)$ and $s^* h_1(\cdot) s = h_2$.

(ii): Let C_k the C^*-algebra which is generated by $h_k(A)$ and a strictly positive element of D_k, $(k = 1, 2)$. Then there are unique epimorphisms ψ_k from C_k onto A with $\psi_k h_k = \mathrm{id}_A$.

Let $T_1 := h_2 \psi_1$ and $T_2 := h_1 \psi_2$. Then C_k and $T_k\colon C_k \to \mathcal{M}(B)$ satisfy the assumptions of Theorem 5.6.2(ii) or of Corollary 5.6.1(ii). Thus we find norm continuous maps $t \mapsto U_k(t)$ into the unitaries of $\mathcal{M}(B)$ such that, for $a \in A$, $t \in \mathbb{R}_+$,

$$(h_k(a) \oplus T_k(h_k(a))) - U_k(t)^* h_k(a) U_k(t) \in B$$

and

$$\lim_{t \to \infty} \|(h_k(a) \oplus T_k(h_k(a))) - U_k(t)^* h_k(a) U_k(t)\| = 0.$$

Since $T_1 h_1 = h_2$ and $T_2 h_2 = h_1$, we get the desired result with the unitaries $U(t) = U_1(t) V_0 U_2(t)^*$, where V_0 is a unitary with $V_0^*(a \oplus b) V_0 = b \oplus a$.

By Chapter 4, (iii) and (iv) are special cases of (i). □

7. W-vN type results for simple p.i. algebras

The Part (ii) of below given Corollary 5.7.1 says among others that the absorption criteria of G. Elliott and D. Kucerovsky [**264**] (cf. [**310**] for the non-unital case) is satisfied for all pi-sun algebras.

Corollary 5.7.1. *Let B a nonzero σ-unital C^*-algebra and t_1, t_2 isometries in $\mathcal{M}(B \otimes \mathbb{K})$ with $t_1 t_1^* + t_2 t_2^* = 1$.*

The following conditions on B are equivalent:

(i) *B is stably isomorphic to a unital purely infinite simple C*-algebra or to* $\mathbb{C}$*;*

(ii) *For every unital separable C*-subalgebra C of $\mathcal{M}(B \otimes \mathbb{K})$ such that $C \cap (B \otimes \mathbb{K})$ contains a strictly positive element of $B \otimes \mathbb{K}$ and for every unital* weakly nuclear *map $V : C \to \mathcal{M}(B \otimes \mathbb{K})$ with $V(C \cap (B \otimes \mathbb{K})) = \{0\}$ there exists a sequence (s_n) of isometries in $\mathcal{M}(B \otimes \mathbb{K})$ with*
   (1) *$V(b) - s_n^* b s_n \in B \otimes \mathbb{K}$ for all $b \in C$,*
   (2) *$\|V(b) - s_n^* b s_n\| \to 0$ for all $b \in C$;*

(iii) *For every unital separable C*-subalgebra C of $\mathcal{M}(B \otimes \mathbb{K})$ such that $C \cap (B \otimes \mathbb{K})$ contains a strictly positive element of $B \otimes \mathbb{K}$, and for every unital weakly nuclear C*-morphism $h : C \to \mathcal{M}(B \otimes \mathbb{K})$ with $h(C \cap (B \otimes \mathbb{K})) = \{0\}$, there exists a sequence (u_n) of unitaries in $\mathcal{M}(B \otimes \mathbb{K})$ with*
   (1) *$b \oplus_{t_1,t_2} \varphi(b) - u_n^* b u_n \in B \otimes \mathbb{K}$ for all $b \in C$,*
   (2) *$\|b \oplus_{t_1,t_2} \varphi(b) - u_n^* b u_n\| \to 0$ for all $b \in C$,*

   for some pair s_1, s_2 of generators of $\mathcal{O}_2$ in $\mathcal{M}(B \otimes \mathbb{K})$.

Proof. The implication (i)⇒(ii) follows immediately from Theorem 5.6.2(i) and Corollary 5.6.1(i).

(ii)⇒(iii): If V is unital and weakly nuclear then $\delta_\infty \circ V$ is again unital and weakly nuclear and has the same kernel as V. Therefore the proof of Lemma 5.1.2(vi) can be modified to obtain a proof of the implication (ii)⇒(iii) here. One has only to replace E there by $\ell_\infty(\mathcal{M}(B \otimes \mathbb{K}))/c_0(B \otimes \mathbb{K})$ here. Then we apply Proposition 4.3.5(i) as there.

(iii)⇒(i): By Corollary 2.2.11(i), it suffices to show that the stable corona $\mathrm{Q}^{\mathrm{s}}(B) := \mathcal{M}(B \otimes \mathbb{K})/(B \otimes \mathbb{K})$ of B is simple. This follows already from Part (iii,1):

Let a be a positive element of $\mathrm{Q}^{\mathrm{s}}(B)$ with $\|a\| = 1$. We want to find an element d such that $d^* a d = 1$. This shows than that $\mathrm{Q}^{\mathrm{s}}(B)$ is not only simple but is also purely infinite.

Let χ be a character on $C^*(a)$ with $\chi(a) = 1$, $b \in \mathcal{M}(B)_+$ a contraction with $\pi_B(a)$ and $q \in B \otimes \mathbb{K}$ a strictly positive element of $B \otimes \mathbb{K}$. Now let $C := C^*(b, q)$ be the C^*-algebra generated by b and q. Then $C \cap (B \otimes \mathbb{K})$ contains q and we can apply Part(iii,1). The quotient map π from $\mathcal{M}(B \otimes \mathbb{K})$ onto $\mathrm{Q}^{\mathrm{s}}(B)$ maps C onto $C^*(a)$ Let us define $h : C \to \mathcal{M}(B \otimes \mathbb{K})$ as $h(c) = \chi(\pi(c)) \cdot 1$. Clearly h is a nuclear C^*-morphism and annihilates $C \cap (B \otimes \mathbb{K})$. Hence by Part (iii, α) there exists a unitary $u \in \mathcal{M}(B \otimes \mathbb{K})$ such that

$$b \oplus_{s_1,s_2} 1 - u^* b u \in B \otimes \mathbb{K}$$

(recall that $h(b) = 1$). Recalling the definition of $\oplus_{s_1,s_2}$, we get $s_1 b s_1^* + s_2 s_2^* - u^* b u \in B \otimes \mathbb{K}$. Hence $1 - s_2^*(u^* b u) s_2$ belongs to $B \otimes \mathbb{K}$. So, by taking $d = \pi(u s_2)$, $d^* a d = 1$ in $\mathrm{Q}^{\mathrm{s}}(B)$ as desired. □

Now we show a result that is stronger than the result pointed out in the proof of the implication (iii)⇒(i) of Corollary 5.7.1:

Corollary 5.7.2. *If B is σ-unital, simple and purely infinite and if A is a simple separable unital C^*-subalgebra of the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ such that the inclusion map $A \hookrightarrow \mathrm{Q}^{\mathrm{s}}(B)$ has a completely positive lift $T\colon A \to \mathcal{M}(B \otimes \mathbb{K})$ which is weakly nuclear, then $A' \cap \mathrm{Q}^{\mathrm{s}}(B)$ is purely infinite.*

In particular, $A' \cap \mathrm{Q}^{\mathrm{s}}(B)$ is purely infinite for every simple separable nuclear C^*-subalgebra $A \subseteq \mathrm{Q}^{\mathrm{s}}(B)$ that contains the unit element of $\mathrm{Q}^{\mathrm{s}}(B)$.

Proof. By Corollary 5.7.1(iii), id_A dominates the restriction of h_0 to $A \cong A \otimes 1$ where $h_0\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}^{\mathrm{s}}(B)$ is a weakly nuclear liftable unital *-monomorphism, e.g. coming from a faithful unital *-representation $k\colon A \otimes \mathcal{O}_2 \to \mathcal{L}(H) \cong \mathcal{M}(\mathbb{K})$.

Thus $A' \cap \mathrm{Q}^{\mathrm{s}}(B)$ is not isomorphic to $\mathbb{C}$.

Let $V\colon A \to \mathcal{M}(B \otimes \mathbb{K})$ be a weakly nuclear unital completely positive map with $\pi \circ V = \mathrm{id}_A$. If C is a unital separable C^*-subalgebra of $\mathrm{Q}^{\mathrm{s}}(B)$ and $h\colon C \to A$ is a unital C^*-morphism, then $V \circ h$ is a weakly nuclear lift of h. Thus by Corollary 5.7.1(ii) there exists an isometry $s \in \mathrm{Q}^{\mathrm{s}}(B)$ with $h(c) = s^* c s$ for $c \in C$.

This shows that $A \subset \mathrm{Q}^{\mathrm{s}}(B)$ satisfies the criteria in Proposition 2.2.5(v), and, therefore, $A' \cap \mathrm{Q}^{\mathrm{s}}(B)$ must be simple and purely infinite. □

Next stable absorption theorem characterizes the unital nuclear purely infinite simple algebras B and $B = \mathbb{C}$ as the only C^*-algebras by the "naive" one-to-one generalization of the classical Weyl–von-Neumann theorem. It shows that a useful more general generalization of the WvN-theorem needs more elaborate assumptions and formulations, e.g. as ours.

Corollary 5.7.3. *For a σ-unital C^*-algebra B the following properties* (i) *and* (ii) *are equivalent:*

(i) *B is stably isomorphic to $\mathbb{C}$ or is stably isomorphic to a simple purely infinite unital* nuclear *C^*-algebra A.*
(ii) *For every unital separable C^*-algebra $C \subset \mathcal{M}(B \otimes \mathbb{K})$ and every unital C^*-morphism $h\colon C \to \mathcal{M}(B \otimes \mathbb{K})$ with $h\big(C \cap (B \otimes \mathbb{K})\big) = 0$ there exists a unitary $U \in \mathcal{M}(B \otimes \mathbb{K})$ such that $c \oplus h(c) - U^* c U \in B \otimes \mathbb{K}$ for all $c \in C$.*

Proof. (i)⇒(ii): Let $E := B \otimes \mathbb{K}$. Every C^*-morphism $k\colon F \to \mathcal{M}(E)$ is weakly nuclear, because E is nuclear. Therefore, in this special case, the composition $C + E \to (C + E)/E \cong C/(C \cap B) \to \mathcal{M}(B)$ is again weakly nuclear if $h(C \cap (B \otimes \mathbb{K})) = 0$, and we can assume w.l.o.g. that C contains a strictly positive element of E. Then (ii) follows from Corollary 5.7.1(iii,1).

(ii)⇒(i): We modify the proof of the implication (iii)⇒(i) of Corollary 5.7.1 as follows. Let $C := C^*(b, 1)$, with b as there. Then we don't use the assumption that B is σ-unital, that we have used in the proof of Corollary 5.7.1 to check the conditions in Corollary 2.2.11(i), because we can do *here* the same with *every* unital separable $C \subset \mathcal{M}(B \otimes \mathbb{K})$ by our assumptions in (ii).

By Corollary 2.2.11(i), this gives that B is σ-unital simple and purely infinite and thus is stably isomorphic to a simple purely infinite unital C^*-algebra A by [**172**] and by L.G. Brown's stable isomorphism theorem (cf. Corollary 5.5.6).

Let A unital with $A\otimes\mathbb{K} \cong B\otimes\mathbb{K}$ and let D be a separable unital C^*-subalgebra of A and $R\colon D \to \mathcal{L}(H) \cong 1_A \otimes \mathcal{M}(\mathbb{K})$ a faithful unital *-representation with $R(D) \cap \mathbb{K}(H) = 0$. Then (ii), for $C := R(D)$, implies that there is an isometry $S \in \mathcal{M}(A \otimes \mathbb{K})$ with

$$S^*(1_A \otimes R(d))S \,-\, d \otimes 1_{\mathcal{M}(\mathbb{K})} \in A \otimes \mathbb{K}\,.$$

If we use a sequence of states ψ_n on $\mathbb{K}$ which converges weakly to zero, then we get that $D \hookrightarrow A$ is a point-wise limit of the unital completely positive maps $d \mapsto (\mathrm{id}_A \otimes\psi_n)(S^*1_A \otimes R(d)S)$, which are nuclear. Here we have to extend $(\mathrm{id}_A \otimes\psi_n)\colon A \otimes \mathbb{K} \to A$ naturally to a unital completely positive map $\mathcal{M}(A \otimes \mathbb{K}) \to A$. Thus A is nuclear. □

REMARK 5.7.4. Above considerations show that the range of generalized Weyl–von Neumann theorems seems to be very limited almost to extensions by $\mathbb{K}$ or stable simple purely infinite algebras.

But if we consider special situations, i.e., additional requirements on C and V or h, then Proposition 5.4.1 allows to deduce other types of generalizations of the Weyl–von Neumann theorem in the spirit of Voiculescu's generalization of BDF-theory, e.g. some equivariant cases. Here we state a general result inspired by E. Blanchard [**89**]. We do not use directly the fundamental conditions (α) and (β) of the generalised Weyl–von-Neumann theorem but apply more comfortable assumptions that imply conditions (α) and (β):

Assume that B is σ-unital and stable and that $C \subset \mathcal{M}(B)$ is unital and separable. Let $T\colon C \to \mathcal{M}(B)$ a unital completely positive map such that:

- (α') *$T(C \cap B) = 0$ and $C \cap B$ contains a strictly positive element of B;*
- (β') *For every (non-degenerate) factorial representation $\rho\colon B \to \mathcal{L}(H)$ there exists a net W_τ of inner unital completely positive maps $W_\tau(a) = \sum d_k^* a d_k$ from $N := \rho(B)''$ into N, such that $\rho(V(c))$ is the point-wise weak limit of $W_\tau(\rho(c))$ for every $c \in C$, where we have extended ρ naturally to a weakly continuous homomorphism from $B^{**} \supset \mathcal{M}(B)$ into N.*
- (β'') *For every $b \in B_+$, $c_1, \ldots, c_m \in C$, $\varepsilon > 0$, $n \in \mathbb{N}$ there exist $s_1, \ldots, s_n \in \mathcal{M}(B)$ such that $s_i^* s_j = \delta_{ij}$ and $\|c_k s_i b - s_i c_k b\| < \varepsilon$ for $i = 1, \ldots, n$, $k = 1, \ldots, m$.*

Then the conclusions (ii), (iii) and (iv) Proposition 5.4.1 hold.

Idea of the proof: Clearly (α') implies (α) of Proposition 5.4.1.

A combination of a Hahn-Banach separation argument (which yields a tensor product characterization of approximately inner completely positive maps, cf. Chapter 3 or [**443**, appendix]) with a modification of the argument in the proof of Proposition 2.2.5(iii) shows that (β') and (β'') together imply (β) of Proposition 5.4.1 for $V := \delta_\infty \circ T$.

8. Extensions and $\mathrm{Ext}(\mathcal{C};\, A, B)$

Parts of the proofs of our theorems in Chapter 1 consists in showing that a certain *strong equivalence* of extensions (related to the theorem in question) can be deduced from – in a "formal sense" – considerably weaker equivalence relations on extensions. In particular, we have to deduce that certain extensions are actually split ([17]). It requires that the extensions that absorb certain split extensions need to be characterized with help of generalized Weyl–von-Neumann–Voiculescu theorems. This will be discussed here in this section.

We need urgently the following cases:

(1) $B = \mathcal{O}_2 \otimes \mathbb{K}$, A unital and separable (and exact), $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ unital (!) and with (automatically weakly nuclear) unital c.c. split $T\colon A \to \mathcal{M}(B)$ with $\pi_B \circ T = \varphi$.

If A is exact, then T is automatically nuclear. Thus, this is then the case if $\varphi\colon A \to \mathrm{Q}(B)$ is nuclear.

Is there an "elementary way" to find a splitting by C^*-morphism?

E.g. it could be that φ extends to $A \otimes \mathcal{O}_2$, i.e., that one can find a unital copy of $\mathcal{O}_2$ in $\varphi(A)' \cap \mathrm{Q}(B)$: Then one can take a faithful unital nuclear *-representation $\rho\colon A \to \mathcal{L}(\ell_2) = \mathcal{M}(\mathbb{K})$ and define $\psi\colon A \to \mathrm{Q}(B)$ by $\psi(a) = \pi_{B\otimes\mathbb{K}}(1_{\mathcal{O}_2} \otimes \rho(a))$. ψ is again nuclear and commutes with a copy of $\mathcal{O}_2$ in $\mathrm{Q}(B)$.

Then ψ, φ 1-step dominate each other (needs a proof). It follows then from results in Chapter 4, that they are unitary equivalent in $\mathrm{Q}(B)$. This defines a unital splitting C^*-morphism for φ.

To get such things there are two ways:

Replace A by $A \otimes \mathcal{O}_2$ (covers the exact case), or try to find a way to prove that $\varphi \oplus \varphi$ is unitary equivalent to φ in $\mathrm{Q}(B)$ if A is nuclear. (this should be necessary to get the desired additional observations in case of nuclear A).

We can attempt to first use that $A = \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ has this property (needs homotopy invariance of $\mathrm{Ext}(A, \mathcal{O}_2)$?).

We find every exact C^*-algebra as sub quotient of $\mathcal{O}_2$ in $\mathrm{Q}(B)$, such that each c.p. lift sits in a copy of $B \subseteq V(\mathcal{O}_2) \cap \mathcal{M}(B)$ for a u.c.p. map $V\colon \mathcal{O}_2 \to \mathcal{M}(B)$.

Get then unital embedding into $\mathcal{O}_2$, but first only of extensions of them by B.

Used later:

(1a) If φ and $\psi\colon A \to \mathrm{Q}(B)$ have unital splits, then ψ and φ are unitarily equivalent by a unitary in $\mathrm{Q}(B)$.

Moreover this should be the case if $B = \mathcal{O}_2 \otimes \mathbb{K}$

[17] I.e., is not only stably split after adding a certain split extension.

One should check if ψ extends to ?????

Because, by Kasparov, Voiculescu, ????

(1b) $\mathcal{U}(\mathrm{Q}(B)) = \mathcal{U}_0(\mathrm{Q}(B))$ deduced from K_1-injectivity and $\mathrm{K}_1(\mathrm{Q}(B)) = 0$. (By property (sq) of $\mathrm{Q}(B)$, $\mathrm{K}_*(\mathcal{M}(B)) = 0$ and $\mathrm{K}_*(\mathcal{O}_2) = 0$. Ref. from Chp. 4)

(2) B arbitrary σ-unital, stable strongly p.i. C^*-algebra. $\mathcal{C}$ given by

$$H_0 \colon A \to \mathcal{M}(B)$$

non-degenerate, A separable and *stable*, with the property that H_0 is unitarily homotopic to $\delta_\infty \circ H_0$.

Are δ_∞ and δ_∞^2 really unitarily homotopic?

Yes, they are, because they are unitary equivalent and each unitary equivalence causes "homotopy" in $\mathcal{M}(B)$ by a norm-continuous path of unitaries in $\mathcal{M}(B)$ if B stable and σ-unital, because then $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}_0(\mathcal{M}(B))$.

It is not clear if one can move them with a path $U(t)$ in the unitaries such that

$$U(t)^*\delta_\infty^2(H_0(a))U(t) - \delta_\infty(H_0(a)) \in B$$

for all $a \in A$.

But this should be possible if $H_0 = \delta_\infty \circ h_0$ for some $h_0 \colon A \to B$???

This could be almost possible with a "long" strictly continuous path $W(t)$ with $\lim W(t) = U$ for $W(t) = \exp(h_1(t)) \cdot \ldots \cdot \exp(h_n(t))$, $U = \exp(k_1) \cdot \ldots \cdot \exp(k_n)$, $h_\ell(t) \to k_\ell$ strictly, $h_\ell(t)^* = -h_\ell(t)$, $k_\ell^* = -k_\ell$.

In the corona it does nothing.

Busby invariants in question are:

$\varphi \colon A \to \mathrm{Q}(B)$

with c.p. splits $V \colon A \to \mathcal{M}(B)$: i.e., $b^*V(\cdot)b \in \mathcal{C}$ for each $b \in B$.

Pre-assumption in trivially graded case:

A and B stable, A separable, B σ-unital, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ point-norm closed, countably generated, non-degenerate m.o.c. cone.

$H_{\mathcal{C}} \colon A \to B$ non-degenerate in general position, in 1-1-relation to $\mathcal{C}$, $h_0 := \pi_B \circ H_{\mathcal{C}}$.

$$\mathrm{Ext}(\mathcal{C};\, A, B) \cong \text{ kernel of } \mathrm{K}_0(h_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(\mathrm{Q}(B))\,.$$

This is, because we consider only the zero-dominating maps in $S(h_0; A, \mathrm{Q}(B))$, i.e., where the extension is stable.

Since the σ-unital B is stable, and A is stable and σ-unital, it means equivalently that we consider only *stable* extensions of A with B, by Proposition **??**.

Alternatively: Classes of projections $p \in \mathcal{M}(B)$ with $[p, h_0(a)] \in B$ for all $a \in A$, equivalence modulo B in $h_0(A)' \cap \mathrm{Q}(B)$ (alternatively: isometries T with $TT^* = p$, $T^*T = p'$, T commutes mod B with $h_0(A)$)

Unitary equivalence classes $[\varphi]$ by unitaries in $\pi_B(\mathcal{U}(\mathcal{M}(B))) = \mathcal{U}_0(\mathrm{Q}(B))$ deserve as elements as elements of $\mathrm{SExt}(\mathcal{C}; A, B)$.

$\mathrm{SExt}(\mathcal{C}; A, B) := S(\pi_B \circ H_0\,;\ A, \mathrm{Q}(B))$.

implies $\mathrm{Ext}(\mathcal{C}; A, B) \cong \mathrm{Gr}(\mathrm{SExt}(\mathcal{C}; A, B))$

Needs "absorbing elements".

Or / Respectively, other (not equivalent) definition ????:
$\mathrm{Ext}(\mathcal{C}; A, B) := [\pi_B \circ H_0] + \mathrm{SExt}(\mathcal{C}; A, B)$

The class $[\pi_B \circ H_0]$ deserves as a zero element.

The H_0 for $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(X; A, B)$, respectively $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$, and (if A and B are both separable) $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ should be given.

(3) Specify the "nuclear cases" for A unital, exact, and separable, and B simple s.p.i., σ-unital, stable and separable.

Then $\mathrm{Q}^{\mathrm{s}}(B) = \mathcal{M}(B)/B$ is simple and p.i. If $\varphi, \psi \colon A \to \mathrm{Q}^{\mathrm{s}}(B)$ are two unital nuclear *-monomorphisms. Then there exist isometries $S, T \in \mathrm{Q}^{\mathrm{s}}(B)$ with $S^*\varphi(\cdot)S = \psi$ and $T^*\psi(\cdot)T = \varphi$.

Need that in case $B = \mathcal{O}_2 \otimes \mathbb{K}$ and separable unital $C \subseteq \mathrm{Q}^{\mathrm{s}}(B)$ that the unit of $C' \cap \mathrm{Q}^{\mathrm{s}}(B)$ is properly infinite in $C' \cap \mathrm{Q}^{\mathrm{s}}(B)$, and $\mathrm{K}_*(C' \cap \mathrm{Q}^{\mathrm{s}}(B)) = 0$.

Can follow form ????

In a first step one could consider $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ in place of A.

Implies that $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ is unitally contained in $\mathcal{O}_2$?

Need: All unital *-endomorphisms of $\mathcal{O}_2$ and of $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ are approximately inner.

Must be integrated in Chapter 3.

(4) Discuss:
Absorption like Elliott/Kucerovsky (+ correction by J. Gabe)

(5) Existence and uniqueness – up to unitary homotopy – of universal $H_0 \colon A \to \mathcal{M}(B \otimes \mathbb{K})$. In which cases of $\mathcal{C}$?

In case of B a pi-sun algebra, there exists only $\mathrm{CP}(A, B) = \mathrm{CP}_{\mathrm{nuc}}(A, B)$.

Then each nuclear c.p. map into $\mathrm{Q}^{\mathrm{s}}(B)$ is 1-compression of H_0 and one has to determine the $\mathrm{K}_1(\pi_{B \otimes \mathbb{K}} \circ H_0(A)' \cap \mathrm{Q}^{\mathrm{s}}(B))$.

The quickest way should be to show that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ and that it causes that for each separable subset X of $Q := \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots)$ there exists a copy of $\mathcal{O}_2$ unitally in $X' \cap Q$.

(6??)

The definition of Elliott/Kucerovsky/Gabe of an "absorbing extension" is:

"An extension that absorbs any weakly nuclear extension"

Ell/Kuc in Pacific MJ: (Unital) "purely large" extensions absorb all (unitally) weakly nuclear extensions.

The extension $0 \to B \to E \to A \to 0$ is "purely large" (by Definition of "purely large") if for each $x \in E \setminus B$ the hereditary C^*-subalgebra $\overline{x^*Bx}$ of B contains a stable hereditary C^*-subalgebra $D \subseteq \overline{x^*Bx}$ that is full in B.

My observation:
Write $B \cong B \otimes \mathbb{K}$ for $\mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))$ The extension E must necessarily absorb the extension given by Busby invariant $\pi_B \circ h$ where $h\colon A \ni a \mapsto 1_{\mathcal{M}(B)} \otimes \delta_\infty(d(a)) \in \mathcal{M}(B) \otimes \mathcal{L}(\ell_2) \subseteq \mathcal{M}(B \otimes \mathbb{K}) \cong \mathcal{M}(B)$.

Here, $d\colon A \to \mathcal{L}(\ell_2)$ is any faithful C^*-morphism that has the property that $d(A)\ell_2$ is *not* dense in ℓ_2.

Full hereditary stable C^*-subalgebras D of $B \otimes \mathbb{K}$ contain an in D strictly positive element $e \in D_+$ that is M-vN equivalent in $B \otimes \mathbb{K}$ to a strictly positive element $b \in (B \otimes \mathbb{K})_+$.

How to use it to show our assumptions for the absorption Theorem ??? for $A \subseteq \mathrm{Q}^{\mathrm{s}}(B)$ and $(\pi_B \circ h)\colon A \to \mathrm{Q}^{\mathrm{s}}(B)$

An (unital) extension $0 \to B \to E \to A \to 0$ is "(unitally) weakly nuclear" if there exists a (unital) weakly nuclear c.p. map $V\colon A \to E$ with the property that $\pi_B \circ V$ is the Busby invariant of the extension.

Here V is "weakly nuclear" if $A \ni a \mapsto b^*V(a)b \in B$ is nuclear for all $b \in B$.

(Kucerovsky/Ng, Houston J.Math.32, 2006): Any "full" extension by a σ-unital stable C^*-algebra B with (CFP) is "purely large".

Here "full" means that each non-zero element of $A \subseteq E/B$ is full in E/B.

Corona factorization property (CFP) (of stable C^*-algebras B):

Each full projection $P \in \mathcal{M}(B)$ is properly infinite in $\mathcal{M}(B)$. (P is range of a isometry in $\mathcal{M}(B)$?)

J. Gabe: Lemma 1.2 :

There exists non-unital purely large extension such that the unitization is not purely large.

Final Th.??

We recall some basic definitions from the theory of extensions: For every short exact sequence $0 \to B \to E \to A \to 0$ of C^*-algebras we consider the Busby diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \mathcal{M}(B) & \underset{\pi}{\longrightarrow} & \mathrm{Q}(B) & \longrightarrow & 0
\end{array}$$

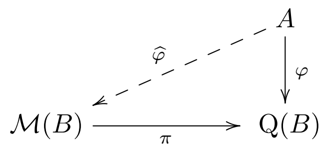
where $\mathrm{Q}(B)$ denotes the **corona algebra** $\mathcal{M}(B)/B$. The right vertical arrow $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ is the **Busby invariant** of the extension. There is a natural *-isomorphism $f \in E \mapsto (\eta(f), \rho(f)) \in E_\varphi$ from E onto the pull-back

$$E_\varphi := A \oplus_{\varphi, \pi_B} \mathcal{M}(B) \subset A \oplus \mathcal{M}(B)\,,$$

where $\eta\colon E \to A$ is the defining epimorphism of the extension E and $\rho(f)b = \iota^{-1}(f\iota(b))$ for $f \in E$, $b \in B$ and the defining monomorphism $\iota\colon B \to E$ of the extension. Thus, the extension is (up to natural *-isomorphisms) uniquely determined by the Busby invariant $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$.

For $\varphi, \psi \in \mathrm{Hom}(A, \mathrm{Q}(B))$, we write $\varphi \approx \psi$ if $\psi(a) = \pi(u)\varphi(a)\pi(u^*)$, $a \in A$, for some unitary $u \in \mathcal{M}(B)$, and call the extensions **strongly equivalent** . Then E_ψ and E_φ are isomorphic as C^*-algebras by an isomorphism that induces the identity map on A and the automorphism $b \mapsto u^*bu$ of B. That will be applied in the proofs of Theorems A and B, and we need to establish strong equivalence in some special cases from essentially weaker definitions of equivalence.

A Busby invariant $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ is called **trivial** if there exists a C^*-morphism $\hat{\varphi} \in \mathrm{Hom}(A, \mathcal{M}(B))$ such that the diagram



commutes, i.e., if φ is **liftable**. Then E_φ is a **split** extension, because the natural epimorphism η from E_φ onto A has a **split morphism** $\widehat{\eta}\colon A \to E_\varphi$ with $\eta \circ \widehat{\eta} = \mathrm{id}_A$. (We do not suppose that the lift is unital if A and φ are unital.)

Now assume that $\mathcal{M}(B)$ contains a copy $C^*(s_1, s_2)$ of $\mathcal{O}_2$ unitally. (This is the case e.g. for stable B, where $\mathcal{M}(B) \cong \mathcal{M}(B \otimes \mathbb{K})$ and $\mathrm{Q}(B) \cong \mathrm{Q}^{\mathrm{s}}(B) := \mathrm{Q}(B \otimes \mathbb{K})$.) Then $\mathrm{Q}(B)$ also contains the copy $C^*(t_1, t_2)$ of $\mathcal{O}_2$ unitally, where $t_k = \pi_B(s_k)$ $(k = 1, 2)$, and
Cuntz addition $\oplus := \oplus_{t_1,t_2}$ *is defined on the classes* $[\varphi]_\approx$:
see Proposition 4.3.2 and notice that only canonical generators $r_1, r_2 \in \mathrm{Q}(B)$ with property $t_1 r_1^* + t_2 r_2^* \in \pi_B(\mathcal{U}(\mathcal{M}(B)))$ are considered.

We write $\varphi \sim \psi$ if there exist trivial elements $\tau_1, \tau_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ such that $\varphi \oplus \tau_1 \approx \psi \oplus \tau_2$. In particular $\varphi \sim (\varphi \oplus 0)$, because $\varphi \oplus 0 = \varphi \oplus (0 \oplus 0) \approx (\varphi \oplus 0) \oplus 0$ by Proposition 4.3.2(iv). Thus, we find for the $\sim$-classes $[\varphi]_\sim$ always a representing element $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ such that the inclusion map of the C^*-subalgebra $\pi_B^{-1}(\varphi(A))$ in $\mathcal{M}(B)$ dominates zero (in the sense of Definition 4.3.3).

The *-morphism $\varphi \oplus_{t_1,t_2} 0$ is unitarily equivalent to $\psi \oplus_{t_1,t_2} 0$ in $\mathrm{Q}(B)$ if and only if $\varphi \oplus 0$ and $\psi \oplus 0$ are conjugate by a unitary in the connected component $\mathcal{U}_0(\mathrm{Q}(B))$ of 1 in the unitaries $\mathcal{U}(\mathrm{Q}(B))$ of $\mathrm{Q}(B)$ (cf. Proposition 4.3.6(iv,c)).

Since $\pi_B\big(\mathcal{U}_0(\mathcal{M}(B))\big) = \mathcal{U}_0(\mathrm{Q}(B))$, it follows that $\varphi \sim \psi$ if and only if, there are “trivial” Busby invariants $\tau_1, \tau_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ such that $\varphi \oplus \tau_1 \oplus 0$ and

$\varphi \oplus \tau_2 \oplus 0$ are unitarily equivalent in $\mathrm{Q}(B)$. Hence, *the relation* $\sim$ *is compatible with Cuntz addition.*

Recall that $\mathrm{Ext}(A, B)$ (or better notated as $\mathrm{SExt}(A, B)$) is the semigroup $\mathrm{Hom}(A, \mathrm{Q}(B))/\sim$, where the zero element is the class of "trivial elements". Let $\mathrm{Ext}^{-1}(A, B)$ denote the group of invertible elements in $\mathrm{Ext}(A, B)$, i.e., of $\sim$-classes of elements $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ such that there are $\psi, \tau_1, \tau_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ such that $(\varphi \oplus \psi) \oplus \tau_1 = \tau_2$ and τ_1, τ_2 are trivial. Notice that $V := ((s_1)^2)^* \widehat{\tau_2}(\cdot)(s_1)^2$ is a completely positive lift of of φ if φ is in $\mathrm{Ext}^{-1}(A, B)$.

In the remaining part of this section we suppose in addition, that B *is stable and* σ*-unital and that* A *is separable and is unital or stable* ([18]).

Then, conversely, *every completely positive liftable* $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ *is "invertible" in* $\mathrm{Ext}(A, B)$, because for a suitable Kasparov-Stinespring dilation $\widehat{\tau} \colon A \to M_2(\mathcal{M}(B)) \cong \mathcal{M}(B)$ of a contractive c.p. lift V of φ there is $\psi \in \mathrm{Hom}(A, \mathrm{Q}(B))$ with $\varphi \oplus \psi \oplus 0 \approx \tau \oplus 0$. Thus, for stable σ-unital B, $\mathrm{Ext}^{-1}(A, B)$ consists of the the $\sim$-classes of completely positive liftable elements of $\mathrm{Hom}(A, \mathrm{Q}(B))$ (with Cuntz addition).

Moreover, for trivial Busby invariants $\tau_1, \tau_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$, there exists trivial $\tau_3 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ with

$$\tau_1 \oplus \tau_3 \approx \tau_3 \approx \tau_2 \oplus \tau_3\,,$$

(Indeed, $\tau_3 := \pi_B \circ \delta_\infty \circ (\widehat{\tau_1} \oplus \widehat{\tau_2})$ does the job, because, with $\widehat{\tau_3} := \delta_\infty \circ (\widehat{\tau_1} \oplus \widehat{\tau_2})$, $\widehat{\tau_1} \oplus \widehat{\tau_3}$ and $\widehat{\tau_2} \oplus \widehat{\tau_3}$ are both unitarily equivalent to $\widehat{\tau_3}$ in $\mathcal{M}(B)$ by Lemma 5.1.2(i,ii).)

It follows that $h_1 \sim h_2$ for $h_1, h_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ if and only if there is trivial $\tau \in \mathrm{Hom}(A, \mathrm{Q}(B))$ with $[h_1] + [\tau] = [h_2] + [\tau]$ (and $[\tau] + [\tau] = [\tau]$) in the semigroup of unitary equivalence classes $([\mathrm{Hom}(A, \mathrm{Q}(B))], +)$.
(Indeed: $h_1 \sim h_2 \Leftrightarrow$ there exists liftable τ with $h_1 \oplus \tau \approx h_2 \oplus \tau \Rightarrow [h_1] + [\tau] = [h_2] + [\tau]$ $\Rightarrow [h_1 \oplus \tau \oplus 0] = [h_2 \oplus \tau \oplus 0] \Rightarrow h_1 \oplus (\tau \oplus 0) \approx h_2 \oplus (\tau \oplus 0)$. The last implication comes from the K_1-injectivity of $\mathrm{Q}(B)$.)

Recall that $\mathrm{Gr}(\mathcal{S})$ denotes the Grothendieck group of an Abelian semigroup $\mathcal{S}$. The above observations imply – for σ-unital stable B – that $\mathrm{Ext}^{-1}(A, B)$ has the following equivalent description as Grothendieck group of a certain semigroup:

Let $\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))$ denote the set of Busby invariants $\varphi \colon A \to \mathrm{Q}(B)$ that have a completely positive lift $V \colon A \to \mathcal{M}(B)$ ([19]). The set of *unitary equivalence classes* $[\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))]$ of elements in $\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))$ by unitaries in $\mathrm{Q}(B)$ are invariant under Cuntz addition $[\varphi] + [\psi] := [\varphi \oplus \psi]$ (where we now have to allow all generators t_1, t_2 of a unital copy of $\mathcal{O}_2$ in $\mathrm{Q}(B)$ for the definition of $\oplus$).

---

[18] Then $\varphi \approx \psi$ if and only if there is $v \in \mathcal{U}_0(\mathrm{Q}(B))$ with $v^* \varphi(\cdot) v = \psi$, because $\mathcal{U}_0(\mathrm{Q}(B)) = \pi_B(\mathcal{U}_0(\mathcal{M}(B)))$ and the unitary group of $\mathcal{M}(B \otimes \mathbb{K}) \cong \mathcal{M}(B)$ is connected by [**180**]. Moreover, one needs only that $[v] = 0$ in $\mathrm{K}_1(\mathrm{Q}(B))$, because $\mathrm{Q}(B)$ is K_1-injective if B is stable and σ-unital, cf. Lemma 5.5.9(ii), or Proposition 4.2.15.

[19] i.e., $\pi_B \circ V = \varphi$, one can always replace V by a completely positive contraction with this property

$\mathrm{Ext}^{-1}(A,B)$ is naturally isomorphic to the Grothendieck group $\mathrm{Gr}([\mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))])$ of the commutative semigroup $([\mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))],+)$. (Indeed: the isomorphism is induced by the semigroup morphisms

$$[\varphi]_{\approx} \mapsto [\varphi] \mapsto [\varphi]_{\mathrm{Gr}}$$

for $\varphi \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))$, because, if $[\varphi]+[\lambda]=[\psi]+[\lambda]$ for $\lambda \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))$ then there exist $\kappa \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B)$ and a *liftable* $\tau \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))$ such that $[\tau]=[\lambda\oplus\kappa]$, and, therefore $\varphi\oplus\tau\oplus 0 \approx \psi\oplus\tau\oplus 0$. I.e., $\varphi\sim\psi$ if $[\varphi]_{\mathrm{Gr}}=[\psi]_{\mathrm{Gr}}$ for $\varphi,\psi \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))$.
Conversely, $\varphi\sim\psi$ implies the existence of a liftable $\tau \in \mathrm{Hom}_{\text{l-cp}}(A,\mathrm{Q}(B))$ with $\varphi\oplus\tau \approx \psi\oplus\tau$, which implies that $[\varphi]+[\tau]=[\psi]+[\tau]$, i.e., that $[\varphi]_{\mathrm{Gr}}=[\psi]_{\mathrm{Gr}}$.)

We introduce the semigroup $\mathrm{SExt}_{\mathrm{nuc}}(A,B)$ and the group $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ in a similar way:
Let $\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ denote the elements $\varphi \in \mathrm{Hom}(A,\mathrm{Q}(B))$ which have weakly nuclear lifts $V\colon A\to\mathcal{M}(B)$. Note that $h_1\approx h_2$ if and only if $u^*h_1(\cdot)u=h_2$ by a unitary $u\in\mathcal{U}_0(\mathrm{Q}(B))$, that $h_1\approx h_2$ implies (by definition) that $u^*h_1(\cdot)u=h_2$ for a unitary $u\in\mathrm{Q}(B)$, and that $u^*h_1(\cdot)u\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ if $h_1\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ and $u\in\mathrm{Q}(B)$ is unitary: Indeed, $V'(a):=d^*V(a)d$ is a weakly nuclear completely positive lift of $u^*h_1(\cdot)u$ if $\pi_B(d)=u$ and $V\colon A\to\mathcal{M}(B)$ is a weakly nuclear lift of h_1. Thus, the subset $\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))\subset\mathrm{Hom}(A,\mathrm{Q}(B))$ is closed under $\approx$, unitary equivalence and under $\oplus$: if $\varphi,\psi\in\mathrm{Hom}(A,\mathrm{Q}(B)$ have weakly nuclear lifts and $\chi\approx\varphi$, then χ and $\varphi\oplus\psi$ have weakly nuclear lifts. It follows that Cuntz addition $\oplus_{t_1,t_2}$ is defined on the set of classes $[\varphi]_{\approx}$ with $\varphi\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ for $t_1=\pi_B(T_1)$, $t_2=\pi_B(T_2)$, where the isometries $T_1,T_2\in\mathcal{M}(B)$ are canonical generators of a copy of $\mathcal{O}_2$ in $\mathcal{M}(B)$. We write $\varphi\sim_{nuc}\psi$ if there exist *weakly nuclear* C^*-morphisms $H_1,H_2\colon A\to\mathcal{M}(B)$ such that $\varphi\oplus(\pi_B\circ H_1)\approx\psi\oplus(\pi_B\circ H_2)$. Note that $h_1\sim_{nuc}h_2$ implies $h_1\sim h_2$, and it happens that $h_1\sim h_2$ for $h_1\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ but $h_2\notin\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ if B is simple and is not nuclear (We do not know if $h_1\sim h_2$ implies $h_1\sim_{nuc}h_2$ for $h_1,h_2\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$.) The relation $\sim_{nuc}$ is compatible with the Cuntz addition on $[\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))]_{\approx}$ and $h_1\sim h_2$ if and only if there is a weakly nuclear C^*-morphism $H\colon A\to\mathcal{M}(B)$ such that $h_1\oplus\pi\circ H\oplus 0$ is unitarily equivalent to $h_1\oplus\pi\circ H\oplus 0$ by a unitary in $\mathrm{Q}(B)$. Since for every $\varphi\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ there exists a contractive weakly nuclear c.p. map $V\colon A\to\mathcal{M}(B)$ and since there is a weakly nuclear C^*-morphism $H\colon A\to\mathcal{M}(B)$ and an isometry $S\in\mathcal{M}(B)$ with $V=S^*H(\cdot)S$, we can see that there is $\psi\in\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))$ such that $\psi\oplus\varphi\oplus 0$ is unitarily equivalent to $\pi_B\circ H\oplus 0$. We get that the quotient $([\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B)]_{\sim_{nuc}},+)$ of the semigroup $[\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))]_{\approx}$ with respect to the relation $\sim_{nuc}$ is a group. One can see that this group is isomorphic to the below defined group $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ (the nuclear Ext-group in the below given Definition 5.8.2). The proof of the isomorphism

$$\mathrm{Hom}_{\text{l-nuc}}(A,\mathrm{Q}(B))/\sim_{nuc}\;\cong\;\mathrm{Ext}_{\mathrm{nuc}}(A,B)$$

is almost verbatim the above outlined proof of the natural isomorphism $\mathrm{Ext}^{-1}(A,B) \cong \mathrm{Gr}([\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))], +)$.

If A is separable and *unital*, then we can consider the set $\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))$ of *unital* morphisms $\varphi\colon A \to \mathrm{Q}(B)$ in $\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))$.

The subset $\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B)) \subset \mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))$ is closed under unitary equivalence (hence under $\approx$-equivalence) and under Cuntz addition $\oplus$. The Grothendieck group

$$\mathrm{Ext}^{\text{u,strong}}_{\text{nuc}}(A,B) := \mathrm{Gr}([\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))]_{\approx}\,, +)$$

can be described equivalently by defining relations $\sim_{u-nuc}$:
We write $\varphi \sim_{u-nuc} \psi$ if there are *unital* weakly nuclear C^*-morphisms $H_1, H_2\colon A \to \mathcal{M}(B)$ such that $\varphi \oplus \pi_B \circ H_1 \approx \psi \oplus \pi_B \circ H_2$.

Then $H_3 := \delta_\infty \circ (H_1 \oplus H_2)$ is unital, weakly nuclear and is unitarily equivalent to $H_3 \oplus H_1$ and to $H_3 \oplus H_2$, and we get: $\varphi \sim_{u-nuc} \psi$, *if and only if, there is a unital weakly nuclear *-morphism* $H\colon A \to \mathcal{M}(B)$ *with* $\varphi \oplus \pi_B \circ H \approx \psi \oplus \pi_B \circ H$. It follows that $\varphi \sim_{u-nuc} \psi$ implies that $[\varphi]_{\mathrm{Gr}} = [\psi]_{\mathrm{Gr}}$ in $\mathrm{Ext}^{\text{u,strong}}_{\text{nuc}}(A,B)$. Conversely, if $[\varphi]_{\mathrm{Gr}} = [\psi]_{\mathrm{Gr}}$ then there exists $\chi \in \mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))$ with $\varphi \oplus \chi \approx \psi \oplus \chi$. By definition of $\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))$, there is a weakly nuclear c.p. lift $W\colon A \to \mathcal{M}(B)$ with $\pi_B \circ W = \chi$. By a standard argument there are $b \in B_+$ with $\|b\| \le 1$ and a unital weakly nuclear c.p. map $U\colon A \to \mathcal{M}(B)$ such that $W'(a) := (1-b^2)^{1/2} W(a) (1-b^2)^{1/2} + bU(a)b$ satisfies $\|W'(1) - 1\| < 1/2$. Thus, $V(a) := W'(1)^{-1/2} W'(a) W'(1)^{-1/2}$ defines a unital weakly nuclear c.p. map with $\pi_B \circ V = \chi$. Let $\mathcal{C} := \mathrm{CP}_{\text{nuc}}(A,B))$ in the (more general) Corollary 5.4.6. It shows that there is a unital weakly nuclear C^*-morphism $H\colon A \to \mathcal{M}(B)$ and isometries $S, T \in \mathcal{M}(B)$ with $SS^* + TT^* = 1$ and $V(a) - S^* H(a) S \in B$ for all $a \in A$. By Lemma 4.3.4(i) (applied to $g := \pi_B(SS^*)$ and $h_1 := \pi_B \circ H$) we get that $SS^* H(a) - H(a) SS^* \in B$ and $SV(a)S^* + TT^* H(a) TT^* - H(a) \in B$ for all $a \in A$, i.e., $\chi \oplus_{s,t} \vartheta = \pi_B \circ H$ for $\vartheta(a) := \pi_B(T^* H(a) T)$ and $s := \pi_B(S)$ and $t := \pi_B(T)$. Hence, $\varphi \oplus \pi_B \circ H \approx \psi \oplus \pi_B \circ H$, i.e., $\varphi \sim_{u-nuc} \psi$.

REMARK 5.8.1. Let E denote a unital C^*-algebra and let $\mathcal{O}_2 \cong C^*(s_1, s_2)$ be unitally contained in E. (Here we use only the unital embedding $\iota\colon \mathcal{O}_2 \hookrightarrow E$ up to unitary equivalence by unitaries in the *connected* component $\mathcal{U}_0(E)$ of 1_E in the unitaries $\mathcal{U}(E)$ of E.)

We say that a C^*-morphism $\varphi\colon A \to E$ **strongly absorbs** $\psi\colon A \to E$ if there is a unitary $u \in \mathcal{U}_0(E)$ with $u^* \varphi(\cdot) u = \varphi \oplus_{s_1,s_2} \psi$, I.e., $\varphi \approx \varphi \oplus \psi$, if $E = \mathrm{Q}(B)$ for stable and σ-unital B.

It allows to formulate a useful lifting criterium (that follows straight from the definitions):
Suppose that a unital C^*-morphism $\varphi\colon A \to \mathrm{Q}(B)$ strongly absorbs every unital C^*-morphism $\pi_B \circ H$, where $H\colon A \to \mathcal{M}(B)$ is unital and weakly nuclear:
There is a unital *weakly nuclear C^*-morphism $\widehat{\varphi}\colon A \to \mathcal{M}(B)$ with $\pi_B \circ \widehat{\varphi} = \varphi$, if and only if, φ is in* $\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))$ *and* $[\varphi]_{\approx} = 0$ *in* $\mathrm{Ext}^{\text{u,strong}}_{\text{nuc}}(A,B)$.

By the formulas in the proof of part (i) of Proposition 4.3.5, one can see that *φ strongly absorbs $\pi_B \circ \delta_\infty \circ H$ if and only if, there is an isometry S in $\mathcal{M}(B)$ such that $\pi_B \circ \delta_\infty \circ H = \pi_B(S)^*\varphi(\cdot)\pi_B(S)$.*

This links unital lifting problems to Weyl–von-Neumann type results and to KK-theory.

DEFINITION 5.8.2. Suppose that A is separable and B is σ-unital and stable. Let

$$\mathrm{SExt}_{\mathrm{nuc}}(A,B) := [\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))] \subset [\mathrm{Hom}(A, \mathrm{Q}(B))]$$

denote the unitary equivalence classes $[\varphi]$ (by all unitaries in $\mathrm{Q}(B)$!) of the *weakly nuclear* liftable C^*-morphisms $\varphi\colon A \to \mathrm{Q}(B)$ equipped with Cuntz addition $[\varphi] + [\psi] := [\varphi \oplus \psi]$.

We define the nuclear Ext-group $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ as the Grothendieck group of $\mathrm{SExt}_{\mathrm{nuc}}(A,B)$:

$$\mathrm{Ext}_{\mathrm{nuc}}(A,B) := \mathrm{Gr}(\mathrm{SExt}_{\mathrm{nuc}}(A,B)) .$$

If C is σ-unital (but not necessarily stable) then we let

$$\mathrm{Ext}_{\mathrm{nuc}}(A,C) := \mathrm{Ext}_{\mathrm{nuc}}(A, C \otimes \mathbb{K})$$

It is naturally isomorphic to the formerly defined group $\mathrm{Ext}_{\mathrm{nuc}}(A,C)$ if C is stable (cf. Lemma 5.8.5 below).

If A is unital, we let $\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)$ denote the sub-semigroup of $\mathrm{SExt}_{\mathrm{nuc}}(A,B)$ of classes $[\varphi] \in \mathrm{SExt}_{\mathrm{nuc}}(A,B)$ with $\varphi(1_A) = 1_{\mathrm{Q}(B)}$, i.e., , thus

$$\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) := [\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))] .$$

Then define the "unital" extension group $\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)$ of unital extensions with weakly nuclear c.p. unital lifts $V\colon A \to \mathcal{M}(B)$ by:

$$\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) := \mathrm{Gr}(\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)) .$$

REMARK 5.8.3. The natural semigroup morphisms

$$[\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))]_{\approx} \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)$$

induces an isomorphism

$$[h_1]_{\approx} + [\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))]_{\approx} \cong \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)$$

for $h_1 := \pi_B \circ \rho$ for any unital faithful *-representation $\rho\colon A \to \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\mathcal{H}) \subset \mathcal{M}(B)$ with $\rho(A) \cap \mathbb{K} = \{0\}$.

If A is unital, then there are natural exact sequences:

$$0 \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) \to \mathrm{Ext}_{\mathrm{nuc}}(A,B) \to \mathrm{K}_0(\mathrm{Q}(B)) \cong \mathrm{K}_1(B)$$

and

$$\mathrm{K}_1(h_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \to \mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A,B) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) \to 0 .$$

The problem is now if every element of $\mathrm{CP}_{\mathrm{nuc}}(A,B)$ can be approximated via compressions of $H_0\colon A \to \mathcal{M}(B)$. It respects no ideals!

REMARK 5.8.4. Suppose that A is separable and that B is σ-unital and stable.

Let $H_0 \colon A \to \mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B)$ a *non-degenerate* *-monomorphism with $H_0(A) \cap \mathbb{K} = \{0\}$.

Then H_0 satisfies the assumptions of Corollary 5.4.10 with $D := \mathbb{K}$, thus, H_0 satisfies the assumptions of Corollary 5.4.9 for the cone $\mathcal{S} = \mathrm{CP}_{\mathrm{nuc}}(A, B)$. Thus, $h_0 := \pi_B \circ H_0 \colon A \to \mathrm{Q}(B)$ satisfies $[h_0 \oplus h_0] = [h_0]$, $S(h_0; A, \mathrm{Q}(A)) \subset S(h_0 \oplus 0; A, \mathrm{Q}(B)) \subset [\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))]$, and $[h] \in S(h_0 \oplus 0;\, A, \mathrm{Q}(B))$ for every $h \in \mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))$, by Corollary 5.4.9. Thus:

$$\mathrm{SExt}_{\mathrm{nuc}}(A, B) = [\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))] = S(h_0 \oplus 0; A, \mathrm{Q}(B))\,.$$

If A is *unital*, then H_0 is unital the above considerations show that (**??**)

$$\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A, B) = S(h_0; A, \mathrm{Q}(B))$$

and

$$\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, B) = G(h_0; A, \mathrm{Q}(B)),$$

and finally,

$$\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, B) = G_{\approx}(h_0; A, \mathrm{Q}(B))\,.$$

If A is *stable* then

$$\mathrm{Gr}([\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))], +) = [h_0] + S(h_0; A, \mathrm{Q}(B)) = G(h_0; A, \mathrm{Q}(B))$$

because then $[h_0 \oplus 0] = [h_0]$ in $\mathrm{Q}(B)$ by Corollary 5.5.14.

But in general $[\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))] \ncong S(h_0; A, \mathrm{Q}(B))$, cf. Remark 5.5.8.

Since $\mathrm{Ext}_{\mathrm{nuc}}(A, B) := \mathrm{Gr}([\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))], +)$, it follows (for separable stable σ-unital A):

$$\mathrm{Ext}_{\mathrm{nuc}}(A, B) \cong [h_0] + S(h_0\,;\, A, \mathrm{Q}(B)) = G(h_0 \oplus 0\,;\, A, \mathrm{Q}(B))$$

and

$$\mathrm{Ext}_{\mathrm{nuc}}(A, B) \cong \mathrm{kernel}(\mathrm{K}_0(h_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(B))$$

by Propositions 4.4.2 and 4.4.3.

In particular, the Corollary 5.7.1 implies the following:
Suppose that B is simple, purely infinite, σ-unital and stable, and that A is separable, stable and exact. Then

$$[\mathrm{Mon}_{\mathrm{nuc}}(A, \mathrm{Q}(B))] = \mathrm{Ext}_{\mathrm{nuc}}(A, B)\,.$$

And this formula is equivalent to the following observations (i)-(iv):

(i) Every nuclear *-monomorphism $h \colon A \hookrightarrow \mathrm{Q}(B)$ dominates $h_0 := \pi_B \circ \rho$ for a non-degenerate *-monomorphism $\rho \colon A \hookrightarrow \mathcal{L}(\mathcal{H}) \subset \mathcal{M}(B)$ with $\rho(A) \cap B = \{0\}$ (i.e., $[\mathrm{Mon}_{\mathrm{nuc}}(A, \mathrm{Q}(B))] = [h_0] + [\mathrm{Hom}_{\mathrm{nuc}}(A, \mathrm{Q}(B))]$).
(Indeed: Theorem 5.6.2 with $D = B$, $C := \pi_B^{-1}(h(A))$ and $T := \rho \circ h^{-1}$ yields the existence of an isometry $S_1 \in \mathcal{M}(B)$ with $S_1^* c S_1 - T(c) \in B$ for $c \in C$. Thus $t_1^* h(\cdot) t_1 = h_0$ for $t_1 := S_1 + B \in \mathrm{Q}(B)$.)

(ii) If a C^*-morphism $h\colon A \to \mathrm{Q}(B)$ has a weakly nuclear lift $V\colon A \to \mathcal{M}(B)$ then h_0 dominates h, and h is nuclear (i.e., $[\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))] \subset S([h_0], A, \mathrm{Q}(B)) \subset \mathrm{Mon}_{\mathrm{nuc}}(A, \mathrm{Q}(B)))$.
(Indeed: it can be managed that the weakly nuclear lift V of h is a contraction, cf. Lemma 5.4.8, then Theorem 5.6.2, with $D = B$, $C := \pi_B^{-1}(h_0(A)) = B + \rho(A)$ and $T := V \circ h_0^{-1}$, yields the existence of an isometry $S_2 \in \mathcal{M}(B)$ with $S_2^*\rho(a)S_2 - V(a) \in B$ for $a \in A$, and $t_2 := S_2 + B \in \mathrm{Q}(B)$ satisfies $t_2^* h_0(\cdot)t_2 = h$.)

(iii) $\mathrm{Hom}_{\mathrm{nuc}}(A, \mathrm{Q}(B)) \subseteq \mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))$, because every nuclear c.p. contraction $V\colon A \to \mathrm{Q}(B)$ has a nuclear lift $T\colon A \to \mathcal{M}(B)$ (by a Theorem of Effros and Choi, cf. [**43**]).

(iv) If $h_1, h_2 \in \mathrm{Hom}(A, \mathrm{Q}(B))$ are unitarily equivalent, then they are unitarily equivalent by a unitary $u = \pi_B(U)$ with $U \in \mathcal{M}(B)$.
(Indeed: h_1 dominates zero by Corollary 5.5.16 and Proposition 5.5.12(ii). The unital C^*-algebra $\mathrm{Q}(B)$ is K_1-injective by Lemmas 5.5.10(iv) and 4.2.6(viii).

`The` K_1`-bijectivity of` $\mathrm{Q}^s(B)$ `for` σ`-unital // has been shown until here 3-times!! // Clear the places !!!`

Thus, Proposition 4.3.6(iv,d). implies that h_1 and h_2 are unitarily equivalent by a unitary $u \in \mathcal{U}_0(\mathrm{Q}(B))$.)

Any two nuclear *-monomorphisms $h, k\colon A \hookrightarrow \mathrm{Q}(B)$ dominate each other (by (i) and (ii)). Therefore, the unitary equivalence classes $[h] \in [\mathrm{Mon}_{\mathrm{nuc}}(A, \mathrm{Q}(B))] = [h_0] + S(h_0\,;\, A, \mathrm{Q}(B))$ of nuclear *-monomorphisms $h\colon A \hookrightarrow \mathrm{Q}(B)$ is the Grothendieck group of $[\mathrm{Hom}_{\mathrm{nuc}}(A, \mathrm{Q}(B))]$ by Proposition 4.4.2(ii).

Recall for applications of the following lemma that $\mathrm{Ext}_{\mathrm{nuc}}(A, B \otimes \mathbb{K})$ onto $\mathrm{Ext}_{\mathrm{nuc}}(A, B) := \mathrm{Ext}(\mathcal{C};\, A, B)$ for the m.o.c. cone $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ and $\mathrm{Ext}_{\mathrm{nuc}}(A, B \otimes \mathbb{K}) := \mathrm{Ext}(\mathcal{C}';\, A, B \otimes \mathbb{K})$ for $\mathcal{C}' := \mathrm{CP}_{\mathrm{nuc}}(A, B)_{\mathbb{K}} := \mathrm{CP}_{\mathrm{nuc}}(A, B \otimes \mathbb{K}) = \mathrm{CP}_{\mathrm{nuc}}(A, B) \otimes \mathrm{CP}(\mathbb{C}, \mathbb{K})$.

Here $\mathcal{C}_{\mathbb{K}} := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathbb{K})$ denotes the (non-algebraic) tensor product of matrix operator-convex cones in sense of Definition **??**, i.e., is the smallest point-norm closed m.o.c. cone in $\mathrm{CP}(A, B \otimes \mathbb{K})$ that contains all tensor products $T \otimes S$ with $T \in \mathcal{C}$ and $S \in \mathrm{CP}(\mathbb{C}, \mathbb{K}) \cong \mathbb{K}_+$.

Lemma 5.8.5. *Suppose that B is stable and σ-unital, that A is separable and $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a countably generated non-degenerate full m.o.c. cone. Let $\lambda\colon B \otimes \mathbb{K} \to B$ be an isomorphism from $B \otimes \mathbb{K}$ onto B such that $b \to \lambda(b \otimes p_{11})$ is unitarily homotopic to id_B (cf. Corollary 5.5.6 and Lemma 5.5.11 for the existence of λ). Denote by $\mathrm{Q}(\lambda)$ the induced isomorphism from $\mathrm{Q}(B \otimes \mathbb{K})$ onto $\mathrm{Q}(B)$.*

Then

$$\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B \otimes \mathbb{K})) \mapsto \mathrm{Q}(\lambda) \circ \varphi \in \mathrm{Hom}(A, \mathrm{Q}(B))$$

defines a isomorphism from $\mathrm{Ext}(\mathcal{C}_{\mathbb{K}};\, A, B \otimes \mathbb{K})$ onto the 0-dominating $\mathrm{Ext}(\mathcal{C};\, A, B)$ or unital version $\mathrm{Ext}(\mathcal{C};\, A, B)$ that is a functor with respect to A.

Check here:
It needs that we consider only zero-dominating
Busby invariants.
Discussion e.g. with $B = M_{2^\infty}$, $B = \mathcal{O}_\infty$, $A = \mathcal{O}_2$.

The inverse is given by

$$\mathrm{Ext}(\mathcal{C};\, A, B) \to \mathrm{Ext}(\,_{\mathbb{K}}\mathcal{C}_{\mathbb{K}}\,;\; A \otimes \mathbb{K}, B \otimes \mathbb{K}) \to \mathrm{Ext}(\mathcal{C}_{\mathbb{K}};\, A, B \otimes \mathbb{K})\,,$$

i.e., is given by forming first the tensor product $E_\varphi \otimes \mathbb{K}$ of the extension E_φ with $\mathbb{K}$ and then by application of the map $a \in A \mapsto a \otimes p_{11} \in A \otimes \mathbb{K}$ (cf. Lemma 5.5.11, Proposition 5.5.12(v)).

In particular, there is a natural isomorphism

$$\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B) \to \mathrm{Ext}_{\mathrm{nuc}}(A, B)$$

from $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B)$ onto $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ that is induced by $\varphi \mapsto \varphi((\cdot) \otimes p_{11})$.

Proof. to be filled in ??
Do we need additional properties on the extensions,
e.g. zero absorption or more relaxed equivalence classes? □

Remark 5.8.6. The natural semigroup homomorphism

$$[\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))]_{\approx} \to \mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A, B)$$

and the induced group homomorphism

$$\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, B) = \mathrm{Gr}([\mathrm{Hom}_{\text{ul-nuc}}(A, B)]_{\approx}) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, B)$$

are in general *not* isomorphisms.

The definitions show that there are natural group and semigroup homomorphisms, e.g.

$$\mathrm{SExt}_{\mathrm{nuc}}(A, B) \to [\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))] \to \mathrm{Ext}^{-1}(A, B)\,,$$

$$\mathrm{Ext}_{\mathrm{nuc}}(A, B) \to \mathrm{Ext}^{-1}(A, B)\,.$$

If A is unital, then there are group homomorphisms

$$\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, B) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, B) \to \mathrm{Ext}_{\mathrm{nuc}}(A, B) \to \mathrm{K}_0(\mathrm{Q}(B)) \cong \mathrm{K}_1(B)\,,$$

induced by $\varphi \in \mathrm{Hom}(A, \mathrm{Q}(B)) \to [\varphi(1)] \in \mathrm{K}_0(\mathrm{Q}(B))$.

Further, there is the group morphism $\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, B) \to \mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$, given by forming tensor products $[E_\varphi \otimes \mathbb{K}]$ of the corresponding extensions $[E_\varphi] \in \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, B)$.

more ?? ?? ?? ??

Remark 5.8.7. Let $\rho\colon A \to \mathcal{L}(\mathcal{H}) \cong \mathcal{M}(\mathbb{K})$ a faithful non-degenerate *-representation of A on a Hilbert space $\mathcal{H} \cong \ell_2(\mathbb{N})$ such that $\rho(A) \cap \mathbb{K} = \{0\}$. Fix a unital strictly continuous *-monomorphism $I\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$, and define $H\colon A \to \mathcal{M}(B)$ by $H := I \circ \rho$. We let $h_0 := \pi_B \circ H\colon A \to \mathrm{Q}(B)$, i.e., consider $H(a)$ modulo B.

Then $\mathrm{SExt}_{\mathrm{nuc}}(A,B) = S(h_0;\, A, \mathrm{Q}(B))$ if A is stable, and $\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) = S(h_0;\, A, \mathrm{Q}(B))$ if A is unital, cf. Corollary 5.6.4(iii).

This implies the natural isomorphisms $\mathrm{Ext}_{\mathrm{nuc}}(A,B) \cong G(h_0, A, \mathrm{Q}(B))$, respectively, in case of unital A, $\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) \cong G(h_0, A, \mathrm{Q}(B))$ and $\mathrm{Ext}_{\mathrm{nuc}}(A,B) \cong G(\pi_B \circ (h_0 \oplus 0), A, \mathrm{Q}(B))$.

There are natural homomorphisms

$$\mathrm{K}_1(\pi_B(h_0(A))' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B)$$

and

$$\mathrm{K}_0(B) \cong \mathrm{K}_1(\mathrm{Q}(B)) \to \mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A,B)\,.$$

induced by $u \in \mathcal{U}(\mathrm{Q}(B)) \to [u^* h_0(\cdot) u]_{\approx} \in [\mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(B))]_{\approx}$.

The following Corollary 5.8.8 is a consequence of two of Kasparov's theorems: Corollary 5.6.1 and [**73**, prop. 17.6.5].

Corollary 5.8.8. *Suppose that B is σ-unital and stable and that A is separable.*

If A or B is nuclear then

$$G(h_0, A \otimes \mathbb{K}, \mathrm{Q}(B)) \;\cong\; \mathrm{Ext}_{\mathrm{nuc}}(A,B) = \mathrm{Ext}^{-1}(A,B)\,.$$

Note that $\mathrm{Ext}^{-1}(A,B) \cong \mathrm{KK}(A, \mathrm{C}_0(\mathbb{R}, B)) \cong \mathrm{KK}(\mathrm{C}_0(\mathbb{R}, A), B)$ by theorems of Kasparov (cf. [**73**, prop. 17.6.5, thm. 17.10.7, cor. 19.2.2.]).

Proof. By Lemma 5.8.5 and Remark 5.8.7, there is a natural isomorphism from $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ onto

$$\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, B) = [h_0] + [\mathrm{Hom}_{\text{l-nuc}}(A \otimes \mathbb{K}, B)] = G(h_0, A \otimes \mathbb{K}, \mathrm{Q}(B))\,.$$

We have $\mathrm{Hom}_{\text{l-cp}}(A \otimes \mathbb{K}, \mathrm{Q}(B)) = \mathrm{Hom}_{\text{l-nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(B))$ and $\sim_{nuc} = \sim$ for nuclear B. If A is nuclear, then (trivially) $\mathrm{Hom}(A \otimes \mathbb{K}, \mathrm{Q}(B)) = \mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(B))$ and $\mathrm{Hom}_{\mathrm{nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(B)) \subset \mathrm{Hom}_{\text{l-nuc}}(A \otimes \mathbb{K}, \mathrm{Q}(B))$ (by the Choi-Effros lifting theorem, cf. [**43**]). In both cases $\mathrm{Ext}^{-1}(A,B) = \mathrm{Gr}([\mathrm{Hom}_{\text{l-cp}}(A, \mathrm{Q}(B))]) = \mathrm{Gr}([\mathrm{Hom}_{\text{l-nuc}}(A, \mathrm{Q}(B))]) = \mathrm{Ext}_{\mathrm{nuc}}(A,B)$. □

Corollary 5.8.9. *Suppose that A is separable and unital, B purely infinite, stable and σ-unital, $h_0 \colon A \to \mathcal{L}(H) \cong \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B)$ a faithful and unital *-representation with $h_0(A) \cap \mathbb{K} = 0$. Then*

(i) $\mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) = G(h_0, A, \mathrm{Q}(B))$*;* **??**
(ii) $0 \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) \to \mathrm{Ext}_{\mathrm{nuc}}(A,B) \to K_1(B)$ *is an exact sequence.*
(iii) *There is a natural exact sequence*

$$\mathrm{K}_1(\pi_B(h_0(A))' \cap \mathrm{Q}^{\mathrm{s}}(B)) \to \mathrm{K}_0(B) \to \mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A,B) \to \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B) \to 0.$$

In particular, $\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A,B) = \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A,B)$ *if* $\mathrm{K}_0(B) = 0$.

Proof. `to be filled in` **??** □

COROLLARY 5.8.10. *If A is a separable unital C^*-algebra, then*

$$\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, \mathcal{O}_2 \otimes \mathbb{K}) = \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, \mathcal{O}_2 \otimes \mathbb{K}) = \mathrm{Ext}^{-1}(A, \mathcal{O}_2) \cong 0\,.$$

*In particular, if $h\colon A \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ is a unital *-monomorphism which has a completely positive lift $V\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$, then there is a unital C^*-morphism $\varphi\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ with $\pi_{\mathcal{O}_2\otimes\mathbb{K}}\varphi = h$.*

PROOF.

$$\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, \mathcal{O}_2) = \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, \mathcal{O}_2) \subset \mathrm{Ext}_{\mathrm{nuc}}(A, \mathcal{O}_2) = \mathrm{Ext}^{-1}(A, \mathcal{O}_2)$$

by Corollary 5.8.9(iii), because $\mathrm{K}_0(\mathcal{O}_2) = 0$ and $\mathcal{O}_2$ is simple, purely infinite and nuclear, cf. [**172**], [**169**].

$$\mathrm{Ext}^{-1}(A, \mathcal{O}_2) \cong \mathrm{KK}^1(A, (\mathcal{O}_2)_{(1)}) \cong \mathrm{KK}(SA, \mathcal{O}_2)$$

by Kasparov's isomorphisms and Bott periodicity, cf. [**73**, prop. 17.6.5, cor. 19.2.2] (see also Chapter 8).

By [**172**], $\mathrm{id}_{\mathcal{O}_2}$ is homotopic to $\mathrm{id}_{\mathcal{O}_2} \oplus_{s,t} \mathrm{id}_{\mathcal{O}_2}$ in the unital endomorphisms of $\mathcal{O}_2$. Thus $\mathrm{KK}(SA, \mathcal{O}_2) = 0$, because $\mathrm{KK}(SA, \,\cdot\,)$ is a homotopy invariant functor such that the Cuntz addition $\oplus$ induces the addition in $\mathrm{KK}(SA, \mathcal{O}_2)$, cf. Chapter 8.

Now let $h\colon A \hookrightarrow \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ a unital *-monomorphism that has a c.p. lift $V\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$. Then $\mathrm{Ext}^{\mathrm{u,strong}}_{\mathrm{nuc}}(A, \mathcal{O}_2) = 0$ implies the existence of unital *-monomorphisms $h_1, h_2\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ and of a unitary $U_1 \in \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ such that $(V(a) \oplus h_1(a)) - U_1^* h_2(a) U_1 \in \mathcal{O}_2 \otimes \mathbb{K}$ for $a \in A$, where $V\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ is a unital completely positive lift of $h\colon A \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$. We get a unitary $U_2 \in \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ with $U_2^*(V(a) \oplus h_1(a))U_2 - V(a) \in \mathcal{O}_2 \otimes \mathbb{K}$ for every $a \in A$ from Theorem 5.6.2, which has to be applied to $C := V(A) + \mathcal{O}_2 \otimes \mathbb{K}$ and the completely positive unital map

$$T \;:=\; h_1 \circ h^{-1} \pi_{\mathcal{O}_2\otimes\mathbb{K}}\colon C \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})\,.$$

Thus, $\varphi\colon A \ni a \mapsto U_3^* h_2(a) U_3$ with unitary $U_3 := U_1 U_2$ is a unital C^*-morphism that is a lift of h. □

COROLLARY 5.8.11. *Suppose that A is separable and unital.*

(i) $\mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A, \mathcal{O}_2 \otimes \mathbb{K}) \cap [\mathrm{Mon}(A, \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2))] = \mathrm{Ext}^{\mathrm{u}}_{\mathrm{nuc}}(A, \mathcal{O}_2 \otimes \mathbb{K}) = 0$.
(ii) *If a C^*-morphism $h\colon A \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ has a unital completely positive lift $T\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ then h has a lift as a unital C^*-morphism $\varphi\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$, i.e., $\pi_{\mathcal{O}_2\otimes\mathbb{K}}\varphi = h$.*

PROOF. (i) follows from Corollary 5.6.4(iv), because $\mathcal{O}_2 \otimes \mathbb{K}$ is purely infinite and simple.

???? **??**

(ii): Let $T\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ a unital completely positive map with $\pi T = h$. Since $\mathcal{O}_2$ is nuclear, T is weakly nuclear, and thus $h \in \mathrm{Hom}_{\text{ul-nuc}}(A, \mathrm{Q}(\mathcal{O}_2 \otimes \mathbb{K}))$, i.e., $[h] \in \mathrm{SExt}^{\mathrm{u}}_{\mathrm{nuc}}(A, \mathcal{O}_2 \otimes \mathbb{K})$ by Definition 5.8.2.

Let $d\colon A \to \mathcal{L}(H) \subset \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ a faithful unital *-representation of A such that $d(A) \cap \mathbb{K} = 0$. Let $k := \pi d$. By Kasparov's generalized Weyl–von Neumann theorem ([**?**], see Corollary 5.6.4), we have $\mathrm{Ext}_u(A, \mathcal{O}_2) := \mathrm{Gr}([Ext_u^{-1}(A, \mathcal{O}_2)]) = G(k, A, \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2))$, where we have used that $\mathcal{O}_2$ is nuclear (cf. [**169**]).

By Corollary 5.8.10, $\mathrm{Ext}_u(A, \mathcal{O}_2) = 0$.

Thus there exists a unitary v in $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ such that $h \oplus k = v^* k(\cdot) v$.

From now on we use that $\mathcal{O}_2$ is purely infinite and simple, [**169**], [**172**].

h absorbs k by Corollary 5.6.4(iv), i.e., there exists a unitary $w \in \mathcal{U}_0(\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2))$ such that $w^* h(\cdot) w = h \oplus k$.

It is easy to see, e.g. with help of Lemma 5.1.2(ii), that multiplier algebras of stable algebras have trivial K-theory. Therefore we get $\mathrm{K}_1(\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)) \cong \mathrm{K}_0(\mathcal{O}_2) = 0$ from the 6-term exact sequence of K-theory, [**73**].

By Corollary 5.7.2, $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ is simple and purely infinite. By a theorem of Cuntz [**172**], the quotient of the unitary group of $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ by its connected component is naturally isomorphic to $\mathrm{K}_1(\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)) = 0$. (More generally stable coronas of σ-unital C^*-algebras are K_1-bijectivity by Proposition 4.2.15.)

Thus vw^* is in the connected component of 1 in the unitary group of $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ and therefore there exists a unitary $U \in \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ with $\pi(U) = vw^*$.

$U^* d(\cdot) U$ is a unital lift of h, i.e., $\pi(U^* d(a) U) = k(a)$. □

Corollary 5.8.12. *Suppose that E is a unital separable C^*-algebra, $\lambda\colon E \to A$ is an epimorphism such that the kernel D of λ is essential in E, $D \cong \mathcal{O}_2 \otimes \mathbb{K}$, and that there exists a unital completely positive map $V\colon A \to E$ such that $\lambda V = \mathrm{id}_A$. Then there exists a unital *-monomorphism $\psi\colon A \to E$ with $\lambda\psi = \mathrm{id}_A$.*

Proof. Expressed in algebraic terminology, we show that a semi-split essential exact sequence $0 \to D \to E \to A \to 0$ splits unitally if $D \cong \mathcal{O}_2 \otimes \mathbb{K}$.

Let $\pi\colon \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}) \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ be the natural epimorphism with kernel $\mathcal{O}_2 \otimes \mathbb{K}$ and let θ be the *-monomorphism from E into $\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ which is defined by the isomorphism from D onto $\mathcal{O}_2 \otimes \mathbb{K}$. It is a monomorphism, because D is an essential ideal of E (by our assumptions). In particular, $\theta(D) = \mathcal{O}_2 \otimes \mathbb{K}$.

Then there is a unique unital *-monomorphism $\tau\colon A \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ with $\pi\theta(E) = \tau(A)$ and $\pi\theta = \tau\lambda$.

$W := \theta V$ is a unital completely positive lift of τ, i.e., $\pi W = \tau$.

By Corollary 5.8.10 there exists a unital lift h of τ, i.e., $\pi(h(a)) = \tau(a)$. But then $h(A) \subset \pi^{-1}(\tau(A)) = \theta(E)$, and $\psi(\cdot) := \theta^{-1}(h(\cdot))$ is a unital *-monomorphism from A into E such that $\lambda\psi = \mathrm{id}_A$, because $\tau\lambda\psi = \pi\theta\psi = \pi h = \tau$ and τ is faithful. Thus ψ is as desired. □

9. The Ψ-residually nuclear case and $\mathrm{Ext}(\mathcal{C};\, A, B)$

Start of collection for Chapter 5:

Needed for Theorem 6.3.1.

But is the here used notation ok?

DEFINITION 5.9.1. Let X be a Dini-space, i.e., a sober second countable locally quasi-compact T_0 space.

Suppose that A and B are stable and σ-unital C^*-algebras, and that $\Psi_A \colon \mathcal{O}_X \to \mathcal{I}(A)$ is a *lower* semi-continuous action of X on A and $\Psi_B \colon \mathcal{O}_X \to \mathcal{I}(B)$ an *upper* semi-continuous action of X on B (cf. Definition 1.2.6). A completely positive map $V \colon A \to \mathcal{M}(B)$ is called **weakly Ψ_A-Ψ_B-residually nuclear**, (or shortly Ψ-residual nuclear) if, for every $b \in B$ the completely positive map $a \in A \mapsto b^* V(a) b \in B$ is Ψ_A-Ψ_B-residually nuclear in the sense of Definition 1.2.8. We denote by $\mathrm{Hom}_{\text{l-nuc}}(X;\, A, \mathrm{Q}(B)) \subset \mathrm{Hom}(A, \mathrm{Q}(B))$ the set of C^*-morphisms h from A into the (stable) corona $\mathrm{Q}(B) = \mathcal{M}(B)/B$ of B which have a weakly Ψ_A-Ψ_B-residually nuclear completely positive lift $V \colon A \to \mathcal{M}(B)$. Then $\mathrm{Hom}_{\text{l-nuc}}(X;\, A, \mathrm{Q}^{\mathrm{s}}(B))$ is invariant under passage to unitary equivalent C^*-morphisms and under Cuntz addition. Let $\mathrm{SExt}_{\mathrm{nuc}}(X;\, A, B)$ denote the semigroup of unitary equivalence classes of morphisms in $\mathrm{Hom}_{\text{l-nuc}}(X;\, A, \mathrm{Q}^{\mathrm{s}}(B))$ with Cuntz addition.

We define:

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B) := \mathrm{Gr}(\mathrm{SExt}_{\mathrm{nuc}}(X;\, A, B)).$$

We generalize the above definitions in the following manner, because all constructions of ideal-system equivariant versions of Kasparov's Ext-theory factorize anyway though the variants of Ext-theory for the corresponding m.o.c. cones:
Let A separable, B σ-unital and stable, and let $\mathcal{C} \subset \mathrm{CP}(A, B)$ a matricial operator-convex cone. We denote by $\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B)) \subset \mathrm{Hom}(A, \mathrm{Q}(B))$ the set of the C^*-morphisms $h \colon A \to \mathrm{Q}(B) \cong \mathrm{Q}^{\mathrm{s}}(B)$ that have a c.p. lift $V \colon A \to \mathcal{M}(B)$ with the property that $b^* V(\cdot) b \in \mathcal{C}$ for all $b \in B$. Again, $\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$ is invariant under unitary equivalence and Cuntz addition, and we can define the semigroups and its Grothendieck groups by

$$\mathrm{SExt}(\mathcal{C};\, A, B) := [\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))] \subset [\mathrm{Hom}(A, \mathrm{Q}(B))]$$

Here $[\cdot]$ means the unitary equivalence classes of morphisms in $\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$ with Cuntz addition.)

$$\mathrm{Ext}(\mathcal{C};\, A, B) := \mathrm{Gr}(\mathrm{SExt}(\mathcal{C};\, A, B))\,.$$

Note that $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B) = \mathrm{Ext}_{\mathrm{nuc}}(\mathcal{C};\, A, B)$ for the m.o.c. cone $\mathcal{C} \subset \mathrm{CP}(A, B)$ of Ψ_A–Ψ_B-residually nuclear c.p. maps from A into B. If A is exact and B is separable, then there is a unique action Ψ_A of $X := \mathrm{Prim}(B)$ such that a given point-norm closed m.o.c. cone $\mathcal{C}$ is the same as $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ with Ψ_B defined as the identity action of X on X.

REMARK 5.9.2. Special cases of the groups $\mathrm{Ext}(\mathcal{C};\, A,B)$ are $\mathrm{Ext}(A,B) = \mathrm{Ext}(\mathrm{CP}(A,B);\, A,B)$, $\mathrm{Ext}_{\mathrm{nuc}}(A,B) = \mathrm{Ext}(\mathrm{CP}_{\mathrm{nuc}}(A,B);\, A,B)$, $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A,B) = \mathrm{Ext}(\mathrm{CP}_{\mathrm{rn}}(X;\, A,B);\, A,B)$, where $\mathrm{CP}_{\mathrm{rn}}(X;\, A,B) \subset \mathrm{CP}(A,B)$ denotes the Ψ_A-Ψ_B-residually nuclear c.p. maps, depending on fixed actions $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B \colon \mathbb{O}(X) \to \mathcal{I}(B)$ of X on A and B.

We can also define

$$\mathrm{Ext}(X;\, A,B) := \mathrm{Ext}(\mathrm{CP}(X;\, A,B);\, A,B)\,,$$

the group of Ψ_A-Ψ_B-equivariant extensions, where $\mathrm{CP}(X;\, A,B) \subset \mathrm{CP}(A,B)$ is the Ψ-equivariant c.p. maps.

Next Def's could be in Chp.3

The cone $\mathcal{C} \subset \mathrm{CP}(A,B)$ is **faithful** (respectively **non-degenerate**) if there is no non-zero ideal J of A with $T(J) = \{0\}$ for all $T \in \mathcal{C}$ (respectively there is no ideal $I \neq B$ of B with $T(A) \subset I$ for all $T \in \mathcal{C}$).

There is also an explanation of $\mathrm{Ext}(\mathcal{C};\, A,B)$ with help of extensions with Busby invariant that have lifts that are limits of maps in $\mathcal{C}$ with respect to the strict topology on $\mathcal{M}(B)$. It can be done in a similar manner as in the explanation given in Section 8 for the case X =point (Note that $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A,B) = \mathrm{Ext}_{\mathrm{nuc}}(A,B)$ if $X = \{p\}$ is a point).

Corollaries 5.4.4 and 5.4.9 tell us:
If $\mathcal{C}$ is countably generated, then there is a "universal" element $H_0 \in \mathrm{Hom}(A, M(B))$ that defines the point-norm closure of $\mathcal{C}$, with $\delta_\infty \circ H_0$ unitarily equivalent to H_0, and $H_0(A)BH_0(A)$ a split corner of B. It can be chosen non-degenerate if $\mathcal{C}$ is *non-degenerate* , i.e., if $\mathcal{C}$ is not contained in $CB(A,J)$ for some closed ideal J of B. Let $h_0 := \pi_B \circ H_0$. Then $\mathrm{Ext}(\mathcal{C};\, A,B) = [h_0] + \mathrm{SExt}(\mathcal{C};\, A,B) = G(h_0;\, A,B)$. The split extension $B + H_0(A) = \pi_B$ is stable if $\mathcal{C}$ is non-degenerate. By Proposition 5.5.12(i+ii), this implies that every extension with Busby invariant in $\mathrm{Ext}(\mathcal{C};\, A,B)$ is a stable C^*-algebra. (Here we can see that the questions about the non-degeneracy of $\mathrm{CP}_{\mathrm{rn}}(X;\, A,B)$ is of importance for actions of X on A and B.)

Recall that a **corner** of a C^*-algebra E is a hereditary C^*-subalgebra F of E such that there is a projection $p \in \mathcal{M}(E)$ with $F = pEp$, and that a subalgebra $G \subset E$ **generates a corner** of E, if $C^*(G)EC^*(G)$ is a corner of E ([20]). The following Ψ-equivariant generalization of Theorem 5.6.2 seems to be the maximal possible, only special cases are used in our applications.

THEOREM 5.9.3. *Suppose that B is a σ-unital C^*-algebra, $D \subset \mathcal{M}(B)$ is strongly purely infinite, separable and stable C^*-subalgebra such that DB is dense in B (and thus $\mathcal{M}(D) \subset \mathcal{M}(B)$ naturally), and that $C \subset \mathcal{M}(D)$ is a separable C^*-subalgebra. Define $\Psi_C := \Psi^{\mathrm{up}}_{D,C}$ and $\Psi_B := \Psi^{D,B}_{\mathrm{down}}$ as in Chapter* 1.

[20] The *set* (!) of products $C^*(G) \cdot E \cdot C^*(G)$ is always a hereditary C^*-subalgebra of E by the Cohen factorization theorem for Banach modules.

Suppose further that

($\alpha 3$) *for every closed ideal J of D, the subalgebra $\mathcal{M}(\pi_J)(C) \cap \pi_J(D)$ of $\pi_J(D) = D/J$ generates a corner of D/J, and*

($\beta 3$) *Let $V\colon C \to \mathcal{M}(B)$ be a weakly Ψ_C-Ψ_B–residually nuclear completely positive map such that, for every $J \in \mathcal{I}(D)$, $\xi \in \mathbb{C}$ and $c \in C \cap (\xi 1 + D + \mathcal{M}(D, J))$ the element $V(c)$ is in the strict closure of $\xi 1 + \mathrm{span}(BJB)$* ([21]).

Then id_C asymptotically dominates V in the sense of Definition 5.0.1.

If, moreover, V is a C^*-morphism then id_C asymptotically absorbs V, i.e., $\mathrm{id}_C \oplus V\colon C \to \mathcal{M}(B)$ and id_C are unitarily homotopic in the sense of Definition 5.0.1.

Proof. Since B and D are σ-unital, there are natural isomorphisms $\mathcal{M}(B)/\mathcal{M}(B, I) \cong \mathcal{M}(B/I)$ and $\mathcal{M}(D)/\mathcal{M}(D, J) \cong \mathcal{M}(D/J)$ for closed ideals I of B and J of D by the non-commutative version of the Tietze extension theorem ([22]).

They allow us to proceed as in the proof of Theorem 5.6.2:

First, ($\beta 3$) implies $V(C \cap D) = \{0\}$ if we take $\xi = 0$ and $J = \{0\}$. Furthermore $C \cap D$ is a corner of D by ($\alpha 3$). The same properties hold for $\mathcal{M}(\pi_J)(C) = C/\Psi_C(J)$ and $\pi_J(D) = D/J$ and the unit element 1 of $\mathcal{M}(D/J) \cong \mathcal{M}(D)/\mathcal{M}(D, J)$ in place of C, D and $1 \in \mathcal{M}(D)$ by ($\beta 3$) and ($\alpha 3$).

Notice that $\mathcal{M}(D, J) \subset \mathcal{M}(D) \cap \mathcal{M}(B, \Psi_B(J))$ and $\Psi_C(J) = C \cap \mathcal{M}(D, J)$. The latter identity is the definition of $\Psi_C = \Psi^{\mathrm{up}}_{D,C}$, and the first inclusion follows from $JB \subset \Psi_B(J)$, i.e., from $J \subset \mathcal{M}(B, \Psi_B(J))$, because $\mathcal{M}(B, \Psi_B(J))$ is strictly closed in $\mathcal{M}(B)$, the inclusion $\mathcal{M}(D) \to \mathcal{M}(B)$ is strictly continuous and $\mathcal{M}(D, J)$ is the strict closure of J in $\mathcal{M}(D)$ (Recall that $\Psi_B(J)$ is the closure of $\mathrm{span}(BJB)$).

It follows that $\mathcal{M}(D) \cap \mathcal{M}(B, \Psi_B(J))$ is a strictly closed ideal of $\mathcal{M}(D)$. Thus $\mathcal{M}(D) \cap \mathcal{M}(B, \Psi_B(J)) = \mathcal{M}(D, J_1)$ for the ideal $J_1 := D \cap \mathcal{M}(B, \Psi_B(J))$ of D. Then $J \subset J_1$ and $\Psi_B(J_1) = \Psi_B(J)$. The arguments show that our J_1 is the biggest ideal of D with $\Psi_B(J_1) = \Psi_B(J)$, and that, for every closed ideal I of B, the ideal $\Psi_D(I) := D \cap \mathcal{M}(B, I)$ of D satisfies $\mathcal{M}(D, \Psi_D(I)) = \mathcal{M}(D) \cap \mathcal{M}(B, I)$, $\Psi_C(\Psi_D(I)) = C \cap \mathcal{M}(B, I)$ and $\Psi_B(\Psi_D(I)) \subset I$. In particular, $V(C \cap \mathcal{M}(B, I)) = V(\Psi_C(\Psi_D(I))) \subset \mathcal{M}(B, I)$ for all ideals I of B.

We let $C_1 := D + C + \mathbb{C}1$ and define $V_e\colon C_1 \to \mathcal{M}(B)$ by $V^e(d + c + \xi 1) := V(c) + \xi 1$ for $d \in D$, $c \in C$, $\xi \in \mathbb{C}$. It is well-defined, because $d' + c' + \xi' 1 = d + c + \xi 1$ implies that $c' - c \in (\xi - \xi')1 + D$, thus, $V(c' - c) = (\xi - \xi')1$ by ($\beta 3$) with $J = \{0\}$, which yields $V(c') + \xi' 1 = V(c) + \xi 1$. By definition $V^e(D) = 0$ and $[V^e]_D\colon C_1/D \to \mathcal{M}(B)$ is the extension of $[V]_{D \cap C}\colon C/(D \cap C) \to \mathcal{M}(B)$ to a

[21] I.e., $V(C \cap (\xi 1 + D + \mathcal{M}(D, J))) \subset \xi 1 + \mathcal{M}(B, \Psi_B(J))$ for $J \in \mathcal{I}(D)$ and $\xi \in \{0, 1\}$.

[22] Cf. [**616**, Prop.3.12.10.] for the case of separable B. It extends to σ-unital C^*-algebras B, because then $\mathcal{M}(B/J)$ is the algebraic inductive limit of its C^*-subalgebras $\mathcal{M}(A/(A \cap J))$ where the C^*-algebras $A \subset B$ are separable and $e \in A$ for some $e \in B_+$ that is strictly positive in B.

unital map of $C_1/D \cong C/(D\cap C) + \mathbb{C}1$ into $\mathcal{M}(B)$. Since V is a c.p. contraction with $V(D\cap C)=0$, it follows that $V^e\colon C_1 \to \mathcal{M}(B)$ is a u.c.p. map with $V^e(D)=0$.

Thus, condition (α) of Proposition 5.4.1 is satisfied for C_1, V^e and a strictly positive contraction $h \in D \subset C_1$.

Let $\Phi(J) := C_1 \cap \mathcal{M}(D,J)$ for ideals J of D. We have that $V^e(\Phi(J)) \subset \mathcal{M}(B, \Psi_B(J))$ by (β3): Indeed, if $d+c+\xi 1 \in \mathcal{M}(D,J)$, then $c \in C\cap((-\xi)1 + D + \mathcal{M}(D,J))$, and, by condition ($\beta$3), $V(c) \in (-\xi)1 + \mathcal{M}(B,\Psi_B(J))$, i.e., $V^e(d+c+\xi 1) = V(c) + \xi 1 \in \mathcal{M}(B,\Psi_B(J))$.

Lemma B.7.7(ii) shows that assumption (β3) implies that V^e is even Φ-Ψ_B–residually nuclear: The (induced class-)map

$$[V^e]_J\colon C_1/\Phi(J) \to \mathcal{M}(B/\Psi_B(J))$$

is weakly nuclear by Lemma B.7.7(ii) for every closed ideal J of D, because $C_1/\Phi(J) = \mathcal{M}(\pi_J)(C) + \pi_J(D) + \mathbb{C}1 \subset \mathcal{M}(D/J)$, $[V^e]_J(d+c+\xi 1) = [V]_J(c) + \xi 1 \in \mathcal{M}(B/\Psi_B(J))$, $\mathcal{M}(\pi_J) \cap \pi_J(D)$ generates a corner F_J of $D/J = \pi_J(D)$ and $[V]_J\colon \mathcal{M}(\pi_J)(C) \to \mathcal{M}(B/\Psi_B(J))$ is weakly nuclear.

It follows that $T := \delta_\infty \circ V^e$ is also a unital Φ-Ψ_B–residual weakly nuclear u.c.p. map from $C_1 \subset \mathcal{M}(D)$ into $\mathcal{M}(B)$. The strictly positive contraction $h \in D_+$ again satisfies condition (α) of Proposition 5.4.1 for $T\colon C_1 \to \mathcal{M}(B)$.

Proposition 3.6.1 applies to T, and shows that the Φ-Ψ_B–residual nuclear maps $b^*T(\cdot)b$ ($b \in B$) are approximately 1-step inner in $\mathcal{M}(B)$. Thus, also condition (β) of 5.4.1 is satisfied, and the conclusions follow from parts (iii) and (iv) of Proposition 5.4.1. □

Remark 5.9.4. The tricky points of the reduction to Proposition 5.4.1 in the proof of Theorem 5.9.3 are contained in the proof of Lemma B.7.7 and in the proof of the local approximation result in Proposition 3.6.1. The latter proof essentially reduces to the case, where there is a weakly residually nuclear *-monomorphism $H_1\colon C \to \mathcal{M}(D)$ such that $H_1(C)\cap \mathcal{M}(D,J) = H_1(C\cap \mathcal{M}(D,J))$ for every closed ideal J of D, as it is the case for the ultrapower D_ω in place of D.

The following Corollary 5.9.5 is an immediate consequence of our above given Weyl–von-Neumann–Voiculescu type Theorem 5.9.3 for weakly residually nuclear maps. We define for $D \subseteq B$ and $X := \operatorname{Prim}(D)$, an action

$$\Psi_E\colon \mathbb{O}(X) \cong \mathcal{I}(D) \to \mathcal{I}(\mathrm{Q}(B))$$

of X on $E := \mathrm{Q}(B)$ by

$$\Psi_E(J) := \pi_B(\mathcal{M}(B, \operatorname{span}(BJB))) \quad \text{for } J \in \mathcal{I}(D)\,.$$

If $\Psi_A\colon \mathbb{O}(X) \to \mathcal{I}(A)$ is an action of X on A, we denote by $\operatorname{Mon}(X;\, A, E)$ the set of *-monomorphisms $\varphi\colon A \to E$ with $\varphi(\Psi_A(U)) = \varphi(A) \cap \Psi_E(U)$ for $U \in \mathbb{O}(X)$. (It might happen that $\operatorname{Mon}(X;\, A, E)$ is empty.)

Corollary 5.9.5. *Suppose that A, and $D \subseteq B$ are stable C^*-algebras, such that A and D are separable, D is strongly purely infinite and DB is dense in B.*

*Let $X := \mathrm{Prim}(D)$. Further let $H_1\colon A \to \mathcal{M}(D)$ be a weakly residually nuclear *-monomorphism such that H_1 is unitarily equivalent to $\delta_\infty \circ H_1$ and that $H_1(A)D$ is dense in D. Denote by $H_0\colon A \to \mathrm{Q}(B) = \mathcal{M}(B)/B \cong \mathrm{Q}^{\mathrm{s}}(B)$ the corresponding monomorphism from A into the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ of B, i.e., $H_0 := \pi_B H_1$.*

Consider the actions $\Psi_B := \Psi^{D,B}_{\mathrm{down}}$ of X on B and the action $\Psi_A := H_1^{-1}\Psi^{\mathrm{up}}_{D,H_1(A)}$ of X on A.

Then there are natural isomorphisms

$$\mathrm{SExt}_{\mathrm{nuc}}(X;\, A, B) \cong S(H_0,\, A, \mathrm{Q}(B))$$

and

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B) \cong G(H_0,\, A, \mathrm{Q}(B))\,.$$

If, in addition, $B = D$ then

$$\mathrm{SExt}_{\mathrm{nuc}}(X;\, A, B) \cap [\mathrm{Mon}(X;\, A, \mathrm{Q}(B))] = G(H_0, A, \mathrm{Q}(B))\,.$$

Recall that $G(H_0, A, \mathrm{Q}(B))$ is naturally isomorphic to the kernel of $\mathrm{K}_0(H_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(\mathrm{Q}(B))$ by Proposition 4.4.3(ii).

PROOF. `to be filled in` ?? □

LEMMA 5.9.6. *Suppose that B is σ-unital, A is a separable C^*-subalgebra of $\mathcal{M}(B)/B$, and that $V\colon A \to \mathcal{M}(B)/B$ is a completely positive contraction.*

If there exist a sequence $d_1, d_2, \ldots$ of contractions in $\mathcal{M}(B)/B$, such that, for $a \in A$,

(i) $\lim_{n\to\infty} \|d_n^* a d_{n+1}\| = 0\,,\ \ \lim_{n\to\infty} \|d_n^* d_{n+1}\| = 0\,,$ *and*
(ii) $\lim_{n\to\infty} \|d_n^* a d_n - V(a)\| = 0,$

then there exists a contraction $d \in \mathcal{M}(B)$, such that $V(a) = \pi_B(d)^* a \pi_B(d)$ for $a \in A$.

PROOF. Since A is separable, there is a strictly positive contraction $c \in A_+$ with $\|c\| = 1$.

The proof is similar to that of Proposition 5.4.1. Let Ω be a compact subset of the contractions in A_+ that linearly generates a dense linear subspace of A, and let $\gamma\colon \Omega \to \mathcal{M}(B)_+$ (respectively $\tau\colon V(\Omega) \to \mathcal{M}(B)_+$) a topological lift of $\Omega \subseteq \mathrm{Q}(B)$ (respectively of $V(\Omega) \subseteq \mathrm{Q}(B)$) into the positive contractions in $\mathcal{M}(B)$. Further let $e \in B_+$ a strictly positive contraction with $\|e\| = 1$, and let $f_1, f_2, \ldots \in \mathcal{M}(B)$ contractive lifts of $d_1, d_2, \ldots$. The unital C^*-subalgebra $D := C^*(1, e, \gamma(\Omega), \tau(V(\Omega)), \{f_n\}_n) \subseteq \mathcal{M}(B)$ generated by $\{e, 1, f_1, f_2, \ldots\} \cup \gamma(\Omega) \cup \tau(V(\Omega))$ is separable.

By Remark 5.1.1(3), we find elements $g_n \in C^*(e)_+$, namely, square roots of suitable differences of elements in a suitable approximate unit of B that is approximately central for D, with the following properties

(1) $\sum_n g_n^2$ converges strictly to $1_{\mathcal{M}(B)}$, $g_n g_m = 0$ for $|m-n| > 1$, $\|e g_n\| < 2^{-n}$

(2) $\|f_k g_n - g_n f_k\| < 2^{-n}$ for $k \leq n$,
(3) $\|g_n\gamma(a) - \gamma(a)g_n\| < 2^{-n}$ and $\|g_n\tau(V(a)) - \tau(V(a))g_n\| < 2^{-n}$ for $a \in \Omega$,
(4) $\|g_n f_n^* b f_{n+1} g_{n+1}\| \leq 2^{-n} + \mathrm{dist}(f_n^* b f_{n+1}, B)$ for $b \in \gamma(\Omega) \cup \{1\}$, and
(5) $\|g_n(f_n^*\gamma(a)f_n - \tau(V(a)))g_n\| \leq 2^{-n} + \mathrm{dist}(f_n^*\gamma(a)f_n - \tau(V(a)), B)$ for $a \in \Omega$.

Let $\Gamma(b_1, b_2, \ldots) := \sum g_n b_n g_n$ for $(b_1, b_2, \ldots) \in \ell_\infty(\mathcal{M}(B))$. By Remarks 5.1.1(2,4) check numbers 2 and 4!, Γ is a unital c.p. map from $\ell_\infty(\mathcal{M}(B))$ into $\mathcal{M}(B)$. It holds $\Gamma(b_1, b_2, \ldots) \in B$ if $\lim \|g_n b_n g_n\| = 0$, because then $\Gamma(b_1, b_2, \ldots)$ is the sum of the two series $\sum g_{2n-1}b_{2n-1}g_{2n-1}$ and $\sum g_{2n}b_{2n}g_{2n}$ which have mutually orthogonal summands by (1). Thus, by (5) and (ii),

$$\Gamma(f_1^*\gamma(a)f_1 - \tau(V(a)), f_2^*\gamma(a)f_2 - \tau(V(a)), \ldots) \in B\,.$$

Moreover, by Remarks 5.1.1(4,5,6), $\tau(V(a)) - \Gamma(\tau(V(a)), \tau(V(a)), \ldots) \in B$ (by (3)), $\sum f_n g_n$ converges strictly to an element $f \in \mathcal{M}(B)$, and, by (2)–(4)

$$f^*bf - \Gamma(f_1^*bf_1, f_2^*bf_2, \ldots) \in B$$

for $b \in \gamma(\Omega) \cup \{1\}$, because $\lim_n \|g_n f_n^* b f_{n+1} g_{n+1}\| = 0$ by (4) and (i).

Thus, $\pi_B(f)$ is a contraction and $\pi_B(f)^* a \pi_B(f) = V(a)$ for $a \in \Omega$. Let $d := h\lambda(h^*h)$, where $\lambda(t) := \min(1, t^{-1/2})$ for $t \in [0, \infty)$. Then d is a contraction in $\mathcal{M}(B)$ with $\pi_B(d)^* a \pi_B(d) = V(a)$ for $a \in \Omega$, and thus for all $a \in A$. □

Remark 5.9.7. If (in addition to the assumptions of Lemma 5.9.6) B is stable and A is contained in a σ-unital stable C^*-subalgebra D of $\mathcal{M}(B)/B$ such that $\pi_B^{-1}(D)$ is stable, then d can be replaced by an isometry $s = \pi_B(S)$ for some isometry $S \in \mathcal{M}(B)$, i.e., $d^*ad = s^*as$ for all $a \in A$.
(Indeed: there are isometries $T_1, T_2 \in \mathcal{M}(B)$ such that $T_1T_1^* + T_2T_2^* = 1$, $ct_2 = 0$ and $t_1c = c = ct_1$ for all $c \in D \supseteq A$ and $t_k = \pi_B(T_k)$,$k = 1, 2$, by Corollary 5.5.14. Let $S := T_1 d + t_2(1 - d^*d)^{1/2} \in \mathcal{M}(B)$.)

Proposition 5.9.8. *Suppose that A is separable, B is stable and σ-unital and* (1-)*purely infinite, $h\colon A \to \mathrm{Q}(B)$ is a C^*-morphism, and that $(V_k\colon \mathrm{Q}(B) \to \mathrm{Q}(B))$ is a sequence of approximately inner completely positive maps.*

If h dominates $h \oplus h$ and $V_n \circ h$ converges point-wise on A to a c.p. contraction $T\colon A \to \mathrm{Q}(B)$, then h dominates T.

Proof. The multiplier algebra $\mathcal{M}(B)$ and the corona $\mathrm{Q}(B)$ contain copies of $\mathcal{O}_2$ unitally, because B is stable. Since h dominates $h \oplus h$, there is a copy of $\mathcal{O}_\infty$ unitally contained in $h(A)' \cap \mathrm{Q}(B)$ (cf. Proposition 4.3.5(iv)). The sequence of c.p. maps $(V_k \circ h)$ (with point-norm limit T) allows to construct a sequence d_n of contractions in $\mathrm{Q}(B)$ such that $d_{n+k}^* b d_n = 0$ for $b \in h(A) + \mathbb{C}1$, $k > 0$, and $\lim_{n\to\infty} \|d_n^* h(a) d_n - T(a)\| = 0$ (cf. Lemma 3.10.5). Thus, Lemma 5.9.6 applies and there is a contraction $d \in \mathrm{Q}(B)$ with $d^*h(\cdot)d = T$. Since A is stable and B is purely infinite, $\pi_B^{-1}(h(A))$ is stable by Corollary 5.5.16. Thus d can be replaced by an isometry (by Remark 5.9.7). □

REMARK 5.9.9.

`Where 5.9.9 is used? ??`

If $B \cong B \otimes \mathcal{D}_\infty$ (with $\mathcal{D}_\infty = \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$), then the latter approximate domination should be the case if and only if (for suitable topological lifts) $\widetilde{h}\colon \Omega \to \mathcal{M}(B \otimes \mathcal{D}_\infty)$ approximately dominates $\widetilde{T}$ modulo a suitable commutative approximate unit $e_1 \leq e_2 \leq \ldots$ of B. More precisely: Let $\widetilde{h}$ and $\widetilde{T}$ denote topological lifts of h and T on a linearly generating compact subset Ω of A. Then, for every $\varepsilon > 0$ there should exist $k_\varepsilon \in \mathbb{N}$, such that, for every $m > n > k_\varepsilon$, there is a contraction $d_{n,m} \in B \otimes \mathcal{D}_\infty$ with

$$\|d_{n,m}^* \widetilde{h}(a) d_{n,m} - e_m(1-e_n)\widetilde{T}(a)(1-e_n)e_m\| < \varepsilon.$$

for $a \in \Omega$. (Also some orthogonality relations of the $d_{n,m}$ are required, coming here from a central sequence of copies of $\mathcal{O}_\infty$ in $\mathcal{M}(B \otimes \mathcal{D}_\infty)$ for $B \otimes \mathcal{D}_\infty$.)

LEMMA 5.9.10. *Let B a stable and σ-unital C^*-algebra and $b \in \mathcal{M}(B)_+$. Denote by $J(b) := \overline{\operatorname{span} BbB}$ the closed ideal of B, and by $I(b) := \overline{\operatorname{span} \mathcal{M}(B)b\mathcal{M}(B)}$ the norm-closed ideal of $\mathcal{M}(B)$ generated by b. Then, for every $b \in \mathcal{M}(B)_+$,*

$$\mathcal{M}\big(B, J((b-\varepsilon)_+)\big) \subseteq I\big(\delta_\infty(b)\big) \subseteq \mathcal{M}\big(B, J(b)\big)\,,$$

and $b \in I\big(\delta_\infty(b)\big) \in \mathcal{I}(\mathcal{M}(B))$.

PROOF. Let e denote a strictly positive element of B_+, $s_1, s_2, \ldots \in \mathcal{M}(B)$ a sequence of isometries with $\sum s_n s_n^* = 1$ strictly, and $b \in \mathcal{M}(B)_+$. Then

$$bB \subseteq J(b) := \overline{\operatorname{span} BbB}\,,$$

because $\lim_{n\to\infty} e^{1/n}bc = bc$ for $c \in B$. Thus, $Bb \cup bB \subseteq J(b)$, i.e., $b \in \mathcal{M}(B, J(b))$ for every $b \in \mathcal{M}(B)_+$.

It follows that $\delta_\infty(b) \in \mathcal{M}(B, J(b))$ for each $b \in \mathcal{M}(B)_+$, because $\overline{\operatorname{span} BbB} = \overline{\operatorname{span}(B\delta_\infty(b)B)}$ by strict convergence of $\sum_n s_n b s_n^* =: \delta_\infty(b)$, i.e., by norm-convergence in B of $\sum_{n=1}^m s_n b s_n^* a$ to $\delta_\infty(b)a$ for all $a \in B$.

Clearly, $b = s_1^* \delta_\infty(b) s_1 \in I\big(\delta_\infty(b)\big)$.

Let $J := J((b-\varepsilon)_+) = \overline{\operatorname{span}(B(b-\varepsilon)_+B)}$, and let $t \in \mathcal{M}(B,J)_+$. If we use functional calculus and convex combination, then we find $e_n \in C^*(e)$ with $0 \leq e_n \leq 1$, $e_n e_{n+1} = e_n$, $\|e_n e - e\| < 2^{-n}$, $\|e_n t - t e_n\| < 2^{-n}$, and $\|g_n t - t g_n\| < 2^{-n}$ for $g_n := (e_n - e_{n-1})^{1/2}$ with $e_0 := 0$, cf. Remark 5.1.1(3).

`Next still to be checked:`

The sum $V(t) := \sum_{0\leq n\leq\infty} g_n t g_n$ converges strictly and the elements tg_n and $V(t) - t$ are in J, cf. Remark 5.1.1(4). Since $g_n t g_n \in J_+$ for $t \in \mathcal{M}(B,J)$ and since J_+ is the closed linear span of $\{b^*(a-\varepsilon)_+ b\,;\ b \in B\,\}$, we get $(a-\varepsilon)_+^\delta \cdot B \subseteq J$ for each $\delta \in (0,1)$.

By definition of J and $\mathcal{M}(B,J)$, there exist $k_n \in \mathbb{N}$ and $b_{n,k} \in B$ where $k \in \{1,\ldots,k_n\}$, with

$$\| g_n t g_n - \sum_{k=1}^{k_n} (b_{n,k})^*(a-\varepsilon)_+ b_{n,k} \| < 2^{-n}.$$

Let $s_1, s_2, \ldots$ a sequence of isometries in $\mathcal{M}(B)$, such that $\sum s_n s_n^*$ strictly converges to 1. We define a bounded continuous function $\varphi\colon [0,\infty)$ by $\varphi(0) := 0$ and $\varphi(\tau) := ((\tau-\varepsilon)_+/\tau)^{1/2}$ for $\tau \in (0,\infty)$. Then $\varphi(\tau)^2 \le \varepsilon^{-1}(\tau-\varepsilon)_+$ because $\tau\varphi(\tau)^2 = (\tau-\varepsilon)_+$. It follows that $\varphi(a)b_{n,k} \in J$ for each $n \in \mathbb{N}$ and $k = 1,\ldots,k_n$.

The elements $f_n := (e_{n+2} - e_{n-1})^{1/2}$ are positive contractions with $g_n f_n = g_n$ if we let $e_0 := 0$.

We define inductively $m_n \in \mathbb{N}$, $d_n \in B$ and $q_n \in \mathcal{M}(B)$, by $m_1 = 0$, $m_{n+1} := m_n + k_n$,

$$d_n := \sum_{k=1}^{k_n} s_{k+m_n}\varphi(a)b_{n,k}f_n \quad \text{and} \quad q_n := \sum_{k=1}^{k_n} s_{k+m_n}s_{k+m_n}^* .$$

The sum $\sum q_n$ of the projections $q_n \in \mathcal{M}(B)$ converges strictly to $1_{\mathcal{M}(B)}$. The d_n satisfy

$$\|g_n t g_n - d_n^*\delta_\infty(a)d_n\| = \|f_n\big(g_n t g_n - \sum_{k=1}^{k_n}(b_{n,k})^*(a-\varepsilon)_+ b_{n,k}\big)f_n\| < 2^{-n}$$

and $d_n = q_n x_n f_n$, where $x_n \in B$ is defined by $x_n := \sum_{k=1}^{k_n} s_{k+m_n}\varphi(a)b_{n,k}$. Notice that x_n and d_n are in J, because $b_{n,k}^*\varphi(a)^2 b_{n,k} \le \varepsilon^{-1}(b_{n,k})^*(a-\varepsilon)_+ b_{n,k} \in J$. The sequence $x_1, x_2, \ldots$ is bounded, because $b_{n,k}^*\varphi(a)^2 b_{n,k} \le \varepsilon^{-1}(b_{n,k})^*(a-\varepsilon)_+ b_{n,k}$ implies that

$$x_n^* x_n = \sum_{k=1}^{k_n} b_{n,k}^*\varphi(a)^2 b_{n,k} \le \varepsilon^{-1}(2^{-n}1 + g_n t g_n) \le \varepsilon^{-1}(1 + \|t\|) .$$

Since also $\sum f_n^2 \le 3\sum g_n^2 = 3 \cdot 1_{M(B)}$ converges strictly to some element $F \in \mathcal{M}(B)_+$ with $\|F\| \le 3$, we get that $\sum_{n=1}^\infty d_n$ is strictly convergent in $\mathcal{M}(B)$, cf. Remark 5.1.1(2). Let $d := \sum_{n=1}^\infty d_n \in \mathcal{M}(B)$. Using that $d_m^*\delta_\infty(a)d_n = \delta_{m,n}d_n^*\delta_\infty(a)d_n$ we get

$$d^*\delta_\infty(a)d = \sum_n d_n^*\delta_\infty(a)d_n .$$

The estimates $\|d_n^*\delta_\infty(a)d_n - g_n t g_n\| < 2^{-n}$ and that $d_n^*\delta_\infty(a)d_n, g_n t g_n \in J$ show that $d^*\delta_\infty(a)d - V(t)$ is in J, because $d_n^*\delta_\infty(a)d_n$, $g_n t g_n \in J$. It implies $V(t) \in J + I(\delta_\infty(a))$. Since $J = J((a-\varepsilon)_+) \subseteq J(a) \subseteq I(\delta_\infty(a))$, the element $t = V(t) + (t - V(t)) \in \mathcal{M}(B,J)_+$ is contained in $I(\delta_\infty(a)) + J(a)$. Clearly $I(\delta_\infty(a)) + J(a) = I(\delta_\infty(a))$. □

The following Proposition 5.9.11 characterizes the *invertible elements* of

$$\mathrm{SExt}_{\mathrm{nuc}}(\mathrm{Prim}(B);\, A, B) \subseteq [\mathrm{Hom}(A, \mathrm{Q}^{\mathrm{s}}(B))]$$

for exact $A \subseteq \mathcal{M}(B)$ with the property that the imbedding map $a \in A \to a \in \mathcal{M}(B)$ is (weakly) nuclear, where B is σ-unital, stable and has the WvN-property. It generalizes the characterization of invertible elements in $\mathrm{SExt}_{\mathrm{nuc}}(A,B)$ given in Remark 5.8.4 (using Theorem 5.6.2). This characterization will be *used for the proof of Theorem K.*

PROPOSITION 5.9.11 (Absorbing elements). *Suppose that B is σ-unital and stable, satisfies the WvN-property, and that $A \subseteq \mathcal{M}(B)$ is a separable stable exact C^*-subalgebra, such that $\mathrm{id}_{\mathcal{M}(B)}|A$ is weakly nuclear and non-degenerate. Let $\varphi\colon A \to \mathrm{Q^s}(B) = \mathcal{M}(B)/B$ a *-monomorphism, and let h_0 the restriction of $\pi_B \circ \delta_\infty$ to A.*

Define the "natural" lower semi-continuous action of $\mathrm{Prim}(B)$ on A by $\Psi_A(J) := A \cap \mathcal{M}(B,J)$ for $J \in \mathcal{I}(B)$.

Then φ defines an element in $G(h_0;\, A, \mathrm{Q}(B)) \cong \mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B);\, A, B)$ (i.e., φ and h_0 dominate in $\mathrm{Q}(B)$ each other), if and only if,

(i) *φ is nuclear, and*
(ii) *$\varphi(\Psi_A(J)) = \pi_B(\mathcal{M}(B,J)) \cap \varphi(A)$ for $J \in \mathcal{I}(B)$.*

PROOF. The weak nuclearity of $\mathrm{id}|A$ and the exactness of A imply, that $H_0 := \delta_\infty|A$ is nuclear, cf. `Chp.3: if` A `is exact then ''weakly nuclear''` H_0 `(norm-)''nuclear'' ??`. Since $\delta_\infty(\mathcal{M}(B)) \cap \mathcal{M}(B,J) = \delta_\infty(\mathcal{M}(B,J))$, we have $H_0(\Psi_A(J)) = H_0(A) \cap \mathcal{M}(B,J)$ for $J \in \mathcal{I}(B)$. Since A is exact, this, together with the nuclearity of H_0, implies that H_0 is Ψ_A-Ψ^{up}-residually nuclear.

Let e denote a strictly positive element of B. Since $E_\varphi := \{b \in \mathcal{M}(B)\colon \pi_B(b) \in \varphi(A)\}$ is stable by Corollary 5.5.16 and Remark 3.10.9(iii), we find a stable separable C^*-subalgebra C of $\mathcal{M}(B)$, such that $e \in C$ and $\pi_B(C) = \varphi(A)$.

Let $h := \varphi^{-1}\pi_B\colon C \to A$, then h is residually equivariant, because $h(C \cap \mathcal{M}(B,J)) \subseteq \Psi_A(J)$, by (ii), because $\pi_B\,(C \cap \mathcal{M}(B,J)) \subset \varphi(A) \cap \pi_B(\mathcal{M}(B,J)) = \varphi(\Psi_A(J))$.

Thus, $T := H_0 \circ h$ is a non-degenerate weakly residually nuclear C^*-morphism from C into $\mathcal{M}(B)$.

By Theorem 5.9.3, there exists an isometry S in $\mathcal{M}(B)$, such that $S^*cS - T(c) \in B$ for $c \in C$. This means that φ dominates $h_0 := \pi_B \circ H_0$.

Let $a \in A_+$ and $\varepsilon > 0$. By Lemma 5.9.10, $(a-\varepsilon)_+$ is in $\Psi_A(J) = A \cap \mathcal{M}(B,J)$ for $J := \overline{\mathrm{span}(B(a-\varepsilon)_+B)}$. By condition (ii), $\varphi((a-\varepsilon)_+) \subseteq \pi_B(\mathcal{M}(B,J))$.

On the other hand, again by Lemma 5.9.10, $\mathcal{M}(B,J)$ is contained in the norm-closed ideal of $\mathcal{M}(B)$ that is generated by $H_0(a) = \delta_\infty(a)$. Thus, $\varphi((a-\varepsilon)_+)$ is contained in the closed ideal of $\mathrm{Q^s}(B)$, that is generated by $h_0(a)$.

If follows that, for every closed ideal I of A, $\varphi(I)$ is contained in the ideal of $\mathrm{Q^s}(B)$ that is generated by $h_0(I)$. Since $h_0 \oplus\ h_0$ is unitarily equivalent to h_0, and since φ is nuclear, it follows that h_0 approximately dominates φ by Corollary 3.10.8(ii). The C^*-algebra $\pi_B^{-1}(h_0(A)) = \delta_\infty(A) + B$ is stable, because $\delta_\infty(A)$ is stable and is a *non-degenerate* C^*-subalgebra. The commutant $h_0(A)' \cap \mathrm{Q^s}(B)$

contains a copy of $\mathcal{O}_\infty = C^*(s_1, s_2, \dots)$ unitally, because $\delta_\infty(A)' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_\infty = C^*(s_1, s_2, \ldots)$ unitally by Lemma 5.1.2(i). If $c_1, c_2, \ldots$ is a sequence of contractions in $\mathrm{Q}^{\mathrm{s}}(B)$ such that $\| c_n^* h_0(a) c_n - \varphi(a) \| \to 0$, then $d_n := s_n c_n$ satisfy the assumptions (i) and (ii) of Lemma 5.9.6 (for $h_0(A)$ and φ in place of A and V there). By Lemma 5.9.6 there exists an isometry $T \in \mathcal{M}(B)$ such that $\pi_B(T)^* h_0(\cdot) \pi_B(T) = \varphi$. Thus $a \mapsto T^* \delta_\infty(a) T$ is a Ψ_A residually nuclear lift of φ. Thus $[\varphi] \in G\big(h_0;\, A, Q^s(B)\big)$. I.e. $[\varphi]$ is the Busby invariant of an element in $\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B);\, A, B)$. □

REMARK 5.9.12. Let $\mathcal{D}_\infty := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$. (It is isomorphic to $\mathcal{O}_\infty$ by Corollary H, or by a result of Lin and Phillips [**531**].)

We consider $\psi_1 \colon a \in \mathcal{D}_\infty \to 1 \otimes a \in \mathcal{D}_\infty \otimes \mathcal{D}_\infty$ and an isomorphism ψ_2 from $\mathcal{D}_\infty$ onto $\mathcal{D}_\infty \otimes \mathcal{D}_\infty$. ($\psi_1$ and ψ_2 are unitarily homotopic by Corollary H.)

It holds $C \cong C \otimes \mathcal{D}_\infty$ for separable hereditary C^*-subalgebras C of $B \otimes \mathcal{D}_\infty$, cf. [**443**].

Suppose that A and B are separable, and that

$$0 \to B \otimes \mathcal{D}_\infty \to E \to A \otimes \mathcal{D}_\infty \to 0$$

is an extension. Then there exists an isomorphism λ from E onto $E \otimes \mathcal{D}_\infty$, cf. [**443**, chp. 8].

If A and B are stable then E is stable by Corollary 5.5.16, because $B \otimes \mathcal{D}_\infty$ is purely infinite.

QUESTION 5.9.13. Suppose that A and B are separable stable C^*-algebras and that

$$\varphi_1, \varphi_2 \in \mathrm{Hom}(A \otimes \mathcal{O}_\infty, \mathrm{Q}(B \otimes \mathcal{O}_\infty))$$

are (*not necessarily c.p. liftable*) Busby invariants, and suppose that φ_1 and φ_2 are unitarily homotopic (by unitaries in $\mathrm{Q}(B \otimes \mathcal{O}_\infty)$). Does it imply $\varphi_1 \approx \varphi_2$?

(Recall that $\varphi_1 \approx \varphi_2$ means the rather strong property that there is a unitary $u \in \mathcal{M}(B \otimes \mathcal{O}_\infty)$ with $\varphi_1(a) = \pi(u) \varphi_2(a) \pi(u^*)$ for all $a \in A \otimes \mathcal{O}_\infty$.

NEXT TRUE?:

The question has a positive answer, if - in addition - φ_1 has a completely positive lift. The latter is e.g. the case, if A is nuclear, cf. [**443**].)

`Collection/ to be sorted/ to be integrated: ??`

`next used in Chapter 8:`

The next lemma shows that the relation introduced by J. Cuntz for an alternative definition of Kasparov's $\mathrm{Ext}(A, B)$ also hold for our $\mathrm{Ext}(\mathcal{C};\, A, B)$ with obvious modifications.

LEMMA 5.9.14. *Suppose that A is separable and that B is σ-unital and stable, and that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a point-norm closed matrix operator-convex cone.*

Let $\mathcal{S}$ denote the set of pairs (φ, p), where $\varphi\colon A \to \mathcal{M}(B)$ is a C^*-morphism with $b^*\varphi(\cdot)b \in \mathcal{C}$ for all $b \in B$ and $p \in \mathcal{M}(B)$ is a projection that satisfies $\varphi(a)p - p\varphi(a) \in B$ for all $a \in A$.

We denote by $[\mathcal{S}]$ the set of unitary equivalence classes $[(\varphi, p)]$ using unitaries in $\mathcal{M}(B)$. An element $(\varphi, p) \in \mathcal{S}$ is ***degenerate*** *if $p\varphi(a) = \varphi(a)p$ for all $a \in A$.*

(i) *$[\mathcal{S}]$ is a commutative semigroup (with Cuntz-Addition on maps and projections).*
(ii) *The map $[\varphi, p] \mapsto \pi_B(\varphi(\cdot)p) \in \mathrm{Hom}(A, \mathrm{Q}(B))$ defines an additive semigroup morphism ϑ from $[\mathcal{S}]$ onto $\mathrm{SExt}(\mathcal{C};\, A, B)$. The image of ϑ contains $[0] + \mathrm{SExt}(\mathcal{C}\,;\; A, B)$.*
(iii) *The map $\mathcal{S} \to \mathrm{Ext}(\mathcal{C};\, A, B)$ induces an equivalence relation $\sim$ on $\mathcal{S}$ that is compatible with Cuntz addition and is generated by the following operations and relations:*
   (a) *unitary equivalence by unitaries in $\mathcal{M}(B)$, i.e., $(\varphi, p) \sim (\psi, q)$ if there is a unitary $u \in \mathcal{M}(B)$ with $\psi = u^*\varphi(\cdot)u$ and $q = u^*pu$,*
   (b) *addition $(\varphi \oplus \psi, p \oplus q)$ of* degenerate *elements $(\psi, q) \in \mathcal{S}$ to elements $(\varphi, p) \in \mathcal{S}$, i.e., $(\varphi \oplus \psi, p \oplus q) \sim (\varphi, p)$, and*
   (c) *unitary perturbation: $(\varphi, p) \sim (\varphi, q)$ if there is $b^* = b \in \mathcal{M}(B)$ such that $\varphi(a)b - b\varphi(a) \in B$ and $(p - e^{-ib}qe^{ib})\varphi(a) \in B$ for all $a \in A$.*
(iv) *The natural semigroup morphism from $\mathrm{SExt}(\mathcal{C};\, A, B)$ into $\mathrm{Ext}(\mathcal{C};\, A, B)$ is an epimorphism.*
(v) *If $\mathcal{C}$ is countably generated and non-degenerate, and if $H\colon A \otimes \mathbb{K} \to \mathcal{M}(B)$ is as in Corollary 5.4.4, then the unitary equivalence classes $[(H, p)] \in [\mathcal{S}]$ build a sub-semigroup $[\mathcal{P}]$ of $[\mathcal{S}]$ such that ϑ maps $[\mathcal{P}]$ onto $\mathrm{Ext}(\mathcal{C}; A, B)$.*

Proof. Let $S, T \in \mathcal{M}(B)$ isometries with $SS^* + TT^* = 1$, and let $s := \pi_B(S) = S + B$ and $t := \pi_B(T) = T + B$ in $\mathrm{Q}(B) := \mathcal{M}(B)/B$.

(i): Unitary equivalence $(\varphi, p) \approx (\psi, q)$ for $(\varphi, p), (\psi, q) \in \mathcal{S}$ means that there is a unitary $U \in \mathcal{M}(B)$ with $\psi = U^*\varphi(\cdot)U$ and $q = U^*pU$. Now straight calculation shows that the Cuntz addition

$$(\varphi, p) \oplus (\psi, q) := (\varphi \oplus_{S,T} \psi, p \oplus_{S,T} q)$$

defines an operation on $\mathcal{S}$. It is associative and commutative on the set $[\mathcal{S}]$ of unitary equivalence classes $[\varphi, p]_\approx$ of elements $(\varphi, p) \in \mathcal{S}$, by Proposition 4.3.2.

(ii): The map

$$\gamma(\varphi, p)\colon a \in A \mapsto \pi_B(\varphi(a)p) \in \mathrm{Q}(B)$$

is a *-morphism from A into $\mathrm{Q}(B)$ if $(\varphi, p) \in \mathcal{S}$, and the c.p. map $V = p\varphi(\cdot)p \in \mathrm{CP}(A, \mathcal{M}(B))$ satisfies $b^*V(\cdot)b = (pb)^*\varphi(\cdot)pb \in \mathcal{C}$ for each $b \in B$, because $\psi(\cdot)$ is in the closure of $\mathcal{C}$ with respect to the strict topology (by definition of $\mathcal{S}$). Thus, $\gamma(\varphi, p)$ is in $\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$ and defines an Element $[\gamma(\varphi, p)]$ of

$$\mathrm{SExt}(\mathcal{C}; A, B) := [\,\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))\,]\,.$$

Clearly, $\gamma(U^*\varphi(\cdot)U, U^*pU^*) = \pi_B(U)^*\gamma(\varphi, p)\pi_B(U)$ for unitary $U \in \mathcal{M}(B)$. One gets $[\gamma(\varphi\oplus\psi, p\oplus q)] = [\gamma(\varphi, p)] + [\gamma(\psi, q)]$ in $\mathrm{SExt}(\mathcal{C};\, A, B)$ by straight calculation. We define $\vartheta\colon [\mathcal{S}] \to \mathrm{SExt}(\mathcal{C}; A, B)$ by

$$\vartheta([(\varphi, p)]_{\approx}) := [\gamma(\varphi, p)]\,.$$

If $\rho \in \mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$, then there is a c.p. contraction $V\colon A \to \mathcal{M}(B)$ with $b^*V(\cdot)b \in \mathcal{C}$ for all $b \in B$ (by definition of $\mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$). By Lemma 3.6.24, there is a *-morphism $\psi\colon A \to \mathcal{M}(B)$ with $b^*\psi(\cdot)b \in \mathcal{C}$ and isometries $s_1, t_1 \in \mathcal{M}(B)$ such that $t_1^*\psi(\cdot)t_1 = V$ and $s_1s_1^* + t_1t_1^* = 1$. Let $U := s_1S^* + t_1T^*$, $\varphi(\cdot) := U^*\psi(\cdot)U$ and $p := SS^*$. Then $(\varphi, p) \in \mathcal{S}$ and $0 \oplus_{s,t} \rho = \gamma(\varphi, p)$. Thus $[0] + \mathrm{SExt}(\mathcal{C};\, A, B) \subseteq \vartheta([\mathcal{S}])$.

(iii) +(iv): Recall that $\mathrm{Ext}(\mathcal{C};\, A, B) := \mathrm{Gr}(\mathrm{SExt}(\mathcal{C}; A, B))$ is the Grothendieck group of the semigroup of unitary equivalence classes of morphisms $\rho \in \mathrm{Hom}_l(\mathcal{C};\, A, \mathrm{Q}(B))$ *by unitaries* $u = \pi_B(U)$ *for unitaries* U *in* $M(B)$.

Let $(\varphi, p) \in \mathcal{S}$, $q \in \mathcal{M}(B)$ a projection and $U \in \mathcal{M}(B)$ a unitary with $(p - q)\varphi(A) \subseteq B$, respectively with $U\varphi(a) - \varphi(a)U \in B$ for all $a \in A$. Then (φ, q) and (φ, U^*pU) are in $\mathcal{S}$, i.e., $q\varphi(a) - \varphi(a)q \in B$ and $U^*pU\varphi(a) - \varphi(a)U^*pU \in B$ for all $a \in A$. Moreover, $\gamma(\varphi, p) = \gamma(\varphi, q)$ and $\pi_B(U)^*\gamma(\varphi, p)\pi_B(U) = \gamma(\varphi, U^*pU)$ in $\mathrm{Hom}_l(\mathcal{C}; A, \mathrm{Q}(B))$. Thus, the relations (a) and (c) on $\mathcal{S}$ are stronger (or equal) to the relation on $\mathcal{S}$ induced by $\vartheta\colon [\mathcal{S}] \to \mathrm{SExt}(\mathcal{C}; A, B)$.

If $(\varphi, p) \in \mathcal{S}$ is degenerate, then $\psi := \delta_\infty \circ \varphi$ and $q := \delta_\infty(p)$ ([23]) satisfy $b^*\psi(\cdot)b \in \mathcal{C}$ for all $b \in B$ and $q\psi(a) = \psi(a)q$ for all $a \in A$. Thus $(\psi, q) \in \mathcal{S}$. Since $\mathrm{K}_*(\mathcal{M}(B)) = 0$ and $s_1^*s_2 = 0$, it follows from Lemma 4.2.6(ii) that there is an isometry $R \in \mathcal{M}(B)$ with $RR^* = 1 - s_1s_1^*$. More explicitly the series $\sum s_{n+1}s_n^*$ converges strictly in $\mathcal{M}(B)$ to an element $R := \sum s_{n+1}s_n^*$ of $\mathcal{M}(B)$ by Remark 5.1.1(2). The element R is an isometry in $\mathcal{M}(B)$ with $RR^* = 1 - s_1s_1^*$ and $Rs_n = s_{n+1}$ for $n = 1, 2, \ldots$.

Then $U := Ss_n^* + TR^*$ is unitary, $U^*(\varphi \oplus_{S,T} \psi)U = \psi$ and $U^*(p \oplus_{S,T} q)U = q$, i.e., $\vartheta([\varphi, p]) + \vartheta([\psi, q]) = \vartheta([\psi, q])$ in $\mathrm{SExt}(\mathcal{C}; A, B)$. Thus, $[\vartheta([\varphi, p])] = 0$ in $\mathrm{Ext}(\mathcal{C}; A, B)$ if (φ, p) is a degenerate element in $\mathcal{S}$. It implies that $(\varphi_1, p_1) \oplus_{S,T} (\varphi, p)$ and (φ_1, p_1) define the same class in $\mathrm{Ext}(\mathcal{C}; A, B)$ for all $(\varphi_1, p_1) \in \mathcal{S}$ if (φ, p) is degenerate.

It follows that the equivalence relation $(\varphi, p) \sim (\psi, q)$ on $\mathcal{S}$ defined by the operations (a), (b) and (c) is stronger or equal to the equivalence relation $(\varphi, p) \sim (\psi, q)$ defined by $[\vartheta([\varphi, p])] = [\vartheta([\psi, q])]$ in $\mathrm{Ext}(\mathcal{C}; A, B)$. If we consider the semigroup $[\mathcal{S}]$ then (a) becomes equality and (b) describes there addition by elements in the sub-semigroup $\mathcal{D}$ of $[\mathcal{S}]$ of unitary equivalence classes of degenerate elements. Thus, relation (b) is compatible with Cuntz addition on $[\mathcal{S}]$. Straight calculations show that also relation (c) is compatible with Cuntz addition on $[\mathcal{S}]$. It follows that the

[23] Here $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ is defined by a sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ with strictly converging sum $\sum s_ns_n^* = 1$, as considered in Remark 5.1.1(8) and Lemma 5.1.2.

relation $\sim$ – coming from (a), (b), (c) – defines on $[\mathcal{S}]$ an equivalence relation that is compatible with addition and that $[(\varphi, p)] \sim [(\psi, q)]$ implies $[(\varphi, p)] \sim_c [(\psi, q)]$.

To see that $\sim_c$ is equal to $\sim$, it suffices now to show that $[\mathcal{S}]/\sim$ is a group and that $\vartheta([\varphi, p]) = \vartheta([\psi, q])$ implies $(\varphi, p) \sim (\psi, q)$.

We show first that $\vartheta([\varphi, p]) = \vartheta([\psi, q])$ implies $(\varphi, p) \sim (\psi, q)$, which is the crucial and most involved part of the whole proof:

By definition of ϑ and of the equivalence classes $[\mathrm{Hom}_l(\mathcal{C}; A, \mathrm{Q}(B))]$, there is a unitary $U \in \mathcal{M}(B)$ such that $\gamma(\varphi, p) = \gamma(U^*\psi(\cdot)U, U^*qU)$. Since $(U^*\psi(\cdot)U, U^*qU) \sim (\psi, q)$ by relation (a). We may rename $U^*\psi(\cdot)U$ and U^*qU by ψ respectively by q.

Thus we have to study the implication of the property

$$\varphi(a)p - \psi(a)q \in B \quad \forall\, a \in A\,.$$

Let $\rho := (\varphi \oplus \psi) \oplus 0$, $p' := (p \oplus 0) \oplus 0$, $q' := (0 \oplus q) \oplus 0$, $V := S(ST^* + TS^*)S^* \oplus TT^*$. Then $\rho(\cdot)T = 0$, i.e., ρ dominates zero, $(\rho, p') \sim (\varphi, p)$ and $(\rho, q') \sim (\psi, q)$ by relation (b), and $\rho(a)p' - V^*\rho(a)q'V \in B$ for all $a \in A$. Let $s_1, s_2, \ldots \in \mathcal{M}(B)$ a sequence of isometries with $\sum s_n s_n^* = 1$, and let $\lambda := \delta_\infty \circ \rho$, $p'' := s_1 p' s_1^*$, $q'' = s_1 q' s_1^*$ and $W := \delta_\infty(V)$. It holds $(\lambda, p'') \sim (\rho, p')$, $(\lambda, q'') \sim (\rho, q')$, and $\lambda(a)p'' - W^*\lambda(a)q''W \in B$ for $a \in A$. Moreover, $\lambda(\cdot)$ dominates zero, the algebra $C^*(\lambda(A), W)$ is contained in $\delta_\infty(\mathcal{M}(B))$ and there is a copy $C^*(S_1, T_1) \cong \mathcal{O}_2$ unitally contained in $\delta_\infty(\mathcal{M}(B))' \cap \mathcal{M}(B)$, cf. Remark 5.1.1(8).

It follows that Lemma 4.6.11 applies to $D := A$, $E := \mathrm{Q}(B)$, $h := \pi_B \circ \lambda$, the unitary $\pi_B(W)$, and the projections $p_1 := \pi_B(p'')$, $q_1 := \pi_B(q'')$, because $\mathrm{K}_*(\mathcal{M}(B)) = 0$ by Lemma 5.5.9(i):
We get the existence of $u \in \mathcal{U}_0(h(A)' \cap \mathrm{Q}(B))$ with $u^*(p_1 \oplus 1 \oplus 0)u = (q_1 \oplus 1 \oplus 0)$ with Cuntz addition $\oplus$ taken with respect to the generators $s_1 := \pi_B(S_1)$ and $t_1 := \pi_B(T_1)$ of a unital copy of $\mathcal{O}_2$ in $\pi_B(\lambda(A)' \cap \mathcal{M}(B)) \subseteq h(A)' \cap \mathrm{Q}(B)$. It follows that there are self-adjoint $h_1, h_2, \ldots, h_m \in \mathcal{M}(B)$ with $h_k\lambda(a) - \lambda(a)h_k \in B$ for $k = 1, \ldots, m$ such that $u = \pi_B(U)$ for $U := e^{ih_1} \cdot \ldots \cdot e^{ih_m}$.

Note that $\lambda \oplus_{S_1,T_1} \lambda = \lambda$. Thus $(\lambda, p'') \sim (\lambda, p'' \oplus 1 \oplus 0) = (\lambda, p'') \oplus (\lambda, 1 \oplus 0)$ and $(\lambda, q'') \sim (\lambda, q'' \oplus 1 \oplus 0)$. Let $p_2 := p'' \oplus 1 \oplus 0$ and $q_2 := q'' \oplus 1 \oplus 0$, then $U^*p_2U - q_2 \in B$.

The equivalence relation $\approx_{(c)}$ generated by (c) is stronger than $\sim$ and yields that $(\lambda, p_2) \approx_{(c)} (\lambda, q_2)$ if there are selfadjoint $h_1, \ldots, h_m \in \mathcal{M}(B)$ with $h_k\lambda(a) - \lambda(a)h_k \in B$ for $k = 1, \ldots, m$ and $U^*p_2U - q_2 \in B$ for $U := e^{h_1} \cdot \ldots \cdot e^{h_m}$.
All together shows that $(\varphi, p) \sim (\psi, q)$.

We check that $[\mathcal{S}]/\sim$ is a group:
For $(\varphi, p) \in \mathcal{S}$, the element $(\varphi, 1 - p)$ is in $\mathcal{S}$ and

$$(\varphi, 1 - p) \oplus (\varphi, p) \sim (\varphi, 1) \oplus (\varphi, 0) \sim (0, 0)\,.$$

This is because $[u, \varphi(a) \oplus_{s,t} \varphi(a)] \in B$ for all $a \in A$ and $u^*(1 \oplus 0)u = (1-p) \oplus p$, where $u := \exp(iH) = s(1-p)s^* + t(1-p)t^* + spt^* - tps^*$ for $H := i(\pi/2)(tps^* - spt^*)$ (respectively with $z = s(1-p)s^* + spt^*$) in (iii,b) ([24]). It follows that the natural image of the semi-group $\mathrm{SExt}(\mathcal{C};\, A, B)$ in $\mathrm{Ext}(\mathcal{C};\, A, B)$ is a group, because the image of $\vartheta\colon [\mathcal{S}] \to \mathrm{SExt}(\mathcal{C};\, A, B)$ contains $[0] + \mathrm{SExt}(\mathcal{C};\, A, B)$ by part (ii) and because $[0] + [0] = [0]$.

If we combine this with the observation that the image of $\mathrm{SExt}(\mathcal{C};\, A, B)$ generates $\mathrm{Ext}(\mathcal{C};\, A, B)$, then we get that the natural semigroup morphism from $\mathrm{SExt}(\mathcal{C};\, A, B)$ into $\mathrm{Ext}(\mathcal{C};\, A, B)$ is a semigroup epimorphism.

(v): Suppose that $\mathcal{C}$ is countably generated and that $h_0\colon A \to \mathrm{Q}(B)$ is defined as in Corollary 5.4.9. If $(\varphi, p) \in \mathcal{S}$, then there is a unitary $u \in \mathcal{M}(B)$ and a projection $q \in \mathcal{M}(B)$ such that $u^*(H(a) \oplus 0)qu - \varphi(a)p \in B$ for all $a \in A$, cf. Corollary 5.4.9(ii). Since there is a suitable copy of $\mathcal{O}_2 = C^*(S_1, T_1)$ unitally contained in $(H \oplus 0)(A)' \cap \cap \mathcal{M}(B)$, it follows that the elements $(H \oplus 0, q)$ build a sub-semigroup of $\mathcal{S}$ under Cuntz-addition. □

next.ref: cor:5.X1.chp12 ??

Corollary 5.9.15. *Let A and B stable and separable C^*-algebras and let $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ a lower semi-continuous action of* Prim(B) *on A, such that $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$.*

*Suppose that there exists a non-degenerate weakly Ψ-residually nuclear *-monomorphism $H_0\colon A \to \mathcal{M}(B)$, such that $\delta_\infty \circ H_0$ is unitarily equivalent to H_0, and, for each $J \in \mathcal{I}(B)$,*

$$\Psi(J) = H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J)) \,.$$

Then the H_0 with this properties is uniquely determined up to unitary homotopy.

Proof. Let $\mathcal{C}(H_0) \subseteq \mathrm{CP}(A, B)$ the point-norm closed m.o.c. cone generated by the maps $a \in A \mapsto b^* H_0(a) b$.

Then $\mathcal{C}(H_0)$ is contained in the m.o.c. cone $\mathcal{C}_{\mathrm{nuc}}(\Psi)$ of all Ψ-residually nuclear c.p. maps $V\colon A \to B$ with $V(\Psi(J)) \subseteq J$ for all $J \in \mathcal{I}(B)$ and $[V]\colon A/\Psi(J) \to B/J$ is nuclear.

But, by assumptions, for each $J \in \mathcal{I}(B)$,

$$\Psi(J) = H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J)) \,,$$

i.e., that $\mathcal{C}(H_0)$ is separating for the action Ψ.

By Corollary 3.9.1 this implies that the m.o.c. cone $\mathcal{C}(H_0)$ is identical with the m.o.c. cone of the – with respect to the l.s.c. action Ψ – residually nuclear c.p. maps.

[24] Use that $C^*(s(1-p)s^*, sps^*, tps^*)$ is naturally isomorphic to $\mathbb{C} \oplus M_2$.

If $H_1 \colon A \to \mathcal{M}(B)$ has the same properties as H_0, – i.e., $\delta_\infty \circ H_1$ is unitary equivalent to H_1 and, for each $J \in \mathcal{I}(B)$,

$$\Psi(J) = H_1^{-1}(H_1(A) \cap \mathcal{M}(B, J)),$$

and $[H_1] \colon A/\Psi(J) \to \mathcal{M}(B/J)$ is nuclear for every closed ideal J of B –, then this implies by Corollary 3.9.1 that the m.o.c. cone $\mathcal{C}(H_1)$ is again the m.o.c. cone of all Ψ-residually nuclear c.p. maps. Thus, $\mathcal{C}(H_0) = \mathcal{C}(H_1)$.

It implies that H_0 and H_1 are unitarily homotopic by Corollary 3.9.2. □

`Ref. to 5.9.16 still exists in Chp. 12!!`

Corollary 5.9.16. *Suppose that A is separable and B is σ-unital and stable.*

If $h_1, h_2 \colon A \to \mathcal{M}(B)$ are non-degenerate nuclear monomorphisms and define the same action of $\mathcal{I}(B)$ on A, then $\delta_\infty \circ h_1$ and $\delta_\infty \circ h_2$ are unitarily homotopic.

Chapter 12 asks also for a Corollary

`cor:5.?.same.lsc.action.gives.unitary.homotop`

that says that for stable σ-unital B the same l.s.c. action of $\mathrm{Prim}(B)$ on $C^*(b) \subset \mathcal{M}(B)$ and (– by pull-back also on $C^*(b)$ –) on $C^*(\lambda(b)) \subset \mathcal{M}(B)$ – for some C^*-morphism $\lambda \colon C^*(b) \to \mathcal{M}(B)$ implies that $\delta_\infty(b)$ and $\delta_\infty(\lambda(b))$ are unitary homotopic in $\mathcal{M}(B)$.

`Have the` h_k `same kernel` J?
`Seems to need/check still that the` $[h_k] \colon A/J \to \mathcal{M}(B)$
`is again nuclear!! Then it reduces to the residually nuclear case with exact` A/J.

The following elementary corollary allows to (in proofs of Chapter 6) the use of results on the generalized KK–theory $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ (cf. Chapters 1 and 8). `next.ref: cor:5.lifting-criterium ?? Is needed that the action is monotonous upper s.c.?`

Corollary 5.9.17. *Suppose that A and B are separable and stable, that A is exact and that B has the WvN–property of Definition 1.2.3. Further suppose that $\psi \colon A \to \mathrm{Q}(B)$ is a nuclear C^*-morphism and $H \colon A \to \mathcal{M}(B)$ is a non-degenerate nuclear *-monomorphism. Furthermore suppose that, for $J \in \mathcal{I}(B)$,*

$$\psi\left(H^{-1}(H(A) \cap \mathcal{M}(B, J))\right) \in \pi_B(\mathcal{M}(B, J)).$$

Then $H_0 := \pi_B \circ \delta_\infty \circ H$ dominates ψ in $\mathrm{Hom}(A, \mathrm{Q}(B))$, i.e.,

$$[\psi] \in \mathrm{SExt}_{\mathrm{nuc}}(X\,;\; A, B) = S(H_0\,,\, A, \mathrm{Q}(B))$$

for the action of $X := Prim(B)$ on A given by

$$\Psi \colon J \in \mathcal{I}(B) \cong \mathbb{O}(X) \mapsto \Psi(J) := H^{-1}\left(H(A) \cap \mathcal{M}(B, J)\right),$$

and the semi-group $S(H_0; A, \mathrm{Q}(B)$ as defined in Chapter 4, Section 4.

If moreover

$$\psi^{-1}\left(\psi(A)\cap\pi_B(\mathcal{M}(B,J))\right) = H^{-1}\left(H(A)\cap\mathcal{M}(B,J)\right)$$

for $J\in\mathcal{I}(B)$, and if there exists a C^*-morphism $\varphi\colon A\otimes\mathcal{O}_2\to \mathrm{Q}(B)$ with $\psi(a)=\varphi(a\otimes 1)$ for $a\in A$, then there exists a unitary $U\in\mathcal{M}(B)$ such that $\pi_B(U^*\delta_\infty\circ H(a)U)=\psi(a)$ for $a\in A$.

next.ref: prop:5.char-Extnuc. needed? ??

Proposition 5.9.18. *Suppose that A and B are separable and stable, that A is exact and that B has the WvN–property (cf. Definition 1.2.3). Further let $\psi\colon A\to \mathrm{Q}(B)$ and $H\colon A\to\mathcal{M}(B)$ a weakly nuclear *-monomorphisms such that $BH(A)B$ generates B.*

Let $\Psi(J):=H^{-1}(H(A)\cap\mathcal{M}(B,J))$ for $J\in\mathcal{I}(B)$, and $h_0:=\pi_B\circ\delta_\infty\circ H\colon A\to \mathrm{Q}(B)$.

Then:

Ψ is a lower semicontinuous action of $\mathrm{Prim}(B)$ on A (corresponding to H).

$$\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B),A,B) = [h_0]+S(h_0;A,\mathrm{Q}(B))$$

$[\psi]\in[h_0]+S(h_0;A,\mathrm{Q}(B))$, if and only if, $\psi(A)\cap\pi_B(\mathcal{M}(B,I))=\psi(\Psi(I))$ for all $I\in\mathcal{I}(B)$.

The following Lemma allows to apply above to the asymptotic case.

next.ref:lem:asympt-absorbtion ??
Chp.7ref:lem:asympt-absorbtion.to.chp.7?
next: check if (1) and (2) are used in chp.7 !!

Lemma 5.9.19. (1) *If B is a C^*-algebra, then*

$$\pi_{SB}(\mathcal{M}(SB,SI))\cap\mathrm{Q}(\mathbb{R},B)=\mathrm{Q}(\mathbb{R},I)$$

for all $I\in\mathcal{I}(B)$, where $SB:=\mathrm{C}_0(\mathbb{R},B)$ and $SI:=\mathrm{C}_0(\mathbb{R},I)$.

(2) *Suppose that A is separable, B is σ-unital and stable, and $h_0\colon A\to B$ satisfies $[h_0]+[h_0]=[h_0\oplus h_0]=[h_0]$. Let $s_1,s_2,\ldots\in\mathcal{M}(B)$ a sequence of isometries such that $\sum s_ns_n^*$ converges strictly to 1 in $\mathcal{M}(B)$, and let $H:=\delta_\infty\circ h_0$ – e.g. realized by $s_1,s_2,\ldots$. Denote by $H_{\mathbb{R}}(a):=\eta(H(a))$, where $\eta\colon\mathcal{M}(B)\to\mathcal{M}(SB)$ denotes the natural embedding that extends the inclusion map $B\subset\mathrm{C_b}(\mathbb{R},B)\subset\mathcal{M}(SB)$.*

If $H_{\mathbb{R}}$ dominates $k\colon A\to\mathrm{Q}(\mathbb{R},I)$ then h_0 dominates k.

Proof. (1): Let $b\in\mathcal{M}(SB,SI))$ and $c\in\mathrm{C_b}(\mathbb{R},B)\supset SB:=\mathrm{C}_0(\mathbb{R},B)$ with $b+SB=c+SB$. There is $a\in SB$ with $b=c+a\in\mathrm{C_b}(\mathbb{R},B)$. Thus $b(t)\in B\cap\mathcal{M}(B,I)=I$ for $t\in\mathbb{R}$, i.e., $b\in\mathrm{C_b}(\mathbb{R},I)$. Since $\mathrm{C_b}(\mathbb{R},I)\cap SB=SI$ it follows $b+SB\in\mathrm{Q}(\mathbb{R},I)\cong\pi_{SB}(\mathrm{C_b}(\mathbb{R},I))$.

(2): Let $d \in \mathcal{M}(SB)$ a contraction with $\pi_{SB}(d^* H^{\mathbb{R}} d) = k$. By assumption, there is a unitary $u \in \mathcal{M}(B)$ with $u^* s_1 h_0(\cdot) s_1^* u + u^* s_2 h_0(\cdot) s_2^* u = h_0$, and $H = \sum_n s_n h_0(\cdot) s_n^*$. Let $t_n = (u^* s_1)^n u^* s_2$. Then $t_n^* t_m = \delta_{n,m} 1$ and $t_n h_0(\cdot) = h_0(\cdot) t_n$.

There exists a positive contraction $e \in \mathrm{C_b}(\mathbb{R}, B)$ and a strictly increasing continuous function $\mu\colon \mathbb{R}_+ \to (0, \infty)$ with $e(t) \leq \sum_{n=1}^{k} s_n s_n^*$ for $k \geq \mu(t)$, $ed - de \in SB$, $H_{\mathbb{R}}(a) e - e H_{\mathbb{R}}(a) \in SB$ for all $a \in A$ and $\pi_{SB}(e) k(a) = k(a) = k(a) \pi_{SB}(e)$ for all $a \in A$. This can be seen with help of a convex commutative approximate unit of B, that approximately commutes with the elements of $H(A)$, $s_1, s_2, \ldots$, $\widetilde{k}(A)_t$ $(t \in [-n, n])$ $d|[-n, n]$. Here $\widetilde{k}\colon A \to \mathrm{C_b}(\mathbb{R}, B)$ is some continuous topological (not necessarily linear) lift of the C^*-morphism $k\colon A \to \mathrm{C_b}(\mathbb{R}, B)/SB$. Then $t \mapsto g(t) := \sum_{1 \leq n \leq [\mu(t)]+1} t_n s_n^* e(t) d(t) e(t)$ is a norm-continuous map from $\mathbb{R}$ into the contractions of B. It satisfies $g^* h_0(a) g - d^* H_{\mathbb{R}}(a) d \in SB$ for all $a \in A$, because $e^2 d^* H_{\mathbb{R}}(a) d e^2 - d^* H^{\mathbb{R}} d \in SB$.

to be filled in: check last 2 formulae ?? □

where is ref.cor:5.to-be-named used? ??

Corollary 5.9.20. *Suppose that A and B are stable C^*-algebras, where A is separable B is σ-unital, and that $H_0\colon A \hookrightarrow \mathcal{M}(B)$ and $\varphi\colon A \hookrightarrow \mathrm{Q}(B) = \mathcal{M}(B)/B$ are nuclear *-monomorphisms with the properties that $H_0(A)B$ is dense in B and, for every $a \in A$, $\pi_B(H_0(a))$ and $\varphi(a)$ generate the same closed ideal of $\mathrm{Q}(B)$.*

Then, if at least one of the following conditions (i) or (ii) is satisfied, there is a unitary $U \in \mathcal{M}(B)$ such that $\varphi = \pi_B(U^* H_0(\cdot) U)$.

(i) *$B \cong C \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \ldots$, or*
(ii) *$\pi_B^{-1}(\varphi(A))$ is stable, and there are C^*-morphisms $H\colon A \otimes \mathcal{O}_2 \hookrightarrow \mathcal{M}(B)$ and $\Phi\colon A \otimes \mathcal{O}_2 \hookrightarrow \mathrm{Q}(B) = \mathcal{M}(B)/B$ with $H_0(a) = H(a \otimes 1)$ and $\varphi(a) = \Phi(a \otimes 1)$ for $a \in A$.*

Proof. to be filled in ?? □

cite ? cor:5.X.1.chp8 ??

Corollary 5.9.21. *Suppose that A and B are stable, A separable and B σ-unital with strictly positive element e. Let H_1 denote the infinite repeat of a faithful *-representation $\rho\colon A \to \mathcal{L}(\mathcal{H}) \cong \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(B)$.*

If $h\colon A \to \mathcal{M}(B)$ is any C^*-morphism such that $a \in A \mapsto eh(a)e$ generates $\mathrm{CP_{nuc}}(A, B)$ as a m.o.c. cone, then $\delta_\infty \circ h$ is unitarily homotopic to $H_1\colon A \to \mathcal{M}(B)$.

It holds $\mathrm{Ext_{nuc}}(A, B) = G(H_0, A, \mathrm{Q}(B))$ for $H_0 := \pi_B \circ H_1$.

Proof. cite ? cor:5.X.1.chp8 ?? □

ref {cor:5.Y?1.chp9} (??) needed in this generality??

COROLLARY 5.9.22. *Suppose that A and B are stable, A is separable and B is σ-unital. Let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a countably generated non-degenerate operator convex cone, and $H_0\colon A \to \mathcal{M}(B)$ the non-degenerate C*-morphism described in Corollary 5.4.4. If Y is a locally compact space, let $H_Y := \pi_{\mathrm{C}_0(Y,B)} \circ \epsilon_X \circ H_0$ and $E_Y := \mathcal{M}(\mathrm{C}_0(Y,B))/\,\mathrm{C}_0(Y,B)$, where $\epsilon_X\colon \mathcal{M}(B) \to \mathcal{M}(\mathrm{C}_0(Y,B))$ is the natural strictly continuous embedding of $\mathcal{M}(B)$ into $\mathcal{M}(\mathrm{C}_0(Y,B)) \cong \mathrm{C}_{\mathrm{b,st}}(Y,\mathcal{M}(B))$.*

Then:

(i)

$$\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C},\mathrm{C}_0(Y));\, A, \mathrm{C}_0(Y,B)) = G(H_Y\,,\, A, E_Y) =: F(\mathcal{C},Y)\,.$$

In particular,

$$\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C},\mathrm{C}_0(\mathbb{R})); A, SB) = G(H_{\mathbb{R}}\,,\, A, E_{\mathbb{R}}) =: F(\mathcal{C},\mathbb{R})$$

for the above defined $H_{\mathbb{R}}\colon A \to E_{\mathbb{R}}$.

(ii) *In particular, if X is a T_0 space that acts on A and B by Ψ_A respectively Ψ_B, and if H_0 (and then $H_{\mathbb{R}}$) is formed with respect to the cone of Ψ_A-Ψ_B–residually nuclear maps from A into B, then*

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, SB) = G(H_{\mathbb{R}}, A, E_{\mathbb{R}})\,.$$

(iii) *If $H_0\colon A \to \mathcal{M}(B)$ is the infinite repeat of a non-degenerate $h_0\colon A \to B$ with $h_0 \oplus h_0 \cong h_0$, then $H_{\mathbb{R}}$ dominates $h_0\colon A \to \mathrm{Q}(\mathbb{R}_+, B) \subset E_{\mathbb{R}} := \mathrm{Q}(SB)$. Every element $h \in S(h_0; A, E_{\mathbb{R}})$ is also in $S(H_{\mathbb{R}}; A, E_{\mathbb{R}})$ and $[h \oplus H_0] \in G(H_{\mathbb{R}}, A, E_{\mathbb{R}})$.*

(iv) *In particular, if A is exact and h_0 is nuclear, then for $h\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ holds $[h \oplus H_0] \in \mathrm{Ext}_{\mathrm{nuc}}(X;\, A, SB)$ if and only if $[h] \in S(h_0, A, E_{\mathbb{R}})$. (Here the action of $X := \mathrm{Prim}(B)$ on A is defined by h_0.)*

We have e.g. $E_{\mathbb{R}} := \mathcal{M}(SB)/SB$ if B is stable $SB := \mathrm{C}_0(\mathbb{R}, B)$.

`next: ref: cor:5.Y5.chp9`

COROLLARY 5.9.23. *Suppose that D is separable and stable, B is σ-unital and stable, and that $k_0\colon D \to B$ is a non-degenerate *-monomorphism such that $k_0 \oplus k_0$ is unitarily homotopic to k_0.*

Let $\mathcal{C} \subseteq \mathrm{CP}(D,B)$ denote the point-norm closed m.o.c. cone of c.p. maps generated by k_0. Let $\mathcal{C}(\mathbb{R}) := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, C_0(\mathbb{R}))$.

*We define a *-monomorphism*

$$H_0 := \delta_\infty \circ k_0\colon D \to \mathcal{M}(B).$$

Let $H_{\mathbb{R}} := IH_0\colon D \to E_{\mathbb{R}} := \mathrm{Q}(SB)$, where $I\colon \mathcal{M}(B) \to \mathrm{Q}(SB)$ is the natural embedding.

Then

$$\mathrm{Ext}(\mathcal{C}(\mathbb{R})\,; D, SB) \cong G(H_{\mathbb{R}}; D, E_{\mathbb{R}}).$$

If D is exact, we have moreover,

$$\mathrm{Ext}_{\mathrm{nuc}}(X; D, SB) = G(H_{\mathbb{R}}, D, E_{\mathbb{R}}) = [H_{\mathbb{R}}] + [\mathrm{Hom}_{\mathrm{nuc}}(X; D, E_{\mathbb{R}})]$$

for $X := \mathrm{Prim}(B)$ *and action* $\Psi_D(J) := k_0^{-1}(k_0(D) \cap J)$ *for* $J \in \mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$.

Let A and B stable C^*-algebras, where A is separable and B is σ-unital. Suppose that the operator convex cone $\mathcal{C}_{\mathrm{rn}}(\Psi_A)$ of all Ψ-residually nuclear maps from A to B for a given lower semi-continuous action $\Psi_A \colon \mathbb{O}(\mathrm{Prim}(B)) \cong \mathcal{I}(B) \to \mathcal{I}(A)$ of $\mathrm{Prim}(B)$ on A is countably generated (as e.g. in the case of separable B).

The universal C^*-morphism $H_{\mathrm{rn}} \colon A \to \mathcal{M}(B)$, that is determined (and defined) by m.o.c. cone $\mathcal{C}_{\mathrm{rn}}(\Psi_A) := \mathcal{C}_{\mathrm{rn}}(\Psi_A, \mathrm{id}; A, B)$ in the sense of Corollary 5.4.4, will be called the **universal weakly residually nuclear map** from A into $\mathcal{M}(B)$.

Recall that H_{rn} is uniquely determined up to unitary homotopy by Corollary 5.4.4. It is now known that $\mathcal{C}_{\mathrm{rn}}(\Psi_A)$ – and therefore also H_{rn} – defines the action Ψ, because B has “weak Abelian separation” which is (formally ?) stronger than residually separation for B. But the proof of “weak Abelian separation” for B (which is a weaker property than the existence of an ideal-separating regular abelian C^*-subalgebra of B) deserves several steps, e.g. that every coherent Dini space X is the primitive ideal space of a separable nuclear C^*-algebra that has Abelian separation, and study some “almost minimal” embedding of of Dini spaces into coherent Dini spaces. It uses some results of this book, but can not summarized here.

PROPOSITION 5.9.24. *If B is stable and separable, then the universal residually nuclear map $H_{\mathrm{rn}} \colon B \to \mathcal{M}(B)$ exists for $\Psi_B = \mathrm{id}_{\mathcal{I}(B)}$.*

*Separable B has residually nuclear separation (cf. Definition 1.2.3), if and only if, H_{rn} is a non-degenerate *-monomorphism and $H_{\mathrm{rn}}(J) = H_{\mathrm{rn}}(B) \cap \mathcal{M}(B, J)$ for all $J \in \mathcal{I}(B)$.*

*If $B \neq \{0\}$ is simple separable and stable, then $H_{\mathrm{rn}} \colon B \to \mathcal{M}(B)$ is unitarily homotopic to $\mathcal{M}(\iota) \circ \rho$ for any non-degenerate *-morphism $\iota \colon \mathbb{K} \mapsto \mathcal{M}(B)$ and any non-degenerate *-representation $\rho \colon B \to \mathcal{L}(\ell_2) = \mathcal{M}(\mathbb{K})$.*

In particular, every *simple separable C^*-algebra has residually nuclear separation.*

PROOF. `to be filled in` ?? □

`Used in Chp.8, ref: prop:5.stabilty.of.Ext.Co :`

`Stability, (partial) Functoriality of` Ext?

PROPOSITION 5.9.25. *Suppose that A and B are σ-unital and that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a point-norm closed m.o.c. cone.*

Then the inclusions $\iota_A \colon A \cong A \otimes p_{11} \subseteq A \otimes \mathbb{K}$ *and* $\iota_B \colon B \cong B \otimes p_{11} \subseteq B \otimes \mathbb{K}$ *induce* natural isomorphisms

$$i(A,B) \;:=\; (\iota_A)^* \colon \operatorname{Ext}(\mathcal{C};\, A \otimes \mathbb{K}, B) \to \operatorname{Ext}(\mathcal{C};\, A, B) \qquad (9.1)$$

$$j(A,B) \;:=\; (\iota_B)_* \colon \operatorname{Ext}(\mathcal{C};\, A, B) \to \operatorname{Ext}(\mathcal{C};\, A, B \otimes \mathbb{K})\,, \qquad (9.2)$$

such that $(\iota_B)_* \circ ((\iota_A)^*)^{-1} = ((\iota_A)^*)^{-1} \circ (\iota_B)_*$ *in the sense*

$$j(A \otimes \mathbb{K}, B) \circ i(A,B) = i(A, B \otimes \mathbb{K}) \circ j(A,B)\,.$$

If, moreover, $\mathcal{C} \subseteq \operatorname{CP}(A,B)$ *is countably generated,* A, A' *are separable,* B, C *are* σ*-unital, then* $\psi \in \operatorname{Hom}(B,C)$*, then there is group morphism*

$$\psi_* \colon \operatorname{Ext}(\mathcal{C};\, A, B) \to \operatorname{Ext}(\mathcal{C}(\psi) \circ \mathcal{C};\, A, C)\,.$$

*A *-morphism* $\varphi \in \operatorname{Hom}(A', A)$ *defines a group morphism*

$$\varphi_* \colon \operatorname{Ext}(\mathcal{C};\, A, B) \to \operatorname{Ext}(\mathcal{C} \circ \varphi; A', B)\,.$$

Proof. **??**

This was outlined in a proof of the corresponding (non-nuclear) Proposition [**73**, 17.6.5]. Note here that the stabilization of any relation that contain at least the (unitary) isomorphisms, automatically allows in the class of each representative also the Cuntz addition of so-called "degenerate" pairs, which can be defined as those with the property that their infinite Hilbert B-module sum defines again a $\mathcal{C}$-compatible Kasparov module. Therefore they have no significance for us, because we pass anyway to the stabilized relations, i.e., to the Grothendieck group. □

`Compare following Remark with Remark 5.9.27`
`Split both Remarks into one ‘‘common Remark’’ and a Proposition`
`Change then all references to them!!!`

Remark 5.9.26. The reader should notice that in general the closed ideal $I(\delta_\infty(B))$ of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$ for a separable *stable* C^*-algebra B is not identical with $\mathcal{M}(B)$ e.g. even for $B = \mathrm{C}_0(0,1] \otimes \mathbb{K}$, because $\mathcal{M}(B) \cong \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}_+, \mathcal{M}(\mathbb{K}))$ is unital and 1 is not in the ideal J of $\mathrm{C}_{\mathrm{b,st}}(\mathbb{R}_+, \mathcal{M}(\mathbb{K}))$ that is generated by $\delta_\infty(\mathrm{C}_0((0,1], \mathbb{K}))$, because it is contained in the ideal generated by $\mathrm{C}_0([0,\infty)) \otimes 1$.

Following properties of σ-unital stable B are equivalent:

(i) $I(\delta_\infty(B)) = \mathcal{M}(B)$.
(ii) $\operatorname{Prim}(B)$ is quasi-compact.

`Equivalent to` $/(B)$ `quasi-compact?`

(iii) $B \otimes \mathcal{O}_2$ contains a full projection.

We can replace in (iii) $B \otimes \mathcal{O}_2$ by $B \otimes \mathcal{O}_\infty$, because $\mathcal{O}_\infty \subseteq \mathcal{O}_2$ (unital) and $\mathcal{O}_2 \subseteq \mathcal{O}_\infty$ (non-unital).

PROOF. Since $\mathcal{L}(\ell_2) \cong \mathcal{M}(\mathbb{K}) \subseteq \delta_\infty(B))' \cap \mathcal{M}(B)$ (by a strictly continuous unital monomorphism), $\delta_\infty(B)$ commutes with a unital copy of $\mathcal{O}_2 \cong C^*(t_1, t_2) \subseteq \mathcal{M}(\mathbb{K})$ and a sequence of isometries $s_1, s_2, \ldots \in \mathcal{M}(\mathbb{K})$ with $\sum_n s_n s_n^*$ strictly convergent to 1 and $t_i s_j = s_j t_i$, $t_i^* s_j = s_j t_i^*$. Let $e \in B_+$ a strictly positive contraction.
(i)⇒(ii): The element $\delta_\infty(e)$ is a strictly positive contraction for $B + \delta_\infty(B)$. And $\delta_\infty(e)$ is properly infinite because $\delta_\infty \circ \delta_\infty$ is unitarily equivalent to δ_∞. Thus, there are $\varepsilon > 0$, $X \in \mathcal{M}(B)$ with $X^* \delta_\infty((e-\varepsilon)_+)X = 1$. It follows that e is in the closed ideal $I((e-\varepsilon)_+)$ of B generated by $(e-\varepsilon)_+$, i.e., $B = I((e-\varepsilon)_+)$. Thus $\|\pi_J((e-\varepsilon)_+)\| = 0$ implies $J = B$ for closed ideals J of B, and $\|J + a\| \geq \varepsilon$ for all $J \neq B$. Since the set $Y_\varepsilon \subseteq \mathrm{Prim}(B)$ of primitive ideals J of B with $\|\pi_J(e)\| = \|J + e\| \geq \varepsilon$ is a quasi-compact G_δ-subset, of $\mathrm{Prim}(B)$, it follows that $\mathrm{Prim}(B) = Y_\varepsilon$ is quasi-compact.
(ii)⇒(iii): The algebra $B \otimes \mathcal{O}_2$ is (strongly) purely infinite by Corollary **??**. If $\mathrm{Prim}(B)$ is quasi-compact, then the purely infinite algebra $B \otimes \mathcal{O}_2$ has quasi-compact primitive ideal space $\mathrm{Prim}(B \otimes \mathcal{O}_2) \cong \mathrm{Prim}(B)$.

By Proposition 2.10.4 $B \otimes \mathcal{O}_2$ contains a full projection.
(iii)⇒(i): If $B \otimes \mathcal{O}_2$ contains a full projection p, then there is a non-degenerate C^*-morphism $h\colon \mathbb{K} \to B \otimes \mathcal{O}_2$.

Above we have seen that we can define by $\mu(b \otimes t_j) = \delta_\infty(b)t_j$ a *-monomorphism $\mu\colon B \otimes \mathcal{O}_2 \to \mathcal{M}(B)$ with $\mu(b \otimes 1) = \delta_\infty(b)$ and with $\delta_\infty \circ \mu = \mu$ (if we take new isometries $s_1, s_2, \ldots$ for the definition of the new infinite repeat δ_∞). It follows, $I(\delta_\infty(B)) = I(\mu(B \otimes \mathcal{O}_2)) = I(\mu \circ h(\mathbb{K}))$ in $\mathcal{M}(B)$. The morphism $\mu \circ h(\mathbb{K}) \subseteq \mathcal{M}(B)$ is non-degenerate, i.e., $\mathcal{M}(\mu \circ h)\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(B)$ exists and is unital.

If we use the sequence $s_1, s_2, \ldots$, then $\delta_\infty(\mu \circ h(x)) = \mu \circ h(x)$ for each $x \in \mathbb{K}$. We find a sequence of isometries $r_1, r_2, \ldots$ in $\mathcal{M}(\mathbb{K})$ with $\sum_n r_n r_n^* = 1$ for the definition of $\delta_\infty\colon \mathbb{K} \to \mathcal{M}(\mathbb{K})$, then $\mathcal{M}(\mu \circ h) \circ \delta_\infty$ is unitarily equivalent to $\delta_\infty \circ \mu \circ h = \mu \circ h$ by Lemma 5.1.2(i), because there is a unitary $U \in \mathcal{M}(B)$ with $U^* \mathcal{M}(\mu \circ h)(r_n) = s_n$ and $U^* \mathcal{M}(\mu \circ h)(\cdot)U$ is strictly continuous. Thus, $I(\mu \circ h(\mathbb{K})) \supset \mathcal{M}(\mu \circ h)(I(\delta_\infty(\mathbb{K})))$. But there is an isometry $S \in \mathcal{M}(\mathbb{K}) \cong \mathcal{L}(\ell_2)$ with $SS^* = \delta_\infty(e_{1,1})$. It follows $1 \in I(\mu \circ h(\mathbb{K})) = I(\delta_\infty(B))$. □

`Compare next with Remark 5.9.26`

REMARK 5.9.27. We consider sometimes elements that are in the ideal generated by $\delta_\infty(B)$ in $\mathcal{M}(B)$, e.g. in some proofs of Chapters 2, 5, 6, 8 and 12. Therefore the following rather strong equivalent properties of the ideal of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$ should be noticed. They show that some care is needed if one establish results with help of elements in the ideal generated by $\delta_\infty(B)$.

Suppose that B is stable, then the following are equivalent:

(a) The ideal $\mathcal{J}$ of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$ is equal to $\mathcal{M}(B)$.
(b) B is σ-unital and prime(B) is quasi-compact.
(c) prime(B) quasi-compact.

(d) There exists a positive contraction $e \in B_+$ such that e is strictly positive and $f := (2e-1)_+$ is "full" in B – in the sense that f generates B as closed ideal –, and that there exists $a_1, \ldots, a_n \in B$ such that

$$g := 2e - (2e-1)_+ = \sum_{k=1}^{n} a_k^* f a_k \,.$$

If B contains a projection $p \in B$ that generates B as a closed ideal, then $\delta_\infty(p)$ is the range of an isometry in $\mathcal{M}(B)$. In particular, then B satisfies the property (a) and $B \cong pBp \otimes \mathbb{K}$.

If B is weakly purely infinite – in addition –, then property (a) implies that B contains a properly infinite projection that generates B as a closed ideal.

Compare proof that stable stable B
with simple $\mathcal{M}(B)/B$ is σ-unital.
Is this proof here applicable?

Proof. For the proof of the existence of a countable approximate unit consisting of contractions $0 \leq b_1 \leq b_2 \leq \cdots$ in B_+ from the existence of $e \in B_+$ and $d_1, \ldots, d_n \in \mathcal{M}(B)$ with $1 - \sum_{k=1}^{n} d_k^* \delta_\infty(e) d_k \in B$, see Chapter 2[Cor.2.1.7(i)?] or proof of Corollary 2.2.11 in case of simple B?

We remind the *definition* of **Dini functions** f on topological T_0 spaces X (by requiring that the generalized classical Lemma of Dini is valid for f):

The function f is lower semi-continuous, and satisfies the Lemma of Dini: *If g_τ is an upward directed net of non-negative lower semi-continuous functions on X that converges point-wise to f on X, then the net converges uniformly on X to f.*

An equivalent definition of bounded Dini functions is: The function f is non-negative, lower semi-continuous and, for every decreasing sequence $F_1 \supseteq F_2 \supseteq \cdots$ of closed subsets $F_n \in X$, holds $\sup f(\bigcap_n F_n) = \inf_n \sup f(F_n)$.

On sober topological T_0 spaces this is equivalent to saying that for each $t > 0$ the set $f^{-1}[t, \infty)$ is a quasi-compact subset of X. (Notice that it is always a G_δ subset of X).

The latter characterization implies that each Dini functions f on $\mathrm{prime}(B)$ is a generalized Gelfand transformations $N(b)\colon \mathrm{prime}(B) \to [0, \infty)$ where $N(b)(J) := \|b + J\| = \|\pi_J(b)\|$ of some $b \in B$, cf. [**447**] ([25]). This is fairly easy to see directly for stable weakly purely infinite C^*-algebras B.

Notice that $\mathrm{prime}(B) \supseteq \mathrm{Prim}(B)$ is the "point-wise completion" of $\mathrm{Prim}(B)$ because both have the same lattice of open subsets. The restriction of a Dini function on $\mathrm{prime}(B)$ to $\mathrm{Prim}(B)$ is again a Dini function, and each bounded Dini function f on $\mathrm{Prim}(B)$ is a generalized Gelfand transformation $N(b)$ by [**447**].

[25] Typo: Replace b by c in the last line of the proof of [**447**, lem.3.3]

(a)⇒(b): Since the (non-zero) elements of $\delta_\infty(B)$ are properly infinite inside the hereditary C^*-subalgebra D of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$, – because $\mathcal{O}_2 \subset \mathcal{M}(\mathbb{K}) \subseteq \delta_\infty(B)' \cap \mathcal{M}(B)$ – there exists a contraction $b \in B_+$ and an operator $T \in \mathcal{M}(B)$ with $T^*\delta_\infty(b)T = 1$.

It follows that there exists $\gamma > 0$ and $S \in \mathcal{M}(B)$ such that $S^*T^*\delta_\infty((b - \gamma)_+)TS = 1$. That implies that $c := (b-\gamma)_+$ generates B as a closed ideal. Thus $\|\pi_J(c)\| > 0$ for all closed ideals of B.

Let $e := \gamma^{-1}(b - (b-\gamma)_+)$. Then $\|e\| \leq 1$ and $ec = c$. It follows $\|\pi_J(e)\| \cdot \|\pi_J(c)\| = \|\pi_J(e)\| > 0$ for every closed ideal $J \neq B$. Thus the function $1 = N(e)$ is a Dini function on prime(B). It implies that prime(B) is quasi-compact (by definition of the Dini functions).

(c)⇒(b): prime(B) is quasi-compact, because $\|\pi_J(g)\| = 1$ for all $J \lhd B$ with $J \neq B$ follows from $gf = f$ and $\|\pi_J(f)\| > 0$ for all $J \lhd B$, $J \neq B$.

(b)⇒(c): The definition of Dini functions implies: prime(B) is quasi-compact, if and only if 1 is a Dini function on prime(B).

Since 1 is the point-wise supremum of the norm functions $N(b)(J) := \|b+J\| = \|\pi_J(b)\|$ (for $J \in$ prime(B)) on prime(B), where $b \in B_+$ with $\|b\| < 1$ and since the positive elements in the open unit ball of B build an upward directed family, the net of this functions $N(b)$ converges point-wise to 1. Since 1 is Dini, it follows that the net converges uniformly to 1. In particular, there exists a contraction $b \in B_+$ with $\|\pi_J(b)\| > 2/3$ for all closed ideals $J \neq B$ of B. Let $a := \min(2b, 1) = 2b - (2b-1)_+$. Then $\|\pi_J(a)\| = 1$ for all closed ideals $J \neq B$ of B.

Indeed: Clearly $0 \leq a$ and $\|a\| \leq 1$. Take a character χ on $C^*(\pi_J(b))$ with $\chi(\pi_J(b)) = \|\pi_J(b)\| > 2/3$. Then $\chi((2\pi_J(b) - 1)_+) = 2\chi(\pi_J(b)) - 1 > 0$ and $\chi(\pi_J(a)) = 2\chi(\pi_J(b)) - (2\chi(\pi_J(b)) - 1)_+ = 1$. Thus, $\|\pi_J(a)\| = 1$.

It follows that ???? and $\|\pi_J(4(a - 1/4)_+)\| = 1$ for all $J \lhd B$ with $J \neq B$.

This implies that $(a - 1/4)_+$ is again full in B and that there exists $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in B$ such that $\|2b - \sum d_k^*(a-1/4)_+ d_k\| < \delta$.

It follows from Lemma 2.1.9

??????

that there is a contraction $e \in B$ with $(2b - \delta)_+ = \sum e^* d_k^*(a - 1/4)_+ d_k e$.

more ???????

(c)⇒(a): We construct a projection $Q \in \delta_\infty(B)\mathcal{M}(B)\delta_\infty(B)$ that majorizes $\delta_\infty(f)$. It follows that $P := \delta_\infty(Q)$ is in the closed ideal generated by $\delta_\infty(B)$ (because δ_∞^2 is unitary equivalent to δ_∞ in $\mathcal{M}(B)$) and PBP is stable full split-corner of B. It follows from the stable isomorphism theorem of L.G. Brown [**107**], respectively from Kasparov absorption theorem [**404**] that P is equivalent to 1 in $\mathcal{M}(B)$.

$\delta_\infty(B)$ commutes with a copy of $C^*(t_1, t_2, \ldots) = \mathcal{O}_\infty \subseteq \mathcal{M}(B)$.

If B contains a projection $p \in B$ that generates B as a closed ideal, then $\delta_\infty(p)$ is the range of an isometry in $\mathcal{M}(B)$. In particular, then B satisfies the property (a) and $B \cong pBp \otimes \mathbb{K}$.

If B is weakly purely infinite –in addition–, then property (a) implies that B contains a properly infinite projection that generates B as a closed ideal.

If B is weakly purely infinite then there exists $n \in \mathbb{N}$ such that $F := s_1 f s_1^* + \cdots + s_n f s_n^*$ is properly infinite in B. Let $E := s_1 e s_1^* + \cdots + s_n e s_n^*$ and $G := s_1 g s_1^* + \cdots + s_n g s_n^*$. Then $G = \min(2E, 1)$, $F = (2E-1)_+$ and $GF = F$. Since $G = \sum_{k=1}^n A_k^* F A_k$ for $A_k := s_1 a_k s_1^* + \cdots + s_n a_k s_n^*$, and F is properly infinite, there exist $T_1, T_2 \in B$ such that $\|T_k^* F T_k - G\| < 1/4$ $(k = 1, 2)$. We can manage that $T_1^* F T_2 = 0$ in addition, because for every $\varepsilon > 0$ there are $D_1, D_2 \in B$ with $D_j^* F D_k = \delta_{jk}(F - \varepsilon)_+$

(by a suitable 2×2-matrix arguments using Lemma 2.1.9 ??). It follows that there exists S_1, S_2 with $(T_j S_j)^* F (T_k S_k) = \delta_{jk} \cdot 2(G - 1/2)_+$. Since $GF = F$ it follows that $2(G - 1/2)_+ F = F$. We get contractions $V_k := F^{1/2} T_k S_k$ with $V_1^* V_2 = 0$ and $V_1^* V_1 = V_2^* V_2$ and $V_j^* V_j V_k = V_k$ The isometry $Z :=$????? satisfies ????. □

`Refer to inventer of this example!`

Remark 5.9.28. An algebra B with properties (a)–(c) of Remark 5.9.27 can be stably projection-less, e.g. there exist simple stable separable nuclear C^*-algebras B that are stably projection-less.

An easy example of a stably projection-less simple nuclear C^*-algebra B is the inductive limit of the C^*-algebras $A_n := \mathrm{C}_0((0,1], M_{3^n})$, with *-monomorphisms $\phi_n \colon A_n \to A_{n+1}$ given for $f \in A_n$ and $t \in (0,1]$ by

$$\phi_n(f)(t) := f(t/2) \oplus f(t^2) \oplus f(t^{1/2}) \in M_{3^{n+1}} .$$

The closed ideals J of A_n are given by $J = \mathrm{C}_0(U) \otimes M_{3^n}$, where U is a countable union of pair-wise disjoint intervals (α_n, β_n) $0 \le \alpha < \beta \le 1$ possibly further united with $(\gamma, 1]$ where $\sup_n \beta_n \le \gamma < 1$.

If the ideal $J(f)$ generated by $f \in A_n^+$ contains $\mathrm{C}_0(\alpha, \beta) \otimes M_{3^n}$ for some $0 < \alpha < \beta < 1$, then for each given $\delta > 0$ there exists (sufficiently big) $k \in \mathbb{N}$ such that the element

$$(\phi_{n+k} \circ \cdots \circ \phi_{n+1} \phi_n)(f)$$

generates an ideal of A_{n+k+1} that contains $\mathrm{C}_0((\delta, 1], M_{3^{n+k+1}})$.

Thus, $\mathrm{indlim}_{n\to\infty}(A_n \to A_{n+1})$ is a simple C^*-algebra. It is stably projection-less, because $A_n \otimes \mathbb{K} \cong \mathrm{C}_0((0,1], \mathbb{K})$ does not contain a non-zero projection.

Remark 5.9.29. Suppose that B is stable and σ-unital, and that A is a pi-sun algebra. Let $\psi \colon A \otimes \mathbb{K} \to \mathrm{Q}(B)$ a C^*-morphism with $0 = [\psi] \in \mathrm{Ext}(A, B)$.

This implies that ψ has a *non-degenerate* lift $h \colon A \otimes \mathbb{K} \to \mathcal{M}(B)$ where h is a homomorphism with $\psi = \pi_B \circ h$ and $h(A \otimes \mathbb{K})B = B$.

It follows that $\mathcal{M}(h) \colon \mathcal{M}(A \otimes \mathbb{K}) \to \mathcal{M}(B)$ is unital and injective ?????

It follows that ψ has a *non-degenerate* lift $h\colon A \otimes \mathbb{K} \to \mathcal{M}(B)$, if an only if, the closed ideal $I(\psi(A \otimes \mathbb{K}))$ generated by $\psi(A \otimes \mathbb{K})$ is equal to $\mathrm{Q}(B)$

????????? why this ???????

and ψ dominates zero

(i.e., there exists an isometry $S \in \mathcal{M}(B)$ with $S^* h(A \otimes \mathbb{K}) S \subseteq B$).

The latter condition holds by Proposition 5.5.12, because $B + h(A \otimes \mathbb{K})$ is (obviously) stable if h is non-degenerate.

`Or is s.p.i. needed?` If B is purely infinite, then ψ always dominates zero.

Thus, the question about existence of non-degenerate split morphisms for ψ reduces to the verification of the necessary and sufficient condition $I(\mathrm{Im}(\psi)) = \mathrm{Q}(B)$.

It is always the case if B is, in addition, simple and $\psi \neq 0$.

There is no useful condition for the existence of possible *degenerate* lifts. But the existence of a properly infinite projection $p \in \mathcal{M}(B)$ with $\pi_B(p)\psi(\cdot)\pi_B(p) = \psi$ and $\pi_B(p) \in I(\mathrm{Im}(\psi))$ seems to be sufficient in the case of purely infinite B.

10. The case of purely large extensions

Definition, from [264] :

Let B be a C^*-algebra, and let C be a C^*-algebra containing B as a closed two-sided ideal.

Let us say that C is *purely large* with respect to B if for every element $c \in C \setminus B$, the C^*-algebra cBc^* (the intersection with B of the hereditary C^*-subalgebra of C generated by cc^*) contains a C^*-subalgebra which is stable (i.e., isomorphic to its tensor product with the C^*-algebra $\mathbb{K}$ of compact operators on an infinite-dimensional separable Hilbert space) and is full in B (i.e., not contained in any proper closed two-sided ideal of B).

`EK comment: It implies that` $\mathrm{Ann}(B, C) = \{0\}$ `(annihilator of` B `in` C`)`,i.e., B `is essential in` C.

Criterium:

Let A and B be C^*-algebras, and let

$$0 \to B \to C \to A \to 0$$

be an extension of B by A (i.e., a short exact sequence of C^*-algebras). Let us say that the extension is **purely large** if the C^*-algebra of the extension, C, is purely large with respect to the image of B in it, in the sense described above.

Note that, if B is non-zero, a purely large extension of B by A is essential (that is, the image of B in the C^*-algebra of the extension C is an essential closed two-sided ideal every non-zero closed two-sided ideal has non-zero intersection with it).

`My comment:` B `is ''essential'' in` C `because` $B \subseteq C$ `is ''purely large'' in` C`.`

Lemma (in Section 7 of [264] ?):

Let C be a C^*-algebra that is purely large with respect to a closed two-sided ideal B (of C ?), in the sense of "Section 1".

Then, for any positive element c of C which is not in B, any $\varepsilon > 0$, and any positive element b of B, there exists $b_0 \in B$ with

$$\|b - b_0 c b_0^*\| < \varepsilon .$$

If b is of norm one, and if the image of c in C/B is of norm one, then b_0 may be chosen to have norm one.

Theorem (in Section 6 of [**264**]).
Let A and B be separable C^*algebras, with B stable and A unital.

A unital extension $B \to C \to A$?? Notation ?? is absorbing, "in the nuclear sense", if, and only if, it is "purely large".

Let us say, correspondingly, that an extension *is absorbing in the nuclear sense* if it absorbs every extension which is *trivial in the nuclear sense.*

Again, "let us say" that a unital extension *is absorbing in the nuclear sense* to mean that " this holds within the semigroup of (equivalence classes of) unital extensions". (With triviality in the nuclear sense the existence of a unital weakly nuclear splitting.)

`EK remark:`
`Precise math. definition is not given!?`
`In the case of non-unital` A`, one has to require in addition`
`that` $\phi\colon A \to \mathcal{M}(B)/B$ `''dominates zero'' in sense of Chapter 4,`
`because only then the map` $a + \alpha \cdot 1 \to \phi(a) + \alpha \cdot 1$
`defines again ???? a purely large extension`
`???? for the unitization (in general).`

EK comments (to Lemma in Section 7 of [**264**]):

One can take (if B is σ-unital) $c_n = (1 - e_n^2)^{1/2} c (1 - e_n^2)^{1/2}$, for a suitable quasi-central unit of B and gets elements $b_n \in B$ with $c - \sum_n b_n^* c b_n \in B + 1$.

When the algebra C/B is simple?

When elements of C/B are properly infinite in $\mathrm{Q}(B)$?

My (or of others ???????????) conjecture:

This criterium is satisfied *for all* nuclear separable unital C^*-subalgebras of the stable corona $\mathrm{Q}^{\mathrm{s}}(B)$ of a σ-unital C^*-algebra B, if and only if, $B = \mathbb{C}$ or B is simple and purely infinite.

One could consider $A := \mathrm{C}[0,1]$ and $A := \mathbb{C}$.

If $B = C \otimes \mathbb{K}$ with pi-sun C, the the result holds for all unital separable $A \subset \mathrm{Q}^{\mathrm{s}}(B)$...?

Questions:

Is it always satisfied for all non-zero stable σ-unital B for given separable simply purely infinite A, $A \subset \mathrm{Q}^{\mathrm{s}}(B)$, that is not contained in an ideal of $\mathrm{Q}^{\mathrm{s}}(B)$?

Has $A \subseteq \mathrm{Q}^{\mathrm{s}}(B)$ the property that for each non-zero $a \in A_+$ there exist $d \in \mathrm{Q}^s(B)$ with $d^*ad = 1$?

What really implies the property that $A \hookrightarrow \mathrm{Q}^s(B)$ absorbs all "hyper-nuclear" and (at the same time) "hyper-inner" c.p. contraction maps $V \colon A \to \mathrm{Q}^s(B)$, in the sense that there exists $T \in \mathrm{Q}^s(B)$ with $T^*(\cdot)T = V$.

Let B σ-unital and stable. It seems that the it would be enough to require that every $a \in A_+ \subset \mathrm{Q}(B)$ is properly infinite in $\mathrm{Q}(B)$. (And full??)

Take the stronger assumption that there exists $x \in \mathcal{M}(B)$ with $\pi_B(x)^* a \pi_B(x) = \|a\| \cdots 1$ (for $a \in A_+$?). It is then equivalent to the property that each $a \in A_+$ is infinite properly infinite in B and ???????????????

Let B σ-unital and stable $c \in \mathcal{M}(B)_+$ with $\pi_B(c^2) = a \in A_+$, $x \in \mathcal{M}(B)$ with $x^*cx \in 1 + B$. $e \in B_+$ strictly positive contraction, $e_1, e_2, \ldots$ suitable quasi-central approximate unit in $C^*(e)$ for $C^*(c)$.

Find $d_{n,1}, d_{n,2}$ with $d_{n,k}d_{n,k}^* \in (c-\delta)_+ B (c-\delta)_+$, $d_{n,1}^* d_{n,2} = 0$, $d_{n,k}^* d_{n,k} = e_n$.

Good? For what?

...

Lemma 5.10.1. *("Kirchberg" cited in [**264**]). Let C be a unital separable C^*-algebra and let B be an essential closed two-sided ideal of C, so that we may view C as a unital subalgebra of $\mathcal{M}(B)$:*

$$B \subseteq C \subseteq \mathcal{M}(B); \quad 1 \in C\,.$$

Let $\phi \colon C \to \mathcal{M}(B)$ be a completely positive map which is zero on B, and suppose that, for every $b_0 \in B$, the map

$$b_0^* \phi(\cdot) b_0 \colon C \to B,$$

given by $c \mapsto b_0^* \phi(c) b_0$, can be approximated (on finite sets) by the maps

$$c \mapsto b^* c b, \quad b \in B\,.$$

It follows that there exists $v \in \mathcal{M}(B)$ such that

$$\phi(c) - v^* c v \in B, \; \; c \in C\,.$$

The element v may be chosen so that the map $c \mapsto v^* c v$ also approximates ϕ on a given finite subset of C (up to $\varepsilon > 0$).

Question 5.10.2. Let $C \subset \mathrm{Q}^{\mathrm{s}}(B)$ a separable unital C^*-subalgebra of $\mathrm{Q}^{\mathrm{s}}(B)$ with σ-unital B.

Suppose that C satisfies the following criteria that is equivalent to the absorption criteria of G. Elliott and D. Kucerovsky in [**264**] (compare J. Gabe [**310**] for the non-unital case):

The hereditary C^*-subalgebra $\overline{d^*(B \otimes \mathbb{K})d}$ contains a stable and full hereditary C^*-subalgebra of $B \otimes \mathbb{K}$ for each $d \in \mathcal{M}(B \otimes \mathbb{K})$ with $\pi_{B\otimes\mathbb{K}}(d) \in C \setminus \{0\}$.

I.e., the ideal $B \otimes \mathbb{K}$ is "purely large" in $\pi^{-1}_{B\otimes\mathbb{K}}(C)$.

Does there exist a non-degenerate stable separable purely infinite simple C^*-subalgebra D of $\mathcal{M}(B \otimes \mathbb{K})$ such that C is contained in $\pi_{B\otimes\mathbb{K}}(\mathcal{M}(D))$?

Is this even an equivalent criteria for absorption?

`An answer is not obvious.`

```
In case that C is non-unital then
one has to require here that the inclusion map
map C ↪ Q^s(A) ''dominates zero''
in sense of Definition ??
see [310].
```

The closing question in [**264**]:
As pointed out above, any extension which is trivial in the nuclear sense has a weakly nuclear splitting – i.e., is weakly nuclear. Is every weakly nuclear trivial extension trivial in the nuclear sense?

```
Fit to better place and make it shorter.
It is for the proof of ''Z-absorption implies (CFP)''. Here CFP means
the corona factorisation property.
```

REMARK 5.10.3. Suppose that $P \in \mathcal{M}(B\otimes\mathbb{K})$ is a projection with the property that $\pi_{B\otimes\mathbb{K}}(P)$ is properly infinite and full in $\mathrm{Q}^s(B)$. Then there exists an isometry $T \in \mathcal{M}(B \otimes \mathbb{K})$ with $TT^* = P$.

Indeed: It says equivalently that there exists a contraction $d \in \mathcal{M}(B\otimes\mathbb{K})$ with $X := 1 - d^*Pd \in (B\otimes\mathbb{K})_+$. On the other hand, exists an isometry $R \in \mathcal{M}(B\otimes\mathbb{K})$ with $\|R^*XR\| < 1/2$ for given $X \in B \otimes \mathbb{K}$. Then R^*D^*PDR is a positive operator with $1 \leq 2R^*D^*PDR \leq 2$. Thus, there exists $Q := (R^*D^*PDR)^{-1/2} \in \mathcal{M}(B \otimes \mathbb{K})_+$ that satisfies $\|Q\| \leq \sqrt{2}$ and $1 = S^*S = S^*PS$ for $S := PDRQ$. This shows that P majorizes a range of an isometry and is – therefore – itself properly infinite in $\mathcal{M}(B \otimes \mathbb{K})$. The existence of an infinite repeat endomorphism δ_∞ on $\mathcal{M}(B \otimes \mathbb{K})$ with $t\delta_\infty(q)t^* + s^*qs$ unitarily equivalent to $\delta_\infty(q)$ for all projections $q \in \mathcal{M}(B \otimes \mathbb{K})$ implies that $0 = [P] \in \mathrm{K}_0(\mathcal{M}(B \otimes \mathbb{K})) \cong \{0\}$. Thus there exists an isometry $T \in \mathcal{M}(B \otimes \mathbb{K})$ with $TT^* = P$.

CHAPTER 6

Exact subalgebras of Purely infinite algebras

We give a proof of Theorem A in this chapter and study a special case of Theorem K. The proof of Theorem A is simple and short, but it looks lengthy, because we carry out every detail.

Let us give a short guide to a proof of Theorem A for the impatient readers: We take Glimm's theorem [**324**] (see [**616**, thm. 6.7.3]) to obtain that $\mathcal{O}_2$ contains M_{2^∞} as a quotient of a C^*-subalgebra of $\mathcal{O}_2$ ([1]). Then we combine it with the result of [**438**] that every separable exact C^*-algebra is a quotient of a C^*-subalgebra of the CAR-algebra M_{2^∞}, in order to realise every separable exact C^*-algebra as a quotient of a C^*-subalgebra of $\mathcal{O}_2$. The embedding into $\mathcal{O}_2$ is then obtained from the KK-triviality of $\mathcal{O}_2$ and from Corollary 5.8.12 of our generalized Weyl–von-Neumann Theorem 5.6.2.

Certainly, for the first part of Theorem A, one could use also the natural embedding of M_{2^∞} into $\mathcal{O}_2$ instead of Glimm's theorem. But then one gets not the additional result in Remark 6.2.2, and one has to manage with additional tools that the below considered extension $0 \to \mathcal{O}_2 \otimes \mathbb{K} \to E \to A \to 0$ (with $E \subset \mathcal{O}_2$) becomes essential. But one could here also first menage to find a sub-quotient C^*-algebra of $s_1(s_1)*\mathcal{O}_2 s_1(s_1)* \cong \mathcal{O}_2$ that is isomorphic to the given separable exact C^*-algebra A (... that can be considered as unital (!) ...) ... If the generated hereditary kernel C^*-subalgebra of the constructed extension is not stable, then the construction is ready, because all non-zero hereditary C^*-subalgebras of $\mathcal{O}_2$ are isomorphic to $\mathcal{O}_2$ or are isomorphic to $\mathcal{O}_2 \otimes \mathbb{K}$ by Lemma 6.1.1(ii). Since the extension algebra is contained in the nuclear C^*-algebra $\mathcal{O}_2$, it is an exact C^*-algebra and as an extension it is locally semi-split. This allows to derive that we have only two cases: A is automatically C^*-subalgebra of $\mathcal{O}_2$, or we get an extension B of A by $\mathcal{O}_2 \otimes \mathbb{K}$ that is contained up to isomorphisms in $\mathcal{O}_2$. The KK-triviality and Ext-triviality of $\mathcal{O}_2 \otimes \mathbb{K}$ implies that the extension is split, as shown below ???ref???

Our proof of Theorem K follows a similar route, but the technical ingredients of the proof of Theorem A have to be improved considerably. Therefore we prove in Section 3 the intermediate result Corollary 6.3.2 (of Theorem 6.3.1), where we assume, in addition to the assumptions on B in Theorem K, that

[1] Alternatively to the general theorem of Glimm we could alternatively use for the first part of the proof the very different natural copy of M_{2^∞} inside $\mathcal{O}_2$ that is just the closed linear span of all elements $TS*$ where T and S are all products of the generating isometries s_1, s_2 of the C^*-algebra $\mathcal{O}_2 := C*(s_1, s_2)$ with relations $(s_1)*s_1 = 1$, $(s_2)*s_2 = 1$, $(s_2)*s_1 = 0$ and $s_1(s_1)* + s_2(s_2)* = 1$.

????

B is separable and that $B \otimes \mathcal{O}_2$ contains a regular Abelian C^*-subalgebra (cf. Definition 1.2.9).

```
Decide:
Below transfer from old Chp.  5 more useful?
```

The generalized Weyl–von Neumann Theorem 5.6.2 will be an ingredient of the proofs of Theorems A and of an important special case of *Theorem M ????* in Chapter 6, under the additional assumption that the algebra B in Theorem M has *residual nuclear separation* (cf. Definition 1.2.3), i.e., that the m.o.c. cone $\mathcal{C}_{rn} \subseteq \mathrm{CP}_{\mathrm{nuc}}(B, B)$ is separating for B and its ideals. This extra assumption will be removed finally in Chapter 12 that uses results from Chapters 7-11.

We assume in Chapter 6, – in addition to the assumptions of Theorem M –, that the universal weakly residually nuclear C^*-morphism $H_{rn} \colon B \to \mathcal{M}(B)$ from B into its multiplier algebra $\mathcal{M}(B)$ is non-degenerate for the in Theorem **??** considered stable separable B and that there exists a C^*-morphism $h \colon A \to \mathcal{M}(B)$ that defines the action of $\mathrm{Prim}(B)$ on A. This additional assumptions will be completely removed in Chapter 12 by using results of Chapters 3 and 7 -11 .

Its generalization to weakly residually nuclear maps – and, more generally, to maps in a given m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ – will be used in Chapters 6 and 12 for the proof of Theorem K.

This additional assumptions will be removed both in Chapter 12.

The here given proof of Theorem 6.3.1 gives also an alternative proof of Theorem A that does *not* use [**438**], but requires to generalise some results of [**437**], see Section 19 of Appendix A.

1. Normalizers of hereditary subalgebras (Part 1)

Recall that a *-subalgebra B of a C^*-algebra A is an **essential subalgebra** of A if it has no non-zero left annihilators in A, i.e., if $aB = \{0\}$ implies $a = 0$ for all a in A. E.g., always a dense *-subalgebra B of a C^*-algebra C is an essential subalgebra of the multiplier $A := \mathcal{M}(C)$ of C.

```
Next red:  Is here a good place?
```

Our Proposition 6.2.1 implies that the classification of (the positions in $\mathcal{O}_2$ of) the essential hereditary C^*-subalgebras D of $\mathcal{O}_2$ – together with a certain semi-splitting property – would in particular create invariants for the classification of all nuclear separable unital C^*-algebras. Unfortunately, up to now there is no method to link the points of this "classifying space" with known invariants of K-theoretic nature.

LEMMA 6.1.1. *Suppose that $D \subseteq J \subseteq E \subseteq B$ are C^*-subalgebras of a C^*-algebra B, and that J is an ideal of E.*

(i) *If D is essential in B and $J \neq E$, then J is an essential ideal of E and is an essential hereditary C^*-subalgebra of B.*

In particular, the C^*-algebra J can not have a unit element.

(ii) *If $D \subset \mathcal{O}_2$ is an essential hereditary C^*-subalgebra of $\mathcal{O}_2$ and $D \neq \mathcal{O}_2$, then D is isomorphic to $\mathcal{O}_2 \otimes \mathbb{K}$.*

PROOF. (i): Every annihilator of J in $B \supset E$ is in an annihilator of D in B. But the annihilators of D in B are zero.

Suppose J has a unit element $e \in J$, i.e., $e = e^*e$ and $ex = xe = x$ for all $x \in J$. The elements of $(1-e)B(1-e)$ are two-sided annihilators of J. Thus $\|(1-e)b^*b(1-e)\| = \|b - be\|^2 = 0$ for every $b \in B$, and e is the unit of B. Since J is an ideal of E, this contradicts our assumption $J \neq E$.

(ii): All this is given by the equivalence of its Parts (ix) and (i) of Proposition 2.2.1, if we use that $\mathcal{O}_2$ is simple and purely infinite in the sense of J. Cuntz.

An alternative proof is the following: Let $E := D + \mathbb{C} \cdot 1 \subset \mathcal{O}_2$. If D is an essential hereditary C^*-subalgebra of $\mathcal{O}_2$ and $D \neq \mathcal{O}_2$, then $D \neq E$ and D has not a unit element.

The non-zero hereditary C^*-subalgebra D of $\mathcal{O}_2$, is simple, purely infinite and separable. It follows that D is either unital or stable by Corollary 5.5.4 (Zhang dichotomy). Thus, $D \cong D \otimes \mathbb{K} \cong \mathcal{O}_2 \otimes \mathbb{K}$ by Corollary 5.5.6 (Brown stable isomorphism theorem, [**107**]). □

Some of the, in Remarks 6.1.2 mentioned, properties are also at other places discussed. Check this!!!

REMARKS 6.1.2. The study of quotients of C^*-subalgebras of a given C^*-algebra C leads to the study of hereditary C^*-subalgebras D of C and of its **normalizer algebra** $\mathcal{N}(D) \subset C$, which is defined as $\mathcal{N}(D) := \{b \in C : bD + Db \subset D\}$ (and is usually very different from the multiplier algebra $\mathcal{M}(D)$ of D).

(1) The set of products CD is a *closed* left ideal of C and the sum $CD + DC$ is a closed linear subspace of C ([2]). We define $C//D := C/(CD + DC)$ (considered as an operator subspace of the W*-algebra C^{**}, see below).

We consider $\mathcal{N}(D)/D$, its natural maps into $C//D$ and into $\mathcal{M}(D)/D$, and the behavior of this maps on the system of closed ideals of C. They lead to the guiding principles for all proofs in this chapter (except some additional arguments on approximation and inductive limits in the Ψ-residually nuclear case).

(2) A hereditary C^*-subalgebra D of C is a closed ideal of $\mathcal{N}(D)$ by definition of $\mathcal{N}(D)$, and, therefore, there is a natural C^*-morphism $\rho\colon \mathcal{N}(D) \to \mathcal{M}(D)$ from

[2] The closed left ideal $L := \overline{\text{span}(CD)}$ is identical with the set of products CD by Cohen factorization theorem, – or simply observe that every element a of a closed left ideal L can be expressed as $a = cd$ for $c \in C$ suitable and $d := (a^*a)^{1/4}$ in $D = L^* \cap L$ –. The set $CD + DC$ is a closed linear subspace of C, because it is the sum of a closed left ideal L and a closed right ideal R, and because $C^{**}p + pC^{**}$ is $\sigma(C^{**}, C^*)$-closed in C^{**}.

$\mathcal{N}(D)$ into the multiplier algebra $\mathcal{M}(D)$ of D. Clearly, the kernel of $\rho\colon \mathcal{N}(D) \to \mathcal{M}(D)$ is the (two-sided) **annihilator** $\mathrm{Ann}(D) := \{c \in C\colon cD + Dc = 0\} \subset \mathcal{N}(D)$ of D in C. From the definition, one sees that $\mathrm{Ann}(D)$ is a closed ideal of $\mathcal{N}(D)$, and that the *-epimorphism $\mathcal{N}(D) \to \mathcal{N}(D)/D$ is faithful on $\mathrm{Ann}(D)$.

(3) A hereditary C^*-subalgebra $D \subset C$ is *essential* in C, if and only if, $\mathrm{Ann}(D) = \{0\}$. Thus, the natural *-morphism $\mathcal{N}(D) \to \mathcal{M}(D)$ is *faithful*, if and only if, D is *essential* in C. The natural morphism $\mathcal{N}(D) \to \mathcal{M}(D)$ is *not necessarily an epimorphism.* Different essential hereditary C^*-subalgebras D and D' have in general non-isomorphic $\mathcal{N}(D)/D$ and $\mathcal{N}(D')/D'$, even if D and D' are isomorphic.

The corona construction $\mathrm{Q}(D) := \mathcal{M}(D)/D$ defines an isomorphism invariant for all algebras D.

(4) Let $F \subset C$ a C^*-subalgebra and J a closed ideal of F, then the set $D := JCJ$ is a hereditary C^*-subalgebra of C such that $F \subset \mathcal{N}(D)$ and $F \cap D = J$, because an approximate unit of J is also an approximate unit of D. Therefore, there is a natural *-monomorphism from F/J into $\mathcal{N}(D)/D$ and the unit of the weak closure of J in the second conjugate C^{**} of C is also the unit of the weak closure of D. Moreover, $F + D$ is closed, because $FD + DF \subset D$, cf. [**767**, Vol. I, chap. I,sec. 8, exercise 2].

(5) Let D be a hereditary C^*-subalgebra of C. Then the unit element p_D of the weak closure of D in C^{**} is called the **support projection** of D. It is a so-called **open** projection, and $q_D := 1 - p_D$ is a **closed** projection.

(6) There are well-known one-to-one relationships between hereditary C^*-subalgebras, open projections, closed left ideals, weakly closed faces of the state space of C, weakly closed right invariant subspaces of C^*, and the (with respect to C) closed projections in C^{**}.
Indeed, let $S(C)$ denotes the state space of C, and let $X \mapsto X^o \mapsto X^{oo}$ the passage to the polar and bipolar of a set, then the one-to-one relations are given by the maps $D \mapsto L := CD$, $L \mapsto D = L^* \cap L$, $D \mapsto K_D = D^o \cap S(C)$, $L \mapsto L^o \mapsto X := L^o \cap S(C) = D^o$, $D \mapsto p_D$, $p \mapsto L_p := C \cap (C^{**}p)$, $K \mapsto L_K := \{a \in C\colon f(a^*a) = 0 \quad \forall f \in K\}$, $p \mapsto q = 1 - p$, $q \mapsto C^*q$, $X \mapsto K_X = S(C) \cap X$, $q \mapsto D_q := \{a \in C\colon q(a^*a + aa^*)q = 0\}$.

We refer the reader to [**767**, Vol. I, chap. III, def. 6.19, cor. 6.20], and to [**616**, prop. 3.11.9, thms. 3.10.7, 3.11.10]. (For our particular needs, the reader can simplify the there given proofs if he uses instead 2×2-matrix arguments.)

(7) It is useful to consider also the quotient $X_D := C//D := C/(CD+DC)$ of C by $CD+DC$ (which is a closed linear subspace of C), and let $\pi_{CD+DC}\colon C \to C//D$ denote the natural epimorphism. The space X_D with its natural matrix order structure and matrix norms ([3]) becomes a (special kind of) ***C*-system*** (defined

[3]It is given on the vector space $M_n(C/(CD + DC))$ by the natural isomorphism with the quotient Banach space norms $M_n(C) \to M_n(C)/M_n(CD + DC)$.

more generally in [**437**]). If C is unital, then the C^*-system X_D is unital with matrix order unit $\pi_{CD+DC}(1)$. The bi-dual operator system of X_D is nothing else $q_D C^{**} q_D$ (under natural identification given by the restriction of π^{**}_{CD+DC} to $q_D C^{**} q_D$), cf. [**437**]. The unital C^*-systems are non-commutative generalizations of the spaces of continuous affine functions on a Choquet simplex, cf. [**437**], [**472**].

The natural isomorphism $q_D C^{**} q_D \to X_D^{**}$ defines a completely isometric isomorphism $\lambda \colon q_D c q_D \to c + (CD + DC)$ from the subspace $q_D C q_D$ of C^{**} onto X_D, Therefore, the kernel of the completely positive map $a \in C \mapsto q_D a q_D$ is just $CD + DC$.

It is easy to see that $\mathcal{N}(D) = \{q_D\}' \cap C$, and therefore, that there is a natural unital C^*-morphism $\mathcal{N}(D) \to \mathcal{N}(D) q_D \subset q_D C q_D$. By definition of $p_D = 1 - q_D$, the kernel of this *-epimorphism onto $q_D \mathcal{N}(D)$ is just D. We get a natural *-isomorphism $\tau(b + D) := b q_D$ from $\mathcal{N}(D)/D$ onto $q_D \mathcal{N}(D) \subset q_D C q_D$, and we have for $a \in \mathcal{N}(D)$:

$$\operatorname{dist}(a, C^{**} p_D + p_D C^{**}) = \operatorname{dist}(a, CD + DC) = \|a + D\| = \|q_D a q_D\| .$$

(8) We use for the proof of Proposition 6.2.1 results from the paper [**438**] which rely on the following (non-trivial) fact [**437**, thm. 1.4(iii)] on normalizer algebras: *If C is unital, then the natural unital C^*-morphism*

$$\pi'_D \colon \mathcal{N}(D) \ni a \mapsto a q_D = q_D a q_D \in q_D C^{**} q_D$$

defines a completely isometric and completely positive isomorphism from $\mathcal{N}(D)/D$ onto *the (two-sided) multiplier algebra $\mathcal{M}(X_D) \subset X_D$ of X_D in $X_D^{**} \cong q_D C^{**} q_D$* (the latter naturally identified by the map λ from (7)).

(9) An obvious consequence of (8) is the following:
Suppose that F is a C^*-subalgebra of $\mathcal{N}(D) \subset \{p_D\}' \cap C = \{q_D\}' \cap C$ and that the image $F q_D$ of the map $c \to q_D c q_D$ is $\sigma(C^{**}, C^*)$-dense in the image $q_D C q_D$ of C (which happens e.g. if $q_D C q_D = q_D F$). Then, $\pi_{CD+DC}(F) = C/(CD + DC)$, because it is a C^*-subalgebra of $\mathcal{M}(C/(CD+DC))$). In particular $F + D = \mathcal{N}(D)$ and $C = \mathcal{N}(D) + DC + CD$.
Notice that, conversely, $q_D C q_D = q_D \mathcal{N}(D)$ if $C = \mathcal{N}(D) + CD + DC$.
From the above collected results, we get that the following properties (i)-(v) of hereditary C^*-subalgebras D of a unital C^*-algebra C are equivalent:

(i) The C^*-system $C//D$ is unitally and isometrically isomorphic to a C^*-algebra.
(ii) $\mathcal{M}(C//D) = C//D$ (i.e., $C//D$ is a C^*-subalgebra of $q_D C^{**} q_D$).
(iii) $C = \mathcal{N}(D) + CD + DC$.
(iv) There is a C^*-algebra A and a completely positive contraction $T \colon C \to A$ such that $T(D) = \{0\}$, $T|\mathcal{N}(D)$ is a *-epimorphism from $\mathcal{N}(D)$ onto A with kernel D, and that there is a projection $q \in C^{**}$ and an W*-isomorphism S from $q C^{**} q$ onto A^{**} with $T^{**}(c) = S(qcq)$ for all $c \in C$.

(v) There is a completely positive map $T\colon C \to \mathcal{N}(D)/D$ with $T|\mathcal{N}(D) = \pi_D$ and $\ker(T) = CD + DC$.

(10) Let $T\colon C \to G$ be a unital completely positive map. We define the **multiplicative domain** $M_T \subset C$ of T as the set of $d \in C$ with $T(db) = T(d)T(b)$ and $T(bd) = T(b)T(d)$ for $b \in C$. M.D. Choi's generalized Kadison inequality $T(a)^*T(a) \le T(a^*a)$ shows that $d \in M_T$, if and only if, $T(d^*d) = T(d)^*T(d)$ and $T(dd^*) = T(d)T(d)^*$. Unfortunately, the multiplicative domain (resp. the kernel) of the unital completely positive map $T^{**}\colon C^{**} \to G^{**}$ is not necessarily contained in the weak closure of the multiplicative domain (resp. the kernel) of T, cf. Part (11).

The multiplicative domain of the unital completely positive map

$$T_D\colon a \in C \mapsto q_D a q_D \in q_D C^{**} q_D$$

is just $\mathcal{N}(D) = \{q_D\}' \cap C$ and $\mathcal{N}(D)^{**} \neq \{q_D\}' \cap C^{**}$.

(11) The above remarks on the quotient spaces $X_D = C//D$ show that the second conjugate $(\pi_{CD+DC})^{**}$ of $\pi_{CD+DC}\colon x \in C \mapsto x + (CD + DC) \in C//D$ is just the normalization $a \in C^{**} \mapsto q_D a q_D$ of T_D in Part (10). The multiplicative domain $p_D C^{**} p_D + q_D C^{**} q_D$ of $(\pi_{CD+DC})^{**}$ is in general much bigger than the weak closure of $\mathcal{N}(D)$ in C^{**}. But the *kernel* of $(\pi_{CD+DC})^{**}$ is just $p_D C^{**} + C^{**} p_D$, which is the weak closure of the kernel $DC + CD$ of T_D and of π_D.

(12) Let $X := X_D := C//D \subset q_D C^{**} q_D$, and let $\eta_1\colon X \to X^{**} \cong q_D C^{**} q_D$, $\eta_2\colon X^* \to X^{***}$ and $\eta_3\colon X^{**} \to X^{****}$ the natural inclusions. Then $(\eta_1)^{**}\colon X^{**} \to X^{****}$ is a normal unital completely isometric map, $(\eta_2)^*\colon X^{****} \to X^{**}$ is a normal *-epimorphism, and η_3 is a (non-normal) *-monomorphism. The map $E := (\eta_1)^{**} \circ (\eta_2)^*$ is a normal u.c.p. map on the fourth conjugate W*-algebra of X with $E^2 = E$.

We say that $y \in X^{**} \cong q_D C^{**} q_D$ is a (two-sided) *multiplier* of X if $yx, xy \in X$ for all $x \in X$, where the multiplication is given by the W*-algebra structure on X^{**}. Let $\mathcal{M}(X) \subset X^{**}$ denote the set of all multipliers of X. Then, clearly, $\mathcal{M}(X)$ is a unital C^*-subalgebra of X^{**}, and is contained in X, because $q_D = 1_{X^{**}} \in X$.

One can see by a separation argument, that $y \in \mathcal{M}(X)$, if and only if, $\eta_3(y)$ is in the multiplicative domain of E (see [**437**] for details). It follows from Remark (10) that $y \in \mathcal{M}(X)$, if and only if, y, y^*y and yy^* are all in X.

(13) By [**437**, cor. 1.5], $a \in \mathcal{N}(D) \mapsto aq_D \in q_D C q_D \cong C//D$ is a *-epimorphism from $\mathcal{N}(D)$ onto $\mathcal{M}(C//D)$ with kernel equal to D. It implies together with Remark (12) the following observation:

Let $x \in q_D C^{**} q_D$. There is $a \in \mathcal{N}(D)$ with $aq_D = x$ if and only if, $x, x^*x, xx^* \in C//D$, if and only if, $x \in \mathcal{M}(C//D)$.

(14) If C is *not unital*, then we can pass to the unitization $\widetilde{C}$ of C, and get immediately the following reformulation of (13):

Let $x \in q_D C^{**} q_D$. There is $a \in \mathcal{N}(D)$ with $aq_D = x$ if and only if, $x, x^*x, xx^* \in C//D$, if and only if, $x \in C//D \cap \mathcal{M}(C//D)$.

In particular, $\mathcal{N}(D)/D \cong (C//D) \cap \mathcal{M}(C//D)$ via identifications of $a + D$, ap_D and $a + (DC + CD)$ for $a \in \mathcal{N}(D)$.

Lemma 6.1.3. *Suppose that $E \subset B$ is a C^*-subalgebra of a* nuclear *C^*-algebra B, D is a σ-unital hereditary C^*-subalgebra of B, $D \subset E$, and that D is an ideal of E.*

Then every completely positive contraction φ from a separable *unital C^*-algebra A into E/D has a completely positive and contractive lifting $V: A \to E$, i.e., $\pi_D \circ V = \varphi$.*

If, in addition, B and A are unital, $1_B \in E$, and $\varphi: A \to E/D$ is unital, then a unital *nuclear c.p. map $V: A \to E$ with $\pi_D \circ V = \varphi$ exists.*

Notice, that a lift $V: A \to E$ of an isomorphism $\varphi: A \to E/D$ from A onto E/D can't be nuclear if A is not a nuclear C^*-algebra. But $V: A \to B \supset E$ is a nuclear c.p. map from A into B, because B is nuclear.

Proof. If A is separable and unital, B is unital with $1_B \in E$ and $\varphi: A \to E/D$ is unital, then we can replace A by the separable unital C^*-subalgebra F of E/D generated by $\varphi(A)$, φ by the identity map of this new algebra F, and finally E by $G := \pi_D^{-1}(F)$. The extension given by the exact sequence

$$0 \to D \to G \to F \to 0$$

has a splitting unital c.p. map $T: F \to E \subseteq B$ for π_D by [**238**, thm. B, prop. 4.3]. It uses that D is nuclear as a hereditary C^*-subalgebra of the nuclear algebra B, i.e., D fulfills assumption (1) of [**238**, thm.B]. The C^*-subalgebra $G \subseteq B$ inherits property (C) from the nuclear B by [**238**, prop. 5.3] (citing [**35**, thms. 3.2, 3.3]).

Property (C″) – which is equivalent to local reflexivity in the operator space sense –, follows from property (C) and allows to verify the assumption (2) of [**238**, thm.B], with help of [**238**, prop. 5.3(3)] and [**238**, prop. 5.5].

(Notice that the properties (C) and (C′) both are equivalent to exactness by [**438**], because every separable exact C^*-algebras is a quotient of some C^*-subalgebras of the CAR algebra M_{2^∞}.)

Check next again: Unit OK?

If one of B, E, A or φ are not unital, or if $1_B \notin E$, then we can adjoin to all algebras (except D) an *outer* unit 1, and extend φ to a unital c.p. map

$$\widetilde{\varphi}: \widetilde{A} \to \widetilde{E}/D \cong \widetilde{E/D},$$

by $\widetilde{\varphi}(a+t1) := \varphi(a)+t1$. There is a unital u.c.p. map $W: \widetilde{A} \to \widetilde{E}$ with $\pi_D \circ W = \widetilde{\varphi}$. Then $W(A) \subset E$ and the map $V := W|A$ is a completely positive contraction from A into E with $\pi_D \circ V = \varphi$. □

Remark 6.1.4. Other methods in Section 3 that are different from those of [**238**] and are more near to the study of the convex cone of c.p. maps $V: A \to E$

like methods of Arveson in the proof of [**43**, thm. 6]. The results are more general lifting results than that stated in Lemma 6.1.3, e.g.

Suppose that B is separable and that $B//D$ is a *nuclear C^*-space* (i.e., that $q_D B^{**} q_D$ is an injective von-Neumann algebra), that D is an *essential* hereditary C^*-subalgebra of B and that $D \subseteq E \subseteq \mathcal{N}(D) \subseteq B$.

Then for every separable C^*-algebra A and every c.p. contraction $\varphi\colon A \to E/D$ there exist a c.p. contraction $V\colon A \to E$ with $\pi_D \circ V = \varphi$ such that $V\colon A \to B$ is nuclear (but is not necessarily nuclear as a map from A to E).

If A and B are unital, $1_B \in E$ and φ is unital, then one can manage that V is unital.

(In fact, the reduction to [**43**, thm. 6] is done be showing that there exists a (not necessarily unique) c.p. contraction $T\colon B//D := B/(BD + DB) \to \mathcal{M}(D)/D$ such that $T|\mathcal{N}(D)/D$ is the natural C^*-morphism from $\mathcal{N}(D)/D$ into $B//D$.)

LEMMA 6.1.5. *Suppose that F is a C^*-subalgebra of a C^*-algebra C, and that $q \in F' \cap C^{**}$ is a projection that is closed in C^{**} and satisfies $Fq = qCq$.*

Let D denote the hereditary C^*-subalgebra of C with open support projection $1 - q$. Then:

(i) *$\mathcal{N}(D) = D + F$.*
(ii) *$T_q\colon c \in C \mapsto qcq \in C^{**}$ is a c.p. contraction and has kernel equal to $DC + CD$.*
(iii) *The multiplicative domain of T_q is given by $\{c \in C\,;\; cq = qc\}$, which is the same as $\mathcal{N}(D)$.*
(iv) *There are natural isomorphisms*

$$Fq \cong F/(F \cap D) \cong \mathcal{N}(D)/D \cong C/(DC + CD)$$

that are given by

$$[T_q]\colon \pi_{DC+CD}(c) \mapsto qcq \in Fq = \mathcal{N}(D)q \cong \mathcal{N}(D)/D\,.$$

(v) *$C = \mathcal{N}(D) + DC + CD$.*

PROOF. `to be filled in` ?? □

LEMMA 6.1.6. *Suppose that $D_1 \subset C$ is a hereditary C^*-subalgebra of a C^*-algebra C such that $C = \mathcal{N}(D_1) + D_1 C + C D_1$. Let $\rho\colon \mathcal{N}(D_1) \to B$ a *-epimorphism from $\mathcal{N}(D_1)$ onto a C^*-algebra B with kernel equal to D_1, and let $D_2 \subset B$ a hereditary C^*-subalgebra of B.*

Denote by J the hereditary C^*-subalgebra $J := \rho^{-1}(D_2)$ of $\mathcal{N}(D_1)$, and define the hereditary C^*-subalgebra $D \subset C$ by $D := JCJ$.

(i) *$D_1 \subset D$, and $\mathcal{N}(D) = \rho^{-1}(\mathcal{N}(D_2)) + D$.*
(ii) *There is a unique positive contraction $T\colon C \to B$ that extends ρ. T is completely positive, has kernel $\ker(T) = D_1 C + C D_1$ and multiplicative domain $\mathcal{N}(D_1)$.*

(iii) $T(D) = D_2$, $T(DC + CD) = D_2B + BD_2$ *and* $T(\mathcal{N}(D)) = \mathcal{N}(D_2)$.

(iv) *The c.p. map* $T|\mathcal{N}(D)$ *defines an isomorphism of* C^**-algebras*

$$\eta\colon \mathcal{N}(D)/D \mapsto \mathcal{N}(D_2)/D_2\,,$$

with $\eta(\pi_D(c)) = \pi_{D_1}(T_1(c))$ *for* $c \in \mathcal{N}(D)$.
In particular, $\eta(c + D) = \rho(c) + D_1$ *for* $c \in \rho^{-1}(\mathcal{N}(D_2))$.

(v) *The class-map* $[T]_{(DC+CD)}$ *is a completely positive and completely isometric isomorphism from the* C^**-system* $C//D$ *onto* $B//D_2$.

PROOF. to be filled in ??

Let $F \subset \mathcal{N}(D_1) \subset C$ denote the inverse image $\rho^{-1}(\mathcal{N}(D_2)) =: F$ of $\mathcal{N}(D_2)$ under the epimorphism ρ from $\mathcal{N}(D_1)$ onto B.

The inverse image $J := \rho^{-1}(D_2)$ of D_2 is the kernel of the epimorphism $\pi_{D_2} \circ \rho\colon F \to \mathcal{N}(D_2)/D_2$. In particular, J is an ideal of F, J contains the kernel D_1 of ρ, $\rho(J) = D_2$. Thus, there is an isomorphism $\mu\colon F/J \to \mathcal{N}(D_2)/D_2$ with $\mu \circ \pi_J = \pi_{D_2} \circ \rho$.

Now we consider the hereditary C^*-subalgebra $D := JCJ = \overline{JCJ}$ of C. Then D is a hereditary C^*-subalgebra of C, and $D_1 \subset D$. Since the ideal J of F contains an approximate unit of D, we get that the hereditary C^*-subalgebra $F \cap D$ of F must be equal to the ideal J of F, and that $F \subset \mathcal{N}(D)$. In particular, F/J is naturally isomorphic to a C^*-subalgebra of $\mathcal{N}(D)/D$. The above defined natural unital monomorphism $F/J \to \mathcal{N}(D)/D$ and the isomorphism $\mu^{-1}\colon \mathcal{N}(D_2)/D_2 \xrightarrow{\sim} F/J$ define together a unital *-monomorphism

$$\xi\colon \mathcal{N}(D_2)/D_2 \hookrightarrow \mathcal{N}(D)/D\,,$$

such that $\xi(\pi_{D_2}(\rho(a)) = \pi_D(a)$ for $a \in F$. □

2. The Proof of Theorem A

PROPOSITION 6.2.1. *Suppose that* A *is a unital separable exact* C^**-algebra. Then there exist a unital* C^**-subalgebra* E *of* $\mathcal{O}_2$*, a closed ideal* D *of* E*, and a* **-isomorphism* φ *from* A *onto* E/D*, such that*

(i) D *is an* essential *hereditary* C^**-subalgebra of* $\mathcal{O}_2$*;*

(ii) φ *has a unital completely positive lifting* $V\colon A \to E$*, i.e.,* $\varphi = \pi_D \circ V$*,*

(iii) $D \cong \mathcal{O}_2 \otimes \mathbb{K}$.

If A *is nuclear, then the subalgebras* $D \subset E \subset \mathcal{O}_2$*, and the isomorphism* $\varphi\colon A \xrightarrow{\sim} E/D$ *can be chosen such that the epimorphism* $\varphi^{-1} \circ \pi_D\colon E \to A$ *extends to a unital completely positive map* $T\colon \mathcal{O}_2 \to A$ *with kernel* $D \cdot \mathcal{O}_2 + \mathcal{O}_2 \cdot D$.

PROOF. Let B denote the CAR-algebra M_{2^∞} and let $C := \mathcal{O}_2$. We define $D \subset E \subset C$ and $\varphi\colon A \xrightarrow{\sim} E/D$ and show later that they satisfy (i)–(iii).

First we consider the case of an *exact* separable A that is not necessarily nuclear:

Since C is not a C^*-algebra of type I, we can apply the improved variant [**616**, thm. 6.7.3] of Glimm's theorem: There exist a unital subalgebra F_1 of C and a *closed* projection q_1 in C^{**} such that q_1 commutes with F_1, $q_1 C q_1 = q_1 F_1$ and that the C^*-subalgebra $q_1 C q_1$ of $q_1 C^{**} q_1$ is isomorphic to the CAR-algebra $B := M_{2^\infty}$.

Let $p_1 := 1 - q_1$ the open complement of q_1, and let D_1 denote the hereditary C^*-subalgebra of B with open support projection $p_1 = p_{D_1}$, cf. Remark 6.1.2(5), i.e., $(D_1)_+ = \{c \in C_+ ;\, q_1 c q_1 = 0\}$. Furthermore, let $h_1 \colon q_1 C q_1 \xrightarrow{\sim} B$ a *-isomorphism from $q_1 C q_1 \cong q_1 F_1$ onto B, and define $T_1 \colon C \to B$ by $T_1(a) := h_1(q_1 a q_1)$.

By Lemma 6.1.5, T_1 is a unital completely positive map, and the kernel of T_1 is $CD_1 + D_1 C$. Moreover, $\mathcal{N}(D_1) = F_1 + D_1$, and $\mathcal{N}(D_1)$ is just the multiplicative domain of T_1. It implies that $C = \mathcal{N}(D_1) + CD_1 + D_1 C$.

Let $\rho := T_1 | \mathcal{N}(D_1)$. Then $\rho \colon \mathcal{N}(D_1) \to B$ is a *-epimorphism from $\mathcal{N}(D_1)$ onto B with kernel $D_1 = \mathcal{N}(D_1) \cap (D_1 C + CD_1)$.

By [**438**, cor. 1.3, 1.4, thm. 4.1], for every separable *unital* exact C^*-algebra A, there exists a hereditary C^*-subalgebra $D_2 \subset B$ and a unital *-monomorphism $\psi \colon A \to \mathcal{N}(D_2)/D_2$.

Since D_2 is an ideal of $\mathcal{N}(D_2)$, the inverse image $J := \rho^{-1}(D_2)$ is an ideal of the C^*-subalgebra $\rho^{-1}(\mathcal{N}(D_2))$ of $\mathcal{N}(D_1)$. We let $D := JCJ$.

The equation $C = \mathcal{N}(D_1) + CD_1 + D_1 C$ and the Lemma 6.1.6 on iterated normalizers show that that $T_1 \colon C \to B$ is the unique completely positive extension of the epimorphism $\rho \colon \mathcal{N}(D_1) \to B$, that $T_1(D) = D_2$, that $T_1(\mathcal{N}(D)) = \mathcal{N}(D_2)$ and that $T_1 | \mathcal{N}(D)$ defines a C^*-isomorphism η from $\mathcal{N}(D)/D$ onto $\mathcal{N}(D_2)/D_2$ with $\eta(\pi_D(c)) = \pi_{D_1}(T_1(c))$ for $c \in \mathcal{N}(D)$.

We define a unital *-monomorphism $\varphi \colon A \to \mathcal{N}(D)/D$ by $\varphi := \eta^{-1} \circ \psi$ and let

$$E := \pi_D^{-1}(\varphi(A)) \subset \mathcal{N}(D) \subset \mathcal{O}_2$$

be the inverse image of $\varphi(A)$ under the quotient map $\pi_D \colon \mathcal{N}(D) \to \mathcal{N}(D)/D$.

Thus, we get *a hereditary C^*-subalgebra D of $C := \mathcal{O}_2$, a C^*-subalgebra $E \subset \mathcal{N}(D) \subset \mathcal{O}_2$ with $D_1 \subset D \subset E \subset \mathcal{N}(D)$, and a unital *-isomorphism $\varphi \colon A \xrightarrow{\sim} E/D$.*

(i,iii): We use Lemma 6.1.1: Since $D_1 \subset D$ and $D \neq \mathcal{O}_2$, it suffices to show that D_1 is essential in $\mathcal{O}_2$, i.e., that $\mathrm{Ann}(D_1) = \{0\}$. Recall, that $\mathrm{Ann}(D_1)$ is a closed ideal of $\mathcal{N}(D_1)$ and is a hereditary C^*-subalgebra of $C := \mathcal{O}_2$. The epimorphism $\rho \colon \mathcal{N}(D_1) \to B := M_{2^\infty}$ has kernel D_1. Thus, $\rho | \mathrm{Ann}(D_1)$ is faithful, and $\rho(\mathrm{Ann}(D_1))$ is a closed ideal of M_{2^∞}. Since every non-zero hereditary C^*-subalgebra of $\mathcal{O}_2$ is stably infinite and since M_{2^∞} is stably finite, we get $\rho(\mathrm{Ann}(D_1)) = \{0\}$ and $\mathrm{Ann}(D_1) = \{0\}$.

(ii): By Lemma 6.1.3, there is a u.c.p. map $V \colon A \to E$ with $\pi_D \circ V = \varphi$.

This completes the proof for general exact A.

We consider now the special *case of nuclear* A:
If A is *nuclear*, then [**438**, cor. 1.5] says that the hereditary C^*-subalgebra $D_2 \subset B := M_{2^\infty}$ and $\psi\colon A \to \mathcal{N}(D_2)/D_2$ can be chosen such that ψ is an isomorphism from A onto $\mathcal{N}(D_2)/D_2$ and, moreover, that $B = \mathcal{N}(D_2) + D_2B + BD_2$, i.e., $\psi(A) = \mathcal{N}(D_2)/D_2 \cong B//D_2$.

As in the case of (not-necessarily nuclear) exact A we define $D \subset E \subset C$, and $\varphi\colon A \to E/D$:
Let $D := JCJ$ for $J := \rho^{-1}(D_2) \supset D_1$ with $\rho := T_1|\mathcal{N}(D_1)$ (defined as in the case of exact A), and use the isomorphism $\eta\colon \mathcal{N}(D)/D \to \mathcal{N}(D_2)/D_2$ to define $\varphi := \eta^{-1} \circ \psi$. Then $\varphi\colon A \to \mathcal{N}(D)/D$ is an isomorphism of A onto $\mathcal{N}(D)/D$. Thus $E := \pi_D^{-1}(\varphi(A)) = \mathcal{N}(D)$.

We apply Lemma 6.1.6 to $(B,\ D_2,\ \psi^{-1} \circ \pi_{D_2},\ A,\ 0)$, in place of the there considered general system $(C,\ D_1,\ \rho,\ B,\ D_2)$, and get the unique u.c.p. extension $T_2\colon B \to A$ of the *-epimorphism

$$\rho_1 := \psi^{-1} \circ \pi_{D_2}\colon \mathcal{N}(D_2) \to A\,,$$

and T_2 has kernel $D_2B + BD_2$ and multiplicative domain $\mathcal{N}(D_2)$.

Since $C = \mathcal{N}(D_1) + CD_1 + D_1C$, we get from Lemma 6.1.6, that

$$T_1(DC + CD) = D_2B + BD_2\,.$$

It implies that the u.c.p. map $T := T_2 \circ T_1\colon C \to A$ has kernel $DC + CD$.

Let $c \in \mathcal{N}(D) \subset C$. Then $V_1(c) \in \mathcal{N}(D_2)$ and $\pi_{D_2}(V_1(c)) = \eta(\pi_D(c))$ by Lemma 6.1.6. Thus,

$$V(c) = \rho_1(V_1(c)) = \varphi^{-1}(\pi_D(c)) \qquad \text{for} \quad c \in \mathcal{N}(D)\,.$$

Thus, the u.c.p. map $V\colon C \to A$ extends the *-epimorphism from $E = \mathcal{N}(D)$ onto A, and is given by $\varphi^{-1} \circ \pi_D$. □

We have seen so far, that there exists, for every separable unital exact C^*-algebra A, an essential semi-split exact sequence

$$0 \to D \to E \to A \to 0$$

where E is a unital C^*-subalgebra of $\mathcal{O}_2$, and where $D \cong \mathcal{O}_2 \otimes \mathbb{K}$ is essential in $\mathcal{O}_2$. This extension splits unital by Corollary 5.8.12. The unital splitting C^*-morphism is a monomorphism from A onto a unital subalgebra of $E \subset \mathcal{O}_2$.

The desired conditional expectation onto the image of A in $\mathcal{O}_2$ is then just the composition of $T\colon \mathcal{O}_2 \to A$ with the splitting C^*-morphism. More details are given below in Remark 6.2.2.

Proof of Theorem A.. Ad(i): Let A be a separable exact C^*-algebra. If A is not unital then the unitization of A is again exact, as our $A \otimes (\cdot)$ -exactness definition of exactness of A in Chapter 3 immediately implies (e.g. use the 3×3-lemma of category theory). Thus we can restrict our considerations to the case of *unital* exact C^*-algebras A.

The unital semi-split essential extension

$$0 \longrightarrow D \longrightarrow E \xrightarrow{\lambda} A \longrightarrow 0$$

with $\lambda := \varphi^{-1} \circ \pi_D$ from Proposition 6.2.1 has a unital split morphism by Corollary 5.8.12, because $D \cong \mathcal{O}_2 \otimes \mathbb{K}$ and E is exact.

The unital split C^*-morphism $\psi \colon A \to E \subseteq \mathcal{O}_2$ with $\lambda \circ \psi = \mathrm{id}_A$ is the desired unital *-monomorphism A into $\mathcal{O}_2$.

Conversely, $\mathcal{O}_2$ is nuclear ([**169**]), nuclear C^*-algebras are exact and C^*-subalgebras of exact C^*-algebras are again exact ([**432**, prop. 7.1(i)] or [**810**, 2.5.1, 2.5.2], or see Remark 3.1.2(ii) and (iii)).

Ad(ii): If A is nuclear and $T \colon \mathcal{O}_2 \to A$ is the completely positive map from Proposition 6.2.1, then $P := \psi \circ T$ is a conditional expectation from $\mathcal{O}_2$ onto $\psi(A) \subset E$.

Indeed: P is a completely positive contraction that maps $\mathcal{O}_2$ into $\psi(A)$. The maps T, ψ and λ satisfy $T(\mathcal{O}_2) = A$, $\psi(A) \subset E$, $T|E = \lambda$, and $\lambda \circ \psi = \mathrm{id}_A$. Therefore, $T \circ \psi = \mathrm{id}_A$, $P^2 = P$ and

$$P(\psi(a)) = \psi(T(\psi(a))) = \psi(\lambda(\psi(a)) = \psi(a) \qquad \forall\ a \in A\,.$$

Conversely, let P be a conditional expectation from a nuclear C^*-algebra B onto a C^*-subalgebra $A \subset B$ and $\eta_A \colon A \hookrightarrow B$ the inclusion map. Then $\mathrm{id}_A = P \circ \mathrm{id}_B \circ \eta_A$ is nuclear because id_B is nuclear and nuclearity is preserved under composition with other completely positive maps. □

Remark 6.2.2. The conditional expectation $P := \psi \circ T \colon \mathcal{O}_2 \to \psi(A)$ that has been defined in the proof of Part (ii) of Theorem A(ii) is an extreme point of the convex set of all linear contractions from $\mathcal{O}_2$ into $\mathcal{O}_2$.

Proof. Let M be a von Neumann algebra, $q \in M$ a projection, and $\mu \colon M \to M$ a *unital* contraction such that $\mu(1-q) = 0$ (i.e., $\mu(q) = 1$), $q\mu(qaq)q = qaq$ for $a \in M$ and $\mu|qMq$ is a C^*-morphism.

Then μ is an extreme point of the convex set of linear contractions from M into M. Indeed:

We use that each unitary $U \in M$ is an extreme point of the closed unit-ball of M, (cf. [**704**, thm. 1.6.4]). If $T, S \colon M \to M$ are contractions and $\theta \in (0,1)$ with $\theta T + (1-\theta)S = \mu$, then $\theta T(1) + (1-\theta)S(1) = 1$. It implies that $T(1) = 1 = S(1)$. Hence T and S are positive unital maps. It follows $T(1-q) \geq 0$, $S(1-q) \geq 0$ and $0 = \mu(1-q) = \theta T(1-q) + (1-\theta)S(1-q)$, thus $T(1-q) = 0 = S(1-q)$ and $T(a) = T(qaq)$, $S(a) = S(qaq)$ for $a \in M$. For unitaries u of qMq we get that $U := \mu(u)$ is unitary and $U = \theta T(u) + (1-\theta)S(u)$. Thus $\mu(u) = T(u) = S(u)$ for all unitaries of qMq. Together it implies $\mu = T = S$.

The $P \colon \mathcal{O}_2 \to \psi(A)$ annihilates D and, therefore, $M := {\mathcal{O}_2}^{**}$, $\mu := P^{**}$ and $q = 1 - p_D$ satisfy the above assumptions on M, μ and q, where p_D denotes the support projection of D. Since P^{**} is an extreme point of the contractions on M, P must be an extreme point of the contractions on $\mathcal{O}_2$. □

next Rem's have overlapping statements

REMARKS 6.2.3. Let $\varphi_j \colon A \to \mathcal{O}_2$ $(j = 1, 2)$ unital *-monomorphisms, then they are unitarily homotopic by Theorem B. In particular, they are approximately unitarily equivalent.

By Remark 6.2.2, one finds always for separable unital *nuclear* A at least one unital embedding $\iota \colon A \hookrightarrow \mathcal{O}_2$ such that there is an extremal conditional expectation P from $\mathcal{O}_2$ onto $\iota(A)$.

For any other embedding $\kappa \colon A \hookrightarrow \mathcal{O}_2$ there is a norm-continuous path $t \in [0, \infty) \mapsto U(t)$ into the unitary group of $\mathcal{O}_2$, such that $U(0) = 1$ and $\lim_{t\to\infty} \|U(t)^*\iota(a)U(t) - \kappa(a)\|$ for all $a \in A$. One can use this to show that for any (more general type of) unital *-monomorphism $\kappa \colon A \hookrightarrow \mathcal{O}_2$ there exists a norm-continuous path $t \in [0, \infty) \mapsto V(t)$ into the *isometries* in $\mathcal{O}_2$ such that $\lim_{t\to\infty} \|V(t)^*\kappa(a)V(t) - \kappa(a)\|$ for all $a \in A$ and $\lim_{t\to\infty} \operatorname{dist}(V(t)^*bV(t), \kappa(A))$ *for all* $b \in \mathcal{O}_2$. (This could replace the existence of conditional expectations in an approximate sense.)

There can not be stronger results, even under very reasonable additional conditions, because *there exist a unital nuclear C^*-algebra A unital *-monomorphisms φ_1 and φ_2 from a unital nuclear C^*-algebra A into $\mathcal{O}_2$, such that there does* not *exist an automorphism γ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with $\gamma(\varphi(a) \otimes 1) = \psi(a) \otimes 1$ for $a \in A$.*

Let us consider two nuclear unital C^*-algebras $A \subset B$ such that there is no conditional expectation from B onto A:

If $h \colon B \to \mathcal{O}_2$ is any unital *-monomorphism, and if $\psi \colon A \to \mathcal{O}_2$ and $P \colon \mathcal{O}_2 \to \psi(A)$ are as in Remark 6.2.2, then there can not exist an automorphism γ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with the property $\gamma \circ (h \otimes \mathrm{id})|A \otimes 1 = (\psi \otimes \mathrm{id})A \otimes 1$.

This is, because, on one side, there is a conditional expectation $Q := P \otimes (\lambda(\cdot)1)$ from $\mathcal{O}_2 \otimes \mathcal{O}_2$ onto $\psi(A) \otimes 1$, (where λ is a pure state on $\mathcal{O}_2$), but there can not be a conditional expectation Q' from $\mathcal{O}_2 \otimes \mathcal{O}_2$ onto $h(A) \otimes 1$, because otherwise $((h|A)^{-1} \otimes 1)((Q'|(h(B) \otimes 1))$ defines an conditional expectation from B onto A.

Suppose that A is a nuclear C^*-subalgebra of a separable unital exact C^*-algebra B such that $1_B \in A$ and such that there does not exist a conditional expectation from B onto A. If $k \colon B \hookrightarrow \mathcal{O}_2$ is a unital embedding and if we define $i := k|A$ as the restriction of k to A, then there does not exist any conditional expectation E from $\mathcal{O}_2$ onto $i(A) = k(A)$, because otherwise $k^{-1} \circ E \circ k$ would be a conditional expectation from B onto A.

Examples:

(o) Suppose that D is a hereditary C^*-subalgebra of a unital C^*-algebra B such that the normalizer algebra $\mathcal{N}(D) := \{b \in B\,;\, bD + Db \subset D\}$ is different from $A := D + \mathbb{C}1$ and that $bD \neq \{0\}$ for all $b \in B_+ \setminus \{0\}$. Then there does not exist a conditional expectation E from B onto A. (Here $\mathbb{C}$ denotes the complex numbers.) Indeed: $(E(b)-b)^*(E(b)-b)d=0$ for all $b\in\mathcal{N}(D)$, $d\in D$, i.e. $E|\mathcal{N}(D)=\mathrm{id}$. It yields e.g. the following two examples (i) and (ii):

(i) Consider $\mathcal{O}_2$ as a unital C*-subalgebra of $\mathcal{L}(\ell_2)$ (by some *-representation). Let $\mathbb{K}$ denote the compact operators, and let $A:=\mathbb{K}+\mathbb{C}1$, $B:=\mathbb{K}+\mathcal{O}_2$.

(ii) Let $B:=\mathcal{T}$, $A:=\mathbb{K}+\mathbb{C}1$, where $\mathcal{T}$ denotes the Toeplitz algebra, generated as C*-algebra by the Toeplitz operator T (i.e. the unilateral shift of ℓ_2).

(iii) Consider the natural continuous epimorphism σ from the Cantor Space $\Omega:=\{0,1\}^\infty$ onto $[0,1]$ given by

$$\sigma\colon (a_1,a_2,...)\mapsto \sum_{n=1}^{\infty} a_n 2^{-n}$$

Then $\gamma\colon f\in \mathrm{C}[0,1]\mapsto \gamma(f):=f\circ\sigma\in \mathrm{C}(\Omega)$ gives a unital embedding of $\mathrm{C}[0,1]$ into $\mathrm{C}(\Omega)=\bigotimes_{n=1}^{\infty}(\mathbb{C}\oplus\mathbb{C})\subset M_{2^\infty}$.

Let $A:=\mathrm{C}[0,1]\subset \mathrm{C}(\Omega)=:B$, where Ω is the Cantor set $\Omega:=\{0,1\}^\infty$, and the monomorphism $A\hookrightarrow B$ is given by $f\mapsto f\circ\rho$ for the continuous map

$$\rho\colon (\alpha_1,\alpha_2,\ldots)\to\sum_{n=1}\alpha_n 2^{-n}$$

from Ω onto $[0,1]$.

The pairs $A:=\gamma(\mathrm{C}[0,1])$ and $B:=\mathrm{C}(\Omega)$ – or $B:=M_{2^\infty}$ – have the property that there does not exist a conditional expectation from B onto A, because the continuous map σ is not open.

Still Question ?: Suppose that $i,j\colon A\hookrightarrow\mathcal{O}_2$ are unital embeddings such that there are extremal conditional expectations from $\mathcal{O}_2$ onto $i(A)$, respectively onto $j(A)$.

Is there an automorphism γ of $\mathcal{O}_2\otimes\mathcal{O}_2$ with $\gamma(i(a)\otimes 1)=j(a)\otimes 1$ for all $a\in A$?

There are nuclear A and unital embeddings $i,j\colon A\hookrightarrow\mathcal{O}_2$ such that there is a conditional expectation P from $\mathcal{O}_2$ onto $i(A)$, but that there does not exist a conditional expectation from $\mathcal{O}_2$ onto $j(A)$.

It follows that there is no automorphism γ of $\mathcal{O}_2\otimes\mathcal{O}_2$ such that $\gamma(i(A)\otimes 1)=j(A)\otimes 1$, because — otherwise — there is an automorphism $\beta\in\mathrm{Aut}(A)$ with $\gamma(i(a)\otimes 1)=j(\beta(a))\otimes 1)$ for $a\in A$ and the the conditional expectation $P\colon\mathcal{O}_2\to i(A)$ (with $P(\mathcal{O}_2)=i(A)$) defines a conditional expectation $Q\colon\mathcal{O}_2\to j(A)$ (with $Q(\mathcal{O}_2)=j(A)=(j\circ\beta)(A))$ by $Q(b):=\mathrm{id}\otimes\lambda\big(\gamma(P\otimes\lambda(\gamma^{-1}(b\otimes 1)))\big)$, where $\lambda(b):=\rho(b)1\in\mathcal{O}_2$ for $b\in\mathcal{O}_2$.

`What the hell is that ?:`

One can show that there exist unital endomorphisms $h_1,h_2\colon\mathcal{O}_2\to\mathcal{O}_2$, such that there is no automorphism γ of $\mathcal{O}_2\otimes\mathcal{O}_2$ such that $\gamma\circ(h_1\otimes\mathrm{id})=h_2\otimes\mathrm{id}$. An example is given by h_1,h_2 as $h_k(a)=1\otimes\iota_k(a)$ defined by isomorphisms ι_1 from

$M_{2^\infty} \otimes \mathcal{O}_2$ onto $\mathcal{O}_2$ and ι_2 from $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ onto $\mathcal{O}_2$: Here such an automorphism γ would also define an isomorphism from M_{2^∞} onto $\mathcal{O}_2$.

???????????

Notice that, moreover, the latter endomorphisms h_k have the property that there is an extremal conditional expectation P from $\mathcal{O}_2 \otimes \mathcal{O}_2$ onto $h_k(\mathcal{O}_2) \otimes \mathcal{O}_2$.

The above examples suggest, that a separable unital exact C^*-algebra A is finite-dimensional if all unital *-monomorphisms $h\colon A \hookrightarrow \mathcal{O}_2$ are conjugate by an automorphism γ of $\mathcal{O}_2$. (The K_0-triviality of $\mathcal{O}_2$ implies immediately that all unital *-monomorphisms $h\colon A \to \mathcal{O}_2$ are unitarily equivalent by a unitary in $\mathcal{O}_2$.)

Does there exist simple separable nuclear unital C^*-subalgebras $A \subset B$ such that $1_B \in A$ and such that there does not exist conditional expectation from B onto A?

Does there exist a unital endomorphism $\iota\colon M_{2^\infty} \hookrightarrow M_{2^\infty}$ such that there is no conditional expectation from M_{2^∞} onto $\iota(M_{2^\infty})$?

Then it would follow that there are unital endomorphisms $i\colon \mathcal{O}_2 \hookrightarrow \mathcal{O}_2$ such that there does not exist a conditional expectation from $\mathcal{O}_2$ onto $i(\mathcal{O}_2)$, because $\mathcal{O}_2 \cong A \otimes \mathcal{O}_2 \subset B \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

3. The residually nuclear case

The proof of a Ψ-residually equivariant version of Proposition 6.2.1 requires some generalizations of the above used ideas. In particular, we must generalize some definitions and results of [**437**] and [**438**]. The following Theorem 6.3.1 is half on the way to the proof of Theorem K. The proof of Theorem K will be completed in Chapter 12. Compare Definitions 1.2.6, 1.2.8 and 1.2.3 and Chapters 3 and 5 for the used notation.

THEOREM 6.3.1. *Suppose that A and B are separable stable C^*-algebras, and $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is an action of* Prim(B) *on A, that has the following properties (i)–(v):*

(i) *B has the WvN-property* ([4]), *and*
(ii) *B has residually nuclear separation* ([5]),
(iii) *A is exact* ([6]).
(iv) *Ψ is non-degenerate, lower semi-continuous, and monotone upper semi-continuous* ([7]).

[4] Def. WvN-property ????

[5] Residually nuclear separation is defined in Definition 1.2.3. Equivalently, the universal weakly residually nuclear *-morphism $H_{\mathrm{rn}}\colon B \to \mathcal{M}(B)$ of Definition **??** is a non-degenerate *-monomorphism and separates the ideals of B by Proposition 5.9.24.

[6] give Def. exact ??

[7] Defs. non-degenerate, lsc, monotone usc.

(v) *The m.o.c. cone* $\mathcal{C} \subset \mathrm{CP}(A,B)$ *of* Ψ*-equivariant completely positive maps* ([8]) *from* A *into* B *is separating for* Ψ*, i.e.,* $\Psi_{\mathcal{C}} = \Psi$ *by Proposition* **??** *I.e. there exists a *-monomorphism* $H\colon A \to \mathcal{M}(B)$*, with the property* $\Psi(J) = H^{-1}(H(A) \cap \mathcal{M}(B,J))$ *for all* $J \in \mathcal{I}(B)$ ([9])*.*

*The above properties lead to a non-degenerate nuclear *-monomorphism* $k\colon A \otimes \mathcal{O}_2 \hookrightarrow B$ *such that* $h(\Psi(J)) = h(A) \cap J$ *for all* $J \in \mathcal{I}(B)$*, where* $h(a) := k(a \otimes 1)$*.*

The infinite repeats $\delta_\infty h$ *and* $\delta_\infty \circ H$ *are unitarily homotopic.*

If, in addition, A *is nuclear,* A *contains a regular Abelian* C^**-subalgebra, and the action* Ψ *is continuous (i.e.,* $\Psi(I) \cup \Psi(J) = \Psi(I \cup J)$ *– in addition), then* h *is unitarily homotopic to a non-degenerate *-morphism* $h_1\colon A \to B$ *such that there exists an approximately inner conditional expectation* P *from* B *onto* $h_1(A)$*.*

([10]).

It follows from [**464**] that $A \otimes \mathcal{O}_2$ contains a regular abelian C^*-subalgebra for every nuclear separable C^*-algebra A.

If B is any separable C^*-algebra, and $B \otimes \mathcal{O}_2$ contains a *regular Abelian* C^**-subalgebra* $C \subset B \otimes \mathcal{O}_2$, then every lower semi-continuous action $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is realized by $\mathrm{CP}_{\mathrm{rn}}(\Psi; A, B) = \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B); \Psi, \mathrm{id}; A, B)$, cf. Proposition **??**. In particular, then B has residually nuclear separation (in the sense of Definition 1.2.3, i.e., $\mathrm{CP}_{\mathrm{rn}}(B,B)$ defines the identity action of $\mathrm{Prim}(B)$ on B), cf. Corollary **??**. A separable C^*-algebra B with residually nuclear separation has the WvN-property, if and only if, B is strongly purely infinite (cf. Proposition **??**). Hence, Theorem 6.3.1 implies:

Corollary 6.3.2. *Theorem K is valid under the* additional *assumptions that* B *is* separable *and that* $B \otimes \mathcal{O}_2$ *contains a* regular Abelian C^**-subalgebra* $C \subset B \otimes \mathcal{O}_2$*.*

Remark 6.3.3. We derive from Corollary 6.3.2 a complete proof of Theorem K in Chapter 12, using the study of scale-invariant morphisms in Chapter 9. The proof relies on the construction of a suitable strongly purely infinite separable C^*-subalgebra B_1 of $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B)$ that contains a regular Abelian C^*-subalgebra and a sufficiently large separable C^*-subalgebra of B as a non-degenerate subalgebra (of B_1).

[8] Give Ref to Def.Psi.equivariant.maps

[9] I.e., $\Psi_{\mathcal{C}} = \Psi$ for the action Ψ of $\mathrm{Prim}(B)$ on A and the m.o.c. cone $\mathcal{C} \subset \mathrm{CP}(A,B)$ of the Ψ-equivariant nuclear maps. Then there is a C^*-morphism $h_1\colon A \to \mathcal{M}(B)$ such that $h_2 := \mathcal{M}(H_{\mathrm{rn}}) \circ h_1$ induces the action Ψ_A of $\mathrm{Prim}(B)$ on A by $\Psi_A(J) = h^{-1}(h(A) \cap \mathcal{M}(B,J))$ for $J \in \mathcal{I}(B)$, cf. Proposition 5.9.24.

[10] We assume, in addition to the assumptions of Theorem K, that B has residually nuclear separation in sense of Definition 1.2.3, i.e., that the universal weakly residually nuclear *-morphism $H_{\mathrm{rn}}\colon B \to \mathcal{M}(B)$ of Proposition 5.9.24

is non-degenerate and is separating for the ideals of B, and that $\Psi_{\mathcal{C}} = \Psi$ for the action Ψ of $\mathrm{Prim}(B)$ on A and the m.o.c. cone $\mathcal{C} \subset \mathrm{CP}(A,B)$ of the Ψ-residually nuclear maps, i.e., that there is a C^*-morphism $h_1\colon A \to \mathcal{M}(B)$ such that $h_2 := \mathcal{M}(H_{\mathrm{rn}}) \circ h_1$ induces the action Ψ_A of $\mathrm{Prim}(B)$ on A by $\Psi_A(J) = h^{-1}(h(A) \cap \mathcal{M}(B,J))$ for $J \in \mathcal{I}(B)$, cf. Proposition 5.9.24. This additional assumptions will be removed both in Chapter 12.

Corollary 6.3.2 gives a nuclear *-monomorphism $k\colon A\otimes\mathcal{O}_2\to B_1\subset \mathrm{Q}(\mathbb{R}_+,B)$ with image in the ideal generated by $B\subset \mathrm{Q}(\mathbb{R}_+,B)$ and with the property that $k(A\otimes 1)\cap J=k(\Psi_A(B\cap J)\otimes 1)$ for any closed ideal J of $\mathrm{Q}(\mathbb{R}_+,B)$.

The latter property induces the existence of a unitary $U\in \mathrm{C_b}(\mathbb{R}_+,\mathcal{M}(B))$ and of a non-degenerate nuclear *-monomorphism $h\colon A\otimes\mathcal{O}_2\to B$ with $k(a\otimes 1)=U^*h(a\otimes 1)U+\mathrm{C}_0(\mathbb{R}_+,B)$ for $a\in A$ (see Chapter 9). The *-morphism h is as stipulated in Theorem K.

Remark 6.3.4. *Since every simple algebra has residually nuclear separation, Theorem 6.3.1 contains Theorem A(i), if we consider $\mathcal{O}_2\otimes A\otimes\mathbb{K}\subset\mathcal{L}(\ell_2)\cong\mathcal{M}(\mathbb{K})\subset\mathcal{M}(B)$ for $B=\mathcal{O}_2\otimes\mathbb{K}$.* Our proof of Theorem 6.3.1 is independent from almost all results on p.i.s.u.n. algebras, because for the existence of the lift $k\colon A\otimes\mathcal{O}_2\to B$ we have only used that $A\otimes\mathcal{O}_2\otimes\mathcal{O}_2$ is exact, and our elementary characterization of absorbing liftable elements in $\mathrm{SExt}_{\mathrm{nuc}}(\mathrm{Prim}(B);A,B)$.

We describe now the steps of the proof of Theorem 6.3.1.

Recall that $H\colon A\to\mathcal{M}(B)$ is weakly nuclear, iff, or every $b\in B$, the map $A\ni a\to b^*H(a)b\in B$ is nuclear.

Since Ψ is non-degenerate, we get that $H\colon A\to\mathcal{M}(B)$ is faithful and $BH(A)B$ is dense in B. We have see in Chapter 3 `where exactly?` that there exists a non-degenerate weakly nuclear *-monomorphism $H_0\colon A\to\mathcal{M}(B)$ with $\delta_\infty\circ H_0$ unitarily equivalent to H_0, such that H_0 is unitarily homotopic to $\delta_\infty\circ H$.

It follows that $H_0(A)\subset\delta_\infty(\mathcal{M}(B))$. The commutant of $\delta_\infty(\mathcal{M}(B))$ in $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally.

We rename $\delta_\infty\big(\overline{\mathrm{span}(H_0(A)\cdot\mathcal{O}_2)}\big)\cong A\otimes\mathcal{O}_2$ by A. Then, from now on, $A\cong A\otimes\mathcal{O}_2$. Then (i)–(v) imply:

(1) A is non-degenerate,i.e., AB is dense in B, and

(2) for every $b\in B$, the map $A\ni a\to b^*H(a)b\in B$ is nuclear.

(3) The action $\Psi\colon\mathcal{I}(B)\to\mathcal{I}(A)$ is given by $\Psi(J):=A\cap\mathcal{M}(B,J)$ and is monotone upper semicontinuous,i.e., $\bigcup_n\Psi(J_n)$ is dense in $\Psi(J)$ for $J:=\overline{\bigcup_n J_n}$, $J_1\subset J_2\subset\cdots\subset B$.

The proof of Theorem 6.3.1 will be given at the end of this Section. But here we describe the steps of the proof:
First step:
With A satisfies also $\delta_\infty(A)$ and $C^*(\delta_\infty(A),\mathcal{O}_2)\cong A\otimes\mathcal{O}_2$ the properties (1)-(3).
Second step:

Next had disappeared and is restored from beamers

Steps of the embedding realization. Step 1:
We start with $\lambda\colon B\to\mathcal{M}(B)$, a weakly continuous non-degenerate C^*-morphism $\lambda\colon B\to\mathcal{M}(B)$ that defines the identity action $J=\lambda^{-1}\big(\lambda(B)\cap\mathcal{M}(B,J)\big)$ on $\mathcal{I}(B)$

and is in "general position" (i.e., $\delta_\infty \circ \lambda = u^*\lambda(\cdot)u$). It exists because our B has in particular "Abelian" factorization (and is separable, stable and s.p.i.).

Then obtain for $A \cong A \otimes \mathcal{D}_2$ our non-degenerate Ψ-realization $H_1 \colon A \to \mathcal{M}(B)$ – coming also from the "Abelian" factorization property of B that reduces it to a classical selection problem. Get non-degenerate

$$H_0 := \mathcal{M}(\lambda) \circ H_1 \colon A \subset \mathcal{M}(\lambda)(\mathcal{M}(B)) \subset \mathcal{M}(B)\,.$$

Here we list the former results that are needed for the "remaining" parts of the proof:

(1) $H_0(A)$ is non-degenerate, i.e., $H_0(A)B$ is dense in B, and is in "general position" (i.e., there exists a unitary $U \in \mathcal{M}(B)$ $U^*H_0(\cdot)U = \delta_\infty \circ H_0$).

(2) The given lower s.c. action $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is realized by $\Psi(J) := H_0^{-1}(H_0(A) \cap \mathcal{M}(B,J))$ and Ψ is monotone upper semi-continuous, i.e., $\bigcup_n \Psi(J_n)$ is dense in $\Psi(J)$ for $J := \overline{\bigcup_n J_n}$, if the sequence $J_n \in \mathcal{I}(B)$ is increasing: $J_1 \subset J_2 \subset \cdots$.

(3) For every $b \in B$, the Ψ_A-compatible map $A \ni a \mapsto b^*H_0(a)b \in B$ is nuclear, and can be approximated by compositions $V_2 \circ V_1$ of the residually nuclear maps $V_1 \colon a \in A \mapsto b_1^*H_1(a)b_1 \in B$ and $V_2 \colon b \in B \mapsto b_2^*\lambda(b)b_2$. I.e. V_2 satisfies $V_2(J) \subset J$ for all $J \in \mathcal{I}(B)$ and that $[V_2]_J \colon B/J \to B/J$ is nuclear for all $J \in \mathcal{I}(B)$.

Step 2:
If A satisfies (1)–(3), then, – with $a \in A$ identified with $H_0(a) \in H_0(A) \subset \mathcal{M}(B)$ – for every $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, there exist completely positive contractions $V \colon \mathcal{M}(B) \to B$ and $W \colon B \to \mathcal{M}(B)$ that satisfy the following conditions (a), (b) and (c):

(a) $\|W \circ V(a_j) - a_j\| < \varepsilon$, for $j = 1, \ldots, n$.
(b) V is strictly continuous and is residually equivariant, i.e., $\lim_n \|V(b_n) - V(b)\| = 0$ if $b_n \to b$ in $\mathcal{M}(B)$ strictly and $V(J) \subset J \cap B$ for $J \in \mathcal{I}(\mathcal{M}(B))$.
(c) $W \colon B \to \mathcal{M}(B)$ is weakly residually nuclear, i.e., $W(J)B \subset J$ for $J \in \mathcal{I}(B)$, and the maps $[W]_J \colon B/J \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B,J)$ satisfy that $([W]_J)_d \colon b \in B/J \mapsto d^*[W]_J(b)d \in B/J$ is a nuclear map for all $d \in B/J$. Here $[W]_J(a+J) := W(a) + \mathcal{M}(B,J)$, i.e., $d^*[W]_J(b)d = \pi_J(f^*W(a)f)$ for $b = a+J$ and $d = f+J$.

More on Step 2:

Consider the set of maps $V := V_c \colon \mathcal{M}(B) \to B$ given by $V_c \colon b \in \mathcal{M}(B) \mapsto c^*\mathcal{M}(\lambda)(b)c$. The point-norm closure is an m.o.c. cone $\mathcal{C}_1$.

Do the same with the maps $W := W_T \colon B \to \mathcal{M}(B)$ given by $W_T(b) := T^*\lambda(b)T$, for $T \in \mathcal{M}(B)$, and we denote this m.o.c. cone by $\mathcal{C}_2$.

Both cones are singly generated (as m.o.c.c.), e.g. $\mathcal{C}_1$ by $e\mathcal{M}(\lambda)(\cdot)e$ where $e \in B_+$ is strictly positive, and $\mathcal{C}_2$ by λ (as u.c.p. map).

The properties (b) and (c) follows from the properties of λ and the fact that $\mathcal{M}(\lambda)$ is the unique strictly continuous extensions of λ.

Continuation 1: More on Step 2:

Notice that $\lambda(b)$ and $\delta_\infty(b)$ for each self-adjoint $b \in B$ must be unitarily homotopic in $\mathcal{M}(B)$ by the generalized W-vN theorem, because both define *-mono-morphisms from $C^*(b)$ into $\mathcal{M}(B)$ that are in "general position" and define the same action of $\mathrm{Prim}(B)$ on $C^*(b)$ by definition of λ, namely the action $J \in \mathcal{I}(B) \mapsto \mathcal{M}(B, J) \cap C^*(b)$.

It follows that each $W \in \mathcal{C}_2$ maps B into the closure I_0 of $\mathcal{M}(B)\delta_\infty(B)\mathcal{M}(B)$, the closed ideal of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$.

Moreover, $b \in \mathcal{M}(B)_+$ will be mapped by all $P \in \mathcal{C}_2 \circ \mathcal{C}_1$ into the ideal of $\mathcal{M}(B)$ that is the norm-closure of the union of the ideals $\mathcal{M}(B, J((ebe - 1/n)_+))$, with $J(b) \in \mathcal{I}(B)$ as defined above, $e \in B_+$ strictly positive contractions.

`A clear statement is now in Chapter 12:`
The point-norm closed m.o.c. cone $\mathcal{C}_1 \subset \mathrm{CP}(A, \mathcal{M}(B))$ generated by $a \in A \to W(V(a))) \in \mathcal{M}(B)$ is contained in $\mathrm{CP}_{\mathrm{nuc}}(A, \mathcal{M}(B))$ and is element-wise approximating. Thus, is point-norm approximating for $H_0 \colon A \to \mathcal{M}(B)$.

Continuation 3: More on Step 2:

The condition (a) is equivalent to $H_0 \in \mathcal{C}_3 := \mathcal{C}_2 \circ \mathcal{C}_1 \circ \mathcal{C}_{H_0}$. The m.o.c. cone $\mathcal{C}_3$ is contained in the cone of (norm-) nuclear c.p. maps from A into $\mathcal{M}(B)$, and the elements of $\mathcal{C}_3$ map A into the norm-closed ideal $I(\delta_\infty(B))$ of $\mathcal{M}(B)$ that is generated by $\delta_\infty(B)$.

More precisely, the elements of $\mathcal{C}_3$ map $a \in A_+$ into the closed ideal of $I(\delta_\infty(B)) \subset \mathcal{M}(B)$ generated by the element $\delta_\infty(e)\delta_\infty^2(e)\delta_\infty^3(a)\delta_\infty^2(e)\delta_\infty(e)$.

If there is $b \in B_+$ such that $\delta_\infty(b)$ and a generate the same closed ideal of $\mathcal{M}(B)$, then there are $T_n \in \mathcal{C}_2 \circ \mathcal{C}_1$ such that $\|T_n(a) - a\| \to 0$.

Step 3:
We use that $A \cong A \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$, ([11]). and select suitable $V_n \colon \mathcal{M}(B) \to B$, $W_n \colon B \to \mathcal{M}(B)$ from Step 2, such that $W_n \circ V_n|A$ converges sufficiently fast in point-norm to the inclusion $A \hookrightarrow \mathcal{M}(B)$ (where $H_0(A)$ and A are naturally identified). Then A is in the multiplier sub-algebra of the separable (non-unital) C^*-system

$$Y := \mathrm{indlim}_n(W_n \circ V_n \colon \mathcal{M}(B) \to \mathcal{M}(B)) \subset \mathcal{M}(B)_\infty\,.$$

By a standard Banach space theory argument, (cf. Section 14 of Appendix B), Y is completely positive and completely isometric isomorphic to the (nuclear) C^*-system

$$X := \mathrm{indlim}_n(V_{n+1} \circ W_n \colon B \to B) \subset B_\infty = \ell_\infty(B)/c_0(B)\,.$$

---

[11]Notice that we do not use that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdot$, because we want to give a proof that is independent from the former results on p.i.s.u.n./ algebras, and gives also a new proof of Theorem A.

Step 4:
An inspection of the natural completely positive and completely isometric isomorphism I from Y onto X shows that

(α) The isomorphism I maps $A \subset Y$ into the intersection $X \cap \mathcal{M}(X)$ of the (two-sided) multiplier algebra $\mathcal{M}(X) \subset X^{**}$ of X with $X \subset X^{**}$.

(β) This happens in an ideal-system equivariant way, i.e.,

$$I(\mathcal{M}(B,J)_\infty \cap Y) = J_\infty \cap X \quad \text{for all } J \in \mathcal{I}(B)\,.$$

(γ) The operator system X is *nuclear* because the maps $T_n\colon V_{n+1} \circ W_n$ are nuclear, i.e., T_n can be approximated point-wise by "decomposable" contractions that factorize approximately through suitable matrix-algebras M_{k_n}.

Use here and above that inductive limits of – not necessarily unital – *C^*-systems* by using c.p. contractions always are *C^*-systems*, and that the second conjugate with the natural matrix-order and matrix-norm is a W*-algebra.

Step 5:
We construct a hereditary C^*-subalgebra $D \subset B$ and a completely positive and completely isometric isomorphism φ from X onto $B//D$, that have the properties

(P1) D is residually essential – in particular D is stable,

(P2) $\varphi(J_\infty \cap X) = \pi_{DB+BD}(J)$,

(P3) $\varphi(X \cap \mathcal{M}(X)) = \pi_D(\mathcal{N}(D))$,

(P4) the natural map $\mathcal{N}(D)/D \to \mathcal{M}(D)/D \cong \mathrm{Q}(B)$ extends to completely positive map γ from $B//D$ into $\mathcal{M}(D)/D$ with the property that $\gamma(J//D) = \gamma(B//D) \cap \mathcal{M}(D, D \cap J)$ for all $J \triangleleft B$.

Idea for finding D in Step 5 is the following
Lemma on replacements of inductive limits:

(See somewhere below)

End of Lemma.

Since B is stable, separable and s.p.i., we get that the ($\mathcal{I}(B)$-) residually nuclear contractions $P\colon B \to B$ can be approximated point-wise by maps $T\colon B \to B$ given by $T_S(b) := S^*bS$ for isometries $S \in \mathcal{M}(B)$ with $1 - SS^*$ properly infinite. Thus we can take a fixed copy of $C^*(s,t) \cong \mathcal{O}_2$ in $\mathcal{M}(B)$ and find the approximating T_S as $S = U^*sU$ for suitable unitary $U \in \mathcal{M}(B)$.

Thus X becomes by the Lemma simply *the same* as $\operatorname{indlim} T_n\colon B \to B$ with $T_n(b) = U_n^* s^* U_n b U_n^* s U_n$ for a suitable sequence of unitaries $U_n \in \mathcal{M}(B)$. It turns then out that X is c.i. and c.p. isomorphic to $B//D$ where D is found in a certain inductive limit of copies of B by inner automorphisms of B.

Step 6:
We apply the Proposition on Ψ-compatible Busby invariants to the Busby invariant $\gamma \circ \varphi \circ I\colon A \to \mathcal{M}(D)/D$. It has image in $\mathcal{N}(D)/D$ and admits a Ψ-equivariant lift $h\colon A \to \mathcal{N}(D) \subset B$, that is nuclear as map from A into B.

(But h is not necessarily nuclear as a map from A into $\mathcal{N}(D)$.)

It turns out that h is unitarily homotopic to a *non-degenerate* nuclear monomorphism $h_0 \colon A \to B$ that is Ψ-equivariant.

Step 7:

????????????????????????

The case of (bi-) continuous action Ψ of $\mathrm{Prim}(B)$ on A.

Target: If A is nuclear and $h_0(A)$ regular in B, then there exists an Ψ-equivariant conditional expectation from B to $h_0(A)$ or at least a sequence of Ψ-equivariant c.p. contractions $V_n \colon B \to h_0(A)$ with $V_n(h_0(a)) \to h_0(a)$ for $n \to \infty$.

If $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is "continuous", then $h_0(A)$ must be a regular C^*-subalgebra of B. It follows, that there is a lower semi-continuous action $\Phi \colon \mathcal{I}(A) \to \mathcal{I}(B)$ with $\Phi(h(J)) :=$ biggest $K \triangleleft B$ with $K \cap h(A) \subset h(J)$.

REF ??????????

Since A contains a regular abelian C^*-subalgebra, the cone $\mathrm{CP}_{\mathrm{rn}}(\Phi; B, A)$ of Φ-residually nuclear maps $W \colon B \to A$ is separating for Φ, i.e., for each $b \in B_+$ and each pure state λ on A with $\lambda(J) \neq \{0\}$, there is $W \in \mathrm{CP}_{\mathrm{rn}}(\Phi; B, A)$ with $\lambda(W(\Phi(J))) \neq \{0\}$.

Since $\Phi(K \cap h(A)) \supset K$ for all ideals $K \triangleleft B$, it follows that $\Phi \circ \Psi$ majorizes Id of $\mathcal{I}(B)$.

??????? ??????????????????? If A is nuclear, ??????????????

??

`Overview on proof Thm. 6.3.1 finished?`

We need some simple general results on residually essential hereditary C^*-subalgebras D of a C^*-algebra B, non-unital C^*-systems and on inductive limits. First we explain some properties of certain subspaces of C^*–algebras.

REMARK 6.3.5. Suppose that B is separable and that $A \subset \mathcal{M}(B)$ is a separable *stable* exact C^*-subalgebra, such that AB is dense in B and that the maps $a \in A \mapsto b^*ab \in B$ are nuclear for all $b \in B$.

Then B is stable, and maps $V \colon \mathcal{M}(B) \to B$ and $W \colon B \to \mathcal{M}(B)$ with properties (a), (b) and (c) for (given) $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ exist, if and only if,

(i) The m.o.c. cone $\mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B), B, B)$ of residually nuclear maps from B to B is non-degenerate, i.e., the universal weakly residually nuclear $H_{rn} \colon B \to \mathcal{M}(B)$ C^*-morphism (given by Corollary **??** for $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(\mathrm{Prim}(B); B, B)$) is non-degenerate.
  And separating for $\mathrm{Prim}(B)$??

(ii) The representations $a \in A \mapsto \mathcal{M}(H_{rn})(a) \in \mathcal{M}(B)$ and $a \in A \mapsto a \in \mathcal{M}(B)$ define the same action of of $\mathrm{Prim}(B)$ on A.

(iii) The (lower semi-continuous) action of $\mathrm{Prim}(B)$ on A is monoton upper semi-continuous.

(See the arguments in Chapter 12 for a proof.)

????????????

`Move the needed arguments of Chapter 12 to here. ??`

The conditions (i)–(iii) are satisfied if the action of $\mathrm{Prim}(B)$ on B defined by H_{rn} is the identity map of $\mathcal{I}(B) \cong \mathbb{O}(\mathrm{Prim}(B))$.

Definition 6.3.6. A hereditary C^*-subalgebra $D \subset B$ is *residually essential* in B, if for every closed ideal $J \in \mathcal{I}(B)$ holds that $\pi_J(D)$ is essential in B/J, i.e., $b \in B_+$ and $bD \subset J$ imply $b \in J$ (equivalently: $\mathrm{Ann}(\pi_J(D)) = 0$ in B/J).

Lemma 6.3.7. *Let $T\colon X \to C$ a positive and isometric linear map from a (not necessarily unital) C^*-system X into a C^*-algebra C.*

Then $T^{-1}(C_+ \cap T(X)) = X_+$.

Proof. We can pass to the second adjoint $T^{**}\colon X^{**} \to C^{**}$. The map T^{**} is still positive and isometric, $T^{**}|X = T$, $X_+ = (X^{**})_+ \cap X$, and X^{**} is unitally order isomorphic to a unital C^*-algebra. Thus, we may suppose that X is a C^*-algebra.

Let $a = h + ik \in X$ with self-adjoint $h, k \in X$ with polar decompositions $h = h_+ - h_-$, $k = k_+ - k_-$, and suppose $T(a) \in C_+$. Then $T(h_+)$, $T(h_-)$, $T(k_+)$ and $T(k_-)$ are all in C_+. Thus $T(k) = 0$, $a = h_+ - h_-$ and $0 \leq T(h_-) \leq T(h_+)$.

Suppose that $h_- \neq 0$. Then $T(h_+) \neq 0$ and $h_+ \neq 0$. Let $x := \|h_-\|^{-1}h_-$ and $y := \|h_+\|^{-1}h_+$. It follows $x, y \in X_+$, $xy = 0$, $T(x), T(y) \in C_+$, $\|T(x)\| = \|x\| = 1$, $\|T(y)\| = \|y\| = 1$ and $\|T(x + y)\| = \|x + y\| = \max(\|x\|, \|y\|) = 1$. Thus, there exists a state ρ on C with $\rho(T(x)) = 1$ (by extending a suitable character of $C^*(T(x))$ to C). Then $1 \leq \rho(T(x)) + \rho(T(y)) = \rho(T(x + y)) \leq 1$. It implies $\rho(T(y)) = 0$, which contradicts $\|h_-\|T(x) \leq \|h_+\|T(y)$ if $\|h_-\| > 0$. Thus $h_- = 0$ and $a = h_+$. □

Lemma 6.3.8. *Suppose that B and C are C^*-algebras, that A is a C^*-subalgebra of C, that $\Psi_C\colon \mathcal{I}(B) \to \mathcal{I}(C)$ is a monotone map, that $D \subset B$ is a hereditary C^*-subalgebra of B.*

If $T\colon B//D \to C$ is a completely positive and completely isometric linear map, such that $A \subset T(B//D)$ and, for $J \in \mathcal{I}(B)$,

$$T(\pi_D(J)) = \Psi_C(J) \cap T(B//D)\,,$$

*then the *-monomorphism $\lambda := T^{-1}|A\colon A \to \mathcal{N}(D)/D$ satisfies, for $J \in \mathcal{I}(B)$,*

$$\lambda(A \cap \Psi_C(J)) = \lambda(A) \cap \pi_D(J)\,.$$

Proof. The map $\lambda\colon A \to (B//D)^{**} \cong q_D B^{**} q_D$ is an isometric completely positive map by Lemma 6.3.7, because

$$T^{**} \otimes \mathrm{id}_n = (T \otimes \mathrm{id}_n)^{**}\colon q_D B^{**} q_D \otimes M_n \to C^{**} \otimes M_n$$

is isometric and positive for each $n \in \mathbb{N}$. Since $\|\lambda\| \leq 1$, we get from the generalized Kadison inequality of Choi that $\lambda(a^*a) \geq \lambda(a)^*\lambda(a)$ for $a \in A$. If we apply T^{**} to this inequality and use the Choi-Kadison inequality for T^{**}, we get

$$a^*a = T^{**}(\lambda(a^*a)) \geq T^{**}(\lambda(a)^*\lambda(a)) \geq T^{**}(\lambda(a))^*T^{**}(\lambda(a)) = a^*a\,.$$

Thus, $T^{**}(\lambda(a^*a) - \lambda(a)^*\lambda(a)) = 0$. Using that T^{**} is faithful on $C^*(\lambda(A))$ we get $\lambda(a^*a) = \lambda(a)^*\lambda(a)$ for all $a \in A$.

Thus $\lambda \colon A \to q_D B^{**} q_D$ is a *-monomorphism with $\lambda(A) \subset q_D B q_D = B/\!/D$. By Remark 6.1.2(14), $\lambda(A) \subseteq \mathcal{M}(B/\!/D) \cap (B/\!/D) = \mathcal{N}(D)/D$.

The equation follows from $A \cap \Psi_C(J) = A \cap T(\pi_D(J))$, because T is isometric. □

LEMMA 6.3.9. *Suppose that D is a σ-unital hereditary C^*-subalgebra of a C^*-algebra B, and that A is a separable C^*-subalgebra of the normalizer algebra $\mathcal{N}(D) \subset B$.*

Let $\pi_0 \colon B \to B/\!/D$ and $\pi_2 \colon \mathcal{M}(D) \to \mathcal{M}(D)/D$ denote the quotient maps, and let ρ denote the natural C^*-morphism from $\mathcal{N}(D)$ into $\mathcal{M}(D)$ with kernel $\mathrm{Ann}(D)$ *(the two-sided annihilator of D in B).*

For a closed ideal J of B let

$$\mathrm{Ann}(D, J) := \{a \in B\,;\, aD + Da \subset J\} \supset J\,.$$

Then there exists a completely positive contraction T from the C^*-system $B/\!/D$ into the C^*-algebra $\mathcal{M}(D)/D$, such that T has the following properties (i)-(ii)*:*

(i) *$T(\pi_0(\mathrm{Ann}(D, J))) \subset \pi_2(\mathcal{M}(D, D \cap J))$ for every $J \in \mathcal{I}(B)$.*
(ii) *$(T \circ \pi_0)|A$ is the C^*-morphism $(\pi_2 \circ \rho)|A$ from A into $\mathcal{M}(D)/D$, and has kernel $A \cap (D + \mathrm{Ann}(D))$.*
(iii) *$T \circ \pi_0(A \cap \mathrm{Ann}(D, J)) = T(\pi_0(A) \cap \pi_0(\mathrm{Ann}(D, J))) = T(\pi_0(A)) \cap \pi_2(\mathcal{M}(D, D \cap J))$ for $J \in \mathcal{I}(B)$.*

*In particular, – if $B/\!/D$ is nuclear –, then $\pi_2 \circ \rho$ defines a nuclear *-monomorphism from $A/(A \cap (D + \mathrm{Ann}(D)))$ into $\mathcal{M}(D)/D$.*

PROOF. It holds $\mathcal{N}(D) \supseteq \mathrm{Ann}(D) = \mathrm{Ann}(D, \{0\}) \cong (\mathrm{Ann}(D) + D)/D$ because $D + \mathrm{Ann}(D) \cong D \oplus \mathrm{Ann}(D)$ naturally, thus $D + \mathrm{Ann}(D)$ is the kernel of $\pi_2 \circ \rho\colon \mathcal{N}(D) \to \mathrm{Q}(D) = \mathcal{M}(D)/D$.

Since D is σ-unital, it contains a strictly positive contraction $e \in D_+$. We can suppose that $e \in A$ (consider otherwise $A_1 := C^*(A, e)$). Let $a_1, a_2, \ldots \in A$ a dense sequence in the unit ball of A. There is an approximate unit $e_1 \leq e_2 \leq \ldots \in C^*(e) \subset D$ with $e_n e_{n+1} = e_n \leq 1$, $\|e_n e - e_n\| < 2^{-n}$ and $\|[a_k, e_n^{1/2}]\| < 2^{-n-2}$ for $k \leq n$. Let $g_1 := e_1^{1/2}$, $g_{n+1} := (e_{n+1} - e_n)^{1/2}$

$b \mapsto W(b) := \sum_n g_n b g_n$ is a completely positive contraction from B into $\mathcal{M}(D)$ with $W(e) = e$, $W(\mathrm{Ann}(D)) = 0$ and $W(D) \subset D$, because $\sum_n g_n b g_n$ strictly converges in $\mathcal{M}(D)$, cf. Remark 5.1.1(4). Moreover,

$$\|(\rho(a)-W(a))(1-e_m)\| \leq \sup_{n\geq m} \|a(e_n-e_m) - \sum_{m-1\leq k\leq n+1} g_k a g_k (e_n-e_m)\| \leq 2^{-m}\,.$$

Thus $\rho(a) - W(a) \in D$ for $a \in A$.

$\pi_2 \circ W \colon B \to \mathcal{M}(D)/D$ is completely positive and contains D in its kernel. It follows $W(BD+DB) \subset D$ and that there is a unique completely positive map $T\colon B/\!/D = B/(BD+DB) \to \mathrm{Q}(D) = \mathcal{M}(D)/D$ with $T\circ\pi_0 = \pi_2\circ W$.

(i): Let $J \in \mathcal{I}(B)$. Then $W(\mathrm{Ann}(D,J)) \subset \mathcal{M}(D, D\cap J)$, because $eW(b)e = \sum eg_n b g_n e \in D\cap J$ for $b \in \mathrm{Ann}(D,J)$. Thus $T(\pi_0(\mathrm{Ann}(D,J))) \subset \pi_2(\mathcal{M}(D,D\cap J))$.

(ii): $T\circ \pi_0|A = \pi_2\circ\rho|A$ because $\pi_2(W(a)) = \pi_2(\rho(a))$ for $a\in A$. The kernel of $\pi_2\circ\rho\colon \mathcal{N}(D) \to \mathcal{M}(D)/D$ is $\rho^{-1}(D) = D + \mathrm{Ann}(D)$.

(iii): $\pi_2 \circ \rho(A\cap \mathrm{Ann}(D,J)) = T\circ\pi_0(A\cap\mathrm{Ann}(D,J)) \subset T(\pi_0(A)\cap \pi_0(\mathrm{Ann}(D,J))) \subset T(\pi_0(A))\cap\pi_2(\mathcal{M}(D,D\cap J)) = \pi_2(\rho(A))\cap\pi_2(\mathcal{M}(D,D\cap J)$, because $W(\mathrm{Ann}(D,J)) \subset \mathcal{M}(D,D\cap J)$, $T\circ\pi_0 = \pi_2\circ W$ and $T\circ\pi_0|A = \pi_2\circ\rho$.

Conversely, if $b\in\mathcal{N}(D)$ and $\rho(b) = c+d$ with $d\in D$ and $c\in\mathcal{M}(D,D\cap J)$ then $c = \rho(b)-d = \rho(b-d) \in \rho(\mathcal{N}(D))$ and $\rho(b-d)\in\mathcal{M}(D,D\cap J)$, i.e., $b-d\in\mathrm{Ann}(D,J)$. Thus $b\in\mathcal{N}(D)\cap\mathrm{Ann}(D,J)+D$ and $\pi_0(b)\in\pi_0(\mathrm{Ann}(D,J))$.

Since $D\subset\mathcal{N}(D)$, $(\mathcal{N}(D)\cap\mathrm{Ann}(D,J))+D = \mathcal{N}(D)\cap(\mathrm{Ann}(D,J)+D) = (\pi_2\circ\rho)^{-1}(\mathcal{N}(D)\cap\mathcal{M}(D,D\cap J))$. □

`Related/re-formulated text (new` φ `good?) for Lemma 6.3.9: Next version better? ??`

Recall that for C^*-subalgebra D of a C^*-algebra B, $\mathcal{N}(D)$ ($=\mathcal{N}(B,D)$) is defined as the set of two-sided normalizers $b\in B$ of D: $bD\cup Db\subseteq D$.

This definition of $\mathcal{N}(B,D)$ gives also a natural C^*-morphism $\varphi_D\colon\mathcal{N}(D)\to\mathcal{M}(D)$ given by $\varphi_D(b)d := bd$ for $b\in\mathcal{N}(D)$ and $d\in D$. The kernel of φ_D is the two-sided annihilator $\mathrm{Ann}(D)$ ($=\mathrm{Ann}(D,0)$) of D in B defined as the set of $b\in B$ with $bD = \{0\} = Db$.

Some of the following are discussed in Chapter 2 ? Ref's??

LEMMA 6.3.10. *Suppose that $D\subset B$ is a σ-unital residually essential* hereditary *C^*-subalgebra of B.*

(i) *If B has no unital quotient, then D has no unital quotient.*
(ii) *The map $J\in\mathcal{I}(B)\mapsto D\cap J\in\mathcal{I}(D)$ is a bijection onto $\mathcal{I}(D)$.*
(iii) *The natural map $\varphi_D\colon\mathcal{N}(D)\to\mathcal{M}(D)$ given is injective, and $\varphi_D(a)\in\mathcal{M}(D,D\cap J)$, if and only if, $a\in\mathcal{N}(D)\cap J$.*
(iv) *For $a\in\mathcal{N}(D)$ and $J\in\mathcal{I}(B)$ holds:*

$$a\in D+(\mathcal{N}(D)\cap J) \;\Leftrightarrow\; \pi_{B/\!/D}(a)\in\pi_{B/\!/D}(J) \;\Leftrightarrow\; \varphi_D(a)\in D+\mathcal{M}(D,D\cap J)\,.$$

(v) *For each separable C^*-subalgebra $A \subset B$ there is a completely positive contraction $T_A \colon A \to \mathcal{M}(D)/D$ such that $T_A|(A \cap \mathcal{N}(D)) = \pi_D \circ \varphi_D$ and $T_A(A \cap J) \subset \pi_D(\mathcal{M}(D, D \cap J))$ for each closed ideal J of B.*

`Sort out: Where `` hereditary in` $B$`'' is needed for` $D$`??`

PROOF. Recall that the definition of "residual essential" (in B) for a (not necessarily hereditary) C^*-subalgebra $D \subseteq B$ is:

For each closed ideal $K \subseteq B$, there is no non-zero element of $(B/K)_+$ that is orthogonal to $\pi_K(D)$.

Give ref. to Definition?

This implies obviously that $D \subseteq K := \overline{\text{span}(BDB)} = B$. Moreover, the hereditary C^*-subalgebra $E := \overline{\text{span}(DBD)}$ of B is again residually essential C^*-subalgebra of B.

(ii): If D is a hereditary C^*-subalgebra of a C^*-algebra B that is not contained in any closed ideal K of B with $K \neq B$, then the map

$$K \in \mathcal{I}(B) \mapsto J_K := K \cap D \in \mathcal{I}(D)$$

is a bijective lattice isomorphism.

`cite TEXT books for this`

(i): Suppose that B has no unital quotient, and suppose that there exists a closed ideal J of D such that D/J is unital.

The closed ideal $K := \overline{BJB}$ of B is the smallest closed ideal of B that contains J.

Since D is hereditary, each ideal J of D is the intersection of D with an ideal L of B, i.e., $J = D \cap L$. Thus $J \subseteq D \cap K \subseteq D \cap L = J$, i.e., $J = D \cap K$.

It implies that $D/J \cong \pi_K(D) = p(B/K)p$ for some projection in $p \in B/K$, because $\pi_K(D)$ is a unital hereditary C^*subalgebra of B by $\pi_K(D)\pi_K(B)\pi_K(D) = \pi_K(DBD) = \pi_K(D)$.

Since $\pi_K(D)$ is "essential" in B/K it follows that $(1-p)(B/K)(1-p) = \{0\}$, i.e., that B/K is unital, a contradiction to the assumption that all non-zero quotients B/K of B are not unital.

(iii): The map $\varphi_D \colon \mathcal{N}(D) \to \mathcal{M}(D)$ is injective because the kernel is $\text{Ann}(D)$ and $\text{Ann}(D) = \{0\}$ for essential C^*-subalgebras D of B (by definition of "essential").

For $a \in \mathcal{N}(D) \subseteq B$ holds that $\varphi_D(a) \in \mathcal{M}(D, D \cap J)$, if and only if, $a \in \mathcal{N}(D) \cap J$:

If $a \in \mathcal{N}(D) \cap J$, then $aD \cup Da \subseteq D \cap J$. Thus, $\varphi_D(a) \in \mathcal{M}(D, D \cap J)$. If $a \in \mathcal{N}(D)$ and $\varphi_D(a) \in \mathcal{M}(D, D \cap J)$, then $aD \cup Da \subseteq D \cap J$. It follows that $\pi_J(a)\pi_J(D) = \{0\}$ and $\pi_J(D)\pi_J(a) = \{0\}$. It says that $\pi_J(a^*a + aa^*)$ is orthogonal to $\pi_J(D)$.

The algebra D is "residually essential", which means that $\pi_J(D)$ is essential in $\pi_J(B)$ for each $J \in \mathcal{I}(B)$.

It follows that $\pi_J(a^*a + aa^*) = 0$ or that $\pi_J(D) = \{0\}$ must happen, i.e., $a \in J$ or $D \subseteq J$. Thus $a \in \mathcal{N}(D) \cap J$ or $D \subseteq J$.

But essential C^*-subalgebras in B generate B as ideal, i.e., if $D \subseteq J$ and D is essential in B imply that $J = B$ and $a \in \mathcal{N}(D) \subseteq J = B$.

(iv): For $a \in \mathcal{N}(D)$ are equivalent

$$a \in D + (\mathcal{N}(D) \cap J) \;\Leftrightarrow\; \pi_{B//D}(a) \in \pi_{B//D}(J) \;\Leftrightarrow\; \varphi_D(a) \in D + \mathcal{M}(D, D \cap J)\,.$$

Let $a \in D + (\mathcal{N}(D) \cap J)$ then $a = d + c$ with $d \in D$ and $c \in \mathcal{N}(D) \cap J$. Then trivially $\pi_{B//D}(a) = \pi_{B//D}(c)$ because the kernel of $\pi_{B//D}$ is $BD + DB \supseteq D$. Thus $\pi_{B//D}(a) \in \pi_{B//D}(J)$.

Notice that $\mathcal{N}(D)/D \cong \mathcal{M}(B//D) \cap (B//D)$ by a natural isomorphism. $\pi_{B//D}$ coincides on $\mathcal{N}(D)$ with the C^*-morphism $a \mapsto a + D$. If $\pi_{B//D}(a) \in \pi_{B//D}(J)$ and $a \in \mathcal{N}(D)$, then $a \in (J + BD + DB) \cap \mathcal{N}(D) = D + (\mathcal{N}(D) \cap J)$.

(To see more look to the second conjugate of B: $\mathcal{N}(D) = \{P_D\}' \cap B \subset \{P_D\}' \cap B^{**}$.)

(v): `To be filled in` ??

For each separable C^*-subalgebra $A \subset B$ there is a completely positive contraction $T_A \colon A \to \mathcal{M}(D)/D$ such that $T_A|(A \cap \mathcal{N}(D)) = \pi_D \circ \varphi_D$ and $T_A(A \cap J) \subset \pi_D(\mathcal{M}(D, D \cap J))$ for each closed ideal J of B:

□

Proposition 6.3.11. *Suppose that D is a residually essential hereditary C^*-subalgebra of a separable stable C^*-algebra B with WvN-property, and that the operator space $B//D := B/(DB + BD)$ is nuclear.*

Let $\pi_1 \colon \mathcal{N}(D) \to \mathcal{N}(D)/D$ and $\pi_2 \colon \mathcal{M}(D) \to \mathcal{M}(D)/D$ denote the natural epimorphisms, and let A is a stable separable C^*-subalgebra of $\mathcal{M}(B)$, such that $A \hookrightarrow \mathcal{M}(B)$ is weakly residually nuclear and AB is dense in B. Then:

(i) *D is stable and there is an isomorphism φ from B onto D, which is unitarily homotopic to id_B, and, therefore, satisfies $\varphi(J) = D \cap J$ for $J \in \mathcal{I}(B)$.*
(ii) *The natural C^*-morphism ρ from $\mathcal{N}(D)/D$ into $\mathcal{M}(D)/D$ is a nuclear *-monomorphism, and satisfies, for closed ideals J of B,*

$$\rho(\pi_1(\mathcal{N}(D) \cap J)) = \rho(\mathcal{N}(D)/D) \cap \pi_2(\mathcal{M}(D, D \cap J)).$$

(iii) *If $\lambda \colon A \to \mathcal{N}(D)/D \subset B//D$ is a *-monomorphism with*

$$\lambda(A \cap \mathcal{M}(B, J)) = \lambda(A) \cap \pi_1(\mathcal{N}(D) \cap J) \qquad \textit{for} \quad J \in \mathcal{I}(B)\,,$$

then an element

$$H \;:=\; \mathrm{Q}(\varphi)^{-1} \circ \rho \circ \lambda \colon\; A \to \mathrm{Q}(B)$$

is nuclear and defines an element $[H]$ *of the semi-group*

$$\mathrm{SExt}_{\mathrm{nuc}}(\mathrm{Prim}(B); A, B) := S(H_0; A, \mathrm{Q}(B))$$

where $H_0 := \pi_B \circ \delta_\infty | A$ *with* $2[H_0] = [H_0]$.
The element $[H]$ *is contained in the subgroup*

$$\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B); A, B) = [H_0] + \mathrm{SExt}_{\mathrm{nuc}}(\mathrm{Prim}(B), A, B)\,.$$

(iv) *If, in addition to* (iii), $[H] = 0$ *in* $\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B), A, B)$, *then there is a non-degenerate nuclear *-monomorphism* h *from* A *to* B, *such that*

$$h(A \cap \mathcal{M}(B, J)) = h(A) \cap J \qquad \textit{for} \quad J \in \mathcal{I}(B)\,.$$

The infinite repeats $\delta_\infty \circ h$ *and* $\delta_\infty | A$ *are unitarily homotopic.*

PROOF. Since D is residually essential, we have in particular that B is the closed span of BDB.

(i): Since B has the WvN-property, B and its hereditary C^*-subalgebra D must be purely infinite in particular. D is separable and can not have a unital quotient, because for every closed ideal I of D there is a closed ideal J of B with $I = D \cap J$, and $D/I \cong \pi_J(D)$ is an essential subalgebra of the stable C^*-algebra $\pi_J(B)$ (cf. assumptions). By Corollary 5.5.4, D is stable. Now use Corollary 5.5.6 to get φ with property(i).

(ii): The C^*-algebras $\mathcal{N}(D)$ and $\mathcal{N}(D)/D$ are separable. The natural C^*-morphism $\rho\colon \mathcal{N}(D)/D \to \mathcal{M}(D)/D$ is faithful because $\mathrm{Ann}(D) = \{0\}$. Let $T\colon B//D \to \mathcal{M}(D)/D$ be as in Lemma 6.3.9 with

$$T\pi_{BD+DB}|\mathcal{N}(D) = \pi_2 \circ \eta|\mathcal{N}(D) = \rho \circ \pi_1\,,$$

where $\pi_2\colon \mathcal{M}(D) \to \mathrm{Q}(D) = \mathcal{M}(D)/D$ is the quotient map and $\eta\colon \mathcal{N}(D) \to \mathcal{M}(D)$ denotes the natural *-morphism with $\eta(d) = d$ for $d \in D$. Then $T = T \circ \mathrm{id}_{B//D}$ is nuclear, because $B//D$ is nuclear. We obtain $\rho = T \circ \gamma$, where γ denotes the natural isomorphism from $\mathcal{N}(D)/D$ onto $\mathcal{M}(B//D) \cap (B//D)$ with $\gamma \circ \pi_1 = \pi_{BD+DB}|\mathcal{N}(D)$ that is given by Lemma A.19.8(i). Thus ρ is a *nuclear* *-monomorphism.

By Lemma 6.3.9, $\mathrm{Ann}(D, J) \subset \pi_J^{-1}(\mathrm{Ann}(\pi_J(D)) = J$ for residually essential D and for $J \in \mathcal{I}(B)$. It follows:

$$\rho\pi_1(\mathcal{N}(D) \cap J) = \rho\pi_1(\mathcal{N}(D)) \cap \pi_2(\mathcal{M}(D, D \cap J)).$$

(iii): The C^*-algebra A is a stable C^*-subalgebra of $\mathcal{M}(B)$, such that $\mathrm{id}\,|A$ is weakly residually nuclear, and AB is dense in B. If there is a *-monomorphism λ from A into $\mathcal{N}(D)/D$ with

$$\lambda(A \cap \mathcal{M}(B, J)) = \lambda(A) \cap \pi_1(\mathcal{N}(D) \cap J)$$

for $J \in \mathcal{I}(B)$, then

$$H := (\mathrm{Q}(\varphi))^{-1} \circ \rho \circ \lambda\colon A \hookrightarrow \mathrm{Q}(B) := \mathcal{M}(B)/B$$

is a monomorphism from A into $\mathrm{Q}(B)$. Here $\mathrm{Q}(\varphi)$ is the isomorphism from $\mathrm{Q}(D) := \mathcal{M}(D)/D$ onto $\mathrm{Q}(B)$ induced by φ, i.e., $\mathrm{Q}(\varphi)(d + D) = \mathcal{M}(\varphi)(d) + B \in \mathrm{Q}(B)$ for

$d \in \mathcal{M}(D)$. Note $\mathrm{Q}(\varphi)^{-1} = \mathrm{Q}(\varphi^{-1})$. Then $H\colon A \to \mathrm{Q}(B)$ is nuclear, because ρ is a nuclear monomorphism. In particular, A is an *exact* C^*-algebra. By Corollary ??, the inclusion map $\mathrm{id}\colon A \hookrightarrow \mathcal{M}(B)$ is *nuclear*, because $\mathcal{M}(B)$ is weakly nuclear and A is exact.

The map H satisfies, by asumptions in (ii),

$$H(A \cap \mathcal{M}(B,J)) = \mathrm{Q}(\varphi)^{-1}\rho(\lambda(A) \cap \pi_1(\mathcal{N}(D) \cap J)) \tag{3.1}$$

$$= H(A) \cap \mathrm{Q}(\varphi)^{-1}(\pi_2(\mathcal{M}(D, D \cap J))). \tag{3.2}$$

But $Q(\varphi)^{-1}(\pi_2(\mathcal{M}(D, D \cap J))) = \pi_B(\mathcal{M}(B,J))$, because $\varphi(B) = D$ and $\mathcal{M}(\varphi)(\mathcal{M}(B,J)) = \mathcal{M}(D, \varphi(J)) = \mathcal{M}(D, D \cap J)$.

The exactness of A gives that H is moreover residually nuclear with respect to the natural actions of $\mathrm{Prim}(B)$ on $A \subset \mathcal{M}(B)$ and $\mathrm{Q}(B)$ (cf. **??**).

Thus $H\colon A \hookrightarrow \mathrm{Q}(B)$, B and $A \subset \mathcal{M}(B)$ satisfy the assumptions of Proposition **??**. **??** Therefore, $[H]$ is in $G(H_0; A, \mathrm{Q}(B)) = \mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B), A, B)$, where $H_0 := \pi_B \circ \delta_\infty | A\colon A \to \mathrm{Q}(B)$.

`to be filled in: show` $[H] \in G(H_0; A, \mathrm{Q}(B))$ **??**

(iv): Since $[H] = 0 = [H_0]$ in $\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B); A, B)$, it follows by Remark **??** **??**). that $H = w^* H_0(\cdot) w$ where $w \in \mathcal{U}_0(\mathrm{Q}(B))$ and $H_0 = \pi_B \circ \delta_\infty | A$, because the extensions given by elements of $\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B), A, B)$ are stable C^*-algebras ([12]).

We find a unitary $u \in \mathcal{M}(B)$, such that

$$H_1 \colon a \in A \mapsto u^* \delta_\infty(a) u \in \mathcal{M}(B)$$

is a residually nuclear lift of $H\colon A \to \mathrm{Q}(B)$ with $H_1(A) \cap \mathcal{M}(B,J) = H_1(A \cap \mathcal{M}(B,J))$ for $J \in \mathcal{I}(B)$. Then

$$H_2 := \mathcal{M}(\varphi) \circ H_1 \colon A \to \mathcal{M}(D)$$

is a residually nuclear lift of $\rho \circ \lambda\colon A \hookrightarrow \mathcal{M}(D)/D$. Necessarily, $H_2(A) \subset \eta(\mathcal{N}(D)) = \pi_2^{-1}\rho(D)$. Thus, the *-monomorphism $H_3 := \eta^{-1} \circ H_2$ maps A into $\mathcal{N}(D) \subset B$. We have $H_3(A) \cap J = \eta^{-1}(H_2(A) \cap \eta(\mathcal{N}(D) \cap J))$, $\mathrm{Ann}(D,J) := \pi_J^{-1}(\mathrm{Ann}(\pi_J(D))) = \pi_J^{-1}(0) = J$, $\mathrm{Ann}(D,J) = \{b \in B\,;\ bD + Db \subset J\}$ **??**

and that $b \in \mathcal{N}(D) \cap \mathrm{Ann}(D,J)$, if and only if $bD + Db \subset D \cap J$. It implies

$$\begin{aligned} \eta(\mathcal{N}(D) \cap J) &= \eta(\mathcal{N}(D) \cap \mathrm{Ann}(D,J)) \\ &= \eta(\mathcal{N}(D)) \cap \mathcal{M}(D, D \cap J). \end{aligned}$$

Thus

$$\begin{aligned} H_3(A) \cap J &= \eta^{-1}(H_2(A) \cap \mathcal{M}(D, D \cap J)) \\ &= \eta^{-1}\mathcal{M}(\varphi)(H_1(A) \cap \mathcal{M}(B,J)) \\ &= \eta^{-1}\mathcal{M}(\varphi) H_1(A \cap \mathcal{M}(B,J)) \\ &= H_3(A \cap \mathcal{M}(B,J)) \end{aligned}$$

---

[12] Use here that A is stable, and that B is σ-unital, stable and purely infinite, cf. Corollary 5.5.16 .

for $J \in \mathcal{I}(B)$.

Further, $H_3^{-1}(0) = \{0\}$, and $H_3(A)$ must generate B as a two-sided closed ideal, because $J_1 := \overline{\operatorname{span} BH_3(A)B}$ satisfies $H_3(A) \cap J_1 = H_3(A)$ and,thus, $A \subset \mathcal{M}(B, J_1)$, $AB \subset J_1 \subset B$ and $J_1 = B\,\overline{AB} = B$.

Since A and B are stable and separable, it follows, by Proposition **?? ??**, that H_3 is unitarily homotopic to a nuclear *non-degenerate* *-monomorphism $h \colon A \hookrightarrow B$. We get $h(A) \cap J = h(A \cap \mathcal{M}(B, J))$ for $J \in B$.

Thus $k_1 := \delta_\infty \circ h$ and $k_2 := \delta_\infty | A$ are both nuclear, non-degenerate *-monomorphisms and satisfy $k_i(A) \cap \mathcal{M}(B, J) = k_i(A \cap \mathcal{M}(B, J))$ for $J \in \mathcal{I}(B)$ and $i = 1, 2$. By Corollary **??**, k_1 and k_2 are unitarily homotopic. □

LEMMA 6.3.12. *Suppose that $E \subset F \subset B$ are hereditary C^*–subalgebras of B.*

(i) *If for every $a \in \mathcal{M}(B//E)_+ \cap (B//E)$ with $a \neq 0$ there is $b \in \pi_{BE+EB}(F)$ with $ab \neq 0$, then F is essential in B.*
(ii) *If $D \subset C$ is hereditary, $B = C \otimes M_{2^\infty}$, $E = D \otimes M_{2^\infty}$ and ρ is a pure state on M_{2^∞}, then $B//E \cong (C//D) \otimes M_{2^\infty}$ by a natural completely positive and completely isometric map V from $B//E$ onto $(C//D) \otimes M_{2^\infty}$.*
The hereditary C^*-subalgebra $F \subset B$ that ist generated by $\{b \in B_+ \,;\, (\mathrm{id}_{C//D} \otimes \rho)(V(\pi_{BE+EB}(b))) = 0\}$ satisfies the assumption of (i). *In particular, F is essential in B.*

PROOF. Recall that the product ab in (i) has to be calculated in the von Neumann algebra $(B//E)^{**}$, that is naturally isomorphic to $(1 - p_E)B^{**}(1 - p_E)$.

(i): Let $c \in B_+$ with $cF = \{0\}$. Then $cE = \{0\}$ and, therefore, $c \in \operatorname{Ann}(E)_+ \subset \mathcal{N}(E)_+$. It follows that $a := \pi_{BE+EB}(c) \in \mathcal{M}(B//E)_+ \cap (B//E)$ and that $a\pi_{BE+EB}(F) = \{0\}$. Thus $a = 0$ by assumption (i), i.e., $(1-p_E)c(1-p_E) = 0$, $c^{1/2}(1 - p_E) = 0$, $c = cp_E$ and $c = p_E c$, where $p_E \in B^{**}$ denotes the open support projection of E. Thus $c \in E$. Since $cE = \{0\}$ and $c \in E_+$, we get $c^2 = 0$ and $c = 0$.

(ii): Let $L := CD$ and $R := DC$ the closed left respectively right ideals of C generated by D. Then $EB + BE = (L + R) \otimes M_{2^\infty}$. Therefore, there is a natural completely positive and completely isometric isomorphism V from $B//E := (C \otimes M_\infty)//(D \otimes M_\infty)$ onto $(C//D) \otimes M_\infty$ that satisfies $V((c \otimes d) + EB + BE) = (c + (CD + DC)) \otimes d$ for $c \in C$ and $d \in M_{2^\infty}$, i.e., $V \circ \pi_{BE+EB} = \pi_{CD+DC} \otimes \mathrm{id}$. Let $A := \mathcal{M}(C//D) \cap (C//D)$. Using slice maps, one can see that $V|\mathcal{M}(B//E) \cap (B//E)$ is a *-isomorphism from $\mathcal{M}(B//E) \cap (B//E)$ onto $A \otimes M_{2^\infty}$, and $\mathcal{N}(E) = \mathcal{N}(D) \otimes M_{2^\infty}$.

Let $G := \{g \in M_{2^\infty} \,;\, \rho(g^*g + gg^*) = 0\}$. The G is the hereditary C^*-subalgebra of M_{2^∞} generated by $\{0 \leq g \in M_{2^\infty} \,;\, \rho(g) = 0\}$, and G is essential in M_{2^∞} because $0 \leq a \in M_{2^\infty}$ and $aG = 0$ implies that $a = 0$ or that the pure state ρ is faithful on the hereditary C^*-subalgebra of M_{2^∞} generated by a is faithful, but the latter can not happen because M_{2^∞} is anti-liminal.

Thus, $\mathcal{N}(D)\otimes G\subset F$. If $a\in(A\otimes M_{2^\infty})_+$ and $aV(F)=\{0\}$, then $a(A\otimes G)=\{0\}$. But this implies that $(f\otimes\mathrm{id})(a)G=\{0\}$ for every positive linear functional f on A, thus $f\otimes\mathrm{id}(a)=0$ for all $f\in A^*$, hence $a=0$.

It follows that $c\in\mathcal{M}(B//E)_+\cap(B//E)$ and $c\pi_{BE+EB}(F)=\{0\}$ together implies $c=0$. □

LEMMA 6.3.13. *Suppose that D_1,D_2 are hereditary C^*–subalgebras of B, that $T\colon B//D_2\to(B//D_1)\otimes M_{2^\infty}$ is a completely positive and completely isometric map from $B//D_2$ onto $(B//D_1)\otimes M_{2^\infty}$ with $T(\pi_2(J))=\pi_1(J)\otimes M_{2^\infty}$ for every closed ideal J of B, where $\pi_k\colon B\to B//D_k:=B/(BD_k+D_kB)$ are the quotient maps for $k=1,2$. Let $\rho\colon M_{2^\infty}\to\mathbb{C}$ a pure state, and let D_3 denote the hereditary C^*-algebra of B that is generated by the elements $f\in B_+$ with $(id\otimes\rho)(T(\pi_2(f)))=0$.*

Then

(i) *$D_3\supset D_2$, D_3 is residually essential in B, and*
(ii) *there is a completely positive and completely isometric isomorphism I from $B//D_3$ onto $B//D_1$ with*

$$I(\pi_3(J))=\pi_1(J)$$

for every $J\in\mathcal{I}(B)$, where $\pi_3\colon B\to B//D_3$ is the quotient map $\pi_3(b):=b+BD_3+D_3B$.

PROOF. Clearly, $V\colon b\in B\mapsto(\mathrm{id}\otimes\rho)(T(\pi_2(b)))\in B//D_1$. is a completely positive contraction from B onto $B//D_1$ with $V(D_3)=0$ and $V(J)=\pi_1(J)$.

Let $A:=M_{2^\infty}$ and $G\subset A$ the hereditary C^*-subalgebra $G:=\{a\in A\,;\ \rho(a^*a+aa^*)=0\}$. It turns out that $I:=[V]\colon B//D_3\to B//D_1$ is a complete isometry: One has $T(\pi_2(D_3))=(B//D_1)\otimes G$ and $T(\pi_2(BD_3+D_3B))=(B//D_1)\otimes(AG+GA)$. The same happens with $T\otimes\mathrm{id}_n$.

to be filled in: new proof coming from Lemma 6.3.12 or 3.1.8

??

Let $G:=\{a\in M_{2^\infty}\,;\ \rho(a^*a)=0=\rho(aa^*)\}$. G is essential in M_{2^∞}. There is no non-zero element $x\in((B//D_1)\otimes M_{2^\infty})_+$ that is orthogonal to the support projections of $(B//D_1)\otimes G$ in $((B//D_1)\otimes M_{2^\infty})^{**}$, because there is no non-zero $x\in(B//D_1)^{**}\otimes M_{2^\infty}$ which is orthogonal to $(B//D_1)^{**}_+\otimes e$ for a strictly positive element e of G.

γ has the property that $\pi_1(J)\otimes M_{2^\infty}$ corresponds to the natural image of $J\otimes M_{2^\infty}$ in $(B\otimes M_{2^\infty})//(D_1\otimes M_{2^\infty})$. Thus, one has natural isomorphisms

$$((B/J)//\pi_J(D_1))\otimes M_{2^\infty}\cong((B//D_1)\otimes M_{2^\infty})/(\pi_1(J)\otimes M_{2^\infty})\,.$$

?? Clearly, $D_2\subset D_3$ and $\pi_2(D_3)$ contains $T((B//D_1)\otimes G)$. Recall that T^{**} is a W*-algebra isomorphism from $((B//D_1)\otimes M_{2^\infty})^{**}$ onto $(B//D_2)^{**}$. If $b\in B_+$ with $bD_3+D_3b=\{0\}$ then $\pi_2(b)$ is orthogonal to $T((B//D_1)\otimes G)$ in $(B//D_2)^{**}$. The element $x:=T^{-1}(\pi_2(b))$ is orthogonal to $(B//D_1)\otimes G$ in $((B//D_1)\otimes M_{2^\infty})^{**}$, which implies (step by step) $x=0$, $\pi_2(b)=0$, $b\in D_1\subset D_3$ and $b^2=0$, $b=0$.

Hence, D_3 is essential in B.

If J is a closed ideal of B, then $\pi_k(J)$ is an M-ideal of $B//D_k$ (i.e., there is a projection z_J in the center of $(B//D_k)^{**}$ with $\pi_k(J)^{**} = z_J(B//D_k)^{**}$) and there are natural completely positive and completely isometric isomorphisms $(B/J)//\pi_J(D_k) \cong (B//D_k)/\pi_k(J)$. Thus T defines a completely isometric and completely positive isomorphism $[T]_J$ from $(B/J)//\pi_J(D_2) \cong (B//D_2)/\pi_2(J)$ onto

$$((B/J)//\pi_J(D_1)) \otimes M_{2^\infty} \cong ((B//D_1) \otimes M_{2^\infty})/(\pi_1(J) \otimes M_{2^\infty}) .$$

Let $[\pi_2]_J$ denote the quotient map $B/J \to (B/J)/\pi_J(D_2)$. $\pi_J(D_3)$ is essential in B/J, because $\pi_J(D_3)$ is the hereditary C^*–subalgebra of B/J which is generated by the elements $g \in (B/J)_+ = \pi_J(B_+)$ with $\mathrm{id} \otimes \rho)([T]_J^{-1}([\pi_2]_J(g))) = 0$, and because we can replace in the above arguments B, $B//D_2$, $(B//D_1)$, D_2, T and π_2 by B/J, $(B/J)//\pi_J(D_2)$, $((B/J)//\pi_J(D_1))$, $\pi_J(D_3)$, $[T]_J$ and $[\pi_2]_J$. □

LEMMA 6.3.14. *Suppose that B is stable and that $T_n\colon B \to B$ is a sequence of completely positive contractions. Then there is a sequence of isomorphisms $\gamma_n\colon B \to B\otimes M_{2^n}$, such that γ_n is unitarily homotopic to the map $b \in B \mapsto b\otimes p_{11} \in B\otimes M_{2^n}$. The sequence (γ_n) defines a *-isomorphism γ_∞ from $B_\infty := \ell_\infty(B)/c_0(B)$ onto*

$$E := \left(\prod_n B \otimes M_{2^n}\right) / \bigoplus_n (B \otimes M_{2^n}) \subset (B \otimes M_{2^\infty})_\infty$$

by

$$\gamma_\infty((b_1, b_2, \ldots) + c_0(B)) := (\gamma_1(b_1), \gamma_2(b_2), \ldots) + c_0(B \otimes M_{2^\infty}) .$$

Let $S_n := \gamma_{n+1}^{-1}(T_n \otimes \iota_n) \circ \gamma_n$, where $\iota_n\colon a \in M_{2^n} \mapsto a \otimes 1_2 \in M_{2^{n+1}}$. Consider $X := \mathrm{indlim}(T_n\colon B \to B)$ and $Y := \mathrm{indlim}(S_n\colon B \to B)$ as subspaces of $B_\infty := \ell_\infty(B)/c_0(B)$, and let

$$Z := \mathrm{indlim}(T_n \otimes \iota_n\colon B \otimes M_{2^n} \to B \otimes M_{2^{n+1}}) \subset E .$$

Then

(i) *Z coincides with the image of $X \otimes M_{2^\infty}$ by the natural *-monomorphism η from $B_\infty \otimes M_{2^\infty}$ into E.*

(ii) *The isomorphism γ_∞ from B_∞ onto E maps Y onto $Z = \eta(X \otimes M_{2^\infty})$ and satisfies*

$$\gamma_\infty(Y \cap J_\infty) = \eta((X \cap J_\infty) \otimes M_{2^\infty})$$

for all closed ideals J of B.

(iii) *If T_n is residually nuclear, then S_n is residually nuclear and X is a nuclear (non-unital) operator system.*

PROOF. Let κ_n an isomorphism from the algebra of compact operators $\mathbb{K}$ onto $\mathbb{K} \otimes M_{2^n}$. Since every endomorphism of $\mathbb{K}$ onto a hereditary C^*–subalgebra D of $\mathbb{K}$ is unitarily homotopic to the identity map on $\mathbb{K}$ (cf. e.g. Corollary 5.5.6), we get that $a \in \mathbb{K} \mapsto a \otimes p_{11} \in \mathbb{K} \otimes M_{2^\infty}$ is unitarily homotopic to κ_n. There is an isomorphism λ from B onto $B \otimes \mathbb{K}$, because B is stable and $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$. Then $\gamma_n := (\lambda \otimes \mathrm{id}_n)^{-1} \circ (\mathrm{id}_B \otimes \kappa_n) \otimes \lambda$ is an isomorphism from B onto $B \otimes M_{2^n}$ that is unitarily homotopic to $b \in B \mapsto b \otimes p_{11} \in B \otimes M_{2^n}$. In particular, $\gamma_n(J) = J \otimes M_{2^n}$ for $J \in \mathcal{I}(B)$, which implies

$$\gamma_\infty(J_\infty) = \prod_n (J \otimes M_{2^n}) / \bigoplus_n (J \otimes M_{2^n}) .$$

(i): Let $I := \bigoplus_n B \otimes M_{2^n} \subset c_0(B \otimes M_{2^\infty})$ and let $V_n \colon M_{2^\infty} \to M_{2^n}$ the natural conditional expectation from M_{2^∞} onto $M_{2^n} \cong M_{2^n} \otimes 1 \otimes 1 \otimes \ldots \subset M_{2^\infty}$ with $V_n(a \otimes b) = \mathrm{tr}(b)a$ for $a \in M_{2^n}$, $b \in M_{2^\infty}$. Then $\eta \colon B_\infty \otimes M_{2^\infty} \to E$ is the C^*-morphism that is defined by

$$\eta\left(((b_1, b_2, \ldots) + c_0(B)) \otimes c\right) := (b_1 \otimes V_1(c), b_2 \otimes V_2(c), \ldots) + I$$

for $(b_1, b_2, \ldots) \in \ell_\infty(B)$ and $c \in M_{2^\infty}$. This defines η, because $B_\infty \otimes M_{2^\infty} = B_\infty \otimes^{\max} M_{2^\infty}$ by nuclearity of M_{2^∞}.

Let $U_{k,\infty} \colon B \otimes M_{2^k} \to Z \subset E$ the canonical maps for the inductive limit Z defined by the maps $T_n \otimes \iota_n$, i.e., starting with $k-1$ zeros one has

$$U_{k,\infty}(d) = (0, \ldots, 0, d, T_k \otimes \iota_k(d), (T_{k+1}T_k) \otimes (\iota_{k+1}\iota_k)(d), \ldots) + I ,$$

where $d \in B \otimes M_{2^k}$.

Then Z is the closed linear span of $U_{k,\infty}(b \otimes c)$ for $b \in B$ and $c \in M_{2^k}$, $k \in \mathbb{N}$. A straight calculation shows that this set coincides with $\eta(X \otimes M_{2^\infty})$.

(ii): We have $(T_n \otimes \iota_n) \circ \gamma_n = \gamma_{n+1} \circ S_n$. Thus,

$$\gamma_\infty \circ S_{m,\infty}(b) = U_{m,\infty}(\gamma_m(b)) ,$$

for $b \in B$, which implies that the *-isomorphism γ_∞ from B_∞ onto E maps the closed subspace Y onto Z.

Clearly, $\eta(J_\infty \otimes M_{2^\infty}) \subset \gamma_\infty(J_\infty)$. Since $\gamma_\infty(J_\infty)$ is an ideal, and since M_{2^∞} is simple and nuclear, there is a unique closed ideal K of B_∞ such that $\eta(K \otimes M_{2^\infty}) = \eta(B_\infty \otimes M_\infty) \cap \gamma_\infty(J_\infty)$. Let $(b_1, b_2, \ldots) \in \ell_\infty(B)$ with $(b_1, b_2, \ldots) + c_0(B) \in K$. Then

$$\eta\left(((b_1, b_2, \ldots) + c_0(B)) \otimes 1\right) := (b_1 \otimes 1_2, b_2 \otimes 1_4, \ldots) + I \in \gamma_\infty(J_\infty) ,$$

i.e., there exists a sequence $c_n \in J \otimes M_{2^n}$ with $\lim \|(b_n \otimes 1_{2^n}) - c_n\| = 0$. If we apply the slice maps $\mathrm{id}_B \otimes \mathrm{tr}_{2^n} \colon B \otimes M_{2^n} \to B$, we get that there is a sequence $d_1, d_2, \ldots \in J$ with $\lim \|b_n - d_n\| = 0$, i.e., $(b_1, b_2, \ldots) + c_0(B) \in J_\infty$ and $K = J_\infty$.

The equation $(X \otimes M_{2^\infty}) \cap (J_\infty \otimes M_{2^\infty}) = (X \cap J_\infty) \otimes M_{2^\infty}$ holds by nuclearity of M_{2^∞}.

(iii): Let J a closed ideal of B. Then $\gamma_n(J) = J \otimes M_{2^n}$ and $(T_n \otimes \iota_n)(J \otimes M_{2^n}) \subset J \otimes M_{2^{n+1}}$. Thus, $S_n(J) = \gamma_{n+1}^{-1}\big((T_n \otimes \iota_n)(\gamma_n(J))\big) \subset J$, i.e., S_n is residually equivariant if T_n is residually equivariant. Since residually nuclear maps are residually equivariant, S_n is residually equivariant if T_n is residually nuclear. Then the completely positive quotient map $[S_n]_J \colon B/J \to B/J$ is given by $[S_n]_J = [\beta_{n+1}] \circ ([T_n]_J \otimes \iota_n) \circ [\beta_n]$, where $[T_n]_J \colon B/J \to B/J$ is nuclear and β_n is an isomorphism from B/J onto $(B/J) \otimes M_{2^n}$. Thus, $S_n \colon B \to B$ is residually nuclear.

The inductive limit X of the residually nuclear contractions $T_n : B \to B$ is a nuclear C^*-system by Lemma A.19.6, because residually nuclear maps in particular are nuclear. $\square$

LEMMA 6.3.15. *Suppose that B is a stable and σ-unital C^*-algebra. Let $d \in \mathcal{M}(B)$ with $\|d\| \leq 1$. Then there is a norm-continuous path $t \in [0, \infty) \mapsto s(t) \in \mathcal{M}(B)$ into the isometries of $\mathcal{M}(B)$, such that $\lim_{t\to\infty} s(t)^* b s(t) = d^* b d$ for all $b \in B$.*

PROOF. By Lemma 5.1.2(iv), there exist norm-continuous maps $t \in [0, \infty) \mapsto T_0(t), T_1(t) \in \mathcal{M}(B)$ with $T_0(t)^* T_0(t) = T_1(t)^* T_1(t) = T_0(t) T_0(t)^* + T_1(t) T_1(t)^* = 1$, and $\lim_{t\to\infty} T_k(t)^* b = kb$ for $b \in B$, $k = 0, 1$. Let $s(t) := T_0(t)(1 - d^* d)^{1/2} + T_1(t) d$. Then $t \mapsto s(t)$ is norm-continuous and $\lim_{t\to\infty} s(t)^* b = d^* b$. Since $B = BB$, it follows $\lim_{t\to\infty} s(t)^* b s(t) = d^* b d$. $\square$

Next has to be checked again ??

LEMMA 6.3.16. *Suppose that B a C^*-algebra and that $s_1, s_2, \ldots \in \mathcal{M}(B)$ is a sequence of isometries in $\mathcal{M}(B)$. Let $S_n(b) := s_n^* b s_n$ for $b \in B$ and $n \in \mathbb{N}$, $t_n := s_1 s_2 \cdots s_n$, $q_n := 1 - t_n t_n^*$, and denote by E the closure of $\bigcup_n q_n B q_n$.*

Then E is a hereditary C^*-subalgebra of B and there is a natural completely positive and completely isometric map V from $B//E$ onto $\mathrm{indlim}(S_n : B \to B) \subset B_\infty$ *that satisfies*

$$V(\pi_{BE+EB}(J)) = V(B//E) \cap J_\infty \qquad \text{for} \quad J \in \mathcal{I}(B)\,.$$

PROOF. Since $q_n \leq q_{n+1}$, the set E is the hereditary C^*-subalgebra of B generated by $\bigcup_n q_n B q_n$.

Let $p_n := 1 - q_n = t_n t_n^*$, $R_n(b) := p_n b p_n$, $\pi_{BE+EB} : B \to B//E$ the quotient map, $X := \mathrm{indlim}(S_n : B \to B)$, $T_n(b) := t_n^* b t_n$ and $\rho_n(b) := t_n b t_n^*$ for $b \in B$. Notice that $T_n \circ \rho_n = \mathrm{id}_B$ and $T_{k+n} \circ \rho_{k-1} = S_n \circ S_{n-1} \circ \cdots \circ S_k$.

Let $\pi_\infty(b_1, b_2, \ldots) := (b_1, b_2, \ldots) + c_0(B)$. We define completely positive contractions $T_\infty : B_\infty \to B_\infty$ and $\rho_\infty : B_\infty \to B_\infty$ by

$$\begin{aligned} T_\infty(\pi_\infty(b_1, b_2, \ldots)) &:= \pi_\infty(T_1(b_1), T_2(b_2), \ldots) \\ \rho_\infty((b_1, b_2, \ldots)) &:= \pi_\infty(\rho_1(b_1), \rho_2(b_2), \ldots)\,. \end{aligned}$$

Notice $T_\infty \circ \rho_\infty = \mathrm{id}$ on B_∞, and that $T_\infty(\pi_\infty(b_1, b_2, \ldots)) = t_\infty^* \pi_\infty(b_1, b_2, \ldots) t_\infty$ for the isometry

$$t_\infty := \pi_\infty(t_1, t_2, \ldots) \in \mathcal{M}(B)_\infty \subset \mathcal{M}(B_\infty)$$

with range projection

$$p_\infty := t_\infty t_\infty^* = \pi_\infty(p_1, p_2, \ldots) \in \mathcal{M}(B)_\infty\,.$$

We introduce a "diagonal" map $\Delta : B \to B_\infty$ by $\Delta(b) := \pi_\infty(b, b, \ldots)$.

The C^*-space $B//E$ is isomorphic to $Y := \operatorname{indlim}(R_n\colon B \to B)$, because $R_m R_n = R_n R_m = R_{\max(m,n)}$ (coming from $p_{n+1}p_n = p_{n+1}$) implies that a completely positive and isometric isomorphism from $B//E$ onto

$$Y := \operatorname{indlim}_n(R_n\colon B \to B) \subset B_\infty$$

is given by

$$b \in B \mapsto P_\infty(b) := \pi_\infty((p_1 b p_1, p_2 b p_2, \ldots)) = p_\infty \Delta(b) p_\infty \in B_\infty\,,$$

i.e., for $y = [y_{ij}] \in M_k(B//E) \cong M_k(B)//M_k(E)$ and any $b := [b_{ij}] \in M_k(B)$ with $\pi(b_{ij}) = y_{ij}$ holds that $\|[P_\infty(b_{ij})]\| = \|[y_{ij}]\|$ for the norms in $M_k(B_\infty) \cong (M_k(B))_\infty$ respectively in $M_k(B//E)$, and $[P_\infty(b_{ij})]$ is positive if and only if y is positive in $M_k(B//E)$. This is because

$$\|[y_{ij}]\| = \operatorname{dist}\big([b_{ij}], M_n(BE + EB)\big) = \limsup_n \|[p_n b_{ij} p_n]\|\,.$$

The argument for the complete positivity of the class-map $\eta := [P_\infty]\colon B//E \to Y$ is similar. The completely isometric and completely positive isomorphism η from $B//E$ onto Y is determined by $\eta \circ \pi_{BE+EB} = P_\infty$.

It follows that $\eta(\pi_{BE+EB}(J)) = P_\infty(J) \subseteq J_\infty \cap Y$ for $J \in \mathcal{I}(B)$. The inclusion "$\supseteq$" needs a less simple argument: If $\pi_\infty(p_1 a p_1, p_2 a p_2, \ldots) \in J_\infty$, then there is a bounded sequence $b_1, b_2, \ldots \in J$ with $\lim_{n\to\infty} \|p_n a p_n - p_n b_n p_n\| = 0$. It follows that $\pi_{BE+EB}(a) = \lim_n \pi_{BE+EB}(p_n b_n p_n)$ in $B//E$, i.e., $\pi_{BE+EB}(a)$ is in the closure of $\pi_{BE+EB}(J)$. But it turns out that $J \cap (BE + EB) = J(E \cap J) + (E \cap J)J$ and that the natural map from $J//(E \cap J)$ to $B//E$ is (completely) isometric. This can be seen from the second conjugate spaces which are naturally isomorphic to $(1 - q_E)B^{**}(1 - q_E)q_J$ and to $(1 - q_E)B^{**}(1 - q_E)$. Here we denote by q_E and q_J the open support projection of E and J in B^{**}. This implies together that

$$\eta(\pi_{BE+EB}(J)) = P_\infty(J) = J_\infty \cap Y = (p_\infty J_\infty p_\infty) \cap Y\,.$$

Since $t_n^* t_n = p_n$, we get – for the isometry $t_\infty := \pi_\infty(t_1, t_2, \ldots) \in \mathcal{M}(B)_\infty \subseteq \mathcal{M}(B_\infty)$ – that the restriction to Y of the c.p. contraction

$$T_\infty(\pi_\infty(b_1, b_2, \ldots)) := t_\infty^* \pi_\infty(b_1, b_2, \ldots) t_\infty$$

is a completely positive complete isometry from Y onto the image of the map $S_{1,\infty}\colon B \to X \subset B_\infty$.

In fact, $S_{1,\infty}(b) = T_\infty(\Delta(b)) = T_\infty(P_\infty(b))$. The defining maps $S_{k,\infty}\colon B \to X \subset B_\infty$ for the inductive limit $X = \operatorname{indlim}_n(S_n\colon B \to B)$ are given by

$$S_{k,\infty}(b) := \pi_\infty(0, \ldots, 0, S_k(b), S_{k+1}(S_k(b)), \ldots)\,.$$

(It is not important if one takes zero at places 1 to $k-1$ or takes other elements of B.) The inductive limit $X = \operatorname{indlim}_n(S_n\colon B \to B)$ is the closure of $\bigcup_k S_{k,\infty}(B)$ in B_∞. But, since $S_{1,\infty} = S_{k,\infty} \circ S_{k-1} \circ \cdots \circ S_1$, $S_{k,\infty} = S_{1,\infty} \circ \rho_{k-1}$, and $\rho_n(B) = q_n B q_n$, it follows that

$$S_{k,\infty}(B) = S_{0,\infty}(B) = \{T_\infty(\Delta(b))\,;\ b \in B\} = T_\infty(Y)$$

are the same linear subspace of B_∞. The image $T_\infty(Y) = S_{1,\infty}(B)$ of the (not necessarily unital) C^*-system $Y \subseteq p_\infty B_\infty p_\infty$ for the projection $p_\infty := \pi_\infty(p_1, p_2, \ldots) \in \mathcal{M}(B)_\infty \subseteq \mathcal{M}(B_\infty)$ is closed, because the restriction of $\rho_\infty \circ T_\infty$ to $p_\infty B_\infty p_\infty$ is the identity map, i.e., $T_\infty | p_\infty B_\infty p_\infty$ must be isometric on Y.

The map $T_\infty \circ \eta \colon B//E \to X$ is induced from $T_\infty \circ \eta \circ \pi_{BE+EB} = T_\infty \circ P_\infty$. Since $T_\infty(P_\infty(B)) = T_\infty(Y) = X$, it follows that $T_\infty \circ \eta = [T_\infty \circ P_\infty] \colon B//E \to X$ defines a complete isometry from $B//E$ onto X.

Thus, $\rho_\infty | X$ is a completely positive and completely isometric map from $X \cong B//E$ onto Y, and the map $(\rho_\infty | X)^{-1}$ is given by the class-map $V := [S_{1,\infty}] \colon B//E \to X$ with $V \circ \pi_{BE+EB} = S_{1,\infty} = T_\infty \circ \Delta$.

Since $T_n(p_n J p_n) = T_n(J) = t_n^* J t_n = J$ and $\rho_n(J) = p_n J p_n = J \cap p_n B p_n$, we get for all $J \in \mathcal{I}(B)$ that

$$T_\infty(p_\infty J_\infty p_\infty) = T_\infty(J_\infty) = J_\infty$$

and

$$\rho_\infty(J_\infty) = p_\infty J_\infty p_\infty = J_\infty \cap p_\infty B_\infty p_\infty \,.$$

$$\eta(B//E) \cap \pi(J)_\infty = \rho_\infty(X \cap J_\infty) = P_\infty(J) == \eta(\pi_{BE+EB}(J)) \,.$$

It follows that $V := [T_\infty \circ \Delta] = T_\infty \circ \eta \colon B//E \to X = \mathrm{indlim}_n(S_n \colon B \to B)$ is a complete isometry from $B//E$ onto X and satisfies

$$V(\pi_{BE+EB}(J)) = J_\infty \cap X = \mathrm{indlim}_n(S_n \colon J \to J) \,.$$

□

Proposition 6.3.17. *Suppose that B is separable and stable and has the WvN-property, and that $T_n \colon B \to B$ is a sequence of residually nuclear completely positive contractions. Consider $X := \mathrm{indlim}(T_n \colon B \to B)$ naturally as operator subspace of $B_\infty := \ell_\infty(B)/c_0(B)$.*

Then X is a nuclear subspace of B_∞ and there exist a residually essential *hereditary C^*-subalgebra D of B and a completely isometric and completely positive isomorphism I from $B//D$ onto $X = \mathrm{indlim}(T_n \colon B \to B)$, such that*

$$I(\pi_{BD+DB}(J)) = J_\infty \cap X$$

for every closed ideal J of B.

Proof. The inductive limit X is nuclear by Lemma 6.3.14(iii), because the maps T_n are residually nuclear contractions. Since B has the WvN-property, the maps T_n are in the point-norm closure of the set of one-step inner maps $b \mapsto c^*bc$. By Lemma 3.1.8 the T_n are in the point-norm closure of the *contractive* inner c.p. maps $b \mapsto d^*bd$, i.e., with $\|d\| \leq 1$. By Lemma 6.3.15, the maps $b \mapsto d^*bd$ with $\|d\| \leq 1$ can be approximated in point-norm topology by maps $b \mapsto t^*bt$ with an isometry $t \in \mathcal{M}(B)$, because B is stable and σ-unital.

Now we can apply Lemma A.14.2 to the sequence of maps $T_1, T_2, \ldots$ and the set of maps $b \mapsto t^*bt$ with isometries $t \in \mathcal{M}(B)$, because B is separable. We get that are isomeries $t_n \in \mathcal{M}(B)$ such that $X = \text{indlim}(L_n \colon B \to B)$ for the maps $L_n(b) := t_n^* b t_n$.

Then, by Lemma 6.3.16, there is a hereditary C^*-subalgebra F of B and completely isometric and completely positive isomorphism φ from $B//F$ onto X with $\varphi(\pi_{BF+FB}(J)) = J_\infty \cap X \subset B_\infty$ for $J \in \mathcal{I}(B)$. By Lemma 6.3.14 there are residually nuclear contractions $S_n \colon B \to B$ such that there is a completely isometric and completely positive isomorphism ψ from $Y := \text{indlim}(S_n \colon B \to B)$ onto $X \otimes M_{2^\infty}$ with $\psi(J_\infty \cap Y) = (J_\infty \cap X) \otimes M_{2^\infty}$ for $J \in \mathcal{I}(B)$. In particular,

$$Y \cong (B//F) \otimes M_{2^\infty} \cong (B \otimes M_{2^\infty})//(F \otimes M_{2^\infty}),$$

If we repeat the above arguments with S_n (in place of T_n), then we get that there is a hereditary C^*-subalgebra $E \subset B$ and a completely isometric and completely positive isomorphism χ from $B//E$ onto Y such that $\chi(\pi_{BE+EB}(J)) = J_\infty \cap Y$. If $\rho \colon M_{2^\infty} \to \mathbb{C}$ is a pure state and $V \colon X \otimes M_{2^\infty} \to X \subset B_\infty$ is the completely positive contraction with $V(c \otimes d) = \rho(c)d$, then there is a hereditary C^*-subalgebra D of B with $E \subset D \subset B$ such that $BD + DB$ is the kernel of the completely positive map $W := V \circ \psi \circ \chi \circ \pi_{BE+EB} \colon B \to X$ from B into X. Let $I := [W]_{BD+DB} \colon B//D \to X$. By Lemma 6.3.13, D is residually essential in B and I is a completely isometric and completely positive isomorphism from $B//D$ onto X. If J is a closed ideal of B then, for $J \in \mathcal{I}(B)$,

$$\begin{aligned} I\pi_{BD+DB}(J) &= V\big(\psi\big(\chi\,(\pi_{BE+EB}(J))\big)\big) = V\big(\psi(Y \cap J_\infty)\big) \\ &= V\big((X \cap J_\infty) \otimes M_{2^\infty}\big) = X \cap J_\infty\,. \end{aligned}$$

□

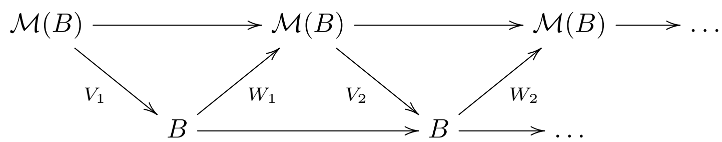
Proof of Theorem 6.3.1: Ad(I): Let a_k be a dense sequence in the unit-ball of A. There are completely positive maps $V_n \colon \mathcal{M}(B) \to B$ and $W_n \colon B \to \mathcal{M}(B)$ that satisfy conditions (b) and (c) of Theorem 6.3.1 and

$$\|W_n V_n(a_j) - a_j\| < 2^{-n} \qquad \text{for } \ 1 \leq j \leq n\,.$$

Then $A \subseteq \mathcal{M}(B) \subset \mathcal{M}(B)_\infty := \ell_\infty(\mathcal{M}(B))/c_0(\mathcal{M}(B))$ is naturally contained in

$$Y := \text{indlim}(S_n \colon \mathcal{M}(B) \to \mathcal{M}(B)) \subset J_\infty$$

with $S_n := W_n V_n$. Indeed, the compositions of the upper rows S_n of the commutative diagram



converge asymptotically to id_A on $A \subset \mathcal{M}(B) \subset \mathcal{M}(B)_\infty$, in the sense, that $S_{n,\infty} \colon \mathcal{M}(B) \to \mathcal{M}(B)_\infty$ converges in point-norm on A to $\text{id}_A \colon A \to \mathcal{M}(B)_\infty$, where

$$S_{n,\infty}(b) := (0, \dots, 0, b, S_n(b), S_{n+1}S_n(b), \dots) + c_0(B).$$

The product maps $\prod_{n\geq 1} V_n$ and $(b_1, b_2, \dots) \mapsto (0, W_1(b_1), W_2(b_2), \dots)$ define complete contractions V_∞ from $\mathcal{M}(B)_\infty$ into B_∞ and $\sigma \circ W_\infty$ from B_∞ into $\mathcal{M}(B)_\infty$. Let $T_n := V_{n+1}W_n \colon B \to B$. Then T_n is a residually nuclear contraction by construction of V_n and W_n, indeed: $U_e \colon B \ni b \mapsto V_{n+1}(eW_n(b)e) \in B$ is a residually nuclear map by conditions (a),(b) and (c) for every $e \in B_+$, and, since V_{n+1} strictly continuous as a map from $\mathcal{M}(B)$ into B (with norm topology), T_n is the point-norm limit of $\{U_{e_\tau}\}$ if $e_\tau \subset B_+$ is an approximate unit of B. V_∞ maps Y into

$$X = \text{indlim}(T_n \colon B \to B) \subset B_\infty$$

and σW_∞ maps X into Y.

We have $W_\infty \circ V_\infty \circ \sigma | Y = (S_\infty \circ \sigma)|Y = \text{id}_Y$ because $S_\infty \circ \sigma \circ S_{m,\infty} = S_{m,\infty}$. Clearly $V_\infty \circ \sigma \circ W_\infty) = T_\infty$. Since $\sigma \circ T_\infty | X = \text{id}_X$, we get $\sigma V_\infty \circ (\sigma W_\infty)|X = \text{id}_X$. If follows that $\sigma W_\infty | X$ is a completely isometric and completely positive map from X *onto* Y.

Let us consider the actions $\Psi_1(J) = J_\infty \subset B_\infty$ and $\Psi_2(J) = \mathcal{M}(B, J)_\infty \subset \mathcal{M}(B)_\infty$ for $J \in \mathcal{I}(B)$ of $\text{Prim}(B)$ on B_∞ and $\mathcal{M}(B, J)_\infty$. Since W_n and V_n are weakly residually equivariant, we get

$$\begin{aligned} V_\infty(\Psi_2(J)) &\subset \Psi_1(J) \quad \text{and} \\ \sigma W_\infty(\Psi_1(J)) &\subset \Psi_2(J). \end{aligned}$$

$$\text{Thus} \quad \begin{aligned} V_\infty(\Psi_2(J) \cap Y) &\subset \Psi_1(J) \cap X \quad \text{and} \\ \sigma W_\infty(\Psi_1(J) \cap X) &\subset \Psi_2(J) \cap Y. \end{aligned}$$

Since $(V_\infty|Y)^{-1} = (\sigma W_\infty | X)$, we get $\sigma W_\infty(\Psi_1(J) \cap X) = \Psi_2(J) \cap Y$. By Proposition 6.3.17 there is a residually essential hereditary C^*-subalgebra D of B and a completely isometric and completely positive map $T \colon B/\!/D \xrightarrow{\sim} X \subset B_\infty$, such that $\overline{BDB} = B$ and

$$T(\pi_D(J)) = \Psi_1(J) \cap X$$

for $J \in \mathcal{I}(B)$. Let $V := (\sigma \circ W_\infty) \circ T$, $C := J_\infty$ and $\Psi_C(J) := \Psi_2(J)$ for J in $\mathcal{I}(B)$.

`to be filled in! ??`

Then Lemma 6.3.8 and Lemma A.19.8 define a *-monomorphism λ from A into $\mathcal{M}(B/\!/D) \cap (B/\!/D) = \pi_1(\mathcal{N}(D))$ such that $\lambda = V^{-1}|A$ and that $\lambda(A \cap \mathcal{M}(B, J)) = \lambda(A) \cap \pi_1(\mathcal{N}(D) \cap J)$, because $V^{-1}(A \cap \mathcal{M}(B, J)) = V^{-1}(A \cap \Psi_C(J)) = V^{-1}(A) \cap \pi_1(J)$.

The operator space X is nuclear, because $T_n \colon B \to B$ is nuclear for every $n \in \mathbb{N}$. Thus $B/\!/D \cong X$ is nuclear, and the assumptions of Proposition 6.3.11 are satisfied for B and $D \subset B$, and $A \subset \mathcal{M}(B)$, $\lambda \colon A \hookrightarrow \pi_1(\mathcal{N}(D)) = \mathcal{M}(B/\!/D) \cap (B/\!/D)$

are as in part (iii) of Proposition 6.3.11. By 6.3.11(iii) we get the class $[H] \in \mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}\, B, A, B)$.

Now suppose that $A \cong E \otimes \mathcal{O}_2$ or $B \cong E \otimes \mathcal{O}_2$ for a stable separable C^*-algebras E. Then we can use the natural isomorphisms

$$\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B); E \otimes \mathcal{O}_2, B) \cong \mathrm{KK}_{\mathrm{nuc}}(\mathrm{Prim}(B); SE \otimes \mathcal{O}_2, B)$$

(for separable stable B with WvN-property [13]) respectively

$$\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B); A, E \otimes \mathcal{O}_2) \cong \mathrm{KK}_{\mathrm{nuc}}(\mathrm{Prim}(E) : SA, E \otimes \mathcal{O}_2)$$

cf. Chapter 8. The left sides of the above isomorphisms are zero because $\mathrm{KK}_{\mathrm{nuc}}(\mathrm{Prim}(B), \cdot, \cdot)$ is homotopy invariant in each variable and id is unitarily homotopic to $\mathrm{id} \oplus \mathrm{id}$ for all C^*-algebras $\cong C \otimes \mathcal{O}_2$.

Hence $[H] = 0$ in this cases, and Proposition 6.3.11(iv) gives the desired non-degenerate nuclear *-monomorphism $h \colon A \hookrightarrow B$ with $h(A \cap \mathcal{M}(B, J)) = h(A) \cap J$. $\delta_\infty \circ h$ is unitarily homotopic to $\delta_\infty | A$ by Corollary **??**.

Ad(II): Suppose, that we can find the W_n such that $W_n(B) \subset A$ (in addition). Then $S_n(A) = W_n V_n(A) \subset A$, $A = \mathrm{indlim}(S_n \colon A \to A) \subset A_\infty \subset B_\infty$, $S_{m,\infty}(B) = S_{m+1,\infty}(S_n(B)) \subset S_{m+1,\infty}(A) \subset A$. It follows that $Y = A$ in B_∞, because Y is the closure of $\bigcap_m S_{m,\infty}(B)$.

Thus, $\lambda = V^{-1} \colon A \hookrightarrow \pi_D(\mathcal{N}(D))$ is an isomorphism from A onto $B//D$ and $\pi_D(\mathcal{N}(D)) = B//D$, i.e., $B = \mathcal{N}(D) + \overline{BD} + \overline{DB}$. In particular, $A \cong B//D$ is nuclear.

If $[H] = 0$ in $\mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(D), A, D) \cong \mathrm{Ext}_{\mathrm{nuc}}(\mathrm{Prim}(B), A, B)$, then again we find a non-degenerate *-monomorphism $H_3 \colon A \to \mathcal{N}(D) \subset B$ with $\pi_D(H_3(a)) = \lambda(a)$ such that

$$H_3(A \cap \mathcal{M}(B, J)) = H_3(A) \cap J$$

(cf. proof of Proposition 6.3.11(iv)). As in the proof of Proposition 6.3.11(iv), it follows that $H_3(A)$ generates B as a two-sided closed ideal. The hereditary C^*-subalgebra $D_1 := \overline{H_3(A) B H_3(A)}$ of B is stable.

There is a *-isomorphism φ from B onto D_1 that is unitarily homotopic to id_B by 5.5.6. In particular, $\varphi(J) = D_1 \cap J$ for $J \in \mathcal{I}(B)$. Let $\pi_D \colon B \to B//D = \lambda(A) = \pi_D(H_3(A))$ the natural quotient map. Then $P_1 := H_3 \lambda^{-1} \pi_D$ satisfies $P_1 H_3 = H_3$ and $P_1(B) \subset H_3(A)$.

It follows $P_1(D_1) \subset D_1$. Thus, $P := \varphi^{-1} \circ P_1 \circ \varphi$ and $h := \varphi^{-1} \circ H_3$ are as desired.

The extremality of the conditional expectation $P \colon B \to h(A)$ follows as in Remark 6.2.2. □

[13] Recall that our nuclear Ext-groups have rather strong equivalence relations that require e.g. that the extensions are *stable* as C^*-algebras.

CHAPTER 7

Groups of asymptotic morphisms $\mathrm{R}(\mathcal{C};\, A, B)$

We consider here "nuclear" (respectively "residually nuclear" , respectively "$\mathcal{C}$-compatible") asymptotic morphisms. Their unitary equivalence classes modulo $\mathrm{C}_0(\mathbb{R}_+, B)$ form a semigroup $\mathrm{SR}(A, B)$ (respectively $\mathrm{SR}(X;\, A, B)$, $\mathrm{SR}(\mathcal{C};\, A, B)$) if the multiplier algebra of B contains a copy of $\mathcal{O}_2$ unitally. The corresponding Grothendieck group $\mathrm{R}(A, B)$ (respectively $\mathrm{R}(X;\, A, B)$, $\mathrm{R}(\mathcal{C};\, A, B)$) is a continuous analog of **Rørdam's group** of asymptotic morphisms over $\mathbb{N}$, cf. [**677**], [**679**], [**680**] [**681**]. This groups are the mediators between the semi-groups $[\mathrm{Hom}(\mathcal{C}; A, B)]$ of unitary equivalence classes of the under Cuntz-addition invariant morphisms in $\mathrm{Hom}(A, B) \cap \mathcal{C}$ and the $\mathcal{C}$-equivariant KK-groups $\mathrm{KK}(\mathcal{C};\, A, B)$. If this is an isomorphism then $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$-equivalence implies Morita equivalence of A and B. It means that we "reduce" some classification problems to questions on the "realization" of KK-theory equivalence by isomorphisms.

The main tools are continuous modifications and generalizations of the Rørdam groups $\mathrm{R}(A, B)$. They play a role in several generalizations, e.g. to the non-simple case as described above, or others, mentioned in the remarks of this Chapter. Some of the known stable invariants for simple separable nuclear C^*-algebras are the same for $\mathrm{R}(.,.)$-equivalent C^*-algebras A and B (e.g. they are KK-equivalent, $F(A)$ and $F(B)$ contain, up to isomorphisms, the same simple separable C^*-subalgebras, the stable rank, the real rank and decomposition rank are the same). A part of further study on classification could be the study of the consequences of $\mathrm{R}(.,.)$-equivalence.

??

1. The semigroups $\mathrm{SR}(\mathcal{C};\, A, B)$

By $\mathrm{CP}(A, B) \subset \mathcal{L}(A, B)$ we denote the matrix operator-convex cone of the completely positive contractions from a C^*-algebra A into a C^*-algebra B.

(We shall later require in addition that A and B are stable, that A is separable and that B is σ-unital.)

Definition 7.1.1. Let $\mathcal{C} \subset CP(A, B)$ denote a point-norm closed matricial operator-convex sub-cone of the cone of completely positive maps from A to B.

Suppose that $V\colon [0, \infty) \ni t \mapsto V(t) \in \mathcal{C} \subset \mathrm{CP}(A, B)$ is a point-norm continuous map (i.e., is continuous with respect to the strong operator topology on $\mathcal{L}(A, B)$). The map V is called an **asymptotic $\mathcal{C}$-morphism** from A to B if V satisfies the following conditions (a), (b) and (c):

(a) For every $t \in \mathbb{R}_+$, $V(t)\colon A \to B$ has norm $\|V(t)\| \leq 1$.
(b) The $V(t)$ are **asymptotically multiplicative** , i.e.,

$$\lim_{t\to\infty} \|V(t)(a^*a) - V(t)(a)^*V(t)(a)\| = 0$$

for every $a \in A$.
(c) for every $a \in A_+$ and $\varepsilon > 0$ there exist $b_1, \ldots, b_n \in B_+$ and $c_1, \ldots, c_n \in \mathrm{C_b}(\mathbb{R}_+, B)$ such that $\limsup_{n\to\infty} \|V(t)(a) - \sum_{i=1}^n c_i(t)^* b_i c_i(t)\| \leq \varepsilon$ ([1]).

Compare next blue with def. of unitary homotopy in Definition 5.0.1 !!

Let V_1, V_2 asymptotic $\mathcal{C}$-morphisms. We call V_1 **unitarily homotopic** to V_2 if there is a map $u\colon t \in \mathbb{R}_+ \mapsto u(t) \in \mathcal{M}(B)$ such that $u(t)$ is unitary, is continuous with respect to the strict topology on $\mathcal{M}(B)$ (Is the footnote really true ?? Likely not. No proof exists so far. [2]), and $\lim \|u(t)^*V_1(t)(a)u(t) - V_2(t)(a)\| = 0$ for every a in A.

Suppose that $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally (i.e., there are isometries $s_1, s_2 \in \mathcal{M}(B)$ with $s_1s_1^* + s_2s_2^* = 1$). It is easy to check that the Cuntz sum

$$(V_1 \oplus V_2)(t) := V_1(t) \oplus_{s_1,s_2} V_2(t) := s_1V_1(t)(\cdot)s_1^* + s_1V_1(t)(\cdot)s_1^*$$

defines again an asymptotic $\mathcal{C}$-morphism $t \mapsto (V_1 \oplus V_2)(t)$ from A to B.

The properties of Cuntz addition (cf. Chapter 4) show that the equivalence classes $[V]$ of unitarily homotopic morphisms V form A to B build a semigroup under Cuntz addition. This semigroup and its Grothendieck group will be denoted by $\mathrm{SR}(\mathcal{C};\, A, B)$, respectively by $\mathrm{R}(\mathcal{C};\, A, B)$.

If $\mathcal{C} = CP_{nuc}(A, B)$ is the cone of all nuclear maps from A to B the we shall write $\mathrm{SR}(A, B)$ for $\mathrm{SR}(CP_{nuc}(A, B); A, B)$ and call $\mathrm{SR}(A, B)$ the **Rørdam semigroup**. Its Grothendieck group $\mathrm{Gr}(\mathrm{SR}(A, B))$ will be denoted by $\mathrm{R}(A, B)$ and call it the **Rørdam group**.

Suppose that X is a T_0 space and that $\Psi_A\colon \mathrm{O}(X) \to \mathcal{I}(A)$ and $\Psi_B\colon \mathrm{O}(X) \to \mathcal{I}(B)$ are actions of X on A respectively B. Let $\mathcal{C}_X \subseteq CP(A, B)$ denote the cone of Ψ_A-Ψ_B-residually nuclear maps from A into B. We call the asymptotic $\mathcal{C}_X$-morphisms **residually nuclear asymptotic morphisms**.

Then we write $\mathrm{SR}(X;\, A, B)$ and $\mathrm{R}(X;\, A, B)$ in place of $\mathrm{SR}(\mathcal{C}_X\,;\, A, B)$ respectively of $\mathrm{R}(\mathcal{C}_X;\, A, B)$. We call $\mathrm{R}(X;\, A, B)$ a **Ψ-equivariant Rørdam group**.

Clearly, all the definition depend from the actions Ψ_A and Ψ_B and not from X alone. The point is that they all come from before chosen non-degenerate point-norm closed matrix operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$.

---

[1] Property (c) means, equivalently, that the morphism $h_V\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ corresponding to $V\colon \mathbb{R}_+ \to \mathcal{C}$ with properties (a) and (b) maps A into the ideal of $\mathrm{Q}(\mathbb{R}_+, B)$ generated by $B \subset \mathrm{Q}(\mathbb{R}_+, B)$.

[2] The map $t \mapsto u(t)$ can be chosen norm-continuous if B is σ-unital and stable and A is separable.

Remark 7.1.2. If $t \to V(t) \in \mathcal{C} \subseteq \mathrm{CP}(A,B)$ is an asymptotic $\mathcal{C}$-morphism, then $V(a)(t) := V(t)(a)$ defines an element $V(a) \in \mathrm{C_b}(\mathbb{R}_+, B)$ and

$$h_V \colon a \in A \mapsto V(a) + \mathrm{C_0}(\mathbb{R}_+, B)$$

is a C^*-morphism from A into $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\mathrm{C_0}(R_+, B)$.

If h_V is unitarily equivalent to a C^*-morphism $k\colon A \to B \subset \mathrm{Q}(\mathbb{R}_+, B)$ by a unitary in $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$ then $t \mapsto V(t)$ and h_V satisfies in particular the following additional continuity conditions:

(d) For all lower semi-continuous 2-quasi-traces $\tau_1, \tau_2 \colon \mathrm{Q}(\mathbb{R}_+, B)_+ \to \mathbb{R}_+ \cup \{+\infty\} = [0, \infty]$ holds $\tau_1 \circ h_V = \tau_2 \circ h_V$ if $\tau_1 | B_+ = \tau_2 | B_+$.

(e) Let $\tau_\gamma, \rho \colon \mathrm{Q}(\mathbb{R}_+, B)_+ \to [0, \infty]$ denote lower semi-continuous 2-quasi-traces on $\mathrm{Q}(\mathbb{R}_+, B)_+$, where the family of quasi-traces $\{\tau_\gamma\}$ are indexed by elements γ is an element of a directed net Γ.

If $\{\tau_\gamma \,;\ \gamma \in \Gamma\}$ and ρ satisfy that

$$\sup_{\delta > 0} \lim_{\gamma \in \Gamma} \tau_\gamma((b - \delta)_+) = \rho(b)$$

for all $b \in B_+$ then we require here the condition of the map V that h_V satisfies in addition that

$$\sup_{\delta > 0} \lim_{\gamma \in \Gamma} \tau_\gamma(h_V((a - \delta)_+)) \;=\; \rho(h_V(a))$$

for all $b \in A_+$.

We call the asymptotic $\mathcal{C}$-morphism V **continuous** if it satisfies (in addition) the above conditions (d) and (e). One can see that the continuous asymptotic $\mathcal{C}$-morphism V define a sub-semigroup $\mathrm{SR}_c(\mathcal{C};\, A, B)$ of of $\mathrm{SR}(\mathcal{C};\, A, B)$.

The condition (d) implies in particular that

$$h_V(A) \cap I_1 = h_V(A) \cap I_2 \qquad \text{if} \quad I_1 \cap B = I_2 \cap B$$

for all closed ideals I_1, I_2 of $\mathrm{Q}(\mathbb{R}_+, B)$, if we consider the lower semi-continuous traces $\tau_k \colon \mathrm{Q}(\mathbb{R}_+, B)_+ \to \{0, \infty\}$, $k = 1, 2$, given by $\tau_k(f) := 0$ if $f \in (I_k)_+$ and by $\tau_k(f) := \infty$ if $f \in \mathrm{Q}(\mathbb{R}_+, B) \setminus I_k$. Thus, conditions (a), (b) and (d) imply the following properties (f) and (g). It shows that the conditions (a,b,d,e) together imply moreover the following property (h):

(f) For every $a \in A$ and every closed ideal J of B there exists

$$\lim_{t \to \infty} \| V(t)(a) + J \| \,,$$

and $\Psi(J) := (h_V)^{-1}(h_V(A) \cap \mathrm{Q}(\mathbb{R}_+, J))$ defines a monotone-continuous action of $\mathrm{Prim}(B)$ on A in the sense of Definition 1.2.6.

(g) $\tau_V(a) := \sup_{\delta > 0} \lim_{t \to \infty} \tau((V(t)(a) - \delta)_+) \in [0, \infty]$ exists for every $a \in A_+$ and for every lower semi-continuous 2-quasi-trace $\tau \colon B_+ \to [0, \infty]$.

The map $\tau_V \colon A_+ \to [0, \infty]$ is necessarily a lower semi-continuous 2-quasi-trace.

(h) The map $\tau \mapsto \tau_V$ is continuous from the set of l.s.c. 2-quasi-traces on B_+ to those on A_+ with respect to the topology of point-wise convergence ([3]).

REMARK 7.1.3. Below we will show that the map $t \mapsto u(t) \in \mathcal{M}(B)$ can be chosen norm-continuous if asymptotic $\mathcal{C}$-morphism V_1 and V_2 are unitarily homotopic and if B is σ-unital and stable, cf. ??.

On the other hand, it is easy to check that $V'(t)(a) := u(t)^*V(t)(a)u(t)$ satisfies conditions (a)-(c) for $\mathcal{C} = CP_{nuc}(A, B)$, if $t \mapsto V(t) \in \mathcal{L}(A, B)$ defines a weakly nuclear contraction $V\colon A \to \mathrm{C_b}(\mathbb{R}_+, B) \subseteq \mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B))$ from A into $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B))$ such that $h_V := \pi_{\mathrm{C_0}(\mathbb{R}_+,B)} \circ V$ is a C^*-morphism from A into $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B))/\,\mathrm{C_0}(\mathbb{R}_+, B)$, and if $t \mapsto u(t)$ is a strictly continuous map from $\mathbb{R}_+$ to $\mathcal{M}(B)$.

It follows, that, equivalently, we can consider $\mathrm{SR}(A, B)$ as a the sub-semigroup of the semigroup of unitary equivalence classes of those C^*-morphisms h from A into

$$\mathrm{Q}(\mathbb{R}_+\,, B) := \mathrm{C_b}(\mathbb{R}_+\,, B)/\,\mathrm{C_0}(\mathbb{R}_+\,, B) \subseteq \mathcal{M}(\mathrm{C_0}(\mathbb{R}_+\,, B))/\,\mathrm{C_0}(\mathbb{R}_+\,, B)$$

that have a completely positive contractive lift $V\colon A \to \mathrm{C_b}(\mathbb{R}_+, B)$ which satisfies the conditions (a)-(c).

By Lemma ??,

the unitaries for the unitary equivalences can be chosen in any of the following unital algebras

$$\mathrm{C_b}(\mathbb{R}_+\,, \mathcal{M}(B))/\,\mathrm{C_0}(\mathbb{R}_+\,, B) \subseteq \mathcal{M}(\mathrm{C_0}(\mathbb{R}_+\,, B))/\,\mathrm{C_0}(\mathbb{R}_+\,, B) \to \mathcal{M}(\mathrm{Q}(\mathbb{R}_+\,, B))\,.$$

REMARK 7.1.4. The conditions (c), (d) and (e) are redundant in some cases, e.g. if A has real rank zero or B is a pi-sun algebra. *But (d) and (e) do not imply (c) in general:*

$\mathbb{K}$ is the only simple C^*-algebra that has an asymptotic morphism into $\mathbb{K}$ with (c).

For example let us consider only the requirements (a)–(d) for $B = \mathbb{K}$:

A non-zero, *simple*, and separable C^*–algebra A must be stably isomorphic to $\mathbb{K}$ or is stably projection-less, if A admits a non-zero C^*-morphism $h\colon A\colon\ \mathrm{Q}(\mathbb{R}_+\,,\mathbb{K})$. (In particular, that $\mathbb{K}$ is the only simple separable stable algebra which is SR-equivalent to $\mathbb{K}$.)

The closed ideal J of $\mathrm{Q}(\mathbb{R}_+;\mathbb{K})$ that is generated by $\mathbb{K}$ is Morita-equivalent to the commutative algebra $\mathrm{Q}(\mathbb{R}_+\,, \mathbb{C}) = \mathrm{C}(\beta\mathbb{R}_+ \setminus \mathbb{R}_+)$. In particular, every separable C^*-subalgebra of J is of type I, in particular it is very different from the huge algebra $\mathrm{Q}(\mathbb{R}_+\,; \mathbb{K})$, that has every separable C^*-algebra as a sub-quotient.

---

[3] Notice that the l.s.c. quasi-traces build a compact set with topology of point-wise convergence, because $[0, \infty]$ is compact.

On the other hand, the cone of every separable C^*-algebra is a C^*-subalgebra of $\mathrm{Q}(\mathbb{R}_+, \mathbb{K})$. In particular, $\mathrm{C}_0(\mathbb{R}_+, \mathcal{O}_2) \subset \mathrm{Q}(\mathbb{R}_+, \mathbb{K})$, and thus, (a), (b), (d), (e) (f), (g), (h) are satisfied for a c.c.p. lift V of the inclusion map, but (c) is not satisfied.

If B is stably isomorphic to a unital C^*-algebra B^{st} (its "Cuntz standard form") which contains unitally a copy of $\mathcal{O}_2$ then B has no semi-finite lower semi-continuous traces and then (c), (f), (g) and (e) follow from (a)-(b), because then $T^+(B) = 0$ and every $b \in \mathrm{C}_\mathrm{b}(\mathbb{R}_+, B)$ is in the ideal which is generated by B: Indeed, by Lemma ??, there exists a projection $p \in \mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathcal{O}_2 \otimes \mathbb{K})$ with $b - pbp \in \mathrm{C}_0(\mathbb{R}_+, B)$. Thus $pb - b \in \mathrm{C}_0(\mathbb{R}_+, B)$, and $p(t) = u(t)p_0u(t)$ for $p_0 \in \mathcal{O}_2 \otimes \mathbb{K} \subseteq B$.

....

Is above OK? Compare Chp. 9 ??

....

But here we add a few comments on the general case (and possible generalizations):

Since (b) implies that $h_V \colon a \in A \mapsto V(a) + \mathrm{C}_0(\mathbb{R}_+, B) \in Q(\mathbb{R}_+, B)$ is a C^*-morphism and that for every $\omega \in (\beta\mathbb{R}_+ \setminus \mathbb{R}_+)$ $\pi_\omega h_V \colon A \to \mathrm{C}_\mathrm{b}(\mathbb{R}_+, B)/J_\omega$ is a C^*-morphism (f) means just that $\pi_\omega h_V$ is a C^*-morphism with the same kernel as h_V, i.e. $\|h_V(a)\|_\omega = \|V(a)\|_\omega = \mathrm{dist}(V(a), \mathrm{C}_0(\mathbb{R}_+, B))$ for every ω in the corona of $\mathbb{R}_+$. E.g. if A is simple then h is zero or $\pi_\omega h_V$ is a monomorphism for every $\omega \in \beta\mathbb{R}_+ \setminus \mathbb{R}_+$.

If A has real rank zero then again (c), (d), (e), (g) and (h) follow from (a), (b) and (f) since every projection of $Q(\mathbb{R}_+, B \otimes \mathbb{K})$ is unitarily equivalent to a projection in $B \otimes \mathbb{K}$ by a unitary in the unitization of $Q(\mathbb{R}_+, B \otimes \mathbb{K})$ and ideals of A are determined by its projections. For $RR(A) = 0$ the topology on $T^+(A)$ is induced by the pairing with $K_0(A)$ thus (h) is satisfied. The same happens with the space of 2-quasi-traces, which gives (d) and (e) from (a),(b) and (f).

The further challenge of the theory of (our type of) asymptotic morphisms are variants of property (f). Here a lot of stronger or weaker versions are possible which will be discussed somewhere else.

Next Cor. cited in Chapter 10.

Corollary 7.1.5. *Suppose that A and B are C^*-algebras, where A is separable and B is a σ-unital and stable.*

Let $\varphi \colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ and $\psi \colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ C^*-morphisms that are unitarily equivalent by a unitary $W \in \mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$.

Then φ and ψ are unitarily equivalent by a unitary $U \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$ that is a finite product of exponentials $\exp(T_j)$, $j = 1, \ldots, n$, where $T_j \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$ with $\|T_j\| < 1$, $T_j^* = -T_j$.

Desired: $T_j \in \mathrm{Q}(\mathbb{R}_+, B)$

PROOF. Let $W \in \mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$ with $W^*\varphi(\cdot)W = \psi(\cdot)$.

Let $a_0 \in A_+$ a strictly positive contraction for A. Then $\varphi(a_0) + \psi(a_0)$ is a strictly positive element of $C^*(\varphi(A) \cup \psi(A))$. There exists contractions $e_0, e_1, e_2 \in \mathrm{C_b}(\mathbb{R}_+, B)_+$ such that, $e_i e_j = e_j e_i$ and $e_1(1 - e_0), e_2(1 - e_1) \in \mathrm{C}_0(\mathbb{R}_+, B)$, $e_3 := e_2 + \mathrm{C}_0(\mathbb{R}_+, B) \in \mathrm{Q}(\mathbb{R}_+, B)$ satisfies $e_3(\varphi(a_0) + \psi(a_0)) = \varphi(a_0) + \psi(a_0)$.

It implies that $e_3\varphi(\cdot) = \varphi(\cdot) = \varphi(\cdot)e_3$ and $e_3\psi(\cdot) = \psi(\cdot) = \psi(\cdot)e_3$. Let $f := e_3 W e_3$. Then $f \in \mathrm{Q}(\mathbb{R}_+, B)$ is a contraction with $f^*\varphi(\cdot)f = \psi(\cdot)$ and $f\psi(\cdot)f^* = \varphi(\cdot)$. It follows that $(1 - ff^*)\varphi(\cdot)ff^* = 0$ and ????

We can lift f to a contraction $g \in \mathrm{C_b}(\mathbb{R}_+, B)$ with $f = g + \mathrm{C}_0(\mathbb{R}_+, B)$ and $e_1 g = g = g e_1$.

Since B is stable and σ-unital, we get from

Lemma **??**

isometries $S, T \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))$ with $SS^* + TT^* = 1$ and $(1 - S)e_0, e_0(1 - S) \in \mathrm{C}_0(\mathbb{R}_+, B)$. It gives $T^*e_0 = 0 = e_0 T$. Let $U_0 := U(g)$ the Halmos unitary of g with respect to (S, T), cf. cf. Remark **??**. Then $U_0 \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))$ and $\pi_B(U_0(t)) = TS^* - ST^*$, i.e., $U_0 \in TS^* - ST^* + \mathrm{C_b}(\mathbb{R}_+, B)$. We get $U_0^* e_1 = U_0^* e_0 e_1 = Sge_0 + T(1 - g^*g)S^*e_0 = Sg = Se_0 g = g$,

Need $U_0^* e_1 = g$???? □

2. The asymptotic corona $\mathrm{Q}(X, A)$

Let us fix some notations: Throughout this section X denotes a locally compact Hausdorff space. In our later applications we use only *closed subsets* X of the plane $\mathbb{R} \times \mathbb{R} \cong \mathbb{C}$ that are not necessarily bounded.

Notice that a locally compact space X is a σ-compact space if and only if the C^*-algebra $\mathrm{C}_0(X)$ contains a strictly positive element $e \in \mathrm{C}_0(X)$: Let $X_1 \subseteq X_2 \subseteq \ldots$ be a sequence of open subsets of X with compact closures such that the closure of X_n is contained in X_{n+1} and $X = \bigcup X_n$.

By Urysohn Lemma, we find continuous functions $\mu_n \in \mathrm{C}_0(X)_+$ with $\|\mu_n\| = 1$, $\mu_n|\overline{X_n} = 1$, $\mu_n|(X \setminus X_{n+1}) = 0$. Then $\mu_n\mu_{n+1} = \mu_n$ and $e(x) := \sum_{n=1}^{\infty} 2^{-n}\mu_n \in \mathrm{C}_0(X)_+$, and $1 \geq e(x) > 0$ for every $x \in X$. Thus, $\mathrm{C}_0(X)$ *is σ-unital with strictly positive element e if X is σ-compact.*

Conversely, suppose that the C^*-algebra $\mathrm{C}_0(X)$ is σ-unital. There exists continuous function $e \in \mathrm{C}_0(X)_+$ that is a strictly positive element of norm one in the C^*-algebra $\mathrm{C}_0(X)$. The sets $U_n := e^{-1}((n^{-1}, 1])$ build a sequence of open subsets of X with $X = \bigcup X_n$, the closure of X_n is contained in compact set $C_n := e^{-1}([(n+1)^{-1}, 1])$. Thus X is σ-compact.

We consider from now on only σ-compact l.c. spaces X.

Let $\mathrm{C_b}(X)$ denote the C^*-algebra of bounded continuous functions on X. By βX we denote the **Stone-Čech compactification** of X, the **corona** $\gamma(X) :=$

$\beta(X) \setminus X$ consists of abstract limit points of the free ultra-filters on X. $\beta(X)$ and $\gamma(X)$ are compact and X is a dense open subset of βX.

In terms of C^*-algebras we have natural isomorphisms $\mathrm{C_b}(X) \cong C(\beta X) \cong \mathcal{M}(\mathrm{C_0}(X))$ and, therefore, $C((\beta X \setminus X)) \cong \mathrm{C_b}(X)/\mathrm{C_0}(X) \cong \mathcal{M}(\mathrm{C_0}(X))/\mathrm{C_0}(X)$. The free ultra-filters on X (i.e. the points of the corona of X) correspond one-to-one to the characters of $\mathrm{C_b}(X)$ which annihilate $\mathrm{C_0}(X)$, i.e. to the characters of $\mathrm{C_b}(X)/\mathrm{C_0}(X)$. In the sequel we consider frequently every continuous bounded function on X also as a continuous function on βX.

It follows that $\gamma(\mathbb{R}) = \beta\mathbb{R}\setminus\mathbb{R}$ is the disjoint union of the open connected subsets $\gamma(\mathbb{R}_-)$ and $\gamma(\mathbb{R}_+)$, where $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, +\infty)$.

More simply expressed:
If f is a bounded continuous function on $\mathbb{R}_-$ and g a bounded continuous function on $\mathbb{R}_+$, then $h(t) := f(t)$ for $t \leq -1$, $j(t) := g(t)$ for $t \geq 0$, $h(t) := |t|f(-1)+(1-|t|)g(0)$ for $t \in [-1, 0]$ is a bounded continuous function with $h|\gamma(\mathbb{R}_-) = f|\gamma(\mathbb{R}_-)$ and $h|\gamma(\mathbb{R}_+) = g|\gamma(\mathbb{R}_+)$.

Now let A be a Banach space. By $\mathrm{C_b}(X, A)$ (resp. $\mathrm{C_0}(X, A)$) we denote the Banach space of bounded continuous functions $f \colon X \to A$ with $\|f\| = \sup\{\|f(x)\| \colon x \in X\}$ (resp. continuous functions $f \colon X \to A$ vanishing at infinity, i.e., on $\gamma(X)$).

If A is a C^*-algebra, then $\mathrm{C_0}(X, A)$ is an ideal of the C^*-algebra $\mathrm{C_b}(X, A)$, $\mathrm{C_b}(X)$ acts on $\mathrm{C_b}(X, A)$ by $(gf)(x) = g(x)f(x)$ for $f \in \mathrm{C_b}(X, A)$ and $g \in \mathrm{C_b}(X)$.

Definition 7.2.1. Let $f \in \mathrm{C_b}(X, A)$. We denote by $N(f) \in \mathrm{C_b}(X)$ the continuous function $N(f)(x) := \|f(x)\|$.

We call the Banach space $\mathrm{Q}(X, A) := \mathrm{C_b}(X, A)/\mathrm{C_0}(X, A)$ the **asymptotic corona of** A **with respect to** X, and $\mathrm{Q}(\mathbb{R}_+, A)$ the **asymptotic corona of** A.

If Y is a closed subset of $\beta(X)$ then

$$J_Y := \{f \in \mathrm{C_b}(X, A) \colon \ N(f)(y) = 0 \ \forall y \in Y\},$$

where we consider $N(f)$ as a function in $\mathrm{C}(\beta X)$. In particular, $J_X = \mathrm{C_0}(X, A)$.

More generally, we define **restrictions** $\mathrm{C_b}(X, A)|Y$ of $\mathrm{C_b}(X, A)$ to closed subsets Y of βX by

$$\mathrm{C_b}(X, A)|Y := \mathrm{C_b}(X, A)/J_Y \,.$$

In particular, $\mathrm{Q}(X, A) = \mathrm{C_b}(X, A)|\gamma(X)$. We write also $\mathrm{Q}(X, A)|Y := \mathrm{C_b}(X, A)|Y$ if $Y \subseteq \gamma(X)$.

The reader can easily check that:

(i) $N(f) \colon x \mapsto \|f(x)\|$ is in $\mathrm{C_b}(X) \cong \mathrm{C}(\beta X)$ if $f \in \mathrm{C_b}(X, A)$ and is $\mathrm{C_b}(X)_+$-homogeneous, i.e. $gN(f) = N(gf)$ if $f \in \mathrm{C_b}(X, A)$, $g \in \mathrm{C_b}(X)_+$.

(ii) $\mathrm{C_0}(X, A) = \mathrm{C_0}(X)\,\mathrm{C_b}(X, A)$, and, more generally,

$$J_Y = \mathrm{C_0}((\beta X) \setminus Y) \cdot \mathrm{C_b}(X, A)$$

if $Y \subseteq \gamma(X)$ ([4]).

Part (ii) can be seen (e.g. in the case of σ-compact X) as follows: Let $f \in J_Y$ and let $e \in \mathrm{C}_0(X)_+ \subseteq \mathrm{C}(\beta(X))$ a fixed continuous function with $e(x) > 0$ for $x \in X \setminus Y$, $e(y) = 0$ for $y \in \gamma(X)$ (i.e., take the above constructed strictly positive element $e \in \mathrm{C}_0(X)$ of the σ-unital C^*-algebra $\mathrm{C}_0(X \setminus Y)_+$ — with X replaced by $X \setminus Y$). $\mathrm{C}_0(X)_+$. Then $g = (N(f) + e)^{1/2} \in \mathrm{C}_0(\beta X \setminus Y)$. The function $f_1(x) := (N(f) + \delta)^{-1/2}(x) f(x)$ for $x \in X$ is continuous and satisfies $N(f_1) \leq N(f)^{1/2}$. In particular $f_1 \in J_Y$ and $g f_1 = f$.

LEMMA 7.2.2. *Let A a C^*-algebra, and X a locally compact, σ-compact and non-compact Hausdorff space.*

There are following natural isomorphisms and properties:

(i) $\mathrm{C}_0(X, A) \cong \mathrm{C}_0(X) \otimes A$ *by the a natural isomorphism that identifies $f(x)a$ and $f \otimes a$ for $a \in A$, $f \in \mathrm{C}_0(X)$. It defines a natural C^*-morphism from* $\mathrm{C_b}(X)$ *into the center of* $\mathcal{M}(\mathrm{C}_0(X, A))$.

(ii) $\mathcal{M}(\mathrm{C}_0(X, A))$ *is naturally isomorphic to the C^*-algebra* $\mathrm{C_{b,st}}(X, \mathcal{M}(A))$ *of bounded strictly (= *-strongly) continuous maps f from X into $\mathcal{M}(A)$.*

(iii) $\mathrm{C_b}(X, A)$ *is identical with the ideal of* $\mathcal{M}(\mathrm{C}_0(X, A))$ *given by*

$$\{\, f \in \mathcal{M}(\mathrm{C}_0(X, A))\,;\; gf \in \mathrm{C}_0(X, A) \quad \forall\; g \in \mathrm{C}_0(X)\}\,.$$

*In particular, the natural *-monomorphism from* $\mathrm{C_b}(X, A)$ *extends to a natural *-isomorphism*

$$\mathcal{M}(\mathrm{C_b}(X, A)) \cong \mathcal{M}(\mathrm{C}_0(X, A))\,.$$

(iv) $\mathrm{Q}(X, A) := \mathrm{C_b}(X, A)/\,\mathrm{C}_0(X, A)$ *is naturally isomorphic to an ideal of* $\mathcal{M}(\mathrm{C}_0(X, A))/\,\mathrm{C}_0(X, A)$.

And $\mathcal{M}(\mathrm{Q}(X, A)) = \mathcal{M}(\mathrm{C}_0(X, A))/\,\mathrm{C}_0(X, A)$.

The algebraic two-sided annihilator $\mathrm{Ann}(\mathrm{Q}(X, A))$ *of* $\mathrm{Q}(X, A)$ *in* $\mathcal{M}(\mathrm{C}_0(X, A))/\,\mathrm{C}_0(X, A) = \mathcal{M}(\mathrm{C_b}(X, A))/\,\mathrm{C}_0(X, A)$ *is identical with*

$$\{T + \mathrm{C}_0(X, A)\,;\; T \in \mathcal{M}(\mathrm{C}_0(X, A))\,,\; Tf, fT \in \mathrm{C}_0(X, A)\;\; \forall\, f \in \mathrm{C_b}(X, A)\}\,.$$

It is identical with the C^*-algebra $\big(\mathrm{C}_0(X) \cdot \mathcal{M}(\mathrm{C}_0(X, A))\big)/\,\mathrm{C}_0(X, A)$.

(v) *For $f, g \in \mathrm{Q}(X, A)_+$, $fg = 0$ there exists contractions $h_1, h_2 \in \mathrm{Q}(X, A)_+$ with $fh_1 = f$, $gh_2 = g$ and $h_1 h_2 = 0$.*

(vi) *For $f \in \mathrm{C_b}(X, A \otimes \mathbb{K})_+$ there exists an isometry $s \in \mathrm{C_b}(X, \mathcal{M}(\mathbb{K})) \subseteq \mathcal{M}(\mathrm{C}_0(X, A \otimes \mathbb{K}))$ such that $fs \in \mathrm{C}_0(X, A \otimes \mathbb{K})$.*

(vii) *If $M_1 \subset \mathcal{M}(\mathrm{C}_0(X, A))$ and $M_2 \subseteq \mathrm{C_b}(X, A)$ are countable subsets then there exist a positive contraction $e \in \mathrm{C_b}(X, A)$ such that $eb - be, c - ec, c - ce \in \mathrm{C}_0(X, A)$ for all $b \in M_1$ and $c \in M_2$.*

(viii) *Suppose that the C^*-algebra B contains a continuous path $t \in \mathbb{R}_+ \mapsto p(t) \in B$ in the projections of B such that $\lim_{t\to\infty} p(t)b = b$ for all $b \in B$.*

[4] The factorization holds for arbitrary l.c. X by Cohen factorization applied to the *non-degenerate* Banach $\mathrm{C}_0(\beta(X) \setminus Y)$-module J_Y.

Let $h\colon B \to \mathcal{M}(A)$ is non-degenerate C^*-morphism, and

$$h_X\colon \mathrm{C_b}(X, B) \to \mathrm{C_{b,st}}(X, \mathcal{M}(A)) \cong \mathcal{M}(\mathrm{C}_0(X, A))$$

its natural extension to $\mathrm{C_b}(X, B)$*.*

Then for every countable subset $M \subset \mathrm{C_b}(X, A)$ there exists a projection $p \in \mathrm{C_b}(X, B)$ such that $(1 - h_X(p))f, f(1 - h_X(p)) \in \mathrm{C}_0(X, A)$ for all $f \in M$.

`It is likely that` $\mathcal{M}(\mathrm{C_b}(X, B))/\,\mathrm{C}_0(X, B) \subseteq \mathcal{M}(\mathrm{Q}(X, B))$ `is not equal to` $\mathcal{M}(\mathrm{Q}(X, B))$`,` e.g. `if` $X = \mathbb{R}_+$ `and` B `is not` σ`-unital?`

`When` $\mathcal{M}(A)/J \subseteq \mathcal{M}(A/J)$ `??`
`Perhaps` $\mathcal{M}(A/J) \cong \mathcal{M}(A)/\mathcal{N}(A, J)$ `if` A `and` J `are` σ`-unital ??`

`Notice that` $J := \mathrm{C}_0(X, B)$ `is not` σ`-unital if` B `is not` σ`-unital!`

PROOF. (i): Easy to see, e.g. [**767**, vol.I,chp.IV.,thm.4.14], [**704**, sec. 1.22].

(ii): For $y \in X$ the $*$-epimorphism $f \in \mathrm{C}_0(X, A) \mapsto f(y) \in A$ defines a C^*-morphism $b \in \mathcal{M}(\mathrm{C}_0(X, A)) \mapsto b(y) \in \mathcal{M}(A)$. If $g \in \mathrm{C}_0(X)$, $a \in A$ then $(b(g \otimes a))(y) = g(y)b(y)a \in \mathrm{C}_0(X, A)$. Thus $y \mapsto b(y)$ is strongly continuous. The same happens for b^*, $y \mapsto (b(y))^* = b^*(y)$ since $\mathrm{C}_0(X) \odot A$ is dense in $\mathrm{C}_0(X, A) = \mathrm{C}_0(X) \otimes A$, for every $b \in \mathcal{M}(\mathrm{C}_0(X, A))$ there exists a strictly continuous function $x \mapsto b(x)$ with $\sup \|b(x)\| \leq \|b\|$ such that $(bf)(x) = b(x)f(x)$. Conversely a bounded strictly continuous map $x \mapsto b(x) \in \mathcal{M}(A)$ defines a multiplier of $\mathrm{C}_0(X, A)$ (respectively of $\mathrm{C_b}(X, A)$), because the A-valued functions $x \to b(x)a(x)$ and $x \to a(x)b(x)$ are in $\mathrm{C}_0(X, A)$ (respectively in $\mathrm{C_b}(X, A)$) if $x \to b(x) \in \mathcal{M}(A)$ is bounded and strictly continuous.

Parts (iii) and (iv) can be seen from (ii) and (i).

(v): It suffices to find a contraction $h_1 \in \mathrm{Q}(X, A)_+$ with $h_1 f = f$ and $h_1 g = 0$, because then one can use (g, h_1) in place of (f, g) to produce in the same way h_2 with $h_2 h_1 = 0$ and and $h_2 g = g$.

Let $b = b^* \in \mathrm{C_b}(X, A)$ with $b + \mathrm{C}_0(X, A) = f - g$. The the positive and the negative parts of b satisfy $b_+(x) = \max(0, b(x))$, $b_-(x) = \max(0, -b(x))$, $b_+ + \mathrm{C}_0(X, A) = f$ and $b_- + \mathrm{C}_0(X, A) = g$. Take a strictly positive contraction $e \in \mathrm{C}_0(X)_+$, i.e., $1 \geq e(x) > 0$ for all $x \in X$ and $\lim_{x \to \infty} e(x) = 0$.

Define $h(x) := e(x)^{-1} \min(b_+(x), e(x) \cdot 1)$, i.e., $h(x) = e(x)^{-1} b_+(x) - e(x)^{-1}(b_+(x) - e(x))_+$. This h is a positive contraction in $\mathrm{C_b}(X, A)$, and with $\|b_+(x)h(x) - b_+(x)\| \leq e(x)$ by spectral calculus, because $(b_+(x) - e(x))_+ \in C^*(b_+(x)) \subseteq A$, $x \mapsto (b_+(x) - e(x))_+$ is continuous and $t(1 - \varepsilon^{-1} \min(t, \varepsilon)) \leq \varepsilon$ for all $t \geq 0$ and $\varepsilon > 0$. Right from the definition we get $b_-(x)h(x) = 0$. Thus $h_1 := h + \mathrm{C}_0(X, A) \in \mathrm{Q}(X, A)$ has the desired properties.

(vi): `Combine proof with (viii). New text in Lemma:`

For $f \in \mathrm{C_b}(X, A \otimes \mathbb{K})_+$ there exists an isometry $s \in \mathrm{C_b}(X, \mathcal{M}(\mathbb{K})) \subseteq \mathcal{M}(\mathrm{C_0}(X, A \otimes \mathbb{K}))$ such that $fs \in \mathrm{C_0}(X, A \otimes \mathbb{K})$.

`Old proof:` Let $f \in \mathrm{C_b}(X, A \otimes \mathbb{K})$ and $X_1 \subseteq X_2 \subseteq \cdots$ a sequence of compact subsets of X such that X_n is contained in the *interior* of X_{n+1} and $\bigcup_n X_n = X$. We find projections $p_1 \leq p_2 \leq \ldots \in \mathbb{K}$ such that $\|f(x)1 \otimes p_n - f(x)\| < \frac{1}{n}$ for $x \in X_n$. Thus it suffices to find a strictly continuous map $x \mapsto s(x) \in \mathcal{L}(H) = \mathcal{M}(\mathbb{K})$ such that $s(x)$ is an isometry and $\|p_n s(x)\| \leq \frac{1}{n}$ for x in the closure of $X_{n+1} \setminus X_n$. Then $x \mapsto 1 \otimes s(x)$ is as desired.

For $X = \mathbb{R}_+$ and $A := \mathbb{C}$ this can be solved by Lemma 5.1.2(iv) with $X_n := [0, n]$: We find a norm-continuous map $t \in \mathbb{R}_+ \mapsto s_0(t)$ into the isometries of $\mathcal{M}(A)$ such that $\|p_n s_0(t)\| \leq \frac{1}{n}$ for $t \in [n, n+1]$.

For general X and A let $s(x) := s_0(t(x))$, where $t \colon X \to \mathbb{R}_+$ is a continuous function with $t(X \setminus X_n) \subseteq (n, \infty)$, e.g. $t(x) := \sum_n (1 - g_n(x))$ with a continuous functions $g_n \colon X \to [0,1]$ with $g_n | X_{n-1} = 1$ and $g_n | (X \setminus X_n^{\circ}) = 0$.

(vii): `stated text:`

If $M_1 \subseteq \mathcal{M}(\mathrm{C_0}(X, A))$ and $M_2 \subseteq \mathrm{C_b}(X, A)$ countable subsets. Then there exist a positive contraction $e \in \mathrm{C_b}(X, A)$ such that $eb - be, c - ec, c - ce \in \mathrm{C_0}(X, A)$ for all $b \in M_1$ and $c \in M_2$.

(viii): `stated text:`

Suppose that the C^*-algebra B contains a continuous path $t \in \mathbb{R}_+ \mapsto p(t) \in B$ in the projections of B such that $\lim_{t \to \infty} p(t)b = b$ for all $b \in B$.

If $h \colon B \to \mathcal{M}(A)$ is a non-degenerate C^*-morphism, and $h_X \colon \mathrm{C_b}(X, B) \to \mathrm{C_{b,st}}(X, \mathcal{M}(A)) \cong \mathcal{M}(\mathrm{C_0}(X, A))$.

Then for every countable subset $M \subseteq \mathrm{C_b}(X, A)$ there exists a projection $p \in \mathrm{C_b}(X, B)$ such that $(1 - h_X(p))f, f(1 - h_X(p)) \in \mathrm{C_0}(X, A)$ for all $f \in M$.

`To be filled in` ?? □

Remark 7.2.3. The following example shows that in general $\mathrm{C_b}(X, A)$ is strictly contained in the C^*-algebra of the bounded strictly continuous functions $f \in \mathrm{C_{b,st}}(X, \mathcal{M}(A))$ with $f(X) \subseteq A$, i.e., a bounded strictly continuous function f on X with values in A is not necessarily continuous. Let $X := \{0\} \cup \{1/n \colon n \in \mathbb{N}\} \subseteq [0,1]$. $A := \mathbb{K}$ and $f(0) := 0$, $f(1/n) = p_{nn}$.

Then X is compact and $\lim_{n \to \infty} p_{nn}$ converges strictly to zero in $\mathcal{M}(A)$. Thus, $f \in \mathrm{C_{b,st}}(X, \mathcal{M}(A))$. Clearly, $f \notin \mathrm{C}(X, \mathcal{M}(A))$, because certainly $f \notin \mathrm{C}(X, A)$.

Thus, f is in $\mathrm{C_{b,st}}(X, \mathcal{M}(A)) \cong \mathcal{M}(\mathrm{C}(X, A))$, has image in A, but f is not in $\mathrm{C_b}(X, A) = \mathrm{C}(X, A)$.

Lemma 7.2.4. *Let (Ω, ρ) be a compact metric space, X a locally compact σ-compact space and F a set of bounded nonnegative continuous functions on $\Omega \times X$, such that*

(i) $|f(\omega_1, x) - f(\omega_2, x)| \leq C\rho(\omega_1, \omega_2)$ *for all* $x \in X$, $f \in F$ *and* $\omega_1, \omega_2 \in \Omega$,

(ii) *For every bounded sequence* $f_1, f_2, \ldots \in F$ *and every countable covering* $\{U_n\}$ *of* X *by open subsets with compact closures (in* X*), there exist* $f \in F$ *and a locally finite decomposition* $\lambda_1, \lambda_2, \ldots \in \mathrm{C}_0(X)_+$ *of* 1*, such that for every* k *there exists* n_k *with support* $\lambda_k \subseteq U_{n_k}$ *and* $f(\omega, x) \leq \sum \lambda_k(x) f_{n_k}(\omega, x)$ *for* $x \in X$.

Let $f^*(x) := \sup\{f(\omega, x) \colon \omega \in \Omega\}$, $g(x) := \inf\{f^*(x) \colon f \in F\}$, h *a bounded continuous function on* X *such that* $g(x) < h(x)$ *for every* $x \in X$ *and* $Y \subset \beta X \setminus X$ *a compact subset of the corona.*

Then there exist $k_1, k_2 \in F$ *with* $k_1^* \leq h$*, and* $k_2^*(y) \leq \sup\{g(y) \colon y \in Y\}$ *for every* $y \in Y$.

It holds $\sup\{g(x) \colon x \in \beta X \setminus X\} \leq \lim_{n\to\infty}(\sup\{g(x) \colon x \in X \setminus X_n\})$ *where* $\overline{X_n} \subseteq X_{n+1} \subseteq \ldots X$ *are open subsets of* X *with compact closures such that* $X = \bigcup X_n$.

Remark 7.2.5. Here we consider only locally finite decompositions of 1 which are finite in a sufficiently small neighborhood every point of X, i.e. for every $x \in X$ there is neighborhood U of x such that only a finite number of the λ_k is nonzero on U.

Proof. By the Lipschitz-condition (i), the f^* are bounded continuous functions on X. $F^* = \{f^* \colon f \in F\}$ satisfies (i) and (ii) for $\Omega =$ point and F replaced by F^*. The conclusions do not require Ω. Therefore it is enough to prove the lemma in case of $\Omega =$ point, $f = f^*$ $\forall f \in F$. Since $g(x) < h(x)$ for $x \in X$ we find for every $x \in X$ an $f \in F$ and a neighborhood $U(x)$ of x such that $f(y) < h(y)$ for every $y \in U(x)$ and $U(x)$ has compact closure. Since X is locally compact we find a countable subsystem $(f_n, U(X_n))$ with $\bigcup U(X_n) = X$. If (λ_k) is the locally finite decomposition of unit subordinated to $\{U(x_n)\}$ such that support of λ_k is contained in some $U(x_{n_k})$ and such that there is $k_1 \in F$ with $k \leq \sum \lambda_k f_{n_k}$, then $g \leq k_1 \leq h$. If $\overline{X_n} \subseteq X_{n+1} \subseteq \ldots X$ is a covering of X by open subsets with compact closure, the a suitable construction of $h \geq g$ shows that $\sup\{g(x) \colon x \in \beta X \setminus X\} \leq \limsup C_n$ where $C_n = \sup\{g(x) \colon x \in \overline{X_{n+2}} \setminus X_n\}$. Now consider $Y \subseteq \beta X \setminus X$, compact, $C = \sup\{g(y) \colon y \in Y\}$.

Then for every $y \in Y$, $n \in \mathbb{N}$ there exists $f \in F \subseteq \mathrm{C_b}(Y) \cong C(\beta X)$, such that $f(y) < C + \frac{1}{n}$. Thus for every point $y \in Y$ there exists a neighborhood $U(y)$ of $y \in \beta X$ and $f \in F \subseteq C(\beta X)$ with $f(x) \leq C + \frac{1}{n}$ for $x \in U(y)$.

Since Y is compact we find a $y_1, \ldots, y_n \in Y$, $f_1, \ldots, f_n \in F$ with $f_k(x) \leq C + \frac{1}{n}$ for $x \in U(y_k)$ and $Y \subseteq U(y_1) \cup \ldots \cup U(y_n)$. In this way we find a sequence $g_1, g_2, \ldots \in F$ such that $\inf\{g_k(y) \colon k = 1, 2, \ldots\} \leq C$ for $y \in Y$. Let $\delta \in \mathrm{C}_0(X)_+$ with $\delta(x) > 0$ for every $x \in X$ (exists because X is σ-compact). By the first part of the proof it is enough to find a bounded continuous function k with $g \leq k \leq \inf\{g_k\} + \delta$ on X. Because then there is $k_1 \in F$ with $k_1 \leq k + \delta$ which implies $k_1(y) \leq g_n(y)$ for $y \in \beta X \setminus X$ and $n = 1, 2, \ldots$. Let $k = \mu_1 g_1 + \sum(\mu_{n+1} - \mu_n) \cdot \inf\{g_1, \ldots, y_n\}$

where $\mu_n(x) \in [0,1]$ for $x \in X_n$, $\mu_n(x) = 1$ on $\overline{X_n}$ and $\mu_n(x) = 0$ on $X \setminus X_{n+1}$ (use Urysohn Lemma). □

REMARK 7.2.6. If $X = \mathbb{N}$ and $\omega \in \gamma(\mathbb{N}) := \beta\mathbb{N} \setminus \mathbb{N}$ is a point of the corona $\gamma(\mathbb{N})$ of $\mathbb{N}$ the ω is called a "free" ultrafilter. $B_\omega = \mathrm{C_b}(\mathbb{N}, B)/J_{\{\omega\}} = l_\infty(B)/c_\omega(B)$ is then the well known (norm-)ultrapower of a Banach space B.

If X is locally compact, non-compact but σ-compact (e.g. $X = \mathbb{R}_+$) and $\omega \in \beta X \setminus X$ then in general there does not (!) exist a sequence $x_1, x_2, \ldots \in X$ and a (free) ultrafilter $\omega_1 \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\omega = \lambda(\omega)$ (where $\lambda\colon \beta\mathbb{N} \to \beta\mathbb{N}$ is induced by the map $n \mapsto x_n$). Therefore in the following proof one needs really our reduction to the bounded continuous functions on X (and not only ultrapower arguments or pure logical partition arguments).

Another technical problem is the following: If Ω is a compact metric space that contain infinitely many points, then the natural continuous epimorphism from $\beta(\mathbb{R}_+ \times \Omega)$ onto $\beta(\mathbb{R}_+) \times \Omega$ is not an isomorphism. Thus $\mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(\Omega, A)) \not\cong \mathrm{C}(\Omega, \mathrm{C_b}(\mathbb{R}_+, A))$ in general. But always $\mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(\Omega, A)) = \mathrm{C_b}(\mathbb{R}_+ \times \Omega, A)$.

. . . .

Suppose that B is a Banach space, $Z \subseteq \beta X$, $f \in \mathrm{C_b}(X, B)$. then $N(f)(y) := \|f(y)\|$ defines an element $N(f)$ of $\mathrm{C}(\beta X) \cong \mathrm{C_b}(X)$. Therefore $\|f\|_Y := \sup\{N(f)(y) \colon y \in Y\}$ is well defined for $Y \subseteq \beta X$ closed and $\|f\|_Y = \|\pi_Y(f)\| = \|f + J_Y\| = \mathrm{dist}(f, J_Y)$.

Suppose, A, B are Banach spaces, T is a set of linear contractions from A into B. T^X denotes the set of strongly continuous maps from X into T. The elements of T^X operate on $\mathrm{C_b}(X, B)$ as linear contractions, indeed: if $L := \{S(y)\}_{y \in X} \in T^X$, $f \in \mathrm{C_b}(X, B)$ then $g(y) := S(y)(f(y))$ defines an element $L(f) = g \in \mathrm{C_b}(X, B)$.

Let Ω be a compact metric space, $\gamma_1 \colon \Omega \to \mathrm{C_b}(X, A)$ and $Z \subseteq \beta X$ a subset of βX. For fixed Ω, γ_1 and γ_2, we define for $Z \subseteq \beta X$ closed, $L \in T^X$ $\mu(L, Z) := \sup\{\|L(\gamma_1(t)) - \gamma_2(t)\|_Z \colon t \in \Omega\}$ $\nu(Z) := \inf\{\mu(L, Z) \colon L \in T^X\}$.

With the above assumptions and notations we have:

PROPOSITION 7.2.7. *Assume that X is σ-compact and T is convex or that X is totally disconnected. Then*

(i) $\nu(Z) \le \sup\{\nu(\{\omega\}) \colon \omega \in Z\}$
(ii) $\nu(\beta X \setminus X) \le \lim_{n\to\infty}(\sup\{\nu(\{y\}) \colon y \in X \setminus X_n\})$
(iii) *If $Z \subseteq \beta X \setminus X$ is closed then there exists $L \in T^X$ with $\nu(Z) = \mu(L, Z)$, i.e. the infimum is attained.*
(iv) $\nu(X) \le \sup\{\nu(\{y\}) \colon y \in X\}$.

PROOF. The functions

$$g(y,t) := \|L(\gamma_1(t)) - \gamma_2(t)\|_y = \|L(y)(\gamma_1(t)(y)) - \gamma_2(t)(y)\|,$$

$y \in X$, $t \in \Omega$ satisfy the assumptions of Lemma **??** where we have to introduce on Ω the metric $\rho(t_1, t_2) = \rho_\Omega(t_1, t_2) + \|\gamma_1(t_1) - \gamma_1(t_2)\| + \|\gamma_2(t_1) - \gamma_2(t_2)\|$. In particular the family $\{g(y,t)\}$ is $\sigma - \mathrm{C}_0(X)$-subconvex in the variable $y \in X$.

But then (i)-(iii) follows from Lemma **??**, where for the proof of (iii) one has to use the family $\max\{0, g(y,t) - \nu(Z)\}$, which again satisfies the assumptions of Lemma **??**. □

Corollary 7.2.8. *Suppose that A and B are C^*-algebras, that X is σ-compact and that T is a set of c.p. contractions from A into B.*

Then, for each $\omega \in \beta(X)$, T^X defines a set of completely positive contractions $T^X|\omega$ from $A_\omega := \mathrm{C}_\mathrm{b}(X, A)/J_\omega$ into $B_\omega := \mathrm{C}_\mathrm{b}(X, B)/J_\omega$, given by the natural map $T^X \mapsto T^X|Y$ from $\mathrm{C}_{\mathrm{b,st}}(X, \mathcal{L}(A, B))$ in $\mathcal{L}(A_Y, B_Y)$ for closed $Y \subseteq \beta(X) \setminus X$.

The restriction of $T^X|Y$ to each separable closed subspace C of A_Y is point-norm closed in $\mathcal{L}(A_Y, B_Y)$.

If $T \subseteq \mathrm{CP}(A, B)$ is convex, (respectively inner-invariant, operator-convex or is a matrix operator-convex cone), then $T^X|Y$ is so as subset of $\mathrm{CP}(A_Y, B_Y)$.

$T^X|Y$ satisfies the assumptions (i)-(iii) of Lemma **??**.

???????

??

Transitivity needed???:

If T is convex, and for every $a \in A_+$ and $b \in B_+$ with $\|a\| = \|b\| = 1$ and $\varepsilon > 0$ there exists $V \in T$ with $\|V(a) - b\| < \varepsilon$, then $T^X|Y$ has this property for $a \in (A_Y)_+$ and $b \in (B_Y)_+$ with $\|a_\omega\| = \|b_\omega\|$ for all $\omega \in Y$.

In particular, if $C \subseteq A_\omega$ is a separable C^*-subalgebra and $V \colon C \to B_\omega$ is a nuclear contraction, then there exists an $L \in T^X$ such that for $V\pi_\omega(a) = \pi_\omega(L(a))$ if $a \in \mathrm{C}_\mathrm{b}(X, A)$ and $\pi_\omega(a) \in C$.

Corollary 7.2.9. *Let $Y \subseteq \beta X \setminus X$ be a compact subset, C a separable subspace of $\mathrm{C}_\mathrm{b}(X, A)/J_Y$ and $S \colon C \to \mathrm{C}_\mathrm{b}(X, B)/J_Y$ a linear map.*

Suppose that for each $\omega \in Y$, $\varepsilon > 0$ and every finite subset $F \subseteq C$ there exists a nuclear completely positive contraction $V_\omega \colon \pi_\omega(C) \to B_\omega$ such that

$$\|V_\omega(\pi_\omega(c)) - \pi_\omega(S(c))\| < \varepsilon$$

for all $c \in F$.

Then there exists a strongly continuous map $L \colon x \in X \mapsto L(x) \in \mathcal{L}(A, B)$ into the nuclear completely positive contractions from A into B such that

$$S(\pi_Y(a)) = \pi_Y(L(a)) \qquad \forall\ a \in \pi_Y^{-1}(C)\,.$$

Proof. Let Ω be a compact subset of C that generates C as a closed linear subspace. and let $\gamma_1 \colon \Omega \to \mathrm{C}_\mathrm{b}(X, A)$ a topological lift, i.e., $\pi_Y \gamma_1 = \mathrm{id}_\Omega$. Let $\gamma_2 \colon \Omega \to \mathrm{C}_\mathrm{b}(X, B)$ be a topological lift of $(S|\Omega) \colon \Omega \to \mathrm{C}_\mathrm{b}(X, B)/J_Y$. Moreover, let T be the convex set of nuclear completely positive contractions from A into B. Define $\nu(L, Y)$ and $\mu(Y)$ as explained above. We have to find $L \in T^X$ with $\nu(L, Y) = 0$ because then L is as desired (since $\pi_Y L$ and S are contractions).

By Proposition **??** (iii) and (i) it suffices to show that $\nu(\{y\}) = 0$ for each $y \in Y$. But that we have seen above in Corollary **??**. □

Now we get our asymptotic Weyl–von Neumann version of the Weyl–von Neumann–Voiculescu–Kasparov theorem:

THEOREM 7.2.10. *Let X be a locally compact σ-compact space, Y a closed subset of the corona $\beta X \setminus X$ of X, B a C^*-algebra and A a purely infinite simple C^*-subalgebra of B such that AB is dense in B. Moreover let C be a separable C^*-subalgebra of $\pi_Y(\mathrm{C_b}(X, A)) = \mathrm{C_b}(X, A)/J_Y$.*

If V is a completely positive contraction from C into $\pi_Y(\mathrm{C_b}(X, B))$ such that for each $\omega \in Y$ there exists a nuclear completely positive contraction $V_\omega \colon \pi_\omega(a) = \pi_\omega(V(a))$ for every $a \in C$, then there exists a contraction $d \in \mathrm{C_b}(X, B)$ such that $V(a) = \pi_Y(d)^* a \pi_Y(d)$ for every $a \in C$.

PROOF. The Theorem is the logical sum of Corollary **??** and Proposition **??**. □

LEMMA 7.2.11. *If $a \in \pi_Y(\mathrm{C_b}(X, A))_+$ satisfies $\|\pi_\omega(a)\| = 1$ for every $\omega \in Y$, then there exists $b \in \mathrm{C_b}(X, A)_+$ such that $\|\pi_x(b)\| = 1$ for every $x \in X$ and $a = \pi_Y(b)$.*

PROOF. Let $d \in A_+$ with $\|d\| = 1$ and let $c \in \mathrm{C_b}(X, A)_+$ be a contractive lift of a, i.e. $\pi_y(c) = a$, $c \geq 0$, $\|c(x)\| = \|\pi_x(c)\| \leq 1$ for every $x \in X$. Now let $g(x) := \|c(x)\|$ then $g \in \mathrm{C}(\beta X) \cong \mathrm{C_b}(X)$, $f(x) := c(x) + (1 - g(x))d$ is in $\mathrm{C_b}(X, A)$ and $\|f\| \leq 1$ and $1/2 \leq \max(g(x), 1 - g(x)) \leq \|f(x)\|$ for $x \in X$. Then $\pi_y(f) = a$ because $g(\omega) = 1$ for $\omega \in Y$. Thus $\|\pi_\omega(f)\| = 1$ for $\omega \in Y$, i.e. $h(\omega) = 1$ for $\omega \in Y$, where $h(x) := \|f(x)\|^{-1}$. □

Next:
Suppose $\mathrm{C_b}(X, A)$ is not s.p.i.

Conjecture (1):
Then there exists a separable C*-subalgebra B of A such that B has the same invariants:

$$\rho(a, b; B) = \rho(a, b; A)$$

for $a, b \in B_+$, where

$$\rho(a, b; C) := \inf\{\|d^* a^2 d - a^2\| + \|e^* b^2 e - b^2\| + \|d^* abe\|\,;\ e, d \in C\}$$

for $a, b \in C$.

Conjecture (2):
Let A a $C*$-algebra. Then
for every separable $C*$-subalgebra B of $\mathcal{M}(A)$
there exists a contraction $d \in A_+$ with
$BD \subseteq D$ for $D := \overline{dAd}$
and $\|b\| = \sup_n \|bd^{1/n}\|$ for all $b \in B$
such that

$$\rho(a, b; \mathcal{M}(A)) = \rho(\lambda(a), \lambda(b); \mathcal{M}(D))$$

for the faithful $C*$-morphism $\lambda \colon B \to \mathcal{M}(\overline{dAd})$
given by $\lambda(b)x := bx$ for $x \in D$ and $b \in B$

The σ-unitality of A and the σ-compactness
of X (used below) can be reduced if Conj. (1) and (2) hold.

If A is strongly p.i. and σ-unital and X is σ-compact,
then $\mathcal{M}(\mathrm{C}_0(X, A))$ is s.p.i. and above defined A_ω strongly p.i.

This special case follows also from:

''B s.p.i.'' and ''C exact'' imply ''$B \otimes C$ s.p.i.'' and
''$\mathcal{M}(\overline{b^*Bb})$ is s.p.i.'' for each $b \in B$
''A s.p.i.'' implies ''$A \otimes \mathrm{C}_0(X) = \mathrm{C}_0(X, A)$ s.p.i.''
implies ''$\mathcal{M}(\mathrm{C}_0(X, A))$ s.p.i.'' if X is σ-compact and A σ-unital

$\mathrm{C_b}(X, A)$ is ideal of $\mathrm{C_{b,st}}(X, \mathcal{M}(A)) \cong \mathcal{M}(\mathrm{C}_0(X, A))$,
because $f \in \mathrm{C_b}(X, A)$, if and only if,
$f \in \mathrm{C_{b,st}}(X, \mathcal{M}(A))$ and $gf \in \mathrm{C}_0(X, A)$ for all
$g \in \mathrm{C}_0(X) \cdot 1 \subseteq$ center of $\mathcal{M}(\mathrm{C}_0(X, A))$.

Property ''s.p.i.'' passes to ideals and quotients.

Corollary 7.2.12. *If A is simple and purely infinite then every fiber $A_\omega = \mathrm{C_b}(X, A)/J_\omega$ of $\mathrm{C_b}(X, A)$ $(\omega \in \beta X)$ is simple and purely infinite.*

In particular, $\mathrm{C_b}(X, A)$ is a continuous field over $\beta(X)$ with simple purely infinite fibres $A_x = A$ for $x \in X$ and A_ω for $\omega \in \beta(X) \setminus X$.

Proof. If $\omega \in X$ then J_ω is the kernel of the epimorphism $\pi_\omega \colon \mathrm{C_b}(X, A) \to A$.

If $\omega \in \beta(X) \setminus X$, then $J_\omega = \mathrm{C}_0(\beta(X) \setminus \{\omega\}) \cdot \mathrm{C_b}(X, A)$, because $f \in J_\omega$ if and only if $N(f) \in \mathrm{C_b}(X) \cong \mathrm{C_b}(\beta X)$ for the $N(f)(x) := \|f(x)\|$ has the property $N(f)(\omega) = 0$.

Consider for a, b in $(A_\omega)_+$, $\|a\| = \|b\| = 1$ the nuclear map $V \colon c \mapsto \rho(c)b$ where ρ is a pure state on A_ω with $\rho(a) = 1$.

By Lemma 7.2.11 and Corollary 7.2.9 there exist elements $c, d \in \mathrm{C_b}(X, A)$ with $\pi_\omega(c) = a$, $\pi_\omega(d) = b$, $\|c(x)\| = \|d(x)\| = 1$ for all $x \in X$.

Need reduction to σ-compact X !
if not supposed before.

For each compact subset $K \subseteq X$ and $\varepsilon > 0$ there is a contraction $f = f(K, \varepsilon) \in \mathrm{C}(K, A)$ with $\|f(x)^* c(x) f(x) - d(x)\| < \varepsilon$ for all $x \in K$

???????????? (cf. Corollary **??**).

Since X is σ-compact, we find continuous functions $g_0 = 0$ and $g_n \colon X \to [0, 1]$ with supports in compact sets K_n of X such that $g_{n+1}(x) = 1$ for $x \in K_n$ and $X = \bigcup K_n$. Let $G_1(x) := \sum_n (g_{2n}(x) - g_{2n-1}(x))^{1/2} f(K_{2n}, 1/n)(x)$ and $G_2(x) := \sum_n (g_{2n+1}(x) - g_{2n}(x))^{1/2} f(K_{2n+2}, 1/n)(x)$. Then $G_j \in \mathrm{C_b}(X, A)$ and $d - G_1^* c G_1 + G_2^* c G_2 \in \mathrm{C_0}(X, A)$. It follows that $y_1^* a y_1 + y_2^* a y_2 = b$ for the contractions $y_j := \pi_\omega(G_j)$. In particular, A_ω is simple.

Since A is naturally contained in A_ω and since A is purely infinite, A_ω is not isomorphic to the compact operators on a Hilbert space. Thus A_ω satisfies the assumptions of part (iv) of Proposition **??**, which implies that A_ω is purely infinite. □

Start of collections for Chapter 7

The *-epimorphism $b \mapsto b(t)$, for $b \in \mathrm{C}([0, 1], B)$, extends naturally to a *-epimorphism $\pi_t \colon \mathrm{C_b}(\mathbb{R}_+ \times [0, 1], B) \to \mathrm{C_b}(\mathbb{R}_+, B)$ by

$$\pi_t((b_s)_{s \in \mathbb{R}_+}) := (b_s(t))_{s \in \mathbb{R}_+} .$$

The π_t define evaluation semi-group morphisms

$$(\pi_t)_* \colon \mathrm{SR}(\mathcal{C}[0, 1]; A, B[0, 1]) \to \mathrm{SR}(\mathcal{C}; A, B)$$

that naturally define group morphisms

$$(\pi_t)_* \colon \mathrm{R}(\mathcal{C}[0, 1]; A, B[0, 1]) \to \mathrm{R}(\mathcal{C}; A, B) .$$

The following proposition confirms the homotopy invariance of the Rørdam groups.

Proposition 7.2.13. *If B is stable and σ-unital $h \colon A \to Q(\mathbb{R}_+, B[0, 1])$, satisfies $[h] \in \mathrm{SR}(\mathcal{C}[0, 1]; A, B[0, 1])$, then $[h_0] = [h_1]$ in $\mathrm{R}(\mathcal{C};\, A, B)$, where $[h_t] := (\pi_t)_*[h] \in \mathrm{R}(\mathcal{C};\, A, B)$ is defined by the evaluation semi-group morphism*

$$(\pi_t)_* \colon \mathrm{R}(\mathcal{C}[0, 1]; A, B[0, 1]) \to \mathrm{R}(\mathcal{C}; A, B) .$$

Lemma 7.2.14. *Suppose that X is a σ-compact locally compact space, B is a stable and σ-unital algebra, and that $D \subseteq \mathcal{M}(B)$ is non-degenerate (i.e. DB is dense in B).*

(i) *For every σ-unital C^*-subalgebra A of $\mathrm{Q}(X, B)$ there exists an contraction $f \in \mathrm{Q}(X, B)_+$ such that $fa = af = a$ for every $a \in A$.*
(ii) *For each $g \in \mathrm{C_b}(X, B)$ there exists a contraction $e \in \mathrm{C_b}(X, D)_+$ with $g - eg, g - ge \in \mathrm{C_0}(X, D)$.*

(iii) *For* $f \in Q(X, B)$ *there exists an isometry* T *in* $C_b(X, \mathcal{M}(B))/\, C_0(X, B)$ *such that* $T^* f T = 0$.
(iv) *Each* C^**-morphism* $h: D \to Q(X, B)$ *dominates zero in* $Q\big(X, \mathcal{M}(B)\big)$ *(and thus in* $C_b(X, \mathcal{M}(B))/\, C_0(X, B) \subseteq E_X$*).*

Recall that $Q(X, B) \subseteq C_b(X, \mathcal{M}(B))/\, C_0(X, B) \subseteq E_X$. Since the annihilator of $Q(X, B)$ in $C_b(X, \mathcal{M}(B))/\, C_0(X, B)$ is just $C_0(X, \mathcal{M}(B))/\, C_0(X, B)$, we have also that $Q(X, B)$ is naturally isomorphic to an ideal of $Q(X, \mathcal{M}(B))$.

PROOF. `to be filled in` ?? □

`Important !`
`Observation:`
`For all` C^*`-algebras` B `and l.c. Hausdorff spaces holds`

$$\mathcal{M}(C_0(X, B)) = \mathcal{M}(C_b(X, B))\,.$$

(1) $C_0(X, B)$ ideal of $C_b(X, B)$, $fg = 0$ for all $g \in C_0(X, B)$ implies that $f = 0$.

And $C_b(X, B)$ is non-degenerate C^*-subalgebra of $\mathcal{M}(C_0(X, B))$.

Induces faithful unital embedding of $\mathcal{M}(C_b(X, B))$ into $\mathcal{M}(C_0(X, B))$.

(2) Need that $C_b(X, B)$ is ideal of $\mathcal{M}(C_0(X, B))$ (then we are ready).

If $f \in C_b(X, B)$, $T \in \mathcal{M}(C_0(X, B))$ and $\psi \in C_0(X)$ then $T.(\psi.f) \in C_0(X, B)$ and $\psi.(T.f) = T.(\psi.f)$. Thus, $\psi T f \in C_0(X, B)$ for all $\psi \in C_0(X, B)$.

$g \in C_{b,st}(X, \mathcal{M}(B)) \cong \mathcal{M}(C_0(X, B))$ is in $C_b(X, B)$ if and only if $\psi \cdot g \in C_0(X, B)$ for all $\psi \in C_0(X)$.

PROPOSITION 7.2.15. *Suppose that* A *and* B *are stable and* σ*-unital, and that* X *is a* σ*-compact, locally compact, non-compact Hausdorff space.*

Then all "natural" definitions of inner equivalence of C^**-morphisms* $h_i: A \to Q(X, B)$, $i = 1, 2$, *define the same classes, i.e., precisely that the following properties* (i)–(v) *of* (h_1, h_2) *are all equivalent:*

(i) *There exists a unitary* u_1 *in the multiplier algebra* $\mathcal{M}(Q(X, B))$ *of* $Q(X, B)$ *such that* $u_1^* h_1(\cdot) u_1 = h_2$ *, i.e.,* $u_1^* h_1(a) u_1 = h_2(a)$ *for every* $a \in A$.
(ii) *There exists a unitary* u_2 *in* $E_X = \mathcal{M}(C_0(X, B))/\, C_0(X, B) = \mathcal{M}(C_b(X, B))/\, C_0(X, B)$ *such that* $u_2^* h_1(\cdot) u_2 = h_2$ *for every* $a \in A$.
(iii) *There exists a unitary* V_3 *in the* C^**-algebra* $C_b(X, \mathcal{M}(B))$ *of bounded norm-continuous functions on* X *with values in* $\mathcal{M}(B)$ *such that* $u_3^* h_1(\cdot) u_3 = h_2$ *for the unitary*

$$u_3 := \pi_{C_0(X,B)}(V_3) = V_3 + C_0(X, B)$$

in $C_b(X, \mathcal{M}(B))/\, C_0(X, B) \subseteq E_X$ *.*
(iv) *If* Y *is a* σ*-compact locally compact Hausdorff space such that the corona* $\beta(X) \setminus X$ *of* X *is an* open *subset of* $\beta(Y) \setminus Y$*, then there exists an unitary* $u_4 \in E_Y$ *such that* $u_4^* h_1(\cdot) u_4 = h_2$*, where we embed* $Q(X, B)$ *naturally as an ideal of* E_Y*.*

(v) *There exists a contraction* e *in* $\mathrm{Q}(X,B)$ *such that* $e^*h_1(\cdot)e = h_2$ *and* $eh_2(\cdot)e^* = h_1$.

In particular, we get for every C^*-morphism $h_0\colon A \to \mathrm{Q}(X,B)$ that the semigroups $S\big(h_0;\, A, \mathrm{Q}(X, \mathcal{M}(B))\big)$ $S\big(h_0;\, A, \mathcal{M}(\mathrm{Q}(X,B))\big)$, and $S(h_0;\, A, E_X)$ become naturally isomorphic semigroups if consider $\mathrm{Q}(X,B)$ as ideals in $\mathrm{Q}(X,\mathcal{M}(B))$ and in E_X.

If especially $X := \mathbb{R}_+$ and $Y := \mathbb{R}$, then $\beta(\mathbb{R}_+)\setminus\mathbb{R}_+$ is naturally homeomorphic to an open subset of $\beta(\mathbb{R})\setminus\mathbb{R} \cong (\beta(\mathbb{R}_-)\setminus\mathbb{R}_-) \biguplus \beta(\mathbb{R}_+)\setminus\mathbb{R}_+$. Let $h_0\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ a C^*-morphism with $[h_0] = [h_0] + [h_0]$ in $\mathrm{Hom}(A, E_{\mathbb{R}_+})$. Then $S\big(h_0\,;\, A, E_{\mathbb{R}}\big)$, $S\big(h_0\,;\, A, E_{\mathbb{R}_+}\big)$ and $S\big(h_0\,;\, A, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))\big)$ are naturally isomorphic semigroups.

Proof. (i)⇒(v): Let $f \in A_+$ a strictly positive contraction. The element $h_0(f) + h_1(f)$ is a strictly positive element of $C^*(h_0(A) \cup h_1(A)) \subseteq \mathrm{Q}(X,B)$. Thus, by Lemma 7.2.14(i) there exists a positive contraction $C \in \mathrm{C_b}(X,B)$ with $cg = g = gc$ for $c := C + \mathrm{C_0}(X,B)$ and $g \in h_0(A) \cup h_1(A)$. Let $e := cu_1c$.

(iii)⇒(ii): Use that $\mathrm{C_b}(X,B) \subseteq \mathrm{C_b}(X,\mathcal{M}(B)) \subseteq \mathcal{M}(\mathrm{C_b}(X,B))$. The restriction of the *-monomorphism

$$\mathcal{M}(\mathrm{C_b}(X,B)) \to \mathcal{M}(\mathrm{Q}(X,B))$$

to the embedding of $\mathrm{C_b}(X,B)$ in $\mathrm{C_b}(X,\mathcal{M}(B))$ induces the quotient map $\mathrm{C_b}(X,B) \to \mathrm{Q}(X,B)$.

(ii)⇒(i): Consider the C^*-algebras

$$\mathrm{C_b}(X,B) \subseteq \mathrm{C_b}(X,\mathcal{M}(B)) \subseteq \mathrm{C_{b,st}}(X,\mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(X,B))\,.$$

The C^*-algebra $\mathrm{C_b}(X,B)$ is an ideal of $\mathcal{M}(\mathrm{C_0}(X,B))$, because $f \in \mathrm{C_b}(X,B)$ if and only if $\psi\cdot f \in \mathrm{C_0}(X)$ for all functions $\psi \in \mathrm{C_0}(X) \cong \mathrm{C_0}(X)1 \subseteq \mathrm{C_{b,st}}(X,\mathcal{M}(B))$.

The induced C^*-morphism from $\mathcal{M}(\mathrm{C_0}(X,B))$ into $\mathcal{M}(\mathrm{C_b}(X,B))$ is faithful and unital and fixes $\mathrm{C_b}(X,B)$. It allows to identify the multiplier algebras $\mathcal{M}(\mathrm{C_0}(X,B))$ and $\mathcal{M}(\mathrm{C_b}(X,B))$ naturally, as

$$\mathrm{C_b}(X,B) \subseteq \mathrm{C_{b,st}}(X,B) = \mathcal{M}(\mathrm{C_0}(X,B)) = \mathcal{M}(\mathrm{C_b}(X,B))\,.$$

Since $\mathrm{Q}(X,B) = \mathrm{C_b}(X,B)/\,\mathrm{C_0}(X,B)$, we get also a natural unital C^*-morphism from $\mathcal{M}(\mathrm{C_b}(X,B))$ into $\mathcal{M}(\mathrm{Q}(X,B))$. The restriction of the corresponding *-morphisms

$$\mathcal{M}(\mathrm{C_0}(X,B)) = \mathcal{M}(\mathrm{C_b}(X,B)) \to \mathcal{M}(\mathrm{Q}(X,B))$$

to the embedding of $\mathrm{C_b}(X,B)$ in $\mathcal{M}(\mathrm{C_0}(X,B))$ induces the quotient map $\mathrm{C_b}(X,B) \to \mathrm{Q}(X,B)$. Thus, define unital C^*-morphisms into $\mathcal{M}(\mathrm{Q}(X,B))$ that is compatible with $\mathrm{C_b}(X,B) \to \mathrm{Q}(X,B)$. This shows that (i) follows from (ii) and (iii).

(v)⇒(iii):

`to be filled in` ??

(iv)⇔(i): In case $Y := X$ it is the same as the implication (ii)⇔(i), by the above proven equivalences of (i),(ii),(iii) and (v).

If the compact set $\beta(X) \setminus X$ is homeomorphic to an open subset of $\beta(Y) \setminus Y$, let $Z := (\beta(Y) \setminus Y) \setminus (\beta(X) \setminus X)$. It follows that Z is open and compact and $\mathrm{Q}(Y, B) = \mathrm{Q}(Y, B)|Z \oplus \mathrm{Q}(X, B)$. Thus $\mathcal{M}(\mathrm{Q}(X, B))$ is a quotient of $\mathcal{M}(\mathrm{Q}(Y, B))$. It shows that (iv) implies (i) in general, because $\mathrm{Q}(X, B) \subseteq \mathrm{Q}(Y, B) \subseteq E_Y$ is an ideal of E_Y, that defines the natural group morphism $\mathcal{U}(E_Y) \to \mathcal{U}(\mathrm{Q}(X, B))$.

`To be filled in`

???? The way from (iii) to (iv) goes over the natural C^*-morphisms

$$\mathrm{C_b}(Y, \mathcal{M}(B))/\,\mathrm{C_0}(Y, B) \to \mathrm{C_b}(X, \mathcal{M}(B))/\,\mathrm{C_0}(X, B)\,.$$

□

COROLLARY 7.2.16. *Suppose that A and B are stable, where B is σ-unital and A is separable.*

Let $h\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ be a C^*-morphism that dominates $h \oplus h$, and let $V\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ a completely positive contraction.

Further let $\Omega \subseteq A$ a (norm-)compact subset that generates a (norm-)dense linear subspace of A, $\gamma_0\colon \Omega \to \mathrm{C_b}(\mathbb{R}_+, B)$ a topological lift of $h|\Omega$ and $\gamma_1\colon \Omega \to \mathrm{C_b}(\mathbb{R}_+, B)$ a topological lift of $V|\Omega$.

The following are equivalent:

(i) *h dominates V by an isometry $S \in \mathrm{C_b}\big(\mathbb{R}_+, \mathcal{M}(B)\big)$.*
(ii) *h approximately 1-dominates V.*
(iii) *h dominates weakly approximately inner V (in the second conjugate of $\mathrm{Q}(\mathbb{R}_+, B)$.*
(iv) *h n-dominates V for some $n \in \mathbb{N}$.*
(v) *There exists a contraction $T \in \mathcal{M}\big(\mathrm{C_0}(\mathbb{R}_+, B)\big)$ such that*

$$T^*\delta_\infty\gamma_0(a)T - \gamma_1(a) \in \mathrm{C_0}(\mathbb{R}_+, B) \qquad \forall\, a \in A\,.$$

The criteria (v) is later needed only the case of constant $h\colon A \to B$.

PROOF. `to be filled in` ?? □

LEMMA 7.2.17. *Let h_1 and h_2 be C^*-morphisms from $A \otimes \mathcal{O}_2$ into a W^*-algebra N such that $h_1(a \otimes 1) = h_2(a \otimes 1)$ for every $a \in A$.*

Then h_1 and h_2 dominate weakly approximately inner each other.

In fact, h_1 and h_2 are unitarily homotopic in N (in norm-topology).

PROOF. Let $H(a) := h_1(a \otimes 1)$ and let M denote the W*-algebra generated by $H(A)$ with unit element $P = 1_M \in N$. Then there are unital C^*-morphisms $g_j\colon \mathcal{O}_2 \to PNP$ such that $H(a)g_j(b) = h_j(a \otimes b)$. It follows that $h_j(\mathcal{O}_2) \subseteq E := M' \cap PNP$. Notice that $\mathrm{K}_1(M' \cap PNP) = 0$ as it is for all W*-algebras, because every unitary is an exponential of a some T with $T^* = -T$ and $\|T\| \leq \pi$.

Any two unital *-morphisms from $\mathcal{O}_2$ into a unital C^*-algebra E with $\mathcal{U}(E) = \mathcal{U}_0(E)$ are point-norm homotopic. The latter implies stable unitary homotopy for copies of $\mathcal{O}_2$. Since id and δ_2 are unitary homotopic in $\mathcal{O}_2$, it follows that stable homotopy of unital copies of $\mathcal{O}_2$ implies that the copies unitarily homotopic in E.

The path of unitaries $t \mapsto U(t) + (1-p)$ defines a unitary homotopy from h_1 to h_2.

`To be filled in: Check again` ?? □

PROPOSITION 7.2.18. *Suppose that A and B are stable, where B is σ-unital and A is separable, X a non-compact locally compact σ-compact Hausdorff space, and that $h\colon A \to \mathrm{Q}(X,B)$ is a C^*-morphism which is unitarily equivalent to $h \oplus h$ by a unitary in $\mathcal{M}(\mathrm{Q}(X,B))$.*

Then there exists a C^*-morphism $k\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(X,B)$ such that

(i) *$k(a \otimes 1) = h(a)$ for all $a \in A$, and*
(ii) *k is unitarily equivalent to $k \oplus k$.*

A C^*-morphism k with properties (i) *and* (ii) *is unique up to unitary equivalence by a unitary in $\mathrm{Q}(X, \mathcal{M}(B))$.*

PROOF. `to be filled in` ?? □

COROLLARY 7.2.19. *Suppose that A and B are stable, where B is σ-unital and A is separable and exact, and that $h_i\colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ are nuclear C^*-morphisms which are unitarily equivalent to $h_i \oplus h_i$, $i = 1, 2$.*

Furthermore, suppose that, for each closed ideal J of A, $h_1(J)$ and $h_2(J)$ generate the same ideal of $\mathrm{Q}(\mathbb{R}_+, B)$.

Then h_1 and h_2 are unitarily equivalent, by a unitary in $\mathrm{Q}\big(\mathbb{R}_+, \mathcal{M}(B)\big)$.

PROOF. `To be filled in` ?? □

`What happens if` $k\colon A \otimes \mathcal{O}_2 \hookrightarrow E$ `exists with` $k((\cdot) \otimes 1) = h_0(\cdot)$? `Is` $h_0 \sim h_0 \oplus h_0$ `in` $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$?

COROLLARY 7.2.20. *Suppose that A is stable, separable and exact, that B is stable and σ-unital, and that $\mathrm{C}_0(\mathbb{R}, B)$ satisfies the WvN-property or that B is strongly purely infinite.*

Let $h_0\colon A \to B$ be a nuclear non-degenerate C^*-morphism from A into B such that $h_0 \oplus h_0$ is unitarily homotopic to h_0. And let $h\colon A \to B$ is a C^*-morphism that satisfies for each $J \in \mathcal{I}(B)$ that

$$h^{-1}(h(A) \cap J) = h_0^{-1}(h_0(A) \cap J)\,,$$

i.e., that $h(a)$ and $h_0(a)$ generate the same closed ideal of B for each $a \in A_+$.

Then h is unitarily homotopic to $h \oplus h_0$.

*In particular, every *-monomorphism $h\colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ is unitarily homotopic to $h \oplus h_0$ if B is simple, purely infinite and σ-unital and A is unital and separable.*

PROOF. The equation shows that $h(A)$ generates B as a closed ideal, because $h_0(A)B$ is dense in B. Since A and B are stable and B is σ-unital, we get from

Proposition **??**

IN CHAPTERS 7 or 5 ??

that h is unitarily homotopic to a non-degenerate C^*-morphism from A into B, i.e., there is a norm-continuous path $t \in \mathbb{R}_+ \to U(t) \in \mathcal{M}(B)$ and a non-degenerate C^*-morphism $H \colon A \to B$ such that $\lim_{t\to\infty} U(t)^* h(a)U(t) = H(a)$ for each $a \in A$.

`to be filled in, check from here` **??**

Therefore, we can assume that $h(A)$ contains a strictly positive element of B.

Since $h_0 \oplus h_0$ is unitarily homotopic to h_0, it follows by

Proposition **??**,

that it suffices to show that h asymptotically absorbs h_0, i.e. that h dominates h_0 in the multiplier algebra $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B)) \cong \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}_+, B)$ of $\mathrm{Q}(\mathbb{R}_+, B)$.

If we want to adapt the proof of

Proposition **??**,

then we have to find a sequence of contractions $d_n \in B$, such that, for $a \in A$ and $j \in \{0, 1\}$,

$$\lim_{n\to\infty} \|d_n^* h(a)d_{n+j} - (1 - j)h_0(a)\| = 0\,.$$

Since $(h_0 \oplus h_0) \circ h^{-1} \colon h(A) \to B$ is residually nuclear by the assumptions, and since B is strongly p.i. (or has WvN-property), we find such a sequence,

cf. Remark **??**.

Thus h absorbs h_0 asymptotically and, therefore $h \oplus h_0$ is unitarily homotopic to h. □

PROOFS OF THEOREMS B(III) AND M(III): First, Theorem B(iii) is a special case of Theorem M(iii):

we have defined $h_0 \colon A\otimes\mathbb{K} \to B\otimes\mathbb{K}$ in Chapter 1 before Theorem B. If we replace A and B by its stabilizations, then with $N := \mathcal{O}_2 \otimes \mathbb{K}$, $\mathrm{Prim}(N) = \mathrm{Prim}(B) =$point, and the assumptions of Theorem M(iii) are satisfied.

We explain the assumptions of Theorem M in the special situation of part (iii): We suppose that A is a separable, stable and exact C^*-algebra, that B is strongly purely infinite, and that N is a strongly purely infinite separable stable C^*-subalgebra of B, such that NB is dense in B and that $\Psi_B := \Psi_0^{N,B}$ is an *isomorphism* from $\mathcal{I}(N) \cong \mathbb{O}(\mathrm{Prim}(N))$ onto $\mathcal{I}(B)$. The latter conditions imply that B is σ-unital and stable, and that, for every $I \in \mathcal{I}(N)$, there is a unique $J \in \mathcal{I}(B)$ with $I = J \cap N$.

Furthermore, there is given a lower semi-continuous action $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ of $X := \mathrm{Prim}(N)$ on A with $\Psi_A(\emptyset) = 0$, $\Psi_A^{-1}(A) = X$, such that Ψ_A satisfies condition (ii) of Definition 1.2.6.

Here we assume the existence of a non-degenerate nuclear *-monomorphism $k\colon A\otimes\mathcal{O}_2 \to N$ such that $h_0 := k((.)\otimes 1)$ induces Ψ_A by $\Psi_A(Z_I) = h_0^{-1}(h_0(A)\cap I)$ for $I \in \mathcal{I}(N)$ and $Z_I \in \mathbb{O}(X)$ corresponding to I. The existence of h_0 follows from Theorem K, which will be shown in Chapter 12. The proof of Theorem K uses results of Chapter 3-6, of the first part of Chapter 9 and of Chapter 7 up to

Corollary ??.

For this h_0 we have $h_0(\Psi_A(Z)) = h_0(A) \cap \Psi_B(Z)$, because $N \cap \Psi_B(Z)$ is the closed ideal of N which corresponds to $Z \in \mathbb{O}(X)$. If $h\colon A \to B$ is a *-monomorphism, such that $h(\Psi_A(Z)) = h(A) \cap \Psi_B(Z)$ for $Z \in \mathbb{O}(X)$, then, for $J \in \mathcal{I}(B)$,

$$h^{-1}(h(A)\cap J) = h_0^{-1}(h_0(A)\cap J),$$

because Ψ_B maps $\mathbb{O}(X)$ onto $\mathcal{I}(B)$.

Thus, Corollary 7.2.20 applies, and h asymptotically dominates h_0. □

3. Remarks on the case of group actions

REMARK 7.3.1. Our Definition 7.1.1 of asymptotic $CP_{nuc}(A, B)$-morphisms does not work well in the case where a T_0 space X acts on A and B, as considered in Section 2 of Chapter 1, or in the case where, moreover, a locally compact group G acts Ψ_A-equivariant on A (via $\gamma_A\colon G \to \mathrm{Aut}(A)$) and Ψ_B-equivariant on B (via γ_B). Then it is useful to replace (a) by the following (a'):

(a') *For every $t \in \mathbb{R}_+$, every open subset $Z \subseteq X$ and every $g \in G$, $V(t)$ is Ψ-residually nuclear and $V(t)$ is asymptotically γ-equivariant.*
That means:
$V(t)(\Psi_A(Z)) \subseteq \Psi_B(Z)$ and the naturally induced completely positive contraction $[V(t)]_Z\colon A/\Psi_A(Z) \to B/\Psi_B(Z)$ is nuclear, and,
for every $g \in G$ and $a \in A$, $V(t)(\gamma_A(g)(a)) - (\gamma_B(g)V(t)(a))$ converges to zero if $t \to \infty$.

This is condition (a) in case of the cone $\mathcal{C}$ of Ψ-residually nuclear maps from A to B if the action of G is trivial on A and B.

If the actions of G are not trivial, i.e., if there exists a $g \in G$ such that $\gamma_A(g) \neq \mathrm{id}$ or $\gamma_B(g) \neq \mathrm{id}$, then also the definition of unitary homotopy should be stronger: We could require that $g \mapsto \mathcal{M}(\gamma_B(g))(u(t))$ is norm-continuous for every t and that $\lim_{t\to\infty} \|u(t) - \mathcal{M}(\gamma_B(g))(u(t))\| = 0$ in the for every $g \in G$.

One has to make some extra assumption on the actions of G on B such that that Cuntz addition makes sense:

There should exist norm-continuous maps $t \mapsto s_1(t)$, $t \mapsto s_2(t)$ from $\mathbb{R}_+$ into the isometries in $\mathcal{M}(B)$ such that $s_1(t)s_1(t)^* + s_2(t)s_2(t)^* = 1$, $\mathcal{M}(\gamma_B(g))(s_i(t)) - s_i(t)$ is in B for every $t \in \mathbb{R}_+$ and every $g \in G$, and tends to zero if $t \to \infty$ for every (fixed) $g \in G$.

The semigroup of asymptotically γ-equivariant Ψ-equivariant nuclear asymptotic morphisms will be denoted by $\mathrm{SR}^G(X,A,B)$. We write $\mathrm{R}^G(X,A,B)$ for its Grothendieck group, and $\mathrm{R}(X,A,B)$ if G is trivial.

Most of the theory ??????????????

`more ??? ??`

4. Collection of needed results.

`PLAN:`

General Plan for Chp. 7:

1. Asymptotic morphisms (def. and classes)

2. lifting of nuclear maps, cases where residually nuclear maps can be lifted, lifting by continuous families of approximately inner maps. When does a continuous family of approximately inner maps define a "locally" approximately inner map of $\mathrm{Q}(X,B)$?. Compare also the invertibility for $\mathrm{SExt}_{\mathrm{nuc}}(X;\, A,B)$ in Chapter 5.

3. "car:" Asymptotic Weyl - von Neumann theorem (for asymptotically residually nuclear maps $V\colon C\to \mathrm{Q}(X,B)$)

4. Absorption theorems for $h\colon A\to \mathrm{Q}(X,B)$, $k\colon A\to \mathrm{C}(Y,B),\ldots$

5. Asymptotic morphism as elements of

$$G(h_0;A,E_{\mathbb{R}})\cong G(h_0;A,\mathrm{Q}(\mathbb{R}_+,\mathcal{M}(B)))\cong G(h_0;A,\mathrm{C_b}(\mathbb{R}_+,\mathcal{M}(B))/\,\mathrm{C}_0(\mathbb{R}_+,B))$$

for $E_{\mathbb{R}}:=\mathcal{M}(\mathrm{C}_0(\mathbb{R},B))/\,\mathrm{C}_0(\mathbb{R},B)$, A separable, B σ-unital and stable.

$\mathrm{R}(\mathrm{Prim}(B);A,B)\cong G(h_0;A,E_{\mathbb{R}})$ if nuclear $h_0\colon A\hookrightarrow B$ with $[h_0\oplus h_0]=[h_0]$ defines the action.

$\mathrm{R}(\mathcal{C}(h_0);A,B)\cong G(h_0;A,E_{\mathbb{R}})$ if $h_0=h((\cdot)\otimes 1)$ for $h\colon A\otimes\mathcal{O}_2\to B$.

6. Existence of zero domination:

For every $h\colon A\to \mathrm{Q}(X,B)$ with stable B and σ-unital A there exist a positive contraction $e\in \mathrm{Q}(X,B)$ and isometries $S,T\in \mathrm{C_b}(X,\mathcal{M}(B))$ with $SS^*+TT^*=1$, $\eta(T)^*e\eta(T)=0$, $h(a)=h(a)e=eh(a)$ for all $a\in A$, where $\eta\colon \mathrm{C_b}(X,\mathcal{M}(B))\to \mathcal{M}(\mathrm{Q}(\mathbb{R}_+,B))$

<== *here* $X=\mathbb{R}_+$?

*denotes the natural *-morphism induced by the natural embedding*

$$\mathrm{C_b}(X,\mathcal{M}(B))\hookrightarrow \mathrm{C_{b,st}}(X,\mathcal{M}(B))\cong\mathcal{M}(\mathrm{C}_0(X,B))\,.$$

In particular, $h\colon A\to \mathrm{Q}(X,B)$ dominates zero in $\mathrm{Q}(X,\mathcal{M}(B))$.

7. Asymptotic unitary equivalence of "stable" monomorphisms to "non-degenerate" monomorphisms (see also Chapter 5)

`to be added:`

`refprop:7.Y1.chp9`

For a C^*-algebra F, let

$$\mathrm{Q}(\mathbb{R}_+, F) := \mathrm{C_b}(\mathbb{R}_+, F)/\,\mathrm{C_0}(\mathbb{R}_+, F)\,.$$

PROPOSITION 7.4.1. *$\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ is in a natural way a C^*-subalgebra of the multiplier algebra of $\mathrm{Q}(\mathbb{R}_+, F)$.*

If F is stable or unital, then, for every separable subset Y of $\mathrm{Q}(\mathbb{R}_+, F)$ and every unitary v in the multiplier algebra of $\mathrm{Q}(\mathbb{R}_+, F)$ there is a unitary $u \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ with $u^*bu = v^*bv$ for every $b \in Y$.

In general (for every σ-unital F), if $A \subseteq \mathrm{Q}(\mathbb{R}_+, F)$ is a separable C^*–subalgebra of $\mathrm{Q}(\mathbb{R}_+, F)$ and v is a unitary in the connected component of 1 in $\mathcal{M}(D)$ for $D := \overline{A\,\mathrm{Q}(\mathbb{R}_+, F)A}$, then there is a unitary $u \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ with $uD + Du \subseteq D$ such that $v = u + \mathrm{Ann}(A)$.

`ref{lem:7.Y2.chp9} or ref{prop:7.Y2.chp9}`

PROPOSITION 7.4.2. *Suppose that $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, that A is a C^*–algebra, and that $h_0 \colon A \hookrightarrow \mathrm{Q}(\mathbb{R}_+, B)$ is a *-monomorphism. If $h_0 \oplus h_0$ is unitarily equivalent to h_0 in $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$ then there is a *-monomorphism $h \colon A \otimes \mathcal{O}_2 \hookrightarrow \mathrm{Q}(\mathbb{R}_+, B)$ with $h(a \otimes 1) = h_0(a)$ for $a \in A$ such that $h \oplus h$ is unitarily equivalent to h by a unitary in $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$.*

If A is σ-unital and stable and $k \colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, B)$ satisfies $k(a \otimes 1) = h_0(a)$ for $a \in A$, then k is unitarily equivalent to h by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

In particular, then $h \oplus h$ is unitarily equivalent to h in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

For the two latter statements it is only needed that A is σ-unital and the unitary group of $D := \overline{h_0(A)\,\mathrm{Q}(\mathbb{R}_+, B)h_0(A)}$ is connected. ¡== for unital A or ???

`or:`

... $\mathcal{U}(\mathcal{M}(\overline{h_0(A)\,\mathrm{Q}(\mathbb{R}_+, B)h_0(A)}))$ is connected ?? **??**

`reflem:7.Y3.chp10?  also:  chp9?`

COROLLARY 7.4.3. *Suppose A is stable and separable, that B is stable, and that h_0 and k_0 are nuclear *-monomorphisms from A into $\mathrm{Q}(\mathbb{R}_+, B)$, such that h_0 and k_0 "extend" to C^*-morphisms h and k from $A \otimes \mathcal{O}_2$ into $\mathrm{Q}(\mathbb{R}_+, B)$.*

Then h and $h \oplus h$ (respectively k and $k \oplus k$) are unitarily equivalent in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

If for every $I \in \mathcal{I}(\mathrm{Q}(\mathbb{R}_+, B))$,

$$h_0^{-1}(h_0(A) \cap I) = k_0^{-1}(k_0(A) \cap I)\,.$$

Then h and k are unitarily equivalent by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

*In particular, if $h, k \colon A \otimes \mathcal{O}_2 \to B$ are nuclear *-monomorphisms that induce the same map from $\mathcal{I}(B)$ to $\mathcal{I}(A)$, then h and k are unitarily homotopic.*

`ref{lem:7.??}(??)`

By Lemma lem:7.??(??), (see ref in chp 9) k_0 comes from $k\colon D\otimes\mathcal{O}_2 \to B$, D, B stable, D separable , B σ-unital *there exists a contraction* $g_1 \in \mathrm{C_b}(\mathbb{R}, 1_B \otimes \mathbb{K})_+$ *such that* $g_1(x) = 0$ *for* $x \leq 0$ *and* $\lim_{x\to+\infty} g_1(x)k_0(a) = k_0(a)$ *for every* $a \in D$, *because* $k_0(D)$ *is separable.*

`refcor:7.Y7.chp9`

Corollary 7.4.4. *Suppose that A is separable and stable, B is σ-unital and stable and that $h, k\colon A \otimes \mathcal{O}_2 \to B$ are *-monomorphisms. Let $h_0 := h((.) \otimes 1)$ and $k_0 := k((.) \otimes 1)$.*

If there are approximately inner completely positive maps $V_n, W_n\colon B \to B$ such that $V_n \circ h_0$ converges in point-norm to k_0 and $W_n \circ k_0$ converges in point-norm to h_0, then h and k are unitarily homotopic.

`next:  reflem:7.Y3.chp9`

Lemma 7.4.5. *There are natural inclusions and isomorphism*

$$\mathcal{M}(B) \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B)) \subseteq \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$$

`ref, cor:7.Y1.chp11`

Corollary 7.4.6. *Suppose that $h_0^u\colon A \to \mathcal{O}_2$ is a unital *-monomorphism, and that the evaluations $\pi_y h$ at $y \in \mathbb{R}_+ \times X$ of $h\colon A \to \mathrm{C_b}(\mathbb{R}_+ \times X, D)$ are unital monomorphisms. Let $\pi\colon \mathrm{C_b}(\mathbb{R}_+ \times X, D) \to \mathrm{Q}(\mathbb{R}_+ \times X, D)$.*

Then $\pi h\colon A \to \mathrm{Q}(\mathbb{R}_+ \times X, D)$ dominates $h_0^u\colon A \to \mathcal{O}_2 \subseteq \mathrm{Q}(\mathbb{R}_+ \times X, D)$.

`ref; prop:7.Y2.chp10`

Proposition 7.4.7. *, Suppose that A and B are stable and σ-unital. and that a C^*-morphism $h_1\colon A \to B$* approximately dominates *a completely positive contraction $h_2\colon A \to B$, (cf. Definition 3.10.1). (i.e., there exists a sequence of contractions $d_n \in B$ such that $\lim \|h_2(a) - d_n^* h_1(a) d_n\| = 0$ for $a \in A$.)*

Then there are isometries $t_n \in \mathcal{M}(B)$ such that $\lim \|h_2(a) - t_n^* h_1(a) t_n\| = 0$ for $a \in A$.

`ref; cor:7.Y4.chp12`

Corollary 7.4.8. *, $h, k\colon A \otimes \mathcal{O}_2 \to B$ nuclear, h is unitarily homotopic to k if $h^{-1}(J) = k^{-1}(J)$ for all closed ideals of B*

`ref; cor:7.Y3.chp12`

Corollary 7.4.9. *Suppose that $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cdots$ and that B is stable. Let $G = \mathrm{Q}(\mathbb{R}_+, B)$ or $G := B_\omega$ and $A \subseteq G$ separable. There is a *-monomorphism h from $A \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ into G with $h(a \otimes 1 \otimes e_{1,1}) = a$.*

`ref; cor:7.Y2.chp12`

COROLLARY 7.4.10. *G as in Corollary 7.4.9 $A \subseteq G$ separable. The approximately inner completely positive contractions $T \colon D \to G$ are one-step inner, i.e., there is a contraction $d \in G$ such that $T(b) = d^* bd$.*

`ref; prop:7.Y1.chp12:`

PROPOSITION 7.4.11. *Suppose that $A_1, A_2, \ldots$ are separable C*-algebras with $A_k \cong A_k \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$, and let $D := \prod_\omega (A_k)$.*

*Then if $B \subseteq D$ is separable, there exists a *-monomorphism φ from $B \otimes \mathcal{O}_\infty$ into D, such that $\varphi(b \otimes 1) = b$ for $b \in B$.*

`ref; lem:7.Y6.chp9`

LEMMA 7.4.12. *Let $a = (a_1, a_2) \in \mathrm{Q}(\mathbb{R}, B) \cong \mathrm{Q}(\mathbb{R}_-, B) \oplus \mathrm{Q}(\mathbb{R}_+, B)$ a positive contraction, then there exist a positive contraction $b = (b_1, b_2) \in \mathrm{Q}(\mathbb{R}, B)$ with $b_1 = 0$ and $b_2 a_2 = a_2$.*

LEMMA 7.4.13. *Every non-zero projection in $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$ is equivalent to $1 \otimes e_{1,1} \in \mathcal{O}_2 \otimes \mathbb{K}$ by unitaries in the unitization of $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$.*

PROPOSITION 7.4.14. ***(Compare also 7.4.18)***

Suppose that $\mathcal{O}_2 \otimes \mathbb{K} \subseteq \mathcal{M}(B)$ non-degenerate with $(\mathcal{O}_2 \otimes \mathbb{K}) \cap B = 0$. Define F by $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K}) \cong F \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B))$. For every projection $q \in F_1 \cong \mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$, and every element $f_1 \in \mathrm{C_b}(\mathbb{R}, B)$ there exists a projection p in F_1 such that $pf_1p - f_1 \in \mathrm{C_0}(\mathbb{R}, B)$, $q \leq p$ and $q \neq p$.

`ref; {prop:7.Y5.chp9}(??), {prop:7.Y5new.chp9}(??),`

PROPOSITION 7.4.15. *Suppose B is stable and σ-unital, and that D separable, stable and* exact.

Let $E_\mathbb{R} := \mathcal{M}(B \otimes \mathrm{C_0}(\mathbb{R}))/B \otimes \mathrm{C_0}(\mathbb{R})$ and $h_0 \colon D \to E_\mathbb{R}$ defined by ...????????????? is nuclear.

??

Let $k \in \mathrm{Hom}(X; D, \mathrm{Q}(\mathbb{R}_+, B))$.

(i) *The map k has unitary equivalence class $[k]$ in $S(h_0, D, E_\mathbb{R})$, if an only if k is nuclear, where $\mathrm{Q}(\mathbb{R}_+, B) \subseteq \mathrm{Q}(\mathbb{R}, B) \subseteq E_\mathbb{R}$ naturally.*
(ii) $\mathrm{R}(X; D, B) = [h_0] + S(h_0, D, E_\mathbb{R}) = [h_0] + S(h_0, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$.
(iii) *If h is nuclear as a map from D to $\mathrm{Q}(\mathbb{R}_+, B)$, and if h is also Ψ-equivariant, then h is dominated by $h_0 \colon D \to \mathrm{Q}(\mathbb{R}_+, B)$.*

The point is, that nuclear asymptotically equivariant h is residually nuclear ??????

`really needed in chp. 9:`

Let $h \colon D \to \mathrm{Q}(\mathbb{R}_+, B)$. Suppose that $H_\mathbb{R}$ dominates $h + \beta h_0$, i.e., there exists an isometry $T \in E_{\mathbb{R}}$ with $T^* H_0(.)T = h + \beta h_0$. Then $h \in S(h_0\,;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$, i.e., h_0 dominates h if and only if $H_{\mathbb{R}}$ dominates $h + \beta h_0$.

(True for any $h_0 \colon D \to B$ non-degenerate with $h_0 \oplus h_0 \sim h_0$, i.e., H_0 pseudo-dominates h, if and only if, H_0 dominates h, if and only if, h_0 dominates h.)

`ref; prop:7.Y5.chp9`

By Proposition 7.4.15, $S(h_0, D, E_{\mathbb{R}})$ is nothing else the semigroup of unitary equivalence classes of Ψ-residually nuclear C^*-morphisms h from D into $\mathrm{Q}(\mathbb{R}_+, B)$, where $\Psi_A(J)$ and $\Psi_{\mathrm{Q}(\mathbb{R}_+,B)}$ are defined as above.

`next: Proof of Thm. B(iii),`
`also Proof of Corollary 7.4.16`

Corollary 7.4.16. *The absorption result (iii) of Theorem B:*

Suppose that B is simple, purely infinite, unital and contains a copy of $\mathcal{O}_2 \subseteq B$ unitally.

*Let A a unital separable exact C^*–algebra and let $k_1 \colon A \to \mathcal{O}_2$, $k_2 \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ unital *-monomorphisms, and $k_0 := k_2(k_1(\cdot) \otimes 1)$.*

*Then, for every unital *-monomorphism $h \colon A \to \mathrm{Q}(\mathbb{R}_+, B)$, $[h \oplus k_0] = [h]$.*

More generally, $\mathrm{R}(D, (\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots) \otimes \mathbb{K}) = 0$ for every stable separable C^*-algebra D.

Lemma 7.4.17. *Let $h \colon D \to B \otimes \mathbb{K}$ a C^*-morphism where D is σ-unital. Then there exists a contraction $g_1 \in \mathrm{C_b}(\mathbb{R}, 1_{\mathcal{M}(B)} \otimes \mathbb{K})_+$ such that $g_1(x) = 0$ for $x \leq 0$ and $\lim_{x \to +\infty} g_1(x)h(a) = h(a)$ for every $a \in D$.*

(because $h(D)$ is σ-unital)

`Next proposition, see chp9, but only the prop is cited:`

Let $G_2 := \epsilon(\mathcal{O}_2 \otimes \mathbb{K}) \subseteq \mathcal{M}(B)$. Then $G_2 \cong \mathcal{O}_2 \otimes \mathbb{K}$, $G_2 \cap B = 0$ and $G_2 \cdot B$ is dense in B. Let $F_1 := \mathrm{C_b}(\mathbb{R}, G_2) \subseteq \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$. Certainly $\tilde{\beta}(F_1) \subseteq F_1$.

$F_1 \cap \mathrm{C_0}(\mathbb{R}, B) = 0$ and therefore $F := \pi_{SB}(F_1) \subseteq E_{\mathbb{R}}$ is a C^*-subalgebra of $E_{\mathbb{R}}$ which is isomorphic to $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$ and satisfies $\beta(F) = F$.

$H_0'(D)G_2 \subseteq B$ implies that $F_1 H_0'(D) \subseteq \mathrm{C_b}(\mathbb{R}, B)$ and thus $F H_0(D) \subseteq J + \beta J$.

Proposition 7.4.18. *For every projection $q \in F_1$, and every element $f_1 \in \mathrm{C_b}(\mathbb{R}, B)$ there exists a projection p in F_1 such that $p f_1 p - f_1 \in \mathrm{C_0}(\mathbb{R}, B)$, $q \leq p$ and $q \neq p$.*

Proof. of Proposition 7.4.18 ?? to be filled in ... □

`Next Corollary 7.4.19 is used in Chp. 9`

Corollary 7.4.19. *(iii) Suppose that D is a stable and separable, that B is stable and σ-unital. Let $k \colon D \otimes \mathcal{O}_2 \to B$ a non-degenerate *-monomorphism,*

$k_0 := k((\cdot)\otimes 1)$ *and let* $\mathcal{C} \subseteq \mathrm{CP}(D,B)$ *denote the point-norm closed matricial operator-convex cone generated by* $k_0\colon D \to B$*. Then:*

(i) $\mathrm{SR}(\mathcal{C}; D, B) = S(k_0; D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ *(as semi-groups). By Corollary* **??***,*

(ii) *If* $k_1 \in \mathrm{Hom}(D,B)\cap\mathcal{C}$ *is non-degenerate and satisfies that* $k_1\oplus k_1$ *is unitarily homotopic to* k_1 *and that* k_1 *generates* $\mathcal{C}$*, then* k_1 *is unitarily homotopic to* k_0*.*

(iii) $\mathrm{R}(\mathcal{C}; D, B) \cong \mathrm{SR}(\mathcal{C}; D, B) + [k_0] \subseteq \mathrm{SR}(\mathcal{C}; D, B)$.

LEMMA 7.4.20. *Suppose that* D *is separable,* B *stable, and let* $I_1\colon \mathrm{Q}(\mathbb{R}_+, B) \to E_{\mathbb{R}} := \mathcal{M}(SB)/SB$ *the canonical embedding.*

The unitary equivalence classes of $k \in \mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))$ *and* $I_1\circ k \in \mathrm{Hom}(D, E_{\mathbb{R}})$ *naturally coincide, i.e.,* $[k]=[k']$ *in* $[\mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))]$ *if and only if* $[I_1\circ k] = [I_1\circ k']$ *in* $[\mathrm{Hom}(D, E_{\mathbb{R}})]$.

PROPOSITION 7.4.21. *For separable stable* D *and* σ*-unital stable* B*, let* $I_1\colon \mathrm{Q}(\mathbb{R}_+, B) \to E_{\mathbb{R}} := \mathcal{M}(SB)/SB$ *the natural embedding. Suppose that* $k\colon D\otimes\mathcal{O}_2 \hookrightarrow B$ *is non-degenerate. Let* $k_0 := k((\cdot)\otimes 1)$ *and* $h_0 := I_1\circ k_0$. *Let* $\mathcal{C}\subseteq \mathrm{CP}(A,B)$ *denote the point-norm closed matricial operator-convex cone generated by* k_0*.*

(i) *A morphism* $k \in \mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))$ *has unitary equivalence class* $[k] \in \mathrm{SR}(\mathcal{C}; D, B)$*, if an only if,* $[I_1\circ k]$ *is in* $S(h_0; D, E_{\mathbb{R}})$*.*

(ii) $[k] \in [k_0] + S(k_0; D, \mathrm{Q}(\mathbb{R}_+, B)) \cong \mathrm{R}(\mathcal{C}; D, B)$*, if and only if,* $[I_1\circ k] \in [h_0] + S(h_0; D, E_{\mathbb{R}}) = G(h_0; D, E_{\mathbb{R}})$*.*
In particular, $[(I_1\circ k)\oplus h_0] \in \mathrm{R}(\mathcal{C}; D, B)$ *for* $[k]\in \mathrm{SR}(\mathcal{C}; D, B)$*.*

We have

$$G(k_0; D, Q(\mathbb{R}_+, B)) \cong \mathrm{R}(\mathcal{C}; D, B) \cong G(I_1\circ k_0; D, E_{\mathbb{R}})$$

if $\mathcal{C} = \mathcal{C}(k_0)$ *for some* $k_0\colon D\to B$ *with* $k_0\oplus k_0$ *unitarily homotopic to* k_0*, if* D *is separable and* B *is stable.*

PROOF. Use Proposition 4.4.3 ?????. □

`Next Lemma {cor:7.YZnew.chp9} is used in Chp. 9`

LEMMA 7.4.22. *Suppose that* D *is stable and separable, that* B *is* σ*-unital and stable, and that* $k\colon D\otimes\mathcal{O}_2 \to B$ *is a non-degenerate *-monomorphism. Let* $k_0 := k((\cdot)\otimes 1)$ *and* $h_0 := I_1\circ k\colon D \to J \subseteq \mathrm{Q}(\mathbb{R}, B) \cong \mathrm{Q}(\mathbb{R}_-, B)\oplus \mathrm{Q}(\mathbb{R}_+, B)$ *for the natural isomorphism* I_1 *from* $\mathrm{Q}(\mathbb{R}_+, B)$ *onto* $J := \pi(K)$ *for* $K := \{f \in \mathrm{C_b}(\mathbb{R}, B)\,;\ f(t)=0 \text{ for } t<0\}$ *(where*

$$\pi := \pi_{\mathrm{C}_0(\mathbb{R},B)}\colon \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B)) \to E_{\mathbb{R}} := \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))/\,\mathrm{C}_0(\mathbb{R}, B)$$

and where we consider B *naturally as a subset* $\mathrm{Q}(\mathbb{R}_+, B)$*). Further, let* $\beta\colon \mathrm{Q}(\mathbb{R}, B) \to \mathrm{Q}(\mathbb{R}, B)$ *the isomorphism with* $\beta(f + \mathrm{C}_0(\mathbb{R}, B)) = f + \mathrm{C}_0(\mathbb{R}, B)$ *for* $f(t) := f(-t)$*, and* $H_0 := \pi\circ(\delta_\infty\circ k_0\otimes 1)\colon D \to E_{\mathbb{R}}$*.*

(i) *If* $h\colon D \to \mathrm{Q}(\mathbb{R}_+, B) \subseteq \mathrm{Q}(\mathbb{R}, B)$ *is a* C^**-morphism then there exists a contraction* $Y \in \mathrm{C_b}(\mathbb{R}, B)_+$ *with* $Y(t) = 0$ *for* $t \leq 0$ *such that* $yhy = y(h + \beta h_0)(\cdot)y = h$ *for* $y := Y + \mathrm{C}_0(\mathbb{R}, B) \in J$.

(ii)

$$H_0(D) \subseteq \pi(\mathcal{M}(B)) \subseteq \pi(\mathrm{Q}(\mathbb{R}, \mathcal{M}(B))) \subseteq E_{\mathbb{R}}\,,$$

by the natural embeddings

$$\mathcal{M}(B) \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B)) \subseteq \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) = \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))\,,$$

(iii) *If* z *is a contraction in the ideal* $\mathrm{Q}(\mathbb{R}, B) = J + \beta(J)$ *of* $E_{\mathbb{R}} := \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))/\,\mathrm{C}_0(\mathbb{R}, B))$*, then there is a contraction* $w \in \mathrm{C_b}(\mathbb{R}, B)$ *with* $w + \mathrm{C}_0(\mathbb{R}, B)) = z$*, and* $w^*(\delta_\infty \circ k_0)(\cdot)w + \mathrm{C}_0(\mathbb{R}, B) = z^*H_0(\cdot)z =: T$.

(iv) *There is a contraction* $v \in \mathrm{C_b}(\mathbb{R}, B)$ *with* $v(t) = 0$ *for* $t < 0$ *and* $v^*k_0(\cdot)v + \mathrm{C}_0(\mathbb{R}, B) = T$*. Thus,* $d^*h_0(\cdot)d = d^*(h_0 + \beta h_0)(\cdot)d = T$ *for the contraction* $d := v + \mathrm{C}_0(\mathbb{R}, B) \in J \cong \mathrm{Q}(\mathbb{R}_+, B)$.

(v) *There is an isometry* $S \in \mathrm{C_b}(\mathbb{R}, M(B)) \subseteq \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ *such that* $s := S + \mathrm{C}_0(\mathbb{R}, B) \in E_{\mathbb{R}}$ *satisfies* $s^*h_0(\cdot)s = d^*h_0(\cdot)d = T$.

Corollary 7.4.23. *Suppose that* $N \subseteq B$ *is a (non-simple) strongly purely infinite* C^**–subalgebra of* B *with* NB *dense in* B*, and that* $k\colon D \otimes \mathcal{O}_2 \to N$ *is a non-degenerate* nuclear **-monomorphism. Let* $k_0 := k((\cdot) \otimes 1) \in \mathrm{Hom}_n\, uc(D, B)$*,* $X := \mathrm{Prim}(N)$ *and define actions* Ψ_A *(respectively* Ψ_B*) of* X *on* A *(respectively on* B*) by* $\Psi_A(J) := k_0^{-1}(k_0(A) \cap J)$ *and* $\Psi_B(J) :=$ *the closed ideal of* B *generated by* $J \in \mathcal{I}(N) \cong \mathbb{O}(\mathrm{Prim}(N))$.

Then $\mathrm{CP_{rn}}(\Psi) = \mathcal{C}(k_0) \subseteq \mathrm{CP}(A, B)$ *and*

$$\mathrm{SR}(X;\, D, B) \;=\; \mathrm{SR}(\mathrm{CP_{nuc}}(\Psi);\, D, B) \;=\; S(k_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))\,.$$

`??`

`Next taken from chap.9.  Need this here ??`

The reader can see, that $X := \mathrm{Prim}(N)$ acts on B upper semi-continuously by Ψ_B where $\Psi_B(J)$ is the closed ideal of B generated by the closed ideal J of N. The T_0 space X acts lower semi-continuously on D by Ψ_D, and, moreover, the action Ψ_D is monotone upper semi-continuous, i.e., satisfies also condition (ii) of Definition 1.2.6. Here $\Psi_D(J)$ for $J \in \mathcal{I}(N)$ is defined by

$$\Psi_D(J) := k_0^{-1}(k_0(D) \cap J)\,.$$

The later in Chapter 12 given proof of Theorem K shows that $k_0\colon D \to B$ (and $k\colon D \otimes \mathcal{O}_2 \to B$) can be constructed up to unitary homotopy from the actions Ψ_D and Ψ_B of X on D, respectively on B.

`Next used in chp.9:`

There is a natural isomorphism $\mathrm{C_{b,st}}(Y, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C}_0(Y, B))$.

`next:  lem:7.ZZnew.chp.9`

LEMMA 7.4.24. *Suppose that D is separable and stable, B is σ-unital and stable and that $h_0 \in \mathrm{Hom}(D,B)$ is non-degenerate and is unitarily equivalent to $h_0 \oplus h_0$.*

Let $\mathcal{C} := \mathcal{C}(h_0) \subseteq \mathrm{CP}(D,B)$, the point-norm closed matrix operator-convex cone generated by h_0.

Then a c.p. contraction $T\colon D \to \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$ is dominated by h_0 in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$, if and only if, there is a point-norm continuous map $t \in \mathbb{R}_+ \mapsto V_t \in \mathcal{C}$ such that $V\colon D \to \mathrm{C}(\mathbb{R}_+, B)$ satisfies $\|V\| \leq 1$, $V(a)(t) := V_t(a)$ and $V(a) + \mathrm{C}_0(\mathbb{R}_+, B) = T(a)$ for all $a \in D$.

If $T \in \mathrm{CP}(D,B)$ is a c.p. contraction, then h_0 dominates T in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$, if and only if, $T \in \mathcal{C}$.

In particular, $h \in \mathrm{Hom}(D,B)$ satisfies $h \in \mathcal{C}$, if and only if, $[h] \in \mathrm{SR}(\mathcal{C};\, D, B)$, if and only if, $[h] \in \mathrm{Gr}\big(h_0;\, D,\, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))\big)$.

The natural group homomorphism

$$G(h_0;\, D, \mathcal{M}(B)) \to G(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$$

is neither surjective nor injective in general, e.g. for $D = \mathcal{O}_2 \otimes \mathbb{K}$, $B = \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and $h_0(\cdot) := 1 \otimes (\cdot)$.

PROOF. $T \in \mathrm{CP}(D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ is dominated by h_0 in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)) \supset B$, if and only if, there is a norm continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ into the isometries in $\mathcal{M}(B)$ with $s^* h_0(\cdot) s = T$ for $s := S + \mathrm{C}_0(\mathbb{R}_+, \mathcal{M}(B))$. Let $V_t(a) := S(t)^* h_0(a) S(t)$, then $V_t \in \mathcal{C} = \mathcal{C}(h_0)$ and $V(a) + \mathrm{C}_0(\mathbb{R}_+, B) = T(a)$ for all $a \in A$ (because $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, B))) \cap \mathrm{C}_0(\mathbb{R}_+, \mathcal{M}(B)) = \mathrm{C}_0(\mathbb{R}_+, B)$ and $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, B)$ is an ideal of $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathcal{M}(B)))$.

If $T \in \mathrm{CP}(D,B)$ (in addition), this means that T is the limit of $V_t := S(t)^* h_0(\cdot) S(t)$ in point-norm. Since $\mathcal{C}$ is closed in point-norm topology, this implies $T \in \mathcal{C}$.

More arguments:
The assumption implies that there are norm continuous paths

$$t \in [0,\infty) \mapsto S_k(t) \in \mathcal{M}(B)$$

(for $k \in \{1,2\}$) of isometries, defining a path of copies of $\mathcal{O}_2$ in $\mathcal{M}(B)$ that commutes with $h_0(D)$ in $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathcal{M}(B))$ modulo $\mathrm{C}_0(\mathbb{R}_+, B)$.

If $T \in \mathcal{C}$ then there exist point-norm approximation $b_n^* h(\cdot) b_n$ of T in $\mathrm{CP}(D,B) \subseteq \mathcal{L}(D,B)$ with suitable b_n. With approximately central unit of D one can manage that $\|b_n\| \leq 1$. This allows to replace the sequence by a norm-continuous path of isometries in $\mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathcal{M}(B))$.

Conversely, if there is contraction $V\colon D \to \mathrm{C}_\mathrm{b}(\mathbb{R}_+, B)$ with $V_t(\cdot) := V(\cdot)(t) \in \mathcal{C}$, then there is a norm-continuous map $t \in \mathbb{R}_+ \mapsto S(t) \in \mathcal{M}(B)$ into the isometries in $\mathcal{M}(B)$, such that $\lim_{t\to\infty} \|V(a)(t) - S(t)^* h_0(a) S(t)\| = 0$ for all $a \in D$.

??

`Give reference for construction of` $S(t)$
`Perhaps as in above blue part?`

Let $h \in \mathrm{Hom}(D, B)$. By Definition **??**of $[h] \in \mathrm{SR}(\mathcal{C};\, D, B)$, if and only if, there is a map $t \in [0, \infty) \mapsto V_t \in \mathcal{C}$, such that $\lim_{t\to\infty} \|V_t(a) - h(a)\| = 0$ for every $a \in D$. Thus $h \in \mathcal{C}$.

If $h \in \mathcal{C}$, then $T := h$ is dominated by h_0 in $\mathrm{Q}(\mathbb{R}_+;\, \mathcal{M}(B))$

(but in general not directly in $\mathcal{M}(B)$ itself, h_0 dominates h only asymptotical inside $\mathcal{M}(B)$). □

`Next cited from chp.8:`
If an element $b \in \mathcal{M}(SB)/SB$ has a representative $c \in \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B))$ such that $c(t) \in B$ for every t. that it even does not say that c is in $\mathrm{C}_{\mathrm{b}}(\mathbb{R}, B)$ (as we have seen in Chapter 7).

`Next definition cited from chp.8:`
$\mathrm{SR}(X; A, B) := \mathrm{Hom}_{\mathrm{nuc}}(X; A, \mathrm{Q}(\mathbb{R}_+, B))$ for stable separable A and σ-unital stable B.
`for the action` Ψ `of` X `on` $\mathrm{Q}(\mathbb{R}_+, B)$
one has to take $\Psi(U) := \mathrm{Q}(\mathbb{R}_+, B)\Psi_B(U)\,\mathrm{Q}(\mathbb{R}_+, B)$

(or less controlled, but NOT $\Psi(U) = \mathrm{Q}(\mathbb{R}_+, \Psi_B(U))$???)

`Somewhere in Chp.7: following should appear for use in Chp.9:`
$\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ is a unital C^*-subalgebra of the multiplier algebra $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, F))$ of $\mathrm{Q}(\mathbb{R}_+, F)$.

5. Related questions

Question:
Let $h_j\colon A \to M$ weakly nuclear C^*-morphisms ($j = 1, 2$), where A is separable and exact and M is a semi-finite W*-algebra and $\|h_1(a)p\| = \|h_2(a)p\|$ for all central projections p of M and all $a \in A$, that the neutral ???? elements e_j of the weak closures of $h_j(A)$ are unitarily equivalent in M, and that $\tau(h_1(a)) = \tau(h_2(a))$ for all $a \in A_+$ and all l.s.c. additive traces $\tau\colon M_+ \to [0, \infty]$ (possibly degenerate).

Is h_1 unitarily homotopic to h_2?

Are they (at least) weakly approximately unitarily equivalent?

(There should be a reduction to the case of countably decomposable M and then to M with separable pre-dual.)

If the image of h_1 has only trivial intersection with the Brauer-ideal of M (i.e., the norm-closed ideal generated by the finite projections in M), then one has only to check that the infinite repeats $\delta_\infty \circ h_j$ are unitarily homotopic to h_j (if $e_1 = e_2 = 1_M$) and then apply Theorem M (here δ_∞ is in $\mathrm{End}(M)$).

CHAPTER 8

The isomorphism of $\mathrm{KK}(\mathcal{C};\, A, B)$ and $\mathrm{Ext}(S\mathcal{C};\, A, SB)$

We have reassuring good news for those readers who are interested *only* in Theorem A and Corollaries C, D, F, G, H and J, and only in the *nuclear* case of Theorem I:
Their proofs only use the triviality of $\mathrm{Ext}^{-1}(A, \mathcal{O}_2)$ and the special case of Theorem B in the introductory Chapter 1, where A is *nuclear*. In this cases we have obviously the identities $\mathrm{Ext}_{\mathrm{nuc}}(A, B) = \mathrm{Ext}^{-1}(A, B)$ and $\mathrm{KK}(A, B) = \mathrm{KK}_{\mathrm{nuc}}(A, B)$. The reader can see that our proof of Theorem B in this special cases involves only the usual KK- and Ext-theory of Kasparov. So, the reader can find the needed facts for this very special case and the here indicated approach in text books, e.g. in [**73**], *except some elementary observations in Sections 3 of this chapter*

`Is Section 3 all?`

needed for the proofs of Theorems B and its generalization M in Chapter 9.

In this Chapter 8 we describe our viewpoint on a (weakly) *nuclear* version of Kasparov's KK-theory, and its isomorphism to $\mathrm{Ext}_{\mathrm{nuc}}(A, SB)$, but in a more general context that covers also our $\mathcal{C}$-equivariant group $\mathrm{Ext}(\mathcal{C}\,;\, A, SB)$ – just by the anyway needed more refined and general terminology. The needed definitions and facts about $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ and generalizations in different directions like $\mathrm{Ext}(X;\, A, B)$ and $\mathrm{Ext}(\mathcal{C};\, A, B)$ can be found in Chapters 3 and 5. We generalize the ordinary KK-functor in the following direction:
We introduce KK-groups $\mathrm{KK}(\mathcal{C};\, A, B)$ depending on (non-degenerate) countably generated point-norm closed matrix operator-convex cones (m.o.c.c.) $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ (cf. Chapter 3). It has the property, that the Kasparov product allows to define a bi-additive map

$$\mathrm{KK}(\mathcal{C}_1\,;\, A, B) \times \mathrm{KK}(\mathcal{C}_2\,;\, B, C) \to \mathrm{KK}(\mathcal{C}_3\,;\, A, C)$$

if $\mathcal{C}_2 \circ \mathcal{C}_1 \subseteq \mathcal{C}_3$ and A and B are both separable, and C is σ-unital. A special case is our *residually nuclear* KK-group $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ that corresponds to a well-defined m.o.c. cone $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X;\, A, B) \subseteq \mathrm{CP}(A, B)$, the cone $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B) := \mathrm{CP}_{\mathrm{rn}}(X;\, \Psi_A, \Psi_B)$ of Ψ-*residually nuclear* c.p. maps for given actions $\Psi_A \colon \mathbb{O}(X) \to \mathcal{I}(A)$ of a T_0 space X on A and Ψ_B of X on B (under the additional assumption that $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ for the actions Ψ_A and Ψ_B is non-degenerate), see Chapters 1 and 3 for the

`give Refs.  for actions and its cones`

definitions of this actions and the related m.o.c. cones. The group $\mathrm{KK}_{\mathrm{nuc}}(A,B)$ of Skandalis [**726**] – even if they are defined directly as the homotopy classes given by elements in $\mathbb{E}(A, B[0,1])$ ...

is the same as our $\mathrm{KK}(\mathcal{C}\,;\, A, B)$ for the special case of the m.o.c. cone $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A,B) \subseteq \mathrm{CP}(A,B)$ (?? `check def in [726] again !` ??),

if A is separable and B is σ-unital, because then $\mathrm{CP}_{\mathrm{nuc}}(A,B)$ is countably generated as matrix operator convex cone.

If A and B are both separable, then $\mathcal{C} := \mathrm{CP}(A,B)$ is itself countably generated and the usual Kasparov group $\mathrm{KK}(A,B)$ is the same as $\mathrm{KK}(\mathrm{CP}(A,B)\,;\, A, B)$.
`Check next! But next is true if` B `is not` σ`-unital!:`
(Notice that $\mathrm{CP}(A,B)$ is not countably generated – which is equal to singly generated – as matrix operator-convex cone if $A := \mathbb{K}(\ell_2(\mathbb{N}))$ or $A := \mathbb{C}$ and B is not separable.)

But $\mathrm{CP}(\mathbb{C}, B) \cong B_+$ is generated as m. o. c. cone by the element $\mathbb{C} \ni z \mapsto ze \in B$ if $e \in B_+$ is a strictly positive element of B. But what happens if $A := \mathrm{C}_0(0,1]$?

All results of this chapter can be obtained by modifying the proofs of the corresponding results for the usual (non-nuclear) KK-theory, but use our observations on operator-convex cones in Chapter 3. We apply the $\mathrm{Ext}(\mathcal{C};\, \cdot, \cdot)$-groups defined in Chapter 5 to give other proofs, which are more "elementary" (or "constructive") and allow to control all identifications and constructions which are needed during our proofs of Theorems B and M, cf. Lemma 8.3.2. In fact, we need to work only with *unitary* homotopy, because this is automatically $\mathcal{C}$-equivariant, and we prove later that $\mathcal{C}$-equivariant homotopy can be realized by unitary homotopy (modulo stabilization). This is also important for the X-equivariant theory $\mathrm{KK}_{\mathrm{nuc}}(X\,;\, A, B) := \mathrm{KK}(\mathcal{C}\,;\, A, B)$ given by the m.o.c. cone for $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X\,;\, \Psi_A, \Psi_B\,;\, A, B)$.

The major results of this chapter are the homotopy invariance of $\mathrm{KK}(\mathcal{C}\,;\, A, B)$, the observations in Section 5 (that are needed in Chapter 9) and the related "controlled" isomorphism

$$\mathrm{Ext}\big(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, A, B \otimes \mathrm{C}_0(\mathbb{R})\big) \cong \mathrm{KK}(\mathcal{C};\, A, B)\,,$$

that gives in particular $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ for actions Ψ_A and Ψ_B of a T_0-space X on A respectively B if the cone $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ is non-degenerate and countably generated, cf. Chapters 1 and 5.

In Chapters 9 and 12 we use results on $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ and $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B)$ in the case where X is often the primitive ideal space of a separable C^*-algebra. The definitions can be found in Chapters 1 and 5. The necessary changes from the theory of $\mathrm{KK}_{\mathrm{nuc}}(A,B)$ and $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ to this general $\mathrm{KK}_{\mathrm{nuc}}(X;\, .,.)$- and $\mathrm{Ext}_{\mathrm{nuc}}(X;\, .,.)$-theories or even to $\mathrm{KK}(\mathcal{C}\,;\, A, B)$ and $\mathrm{Ext}(\mathcal{C}\,;\, A, B)$ are often straight-forward, up to more complicate notations:
Then one has only to replace the emphasized words *"nuclear"* by the words *"Ψ-residually nuclear"*, respectively by *"$\mathcal{C}$-compatible"*, if we consider the more flexible

theory of the groups $\mathrm{KK}(\mathcal{C}\,;\, A, B)$. Therefore we discuss sometimes only properties of the $\mathrm{KK}_{\mathrm{nuc}}$-theory because of simpler notation. All differences to the basic theory appears by a pre-selection of suitably chosen sub-classes of Kasparov modules. If this choice has been made, only two general conditions define the appropriate equivalence relation that makes the direct module-sum to the addition on the corresponding classes that build the elements of the KK-groups. The formed Grothendieck group defines in the case of $\mathcal{C} = \mathrm{CP}(A, B)$ the relation $\sim_c$ on the Kasparov modules considered by G. Kasparov and J. Cuntz.

The prove of the (operator-) homotopy invariance of J. Cuntz and G. Skandalis carries over almost verbatim to $B \mapsto \mathrm{KK}(\mathcal{C}; A, B)$. The full homotopy invariance can then derived by the ideas of A. Connes ...

`Check next statement!!`

A modification of parts of our proof of Theorem B can be used to show that $\mathcal{O}_\infty$*-stabilized unsuspended weakly nuclear* $\otimes\mathcal{O}_\infty$ `necessary?`

E-theory is equivalent to $\mathrm{KK}_{\mathrm{nuc}}$-theory in separable case with trivial grading (an observation of N.Ch. Phillips, cf. Remark 8.3.6).

`Check ???  At some points we could use that` $A' \cap \mathrm{Q}(B)$ `with` $\mathrm{Q}(B) := \mathcal{M}(B)/B$
`is s.p.i. if` $A \hookrightarrow \mathrm{Q}(B)$ `is nuclear`
`and` B `is s.p.i. and` σ`-unital.`

`Then` $A' \cap \mathrm{Q}(B)$ `should be` K_1`-bijective.`

`VERY IMPORTANT!!!`
`We need !!!  and use later that:`

$\mathrm{KK}(\mathcal{C};\, A, B)$ is naturally isomorphic to the kernel of the natural map

$$\mathrm{K}_1\big((\pi_B \circ H_{\mathcal{C}})(A)' \cap \mathrm{Q}(B)\big) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B)$$

if A is separable and stable and B is σ-unital and stable (and all is trivially graded).

And that the "natural" isomorphisms $\mathrm{KK}(\mathcal{C}[0,1]\,;\, A, B[0,1]) \cong \mathrm{KK}(\mathcal{C}\,;\, A, B)$ and $\mathrm{KK}([0,1]\mathcal{C}\,;\, A[0,1], B) \cong \mathrm{KK}(\mathcal{C}\,;\, A, B)$. given by evaluations at points $t \in [0,1]$.

(– The latter are the homotopy invariances of $\mathrm{KK}(\mathcal{C};\cdot,\cdot)$ and works as in the case $\mathcal{C} = \mathrm{CP}(\cdot,\cdot)$ using the G. Kasparov product construction modified by A. Connes ... –).

Claimed in Chapter 5:
There are natural isomorphisms $\mathrm{KK}(A, B) \cong \mathrm{Ext}(A, SB)$ and $\mathrm{KK}(A, B) \cong \ker\big(\mathrm{K}_1(H_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{Q}(B) \cong \mathrm{K}_0(B)\big)$ for separable stable σ-unital A and B.

This should come from Cuntz picture for trivially graded A and B: $\mathrm{KK}^1(\mathcal{C};\, A, B)$ is defined for separable A and σ-unital B and countably generated m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ as equivalence classes $[(\psi, P)]$ in the set of all pairs

$\{(\psi, P)\,;\ \psi\colon A \to \mathcal{M}(B\otimes\mathbb{K}), P = P^* = P^2 \in \mathcal{M}(B\otimes\mathbb{K})\}$ with the properties: ψ is a C^*-morphism, $d^*\psi(\cdot)d \in \mathcal{C}$ for all $d \in B\otimes\mathbb{K}$ and $P\psi(a) - \psi(a)P \in B\otimes\mathbb{K}$.

Can all reduce to case $E := \widehat{\mathbb{H}}_B := \mathbb{H}_B \oplus \mathbb{H}_B^{op}$. It has the "standard even" grading if B is trivially graded. And this induces on $\mathcal{L}(\widehat{\mathbb{H}}_B) \cong \mathcal{M}(B\otimes\mathbb{K})$ the standard even grading.

If E is any trivially graded Hilbert B-module then $E\oplus E^{op}$ induces the standard even grading on $M_2(\mathcal{L}(E))$.

We should start with "Fredholm picture" in case of trivially graded separable A and stable σ-unital trivially graded B:

Triples $(E^{(0)}\oplus E^{(1)}, \phi_0\oplus\phi_1, T)$ with $T \in \mathcal{L}(E^{(0)}, E^{(1)})$ and relations:

$T\phi_0(a) - \phi_1(a)T \in \mathbb{K}(E^{(0)}, E^{(1)})$ for all $a \in A$, $1 - T^*T \in \mathbb{K}(E^{(0)})$ and $1 - TT^* \in \mathbb{K}(E^{(1)})$.

The ϕ_* should satisfy $\langle \phi_*(\cdot)e, e\rangle \in \mathcal{C}$.

Then try to "absorb" the ϕ_* and $E^{(*)}$ by $H_{\mathcal{C}}\colon A \to \mathcal{M}(B)$ and $E \cong B$ in case of stable σ-unital B.

Should get new $T \in \mathcal{M}(B)$ with $[T, H_{\mathcal{C}}(a)] \in B$, $(1 - T^*T)H_{\mathcal{C}}(a) \in B$ $(1 - TT^*)H_{\mathcal{C}}(a) \in B$ for all $a \in A$. ... ???

Is this all at the beginning?

If one can make ????????????

Also very important:

There are natural isomorphisms

$$\mathrm{Ext}(\mathcal{C};\, A, B) \cong \ker(\mathrm{K}_0(H_0(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(\mathrm{Q}(B)) \cong \mathrm{K}_1(B))$$

for separable stable σ-unital A and σ-unital stable B.

$\Leftarrow$ `This has been shown in Chp. 3 or 5 ??? Where? Ref.?`

1. Some basics on KK and Ext

`Think about interchanging the role of` $\mathcal{L}(E)$ `and` $\mathcal{B}(E)$`,`
`or use` $\mathbb{B}(E)$ `and` $\mathbb{L}(E)$ `!!??`

The nuclear KK-theory $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ was defined by Skandalis [**726**]. But we use at the beginning a *different* definition that is more algebraic and refers not explicit to any sort of homotopy invariance. It allows us to give later a "constructive" proof of the existence of a functor isomorphism:

$$\mathrm{Ext}(\mathcal{C};\, A\otimes\mathbb{K}, B) \cong \mathrm{KK}(\mathcal{C};\, A, B_{(1)}) \tag{1.1}$$

in case where A and B are *trivially graded*, i.e., the corresponding $\mathbb{Z}_2$-actions on A and B are given by the identity map, A is separable, B is σ-unital, stable and $B_{(1)}$ denotes the graded algebra $B_{(1)} := B\oplus B$ with the **standard odd grading** $\beta((a, b)) := (b, a)$. In particular $\mathrm{Ext}_{\mathrm{nuc}}(A\otimes\mathbb{K}, B) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B_{(1)})$.

It implies then that $\mathrm{Ext}(S\mathcal{C};\, A, SB) = \mathrm{KK}(\mathcal{C};\, A, B)$ for trivially graded separable stable C^*-algebras A and B.

The natural isomorphism $B_{(1)} \cong B \otimes (\mathbb{C}_{(1)})$ of graded C^*-algebras allows to extend an operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ naturally to a cone $\mathcal{C} \subseteq \mathrm{CP}(A, B_{(1)})$: Simply take the m.o.c. cone-tensor product $\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathbb{C} \oplus \mathbb{C})$, i.e., the smallest point-norm closed hereditary sub-cone of $\mathrm{CP}(A, B_{(1)})$ containing the c.p. maps $V \otimes W$ with $V \in \mathcal{C}$ and $W \in \mathrm{CP}(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) \cong \mathbb{R}_+ \oplus \mathbb{R}_+$.

Recall that the "tensor"-cones are in general not completions of algebraic tensor products – see Definition 3.6.16: Here this "m.o.c.cone-tensor-product" is – as a set – the same as the set of maps $a \in A \mapsto (V_1(a), V_2(a)) \in B_1$ with $V_1, V_2 \in \mathcal{C} \subseteq \mathrm{CP}(A, B)$.

We have to consider $\mathbb{Z}_2$-graded C^*-algebras, because we need formula (1.1) and its consequences concerning the very important homotopy invariance. But the reader should notice that the groups $\mathrm{Ext}_{\mathrm{nuc}}(A, SB)$ and $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ are in general not isomorphic for non-trivially graded separable C^*-algebras A and B.

`Check again ??:`

G. Skandalis [**726**] requires homotopy invariance as a part of his *definition* of $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ right from the beginning, so he was in the comfortable situation that he didn't need to *prove* the formula (1.1). But for us this is a non-obvious key observation, that must be proven, because it is then often used in our proofs, e.g. in Section 5 that establish the applicability the abstract results in Chapter 4 and allow with them together to show in Chapter 9 that

`?? in the general case that we consider ??`

Need: For stable separable C^*-algebras A and B and countable generated non-degenerate m.o.c. cone $\mathcal{C}$ for the canonical $H_{\mathcal{C}} \colon A \to \mathcal{M}(B)$ holds (in the trivially graded case):

The group $\mathrm{KK}(\mathcal{C}\,;\; A, B)$ is naturally isomorphic to the kernel of the natural morphism

$$\mathrm{K}_1\big((\pi_B \circ H_{\mathcal{C}})(A)' \cap \mathrm{Q}^{\mathrm{s}}(B)\big) \;\to\; \mathrm{K}_1\big(\mathrm{Q}^{\mathrm{s}}(B)\big) \cong \mathrm{K}_0(B)\,.$$

A hint is: [**73**, subsec. 17.5.1] but then add the infinite repeat of $\phi_0 \oplus \phi_1$ to it and use suitable unitaries in $\mathcal{M}(B)$ to correct the formula ...

`To be carried out in detail.`

In case of $\mathrm{Ext}(\mathcal{C}\,;\; A, B)$ it is not difficult to see that $\mathrm{Ext}(\mathcal{C}\,;\; A, B)$ is isomorphic to the kernel of

$$\mathrm{K}_0\big((\pi_B \circ H_{\mathcal{C}})(A)' \cap \mathrm{Q}^{\mathrm{s}}(B)\big) \;\to\; \mathrm{K}_0\big(\mathrm{Q}^{\mathrm{s}}(B)\big) \cong \mathrm{K}_0(B)\,.$$

We define $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ as one of our more general groups $\mathrm{KK}(\mathcal{C};\, A, B)$ – specialized to the case where $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$. They have a completely algebraic definition and that does not impose any sort of homotopy invariance *in itself*. We shall later *prove* the homotopy invariance of our $\mathrm{KK}(\mathcal{C};\, A, B)$ by modifications of ideas of G. Kasparov (discussed and modified by A. Connes, G. Skandalis, J. Cuntz, N. Higson and others).

We work with an "algebraic" definition that does not refer to "homotopy invariance" because the homotopy invariance of the purely algebraic defining relations proves also the homotopy invariance of *our* $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ groups (and of our groups $\mathrm{Ext}(\mathcal{C};\, A, B)$, $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B)$ etc.) with help of *our* definition of $\mathrm{KK}(\mathcal{C};\, A, B)$. It is not a triviality.

Even the much weaker (since as a relation stronger) *operator-homotopy invariance* of the equivalence classes that build the elements of in $\mathrm{KK}(\mathcal{C};\, A, B)$ is not evident. The proof of the homotopy invariance from stronger more algebraic definitions is the core of the application of generalized $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$ to classification of non-simple C^*-algebras.

The so-called "degenerate" Kasparov modules represent zero automatically in any version of KK-theory that is a quotient of the Grothendieck group of a sub-semigroup of the semigroup of unitary equivalence classes of Kasparov modules, because the "degenerate" Kasparov modules can be defined equivalently by its property that *their infinite direct sum is again a Kasparov module* ([1]). The proof of the operator homotopy can be done by modifications of the techniques of G. Kasparov (respectively of J. Cuntz and G. Skandalis) in the proof of the homotopy invariance of the below defined $\mathrm{KK}_c(A, B)$-functor for graded separable A and σ-unital graded B, where KK_c can be defined equivalently by the *cobordism*-picture, cf. [**73**, def. 17.10.2, thm. 17.10.7] ([2]). After introduction of the Kasparov products up to operator homotopy (!), the proof of [**73**, thm. 18.5.3] carries over to our situation and establishes the general homotopy invariance, where one can use that Kasparov products define the functor $\mathrm{KK}_{\mathrm{nuc}}$ to as an KK-"ideal" in the KK-category. See below lemmata for a more details.

The way *from* operator homotopy invariance to the existence of Kasparov products (and then to homotopy invariance) can be found also in [**389**, chp. 2].

We use the approach of Kasparov to show that our $\mathrm{KK}_{\mathrm{nuc}}$-bi-functor (respectively our $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$ bi-functor – generalizing our $\mathrm{KK}_{\mathrm{nuc}}(X;\, \cdot, \cdot)$ bi-functor for actions Ψ of $\mathbb{O}(X)$ on ideal lattices) is homotopy invariant. But one has to be careful because *in general the identity maps* id_A *does not represent an element of* $\mathrm{KK}_{\mathrm{nuc}}(A, A)$, and the relations are calculated up to stabilization with *nuclear* Kasparov modules and not up to stabilization with general Kasparov modules. But with this in mind, one can go through the textbooks and over-carry step by step every argument to the case of $\mathrm{KK}_{\mathrm{nuc}}$ (or even to $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$, and thus especially to $\mathrm{KK}_{\mathrm{nuc}}(X;\, \cdot, \cdot)$, if one uses the natural extension of the actions Ψ of X on A to an action $\Psi \otimes C\colon U \to \Psi(U) \otimes C$ on $A \otimes C$, for $C := \mathrm{C}_0((0, 1])$, $\mathrm{C}_0(\mathbb{R})$, $\mathbb{K}$, $\mathrm{C}([0, 1])$ or $\mathcal{O}_\infty$ –).

---

[1] Note that the infinite sums are again weakly *nuclear*, or more generally $\mathcal{C}$-*compatible*.

[2] Note that [**73**, def. 17.10.1] must require $[p, F]\phi(A) \subseteq \mathbb{K}(E)$ in addition, cf. [**187**, def. 3.1].

Let us recall some *general definitions* concerning graded Hilbert modules used in the "algebraic" definition of $\mathrm{KK}(\mathcal{C};\cdot,\cdot)$-theory:

We prefer a picture (– the original picture of Kasparov himself –) that requires to use *graded* σ-unital algebras for the most basic results, in particular, *all* C^*-*algebras* $A, B, \ldots$ *in question are supposed to be* σ-*unital*, of course except the related algebras $\mathcal{M}(A)$, ... , etc.

Suppose that A and B are $\mathbb{Z}_2$-graded C^*-algebras, with grading automorphisms $\beta_A \in \mathrm{Aut}(A)$ and $\beta_B \in \mathrm{Aut}(B)$. We drop the indices A and B, if it is obvious where β acts. Thus, always $\beta^2 = \mathrm{id}$. Further we suppose that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a (non-degenerate) point-norm closed matrix operator-convex (m.o.c.) cone with the (minimally necessary) property that $\beta_B \circ \mathcal{C} \circ \beta_A \subseteq \mathcal{C}$.

In general a $\mathbb{Z}_2$-grading of a vector space V is given by a linear operator $\beta \colon V \to V$ with $\beta^2 = \mathrm{id}_V$.

`compare def.s with book:`

An element $v \in V$ has *degree* $\deg(v) = 0$ if $v = \beta(v)$ and $\deg(v) = 1$ if $v = -\beta(v)$. We use also the notation ∂v for $\deg(v)$.

`Is notation` ∂v `really better?  Not misleading?`

The vector space V is the direct sum $V = V^{(0)} + V^{(1)} \cong V^{(0)} \oplus V^{(1)}$ of its linear subspaces $V^{(0)} = \{v \in V\,;\ \deg(v) = 0\}$ and $V^{(1)} = \{v \in V\,;\ \deg(v) = 1\}$. Thus $\beta(v) = \beta(v_0 + v_1) = v_0 - v_1$ for the unique decomposition $v = v_0 + v_1$ with $\deg(v_0) = 0$ and $\deg(v_1) = 1$.

The graded tensor product $V \otimes W$ of graded vector spaces V, W is usually given by $\beta_V \otimes \beta_W$? **??**

A **Hilbert** B**-module** E is a complete normed right B-module together with positive definite hermitian B-module form $(x, y) \in E \times E \mapsto \langle x, y\rangle \in B$ that is linear in the second variable and anti-linear (conjugate-linear) in the first variable and satisfies $\|x\|^2 = \|\langle x, x\rangle\|$ for all $x \in E$.

We denote by $\mathbb{K}(E)$ the closed linear span of the "B-rank-one" operators:

$$x \in E \mapsto y\langle z, x\rangle \in E\,.$$

It is a C^*-algebra and an essential ideal in the C^*-algebra $\mathcal{L}(E)$ of bounded and $\langle .,.\rangle$-adjoint-able B-module endomorphisms of E. Moreover $\mathcal{L}(E)$ is naturally isomorphic to the multiplier C^*-algebra (real or complex) $\mathcal{M}(\mathbb{K}(E))$ of $\mathbb{K}(E)$. During Chapter 8 we denote by $\mathcal{B}(E)$ the Banach algebra of *all* bounded linear operators on E with the usual operator norm. Then our $\mathcal{L}(E)$ is the subalgebra of $T \in \mathcal{B}(E)$ with $T(x \cdot b) = (Tx) \cdot b$ for $x \in E$ and $b \in B$. The norms on $\mathcal{L}(E)$ are the usual operator norms. $\mathcal{L}(E)$ does not coincide with $\mathcal{B}(E)$ if $B \neq \mathbb{C}$.

Assume that B is $\mathbb{Z}_2$-graded by a C^*-algebra automorphism β_B of order 2. A Hilbert B-module E is **graded** by β_E, if β_E is an isometry with $\beta_E^2 = \mathrm{id}$ and

$\beta_E(xb) = \beta_E(x)\beta_B(b)$ for $x \in E$, $b \in B$. One can easily see that the isometry property $\|\beta_E(x)\| = \|x\|$ ($x \in E$) of β_E is equivalent to invariance property $\beta_B(\langle x, y\rangle) = \langle \beta_E(x), \beta_E(y)\rangle$ for $x, y \in E$, e.g. simply by composing both sides with positive functionals on B, and using the trapezoid rule and that $\|\langle x, x\rangle\| = \|x\|^2$.

The grading β_E *induces* on the C^*-algebra $\mathcal{L}(E)$ the grading $F \to \beta_E \circ F \circ \beta_E$. By E^{op} we denote the **opposite** Hilbert B-module of E, that is the same B-module as E with the new "opposite" grading $-\beta_E$.

An **isomorphism** of graded Hilbert B-modules $\lambda\colon E_1 \to E_2$ is an isometric B-module map from E_1 onto E_2 that satisfies $\lambda \circ \beta_{E_1} = \beta_{E_2} \circ \lambda$ ([3]). The *isomorphism classes* of (over B) countably generated graded Hilbert B-modules build a *set*, if B is σ-unital, because they are isomorphic to orthogonally complemented grading invariant closed subspaces of $\widetilde{\mathcal{H}}_B := \mathcal{H}_B \oplus_B (\mathcal{H}_B)^{op}$, cf. [**73**, thm.14.6.1], by Kasparov's stabilization theorem [**?**], (or, by the almost equivalent Brown stable isomorphism theorem, [**107**]).

The **direct sum** $E \oplus_\infty E'$ of Hilbert B-modules E and E' with norm $\|(x, y)\|_\infty := \max(\|x\|, \|y\|)$ is a Hilbert $B \oplus B$-module, and therefore has also the *diagonal* right B-module structure. If we consider $E \oplus_\infty E'$ as a right *Banach* B-module, then $E \oplus_\infty E'$ is naturally and bounded (but not isometrically!) B-module isomorphic to the **Hilbert B-module sum** $E \oplus_B E'$ with B-valued form $\langle (x_1, y_1), (x_2, y_2)\rangle := \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$. Up to unitary equivalence, $\oplus_B$ is associative and commutative, $\mathcal{L}(E) \oplus_\infty \mathcal{L}(E') \subset \mathcal{B}(E \oplus_\infty E')$ and $\mathcal{L}(E) \oplus_\infty \mathcal{L}(E')$ is a unital C^*-subalgebra of $\mathcal{L}(E \oplus_B E')$.

Some minimal basics on *graded* tensor products of operator-convex cones and on natural extension are needed during this Chapter 8:
We denote by $A \mathbin{\widehat{\otimes}} B$ the (outer) **graded tensor product** of $\mathbb{Z}_2$-graded C^*-algebras A and B. The **outer graded tensor product** of graded Hilbert A-modules E_1 and graded Hilbert B-modules E_2 will be denoted by $E_1 \widehat{\otimes} E_2$ and is a Hilbert $A \widehat{\otimes} B$-module.

Since we have *to define* the groups KK($\mathcal{C}$; $A\widehat{\otimes}C$, $B\widehat{\otimes}D$) for $\mathbb{Z}_2$*-graded separable nuclear* C^*-algebras C and D, we must "extend" a given matrix operator-convex cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ to an m.o.c. cone $\mathcal{C}_{C,D} \subseteq \mathrm{CP}(A\widehat{\otimes}C, B\widehat{\otimes}D)$:
Simply, let $\mathcal{C}_{C,D} := \mathcal{C} \mathbin{\widehat{\otimes}} \mathrm{CP}(C, D)$, where $\mathcal{C} \mathbin{\widehat{\otimes}} \mathrm{CP}(C, D) \subseteq \mathrm{CP}(A\widehat{\otimes}C, B\widehat{\otimes}D)$ denotes the point-norm closed m.o.c. cone that is generated by the graded Hilbert $(A\widehat{\otimes}C, B\widehat{\otimes}D)$-module $(E_1\widehat{\otimes}E_2, \phi_1\widehat{\otimes}\phi_2\colon A\widehat{\otimes}C \to \mathcal{L}(E_1\widehat{\otimes}E_2))$ for a graded Hilbert (A, B)-module $(E_1, \phi_1\colon A \to \mathcal{L}(E_1))$ and a graded Hilbert (C, D)-module $(E_1, \phi_2\colon C \to \mathcal{L}(E_2))$ that generates $\mathcal{C}$ (respectively generates $\mathrm{CP}(C, D)$) in the sense of Definition 3.6.25. (Please observe that the notation $\mathcal{C} \mathbin{\widehat{\otimes}} \mathrm{CP}(C, D)$ does not say that it is a sort of "completion" of the linear span of some kind of tensor

[3] The grading λ is $\mathbb{C}$-linear by the Cohen factorization theorem, because every Hilbert B-module is non-degenerate, and it holds automatically $\langle \lambda(x), \lambda(y)\rangle = \langle x, y\rangle$ for $x, y \in E_1$, because, if $a, b \in B_+$ and $\|c^*ac\| = \|c^*bc\|$ for all $c \in B$ then $a = b$, – as application of pure states to $b - a$ shows by using the "excision" Lemma **??**.

products: It is only "generated" as point-norm closed m.o.c. cone by tensor products $V \otimes W$ with $V \in \mathcal{C}$ and $W \in \mathrm{CP}(C, D)$. See Chapter 3 for more details.

Let us consider an easy example:

Example 8.1.1. We consider the groups $\mathrm{KK}(\mathcal{C};\, \mathrm{C}[0,1], \mathbb{C})$ that are used in the proof of the later considered "homotopy invariance". Where $\mathcal{C} \subseteq \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C}) \cong \mathrm{C}[0,1]^*_+$ is any point-norm closed m.o.c. cone ...

The minimal point-norm closed *non-degenerate*

??????

$\mathrm{CP}(\mathbb{C}, \mathbb{C}) = \mathbb{R}_+ \cdot \mathrm{id}_{\mathbb{C}} \cong \mathbb{R}_+$

matrix operator-convex cone $\mathcal{C}_{\min} \subseteq \mathrm{CP}(\mathrm{C}[0,1]) := \mathrm{CP}_{[0,1]}(\mathrm{C}[0,1], \mathrm{C}[0,1])$ on $\mathrm{C}[0,1]$ are the "inner" c.p. maps, i.e., is here – with other words – simply the set of all multiplication maps $V_g(f) := gf$ with positive elements $g \in \mathrm{C}[0,1]_+$. Thus $\mathcal{C}_{\min}(\mathrm{C}[0,1]) \cong \mathrm{C}[0,1]_+$ by an isometric identification $\|V_g\| = \|g\|$.

Indeed ????: $\mathrm{CP}(\mathrm{C}[0,1], \mathrm{C}[0,1])$ is natural isomorphic to the set of weakly continuous maps

$$[0,1] \ni t \mapsto \mu_t \in \mathrm{C}[0,1]^*_+$$

from $[0,1]$ into the finite positive measures on $[0,1]$.

Let $\mathcal{C} \subseteq \mathrm{CP}(\mathrm{C}[0,1], \mathrm{C}[0,1])$ is a point-norm closed m.o.c. cone, then for each $t \in [0,1]$ there exists $V \in \mathcal{C}$ and $f \in \mathrm{C}[0,1]_+$ such that $V(f)(t) > 1$. By continuity of $V(f)(\cdot)$, order monotony of V and we find an open intervall $I = (\alpha, \beta) \subseteq [-1, 2]$ and $\gamma > 0$ such that $t \in I$ and $V(1)(s) > \gamma$ for $s \in I$. The compactness of $[0,1]$ shows then that there is $\mu > 0$ finite sum of such kind of $V \in \mathcal{C}$ is a $W_0 \in \mathcal{C}$ with $W_0(1)(t) \geq \mu$. It shows that there exists $W \in \mathcal{C}$ with with $W(1) = 1$.

The only point-norm closed non-degenerate m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C}) = \mathrm{C}[0,1]^*_+$ is $\mathrm{C}[0,1]^*_+$ itself ??

Seems to be ... wrong ...????

It implies that $\mathrm{KK}(\mathcal{C};\, \mathrm{C}[0,1], \mathbb{C}) = \mathrm{KK}(\mathrm{C}[0,1], \mathbb{C}) \cong \mathbb{Z}$ and then for all non-degenerate point-norm m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ and, – as we see later – that

$$\mathrm{KK}(\mathcal{C}[0,1]\,;\; A, B[0,1]) \cong \mathrm{KK}(\mathcal{C}\,;\; A, B)$$

via any of the evaluation maps $\{V_t\} \in \mathcal{C}[0,1] \to V_0 \in \mathcal{C}$ and $\{b(t)\} \in B[0,1] := \mathrm{C}([0,1], B) \mapsto b(0) \in B$.

(Compare Definition 3.6.16 for the notations.)

For our later applications on the study of $\mathrm{KK}(\mathcal{C}; A, B)$ and $\mathrm{Ext}(\mathcal{C}; A, B)$ in the trivially graded cases, we need only the below listed cases (I) and (II), where the graded tensor products of algebras and the outer graded tensor-products of its Hilbert modules become naturally isomorphic to the usual spatial C^*-algebra- and Hilbert module- tensor products, then with grading automorphisms on the tensor product given by the tensor product of the grading automorphisms:

(I) A and B are graded, $\mathcal{C} \circ \beta_A \subseteq \mathcal{C}$ and $\beta_B \circ \mathcal{C} \subseteq \mathcal{C}$,
and C, D are trivially graded separable nuclear algebras.

(II) A and B are trivially graded and C, D are graded separable nuclear algebras.

In case (I) we need only $\mathrm{C}[0,1]$, $\mathrm{C}_0(\mathbb{R}_+)$, $\mathrm{C}_0(\mathbb{R})$, $\mathrm{C}_0(\mathbb{R}^2)$, $\mathbb{C}$ and $\mathbb{K}$ in place of C or D for our applications, and in case of (II) the later application for the proofs of Theorems **??** require only the cases $(C, D) = (\mathbb{C}_{(1)}, \mathrm{C}_0(\mathbb{R}))$ and $(C, D) = (\mathrm{C}_0(\mathbb{R}), \mathbb{C}_{(1)})$.

The below given construction of the group morphisms

$$\tau_B\colon \mathrm{KK}(C, D) \to \mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B, B) \otimes \mathrm{CP}(C, D)\,;\; B\widehat{\otimes}C, B\widehat{\otimes}D)$$

becomes natural for all graded B, C and D.

In general, $A\widehat{\otimes}\mathbb{C}_{(1)} \cong A \rtimes_\beta \mathbb{Z}_2 \subseteq A \otimes M_2$ for the $\mathbb{Z}_2$-action defined by the β and with grading $\beta \circ \widehat{\beta}$ on $A \rtimes_\beta \mathbb{Z}_2$ on the right side, where $\widehat{\beta}$ is just the dual $\mathbb{Z}_2$-action on $A \rtimes_\beta \mathbb{Z}_2$ of the action $\beta\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$. It is a C^*-morphism that maps $a \in A$ to $a \in A$ and $\varepsilon := (1, -1)$ to the $\mathbb{Z}_2$-generator in $\mathbb{Z}_2 \subseteq \mathcal{M}(A \rtimes_\beta \mathbb{Z}_2)$. In the special case $\mathcal{C} = \mathrm{CP}_{\mathrm{rn}}(X;\, A, B) := \mathrm{CP}_{\mathrm{rn}}(X;\, \Psi_A, \Psi_B;\, A, B)$ we define a matrix o.c. cone by

$$\mathcal{C}_{C,D} := \mathrm{CP}_{\mathrm{rn}}(X\,;\; A\,\widehat{\otimes}\, C\,,\; B\,\widehat{\otimes} D) := \mathrm{CP}_{\mathrm{rn}}(X;\, \Psi_{A\widehat{\otimes}C}, \Psi_{B\widehat{\otimes}D})$$

for the actions $\Psi_{A\widehat{\otimes}C}(U) := \Psi_A(U)\widehat{\otimes}C$ with $U \in \mathbb{O}(X)$ and (similar) $\Psi_{B\widehat{\otimes}F}$ For simplicity, we assume here that actions Ψ_A and Ψ_B are invariant under the gradings on A and B, e.g. $\beta_A(\Psi_A(U)) = \Psi_A(U)$ for $U \in \mathbb{O}(X)$.

We know from Chapter 3 that for *nuclear* C and D, a c.p. map $V \in \mathrm{CP}(A\,\widehat{\otimes}\, C\,,\; B\,\widehat{\otimes}\, D)$ is in $\mathcal{C}_{C,D} := \mathcal{C}\widehat{\otimes}\, \mathrm{CP}(C, D)$ if and only if

$$a \in A \mapsto (\mathrm{id}_B \,\widehat{\otimes} \rho)(V(a\widehat{\otimes}e)) \in B$$

is in $\mathcal{C}$ for each $e \in C_+$ with $\beta_C(e) = e$ and for each positive functional $\rho \in D^*$ with $\rho \circ \beta_D = \rho$:
See Proposition **??** for the case of trivially graded C and D. The graded case can be reduced to the case of an even grading by the embedding of C and D into $C \rtimes \mathbb{Z}_2$ respectively $D \rtimes \mathbb{Z}_2$ (with the grading given by the self-adjoint unitary that generates the copy of $\mathbb{Z}_2$ in $\mathcal{M}(C \rtimes \mathbb{Z}_2)$). Then the arguments are similar to those in the proof of Proposition **??** (using that the point-norm closed m.o.c. cones are just the point-norm closed convex cones that are hereditary in $\mathrm{CP}(A\widehat{\otimes}C, B\widehat{\otimes}D)$, then properties of hereditary cones).

In case $C := \mathbb{C}$, $D := C_0(Y)$ (with Y a locally compact Hausdorff space), a map $V \in \mathrm{CP}(A, \mathrm{C}_0(Y, B))$ is in $\mathcal{C}_{C,D}$, if and only if, there is a point-strongly continuous map $W\colon y \in Y \mapsto \mathrm{CP}(A, B) \subseteq \mathcal{L}(A, B)$ with $W(y) \in \mathcal{C}$, the map $y \mapsto \|W(y)(a)\|$ is in $\mathrm{C}_0(Y)$ for every $a \in A$ and $W(y)(a) = V(a)(y)$. In this case we write also $\mathcal{C}(Y)$ ($\mathcal{C}[Y]$ or simply $\mathcal{C}$) if $\mathcal{C}_{C,D} = \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(Y))$ for $C = \mathbb{C}$ and $D = \mathrm{C}_0(Y)$. Thus, the extensions $\mathcal{C} \curvearrowright \mathcal{C}_{C,D}$ are rather natural ([4]). $\mathrm{KK}_{\mathrm{nuc}}$-theory is the special case of $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$-theory where $\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(A, B)$,

Check if this is equivalent to the Def. of Skandalis [726]

and it is the special case of $\mathrm{KK}(X;\, \cdot, \cdot)$-theory where X is a singleton, i.e., $X = \{p\}$ with action on A given by $\Psi_A(\emptyset) = 0$, $\Psi_A(\{p\}) = A$ (and similar on B). In the same way $\mathrm{Ext}_{\mathrm{nuc}}(\cdot, \cdot)$ is a special case of $\mathrm{Ext}_{\mathrm{nuc}}(X;\, \cdot, \cdot)$ or of $\mathrm{Ext}(\mathcal{C};\, \cdot, \cdot)$. If A is separable and B is nuclear and separable, then $\mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ for $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$

read again !!!!!! more precise? ??

(where X and non-degenerate actions Ψ_A and Ψ_B of X on A and B are given) and $X := \mathrm{Prim}(B)$, with natural action on B and lower semi-continuous action of $\mathrm{Prim}(B)$ defined by a "universal" Hilbert A-B-module $H_0\colon A \to \mathcal{L}(\mathcal{H}_B)$ (as defined in Chapters 3 and 5, where we have to use that the corresponding cone is separating and non-degenerate by the existence of a *regular* Abelian subalgebra in B (in the sense of Definition B.4.1) using [**464**] and [**359**]).

A triple $\mathcal{E} = (E, \phi, F)$ is called a **Kasparov module** (more precisely: Kasparov (A, B)–module) if E is a graded right Hilbert B-module that is *countably generated* over B, $\phi\colon A \to \mathcal{M}(\mathbb{K}(E)) \cong \mathcal{L}(E)$ is a grading preserving C^*-morphism, and $F \in \mathcal{L}(E)$ is an operator of degree 1 such that $F\phi(a) - \phi(\beta_A(a))F$ ([5]), $(1 - F^2)\phi(a)$ and $(F^* - F)\phi(a)$ are in $\mathbb{K}(E)$ for every $a \in A$. The Kasparov module (E, ϕ, F) is **degenerate** if $F\phi(a) - \phi(\beta_A(a))F$, $(1 - F^2)\phi(a)$ and $(F^* - F)\phi(a)$ are all zero.

If the B-module E is trivially graded, then this definition implies that $F = 0$ and $\phi(A) \subseteq \mathbb{K}(E)$ and that $(E, 0, 0)$ is the only degenerate Kasparov module.

The Kasparov module $\mathcal{E} = (E, \phi, F)$ is **isomorphic** to $\mathcal{E}' = (E', \phi', F')$ (denoted by $\mathcal{E}' \approx_u \mathcal{E}$) if there is an isometric B-module map $U\colon E \to E'$ from E onto E' such that $U\beta_E = \beta_{E'}U$ (i.e., U is of even degree), $\phi'(\cdot)U = U\phi(\cdot)$ and $F'U = UF$. It is easy to see that the isometric B-module morphism U preserve the values of the B-valued hermitian form: $\langle Ux, Uy \rangle = \langle x, y \rangle$ for $x, y \in E$. By definition, the Kasparov modules $\mathcal{E}$ and $\mathcal{E}'$ are **unitary equivalent** if they are isomorphic in this way.

The **sum** $\mathcal{E}' \oplus \mathcal{E}$ of Kasparov A-B-modules $\mathcal{E}'$ and $\mathcal{E}$ is defined by the B-module sum

$$\mathcal{E}' \oplus \mathcal{E} := (E' \oplus_B E, \phi' \oplus \phi, F' \oplus F).$$

A Kasparov module (E_1, ϕ_1, F_1) is a **compact perturbation** of the Kasparov module (E, ϕ, F) if $E_1 = E$, $\phi_1 = \phi$ and $(F - F_1)\phi(a)$ is in $\mathbb{K}(E)$ for every $a \in A$. It suffices here to consider only $a \in A$ of degree $\partial(a) = 0$, because it is obvious that

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[4] See Chapter 3 for a Definition of tensor products of matrix operator-convex cones. We write also $\mathcal{C}$ instead of $\mathcal{C}_{C,D}$ to relax notation.

[5] Because $\partial(F) = 1$, $F\phi(a) - \phi(\beta_A(a))F = [F, \phi(a)]_{gr}$ for the graded commutator $[\cdot, \cdot]_{gr}$.

$A^{(0)}$ contains an approximate unit of A. Occasionally we use then the terminology "**ϕ-compact perturbation**" to underline the difference to perturbations by elements of $\mathbb{K}(E)$, or by generalized "compact" operators on the Banach space E.

The equivalence relation $\sim_{scp}$ which is generated by (unitary) isomorphisms and compact perturbations is compatible with the addition of Kasparov modules. Therefore the addition ?????? defines on the $\sim_{scp}$-classes of Kasparov modules the structure of a semigroup $\mathbb{E}(A, B)/\sim_{scp}$. The relation $\sim_{cp}$ of [**73**, def.17.2.4] is weaker, because the $\sim_{cp}$-classes allow addition of degenerate Kasparov modules. But $\sim_{cp}$ and $\sim_{scp}$ generate the same "stable" relation on $\mathbb{E}(A, B)$.

2. Isomorphism of Ext($\mathcal{C}$; A, B) and KK($\mathcal{C}$; $A, B_{(1)}$)

Notice that the below considered natural map from KK($\mathcal{C}$; $A, B_{(1)}$) to Ext($\mathcal{C}$; A, B)

`Or: from` Ext($\mathcal{C}$; A, B) `to` KK($\mathcal{C}$; $A, B_{(1)}$)?

becomes in general only in the special case of separable trivially graded A and trivially graded σ-unital B an isomorphism.

The here first considered algebras $A, A_1, \ldots, B, C, D, \ldots$ are σ-unital C^*-algebras. We start with some general considerations that require to consider also graded algebras. Thus, the algebras may have grading automorphisms $\beta_A, \beta_{A_1}, \ldots, \beta_B, \ldots$, and then we require that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a point-norm closed matrix operator-convex cone with $\beta_B \circ \mathcal{C} \circ \beta_A \subseteq \mathcal{C}$ (cf. Chapter 3, Section on m.o.c.c. operations and equivariant m.o.c. cones **??** for notations).

Definition 8.2.1. A Kasparov module $(E, \phi, F) \in \mathbb{E}(A, B)$ is called **nuclear** (respectively **Ψ-residually nuclear**, **$\mathcal{C}$-compatible**) if the C^*-morphism

$$\phi \colon A \to \mathcal{M}(\mathbb{K}(E)) = \mathcal{L}(E)$$

is weakly *nuclear* in the sense Definition B.7.6 (respectively is weakly Ψ-*residually nuclear*, respectively is weakly $\mathcal{C}$-*compatible*). Equivalently, this means that the c.p. maps $a \in A \mapsto \langle \phi(a)x, x\rangle \in B$ are *nuclear* (respectively are Ψ-residually nuclear, respectively are in $\mathcal{C}$) for every $x \in E$.

In fact all KK-groups come from first selecting an m.o.c. cone $\mathcal{C}$ and then considering only the $\mathcal{C}$-compatible Kasparov modules, because $\mathrm{KK}(A, B) = \mathrm{KK}(\mathcal{C}\,;\ A, B)$ with $\mathcal{C} = \mathrm{CP}(A, B)$, $\mathrm{KK}_{\mathrm{nuc}}(A, B) = \mathrm{KK}(\mathrm{CP}_{\mathrm{nuc}}\,;\ A, B)$, $\mathrm{KK}(\Psi\,;\ A, B) = \mathrm{KK}(\mathrm{CP}(\Psi\,;\ A, B)\,;\ A, B)$, where $\mathrm{CP}(\Psi\,;\ A, B) \subseteq \mathrm{CP}(A, B)$ denotes the m.o.c. cone of Ψ-equivariant c.p. maps, and, finally, the Ψ-residually nuclear KK-groups are $\mathrm{KK}(\mathcal{C}\,;\ A, B)$ with m.o.c. cone $\mathcal{C}$ defined by the "Ψ-residually nuclear" maps

$$\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(\Psi\,;\ A, B) \subseteq \mathrm{CP}(\Psi\,;\ A, B) \subset \mathrm{CP}(A, B)\,.$$

This theory can be considered as a refinement of the original Kasparov theory.

Examples of *nuclear* Kasparov (A, B)-modules (respectively of $\mathcal{C}$-compatible Kasparov modules) are the modules $(E, 0, F)$ for arbitrary Hilbert B-modules E

and arbitrary $F \in \mathcal{L}(E)$ of degree one, and the **difference construction** $(B, \phi, 0)$ for a *nuclear* C^*-morphism $\phi\colon A \to B$ (respectively for $\psi \in \mathrm{Hom}(A, B) \cap \mathcal{C}$), where B is considered as right B-module.

The category of *nuclear* (respectively of *$\mathcal{C}$-compatible*) Kasparov (A, B)-modules is invariant under Hilbert B-module addition, because

$$\phi \oplus \phi'\colon A \to \mathcal{M}\big(\mathbb{K}(E \oplus_B E')\big)$$

is again weakly nuclear (resp. is again weakly $\mathcal{C}$-compatible) if ϕ and ϕ' are weakly nuclear (resp. are weakly $\mathcal{C}$-compatible). If (E, ϕ, F) is nuclear (respectively $\mathcal{C}$-compatible) and $(E, \phi, F) \sim_{scp} (E', \phi', F')$, then (E', ϕ', F') is also nuclear (respectively $\mathcal{C}$-compatible). It follows that the $\sim_{scp}$-classes of *nuclear* (respectively $\mathcal{C}$-compatible) Kasparov A–B-modules build a sub-semigroup $\mathbb{E}_{\mathrm{nuc}}(A, B)/\sim_{scp}$ (respectively a sub-semigroup $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}$) of the semi-group $\mathbb{E}(A, B)/\sim_{scp}$ (with Hilbert B-module sum as addition).

Definition 8.2.2. We define the **nuclear Kasparov group** $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ as the *Grothendieck group of* $\mathbb{E}_{\mathrm{nuc}}(A, B)/\sim_{scp}$:

$$\mathrm{KK}_{\mathrm{nuc}}(A, B) := \mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{Gr}(\mathbb{E}_{\mathrm{nuc}}(A, B)/\sim_{scp}) = \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp})\,,$$

for $\mathcal{C} := \mathrm{CP}_{nuc}(A, B)$. The more general **$\mathcal{C}$-equivariant Kasparov group** $\mathrm{KK}(\mathcal{C};\, A, B)$ is the *Grothendieck group of* of $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}$:

$$\mathrm{KK}_{\mathrm{nuc}}(A, B) := \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp})\,.$$

The X-equivariant KK-groups and residually nuclear KK-groups are defined by

$$\mathrm{KK}(X;\, A, B) := \mathrm{KK}(\mathcal{C};\, A, B) \quad \text{for } \mathcal{C} := \mathrm{CP}(X;\, A, B)\,,$$

and

$$\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B) := \mathrm{KK}(\mathcal{C};\, A, B) \quad \text{for } \mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X;\, A, B)\,.$$

More precisely, $\mathrm{KK}(X;\, A, B)$ and $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ depend on the actions $\Psi_A\colon \mathbb{O}(X) \to \mathcal{I}(A)$ and $\Psi_B\colon \mathbb{O}(X) \to \mathcal{I}(B)$, that define the related point-norm closed m.o.c. cones $\mathrm{CP}(X, \Psi_A, \Psi_B;\, A, B)$ and $\mathrm{CP}_{\mathrm{rn}}(X, \Psi_A, \Psi_B;\, A, B)$, cf. Chapter 3. But there are other natural and functorial constructions of m.o.c. cones $\mathcal{C}$ from "actions" Ψ_A and Ψ_B of X on A and B. The m.o.c. cones $\mathrm{CP}_{\mathrm{rn}}(X, \Psi_A, \Psi_B;\, A, B) \subseteq \mathrm{CP}_{\mathrm{nuc}}(A, B)$ of Ψ_A-Ψ_B–residually nuclear maps play an important role in our applications.

If A and B are trivially graded and if $h\colon A \to B \otimes \mathbb{K} = B \widehat{\otimes} \mathbb{K}$ is a *nuclear* C^*-morphism (respectively if $h \in \mathrm{Hom}(A, B \otimes \mathbb{K}) \cap \mathcal{C}$), we can define $[h - 0] \in \mathrm{KK}_{\mathrm{nuc}}(A, B)$ (respectively $[h - 0] \in \mathrm{KK}(\mathcal{C};\, A, B)$) as the class which is represented by the **difference construction**

$$(\mathcal{H}_B, h\colon A \to B \otimes \mathbb{K}, 0)\,,$$

where $\mathcal{H}_B$ denotes the Hilbert B-module

$$\mathcal{H}_B := \{(b_1, b_2, \ldots)\colon\, b_n \in B, \sum_n b_n^* b_n \in B\}.$$

with trivial grading, and where we use the natural isomorphism $B \otimes \mathbb{K} \cong \mathbb{K}(\mathcal{H}_B)$.

It can be shown (as in the non-nuclear theory) that the bi-functor $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ is stable, and with help of Kasparov products, that our $\mathrm{KK}_{\mathrm{nuc}}$ groups are *homotopy invariant*, hence, are the same as the $\mathrm{KK}_{\mathrm{nuc}}$-groups of Skandalis [**726**] (at least if A is exact). Therefore, they satisfy variants of *Bott Periodicity, 6-term exact sequences for semi-split exact sequences, ...* But in case of the $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$-theory the situation is more complicate, because this are in essence functors from a "category" of point-norm closed m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ into Abelian groups, and the C^*-algebras play only a formal role as a sort of indices.

The way to homotopy invariance for *our* functors $\mathrm{KK}_{\mathrm{nuc}}(A, B) := \mathrm{KK}(\mathcal{C};\, A, B)$ in case that $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ and the much more general $\mathrm{KK}(\mathcal{C};\, A, B)$ is the following:
Our definitions of $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ is the *weakly nuclear* variant (respective $\mathcal{C}$-compatible variant) of the cobordism-class $\sim_c$ definition for KK, cf. [**73**, defs. 17.2.4, 17.3.1, 17.10.2]. Indeed, weakly *nuclear* (respectively $\mathcal{C}$-*compatible*) Kasparov modules $\mathcal{E}_1$ and $\mathcal{E}_2$ define the same element of $\mathrm{KK}_{\mathrm{nuc}}(A, B)$, if and only if, there exists a *nuclear* (respectively $\mathcal{C}$-*compatible*) Kasparov module $\mathcal{E}_3$ such that $\mathcal{E}_1 \oplus \mathcal{E}_3$ is isomorphic to a compact perturbation of $\mathcal{E}_2 \oplus \mathcal{E}_3$. This is the equivalence relation $\sim_c$ of [**73**, def. 17.2.4] (in case where $\mathcal{C} := \mathrm{CP}(A, B)$). More precisely, the relation $\sim_c$ in [**73**] is the stabilization of the relation $\sim_{cp}$ that allows also addition of degenerate elements. "Degenerate" Kasparov modules become zero even for the stabilization of the relation $\approx_u$ of unitary equivalence in [**73**, def.17.2.1], but here automatic *inside* $\mathbb{E}_{\mathrm{nuc}}(A, B)$ (respectively inside the semigroup $\mathbb{E}(\mathcal{C};\, A, B)$): The difference between the *nuclear* (respectively our more general $\mathcal{C}$-compatible) case and the usual KK-theory relation $\sim_c$ is, that the added module $\mathcal{E}_3$ has to be a *nuclear* (respectively a $\mathcal{C}$-compatible) Kasparov module as well.

We say that Kasparov modules (E, ϕ, F_0) and (E, ϕ, F_1) are **operator homotopic** if there exists a norm-continuous map $t \in [0, 1] \mapsto F(t) \in \mathcal{L}(E)$ such that $F(0) = F_0$, $F(1) = F_1$ and that $(E, \phi, F(t))$ is a Kasparov module for every $t \in [0, 1]$.

The same arguments as in [**73**, sec. 17.10] show that operator homotopic *nuclear* (respectively $\mathcal{C}$-compatible) Kasparov modules define the same element of *our* group $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ (respectively of $\mathrm{KK}(\mathcal{C};\, A, B)$).

The result that operator homotopic $\mathcal{C}$-compatible Kasparov modules are stably equivalent, implies that the Kasparov product is well-defined and is invariant under operator homotopy. In fact, the proofs in [**73**] or in [**389**, sec. 2.2] show that Kasparov products $\mathcal{E}_1 \otimes \mathcal{E}_2$ of representatives $\mathcal{E}_1$ and $\mathcal{E}_2$ of elements $[\mathcal{E}_1] \in \mathrm{KK}_{\mathrm{nuc}}(B, C)$ and $[\mathcal{E}_2] \in \mathrm{KK}(C, D)$ define up to operator homotopy an element of $\mathrm{KK}_{\mathrm{nuc}}(B, C)$ if B and C are separable. One has also that the Kasparov product defines a bi-additive map

$$\mathrm{KK}(A, B) \times \mathrm{KK}_{\mathrm{nuc}}(B, C) \to \mathrm{KK}_{\mathrm{nuc}}(A, C)$$

for σ-unital A, B and C and separable B. Its values are in $\mathrm{KK}_{\mathrm{nuc}}(B, D)$ respectively $\mathrm{KK}_{\mathrm{nuc}}(A, C)$, and if $A = B$ or $D = C$ the multiplication with $[\mathrm{id}_B - 0]$ or $[\mathrm{id}_C - 0]$

gives the identity homomorphism of $\mathrm{KK}_{\mathrm{nuc}}(B, C)$. Therefore, *our* $\mathrm{KK}_{\mathrm{nuc}}$-functor *is homotopy* invariant and coincides with the $\mathrm{KK}_{\mathrm{nuc}}$-functor of Skandalis [**726**].
There are natural homomorphisms from $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ into $\mathrm{KK}(A, B)$, as the definition tells us ([6]).
Certainly, $\mathrm{KK}_{\mathrm{nuc}}(A, B) = \mathrm{KK}(A, B)$ if A or B is nuclear, because then every Kasparov (A, B)-module is nuclear.

We outline now the changes of the Kasparov approach, that are necessary in the case of $\mathrm{KK}(\mathcal{C};\, A, B)$. Recall the other equivalence relations $\sim_c$, $\sim_{cp}$, $\sim_s$, $\sim_{soh}$, $\sim_{oh}$ and $\sim_h$ and on $\mathbb{E}(\mathcal{C};\, A, B)$ that are defined as in [**73**, chp. 17] on all elements of $\mathbb{E}(A, B)$ by

($\sim_{cp}$:) $\mathcal{E}_0 \sim_{cp} \mathcal{E}_1$ if there are *degenerate* $\mathcal{E}_2, \mathcal{E}_3 \in \mathbb{E}(\mathcal{C};\, A, B)$, such that $\mathcal{E}_0 \oplus \mathcal{E}_2 \sim_{scp} \mathcal{E}_1 \oplus \mathcal{E}_3$.

($\sim_c$:) $(E_0, \phi_0, F_0) \sim_c (E_1, \phi_1, F_1)$ if $[(E_0, \phi_0, F_0)] = [(E_1, \phi_1, F_1)]$ in

$$\mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp})\,.$$

`In [73, chp. 17]: ''stabilized version of` $\sim_{cp}$`''`

($\sim_{soh}$:) $(E_0, \phi_0, F_0) \sim_{soh} (E_1, \phi_1, F_1)$ if there is a C^*-morphism $\psi\colon A \to \mathcal{L}(E)$ and an operator-norm continuous map $t \in [0, 1] \mapsto F(t) \in \mathcal{L}(E)$ such that $(E, \psi, F(t)) \in \mathbb{E}(\mathcal{C};\, A, B)$ for every $t \in [0, 1]$ and that (E_i, ϕ_i, F_i) is (unitarily) isomorphic to $(E, \phi, F(i))$ for $i = 0, 1$.

($\sim_{oh}$:) $(E_0, \phi_0, F_0) \sim_{oh} (E_1, \phi_1, F_1)$ if there are *degenerate* $\mathcal{E}_2, \mathcal{E}_3 \in \mathbb{E}(\mathcal{C};\, A, B)$, a Hilbert B-module E, a C^*-morphism $\psi\colon A \to \mathcal{L}(E)$ and a norm-continuous map $t \in [0, 1] \mapsto F(t) \in \mathcal{L}(E)$ such that $(E, \psi, F(t)) \in \mathbb{E}(\mathcal{C}\,;\, A, B)$ for every $t \in [0, 1]$ and that $(E_i, \phi_i, F_i) \oplus \mathcal{E}_{i+2}$ is (unitarily) isomorphic to $(E, \phi, F(i))$ for $i = 0, 1$.

($\sim_h$:) It is the equivalence relation that is *generated* by the reflexive, but not necessarily transitive ???, relation $\sim_{h_0}$:
$(E_0, \phi_0, F_0) \sim_{h_0} (E_1, \phi_1, F_1)$ if there exists

$$(E, \psi, F) \in \mathbb{E}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}[0, 1])\,;\, A,\, B \otimes \mathrm{C}([0, 1]))$$

such that (E_i, ϕ_i, F_i) is isomorphic to $(E(i), \psi_i, F(i))$ for $i = 0, 1$.

Straight calculations show that all this relations are *compatible with addition* in $\mathcal{E}(\mathcal{C};\, A, B)/\approx_u$.

The relation $\sim_{cp}$ is an equivalence relation on $\mathbb{E}(\mathcal{C};\, A, B)$, because the degenerate Kasparov modules in $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}$ build a sub-semigroup of $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}$.

The relation $\sim_{oh}$ is an equivalence relation, because the degenerate Kasparov modules in $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{soh}$ build a sub-semigroup of $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{soh}$.

[6] We don't know if this homomorphisms are injective in general and if their images are naturally complemented in $\mathrm{KK}(A, B)$.

Notice that the Kasparov module (E, ψ, F) in the definition of $\sim_h$ can be considered as a "continuous" family $\{(E(t), \psi_t, F(t))\}_{0\leq t\leq 1}$ in $\mathbb{E}(\mathcal{C};\, A, B)$, if we consider the $B[0,1]$-module E as a complemented $B[0,1]$-submodule of $\widehat{\mathcal{H}}_{B[0,1]} \cong \mathrm{C}([0,1], \widehat{\mathcal{H}})$

but the unitary equivalences at the endpoints can not be specified ?????????

See proofs of part (vii) of Lemma 8.2.3.

Temporary, we define semi-groups

$$\mathrm{KK}_*(\mathcal{C};\, A, B) := \mathbb{E}(\mathcal{C};\, A, B)/\sim_*$$

for $* = h, oh, cp, c$.

Lemma 8.2.3. *Suppose that A and B are σ-unital and graded, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is countably generated point-norm closed operator convex cone, with $\mathcal{C} \circ \beta_A \subseteq \mathcal{C}$ and $\beta_B \circ \mathcal{C} \subseteq \mathcal{C}$.*

Then:

(i) *The unitary equivalence classes $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ build an Abelian semi-group under addition of Kasparov modules.*

(ii) *The relations $\sim_h$, $\sim_{oh}$, $\sim_c$, $\sim_{cp}$, $\sim_{scp}$ and $\approx_u$ (respectively the relations $\sim_{oh}$, $\sim_{soh}$ and $\sim_{scp}$) are successively stronger, and are equivalence relations on $\mathbb{E}(\mathcal{C};\, A, B)$ that are all compatible with addition in $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$.*

Thus, the set of equivalence classes $\mathbb{E}(\mathcal{C};\, A, B)/\sim_*$ are semi-groups, and there are natural semi-group epimorphisms from $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ onto $\mathbb{E}(\mathcal{C};\, A, B)/\sim_*$ for each of the above listed equivalence relation $\sim_*$.

If $\mathcal{C}' \subseteq \mathcal{C}$, then there are natural morphisms from $\mathbb{E}(\mathcal{C}';\, A, B)/\sim_*$ into $\mathbb{E}(\mathcal{C};\, A, B)/\sim_*$.

(iii) *$\mathrm{KK}_*(\mathcal{C};\, A, B) := \mathbb{E}(\mathcal{C};\, A, B)/\sim_*$ are groups for $* = oh, h$ and there are natural semigroup epimorphisms*

$$\mathrm{KK}_c(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C};\, A, B) \to \mathrm{KK}_h(\mathcal{C};\, A, B)\,.$$

The natural epimorphism $\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp} \to \mathbb{E}(\mathcal{C};\, A, B)/\sim_{cp}$ defines a group isomorphism $\mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}) \cong \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{cp})$.

$\mathrm{KK}_c(\mathcal{C};\, A, B)$ is a sub-semigroup of $\mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{cp}))$, and the kernel of $\mathrm{KK}_c(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C};\, A, B)$ consists of the classes $[(E, \phi, F)]_c$ such that there is a degenerate element $(E_0, \phi_0, F_0) \in \mathbb{E}(\mathcal{C};\, A, B)$ with $(E, \phi, F) \oplus (E_0, \phi_0, F_0) \sim_{soh} (E_0, \phi_0, F_0)$.

(iv) *If $f \in \mathrm{Hom}(B, C)$ is a grading preserving epimorphism, let $f_*\mathcal{C} := \mathcal{C}(f)\circ\mathcal{C}$ the point-norm closed matrix operator-convex cone generated by $f \circ \mathcal{C} := \{f \circ V\,;\ V \in \mathcal{C}\}$. Then*

$$f_*[(E, \phi, F)] := [(E\widehat{\otimes}_f C, \phi(\cdot)\widehat{\otimes}_f 1, F\widehat{\otimes}_f 1)]$$

defines semigroup morphism $f_*\colon \mathbb{E}(\mathcal{C}\,;\, A, B)/\approx_u \to \mathbb{E}(f_*\mathcal{C}\,;\, A, C)/\approx_u$ *that is compatible with all relations* $\sim_*$.

(v) *If* $g \in \mathrm{Hom}(D, A)$ *is grading preserving let* $g^*\mathcal{C} := \{V \in \mathrm{CP}(D, B)\,;\ V \circ g \in \mathcal{C}\}$, *then*

$$g^*[(E, \phi, F)] := [(E, \phi \circ g, F)]$$

defines semigroup morphism $g^*\colon \mathbb{E}(g^*\mathcal{C}\,;\, D, B)/\approx_u \to \mathbb{E}(\mathcal{C}\,;\, A, B)/\approx_u$ *that is compatible with all relations* $\sim_*$.

(vi) *Let* $\mathrm{CP_{in}}(B, B) \subseteq \mathrm{CP}(B, B)$ *denote the cone of all approximately inner c.p. maps of* B.

If C *and* D *are separable nuclear graded* C^**-algebras, then the map*

$$[(E, \phi, F)] \mapsto \big[(B \widehat{\otimes} E, id_B \widehat{\otimes} \phi, 1 \widehat{\otimes} F)\big]$$

maps $\mathbb{E}(C, D)/\approx_u$ *into* $\mathbb{E}(\mathrm{CP_{in}}(B, B) \widehat{\otimes}\, \mathrm{CP}(C, D);\ B \widehat{\otimes} C,\ B \widehat{\otimes} D)$ *and defines a morphism*

$$\tau_B\colon\ \mathrm{KK}(C, D) \to \mathrm{KK}_{oh}(\mathrm{CP_{in}}(B, B) \widehat{\otimes}\, \mathrm{CP}(C, D);\ B \widehat{\otimes} C, B \widehat{\otimes} D)$$

with $\tau_B([k - 0]) = [(\mathrm{id}_B \widehat{\otimes} k) - 0]$ *for grading preserving* $k \in \mathrm{Hom}(C, D)$.

In particular, $[\chi_t] = [\chi_t - 0] \in \mathrm{KK}(\mathrm{C}([0, 1]), \mathbb{C})$ *for* $\chi_t(f) := f(t)$ *maps to*

$$\tau_B([\chi_t]) = [\chi_t^B] \in \mathrm{KK}_{oh}(\mathrm{CP_{in}}(B, B) \widehat{\otimes}\, \mathrm{CP}(\mathrm{C}[0, 1], \mathbb{C});\ B \widehat{\otimes}\, \mathrm{C}([0, 1]), B)\,,$$

where χ_t^B *denotes the epimorphism* $\chi_t^B\colon f \in \mathrm{C}([0, 1], B) \mapsto f(t) \in B$ *(*$t \in [0, 1]$*).*

Since $[\chi_t] = [\chi_0]$ *by* [**73**, lem.18.5.2]*,it follows that* $[\chi_0^B] = [\chi_t^B]$ *for* $t \in [0, 1]$.

(vii) *Let* $\widehat{\mathcal{H}}_B := \mathcal{H}_B \oplus_B \mathcal{H}_B$ *with grading* $\beta_\infty \oplus (-\beta_\infty)$*, where* $\beta_\infty(x_1, x_2, \ldots) := (\beta_B(x_1), \beta_B(x_2), \ldots)$ *for* $(x_1, x_2, \ldots) \in \mathcal{H}_B \cong (B \otimes \mathbb{K})(1 \otimes p_{11})$.

There is a construction $\lambda =: \lambda_B$ *that assigns to each class* $[\mathcal{E}] = [(E, \phi, F)] \in \mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ *a special class* $\lambda([\mathcal{E}]) \in \mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ *that has representative* $(\widehat{\mathcal{H}}_B, \psi, U)$ *with* $U = U^* = U^{-1}$*, such that* λ *is additive,* $\mathcal{E} \sim_{cp} \lambda(\mathcal{E})$ *in* $\mathbb{E}(\mathcal{C};\, A, B)$*,* $\lambda(\mathcal{E})$ *is degenerate if* $\mathcal{E}$ *is degenerate,* $\lambda(\mathcal{E}_1) \sim_{scp} \lambda(\mathcal{E}_2)$ *if* $\mathcal{E}_1 \sim_{scp} \mathcal{E}_2$*, and* $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(\mathcal{C}[0, 1];\, A, B[0, 1])$ *implies*

$$(p_t)_*(\lambda_{B[0,1]}[\mathcal{E}]) = \lambda_B((p_t)_*[\mathcal{E}])\,.$$

Moreover, if $E \cong \widehat{\mathcal{H}}_B$ *then* ψ *is unitarily equivalent to* $\phi \oplus 0$ *by an unitary* $W \in \mathcal{L}(\widehat{\mathcal{H}}_B)$ *of degree zero, and, if* $F = F^* = F^{-1}$ *then* $\lambda(\widehat{\mathcal{H}}_B\,, \phi, F)$ *is isomorphic to* $(\widehat{\mathcal{H}}_B\,, \phi \oplus_{s,t} 0, F \oplus_{s,t} (-F))$ *for (any) isometries* $s, t \in \mathcal{L}(\widehat{\mathcal{H}}_B)$ *of even degree.*

In particular, all equivalence-operations for the relations $\sim_*$ *can be elaborated inside the class of special Kasparov modules* $(\widehat{\mathcal{H}}_B, \psi, F)$ *with* $F = F^* = F^{-1}$.

(viii) *If* $\mathcal{M}(B)$ *contains isometries* s, t *of even degree with* $ss^* + tt^* = 1$*, then addition in* $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ *is compatible with*

$$\epsilon\colon B \oplus_\infty B \to B\,,\quad \epsilon(a, b) \mapsto sas* + tbt^*\,,$$

i.e., the composition $\epsilon^* \circ \Sigma$ *of the map*

$$\Sigma\colon [\mathbb{E}(A,B)] \times [\mathbb{E}(A,B)] \to [\mathbb{E}(A, B\oplus B)]$$

given by

$$\Sigma\colon ((E_1,\phi_1,F_1),(E_2,\phi_2,F_2)) \mapsto (E_1 \oplus_\infty E_1, \phi_1(\cdot)\oplus\phi_2(\cdot), F_1\oplus F_2)$$

with ϵ^* *defines the addition in* $\mathbb{E}(A,B)/\!\approx_u$*. In particular,* $2[\mathcal{E}] = \delta^*[\mathcal{E}]$ *in* $\mathbb{E}(\mathcal{C};\, A,B)/\approx_u$ *for* $\mathcal{E}\in\mathbb{E}(\mathcal{C};\, A,B)$ *and* $\delta(b) := sbs^* + tbt^*$*.*

Proof. (i): The unitary equivalence classes $\mathbb{E}(\mathcal{C};\, A,B)/\approx_u$ build an Abelian semigroup under addition of Kasparov modules, because $\mathbb{E}(A,B)/\approx_u$ is an Abelian semigroup (is in particular a *set*), the sum of $\mathcal{C}$-compatible Kasparov (A,B)-modules is $\mathcal{C}$-compatible, and the class of $\mathcal{C}$-compatible Kasparov modules is invariant under unitary equivalence.

(ii): It is easy to check that $\approx_u$, $\sim_{scp}$ and $\sim_{soh}$ are equivalence relations on $\mathbb{E}(\mathcal{C}\,;\, A,B)$ that are compatible with sums of Kasparov modules.

If $(E,\phi,F')\in\mathbb{E}(A,B)$ is a ϕ-compact perturbation of $(E,\phi,F)\in\mathbb{E}(\mathcal{C};\, A,B)$, then $(E,\phi,F(t))$ is in $\mathbb{E}(\mathcal{C};\, A,B)$ for $F(t) := tF + (1-t)F'$, thus $(E,\phi,F')\sim_{soh}$ (E,ϕ,F) in $\mathbb{E}(\mathcal{C};\, A,B)$, i.e., $\sim_{scp}$ is stronger than $\sim_{soh}$ on $\mathbb{E}(\mathcal{C};\, A,B)$.

Thus $\sim_{soh}$, $\sim_{scp}$ and $\approx_u$ are successively stronger equivalence relations $\sim_*$ on $\mathbb{E}(A,B)$ that are compatible with the property that $(E,\phi,F)\sim_* (E',\phi',F')$ and $(E,\phi,F)\in\mathbb{E}(\mathcal{C};\, A,B)$ implies $(E',\phi',F')\in\mathbb{E}(\mathcal{C};\, A,B)$.

It is easy to see, that the *infinite repeat* $\mathcal{E}_\infty := (E\otimes\ell_2;\, \phi(\cdot)\otimes 1, F\otimes 1)$ is a Kasparov (A,B)-module, if and only if, $\mathcal{E} = (E,\phi,F)$ is a *degenerate* Kasparov module. Then $\mathcal{E}_\infty\oplus\mathcal{E}\approx_u\mathcal{E}_\infty$, and $\mathcal{E}$ is a degenerate element in $\mathbb{E}(\mathcal{C};\, A,B)$ if and only if $\mathcal{E}_\infty$ is in $\mathbb{E}(\mathcal{C};\, A,B)$.

compare above and below! ??

Infinite (B-module) sums $(E,\phi,F)\oplus(E,\phi,F)\oplus\cdots\cong(\mathcal{H}\otimes E, 1\otimes\phi, 1\otimes F)$ of *degenerate* Kasparov modules $(E,\phi,F)\in\mathbb{E}(\mathcal{C};\, A,B)$ are degenerate Kasparov modules in $\mathbb{E}(\mathcal{C};\, A,B)$, and $(E,\phi,F)\oplus(\mathcal{H}\otimes E,1\otimes\phi,1\otimes F)$ is unitary isomorphic to $(\mathcal{H}\otimes E,1\otimes\phi,1\otimes F)$, i.e., $[(E,\phi,F)] = 0$ in $\mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A,B)/\sim_{scp})$ Thus $\mathcal{E}\oplus(E,\phi,F)\sim_c\mathcal{E}$ for every $\mathcal{E}\in\mathbb{E}(\mathcal{C};\, A,B)$. If follows that $\sim_c$ is stronger than $\sim_{oh}$. Hence, there is a natural semigroup epimorphisms $\mathrm{KK}_c(\mathcal{C};\, A,B)\to\mathrm{KK}_{oh}(\mathcal{C};\, A,B)$.

Since the classes of degenerate modules in $\mathbb{E}(\mathcal{C};\, A,B)$ build the sub-semigroups of $\mathbb{E}(\mathcal{C};\, A,B)/\sim_{scp}$ and $\mathbb{E}(\mathcal{C};\, A,B)/\sim_{sop}$ of elements that are additively absorbed by some other element, we get that $\sim_{op}$ and $\sim_{cp}$ are successively stronger additive equivalence relations on $\mathbb{E}(\mathcal{C};\, A,B)$ that are weaker than $\sim_{sop}$ respectively $\sim_{scp}$. Furthermore, $\mathrm{KK}(\mathcal{C}\,;\, A,B) = \mathrm{Gr}(\mathbb{E}(\mathcal{C}\,;\, A,B)/\sim_{cp})$,

$$\mathrm{Gr}(\mathbb{E}(\mathcal{C}\,;\, A,B)/\sim_{soh}) = \mathrm{Gr}(\mathbb{E}(\mathcal{C}\,;\, A,B)/\sim_{oh})\,,$$

and there is semigroup monomorphism $\mathrm{KK}_c(\mathcal{C};\, A, B) \hookrightarrow \mathrm{KK}(\mathcal{C};\, A, B)$, and semigroup epimorphisms

$$\mathbb{E}(\mathcal{C};\, A, B)/\sim_{cp} \to \mathrm{KK}_c(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C};\, A, B)$$

and a semigroup morphism $\mathrm{KK}_{oh}(\mathcal{C};\, A, B) \to \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{soh})$ such that its image generates $\mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{soh})$. All its images are sub-semigroups that generate the groups.

??????????????????????

The relation $\sim_{soh}$ is stronger than $\sim_{oh}$ by definition, and $\mathrm{KK}_{oh}(\mathcal{C};\, A, B) = \mathbb{E}(\mathcal{C};\, A, B)/\sim_{oh}$ is a group ???? (if we include unitary equivalence and calculate modulo addition of a degenerate module in $\mathbb{E}(\mathcal{C};\, A, B)$), because $\sim_{oh}$ is compatible with unitary equivalence, compact perturbation and addition of degenerate elements.

???????????????????.

??

The relation $\sim_h$ is an equivalence relation, because homotopy $\sim_{h_0}$ is reflexive and $\approx_u$ is an equivalence relation. Since homotopy "inside" $\mathbb{E}(\mathcal{C};\, A, B)$ are additive as elements of $\mathbb{E}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}[0,1]);\, A, B \otimes \mathrm{C}[0,1])$, and since unitary equivalence is compatible with addition of Kasparov modules, we get that $\sim_h$ is compatible with addition in $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$. Thus $\mathrm{KK}_h(\mathcal{C};\, A, B) := \mathbb{E}(\mathcal{C};\, A, B)/\sim_h$ is a commutative semi-group.

The relation $\sim_{soh}$ is stronger than $\sim_h$ on $\mathbb{E}(\mathcal{C};\, A, B)$, because an operator homotopy $t \mapsto (E, \phi, F(t))$ in $\mathbb{E}(\mathcal{C};\, A, B)$ defines an element $(E \otimes \mathrm{C}[0,1], \phi(\cdot) \otimes 1, F)$ in $\mathbb{E}(\mathcal{C}([0,1]);\, A, \mathrm{C}([0,1], B))$ that defines a homotopy between $(E, \phi, F(0))$ and $(E, \phi, F(1))$. Here:

$$F := \{F(t)\} \in \mathrm{C}([0,1], \mathcal{M}(\mathbb{K}(E))) \subseteq \mathrm{C}_{\mathrm{b,st}}([0,1], \mathcal{M}(\mathbb{K}(E))) \cong \mathcal{L}(E \otimes \mathrm{C}[0,1])$$

By definition of the relation $\sim_{oh}$, $\mathcal{E} \sim_{oh} (\{0\}, 0, 0)$ if $\mathcal{E} \in \mathbb{E}(\mathcal{C}\,;\, A, B)$ is *degenerate*.

The same happens also for $\sim_h$, because the homotopy connecting degenerate $\mathcal{E} := (E, \phi, F) \in \mathbb{E}(A, B)$ and the 0-module $0 := (\{0\}, 0, 0)$, that is given the proof of [**73**, prop.17.2.3], is in $\mathbb{E}(\mathcal{C}([0,1]);\, A, \mathrm{C}([0,1, B))$ if $\mathcal{E} \in \mathbb{E}(\mathcal{C};\, A, B)$. Thus $\sim_{oh}$ is stronger than $\sim_h$.
Finally, there are natural semigroup epimorphisms

$$\mathbb{E}(\mathcal{C};\, A, B)/\approx_u \to \mathrm{KK}_c(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C}\,;\, A, B) \to \mathrm{KK}_h(\mathcal{C};\, A, B)\,.$$

If $(E, \phi, F) \in \mathbb{E}(\mathcal{C}\,;\, A, B)$ then $(E, \phi \circ \beta_A, -F) \in \mathbb{E}(\mathcal{C};\, A, B)$, because $\mathcal{C} \circ \beta_A \subseteq \mathcal{C}$ by our assumptions.

The sum $(E, \phi, F) \oplus (E, \phi \circ \beta_A, -F)$ is operator-homotopic to a degenerate element (cf. [**73**, prop.17.3.3]). Thus, the semigroups $\mathrm{KK}_{oh}(\mathcal{C}\,;\, A, B)$ and $\mathrm{KK}_h(\mathcal{C}\,;\, A, B)$ are groups, and the above natural semigroup morphism $\mathrm{KK}_{oh}(\mathcal{C}\,;\, A, B) \to \mathrm{Gr}(\mathbb{E}(\mathcal{C}\,;\, A, B)/\sim_{sop})$ is an isomorphism of groups.

`to be filled in: proof of 8.2.3`

Let $\sim_{soh}$ denote the equivalence relation generated by $\approx_u$ and operator homotopies $t \in [0,1] \mapsto (E, \phi, F(t)) \in \mathbb{E}(\mathcal{C}_2\,; B, C)$. Then $\sim_{soh}$ is compatible with addition and is stronger than $\sim_{oh}$.

Since $\mathrm{KK}_{oh}(\mathcal{C}\,; B, C) = \mathbb{E}(\mathcal{C}_2\,; B, C)/\sim_{oh}$ is a group, we get that there is a natural epimorphism from $\mathrm{Gr}(\mathbb{E}(\mathcal{C}_2\,; B, C)/\sim_{soh})$ onto $\mathrm{KK}_{oh}(\mathcal{C}; B, C)$. This must be an isomorphism, because $0 = [\mathcal{E}] \in \mathrm{Gr}(\mathbb{E}(\mathcal{C}_2\,; B, C)/\approx_u)$ for every "degenerate" $\mathcal{E} \in \mathbb{E}(\mathcal{C}_2\,; B, C)$.

If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then (obviously) $\mathbb{E}(\mathcal{C}_1\,; B, C) \subseteq \mathbb{E}(\mathcal{C}_2\,; B, C)$.

For the relations $\sim_* = \approx_u, \sim_{scp}$ or $\sim_{soh}$ It is easy to see, that elements $\mathcal{E}_j \in \mathbb{E}(\mathcal{C}_1\,; B, C)$ $(j = 1, 2)$ satisfy $\mathcal{E}_1 \sim_* \mathcal{E}_2$ in $\mathbb{E}(\mathcal{C}_1\,; B, C)$, if and only if, $\mathcal{E}_1 \sim_* \mathcal{E}_2$ in $\mathbb{E}(\mathcal{C}_2\,; B, C)$.

Thus $\mathbb{E}(\mathcal{C}_1\,; B, C)/\sim_*$ is a subgroup of $\mathbb{E}(\mathcal{C}_2\,; B, C)/\sim_*$. If we apply the Grothendieck functor, we get natural semigroup morphisms $\mathrm{KK}_c(\mathcal{C}_1\,; B, C) \to \mathrm{KK}_c(\mathcal{C}_2\,; B, C)$ and $\mathrm{KK}_{oh}(\mathcal{C}_1\,; B, C) \to \mathrm{KK}_{oh}(\mathcal{C}_2\,; B, C)$.

In the case of the relation $\sim_* = \sim_h$, one has at least that $\mathcal{E}_1 \sim_h \mathcal{E}_2$ in $\mathbb{E}(\mathcal{C}_1\,; B, C)$ implies that $\mathcal{E}_1 \sim_h \mathcal{E}_2$ in $\mathbb{E}(\mathcal{C}_2\,; B, C)$, because $\mathbb{E}(\mathcal{C}_1 \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}[0,1])\,; B, \mathrm{C}([0,1], C))$ is contained in $\mathbb{E}(\mathcal{C}_2 \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}[0,1])\,; B, \mathrm{C}([0,1], C))$.

??

Thus, if $\mathcal{C}_1 \subseteq \mathcal{C}_2$, there are natural semigroup morphisms

$$\mathrm{KK}_*(\mathcal{C}_1\,; B, C) \to \mathrm{KK}_*(\mathcal{C}_2\,; B, C)\,.$$

This semi-group morphisms are in general neither injective nor surjective.

??

semigroup epimorphisms

$$\mathrm{KK}_c(\mathcal{C}_2\,; B, C) \to \mathrm{KK}_{oh}(\mathcal{C}_2\,; B, C) \to \mathrm{KK}_h(\mathcal{C}_2\,; B, C)\,.$$

The natural injections $B \cong C \otimes p_{11} \subseteq B$ and $C \cong C \otimes p_{11} \subseteq C \otimes \mathbb{K}$ define isomorphisms from $\mathrm{KK}_{oh}(\mathcal{C}_3\,; B \otimes \mathbb{K}, C)$ onto $\mathrm{KK}_{oh}(\mathcal{C}_2\,; B, C)$ respectively from $\mathrm{KK}_{oh}(\mathcal{C}_2\,; B, C)$ onto $\mathrm{KK}_{oh}(\mathcal{C}_4\,; B, C \otimes \mathbb{K})$, where the m.o.c. cones $\mathcal{C}_3 \subseteq \mathrm{CP}(B \otimes \mathbb{K}, C)$ and $\mathcal{C}_4 \subseteq \mathrm{CP}(B, C \otimes \mathbb{K})$ are the natural extensions of $\mathcal{C}_2 \subseteq \mathrm{CP}(B, C)$ to the stabilizations.

`more???`

(vi): Let χ_t^B denote the epimorphisms $\chi_t^B \colon f \in \mathrm{C}([0,1], B) \mapsto f(t) \in B$ $(t \in [0,1])$, and let $\mathrm{CP}_{\mathrm{in}}(B, B) \subseteq \mathrm{CP}(B, B)$ denote the cone of approximately inner c.p. maps of B, and $\mathcal{S} := \mathrm{CP}_{\mathrm{in}}(B, B) \otimes \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C})$. Then the difference construction $(B\,, \chi_t^B\,, 0)$ is an Element in ([7])

$$\mathbb{E}\big(\mathcal{S};\, \mathrm{C}([0,1], B), B\big) = \mathbb{E}\big(\mathrm{CP}_{\mathrm{in}}(B, B) \otimes \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C})\,; B \otimes \mathrm{C}[0,1],\, B\big)\,.$$

We denote by $[\chi_t^B - 0] := [(B, \chi_t^B\,, 0)]$ its class in $\mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B, B); B \otimes \mathrm{C}([0,1]), B)$.

[7]The equation expresses only our conventions on notations, i.e., it is a trivial identity!

Since by ??? **??**

$$\mathcal{C} = (\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}[0,1])) \circ (\mathrm{CP}_{\mathrm{in}}(B,B) \otimes \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C}))$$

holds for all point-norm closed matrix operator-convex cones $\mathcal{C} \subseteq \mathrm{CP}(A,B)$, one has only to check that, for all σ-unital B and all $t \in [0,1]$ holds $[\chi_0^B - 0] = [\chi_t^B - 0]$ in $\mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B,B);\, B \otimes \mathrm{C}[0,1], B)$.

The cone $\mathrm{CP}_{\mathrm{in}}(B,B)$ is singly generated by $\{\mathrm{id}_B\}$. Thus, our theory applies to $\mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B,B);\, B \otimes \mathrm{C}[0,1], B)$.

$$(E, \phi, F) \mapsto (B \widehat{\otimes} E, \mathrm{id}_B \widehat{\otimes} \phi, 1_{\mathcal{M}(B)} \widehat{\otimes} F)$$

defines an additive map from $\mathbb{E}(C,D)$ into $\mathbb{E}(\mathrm{CP}_{\mathrm{in}}(B,B) \otimes \mathrm{CP}(C,D);\, B \otimes C, B \otimes D)$ for *trivially graded* nuclear C and D (where e.g. the graded tensor product $(B \widehat{\otimes} C, \beta)$ coincides with $(B \otimes C, \beta_B \otimes \mathrm{id}_C)$). It induces a group morphism $\tau_B \colon \mathrm{KK}(C,D) \to \mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B,B);\, B \otimes C, B \otimes D)$. Obviously, $\tau_B([\psi - 0]) = \tau_B([(D,\psi,0)]) = [(B \otimes D;\, \mathrm{id} \otimes \psi, 0)] = [(\mathrm{id} \otimes \psi) - 0]$ if $\psi \in \mathrm{Hom}(C,D)$.

We have the identities $[(\mathbb{C}, \chi_t, 0)] = [\chi_t - 0] = [\chi_0 - 0]$ in $\mathrm{KK}_{oh}(C[0,1], \mathbb{C})$ for all $t \in [0,1]$, where $\chi_t \colon f \in \mathrm{C}[0,1] \mapsto f(t) \in \mathbb{C}$, cf. [**73**, lem.18.5.2].

It is not difficult to see that the original proof of Kasparov of $[\chi_0^B - 0] = [\chi_t^B - 0]$ in $\mathrm{KK}(B \otimes \mathrm{C}[0,1], B)$ works also in $\mathrm{KK}(\mathrm{CP}_{\mathrm{in}}(B,B);\, B \otimes \mathrm{C}[0,1], B)$, because $(B, \chi_t^B, 0) = (B \otimes \mathbb{C}, \mathrm{id}_B \otimes \psi_t, 0)$ for the morphism $\psi_t \colon g \in C[0,1] \mapsto g(t) \in \mathbb{C}$.

(vii): Let E^{op} denote the Hilbert B-module E with grading $-\beta_E$. Then $\widehat{\mathcal{H}}_B \cong \mathcal{H}_B \oplus_B \mathcal{H}_B$. We use that there are unitary grading-preserving B-module isomorphisms $E \oplus_B \widehat{\mathcal{H}}_B \cong \widehat{\mathcal{H}}_B$ (Kasparov stabilization, [**73**, thm.14.6.1], [**389**, thm.1.2.12], $\widehat{\mathcal{H}}_B \oplus_B (\widehat{\mathcal{H}}_B)^{op} \cong \widehat{\mathcal{H}}_B$.

We define $\lambda[(E,\phi,F)]$ for $[\mathcal{E}] = [(E,\phi,F)] \in \mathbb{E}(\mathcal{C};\, A, B)/\approx_u$ by the operations $\lambda = \lambda_2 \circ \lambda_1$:

$$\lambda_1([E,\phi,F]) := (E \oplus_B \widehat{\mathcal{H}}_B\,, \phi \oplus 0\,, G_1(F) \oplus 0)\,,$$

where $G_1(F) := g((F^* + F)/2)$ for the function $g(t) = t$ on $|t| \le 1$ and $g(t) := t/|t|$ for $|t| > 1$. Then $G_1(F) = G_1(F)^*$ and $\|G_1(F)\| \le 1$.

Note that $(E, \phi, G_1(F))$ is degenerate if (E,ϕ,F) is degenerate, $G_1(F_1 \oplus F_2) = G_1(F_1) \oplus G_1(F_2)$, $G_1(U^*FU) = U^* G_1(F) U$ for unitaries U, $G_1(F_2)$ is a ϕ-compact perturbation of $G_1(F_1)$ if F_2 is a ϕ-compact perturbation of F_1. and $G_1(F)(t) = G_1(F(t))$ for $F = \{F(t)\} \in \mathcal{L}(E)$ if $E = \{E(t)\}$ is a Hilbert $B[0,1]$-module. Since $\mathcal{E} \oplus (\widehat{\mathcal{H}}_B, 0, 0) \in \mathbb{E}(\mathcal{C};\, A, B)$, it follows from the arguments in [**73**, 17.4] that $\lambda =: \lambda_B$ has the quoted properties.

One can now go a step further and build from $(E, \phi, G) \in \mathbb{E}(\mathcal{C};\, A, B)$ with $E := \widehat{\mathcal{H}}_B$, $G = G^*$ and $\|G\| \le 1$ the Kasparov module

$$\lambda_2\big(E, \phi, (E \oplus_B E^{op}, \phi \oplus 0, U(G)\big) \in \mathbb{E}(\mathcal{C};\, A, B)$$

with $U(G) \in \mathcal{M}_2(\mathcal{L}(E)) \cong \mathcal{L}(E \oplus_B E^{op})$ where $U(G)$ is the (selfadjoint) Halmos unitary $U(G)$ of G with $U(G)_{11} := G$, $U(G)_{22} := -G$ and $U(G)_{12} = U(G)_{21} := (1 - G^2)^{1/2}$. Straight calculations show $U(G)(\beta_E(x), -\beta_E(y)) = ((-\beta_E) \oplus$

$\beta_E)(U(G)(x,y))$ (i.e., $U(G)$ has odd degree), and $U(G)$ is a $(\phi \oplus 0)$-compact perturbation of $G \oplus -G$. The graded commutator $U(G)(\phi(a) \oplus 0) - (\phi(\beta_A(a)) \oplus 0)U(G)$ is given by the 2×2-matrix with diagonal entries $G\phi(a) - \phi(\beta_A(a))G$ and 0 and with off-diagonal entries $(1 - G^2)^{1/2}\phi(a)$ and $-(\phi(\beta_A(a))(1 - G^2)^{1/2})$, thus is in $\mathbb{K}(E \oplus E^{op}) \cong M_2(\mathbb{K}(E))$ (and is zero if (E, ψ, G) is degenerate). Now one can use again that $E \oplus E^{op} \cong \widehat{\mathcal{H}}_B$. The operation λ_2 also plays nicely together with unitary equivalence, (Cuntz-)addition, ϕ-compact perturbation, operator homotopy and homotopy (all inside the class of Kasparov $\mathcal{C}$-modules with self-adjoint contractive ψ-derivation G).

??

(viii):

(ix): ?????????????????? □

LEMMA 8.2.4. *Suppose that A and B are trivially graded and stable, A is separable, B is σ-unital, and that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a non-degenerate point-norm closed m.o.c. cone that is countably generated.*

Let $B_{(1)}$ denote $B \oplus B$ with odd grading $\beta_{B \oplus B}(a, b) = (b, a)$.

Let $H\colon A \to \mathcal{M}(B)$ defined by $\mathcal{C}$ such that $\delta_\infty \circ H$ is unitarily equivalent to H and is non-degenerate.

Let s, t isometries in $H(A)' \cap \mathcal{M}(B)$ with $ss^* + tt^* = 1$.

$H_1 := H(\cdot) \oplus H(\cdot)\colon A \to \mathcal{M}(B \oplus B) = \mathcal{M}(B) \oplus \mathcal{M}(B)$.

(o) $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ *is countably generated if A is separable and B is σ-unital.*

(i) *There is a natural grading preserving isomorphism $B_{(1)} \cong B \widehat{\otimes} \mathbb{C}_{(1)}$, and the graded standard Hilbert $B_{(1)}$-module $\widehat{\mathcal{H}}_{B_{(1)}}$ is isomorphic to the $B \oplus B$-module $B \oplus B$ with standard odd grading $S(x, y) = (y, x)$.*

Furthermore, $\mathbb{K}(B_{(1)}) \cong B_{(1)}$, and $\mathcal{L}(B_{(1)}) = \mathcal{M}(B_{(1)}) = \mathcal{M}(B)_{(1)}$. (i.e., $\mathcal{M}(B) \oplus \mathcal{M}(B)$ with standard odd grading).

There is a grading-preserving unitary $B_{(1)}$-module monomorphism γ from $B_{(1)} \oplus_{B_{(1)}} B_{(1)}$ onto $B_{(1)}$ that induces the Cuntz addition $\oplus$ on $\mathcal{M}(B) \oplus \mathcal{M}(B)$, i.e.,

$$\mathcal{M}(\gamma)((F_1, G_1) \oplus_{B \oplus B} (F_2, G_2)) = (sF_1s^* + tF_2t^*, sG_1s^* + tG_2t^*)$$

for $(F_1, G_1), (F_2, G_2) \in \mathcal{M}(B) \oplus \mathcal{M}(B)$.

The morphism $H_1\colon a \in A \mapsto H(a) \oplus H(a) \in \mathcal{M}(B) \oplus \mathcal{M}(B)$ defines a degenerate element $(B_{(1)}, H_1, 1 \oplus (-1)) \in \mathbb{E}(\mathcal{C};\, A, B_{(1)})$.

(iii) *The isomorphism classes of the Kasparov $B_{(1)}$-modules $(B_{(1)}, H_1, U)$, with self-adjoint unitary U define a sub-semigroup $\mathcal{S}_p$ of $\mathbb{E}(\mathcal{C};\, A, B_{(1)})/\approx_u$, such that the in $\mathbb{E}(\mathcal{C};\, A, B_{(1)})/\approx_u$ defined addition*

$$[(B_{(1)}, H_1, U_1)] + [(B_{(1)}, H_1, U_2)] := [(B_{(1)}, H_1, U_3)]$$

is realized by Cuntz addition $(U_1, U_2) \mapsto U_3 := SU_1S^* + TU_2T^*$, where $S := (s, s)$ and $T = (t, t)$.

It holds $U = (1-2p, 2p-1)$ *with* $(1-U)/2 =: p = p^* = p^2$ *if and only if* $[p, H(A)] \subseteq B$.

The semigroup morphism from $\mathcal{S}_p$ *into*

$$\mathrm{KK}_h(\mathcal{C};\, A, B_{(1)}) := \mathbb{E}(\mathcal{C};\, A, B_{(1)})/\sim_h$$

is an epimorphism.

(iv) *If we consider* B *as graded Hilbert* B*-module with even grading given by the symmetry* $I = ss^* - tt^* \in \mathcal{M}(B)$*, then* $\widehat{\mathcal{H}}_B \cong B$*. If* $V \in \mathcal{M}(B)$ *is a selfadjoint unitary, then* (B, H, V) *is in* $\mathbb{E}(\mathcal{C};\, A, B)$ *if and only if* $U := s^*Vt$ *is a unitary with* $[U, H(A)] \subseteq B$*. Then* $V = sUt^* + tU^*s^*$.

The isomorphism classes of the Kasparov B*-modules*

$$(B, H, sUt^* + tU^*s^*)\,,$$

where U *is a unitary in* $\mathcal{M}(B)$ *with* $[U, H(A)] \subseteq B$ *build a sub-semigroup* $\mathcal{S}_u$ *of* $\mathbb{E}(\mathcal{C};\, A, B)/\approx_u$*, such that the addition*

$$[(B, H, sU_1t^* + tU_1^*s^*)] + [(B, H, sU_2t^* + tU_2^*s^*)] = [(B, H, sU_3t^* + tU_3^*s^*)]$$

in $\mathbb{E}(\mathcal{C};\, A, B_{(1)})/\approx_u$ *is realized by Cuntz addition*

$$(U_1, U_2) \mapsto U_3 := sU_1s^* + tU_2t^*\,.$$

The semigroup morphism from $\mathcal{S}_u$ *into*

$$\mathrm{KK}_h(\mathcal{C};\, A, B) := \mathbb{E}(\mathcal{C};\, A, B)/\sim_h$$

is an epimorphism.

Proof. (o): ?????????????

The m.o.c. cone $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ is countably generated if A is separable and B is σ-unital, because it is generated by the c.p. map $V_{\varphi,e}(a) := \varphi(a)e$, where $e \in B_+$ is a strictly positive contraction and $\varphi \in A_+^*$ is a faithful positive functional.

(i): Since B is stable, there are isometries $s_1, s_2, \ldots \in \mathcal{M}(B)$ with $\sum s_n s_n^*$ strictly convergent to 1, cf. Remark 5.1.1(8). Then $\mu(b) := (s_1 b, s_2 b, \ldots)$ defines an isometric B-module isomorphism from B onto $\mathcal{H}_B$ with inverse $\mu^{-1}(b_1, b_2, \ldots) = \sum_n s_n^* b_n$.

$\mathcal{H}_{B_{(1)}}$ is $\mathcal{H}_{B\oplus B}$ with grading given by

$$((x_1, y_1), (x_1, y_2), \ldots) \mapsto ((y_1, x_1), (y_1, x_2), \ldots)\,,$$

and $\mathcal{H}_{B\oplus B}$ is $B \oplus B$-module isomorphic to $\mathcal{H}_B \oplus_\infty \mathcal{H}_B$ by

$$((x_1, y_1), (x_1, y_2), \ldots) \mapsto ((x_1, x_2, \ldots), (y_1, y_2, \ldots))$$

This isomorphism becomes grading-preserving if we take on $\mathcal{H}_B \oplus_\infty \mathcal{H}_B$ the standard odd grading $(x, y) \mapsto (y, x)$. It follows that $\mathcal{H}_{B_{(1)}}$ is isomorphic to $B \oplus B$ with odd grading $\beta(x, y) = (y, x)$.

Thus $\widehat{\mathcal{H}}_{B_{(1)}} \cong (B \oplus B) \oplus_{B\oplus B} (B \oplus B)$ with grading $\beta((x_1, y_1), (x_2, y_2)) := ((y_1, x_1), (-y_2, -x_2))$.

Since B is stable, $\mathcal{M}(B)$ contains a copy

$$C^*(s,t\,;\, ss^* + tt^* = s^*s = t^*t = 1)$$

of $\mathcal{O}_2$ unitally. The map $\nu((x_1, y_1), (x_1, y_2)) := (sx_1 + tx_2, sy_1 - ty_2)$ defines a grading preserving $B_{(1)}$-module map from $(B \oplus B) \oplus_{B \oplus B} (B \oplus B)$ with grading β onto $B_{(1)}$.

Clearly $\mathcal{L}(B_{(1)}) = \mathcal{M}(B_{(1)}) = \mathcal{M}(B)_{(1)}$ and $\mathbb{K}(B_{(1)}) = B_{(1)}$.

Since B is trivially graded, the graded tensor product $B \widehat{\otimes} \mathbb{C}_{(1)}$ is isomorphic to $B \otimes (\mathbb{C} \oplus \mathbb{C})$ with grading $\mathrm{id}_B \otimes \alpha$, where $\alpha(z_1, z_2) = (z_2, z_1)$ is the grading of $\mathbb{C}_{(1)}$. Thus, $B \widehat{\otimes} \mathbb{C}_{(1)} \cong B_{(1)}$ in this case.

(ii):

(iii):

(iv): □

The next proposition implies that relations $\sim_c$, $\sim_{oh}$ and $\sim_h$ on $\mathbb{E}(\mathcal{C};\, A, B)$ are the same if A is separable and B is σ-unital. We use the above introduced conventions on the extensions of the m.o.c. cone $\mathcal{C}$.

PROPOSITION 8.2.5. *Suppose that A, B, C are σ-unital graded C^*-algebras, where A is* separable, *and let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ and $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathrm{CP}(B, C)$ denote non-degenerate countably generated point-norm closed matricial operator-convex cones with $\mathcal{C} \circ \beta_A \subseteq \mathcal{C}$, $\beta_B \circ \mathcal{C} \subseteq \mathcal{C}$, $\mathcal{C}_j \circ \beta_B \subseteq \mathcal{C}_j$ and $\beta_C \circ \mathcal{C}_j \subseteq \mathcal{C}_j$ for $j = 1, 2$.*

(i) *The Kasparov product*

$$[(E', \phi', F')] \circ [(E, \phi, F)] = [(E' \widehat{\otimes}_B E\,,\ \phi(\cdot) \widehat{\otimes}_B 1\,,\,, F' \sharp F)]$$

defines a bi-additive map

$$\mathrm{KK}_{oh}(\mathcal{C};\, A, B) \times \mathrm{KK}_{oh}(\mathcal{C}_2, B, C) \mapsto \mathrm{KK}_{oh}(\mathcal{C}_2 \circ \mathcal{C};\, A, C)\,.$$

The natural injections $B \cong C \otimes p_{11} \subseteq B$ and $C \cong C \otimes p_{11} \subseteq C \otimes \mathbb{K}$ define isomorphisms from $\mathrm{KK}_{oh}(\mathcal{C}_2\,;\ B \otimes \mathbb{K}, C)$ onto $\mathrm{KK}_{oh}(\mathcal{C}_2\,;\ B, C)$ respectively from $\mathrm{KK}_{oh}(\mathcal{C}_2\,;\ B, C)$ onto $\mathrm{KK}_{oh}(\mathcal{C}_2\,;\ B, C \otimes \mathbb{K})$.

(ii) *The Kasparov $\mathcal{C}$-group $\mathrm{KK}(\mathcal{C};\, A, B) := \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/ \sim_{scp})$ is homotopy invariant in the – stronger – sense that the natural epimorphisms*

$$\mathrm{KK}_c(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C};\, A, B) \to \mathrm{KK}_h(\mathcal{C};\, A, B)$$

and the natural monomorphism $\mathrm{KK}_c(\mathcal{C};\, A, B) \hookrightarrow \mathrm{KK}(\mathcal{C};\, A, B)$ are isomorphisms.

The groups $\mathrm{KK}(\mathcal{C} \otimes \mathrm{CP}(\mathrm{C}[0,1], \mathbb{C});\, \mathrm{C}([0,1], A), B)$, $\mathrm{KK}(\mathcal{C};\, A, B)$ and $\mathrm{KK}(\mathcal{C}([0,1]);\, A, \mathrm{C}([0,1], B))$ are naturally isomorphic.

In particular, our above defined bi-functor

$$\mathrm{KK}_{\mathrm{nuc}}(A, B) = \mathrm{KK}(\mathrm{CP}_{\mathrm{nuc}}(A, B);\, A, B)$$

is homotopy invariant, and coincides with the $\mathrm{KK}_{\mathrm{nuc}}$*-functor of Skandalis* [**726**]. *Therefore* $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ *is stable and is half exact (on semi-split short exact sequences) in each variable.*

(iii) *There are natural isomorphisms*

$$\mathrm{KK}(\mathcal{C};\, A, B_{(1)}) \cong \mathrm{KK}(\mathcal{C};\, A, SB) \cong \mathrm{KK}(\mathcal{C};\, SA, B),$$
$$\mathrm{KK}(\mathcal{C};\, A \otimes \mathbb{K}, B) \cong \mathrm{KK}(\mathcal{C};\, A, B) \cong \mathrm{KK}(\mathcal{C};\, A, B \otimes \mathbb{K})\,.$$

In particular,

$$\mathrm{KK}_{\mathrm{nuc}}(A, B_{(1)}) \cong \mathrm{KK}_{\mathrm{nuc}}(A, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(SA, B)\,.$$

PROOF. To deduce the ???????

??

By the Kasparov product construction of an connection $F_1 \sharp F_2$, the Kasparov module $(E_1 \widehat{\otimes}_B E_2\,,\ \phi_1 \widehat{\otimes}_B \mathrm{id}_{E_2}\,,\ F_1 \sharp F_2)$ exists (in $\mathbb{E}(A, C)$) for $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$, because A is separable. It is an element in $\mathbb{E}(\mathcal{C}_2 \circ \mathcal{C};\, A, C)$, if $(E_1, \phi_1, F_1) \in \mathbb{E}(\mathcal{C};\, A, B)$ and $(E_2, \phi_2, F_2) \in \mathbb{E}(\mathcal{C}_2\,;\, B, C)$, cf. Proposition **??** where $\mathcal{C}_2 \circ \mathcal{C}$ denotes the m.o.c. cone that is generated by the compositions $W \circ V$ for $V \in \mathcal{C}$ and $W \in \mathcal{C}_2$.

Since the connection $F_1 \sharp F_2$ is uniquely defined up to operator homotopy, it defines the desired bi-additive map from $\mathrm{KK}_{oh}(\mathcal{C};\, A, B) \times \mathrm{KK}_{oh}(\mathcal{C}_2\,;\, B, C)$ into $\mathrm{KK}_{oh}(\mathcal{C}_2 \circ \mathcal{C};\, A, C)$.

(ii): By definition two elements $\mathcal{E}_j = (E_j, \phi_j, F_j) \in \mathbb{E}(\mathcal{C};\, A, B)$ $(j = 0, 1)$ are **cobordant** in $\mathbb{E}(\mathcal{C};\, A, B)$, if there are (E, ϕ, G) in $\mathbb{E}(\mathcal{C};\, A, B)$ and a partial isometry $v \in \mathcal{L}(E) = \mathcal{M}(\mathbb{K}(E))$ of degree 0, commuting with $\phi(A)$, such that $[v, G]\phi(A) \subseteq \mathbb{K}(E)$ and that the "restricted" Kasparov modules $(E, \phi, G)_{1-vv^*} := (pE, \phi(\cdot)|pE, pG|pE)$ and $(E, \phi, G)_{1-v^*v}$ are (unitary) equivalent to $\mathcal{E}_0$ respectively $\mathcal{E}_1$.
Then the system (E, ϕ, G, v) is a $\mathcal{C}$–**cobordism** between $\mathcal{E}_0$ respectively $\mathcal{E}_1$ (inside $\mathbb{E}(\mathcal{C};\, A, B)$).

Inspections of the arguments in the proofs of [**73**, sec.17.10] shows that the relation $\sim_c$ on $\mathbb{E}(\mathcal{C};\, A, B)$ is equivalent to the relation "*cobordant*" on $\mathbb{E}(\mathcal{C};\, A, B)$, and that the relations $\sim_c$ is the same as $\sim_{oh}$.

Indeed, the cobordism between 0 and (E, ϕ, F) for a degenerate $(E, \phi, F') \in \mathbb{E}(\mathcal{C};\, A, B)$ that is operator homotopic to (E, ϕ, F) is given by $(E \widehat{\otimes} \ell_2, \phi \widehat{\otimes} 1, G, 1 \widehat{\otimes} T)$ (where $T \in \mathcal{L}(\ell_2)$ is the Toeplitz operator and G is suitable). It is also $\mathcal{C}$–cobordism in $\mathbb{E}(\mathcal{C};\, A, B)$ between 0 and (E, ϕ, F).

(Note that it suffices to prove that elements $\mathcal{E} \in \mathbb{E}(\mathcal{C};\, A, B)$ which are operator homotopic to a degenerate element in $\mathcal{E} \in \mathbb{E}(\mathcal{C};\, A, B)$ are also cobordant to $(0, 0, 0)$, because this implies that $\mathrm{KK}_c(\mathcal{C};\, A, B)$ is a group with the same representatives $(-\mathcal{E})$ for the inverse elements of $[\mathcal{E}]_c$ as for $[\mathcal{E}]_{oh}$, and then, that $\mathcal{E}' \sim_{oh} \mathcal{E}$ implies that $\mathcal{E}' \oplus (-\mathcal{E}) \oplus \mathcal{E}_1 \sim_{soh} \mathcal{E}_2$ for degenerate $\mathcal{E}_1$ and $\mathcal{E}_2$. Thus $\mathcal{E}' \oplus (-\mathcal{E}) \sim_c 0$ and $\mathcal{E} \oplus (-\mathcal{E}) \sim_c 0$. Hence $\mathcal{E}' \sim_{oh} \mathcal{E}$ implies $\mathcal{E}' \sim_c \mathcal{E}$ by the cancelation property of $\sim_c$.

The relation $\sim_{op}$ is different from $\sim_c$ because $\mathbb{E}_{\mathrm{nuc}}(\mathcal{C};\, A, B)/\sim_{op}$ has in general not cancelation.)

Since $\mathrm{KK}_c(\mathcal{C};\, A, B)$ is a sub-semigroup of

$$\mathrm{KK}(\mathcal{C};\, A, B) := \mathrm{Gr}\big(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}\big)$$

that generates $\mathrm{KK}(\mathcal{C};\, A, B)$, it follows that the natural group morphism

$$\mathrm{KK}(\mathcal{C};\, A, B) \to \mathrm{KK}_{oh}(\mathcal{C};\, A, B)$$

is an isomorphism, i.e., our *maximal* group $\mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{Gr}\big(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{scp}\big)$ is naturally isomorphic to

$$\mathrm{KK}_{oh}(\mathcal{C};\, A, B) := \mathrm{Gr}(\mathbb{E}(\mathcal{C};\, A, B)/\sim_{oh}))$$

if A is separable and B is σ-unital.

If the invariance of the classes in $\mathrm{KK}(\mathcal{C};\, A, B)$ under unitary homotopy of its representatives in $\mathbb{E}(\mathcal{C};\, A, B)$ is established for all separable A and all σ-unital B and all (point-norm closed non-degenerate) operator-convex cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$, then Kasparov's proof of the homotopy-invariance works also for $\mathrm{KK}(\mathcal{C};\, A, B)$ and its special cases $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ and $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$:

Then one can use the (formally larger) classes of Kasparov modules $(E, \phi, F) \in \mathbb{E}(\mathcal{C};\, A, B)$ given by by the equivalence relation $\sim_{oh}$ (i.e., given by unitary isomorphisms and operator homotopy) on $\mathbb{E}(\mathcal{C};\, A, B)$.

This allows to use Kasparov products to show finally the homotopy invariance. The Kasparov product reduces its proof to Lemma 8.2.3(vi) as follows:

`shorten/complete above and below given arguments` ??

The homotopy invariance of $\mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{KK}_{oh}(\mathcal{C};\, A, B)$ finally follows from Lemma 8.2.3(vi). Indeed, let

$$\mathcal{E} := (E, \phi, F) = \{(E_t, \phi_t, F_t)\} \in \mathbb{E}(\mathrm{CP}(\mathbb{C}, \mathrm{C}[0,1]) \otimes \mathcal{C};\, A, \mathrm{C}([0,1], B))\,.$$

The Kasparov modules $(E_t, \phi_t, F_t) = \chi_t^B(E, \phi, F)$ define the same element $[(E_t, \phi_t, F_t)] = (\chi_t^B)_*[(E, \phi, F)]$ of $\mathrm{KK}(\mathcal{C};\, A, B)$ for $t \in [0,1]$.

This is because, for $t \in [0,1]$, $[\chi_t - 0] = [\chi_1 - 0] \in \mathrm{KK}(\mathrm{C}[0,1], \mathbb{C})$ and $(\chi_t^B)_*[(E, \phi, F)]$ is the the same as the Kasparov product

$$[\mathcal{E}] \otimes_{B \otimes \mathrm{C}[0,1]} [(B, \chi_t^B, 0)] = [\mathcal{E}] \otimes \tau_B([\chi_t - 0])\,.$$

Notice here that the smallest point-norm closed m.o.c. cone $\mathcal{C}_{\min}(\mathrm{C}[0,1], \mathbb{C})$ that contains all characters

$$\chi_t \colon f \in \mathrm{C}[0,1] \mapsto f(t) \in \mathbb{C}$$

is nothing else all of $\mathrm{CP}(\mathrm{C}[0,1], \mathbb{C}) \cong \mathrm{C}[0,1]^*_+$

The m.o.c. cone $\mathrm{CP}_{in}(B, B)$ of all approximately inner c.p. maps V from B to B is generated by the identity map id_B of B, and has the property that for each point-norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ holds $\mathrm{CP}_{in}(B, B) \circ \mathcal{C} = \mathcal{C}$.

check above and below formula!

Let $\mathcal{C}^{(1)} := \mathrm{CP}_{in}(B,B)\widehat{\otimes}\,\mathrm{CP}(\mathrm{C}[0,1],\mathbb{C}) \subseteq B\otimes$?????? the m.o.c. cone tensor product generated by $\{\, \mathrm{id}_B \otimes \chi_t \,\}$ in $\mathrm{CP}(B\otimes \mathrm{C}[0,1], B)$, i.e., the point-norm closed m.o.c. cone that is generated by $\mathrm{id}_B \otimes \mathrm{CP}(\mathrm{C}[0,1],\mathbb{C})$ and $\mathrm{CP}(\mathrm{C}[0,1],\mathbb{C})$.

This construction reduces the proof of homotopy invariance of general $\mathrm{KK}(\mathcal{C};\, A, B)$ to the check of the homotopy invariance of $\mathrm{KK}(\mathcal{C}^{(1)};\, B[0,1], B)$, which in turn reduces to $\mathrm{KK}(\mathrm{C}[0,1],\mathbb{C}) \cong \mathbb{Z}$.

(iii): ????????

Part (i????) of ??????????

implies in particular, that our $\mathrm{KK}_{\mathrm{nuc}}$-groups are the same as those of Skandalis, [**726**]. Then periodicity (ii) comes now in the same way as it comes for Kasparov's KK-functor.

But there is ???????????? □

The following Proposition 8.2.6 is crucial for the later needed corollaries of the homotopy invariance of *our* groups $\mathrm{Ext}(\mathcal{C};\, A, B)$. Therefore, we give a detailed proof, that uses the natural isomorphism $\mathrm{KK}_c(\mathcal{C};\, A, B_{(1)}) \cong \mathrm{KK}_h(\mathcal{C};\, A, B_{(1)})$ and the isomorphism from $\mathrm{Ext}(\mathcal{C};\, A, B)$ onto the *kernel* ([8]) of the group homomorphism

$$\mathrm{K}_0(H_0(A\otimes\mathbb{K})'\cap \mathrm{Q}^{\mathrm{s}}(B)) \to \mathrm{K}_0(\mathrm{Q}^{\mathrm{s}}(B))\,.$$

PROPOSITION 8.2.6. *Suppose that A is separable and stable, that B is σ-unital and stable, that both are trivially graded, and let $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a point-norm closed matrix operator-convex cone that is countably generated, non-degenerate and faithful.*

(i) $\mathrm{Ext}(\mathcal{C};\, A, B)$ *is* naturally *isomorphic to* $\mathrm{KK}(\mathcal{C};\, A, B_{(1)})$, *such that the isomorphism $\Phi(\mathcal{C};\, A, B)$ from $\mathrm{Ext}(\mathcal{C};\, A, B)$ onto $\mathrm{KK}(\mathcal{C};\, A, B_{(1)})$ satisfies*

$$\Phi(\delta\circ\mathcal{C};\, A, C)\circ \delta_* = (\delta_{(1)})_* \circ \Phi(\mathcal{C}\,; A, B)$$

and

$$\Phi(\mathcal{C}\circ\gamma;\, A_1, B)\circ \gamma^* = \gamma^* \circ \Phi(\mathcal{C}\circ\gamma;\, A_1, B)$$

for isomorphisms $\delta\colon B\to C$ and $\gamma\colon A_1 \to A$.

In particular, $\mathrm{Ext}_{\mathrm{nuc}}(A,B)$ *is naturally isomorphic to* $\mathrm{KK}_{\mathrm{nuc}}(A, B_{(1)})$.

(ii) $\mathrm{Ext}(\mathcal{C}(\mathbb{R}_+);\, A, \mathrm{C}_0(\mathbb{R}_+, B)) \cong 0$.

In particular, $\mathrm{Ext}_{\mathrm{nuc}}(A, \mathrm{C}_0(\mathbb{R}_+, B)) \cong 0$.

(iii) $\mathrm{KK}(\mathcal{C}(\mathbb{R}_+);\, A, \mathrm{C}_0(\mathbb{R}_+, B)) \cong 0$.

(iv) *Let $H_0\colon A \to \mathrm{Q}^{\mathrm{s}}(B)$ the canonical C^*-morphism associated to $\mathcal{C}$ and $S, T \in H(D)'\cap \mathcal{M}(B)$ isometries with $SS^* + TT^* = 1$. Consider the isometries $s := \pi_B(S)$ and $t := \pi_B(T)$ in $H_0(A)'\cap \mathrm{Q}^{\mathrm{s}}(B)$.*

[8] Notice that $\mathrm{Ext}(A,B)$ is not equal to $\mathrm{K}_0(H_0(A\otimes\mathbb{K})'\cap \mathrm{Q}^{\mathrm{s}}(B))$ itself. But one can calculate $\mathrm{Ext}(A,B)$ as quotient by its subgroup $\mathrm{K}_0(\mathrm{Ann}(H_0(A\otimes\mathbb{K}), \mathrm{Q}^{\mathrm{s}}(B)))$. Use Lemma 4.2.20(o) to compare it with our definition.

Then there exists a constant $\gamma(\mathcal{C}) < \infty$ *such that for each unitary* $u \in \mathcal{U}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$ *the geodesic distance* $\mathrm{cel}(sus^* + tt^*)$ *of the unitary* $sus^* + tt^*$ *to* 1 *inside* $\mathcal{U}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$*, satisfies*

$$\mathrm{cel}(sus^* + tt^*) \leq \gamma(\mathcal{C})\,.$$

PROOF. Recall that $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ is singly generated, non-degenerate and faithful if A is separable and B is σ-unital, because $V := \rho(\cdot)b_0$ is a generates $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ as m.o.c. cone if $\rho \in A^*_+$ is a faithful positive functional on A and $b_0 \in B_+$ is a strictly positive element. (The more general m.o.c. cones $\mathrm{CP}_{\mathrm{rn}}(X; A, B)$ are often not countably generated despite $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ itself is a matrix-hereditary sub-cone of $\mathrm{CP}_{\mathrm{nuc}}(A, B)$.)

Notice that $\mathrm{CP}_{\mathrm{nuc}}(A, B) \otimes \mathrm{CP}(C, D) = \mathrm{CP}_{\mathrm{nuc}}(A \otimes C, B \otimes D)$ for all nuclear C^*-algebras C and D by the local characterization in Proposition ??, and that $\mathrm{KK}_{\mathrm{nuc}}(A, B \widehat{\otimes} C) = \mathrm{KK}\big(\mathrm{CP}_{\mathrm{nuc}}(A, B \otimes C);\, A, B \widehat{\otimes} C\big)$ for all (graded) separable nuclear C. Further recall that

$$\mathrm{Ext}_{\mathrm{nuc}}(A, B) := \mathrm{Gr}(\mathrm{SExt}_{\mathrm{nuc}}(A, B)) \cong \mathrm{Ext}(\mathrm{CP}_{\mathrm{nuc}}(A, B);\, A, B)\,,$$

with $\mathrm{SExt}_{\mathrm{nuc}}(A, B) := [\mathrm{Hom}_{\mathrm{l\text{-}nuc}}(A, \mathrm{Q}(B \otimes \mathbb{K}))] \cong \mathrm{SExt}(\mathrm{CP}_{\mathrm{nuc}}(A, B);\, A, B)$, and $\mathrm{KK}_{\mathrm{nuc}}(A, B) = \mathrm{KK}(\mathrm{CP}_{\mathrm{nuc}}(A, B);\, A, B)$ right from the definitions.

We proceed with the $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$ and $\mathrm{Ext}(\mathcal{C};\, \cdot, \cdot)$ for general countably generates point norm closed m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$.

It follows that there is a natural isomorphism from $\mathrm{KK}(\mathcal{C};\, A, B_{(1)})$ onto $\mathrm{Ext}(\mathcal{C};\, A, B)$ for separable stable A, σ-unital stable B, (both trivially graded), with countably generated non-degenerate and faithful m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$. It covers also the cases $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ and $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(X; A, B)$, i.e., it suffices to consider the (more general) case of a non-degenerate faithful countably generated m.o.c. cone $\mathcal{C}$ in place of $\mathrm{CP}_{\mathrm{nuc}}(A, B)$.

(i): Life could be easier if a simply and elementary argument would show that the stabilization $\sim$ of the relation that are given by unitary equivalence and by ψ-"compact" perturbations on the below considered pairs (ψ, q) with $\psi\colon A \to \mathcal{M}(B)$ weakly $\mathcal{C}$-compatible and $q \in \mathcal{M}(B)$ a projection with $q\psi(a) - \psi(a)q \in B$ (equipped with with Cuntz addition $\oplus := \oplus_{S,T}$ with a suitable copy $C^*(S, T)$ of $\mathcal{O}_2$) are the same as the stabilization of relations induced on them by considering them *modulo* $B \otimes \mathbb{K}$ (up to unitary equivalence), i.e., as the corresponding classes of elements that build *our* $\mathrm{Ext}(\mathcal{C};\, A, B)$ defined in Definition 5.8.2. The critical point is, that one has to make sure that the relation $\sim$ is weak enough such that $\pi_B(p_1\phi_1(\cdot)p_1) = \pi_B(p_2\phi_2(\cdot)p_2)$ implies $(\phi_1, p_1) \sim (\phi_2, p_2)$.

In our approach we use that $\mathrm{KK}_c(\mathcal{C};\, A, B_{(1)}) = \mathrm{KK}_h(\mathcal{C};\, A, B_{(1)})$ to get the relation $(\psi_1, p_1) \sim (\psi_2, p_2)$ from the homotopy invariance of $\mathrm{KK}_c(\mathcal{C};\, A, B_{(1)})$.

`check next remark, give cross-ref`

By Lemma 5.9.14, the general defining relations for classes of representing elements (ϕ, p) that build the elements of $\mathrm{Ext}(\mathcal{C};\, A, B)$ are:

$$(\phi_0, p_0) \sim (\phi_1, p_1) \quad \Leftrightarrow \quad \exists\, (\psi_0, 1), (\psi_1, 1),\, \exists\, U \in \mathcal{M}(B): \tag{2.1}$$

$$\forall a \in A: \quad p_0\phi_0(a)p_0 \oplus \psi_0(a) - U^*(p_0\phi_1(a)p_0 \oplus \psi_1(a))U \in B\,. \tag{2.2}$$

Here $\oplus$ means Cuntz-addition in $\mathcal{M}(B)$.

`From Lemma 5.9.14:` Recall:

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a countably generated point-norm closed matrix operator-convex cone. We denote by $\mathcal{S}$ the set of pairs (φ, p), where $\varphi\colon A \to \mathcal{M}(B)$ is a C^*-morphism with $b^*\varphi(\cdot)b \in \mathcal{C}$ for all $b \in B$ and $p \in \mathcal{M}(B)$ is a projection that satisfies $\varphi(a)p - p\varphi(a) \in B$ for all $a \in A$.

We denote by $[\mathcal{S}]$ the set of unitary equivalence classes $[(\varphi, p)]$ by unitaries in $\mathcal{M}(B)$.

An element $(\varphi, p) \in \mathcal{S}$ is **degenerate** if $p\varphi(a) = \varphi(a)p$ for all $a \in A$.

(i) $[\mathcal{S}]$ is a commutative semigroup (with Cuntz-Addition on maps and projections).
(ii) The map $[\varphi, p] \mapsto \pi_B(\varphi(\cdot)p) \in \mathrm{Hom}(A, \mathrm{Q}(B))$ defines an additive semigroup morphism ϑ from $[\mathcal{S}]$ onto $\mathrm{SExt}(\mathcal{C};\, A, B)$. The image of ϑ contains $[0] + \mathrm{SExt}(\mathcal{C};\, A, B)$.
(iii) The map $\mathcal{S} \to \mathrm{Ext}(\mathcal{C};\, A, B)$ induces an equivalence relation $\sim$ on $\mathcal{S}$ that is compatible with Cuntz addition and is generated by the following operations and relations:
  (a) unitary equivalence by unitaries in $\mathcal{M}(B)$, i.e., $(\varphi, p) \sim (\psi, q)$ if there is a unitary $u \in \mathcal{M}(B)$ with $\psi = u^*\varphi(\cdot)u$ and $q = u^*pu$,
  (b) addition $(\varphi \oplus \psi, p \oplus q)$ of *degenerate* elements $(\psi, q) \in \mathcal{S}$ to elements $(\varphi, p) \in \mathcal{S}$, i.e., $(\varphi \oplus \psi, p \oplus q) \sim (\varphi, p)$, and
  (c) unitary perturbation: $(\varphi, p) \sim (\varphi, q)$ if there is $b^* = b \in \mathcal{M}(B)$ such that $\varphi(a)b - b\varphi(a) \in B$ and $(p - e^{-ib}qe^{ib})\varphi(a) \in B$ for all $a \in A$.
(iv) The natural semigroup morphism from $\mathrm{SExt}(\mathcal{C};\, A, B)$ into $\mathrm{Ext}(\mathcal{C};\, A, B)$ is an epimorphism.
(v) If $\mathcal{C}$ is countably generated and non-degenerate, and if $H\colon A \otimes \mathbb{K} \to \mathcal{M}(B)$ is as in Corollary 5.4.4, then the unitary equivalence classes $[(H(\cdot), p)] \in [\mathcal{S}]$ with $p = TT^*$ for some isometry $T \in \mathcal{M}(B)$ build a sub-semigroup $[\mathcal{P}]$ of $[\mathcal{S}]$ such that ϑ maps $[\mathcal{P}]$ onto $\mathrm{Ext}(\mathcal{C}; A, B)$.

Recall that almost directly from the definitions one can see that there are natural isomorphisms $\mathrm{Ext}(\mathcal{C};\, A, B) \cong \mathrm{Ext}(\mathcal{C};\, A \otimes \mathbb{K}, B \otimes \mathbb{K})$ (in particular this happens for $\mathrm{Ext}_{\mathrm{nuc}}$), see cf. Proposition 5.9.25.

One direction of the proof (such as in [**73**, 17.6]) is almost trivial, as the reader can see from the last part of our proof. But for the other direction we make two times very essential use of the homotopy invariance of $\mathrm{KK}(\mathcal{C};\, A, B_{(1)})$, and use the trivial isomorphism $\mathrm{C}([0,1], B_{(1)}) \cong \mathrm{C}([0,1], B)_{(1)}$. Our proof is a compromise between the suggestions in [**73**] before [**73**, prop. 17.6.5] and the detailed exposition in [**389**, chp. 3] (both for ordinary KK and Ext).

We make use of the *stability* of the bi-functors $\mathrm{KK}(\mathcal{C};\,\cdot,\cdot)$ and $\mathrm{Ext}(\mathcal{C};\,\cdot,\cdot)$ to get a constructive picture, that allows to see the $\mathcal{C}$-compatibility of all constructions. In particular, *we suppose from now on that B is stable and σ-unital*, and let $D := A\otimes\mathbb{K}$, $E := \mathrm{Q}(B) := \mathcal{M}(B)/B$ that is naturally isomorphic to $\mathrm{Q}^{\mathrm{s}}(B)$ (because B is stable).

Remark 5.8.7 and ?????????????????? **??**

show that there are natural isomorphisms

$$\mathrm{Ext}_{\mathrm{nuc}}(A,B)\cong\mathrm{Ext}_{\mathrm{nuc}}(D,B)\cong G(H_0\,;\,D,E)\,,$$

respectively,

$$\mathrm{Ext}(\mathcal{C};\,A,B)\cong\mathrm{Ext}(\mathcal{C}\otimes\mathrm{CP}(\mathbb{K},\mathbb{C});\,D,B)\cong G(H_0\,;\,D,E)\,,$$

where $\mathcal{C}$ is naturally extended to $D = A\otimes\mathbb{K}$ as $\mathcal{C}\otimes\mathrm{CP}(\mathbb{K},\mathbb{C})\subseteq\mathrm{CP}(D,B)$, $H_0 = \pi_B\circ H_1$, and $H_1\colon D\to\mathcal{M}(B)$ is an infinite repeat of a faithful non-degenerate representation of D in $\mathcal{M}(\mathbb{K})\subseteq\mathcal{M}(B)$ (compare Corollary 5.4.10 in case $\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(D,B)$), or where $H_1\colon D\to\mathcal{M}(B)$ is constructed from the countably generated cone $\mathcal{C}$ by Corollary 5.4.4. Thus, there is a copy of $\mathcal{O}_2$ unitally contained in the commutant of $H_1(D)$ in $\mathcal{M}(B)$ by Lemma 5.1.2(ii), i.e., $H_0(D)'\cap E$ contains a unitally liftable copy of $\mathcal{O}_2$. By Proposition 4.4.3(i), $G(H_0\,;\,D,E)$ is naturally isomorphic to the kernel of $\mathrm{K}_0(H_0(D)'\cap E)\to\mathrm{K}_0(E)$.

Furthermore, since $C := H_0(D)'\cap E$ contains a unitally liftable copy of $\mathcal{O}_2$, every element of this kernel has a representative $q = p+B\in C$ which comes from a projection $p\in\mathcal{M}(B)$, because $q = q'\oplus 1\oplus 0$ is unitarily equivalent to $1\oplus 0$ by a unitary $u\in\mathcal{U}_0(E) = \pi_B(\mathcal{U}(\mathcal{M}(B)))$ if $[q'] = 0$ in E, but $[q] = [q']$ in $\mathrm{K}_0(C)$, cf. Lemma 4.2.6(iv,b).

Thus the representatives are projections $p\in\mathcal{M}(B)$ with $pH_1(a)-H_1(a)p\in B$ for every $a\in D$. By Lemma 4.2.6(i), the addition in $\mathrm{K}_0(H_0(D)'\cap E)$ coincides with the Cuntz addition (if we use a unital copy of $\mathcal{O}_2$ in E that commutes element-wise with $H_0(D)$). Let

$$\mathcal{P} := \{p\in\mathcal{M}(B)\,;\;\, p = p^* = p^2\,,\;\,[p,H_1(a)]\in B\;\;\forall\, a\in A\,\}\,.$$

The set $\mathcal{P}$ is closed under Cuntz addition defined by a unital copy of $\mathcal{O}_2$ in $H_1(D)'\cap\mathcal{M}(B)$, and it is invariant under unitary equivalence by unitaries in the commutant of $H_1(D)$ in $\mathcal{M}(B)$ and, more generally, by unitaries in $\pi_B^{-1}(H_0(D)'\cap\mathrm{Q}^{\mathrm{s}}(B))$. Therefore, the unitary equivalence classes of projections in $\mathcal{P}$ by unitaries in $H_1(D)'\cap\mathcal{M}(B)$ build a semigroup by Proposition 4.3.2. This semigroup maps additively *onto* $\mathrm{Ext}(\mathcal{C};\,D,B)$ (respectively onto $\mathrm{Ext}_{\mathrm{nuc}}(D,B)$) by the above consideration.

We are now going to describe the equivalence relation which is induced by this additive map into $\mathrm{K}_0(C)$:

By Lemma 4.2.6(iv,b), two representatives p and p' define the same element of $\mathrm{K}_0(H_0(D)' \cap E)$ if and only if the Cuntz sums $0 \oplus p \oplus 1$ and $0 \oplus p' \oplus 1$ are unitarily equivalent modulo B by a unitary $u \in \mathcal{M}(B)$ that can be connected to 1 by a norm-continuous path $\xi \in [0,1] \mapsto u(\xi) \in \mathcal{M}(B)$ of unitaries $u(\xi)$ which all commute element-wise with the image of H_1 modulo B. Another *equivalent condition* is: $0 \oplus p \oplus 1$ and $0 \oplus p' \oplus 1$ are (norm-)homotopic in the set $\mathcal{P}$ (the projections q of $\mathcal{M}(B)$ with $qH_1(a) - H_1(a)q \in B$ for every $a \in D$). (Because the latter implies that their images in $C := H_0(D)' \cap E$ are homotopic inside the projections of C.)

But a more delicate description of the equivalence relation on $\mathcal{P}$ induced by $\mathcal{P} \to \mathrm{K}_0(H_0(D)' \cap E)$ comes from Propositions 5.5.12 and 4.4.3:
The *-morphism $H_0\colon D \to E$ dominates zero, because $H_1(D) + B$ is stable, cf. Proposition 5.5.12(ii). Projections $p, p' \in \mathcal{P}$ define the same class in $\mathrm{K}_0(H_0(D)' \cap E)$, if and only if, there is a unitary $u \in \mathcal{M}(B)$ such that $[u, H_1(a)]$ and $(u^*(p' \oplus 1 \oplus 0)u - (p \oplus 1 \oplus 0))H_1(a)$ are in B for every $a \in D$, cf. Proposition 4.4.3(iii).

By obvious reasons (e.g. consider the given map from $\mathcal{P}$ into $\mathrm{K}_0(H_0(D)' \cap E)$ itself), the corresponding equivalence relations on $\mathcal{P}$ are *compatible* with Cuntz addition.

Now we define a semigroup epimorphism from $([\mathcal{P}], +)$ onto $\mathrm{KK}_{\mathrm{nuc}}(A, B_{(1)}) = \mathrm{KK}_{\mathrm{nuc}}(D, B_{(1)})$. (We show later that this map induces the same equivalence relations as the map from $[\mathcal{P}]$ to $\mathrm{K}_1(C)$.)

As in [**73**, 17.6.4], one can see that the elements of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ (respectively of $\mathrm{KK}_{\mathrm{nuc}}(D, B_{(1)})$) can be represented by $(\mathcal{H}_B \oplus_\infty \mathcal{H}_B, \phi, G)$ where

$$\mathcal{H}_B := \{(b_1, b_2, \ldots)\,;\ b_n \in B, \sum b_n^* b_n \in B\}$$

and the grading is given by $\beta(x, y) = (y, x)$ for $x, y \in \mathcal{H}_B$, ϕ is weakly $\mathcal{C}$-compatible (respectively is *nuclear*), and G is a selfadjoint unitary in $\mathcal{L}(\mathcal{H}_B \oplus_\infty \mathcal{H}_B) = \mathcal{M}(B) \oplus \mathcal{M}(B) = \mathcal{M}(B)_{(1)}$ of degree one. Thus, $G = (U, -U)$, $U = U^* = U^{-1} \in \mathcal{M}(B)$, and $\phi(a)(x, y) = (\phi_1(a)x, \phi_2(a)y)$ satisfies $\psi := \phi_1 = \phi_2$.

The passage to a *selfadjoint* unitary $G \in \mathcal{M}(B)_{(1)}$ respects unitary isomorphism classes and is additive on $\mathbb{E}(\mathcal{C};\, D, B_{(1)})/\approx_u$ (respectively in $\mathbb{E}_{\mathrm{nuc}}(D, B_{(1)})/\approx_u$), the operation is compatible with "compact" perturbations and transforms each homotopy into a homotopy. The same happens with the passage from a graded $B_{(1)}$-module E and $\phi\colon D \to \mathcal{L}(E)$ to $\psi\colon D \to \mathcal{H}_B \oplus \mathcal{H}_B$ with help of Kasparov stabilization: $(E, \phi, F) \oplus ((\mathcal{H}_B)_{(1)}, 0, 1) \approx ((\mathcal{H}_B)_{(1)}, \phi', F')$ for suitable weakly $\mathcal{C}$-compatible $\phi'\colon D \to \mathcal{L}((\mathcal{H}_B)_{(1)})$

???????.

??

But then Hilbert B-module addition becomes Cuntz addition with a unital copy of $\mathcal{O}_2$ with degree zero generators. Thus the equivalence relations from our definition becomes unitary equivalence with an unitary of degree zero and compact perturbation, i.e., passage to an selfadjoint unitary F' with degree $= 1$ and $(F' -$

$F)\phi(D) \subseteq B$. $\mathrm{KK}(\mathcal{C};\, D, B)$ is then again the Grothendieck group of the semigroup which is defined by this equivalence classes.

It follows $\mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B) = \mathcal{M}(B) \oplus \mathcal{M}(B)$, $\phi = \psi \oplus \psi$ and $F = (1-2q, 2q-1)$, where $\psi\colon D \to \mathcal{M}(B)$ is a weakly *nuclear* (possibly degenerate) C^*-morphism and q is a projection in $\mathcal{M}(B)$ with $q\psi(a) - \psi(a)q$ for all $a \in D$. Conversely (E, ϕ, F) is in $\mathbb{E}_{\mathrm{nuc}}(D, B)$ if $E := \mathcal{H}_B \oplus \mathcal{H}_B$, $\phi := \psi \oplus \psi$ and $F = (1-2q, 2q-1)$ for every pair (ψ, q) with *weakly nuclear* (respectively $\psi \in \mathcal{C}$) $\psi\colon D \to \mathcal{M}(B)$ and projection $p \in \mathcal{M}(B)$ such that $q\psi(a) - \psi(a)q$ for all $a \in D$.

The addition in $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ agrees under this identification with the Cuntz addition of the projections q and of the homomorphisms ψ. Isomorphisms become unitary equivalences and "compact" perturbation means passage to a projection $p' \in \mathcal{M}(B)$ with $(p' - p)\psi(D) \in B$. Therefore $(H_1, 1)$ and $(H_1, 0)$ represent the zero element, and we can represent the elements of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ by the elements $(\psi \oplus H_1, q \oplus 1)$. Let $\psi_\tau\colon D \to \mathcal{M}(B)$ be a point-norm continuous family of *weakly nuclear* C^*-morphisms and let $p \in \mathcal{M}(B)$ be a projection such that $\psi_\tau(a) - \psi_0(a)$ and $[\psi_0(a), p]$ are in B for all $a \in D$ and $\tau \in [0,1]$. Then the homotopy invariance of the $\mathrm{KK}_{\mathrm{nuc}}$-functor implies that (ψ_0, p) and (ψ_1, p) represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$.

By Theorem 5.6.2(ii) – applied to $C := H_1(A)$ and $T = \psi \circ (H_1^{-1})$ –, we find a norm-continuous map $t \mapsto U(t)$ into the unitaries of $\mathcal{M}(B)$ such that $U(t)^* H_1(a) U(t) - (\psi(a) \oplus H_1(a)) \in B$ for $t \in \mathbb{R}_+$ and

$$\lim_{t\to\infty} \|U(t)^* H_1(a) U(t) - (\psi(a) \oplus H_1(a))\| = 0$$

for $a \in A$. Thus $(U(0)^* H_1 U(0), q \oplus 1)$ and $(\psi \oplus H_1, q \oplus 1)$ represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)}) \cong \mathrm{KK}(\mathcal{C};\, A, B_{(1)})$. But $(U(0)^* H_1 U(0), q \oplus 1)$ is unitarily equivalent to (H_1, p) where $p := U(0)(q \oplus 1)U(0)^*$, and p is in $\mathcal{P}$. Thus every element of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ can be represented by (H_1, p), where $p \in \mathcal{P}$.

Conversely, the maps

$$p \in \mathcal{P} \mapsto (H_1, p) \mapsto (\mathcal{H}_B \oplus \mathcal{H}_B, H_0 \oplus H_0, (1-2p, 2p-1))$$

define together an additive map from $\mathcal{P}$ onto $\mathbb{E}(\mathcal{C};\, D, B_{(1)})$.

If we use the copy of $\mathcal{O}_2$ in the commutant of $H_1(D)$, then $(H_1, p \oplus 1 \oplus 0)$ is nothing else $(H_1 \oplus H_1 \oplus H_1, p \oplus 1 \oplus 0)$, and represents therefore the same element of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ as it (H_1, p) does.

Let $p, p' \in \mathcal{P}$. If $t \in [0,1] \mapsto u(t)$ is a norm-continuous map into the unitaries which element-wise commutes with $H_1(D)$ modulo B, and if $u(0) = 1$ $p' - u(1)^* p u(1) \in B$ then (H_1, p) and (H_1, p') define the same element of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$ because it defines a homotopy in the *nuclear* Kasparov modules followed by a "compact" perturbation.

So far we have seen that $p \in \mathcal{P} \mapsto (\mathcal{H}_B \oplus \mathcal{H}_B, H_1 \oplus H_1, (1-2p, 2p-1))$ defines a group epimorphism from $\mathrm{Ext}_{\mathrm{nuc}}(D, B)$ onto $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$.

It remains to show that the kernel is trivial, i.e., that $[p] = 0$ in $\mathrm{K}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$ if (H_1, p) represents the zero of $\mathrm{KK}(\mathcal{C};\, D, B_{(1)})$.

If (H_1, p) represents zero, then there is a pair (ψ, q) and a unitary $u_0 \in \mathcal{M}(B)$ such that $H_1 \oplus \psi = u_0 \psi u_0^*$ and $u_0 q u_0^*$ is a $H_1 \oplus \psi$-"compact" perturbation of $p \oplus q$. We add to this relations the pair $(H_1, 1)$. By Proposition 4.3.2 and Theorem 5.6.2(ii) (applied to $C := H_1(A)$ and $T := \psi \circ H_1^{-1}$), we find then unitaries u_1 and u_2 such that $H_1 \oplus (\psi \oplus H_1) = u_1(\psi \oplus H_1)u_1^*$, $u_2^* H_1(a) u_2 - (\psi \oplus H_1)(a) \in B$ for $a \in D$, and such that $u_1(q \oplus 1)u_1^*$ is a $H_1 \oplus (\psi \oplus H_1)$-"compact" perturbation of $p \oplus (q \oplus 1)$.

Now we consider the relations modulo B in $E := \mathrm{Q}^{\mathrm{s}}(B) := \ cM(B)/B$, and let $v_j := \pi_B(u_j)$, $\varphi := \pi_B \circ \psi$, $p' := \pi_B(p)$ and $q' := \pi_B(q)$. Note that $H_0 \oplus H_0 = H_0$, if we use a copy of $\mathcal{O}_2$ in $H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B)$. We get $v_2^* H_0 v_2 = \varphi \oplus H_0$, $v_1^*(1 \oplus v_2)^* H_0 (1 \oplus v_2) v_1 = v_2^* H_0 v_2$ and

$$\big((q' \oplus 1) - v_1^*(p' \oplus (q' \oplus 1))v_1\big) \cdot v_2^* H_0 v_2 = \{0\}\,.$$

It follows that the unitary $w := (1 \oplus v_2) v_1 v_2^*$ and the projection $r := v_2(q' \oplus 1)v_2^*$ are in $H_0(D)' \cap E$, and that $(r - w^*(p' \oplus r)w) H_0(D) = \{0\}$. Let $p_1 := w^*(p' \oplus r)w$. If π denotes the quotient map, then It gives $[p_1] = [r] = 0$ in $K_0(\mathrm{Q}^{\mathrm{s}}(B))$ and $(p_1 - r)) H_0(D) = 0$, $[\pi(p)] = [p'] = [p_1] - [r]$ in $\mathrm{K}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$.

By Proposition 4.4.3(iii), $[p_1] = [r]$ in $\mathrm{K}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$, because H_0 dominates zero. Thus $[\pi(p)] = 0$ in $\mathrm{K}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$.

(ii): The functor

$$Y \mapsto \mathrm{Ext}(\mathcal{C}(Y);\, A, \mathrm{C}_0(Y, B)) \cong \mathrm{KK}(\mathcal{C}(Y);\, A, \mathrm{C}_0(Y, B)_{(1)})$$

is homotopy invariant, because $\mathrm{C}_0(Y, B)_{(1)} = \mathrm{C}_0(Y, B_{(1)})$ and the map $Y \mapsto \mathrm{KK}(\mathcal{C}(Y);\, A, \mathrm{C}_0(Y, C))$ is homotopy invariant for each non-degenerate point-norm closed m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, C)$ for any graded σ-unital graded C^*-algebra C, by Proposition 8.2.5.

(iii): $\mathrm{Ext}(\mathcal{C}(\mathbb{R}_+);\, A, \mathrm{C}_0(\mathbb{R}_+, B)) \cong 0$ follows from (ii), because $\mathrm{C}_0(\mathbb{R}_+, B_{(1)})$ is a contractible.

(iv):

TEXT of (iv):

Let $H_0 \colon A \to \mathrm{Q}^{\mathrm{s}}(B)$ denote the canonical C^*-morphism associated to $\mathcal{C}$ and $S, T \in H(D)' \cap \mathcal{M}(B)$ isometries with $SS^* + TT^* = 1$. Consider the isometries $s := \pi_B(S)$ and $t := \pi_B(T)$ in $H_0(A)' \cap \mathrm{Q}^{\mathrm{s}}(B)$.

Then there exists a constant $\gamma(\mathcal{C}) < \infty$ such that for each unitary $u \in \mathcal{U}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$ the geodesic distance $\mathrm{cel}(sus^* + tt^*)$ of the unitary $sus^* + tt^*$ to 1 inside $\mathcal{U}_0(H_0(D)' \cap \mathrm{Q}^{\mathrm{s}}(B))$, satisfies

$$\mathrm{cel}(sus^* + tt^*) \leq \gamma(\mathcal{C})\,.$$

Proof, To be filled in ?? □

Next Corollary 8.2.7 is important and has to be checked: ?? Move parts of the Cor. (8.b2) to proof !!!

Recall Definitions (4.2) and (4.3) for a C^*-subalgebra $C \subseteq D$ of a C^*-algebra D and an ideal $I \lhd D$ of D: The *derivation C^*-subalgebra* $\mathrm{Der}(C, I) \subseteq E$ is defined by

$$\mathrm{Der}(C, I) := \{e \in D\, ;\; ea - ae \in I \;\; \forall\, a \in C\}\,,$$

and the *normalizer C^*-subalgebra* $\mathcal{N}(C, I) \subseteq E$ is defined as the hereditary C^*-subalgebra

$$\mathcal{N}(C, I) := \{e \in D\, ;\; ea, ae \in I \;\; \forall\, a \in C\}\,.$$

The definitions show that $I \subseteq \mathcal{N}(C, I)$ and that $\mathcal{N}(C, I)$ is a closed ideal of $\mathrm{Der}(C, I)$.

Corollary 8.2.7. *Suppose that A and B are stable C^*-algebras (with trivial gradings $\beta_A = \mathrm{id}_A$ and $\beta_B = \mathrm{id}_B$), A is separable, B is σ-unital and that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a non-degenerate m.o.c. cone that is countably generated.*

Let $H\colon A \to \mathcal{M}(B)$ a non-degenerate C^*-morphism with $\delta_\infty \circ H$ unitarily equivalent to H such that $V_b\colon\; a \in A \mapsto b^* H(a) b$ is in $\mathcal{C}$ for each $b \in B$ and the c.p. maps $\{V_b\, ;\; b \in B\}$ generate $\mathcal{C}$, existing by Corollary 5.4.4.

Further let $S, T \in H(A)' \cap \mathcal{M}(B)$ two isometries with $SS^* + TT^* = 1$ that exist by Remark 5.1.1(8).

Define $H_0\colon A \to \mathcal{M}(B)/B = \mathrm{Q}^{\mathrm{s}}(B)$ by $H_0 := \pi_B \circ H$.

The group $\mathrm{KK}(\mathcal{C};\, A, B)$ is natural isomorphic to the kernel of the group morphism

$$\mathrm{K}_1\big(H_0(A)' \cap \mathrm{Q}^{\mathrm{s}}(B)\big) \to \mathrm{K}_1\big(\mathrm{Q}^{\mathrm{s}}(B)\big) \cong \mathrm{K}_0(B)\,.$$

Alternatively, with $D := \mathcal{M}(B)$ and $C := H(A)$ and $I := B$, we get

$$\mathrm{KK}(\mathcal{C};\, A, B) \cong \mathrm{K}_1\big(\mathrm{Der}(H(A), B)\big)\,.$$

The elements of $\mathrm{KK}(\mathcal{C};\, A, B)$ can be represented by unitaries in $u \in \mathcal{M}(B)$ that satisfy $u \in \mathrm{Der}(H(A), B)$.

Next has to be checked again!
It would be a dream. But not reality?

Two unitaries $u, v \in \mathrm{Der}(H(A), B)$ define the same element of $\mathrm{KK}(\mathcal{C}\,;\, A, B)$, if and only if, there exist $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \mathrm{Der}(H(A), B)$ with $f_j^* = -f_j$ $(j = 1, \ldots, n)$ and

$$(u^* v) \oplus_{S,T} 1 = \exp(f_1) \cdot \ldots \cdot \exp(f_n)\,.$$

In particular, $u \in \mathcal{U}\big(\mathrm{Der}(H(A), B)\big)$ represents the zero element, if and only if, there exist $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \mathrm{Der}(H(A), B)$ with $f_j^* = -f_j$ $(j = 1, \ldots, n)$ and

$$u \oplus_{S,T} 1 = \exp(f_1) \cdot \ldots \cdot \exp(f_n)\,.$$

Proof. A description that is a straight reformulation of the picture coming from the use of Kasparov modules and generalized Fredholm operators is given by classes of elements $F \in \mathrm{Der}(H(A), B)$ with $1 - F^*F, 1 - FF^* \in \mathcal{N}(H(A), B)$, i.e., $F + \mathcal{N}(H(A), B)$ is a unitary in $\mathrm{Der}(H(A), B)/\mathcal{N}(H(A), B)$ and $[F] = [G]$ for $G \in \mathrm{Der}(H(A), B)$ with $1 - G^*G, 1 - GG^* \in \mathcal{N}(H(A), B)$ if and only if $(G^*F \oplus_{S,T} 1) + \mathcal{N}(H(A), B)$ is in $\mathcal{U}_0(\mathrm{Der}(H(A), B)/\mathcal{N}(H(A), B))$. The addition is defined by $[F_1] + [F_2] := [F_1F_2]$.

Equivalently expressed:
$V_j := F_j + \mathcal{N}(H(A), B)$ are unitaries in $\mathrm{Der}(H(A), B)/\mathcal{N}(H(A), B)$.

And $[F_1] = [F_2]$ if and only if $(1 \oplus V_2) - (1 \oplus V_1) \in \mathcal{N}(H(A), B)$

???

Other approach ???

$F_1, F_2 \in \mathrm{Der}(H(A), B)$ with $1 - F_j^*F_j, 1 - F_jF_j^* \in \mathcal{N}(H(A), B)$ for some unitary $F_j + B = V_j \in \mathcal{M}(B)/B$, and $[F_1] = [F_2]$ if and only if

$$1 - (V_1^*V_2 \oplus 1) \in \mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$$

with $\oplus = \oplus_{\pi_B(S),\pi_B(T)}$, i.e., $(V_1 \oplus 1) - (V_2 \oplus 1) \in \mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$.

The equivalence follows from the fact that the hereditary C^*-subalgebra $\mathcal{N}(H(A), B)$ contains B and $\pi_B(\mathcal{N}(H(A), B)) = \mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$.

`to be filled in` ??

By assumptions, A and B are σ-unital, A is stable and H_0 is non-degenerate. It follows that the σ-unital C^*-subalgebras B, $H(A)$ and $H(A) + B$ of $\mathcal{M}(B)$ are stable by Remark 5.1.1(9).

It follows from ???? Prop. **?? ??** that there exist an isometry $S_1 \in \mathcal{M}(B)$ with $S_1^*H(A)S_1 \subseteq B$. Thus, $V := \pi_B(S_1) \in \mathrm{Q^s}(B)$ is an isometry, $V^*V = 1$, that satisfies $V^*H_0(A)V = \{0\}$. In particular, $\mathrm{Ann}(H_0(A)) := \mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$ is a full hereditary C^*-subalgebra of $\mathrm{Q^s}(B)$. It follows

$$\mathrm{K}_*\big((H_0(A)' \cap \mathrm{Q^s}(B))/\,\mathrm{Ann}(H_0(A))\big) \cong \mathrm{kernel}\{\mathrm{K}_*\big(H_0(A)' \cap \mathrm{Q^s}(B)\big) \to \mathrm{K}_*(\mathrm{Q^s}(B))\}\,.$$

We use that $\mathcal{N}(H(A), B)$ is a full hereditary C^*-subalgebra of $\mathcal{M}(B)$ and contains B. If we apply Lemma 4.2.20 to them we obtain that there is a natural isomorphism from $\mathrm{K}_*\big(\mathrm{Der}(H(A), B)/\mathcal{N}(H(A), B)\big)$ onto the *kernel* of the homomorphism $\mathrm{K}_*(\mathrm{Der}(H(A), B) \to \mathrm{K}_*(\mathcal{M}(B)))$ induced by the inclusion $\mathrm{Der}(H(A), B) \subseteq \mathcal{M}(B)$.

Using that $B \subseteq \mathcal{N}(H(A), B) \subseteq \mathrm{Der}(H(A), B)$ we obtain that

$$\mathrm{Der}(H(A), B)/B = H_0(A)' \cap \mathrm{Q^s}(B) \tag{2.3}$$

$$\mathcal{N}(H(A), B)/B = \mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))\,. \tag{2.4}$$

Above we have seen that $\mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$ is full in $\mathrm{Q^s}(B)$. All this together yields

$$\mathrm{Der}(H(A), B)/\mathcal{N}(H(A), B) \cong (H_0(A)' \cap \mathrm{Q^s}(B))/\,\mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$$

and that $\mathrm{Ann}(H_0(A), \mathrm{Q^s}(B))$ is a full hereditary C^*-subalgebra of $\mathrm{Q^s}(B)$ ... MORE !!! ??? □

3. $\mathrm{Ext}(S\mathcal{C};\, A, SB)$ and $\mathrm{KK}(\mathcal{C};\, A, B)$ considered as K_1-groups

We fix now a σ-unital C^*-algebra B – later we suppose in addition that B is stable (which plays in the formal *notation* no role). The separability is then needed for emphour interpretation of the Kasparov product.

$\mathcal{C}$-compatible we need ????

Let A an arbitrary *separable* C^*-algebra and $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a non-degenerate *countably generated* point-norm closed matricial operator-convex cone ("m.o.c. cone"). We define point-norm closed m.o.c. cones $\mathcal{C}(Y) := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(Y))$ and $\mathcal{C}(Z, Y) := \mathcal{C} \otimes \mathrm{CP}(\mathrm{C}_0(Z), \mathrm{C}_0(Y)) \subseteq \mathrm{CP}(\mathrm{C}_0(Z, A), \mathrm{C}_0(Y, B))$ for the *natural extensions* of $\mathcal{C}$ to the tensor products $\mathrm{C}_0(Z, A) = A \otimes \mathrm{C}_0(Z)$ with help of the algebras $\mathrm{C}_0(Z)$. This is compatible with our in Section 1 introduced canonical extensions $\mathcal{C}_{D,E} := \mathcal{C} \otimes \mathrm{CP}(D, E)$ of $\mathcal{C}$ to a m.o.c. cone in $\mathrm{CP}(A \otimes D, B \otimes E)$. In case $Y = \mathbb{R}$ we write sometimes $S\mathcal{C}$ in place of $\mathcal{C}(\mathbb{R})$, but mostly we simply drop the specification to the spaces Y and Z and write simply $\mathcal{C}$ in place of $\mathcal{C}(Y)$ or $\mathcal{C}(Z, Y)$, – if it is visible to which considered spaces the m.o.c. cone has to be canonically extended.

Recall from Chapter 3, Section **??**, that the possible extension of $\mathcal{C}$ to a point-norm closed m.o.c. cone in $\mathrm{CP}(A \otimes M, B \otimes N)$ is *unique* if M and N are simple and nuclear (cf. Chapter 3 `exact refs ??`). This applies in particular to the case where $M = \mathbb{K}$ and $N = \mathbb{K}$. We write also $\mathcal{C}$ for this extension $\mathcal{C} \otimes \mathrm{CP}(\mathbb{K}, \mathbb{K}) \subseteq \mathrm{CP}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$.

It is important to understand clearly that the definitions in Chapter 3 give not "tensor products" in the usual algebraic or topological sense, because it are only point norm closed m.o.c. cones that are "generated" by a set $\mathcal{S}$ of tensor products, and the algebraic hull of this "tensors" is usually not dense in $\mathcal{C}(\mathcal{S})$.

Our below given results show that we only need to know that our functors $Y \mapsto \mathrm{KK}_{\mathrm{nuc}}(A, \mathrm{C}_0(Y, B))$, or $Y \mapsto \mathrm{KK}_{\mathrm{nuc}}(X;\, A, \mathrm{C}_0(Y, B))$ respectively $Y \mapsto \mathrm{KK}(\mathcal{C}(Y);\, A, \mathrm{C}_0(Y, B))$ exists (and are homotopy invariant with respect to the locally compact spaces Y) to get the needed facts for our later applications of $\mathrm{Ext}_{\mathrm{nuc}}$, $\mathrm{Ext}_{\mathrm{nuc}}(X;\, \cdot, \cdot)$, $\mathrm{Ext}(\mathcal{C};\, A, \cdot)$ and to the Rørdam groups $\mathrm{R}(\mathcal{C};\, A, \cdot)$ in the proofs of Theorems B and M.

For the proofs of the general results in Chapter 9 we even need only that $\mathrm{Ext}(\mathcal{C}(\mathbb{R}_+);\, A, \mathrm{C}_0(\mathbb{R}_+, B)) = 0$ (respectively $\mathrm{Ext}_{\mathrm{nuc}}(A, \mathrm{C}_0(\mathbb{R}_+, B)) = 0$, respectively $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, \mathrm{C}_0(\mathbb{R}_+, B)) = 0$). In fact, it would be enough to know that

$$\mathrm{Ext}\big(\mathcal{C}(S^1, \mathbb{R}_+)\, ;\, D \otimes \mathrm{C_b}(S^1)\, ,\, B \otimes \mathrm{C}_0(\mathbb{R}_+)\big) = 0$$

for $D = A \otimes \mathbb{K}$, because e.g. $\mathrm{Ext}_{\mathrm{nuc}}(A, B) = \mathrm{Ext}(\mathcal{C};\, A, B)$ take $\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(A, B)$, and for $\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B)$ take the cone $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ of Ψ_A-Ψ_B-residually nuclear maps from A to B, where A is separable, B is σ-unital, and the action Ψ_B is (downward !) induced by an action of X on a separable non-degenerate C^*-subalgebra $N \subseteq \mathcal{M}(B)$ (which makes sure that $\mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$ is countably generated, cf. Chapter 3).

We know from Prop./Cor. in Chapter 3

`Give exact references !!! ??`

and Corollary 5.4.4 how to find a non-degenerate C^*-morphism $H\colon A \otimes \mathbb{K} \to \mathcal{M}(B \otimes \mathbb{K})$ which determines the point-norm closure of $\mathcal{C}$ – in the case of, closed with respect to the point-norm topology, countably generated and non-degenerate matrix operator-convex cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ and of separable A and σ-unital B.

`find ref's for above ??? ??`

For example, if A and B are both separable, then $\mathcal{C} = \mathrm{CP}(A, B)$ satisfies our requirements on $\mathcal{C}$, but $\mathrm{CP}(A, B)$ is in general not singly generated if A is separable and B is only σ-unital.

We know from Corollary 5.6.1, cf. also Remark 5.1.1(8) and Lemma 5.1.2, that we can find a strictly continuous unital *-monomorphism $\mathcal{M}(\mathbb{K}) \hookrightarrow \mathcal{M}(B \otimes \mathbb{K})$ and

Next H_1 is only related to $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(D, B)$.
we use also the universal $H_1\colon D := A \otimes \mathbb{K} \to\to \mathcal{M}(B)$ in ''general position'' in the sense of Definition 3.3.1,
related to some specified $\mathcal{C} \subseteq \mathrm{CP}(A, B)$
that is ''non-degenerate'' and countably generated.

(... and) for $D := A \otimes \mathbb{K}$ a faithful *-monomorphism $H_1\colon D \to \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B \otimes \mathbb{K})$ such that H_1 is universal for the cone $\mathrm{CP}_{\mathrm{nuc}}(D, B)$ and that there is a copy of $\mathcal{O}_2$ unitally contained in $H_1(D)' \cap \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B \otimes \mathbb{K})$. Here $D := A \otimes \mathbb{K}$.

The general bi-module to be considered for non-degenerate m.o.c. cones $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is given by the universal $H_0\colon D \to \mathcal{M}(B \otimes \mathbb{K})$ in general position – in the sense of Definition 3.3.1, with the property the c.p. maps $d \in D \mapsto e^* H_0(d) e \in B \otimes \mathbb{K}$ generates the natural extension of $\mathcal{C}$ to $\mathcal{C} \otimes \mathrm{CP}(\mathbb{K}, \mathbb{K}) \subseteq \mathrm{CP}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$.

To simplify notation, let us assume again that B is stable in this section, i.e., that $B \cong B \otimes \mathbb{K}$. Further we suppose from now on that B is σ-unital, that $A \cong D := A \otimes \mathbb{K}$ is stable and separable.

We know that there is a given non-degenerate C^*-morphism $H\colon D \to \mathcal{M}(B)$ with $\delta \circ H$ unitarily equivalent to H, such that H corresponds to a non-degenerate countably generated m.o.c. cone $\mathcal{C}$ in the sense of Corollary 5.4.4. We fix H and $\mathcal{C}$ from now on.

Suppose from now on that Y is a closed subspace of $\mathbb{R}^2$. There is a natural unital strictly continuous embedding

$$\mathcal{M}(B) \cong 1 \otimes \mathcal{M}(B) \subseteq \mathrm{C}_{\mathrm{b,st}}(Y \cup \{\infty\}, \mathcal{M}(B)) \subseteq \mathcal{M}(\mathrm{C}_0(Y, B)) .$$

Let $E_Y := \mathcal{M}(\mathrm{C}_0(Y, B))/\, \mathrm{C}_0(Y, B)$ and $H_Y(a) := \pi_{\mathrm{C}_0(Y,B)}(H(a)) \in E_Y$ for $a \in D$. Then $B \cong \mathrm{C}_0(\{0\}, B)$, $H_0 := H_{\{0\}} = \pi_B \circ H$, $H_Y(D)' \cap E_Y$ contains a unital copy of $\mathcal{O}_2$ that comes from a copy of $\mathcal{O}_2$ which is unitally contained in $(1 \otimes H(D))' \cap \mathcal{M}(\mathrm{C}_0(Y, B))$. In particular, $[H_Y \oplus H_Y] = [H_Y]$ for the unitary equivalence classes. Further let $\mathcal{C} \subseteq \mathrm{CP}(D, B)$ denote the point-norm closed m.o. convex cone, that is generated by the maps $V_b \colon a \in D \mapsto b^* H(a) b$ ($b \in B$), and let

$$\mathcal{C}(Y) := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(Y)) \;\subseteq\; \mathrm{CP}(D, B \otimes \mathrm{C}_0(Y))$$

denote the natural extension of $\mathcal{C}$ to a cone in $\mathrm{CP}(D, \mathrm{C}_0(Y, B))$ (cf. Definition 3.6.16 and Corollary 3.6.20. Don't mix it up with a sort of completion of algebraic sums of tensors!).

Then $\mathrm{Ext}(\mathcal{C}(Y);\, A, \mathrm{C}_0(Y, B)) \cong G(H_Y\,; D, E_Y\,)$ by Proposition **??** and Corollary 5.4.4 by putting $\mathcal{C}(Y)$ and $H_Y \colon D \to E_Y$ in place of $\mathcal{C}$ and $H \colon D \to \mathcal{M}(B)$.

Now we introduce a construction that is needed in the proof of Lemma 8.3.2 and Corollary 8.3.3. The Lemma 8.3.2 will be used in the proof of Theorems B(i,ii) and M(i,ii) in Chapter 9.

Note that $\mathrm{Q^s}(SB) = \mathrm{Q^s}(\mathrm{C}_0(\mathbb{R}, B)) = E_{\mathbb{R}}$ is naturally isomorphic to the pull-back construction of the split epimorphisms $\pi_0^+ \colon E^{(+)} \to E^{(0)}$ and $\pi_0^- \colon E^{(-)} \to E^{(0)}$ at $t = 0$, where $E^{(+)} := E_{\mathbb{R}_+} = \mathrm{Q^s}(\mathrm{C}_0(\mathbb{R}_+, B))$, $E^{(-)} := E_{\mathbb{R}_-} = \mathrm{Q^s}(\mathrm{C}_0(\mathbb{R}_-, B))$ and $E^{(0)} := E_{\{0\}} = \mathrm{Q^s}(B)$:

This results from the natural isomorphism $\mathrm{C}_{\mathrm{b,st}}(Y, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C}_0(Y, B))$ in cases $Y = \mathbb{R}$, $Y = \mathbb{R}_+$, $Y = \mathbb{R}_-$ and $Y = \{0\}$, by dividing by $\mathrm{C}_0(Y, B)$. The restriction and evaluation maps are induced by the corresponding restriction and evaluation of strictly continuous functions:

$H \colon D \to \mathcal{M}(B) \cong 1 \otimes \mathcal{M}(B)$ goes into the constant functions of $\mathrm{C}_{\mathrm{b,st}}(Y, \mathcal{M}(B))$ and then down to $E_Y = \mathrm{Q^s}(\mathrm{C}_0(Y, B))$. Therefore we must not always distinguish the various restrictions of "constant" elements in $\mathcal{M}(B) \subseteq \mathrm{C_b}(Y, \mathcal{M}(B)) \subseteq \mathrm{C}_{\mathrm{b,st}}(Y, \mathcal{M}(B))$.

But the reader should not misunderstand the terminology "constant", "restriction" and "evaluation": The point-"evaluations" $\pi_t(b)$ can all be zero but $b \in \mathrm{Q^s}(SB)$ is non-zero, because that says only that b has a representative $c \in \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B))$ such that $c(t) \in B$ for every t. In particular this happens for all $c \in \mathrm{C_b}(\mathbb{R}, B)$.

Notice that the C^*-subalgebra of elements

$$c \in \mathcal{M}(\mathrm{C_b}(X, B)) \cong \mathrm{C}_{\mathrm{b,st}}(X, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C}_0(X, B))$$

with $\pi_x(c) \in B$ for all $x \in X$ for a non-discrete second countable locally compact Hausdorff spaces X is always bigger than $\mathrm{C_b}(X, B)$ if B is σ-unital and stable.

The criterium for $c \in \mathrm{C_b}(X, B)$ is $\pi_x(c) \in B$ for all $x \in X$ *and* $x \mapsto \|\pi_x(c)\|$ is an upper semi-continuous

As we have seen in Chapter 7, this does not imply that c is in $\mathrm{C_b}(\mathbb{R}, B)$, and, in particular, it does not mean that c is in $\mathrm{C_0}(\mathbb{R}, B)$. For example, the natural embedding $\epsilon\colon \mathcal{M}(B) \to \mathrm{Q^s}(\mathrm{C_0}(Y, B))$ of the constant functions into the stable corona of $\mathrm{C_0}(Y, B)$ will be defined as $b \in \mathcal{M}(B) \mapsto b + \mathrm{C_0}(Y, B)$, i.e. $\epsilon(\mathcal{M}(B)) = \mathcal{M}(B) + \mathrm{C_0}(Y, B)$. It is faithful for $Y = \mathbb{R}$ and $Y = \mathbb{R}_+$, but the kernel of $\pi_t \circ \epsilon$ is B for every $t \in Y$. But this is just the nice property that we can *explore* now:

Remark 8.3.1. The algebra $E_{\mathbb{R}} := \mathrm{Q^s}(SB)$ is a weak variant of a sort of "pull-back" of epimorphisms $\pi_0^+\colon E^{(+)} \to E^{(0)}$ and $\pi_0^-\colon E^{(-)} \to E^{(0)}$ if B is σ-unital, where we use the natural identities and epimorphisms from $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_-, B)) = \mathcal{M}(\mathrm{C_b}(\mathbb{R}_-, B)) \to \mathcal{M}(B)$ and $\mathcal{M}(\mathrm{C_0}(\mathbb{R}_+, B)) = \mathcal{M}(\mathrm{C_b}(\mathbb{R}_-, B)) \to \mathcal{M}(B)$

We can define for unitaries U in $\mathcal{M}(B)$ that commute element-wise modulo B with $H(D)$, i.e., with $UH(d) - H(d)U \in B$ for all $d \in D$, in an almost trivial way a weakly *nuclear* (respectively weakly $\mathcal{C}$-*compatible*) extensions $h_U\colon D \to E_{\mathbb{R}}$ if H is weakly *nuclear* (respectively H is weakly $\mathcal{C}$-*compatible*).

More generally, we find in $\mathrm{Ext}(\mathcal{C}_H\,;\, D, SB) = G(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$ extensions by the following method:

$E_{\mathbb{R}}$ is in a natural way isomorphic to the subalgebra of $E^{(-)} \oplus E^{(+)}$ which consists of the pairs (e, f) with $\pi_0^-(e) = \pi_0^+(f)$, i.e., $e = \pi_-(g_-) := \pi_B(g_-(0))$, $f = \pi_+(g_+) := \pi_B(g_+(0))$ for

$$g_\alpha \in \mathrm{C_{b,st}}(\mathbb{R}_\alpha, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(\mathbb{R}_\alpha, B))$$

$(\alpha \in \{+,-\})$, and the restriction maps $g_\alpha \mapsto g_\alpha|0 := g_\alpha(0)$. Notice that $\pi_B(g_-(0)) = \pi_B(g_+(0))$ is equivalent to $g_+(0) - g_-(0) \in B$.

Is $E_{\mathbb{R}}$ really the pull-back of $E^{(-)}$ and $E^{(+)}$?
Is $C^{(0)} := H_{\mathbb{R}}(D)' \cap \mathcal{M}(SB)$ the pull-back of C_+ and C_- with restriction epimorphisms $C_\alpha \to C|0$ $(\alpha \in \{-,+\})$, for

$$C_\alpha^{(0)} := H_{\mathbb{R}_\alpha}(D)' \cap \mathcal{M}(\mathrm{C_0}(\mathbb{R}_\alpha, B))$$

and

$$C^{(0)}|0 = H(D)' \cap \mathcal{M}(B)\,.$$

The map $c \in C^{(0)} \to c(0) \in C^{(0)}|0$ is a splitting epimorphism via $\lambda\colon C^{(0)}|0 \to C^{(0)}$ defined by $\lambda(d)(t) := d \in C^{(0)}|0$ for all $t \in \mathbb{R}$.

Is it the same for $\mathcal{D}_X := \mathrm{Der}(H_X(D); \mathrm{C_0}(X, B))$, with $X = \mathbb{R}$, $X = \mathbb{R}_+$ $X = \mathbb{R}_-$ or $X = \{0\}$?

What about $\mathrm{Der}(H_{\mathbb{R}}(D);K)/\,\mathrm{C}_0(\mathbb{R},B)$ with

$$K := \{f \in \mathrm{C_b}(\mathbb{R},B)\,;\, \lim_{t\to-\infty} \|f\| = 0\}\,.$$

Let $U \in \mathrm{Der}(H(D),B) \subseteq \mathcal{M}(B)$ a unitary, and let $h_U^{\alpha}\colon D \to E^{(\alpha)}$, for $\alpha \in \{+,-\}$, be defined by $h_U^{(-)}(a) := H(a)|\mathbb{R}_-$ and $h_U^{(+)}(a) := UH(a)U^*|\mathbb{R}_+$ for $a \in D$, where we use the natural inclusions of $\mathcal{M}(B)$ into $E_{\mathbb{R}}$, $E^{(-)}$ and $E^{(+)}$.

Then $\pi_0^+ h_U^{(+)}(a) = H(a) = \pi_0^- h_U^{(-)}(a)$ for $a \in D$. Thus, $h_U^{(+)}$ and $h_U^{(-)}$ define a homomorphism $h_U\colon D \to E_{\mathbb{R}}$.

If we use the – a bit inaccurate – restriction notation, we can express this by $h_U(a)|\mathbb{R}_- = H(a)|\mathbb{R}_-$ and $h_U(a)|\mathbb{R}_+ = UH(a)U^*|\mathbb{R}_+$ for $a \in D$.

It turns out that h_U has a completely positive contractive lift $T\colon D \to \mathcal{M}(SB) \cong \mathrm{C_{b,st}}(\mathbb{R},\mathcal{M}(B))$ that is 2-step-dominated by $H\colon D \to \mathcal{M}(B) \subseteq \mathrm{C_{b,st}}(\mathbb{R},\mathcal{M}(B))$:
Let $f \in \mathrm{C_b}(\mathbb{R})$ be defined by $f(t) := 0$ for $t \le 0$, $f(t) := t$ on $[0,1]$ and $f(t) := 1$ for $t \ge 1$, further let $T(d)(t) := (1-f(t))H(d) + f(t)UH(d)U^*$.

Thus $T(\cdot) = e_1^* H(\cdot) e_1 + e_2^* H(\cdot) e_2$ for $e_1 := \sqrt{1-f}$ and $e_2 := U^*\sqrt{f}$, and T is weakly *nuclear* if H is weakly *nuclear* (respectively T is weakly $\mathcal{C}$-*compatible* if H is weakly $\mathcal{C}$-*compatible*).

If $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally and if $[H] = [H] + [H]$ (which is the case by our assumptions), then there are isometries $s_1, s_2 \in H(D)' \cap \mathcal{M}(B)$ which are canonical generators of $\mathcal{O}_2$, cf. Proposition 4.3.5(iii). Then $T(\cdot) = I_U^* H(\cdot) I_U$ for the isometry $I_U = (1-f)^{1/2}s_1 + f^{1/2}s_2U^*$.

Since $UH(d)U^* - H(d) \in B$, $T(d) - H(d)$ is an element of the ideal $I_1 := \{b \in \mathrm{C_b}(\mathbb{R},B)\colon b(t) = 0\,\forall t \le 0\}$, and $T(d) - UH(d)U^*$ is a element of the ideal $I_2 := \{b \in \mathrm{C_b}(\mathbb{R},B)\colon b(t) = 0\,\forall t \ge 1\}$. But $(I_1 + SB)/SB = \mathrm{Q}(\mathbb{R}_+,B)$ and $(I_2+SB)/SB = \mathrm{Q}(\mathbb{R}_-,B)$, where we identify $\mathrm{Q}(\mathbb{R},B)$ naturally with $\mathrm{Q}(\mathbb{R}_-,B) \oplus_{\ell_\infty} \mathrm{Q}(\mathbb{R}_+,B)$ and consider it as an ideal of $E_{\mathbb{R}}$. The natural epimorphism $E_{\mathbb{R}} \to E^{(-)}$ maps $\mathrm{Q}(\mathbb{R}_+,B)$ to zero and $E_{\mathbb{R}} \to E^{(+)}$ maps $\mathrm{Q}(\mathbb{R}_-,B)$ to zero.

One can see that $T(d) + SB = h_U(d)$, because this equality can be calculated separately in $E^{(-)}$ and $E^{(+)}$.

So, if $H(D)' \cap \mathcal{M}(B)$ contains a unital copy of $\mathcal{O}_2$ and if the unitary group of $\mathcal{M}(B)$ is connected, we get a map $U \mapsto h_U$ from the intersection $\mathcal{U}_0(\mathrm{Q}(B)) \cap H_0(D)'$ into $S(H_{\mathbb{R}}\,;\,D,E_{\mathbb{R}})$, where $H_0 := \pi_B \circ H$.

Thus we have seen that, in the case where B is stable and σ-unital and D is stable and separable, for every unitary $U \in \mathcal{M}(B)$ with $UH(d) - H(d)U \in B$ for all $d \in D$,
the homomorphism h_U defines an element $[h_U]$ of $S(H_{\mathbb{R}}\,;\,D,E_{\mathbb{R}})$ and satisfies $h_U(d) - H(d) \in J$ for every $d \in D$, where J is the ideal of $\mathrm{Q^s}(SB)$ which is defined by $\mathrm{Q}(\mathbb{R}_+,B)$.

It follows that $[h_U \oplus H_{\mathbb{R}}]$ is an element of $\mathrm{Ext}(\mathcal{C}(H);\, D, SB) = G(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$.

With the above introduced notations J, E_Y, H_Y, $F(Y)$ and H_0 we get the following lemma.

LEMMA 8.3.2. *Suppose that D is separable and stable, that B is σ-unital and stable, and that $H\colon D \to \mathcal{M}(B)$ is a non-degenerate faithful C^*-morphism with $\delta_\infty \circ H$ unitarily equivalent to H.*

Furthermore, suppose that the following groups for $\mathbb{R}_+ := [0,\infty)$ are trivial:

$$\mathrm{Ext}(\mathcal{C};\, D, \mathrm{C}_0(\mathbb{R}_+, B)) \;=\; G(H_{\mathbb{R}_+}\,;\, D, E_{\mathbb{R}_+}) \;=\; \{[H_{\mathbb{R}_+}]\} \;\cong\; 0\,.$$

Let $[g], [T] \in S(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$ with $[T] = [H_{\mathbb{R}}]$, and let $s_1, s_2 \in E_{\mathbb{R}}$ canonical generators of a copy of $\mathcal{O}_2$ that is unitally contained in $E_{\mathbb{R}}$. Then:

(i) *There exists a unitary $u \in E_{\mathbb{R}}$ such that*

$$u^*T(a)u - (g(a) \oplus_{s_1,s_2} T(a)) \in J \quad \textit{for every } a \in D\,,$$

where J is the image of the natural embedding of $\mathrm{Q}(\mathbb{R}_+, B)$ into

$$\mathrm{Q}(\mathbb{R}, B) \subseteq E_{\mathbb{R}}\,.$$

(ii) *If $\sigma\colon \mathbb{R} \to \mathbb{R}$ is an order preserving homeomorphism of $\mathbb{R}$, then $[\widehat{\sigma} \circ (g \oplus T)] = [g \oplus T]$ in $S(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$.*

This applies to the case where $H\colon D \to \mathcal{M}(B)$ corresponds to a countably generated non-degenerate m.o.c. cone $\mathcal{C}$ – as e.g. to $\mathcal{C} := \mathrm{CP}_{\mathrm{nuc}}(A, B)$ if A is separable and B is σ-unital – because then $\mathrm{Ext}(\mathcal{C};\, D, \mathrm{C}_0(\mathbb{R}_+, B)) \cong 0$, by Corollary 5.9.21 and Proposition 8.2.6.

PROOF. We keep the notations of Remark 8.3.1. The Cuntz addition will be taken at first with respect to fixed isometries s_1 and s_2 in $H_{\mathbb{R}}(D)' \cap E_{\mathbb{R}}$ which generate a copy of $\mathcal{O}_2$ in $H_{\mathbb{R}}(D)' \cap E_{\mathbb{R}}$. Then $H_{\mathbb{R}} \oplus H_{\mathbb{R}} = H_{\mathbb{R}}$.

We show at the end of the proof of Part (i) the independence of the result from the chosen representatives H for for $[H_{\mathbb{R}}]$ in $G(H_{\mathbb{R}};\, D, E_{\mathbb{R}})$ and from the chosen isometries s_1, s_2 in $H_{\mathbb{R}}(D)' \cap E_{\mathbb{R}}$ with $s_1s_1^* + s_2s_2^* = 1$.

(i): Recall that $\mathrm{C}_0(\mathbb{R}_\alpha, B) \cong \mathrm{C}_0((0,1], B)$,

$$E^{(\alpha)} := E_{\mathbb{R}_\alpha} = \mathrm{Q}^{\mathrm{s}}(\mathrm{C}_0(\mathbb{R}_\alpha, B))$$

for $\alpha \in \{-,+\}$ and $E^{(0)} = \mathrm{Q}^{\mathrm{s}}(B)$.

If $g\colon D \to E_{\mathbb{R}} := \mathrm{Q}^{\mathrm{s}}(SB)$ defines an element in $S(H_{\mathbb{R}}\,, D, E_{\mathbb{R}})$, then this means that there is an isometry $s \in E_{\mathbb{R}}$ with $s^*H_{\mathbb{R}}(d)s = g(d)$ for all $d \in D$, i.e., such $ss^* \in H_{\mathbb{R}}(D)' \cap E_{\mathbb{R}}$.

In fact we know at the beginning only that, by definition of $\mathrm{SExt}(\mathcal{C}[\mathbb{R}];\, D, SB)$,

unify notations for $\mathcal{C}[X]$!!!

there is a c.p. contraction $V \colon D \to \mathcal{M}(SB)$ with $\pi_{SB} \circ V = g$ and $a^* V(\cdot) a \in \mathcal{C}[\mathbb{R}]$ for each $a \in \mathrm{C}_0(\mathbb{R}, B) =: SB$. From that we have obtained in Chapters 3 and 4 that there exists an isometry $T \in \mathcal{M}(SB)$ with $T^* H(d) T - V(d) \in SB$ for all $d \in D$. Thus $s = \pi_{SB} \circ T$ has the desired property.

If we restrict now $g(d)$ and s to $\mathbb{R}_-$ and $\mathbb{R}_+$, we get isometries $s^- \in E^{(-)}$, $s^+ \in E^{(+)}$ and homomorphisms $g^- \colon D \to E^{(-)}$ and $g^+ \colon D \to E^{(+)}$ such that $(s^\alpha)^* H_\alpha s^\alpha = g^\alpha$ for $\alpha \in \{+,-\}$ where $H_+ := H_{[0,\infty)}$ and $H_- := H_{(-\infty,0]}$ are the "restrictions" of $H_\mathbb{R}$. Thus $[g^\alpha] \oplus [H_\alpha] \in G(H_\alpha, D, E^{(\alpha)})$ for $\alpha \in \{+,-\}$. By the definition of $E^{(+)}$ and $E^{(-)}$ and by assumption, we get $[g^\alpha \oplus H_\alpha] = [H_\alpha]$. But this means that there are unitaries $U^+ \in E^{(+)}$ and $U^- \in E^{(-)}$ such that $g^\alpha \oplus H_\alpha = (U^\alpha)^* H_\alpha U^\alpha$ for $\alpha \in \{+,-\}$.

If we let $H_0 := \pi_B \circ H$ we get in particular

$$(\pi_0^+(U^+))^* H_0 \pi_0^+(U^+) = (\pi_0^-(U^-))^* H_0 \pi_0^-(U^-)$$

at zero. From now on we use the shorter (but by Remark 8.3.1 somehow inaccurate) notation

$$U^+(0) := \pi_0^+(U^+) \quad \text{and} \quad U^-(0) := \pi_0^-(U^-).$$

Thus $U^+(0) U^-(0)^* \in \mathrm{Q^s}(B)$ is a unitary in the commutant $H_0(D)' \cap \mathrm{Q^s}(B)$ of $H_0(D)$ in $\mathrm{Q^s}(B)$. $U^-(0)$ is in the image of the unitaries of $E^{(-)}$ by π_0^- and $U^+(0)$ is in the image of the unitaries of $E^{(+)}$ by π_0^+.

We have $\mathrm{K}_*(\mathcal{M}(\mathrm{C}_0(\mathbb{R}_+, B))) = 0$, because B is stable. This follows from Lemma 5.1.2(ii) as in the proof of Lemma 4.6.5(ii), or from the contractibility of the unitary groups of multiplier algebras of stable algebras, [**557**], [**180**]. Since $\mathrm{C}_0(\mathbb{R}_+, B) \cong \mathrm{C}_0(\mathbb{R}_-, B)$ is contractible, we get $\mathrm{K}_1(E^{(+)}) = 0$ and $\mathrm{K}_1(E^{(-)}) = 0$ from the 6-term exact sequence of K-theory. Therefore the epimorphisms π_0^+, respectively π_0^-, maps unitaries of $E^{(+)}$, respectively of $E^{(-)}$, into the unitaries u of $E^{(0)} := \mathrm{Q^s}(B)$ with $[u] = 0$ in $\mathrm{K}_1(\mathrm{Q^s}(B))$. Thus $[U^+(0) U^-(0)^*] = [U^+(0)] - [U^-(0)] = 0$ in $\mathrm{K}_1(\mathrm{Q^s}(B))$, and, therefore, by Lemma 4.2.6(v,2), $(U^-(0) U^-(0)^*) \oplus 1$ is in the connected component of 1 in the unitaries of $\mathrm{Q^s}(B)$. Let $V \in \mathcal{M}(B)$ a unitary lift of $(U^+(0) U^-(0)^*) \oplus 1$. Then $H(a) V - V H(a) \in B$ for $a \in D$, and we can form the *-monomorphism $h_V \colon D \to E_\mathbb{R}$ with $[h_V] \in S(H_\mathbb{R}, D, E_\mathbb{R})$ and $h_V(a) - H_\mathbb{R}(a) \in J$.

Let $W := V + \mathrm{C}_0(\mathbb{R}_+, B)$ in $E^{(+)}$, then

$$\pi_0^+(W) = \pi_B(V) = (U^+(0) U^-(0)^*) \oplus 1\,.$$

The unitaries $(U^- \oplus 1) \in E^{(-)}$ and $W^*(U^+ \oplus 1) \in E^{(+)}$, satisfy

$$U^-(0) \oplus 1 = \pi_0^-(U^- \oplus 1) = \pi_0^+(W)^* (U^+(0) \oplus 1) = \pi_0^+(W^*(U^+ \oplus 1)).$$

Therefore they define a unitary $u_0 \in E_\mathbb{R} \subseteq E^{(-)} \oplus E^{(+)}$.

Let $w \in \mathcal{O}_2$ with $w^*((a \oplus b) \oplus c) w = a \oplus (b \oplus c)$, and let $u := u_0 w$. For the Cuntz addition $\oplus$ and its transformation rules we refer to Chapter 4 for every details.

Since $H_0 \oplus H_0 = H_0$, we get from our definition of h_V in Remark 8.3.1 that

$$g^-(a) \oplus H_-(a) = w^*(U^- \oplus 1)^*(h_V(a)|\mathbb{R}_-)(U^- \oplus 1)w$$

and

$$g^+(a) \oplus H_+(a) = w^*(U^+ \oplus 1)^* V^*(h_V(a)|\mathbb{R}_+)V(U^+ \oplus 1)w\,.$$

This means that $g \oplus H_0 = u^* h_V u$ with $u \in \mathcal{U}(E_{\mathbb{R}})$ as above defined. Therefore, $(g(a) \oplus H_0(a)) - u^* H_0(a) u = u^*(h_V(a) - H_0(a))u$ is in J for $a \in D$.

We show the independence from the particular choice of $T \in [H_{\mathbb{R}}]$ and from the generators s_1, s_2 of $\mathcal{O}_2$: Let $T\colon D \to \mathrm{Q}^{\mathrm{s}}(SB)$ another representative of $[H_{\mathbb{R}}]$ and t_1, t_2 isometries in $E_{\mathbb{R}} = \mathrm{Q}^{\mathrm{s}}(SB)$ which generate a copy of $\mathcal{O}_2$. Then there are unitaries $u_1, u_2 \in E_{\mathbb{R}}$ such that $T = u_1^* H_{\mathbb{R}} u_1$ and $u_2 t_j = s_j$, $j = 1, 2$. Thus, with $\oplus = \oplus_{s_1,s_2}$, $g \oplus_{t_1,t_2} T = (1 \oplus u_1)^* u_2 (g \oplus H_{\mathbb{R}}) u_2^* (1 \oplus u_1)$. If $u^* h_V u = g \oplus H_{\mathbb{R}}$, then $u_3^*(u_1^* h_V u_1) u_3 = g \oplus_{t_1,t_2} T$ for $u_3 := u_1^* u u_2^* (1 \oplus u_1)$, and $T(a) - u_1^* h_V(a) u_1 \in J$ for $a \in D$, because J is an ideal.

(ii): Let $\sigma_s(t) := st + (1-s)\sigma(t)$ for $s \in [0,1]$ and $t \in \mathbb{R}$. Since $\sigma\colon \mathbb{R} \to \mathbb{R}$ is order preserving, the σ_s define a homotopy $s \to \rho_s := \widehat{\sigma_s}|\,\mathrm{C}_0(\mathbb{R}, B) \in \mathrm{Aut}(\mathrm{C}_0(\mathbb{R}, B))$ between $\widehat{\sigma}|\,\mathrm{C}_0(\mathbb{R}, B) = \rho_0$ and $\mathrm{id} = \rho_1$, where $\widehat{\sigma_s}(f) := f \circ \sigma_s$ for $f \in \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$. Then $[\widehat{\sigma_s} \circ h] = (\rho_t)_*([h])$ in $\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB) \cong \mathrm{KK}(\mathcal{C};\, D, (SB)_{(1)})$, by definition of $(\rho_t)_* \in \mathrm{Aut}(\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, A, B))$.

Let $\tau_s := (\rho_s)_{(1)}$, i.e., $\tau_s(f, g) := (f \circ \sigma_s, g \circ \sigma_s)$. Then $t \to \tau_s \in \mathrm{Aut}(\mathrm{C}_0(\mathbb{R}, B)_{(1)})$ is a (grading-preserving) homotopy between id and τ_1.

If $\Phi = \Phi(\mathcal{C};\, D, SB)$ denotes the natural isomorphism from $\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB)$ onto $\mathrm{KK}(\mathcal{C};\, D, (SB)_{(1)})$, then $(\tau_s)_* \circ \Phi = \Phi \circ (\rho_s)_*$.

Since $(\tau_1)_* = (\tau_0)_*$ by homotopy invariance of $\mathrm{KK}(\mathcal{C};\, D, (SB)_{(1)})$, we get $[\widehat{\sigma} \circ h] = (\rho_0)_*([h]) = (\rho_1)_*([h]) = [h]$ for $[h] \in G(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$. Since $h = g \oplus T$ satisfies $[h] \in G(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$, we can see that $[\widehat{\sigma} \circ (g \oplus T)] = [g \oplus T]$ in $S(H_{\mathbb{R}}\,;\, D, E_{\mathbb{R}})$. □

In principle, Proposition 8.2.6, and Remark 5.8.7

Check ref. to 5.8.7

imply our below given Corollary 8.3.3 via Bott periodicity.

But we give a proof which uses only the homotopy invariance of $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$ alone to derive that $\mathrm{Ext}(\mathcal{C}(\mathbb{R}_+);\, A, \mathrm{C}_0(\mathbb{R}_+, B)) = 0$ and use the simple construction of Remark 8.3.1, because our explicit constructions on the level of Busby invariants and unitaries are needed in Chapter 9 to show that Theorem 4.4.6 yields the proofs of Theorems B(i,ii) and M(i,ii).

Corollary 8.3.3. *Suppose that D is separable, B is σ-unital and that both are stable and trivially graded. Let $\mathcal{C} \subseteq \mathrm{CP}(D, B)$ a non-degenerate point-norm closed countably generated m.o.c. cone.*

*Let $E := \mathrm{Q}(B) := \mathcal{M}(B)/B \cong \mathrm{Q}^{\mathrm{s}}(B)$, and let $H_0 := \pi_B \circ H\colon D \to E$ be a *-monomorphism that comes from a faithful non-degenerate *-representation $H\colon D \to$*

$\mathcal{M}(B)$ with the property that $\delta_\infty \circ H$ is unitarily equivalent to H and that the maps $a \in D \mapsto b^* H(a) b \in B$ generate $\mathcal{C}$ ([9]).

(i) *There is a natural isomorphism θ from the kernel of $\mathrm{K}_1(H_0(D)' \cap E) \to \mathrm{K}_1(E)$ onto $\mathrm{KK}(\mathcal{C};\, A, B)$.*

(ii) *In particular, $\mathrm{K}_*(H_{\mathbb{R}_+}(D)' \cap \mathrm{Q^s}(\mathrm{C}_0(\mathbb{R}_+, B))) = 0$ where $H_{\mathbb{R}_+}(a)$ means the "restriction" of $H_0(a) \in \mathcal{M}(B) \subseteq E_{\mathbb{R}} = \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))/\mathrm{C}_0(\mathbb{R}, B)$ to $\mathbb{R}_+$.*

(iii) *The map*

$$[g] \in G(H_0\,,\, D, \mathrm{Q^s}(SB)) \to [U^+(0)U^-(0)^*] \in \mathrm{K}_1(H_0(D)' \cap E),$$

where $U^-(0), U^+(0) \in E$ are as in the proof of Lemma 8.3.2, defines a group isomorphism ι from $\mathrm{Ext}(\mathcal{C};\, D, SB) = G(H_0\,;\, D, \mathrm{Q^s}(SB))$ onto the kernel of $\mathrm{K}_1(H_0(D)' \cap E) \to \mathrm{K}_1(E)$.

(iv) *Let $h\colon D \to B$ be a C^*-morphism in $\mathcal{C}$, C_h the extension of D by SB given by the mapping cone construction for h, and let $[h - 0]$ denote the element of $\mathrm{KK}(\mathcal{C};\, A, B)$ that is represented by the difference construction $(B, h, 0)$.*

Then the isomorphism $\theta \circ \iota$ from $\mathrm{Ext}(\mathcal{C};\, A, SB)$ onto $\mathrm{KK}(\mathcal{C};\, A, B)$ maps the class $[C_h] \in \mathrm{Ext}(\mathcal{C};\, A, SB)$ of C_h to $[h-0] \in \mathrm{KK}(\mathcal{C};\, A, B)$, i.e., $\theta \circ \iota([C_h]) = [h - 0]$.

Proof. (i): Let $H_0(a) := H_1(a) + B$, $\mathrm{Q}(B) := \mathcal{M}(B)/B \cong \mathrm{Q^s}(B)$ and let $C := H_0(D)' \cap \mathrm{Q^s}(B)$. Recall that $H_1(D)' \cap \mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, and that $\mathrm{K}_*(H_1(D)' \cap \mathcal{M}(B)) = 0$.

By E_0 we denote the Hilbert B-module $\mathcal{H}_B \oplus \mathcal{H}_B^{op}$, i.e., $\mathcal{H}_B \oplus \mathcal{H}_B$ with grading $\beta_{E_0}\colon (x, y) \mapsto (x, -y)$. Then naturally $\mathcal{L}(E_0) \cong M_2(\mathcal{M}(B))$ with grading $Z \mapsto \beta Z \beta$, $\beta := \mathrm{diag}(1, -1)$ and $\mathbb{K}(E_0) \cong M_2(B)$ under this isomorphism. We define $\phi_0\colon D \to \mathcal{L}(E_0)$ by $\phi_0(a) := \mathrm{diag}(H_1(a), H_1(a))$ for $a \in D$.

For a unitary $U \in \mathcal{M}(B)$ let $F(U) \in \mathcal{L}(E_0)$ denote the selfadjoint unitary which is given by the off-diagonal 2×2-matrix with entries $F(U)_{11} = F(U)_{22} = 0$, $F(U)_{12} = U$ and $F(U)_{21} = U^*$. Then

$$F(U)\phi_0(a) - \phi_0(a)F(U) = F(1)\,\mathrm{diag}\big(U^*H_1(a) - H_1(a)U^*, UH_1(a) - H_1(a)U\big)\,.$$

Thus, $(E_0, \phi_0, F(U))$ is a $\mathcal{C}$-compatible Kasparov module, if and only if, $U \in \mathcal{U}(H_1(D), B)$, with notation of Lemma 4.6.6. In particular, the product $U_0U_2U_1^*$ is in $\mathcal{U}(H_1(D), B)$ if $U_0, U_1, U_2 \in \mathcal{M}(B)$ are unitaries and if

$$(E_0, \mathrm{diag}(U_0^*H_1U_0, U_1^*H_1U_1), F(U_2))$$

is a Kasparov module. The latter is isomorphic to $(E_0, \phi_0, F(U_0U_2U_1^*))$ via $\mathrm{diag}(U_0, U_1)$.

[9] See Corollary 5.4.4. In the special $\mathcal{C} = \mathrm{CP_{nuc}}(A, B)$, $H = \mathcal{M}(\gamma) \circ \rho$ for a non-degenerate *-morphism $\gamma\colon \mathbb{K} \to \mathcal{M}(B)$ and any faithful $\rho\colon D \hookrightarrow \mathcal{M}(\mathbb{K})$ with $\rho(D) \cap \mathbb{K} = 0$.

We define a Hilbert B-module isomorphism $T_0\colon E_0 \oplus_B E_0 \to E_0$ from the Hilbert B-module sum $E_0 \oplus_B E_0$ onto E_0, by

$$T_0\colon ((x_1, y_1), (x_2, y_2)) \mapsto (s_1 x_1 + s_2 x_2, s_1 y_1 + s_2 y_2)\,.$$

Then $T_0^*(\phi_0 \oplus \phi_0)T_0 = \phi_0$ and $T_0^*(F(U_1) \oplus F(U_2))T_0 = F(U_1 \oplus_{s_1,s_2} U_2)$ for unitaries U_1 and U_2 in $\mathcal{M}(B)$. By Lemma 4.6.6, $\mathcal{U}(H_1(D), B)$ is closed under Cuntz addition.

It follows that $\theta_0\colon U \mapsto (E_0, \phi_0, F(U))$ is an additive map from $\mathcal{U}(H_1(D), B)$ with Cuntz addition into the semigroup $\mathbb{E}(\mathcal{C};\, D, B)/\approx$ of isomorphism classes of elements of $\mathbb{E}(\mathcal{C};\, D, B)$.

Since $1 \oplus 1 = 1$, we get that $\theta_0(1)$ represents the zero of $\mathrm{KK}(\mathcal{C};\, D, B)$. Therefore, $\theta_0(U)$ and $\theta_0(U \oplus 1)$ represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B)$.

A simple calculation shows that $\theta_0(U)$ is a compact perturbation of $\theta_0(U')$ in the sense of Definition 8.2.1 if and only if $(U' - U)H_1(D) \subseteq B$.

If $t \mapsto U(t)$ is a continuous map from $[0, 1]$ into $\mathcal{U}(H_1(D), B)$, then $t \mapsto \theta_0(U(t))$ defines an operator homotopy in $\mathbb{E}(\mathcal{C};\, D, B)$, and, therefore, $\theta_0(U(0))$ and $\theta_0(U(1))$ represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B)$ by the (operator) homotopy invariance of $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$.

By Lemma 4.6.6, it follows that $\theta_0(U_1)$ and $\theta_0(U_2)$ represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B)$ if $[U_1 + B] = [U_2 + B]$ in $\mathrm{K}_1(H_0(D)' \cap \mathrm{Q^s}(B))$.

This means that $\theta([U + B]) := [\theta_0(U)]$ defines a group homomorphism θ from the kernel of $\mathrm{K}_1(H_0(D)' \cap E) \to \mathrm{K}_1(E)$ into $\mathrm{KK}(\mathcal{C};\, D, B) \cong \mathrm{KK}(\mathcal{C};\, A, B)$. It remains to show:

(1) θ is an epimorphism, i.e., every element of $\mathrm{KK}(\mathcal{C};\, D, B)$ can be represented by $\theta_0(U)$ for a suitable $U \in \mathcal{U}(H_1(D), B)$.
(2) $[\theta_0(U)] = 0$ in $\mathrm{KK}(\mathcal{C};\, D, B)$ implies $[U + B] = 0$ in $\mathrm{K}_1(C)$.

(1): For trivially graded D and σ-unital trivially graded B, one can use Kasparov's stabilization theorem to obtain a representative (E, ϕ, F) of an element of $\mathrm{KK}(\mathcal{C};\, D, B)$, such that the Hilbert B-module E is isomorphic to the above considered "universal" graded Hilbert B-module E_0, cf. . [**73**, 17.4.1].

An element of $\mathrm{KK}(\mathcal{C};\, D, B) \cong \mathrm{KK}(\mathcal{C};\, A, B)$ with representing element (E, ϕ, F) can be also represented by $\mathcal{C}$-compatible Kasparov modules $(E \oplus E^{op}, \phi \oplus 0, G(F))$ with $G(F) = G(F)^* = G(F)^{-1}$. Here E^{op} is the Hilbert B-module E with the new grading operator $-\beta_E$, and $G(F) \in \mathcal{L}(E \oplus E^{op}) \cong M_2(\mathcal{L}(E))$ is given by the matrix with entries $G(F)_{11} = F_1$, $G(F)_{22} = -F_1$ and $G(F)_{12} = G(F)_{21} = (1 - F_1^2)^{1/2}$, where $F_1 := f((F^* + F)/2)$ is a contractive selfadjoint ϕ-"compact" perturbation of F, and $f \in \mathrm{C_b}(\mathbb{R})$ is given by $f(t) = t$ for $t \in [0, 1]$, $f(t) = -1$ for $t \leq -1$ and $f(t) = 1$ for $t \geq 1$.

$(E \oplus_B E^{op}, \phi \oplus 0, G(F))$ represents the same element of $\mathrm{KK}(\mathcal{C};\, D, B)$ because $(E \oplus_B E^{op}, \phi \oplus 0, G(F))$ is a compact perturbation of the sum of (E, ϕ, F) and $(E^{op}, 0, -F)$ (see [**73**, 17.4.2, 17.4.3, 17.6] for more details).

The passage to this special representatives is functorial in $\mathbb{E}(\mathcal{C};\, D, B)$ in the sense that it is operator homotopy preserving and is additive modulo addition of trivial elements of form $(E', 0, F')$ and compact perturbation. If (E, ϕ, F') is a ϕ-"compact" perturbation of of (E, ϕ, F), then

Is this correct needed?:

$(E \oplus_B E^{op}, \phi \oplus 0, G(F')) \sim_{cp} (E \oplus_B E^{op}, \phi \oplus 0, G(F))$.

Since we did start with a *$\mathcal{C}$-compatible* Kasparov module (E, ϕ, F) such that E is isomorphic to our above defined E_0 by a grading preserving isometric B-module isomorphism, we have that $E \oplus_B E^{op}$ is isometrically and grading preserving B-module isomorphic to E_0.

Thus we find for each element of $\mathrm{KK}(\mathcal{C};\, D, B)$ a representing *$\mathcal{C}$-compatible* Kasparov module $(E_0\,, \phi, G)$ with $G = G^* = G^{-1}$.

Since the degree of $\phi(a)$ is zero and the degree of G is one, the representative has the form $\phi(a) = \mathrm{diag}(\psi_0(a), \psi_1(a))$ with *weakly $\mathcal{C}$-compatible* *-morphisms $\psi_0, \psi_1 \colon D \to \mathcal{M}(B)$ and $G = F(U)$ for a unitary $U \in \mathcal{M}(B)$.

T_0 defines an isomorphism from the sum of $(E_0\,, \mathrm{diag}(\psi_0, \psi_1), F(U))$ and of $\theta_0(1)$ onto

$$(E_0\,, \mathrm{diag}(\psi_0 \oplus H_1, \psi_1 \oplus H_1), F(U \oplus 1)). \tag{*}$$

Since $\theta_0(1)$ represents the zero, every element of $\mathrm{KK}(\mathcal{C};\, D, B)$ is represented by a Kasparov module of form (*).

The *homotopy invariance* of $\mathrm{KK}(\mathcal{C};\, D, B)$ yields the following:
Let $\tau \mapsto \psi_0^{(\tau)} \colon D \to \mathcal{M}(B)$ and $\tau \mapsto \psi_1^{(\tau)} \colon D \to \mathcal{M}(B)$ be a point-norm continuous families of weakly *$\mathcal{C}$-compatible* C^*-morphisms, and let $U \in \mathcal{M}(B)$ be a unitary such that $U\psi_1^{(0)}(a) - \psi_0^{(0)}(a)U \in B$ for all $a \in D$. Suppose that $\psi_0^{(\tau)}(a) - \psi_0^{(0)}(a)$ and $\psi_1^{(\tau)}(a) - \psi_1^{(0)}(a)$ are in B for all $a \in D$ and $\tau \in [0, 1]$. Then $(E_0, \mathrm{diag}(\psi_0^{(0)}, \psi_1^{(0)}), U)$ and $(E_0, \mathrm{diag}(\psi_0^{(1)}, \psi_1^{(1)}), U)$ represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B)$.

By Theorem 5.6.2(ii) (applied to $C := H_1(D)$ and $T := \psi \circ H_1^{-1}$), we find norm-continuous maps $t \mapsto U_0(t)$ and $t \mapsto U_1(t)$ into the unitaries of $\mathcal{M}(B)$ such that, for $k \in \{0, 1\}$, $t \in \mathbb{R}_+$ and $a \in A$,

$$U_k(t)^* H_1(a) U_k(t) - (\psi_k(a) \oplus H_1(a)) \in B$$

and

$$\lim_{t \to \infty} \| U_k(t)^* H_1(a) U_k(t) - \big(\psi_k(a) \oplus H_1(a)\big) \| = 0\,.$$

Thus

$$\big(E_0, \mathrm{diag}(U_0(0)^* H_1 U_0(0), U_1(0)^* H_1 U_1(0)), F(U \oplus 1)\big) \tag{**}$$

and (*) represent the same element of $\mathrm{KK}(\mathcal{C};\, D, B) = \mathrm{KK}(\mathcal{C};\, A, B)$.

(**) is isomorphic to $\theta_0(U_1)$, where $U_1 := U_0(0)(U \oplus 1)U_1(0)^*$, and U_1 is a unitary in $\mathcal{U}(H_1(D), B)$, because (**) is a Kasparov module. Thus, every element of $\mathrm{KK}(\mathcal{C};\, D, B) \cong \mathrm{KK}(\mathcal{C};\, A, B)$ can be represented by a *$\mathcal{C}$-compatible* Kasparov module $\theta_0(U)$, where $U \in \mathcal{U}(H_1(D), B)$, and θ is an epimorphism.

(2): Suppose that $[\theta_0(U)] = 0$ in KK($\mathcal{C}$; D, B).

By Definition 8.2.1, $[\theta_0(U)] = 0$ means that there exist a *$\mathcal{C}$-compatible* Kasparov module (E, ϕ, F) and a compact perturbation (E, ϕ, F') of it, such that $\theta_0(U) \oplus (E, \phi, F)$ is isomorphic to (E, ϕ, F') via a grading preserving B-module unitary I. In particular $F' = I^{-1}(F(U) \oplus F)I$. If we add trivial Kasparov modules, take the above listed $\phi \oplus 0$-"compact" perturbations and pass to isomorphic Kasparov modules (isomorphic in the strong sense of Definition 8.2.1), then we can assume moreover that $E = E_0$, $F = F(V_1)$, $F' = F(V_2)$ for unitaries V_1 and V_2 in $\mathcal{M}(B)$, as we have shown above. Then $\phi = \mathrm{diag}(\psi_0, \psi_1)$ for weakly *$\mathcal{C}$-compatible* C^*-morphisms ψ_0 and ψ_1 from D into $\mathcal{M}(B)$.

By stability of B, there exists a sequence of isometries $t_1, t_2, \ldots$ in $\mathcal{M}(B)$ such that $\sum t_n t_n^*$ converges strictly to 1, cf. Remark 5.1.1(8). Let λ be the infinite repeat of the Cuntz sum $(\psi_0 \oplus \psi_1) \oplus H_1$ i.e., $\lambda := \delta_\infty((\psi_0 \oplus \psi_1) \oplus H_1)$, where $\oplus = \oplus_{s_1,s_2}$.

Then λ is a strictly continuous C^*-morphism from D into $\mathcal{M}(B)$.

The series $\sum((t_n \oplus t_n) \oplus t_n)t_n^*$ and converges strictly to a unitary in $\mathcal{M}(B)$. Therefore λ is unitarily equivalent to the Cuntz sum of the infinite repeats of ψ_0, of ψ_1 and of H_1. It follows that $\psi_0 \oplus \lambda$, $\psi_1 \oplus \lambda$ and $H_1 \oplus \lambda$ are unitarily equivalent to λ.

Now Corollary 5.6.1 applies to the C^*-subalgebra $C := H_1(D)$ of $\mathcal{M}(B)$ and to $T = \lambda \circ H_1^{-1}$. Thus there is a unitary v in $\mathcal{M}(B)$ such that $v^*\lambda(a)v - H_1(a) \in B$ for $a \in D$.

Let $\chi := v^*\lambda v$. Then again $\chi \oplus \psi_0$ and $\chi \oplus \psi_1$ are unitarily equivalent to χ. By Lemma 4.6.5(ii), the relative commutant of $\chi(D)$ in $\mathcal{M}(B)$ has trivial K-theory, in particular, $[V + B] = 0$ in $\mathrm{K}_1(C)$ if V is a unitary in $\mathcal{M}(B)$ which commutes element-wise with $\chi(D)$. By Lemma 5.1.2, χ is unitarily equivalent to $\chi \oplus \chi$, because χ is an infinite repeat.

We add the Kasparov module $(E_0, \mathrm{diag}(\chi, \chi), F(1))$ to $(E_0, \phi, F(V_j))$, $j = 1, 2$. Then we use our above defined isomorphism T_0 from $E_0 \oplus E_0$ onto E_0, and pass to equivalent Kasparov modules, where we use the above described transformation rules for $U \mapsto F(U)$.

Then we get unitaries U_1, U_2, U_3 and U_4 in $\mathcal{M}(B)$ such that

$(E_0, \mathrm{diag}(\chi, \chi), F(U_j))$ are Kasparov modules for $j = 1, 2$,

$\chi(D)(U_1 - U_2) \subseteq B$,

$U_3^*(H_1 \oplus \chi)U_3 = \chi$, $U_4^*(H_1 \oplus \chi)U_4 = \chi$ and $U_3^*(U \oplus U_1)U_4 = U_2$.

In particular $V := U_3^* U_4$ commutes element-wise with the image of χ, and, therefore, $[V + B] = 0$ in $\mathrm{K}_1(C)$.

Since $H_1 \oplus H_1 = H_1$ and $\chi(d) - H_1(d) \in B$ for $d \in D$, we get that the U_k are in $\mathcal{U}(H_1(D), B)$ for $k = 1, 2, 3, 4$.

Let u_0, u_1, u_2, u_3, u_4 denote the images of U, U_1, U_2, U_3, U_4 in $C = H_0(D)' \cap E$.

Then $[u_1] = [u_2]$ in $\mathrm{K}_1(C)$ by Lemma 4.6.6.

$$[u_0] + [u_1] = [u_3^*(u_0 \oplus u_1)u_3] + [u_3^* u_4] = [u_2]\,.$$

Thus $[U + B] = [u_0] = 0$ in $\mathrm{K}_1(H_0(D)' \cap E)$.

(ii): Let $E^{(+)} := \mathrm{Q}^{\mathrm{s}}(\mathrm{C}_0(\mathbb{R}_+, B))$. In the proof of Lemma 8.3.2 we have seen that $\mathrm{K}_*(E^{(+)}) = 0$.

`Is this CONCLUSION ‘‘thus’’ correct? ??`

Thus,

by Propositions 4.4.3(i) and 8.2.6,

$$\mathrm{K}_0(H_{\mathbb{R}_+}(D)' \cap E^{(+)}) = \mathrm{KK}(\mathcal{C};\, D, \mathrm{C}_0(\mathbb{R}_+, B)_{(1)}) = 0\,.$$

By Part (i) and Proposition 8.2.5(ii),

$$\mathrm{K}_1(H_{\mathbb{R}_+}(D)' \cap E^{(+)}) = \mathrm{KK}(\mathcal{C};\, D, \mathrm{C}_0(\mathbb{R}_+, B)) = 0\,.$$

(iii): Let $E := \mathrm{Q}^{\mathrm{s}}(B)$, $C := H_0(D)' \cap E$, $E_{\mathbb{R}} := \mathrm{Q}(SB)$ $E^{(\alpha)} := \mathrm{Q}^{\mathrm{s}}(\mathrm{C}_0(\mathbb{R}_\alpha, B))$ for $\alpha \in \{+, -\}$.

Since $G(H_{\mathbb{R}}, D, E_{\mathbb{R}}) = [H_{\mathbb{R}}] + S(H_{\mathbb{R}}, D, E_{\mathbb{R}})$, from the proof of Lemma 8.3.2 we get the existence of unitaries $U^+ \in E^{(+)}$ and $U^- \in E^{(-)}$ such that

$$g^\alpha = (U^\alpha)^* H_\alpha U^\alpha\,,$$

where $g^\alpha(a) := g(a)|\mathbb{R}_\alpha$, $H_\alpha := H_{\mathbb{R}_\alpha}$ for $\alpha \in \{+, -\}$. Here we use the "restriction" notation as explained in Remark 8.3.1.

We have seen in the proof of Lemma 8.3.2 that $U^+(0)(U^-(0))^*$ is a unitary in C such that $[U^+(0)(U^-(0))^*]$ is in the kernel of the natural group homomorphism from $\mathrm{K}_1(C)$ into $\mathrm{K}_1(E)$. Here $U^+(0) := \pi_0^+(U^+)$ and $U^-(0) := \pi_0^-(U^-)$ for the natural epimorphisms $\pi_0^\alpha \colon E^{(\alpha)} \to E$, where $\alpha \in \{+, -\}$.

If $U_1^+ \in E^{(+)}$ and $U_1^- \in E^{(-)}$ are other unitaries with $g^\alpha = (U_1^\alpha)^* H_\alpha U_1^\alpha$ for $\alpha \in \{+, -\}$. Then $U_1^+(U^+)^*$ is in ????? $H_+(D)' \cap E^{(+)}$. By part (ii), the K_1-images of the natural epimorphism from $H_+(D)' \cap E^{(+)}$ into $C = H_0(D)' \cap E$ is zero. This implies that $[U_1^+(0)(U^+(0))^*] = 0$ in $\mathrm{K}_1(C)$. Similarly, $[U^-(0)(U_1^-(0))^*] = 0$ in $\mathrm{K}_1(C)$. Thus $[U^+(0)(U^-(0))^*] = [U_1^+(0)(U_1^-(0))^*]$ in $\mathrm{K}_1(C)$.

Therefore $\iota([g]) := [U^+(0)(U^-(0))^*]$ is a well-defined map from $G(H_{\mathbb{R}}, D, E_{\mathbb{R}}) = \mathrm{Ext}(\mathcal{C};\, D, SB)$ into the kernel of $\mathrm{K}_1(C) \to \mathrm{K}_1(E)$.

If g and h are representatives of elements in $G(H_{\mathbb{R}}, D, E_{\mathbb{R}})$ and if $U_1^+, U_2^+ \in E^{(+)}$ and $U_1^-, U_2^- \in E^{(-)}$ are unitaries with $h^\alpha = (U_2^\alpha)^* H_\alpha U_2^\alpha$ and $g^\alpha = (U_1^\alpha)^* H_\alpha U_1^\alpha$ for $\alpha \in \{+, -\}$. Then $U^\alpha := U_1^\alpha \oplus U_2^\alpha \in E^{(\alpha)}$ are unitaries with $(g \oplus h)^\alpha = (U_1^\alpha)^* H_\alpha U_1^\alpha$. Thus $\iota([g] + [h]) = [(U_1^+(0)(U_1^-(0))^*) \oplus (U_2^+(0)(U_2^-(0))^*)]$, which is $\iota([g]) + \iota([h])$ in $\mathrm{K}_1(C)$. Therefore, ι is a group homomorphism.

By Lemma 4.6.6, ι is an epimorphism, because for $U \in \mathcal{U}(H_1(D), B)$ and h_U in Remark 8.3.1 we have, for $g = h_U \oplus H_{\mathbb{R}}$, $U^+ = U \oplus 1$ and $U^- = 1$, and therefore $\iota[h_U \oplus H_{\mathbb{R}}] = [U + B]$.

Suppose that $\iota([g]) = [U^+(0)U^-(0)^*] = 0$ in $\mathrm{K}_1(H_0(D)' \cap E)$. By Lemma 4.2.6(v,2), the unitary $V := (U^+(0)U^-(0)^*) \oplus 1$ is in the connected component of 1 in the unitaries of C. Thus there is a continuous map $t \in [0,1] \to U(t)$ into the unitaries of $\mathcal{M}(B)$ such that $U(0) = 1$, $U(1)+B = V$ and $U(t)+B \in C$ for $t \in [0,1]$. Thus $U(t) \in \mathcal{U}(H_1(D), B)$ for $t \in [0,1]$. Let $U := U(1)$ and $W \in \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B)) \subseteq \mathcal{M}(SB)$ the unitary which is defined by $W(t) := 1$ for $t \le 0$, $W(t) := U(t)$ for $t \in [0,1]$, and $W(t) = U$ for $t \ge 1$. Then, for the lift T of h_U in Remark 8.3.1, we get that $T(a)(t) - W(t)^* H_1(a) W(t) \in B$ for $t \in \mathbb{R}$ and $T(a)(t) - W(t)^* H_1(a) W(t) = 0$ for t not in $[0,1]$. Thus $h_U = w^*(H_\mathbb{R})w$ for $w := W + \mathrm{C}_0(\mathbb{R}, B)$. In the proof of Lemma 8.3.2 we have seen that $[g \oplus H_\mathbb{R}] = [h_U]$. Since $[g] \in G(H_\mathbb{R}, D, E_\mathbb{R})$, we get $[g] = [g \oplus H_\mathbb{R}] = [h_U] = [w^* H_\mathbb{R} w] = [H_\mathbb{R}] = 0$.

Thus ι is an isomorphism from $\mathrm{Ext}(\mathcal{C};\, A \otimes \mathbb{K}, SB) \cong G(H_\mathbb{R}, D, \mathrm{Q^s}(SB))$ onto the kernel of natural homomorphism from $\mathrm{K}_1(H_0(D)' \cap \mathrm{Q^s}(B))$ into $\mathrm{K}_1(\mathrm{Q^s}(B))$.

(iv): To simplify notation, we assume that B is stable. We keep the notations $E^{(-)}$, $E^{(+)}$ of Remark 8.3.1, but don't use the notation $E_\mathbb{R}$ used there for $\mathrm{Q^s}(SB) = \mathcal{M}(SB)/SB$. Let $C := H_0(D)' \cap E$.

If $h\colon D \to B$ is a *-morphism in $\mathcal{C}$, then the mapping cone C_h is represented as an element of $S(H_\mathbb{R}\,;\, D, \mathrm{Q^s}(SB))$ by

$$g_h\colon a \in D \mapsto (0, h(a)) \in 0 \oplus B \subset \mathrm{Q}(\mathbb{R}_-, B) \oplus \mathrm{Q}(\mathbb{R}_+, B) = \mathrm{Q}(\mathbb{R}, B) \subseteq E^{(-)} \oplus E^{(+)}.$$

The corresponding element of $G(H_\mathbb{R}, D, \mathrm{Q^s}(SB)) = \mathrm{Ext}(\mathcal{C};\, A \otimes \mathbb{K}, SB)$ is $[g_h \oplus H_\mathbb{R}]$. If we apply Theorem 5.6.2(ii) to $C := H_1(D)$ and separately to $T = 0$ and $T = hH_1^{-1}$, we get norm-continuous maps $t \mapsto U_0(t)$ and $t \mapsto U_1(t)$ from $\mathbb{R}_+$ into the unitaries of $\mathcal{M}(B)$, such that $\lim_{t\to\infty} \|(h \oplus H_1)(a) - U_0(t)^* H_1(a) U_0(t)\| = 0$, $(h \oplus H_1)(a) - U_0(t)^* H_1(a) U_0(t) \in B$, $\lim_{t\to\infty} \|(0 \oplus H_1)(a) - U_1(t)^* H_1(a) U_1(t)\| = 0$ and $(0 \oplus H_1)(a) - U_1(t)^* H_1(a) U_1(t) \in B$ for $a \in D$, $t \in \mathbb{R}_+$.

Let us consider U_0 as a unitary in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B)) \subseteq \mathcal{M}(\mathrm{C}_0(\mathbb{R}_+, B))$ and $V(t) := U_1(-t)$ for $t \le 0$ as a unitary V in $\mathcal{M}(\mathrm{C}_0(\mathbb{R}_-, B))$.

Let $U^+ := \pi^+(U_0)$, $U^- := \pi^-(V)$. Then $(g_h \oplus H_\mathbb{R})|\mathbb{R}_\alpha = (U^\alpha)^* H_\alpha U^\alpha$ for $\alpha \in \{+,-\}$, $U^+(0) = \pi_0^+(U^+) = U_0(0) + B$ and $U^-(0) = \pi_0^-(U^-) = U_1(0) + B$. Thus $\iota([g_h \oplus H_0]) = [U_0(0)U_1(0)^* + B]_C \in \mathrm{K}_1(C)$.

By construction of θ, we have that $\theta([U_0(0)U_1(0)^* + B]_C)$ is represented by $(E_0, \phi_0, F(U_0(0)U_1(0)^*))$.

Let $(\mathcal{H}_B, h, 0)$ the *$\mathcal{C}$-compatible* Kasparov module obtained by the difference construction. Since the derivative F of the module is zero, the construction in the proof of part (i) produces for the class of $(\mathcal{H}_B, h, 0)$ in $\mathrm{KK}(\mathcal{C};\, D, B)$ the representative $(E_0, \mathrm{diag}(h, 0), F(1))$, because $F(1) = G(0)$, where the notation is as in the proof of part (i).

In the proof of part (i) we have seen, that the Kasparov module $(E_0, \phi_0, F(1))$ represents zero. If we add it to $(E_0, \mathrm{diag}(h, 0), F(1))$, we get $(E_0, \mathrm{diag}(h \oplus H_1, 0 \oplus H_1), F(1))$.

By homotopy invariance of $\mathrm{KK}(\mathcal{C};\cdot,\cdot)$, $(E_0, \mathrm{diag}(U_0^* H_1 U_0, U_1^* H_1 U_1), F(1))$ represents the same element, where $U_0 := U_0(0)$ and $U_1 := U_1(0)$ are the unitaries in $\mathcal{M}(B)$ which come from the above considered paths $U_0(t)$ and $U_1(t)$ in $\mathcal{U}(H_1(D), B)$.

In the proof of part (i) we have seen that the latter Kasparov module is isomorphic to $(E_0, \phi_0, F(U_0 U_1^*))$, which represents $\iota([U_0 U_1^*])$. Hence, $\iota \circ \theta([C_h]) = [h - 0]$. □

`Change following to` $\mathcal{C}$`-picture !?`

Corollary 8.3.4. *Suppose that A is separable, that B is σ-unital and stable, and that both are trivially graded. Let $D := A \otimes \mathbb{K}$, $E_{\mathbb{R}} := \mathrm{Q}^{\mathrm{s}}(SB)$, and let $H_{\mathbb{R}} \colon D \to E_{\mathbb{R}}$ be a *-monomorphism which comes from a faithful non-degenerate *-representation of $H \colon D \hookrightarrow \mathcal{M}(B) \subseteq E_{\mathbb{R}}$ in "general position", i.e., with $\delta_\infty \circ H$ is unitarily homotopic to H.*

(i) *For every $t \in [0,1]$, the natural group homomorphism*

$$\pi_t \colon G(H_{\mathbb{R}\times[0,1]}, D, \mathrm{Q}^{\mathrm{s}}(S\,\mathrm{C}([0,1], B))) \to G(H_{\mathbb{R}}, D, E_{\mathbb{R}}),$$

which is induced by the natural epimorphism from $\mathrm{Q}^{\mathrm{s}}(S\,\mathrm{C}([0,1],B))$ onto $E_{\mathbb{R}}$ is an isomorphism and is therefore independent of $t \in [0,1]$.

(ii) *If σ is an orientation preserving homeomorphism of $\mathbb{R}$ and $\widehat{\sigma}$ the automorphism of $E_{\mathbb{R}} := \mathrm{Q}^{\mathrm{s}}(SB)$ which is naturally induced by σ, then $[\widehat{\sigma} \circ h] = [h]$ in $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$ for every representing C^*-morphism $h \colon D \to E_{\mathbb{R}}$ of an element of $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$.*

Proof. (i): It follows from the natural isomorphisms $G(H_0, D, E_{\mathbb{R}}) = \mathrm{Ext}_{\mathrm{nuc}}(D, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(D, B)$ (including $G\big(H_{\mathbb{R}\times[0,1]}, D, \mathrm{Q}^{\mathrm{s}}(S\,\mathrm{C}([0,1],B))\big) \cong \mathrm{KK}_{\mathrm{nuc}}(D, \mathrm{C}([0,1],B)))$ and the homotopy invariance of $\mathrm{KK}_{\mathrm{nuc}}$.

(ii): Since σ is a strictly increasing continuous function from $\mathbb{R}$ into $\mathbb{R}$, for every $f \in SB \cong \mathrm{C}_0(\mathbb{R}, B)$, the definition $g(t,s) := f(st + (1-s)\sigma(t))$ defines an element g in

$$S\,\mathrm{C}([0,1], B) \cong \mathrm{C}_0(\mathbb{R} \times [0,1], B)\,.$$

Let $V \colon D \to \mathcal{M}(SB)$ be a completely positive

weakly *nuclear*, respectively $\mathcal{C}$-compatible,

contractive lift of $h \colon D \to \mathrm{Q}^{\mathrm{s}}(SB)$.

The curvature $d \mapsto V(d^*d) - V(d)^*V(d)$ of V maps D into $SB = \mathrm{C}_0(\mathbb{R}, B)$. Under the natural identifications of $\mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B))$ with $\mathcal{M}(SB)$, respectively of $\mathrm{C}_{\mathrm{b,st}}(\mathbb{R} \times [0,1], \mathcal{M}(B))$ with $\mathcal{M}(S\,\mathrm{C}([0,1],B))$, we let

$$\widetilde{V}(t,s)(d) := V(st + (1-s)\sigma(t))(d)$$

for $t \in \mathbb{R}$, $s \in [0,1]$, $d \in D$. Then $\widetilde{V}$ is

`check again?:`

weakly *nuclear*, respectively $\mathcal{C}$-compatible,

and the curvature $d \mapsto \widetilde{V}(d^*d) - \widetilde{V}(d)^*\widetilde{V}(d)$ of $\widetilde{V}$ maps D into $\mathrm{C}_0(\mathbb{R}\times[0,1], B) = S\,\mathrm{C}([0,1], B)$. Thus it defines a

weakly *nuclear*, respectively $\mathcal{C}$-compatible,

liftable C^*-morphism $k\colon D \to \mathrm{Q^s}(S\,\mathrm{C}([0,1], B))$.

The natural epimorphisms from $\mathrm{Q^s}(S\,\mathrm{C}([0,1], B))$ onto $\mathrm{Q^s}(SB)$ at zero and at one map k into $\widehat{\sigma}h$ and h, respectively. Thus k defines an element $[k]$ of $\mathrm{Ext}_{\mathrm{nuc}}(D, S\,\mathrm{C}_0([0,1], B))$. with $\pi_1([k]) = [h]$ and $\pi_0([k]) = [\widehat{\sigma}h]$. Therefore, by (i), $[h] = [\widehat{\sigma} \circ h]$ in $\mathrm{Ext}_{\mathrm{nuc}}(D, SB)$, respectively is in $\mathrm{Ext}(\mathcal{C};\, D, SB)$. □

Remark 8.3.5. The natural map γ from $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ to $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ is defined as $(\gamma(x))(y) := y \otimes_A x \in \mathrm{K}_*(B)$ for $y \in \mathrm{K}_*(A)$, $x \in \mathrm{KK}_{\mathrm{nuc}}(A, B)$, where we denote by $y \otimes_A x$ the Kasparov product of x and y, and $\mathrm{K}_0(.)$ and $\mathrm{K}_1(.)$ are naturally identified with $\mathrm{KK}(\mathbb{C}, .)$ and $\mathrm{KK}(\mathrm{C}_0(\mathbb{R}), .)$ respectively.

In terms of $\mathrm{Ext}_{\mathrm{nuc}}(A, SB)$ this means equivalently, that γ maps the class with representative $g\colon D \to \mathrm{Q^s}(SB)$ into the connecting maps

$$\delta_*\colon\ \mathrm{K}_*(A) \to \mathrm{K}_{(*+1)\mathrm{mod}2}(SB)$$

of the six term exact sequence for the extension of $A \otimes \mathbb{K}$ by SB which is defined by g. Then use $\mathrm{K}_*(B) \cong \mathrm{K}_{(*+1)\mathrm{mod}2}(SB)$.

Both descriptions of γ allow to see that $\gamma([h - 0]) = \mathrm{K}_*(h)$ if $h\colon A \otimes \mathbb{K} \to B$ is a *nuclear* C^*-morphism.

Remark 8.3.6. By Bott periodicity of $\mathrm{KK}_{\mathrm{nuc}}$ and by the isomorphism (1.1), the similar notion of a *strictly nuclear* E-theory is functorial equivalent to $\mathrm{KK}_{\mathrm{nuc}}$ (See further below). But E-theory is not useful for the proof of (1.1) and its consequences. Note that $\mathrm{E}_{\mathrm{nuc}}(A, B) = \mathrm{E}(A, B)$ if A is nuclear, but in general this equation does hold for nuclear B and non-nuclear A.

Even in the trivially graded case there is no direct proof of the isomorphism

$$\mathrm{Ext}_{\mathrm{nuc}}(A, SB) \cong \mathrm{E}_{\mathrm{nuc}}(A, B)$$

by using directly the weakly nuclear version $\mathrm{SR}(\mathrm{CP}_{\mathrm{nuc}};\, SA, S^2B)$ of $[[SA \otimes \mathbb{K}, S^2B]]$ as defined in the beginning of Chapter 7. Therefore we outline here *another proof*:

Suppose that B is a stable σ-unital C^*-algebra and denote by π_s ($s \in [0,1]$) the natural epimorphism

$$\pi_s\colon\ \mathrm{Q}(\mathbb{R}_+, S\,\mathrm{C}([0,1], B)) \to \mathrm{Q}(\mathbb{R}_+, SB)\,.$$

But notice that a weakly nuclear asymptotic morphism $h = \{V(t)\colon A \otimes \mathbb{K} \to B\}$ in the sense of of Definition 7.1.1(a,b) represents in general *not* an element of the Rørdam semigroup (with the additional requirement (c) of Definition 7.1.1 and with $\mathcal{C} := CP(A \otimes \mathbb{K}, B \otimes \mathbb{K})$). This can be seen easily in the case $A = \mathrm{C}_0(\mathbb{R}, \mathcal{O}_\infty)$ and $B = \mathrm{C}_0(\mathbb{R}) \otimes \mathbb{K}$. Note, that the nuclearity of $V(t)$ is required for each $t \in \mathbb{R}_+$, but the uniform nuclearity on $\mathbb{R}_+$ is *not* required.

We say that weakly nuclear asymptotic morphisms $h_0 = \{V_0(t)\}$ and $h_1 = \{V_1(t)\}$ from $SA \otimes \mathbb{K}$ into SB are **asymptotical homotopic** if there is a weakly nuclear asymptotic morphism $h = \{V(t)\}$ from $SA \otimes \mathbb{K}$ into $\mathrm{C}([0,1], SB)$ such that $h_0 = \pi_0 \circ h$ and $h_1 = \pi_1 \circ h$.

As in [**165**] or [**163**, chp.II.B], one can see that the classes of asymptotically homotopic weakly nuclear asymptotic homomorphisms from $SA \otimes \mathbb{K}$ into $SB \otimes \mathbb{K}$ form a group $E_{nuc}(A, B)$ under Cuntz addition.

The generalized mapping cone construction, cf. Chapters 1 and 9, yields a group homomorphism from $E_{nuc}(A, B)$ into $\mathrm{Ext}_{\mathrm{nuc}}(SA \otimes \mathbb{K}, S^2B)$.

Conversely the natural asymptotic morphism from $SA \otimes \mathbb{K}$ into SB which corresponds to an extension (extension-class) $0 \to SB \to E \to A \otimes \mathbb{K} \to 0$ in $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$ is a weakly nuclear asymptotic morphism. It defines a group homomorphism from $\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB)$ to $E_{nuc}(A, B)$.

The composition maps are equivalent (i.e. are element-wise homotopic) to the natural map $E_{nuc}(A, B) \to E_{nuc}(SA, SB)$, respectively to

$$\mathrm{Ext}_{\mathrm{nuc}}(A \otimes \mathbb{K}, SB) \to \mathrm{Ext}_{\mathrm{nuc}}(SA \otimes \mathbb{K}, S^2B)\,.$$

Thus Proposition 8.2.5(iii) and Corollary 8.3.3(iii) imply that *there is a natural isomorphism* $\mathrm{KK}_{\mathrm{nuc}}(A, B) \cong E_{nuc}(A, B)$ *for separable* A *and* σ*-unital* B.

4. The residually nuclear case

REMARK 8.4.1. Suppose that A, B are stable, where A is separable and B is σ-unital. They are assumed to be trivially graded.

Furthermore, we suppose that X is the primitive ideal space of a separable C^*-algebra C. We can then replace C by $F \otimes C$ with same primitive ideal space, where we let $F := \mathbb{K} \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \ldots$. Then, in addition, C is stable and $C \cong C \otimes \mathcal{O}_2$.

Suppose now that there are given monotonous actions Ψ_A and Ψ_B of X on A and B, respectively, cf. Definition 1.2.6.

The group $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ is defined in Definition 1.2.10 as $\mathrm{KK}(\mathcal{C};\, A, B)$ for the cone $\mathcal{C}$ of Ψ_A-Ψ_B–residually nuclear maps.

$\mathrm{Ext}_{\mathrm{nuc}}(X;\, A, B) \cong G(H_0;\, A, B)$ and $S(H_0, A, B)$ are defined in Chapter 5.

Note that here $H_0 \colon A \to \mathcal{M}(B)$ denotes the “universal” non-degenerate weakly Ψ-residually nuclear C^*-morphism from A to $\mathcal{M}(B)$ such that $\delta_\infty H_0$ is unitarily equivalent to H_0.

`or is it only approximately unitary equivalent`
`with respect to the strict topology on` $\mathcal{M}(B)$ `...?`
`See def. in chp. 5 !!`

The existence and uniqueness of H_0 is assured if the m.o.c. cone of Ψ-residually maps from $A \to \mathcal{M}(B)$ has a countable generating set (as it is e.g. in the case where A and B are separable).

We list now the non-trivial changes of the above considered case X =point (with trivial action $\Psi_A(X) = A$, $\Psi_A(\emptyset) = \{0\}$).

In the beginning of the proof of Proposition 8.2.5(iii):

By Section 9, for D separable and stable and B σ-unital and stable, $\mathrm{Ext}_{\mathrm{nuc}}(X; D, B) = G(H_0; D, E)$ where $E := Q(B)$, $H_0 = \pi_B \circ H_1$ and $H_1 \colon D \to \mathcal{M}(B)$ is an infinite repeat of a sufficiently general non-degenerate weakly Ψ_A-Ψ_B–residually nuclear C^*-morphism from D in $\mathcal{M}(B)$ (The existence of such H_1 for *exact* A follows later from results in Chapter 12).

The unitary homotopy between $H_0 \oplus \psi$ and H_0 comes in the residually nuclear case from Theorem 5.9.3 in the same way as Theorem 5.6.2(ii) has been used in the proof.

`To-do-list: corollaries for chapter 9:`

`next:lem:8.Ynew.chp9`

LEMMA 8.4.2. *There are natural isomorphisms*

$$\mathrm{Ext}\big(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))\,;\, D, B \otimes C_0(\mathbb{R})\big) \;\cong\; \mathrm{KK}(\mathcal{C};\, D, B)\,.$$

The isomorphisms are compatible with respect to scaling of $\mathbb{R}$.

The classes in $\mathrm{KK}(\mathcal{C}\,;\; A, B)$ are invariant under homotopy.

There are natural isomorphisms

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)\,.$$

`Chp:9: By generalized Cor.8.3.3(iii) or Lem. 8.4.2:`

Suppose that ???????

For a C^*-morphism h from D to B with $h \in \mathcal{C}$ the mapping cone construction defines an element $\alpha[h] = [C_h]$ of $\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, D, SB)$ which is mapped to the element $[h - 0] \in \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)$ under the natural isomorphism from $\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, D, SB)$ onto $\mathrm{KK}(\mathcal{C};\, D, B)$.

`next lem:8.4new used in chp:9`

LEMMA 8.4.3. *, For every homomorphism $k \colon D \to E_{\mathbb{R}}$ that is dominated by H_0, there exists a unitary $u \in E_{\mathbb{R}}$ such that $u^*(k \oplus H_0)(a)u - H_0(a) \in J$ for each $a \in D$.*

`next mentioned in chp:1`

There is a natural group homomorphism for tensor products:

$$\mathrm{KK}(X;\, A, B) \times \mathrm{KK}(C, D) \to \mathrm{KK}(X;\, A \otimes C, B \otimes D)\,,$$

where the action is given, e.g. on $A \otimes C$, by

$$Z \in \mathbb{O}(X) \mapsto \Psi_A(Z) \otimes C \in \mathcal{I}(A \otimes C)\,.$$

Is it the same as

$$\mathrm{KK}(\mathcal{C};\, A, B) \times \mathrm{KK}(C, D) \to \mathrm{KK}(\mathcal{C} \otimes \mathrm{CP}(C, D);\, A \otimes C, B \otimes D)\,?$$

`my view on split extensions:`

REMARK 8.4.4. Suppose that

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

is a short exact sequence of graded C^*-algebras, and that $\mathcal{C}_1 \subseteq \mathrm{CP}(A, C)$,

?????

??

We define a map from the locally compact subspaces Y of $\mathbb{R}^2$ into the groups by

$$F(Y) := G(H_Y\,;\, D, E_Y\,) \cong \mathrm{Ext}(\mathcal{C}(Y);\, A, B \otimes \mathrm{C}_0(Y)) \cong \mathrm{KK}(\mathcal{C}(Y), A, SB \otimes \mathrm{C}_0(Y))\,.$$

Then $F(\mathbb{R}_+) = 0$ because $\mathrm{KK}(\mathcal{C};\, \cdot, \cdot)$ is homotopy invariant and the identity map of $B \otimes \mathrm{C}_0(\mathbb{R}_+)$ is homotopic to 0.

PROPOSITION 8.4.5. *With the above notation holds that*

$$\mathrm{K}_*(H_{[0,\infty)}(D)' \cap E_{[0,\infty)}) = 0\ ,$$

that there is a natural isomorphism from $F(\mathbb{R})$ onto the kernel of $\mathrm{K}_1(H_0(D)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B)$,
and that $[\widehat{\alpha} \circ h] = [h] \in \mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, B)$ for every $h\colon D \to E_{\mathbb{R}}$ with $[h] \in F(\mathbb{R})$ and every order-preserving homeomorphism α from $\mathbb{R}$ to $\mathbb{R}$ with $\alpha(0) = 0$.

The proof is contained in the further below given more explicit constructions,

`The reduction to Mayer--Vietoris does not work??`

but we give an independent proof, that reduce the proof to an application of the K_*-theory Mayer–Vietoris sequence to the Y-equivariant isomorphism of $F(Y)$ with the kernel of $\mathrm{K}_0(H_Y(D)' \cap E_Y) \to \mathrm{K}_0(E_Y) \cong \mathrm{K}_1(B \otimes \mathrm{C}_0(Y))$, (that we have seen in Chapters 5 and 4).

LEMMA 8.4.6. *Let $\mathcal{L}(L_2(S^1)) \cong \mathcal{M}(\mathbb{K}) \subseteq H(D)' \cap \mathcal{M}(B)$ given by a non-degenerate map from $\mathbb{K}$ into $H(D)' \cap \mathcal{M}(B)$, and let*

$$\mathrm{C}_0(\mathbb{R}) + \mathbb{C} \cdot 1 \cong \mathrm{C}(S^1) \subseteq \mathcal{L}(L_2(S^1))$$

given by multiplication of functions. Then the C^*-morphism

$$H_1 \colon D \otimes \mathrm{C}_0(\mathbb{R}) \;\to\; H(D)' \cap \mathcal{M}(B)$$

with $H_1(a \otimes f) := D(a)f$ satisfies the assumptions of Corollary 5.4.4 **??**. *for the cone $\mathcal{C} \otimes \mathrm{CP}(\mathrm{C}_0(\mathbb{R}), \mathbb{C})$. Thus*

$$\mathrm{Ext}(\mathcal{C}(\mathbb{R}, Y);\, D \otimes \mathrm{C}_0(\mathbb{R}), B \otimes \mathrm{C}_0(Y)) \cong G(\pi_B \circ H_1;\, D \otimes \mathrm{C}_0(\mathbb{R}), E_Y)$$

and is isomorphic to the kernel of

$$\mathrm{K}_0(T) \to \mathrm{K}_0(E_Y) \cong \mathrm{K}_1(B \otimes \mathrm{C}_0(Y))$$

where $T := \pi_B(H_1(D \otimes \mathrm{C}_0(\mathbb{R})))' \cap E_Y$.

There is an epimorphism *from* $\mathrm{Ext}(\mathcal{C}(\mathbb{R}, \mathbb{R}_+)\,;\, D \otimes \mathrm{C}_0(\mathbb{R}), B \otimes \mathrm{C}_0(\mathbb{R}_+))$ *onto the kernel of the group morphism*

$$\mathrm{K}_1(H_{[0,\infty)}(D)' \cap E_{[0,\infty)}) \to \mathrm{K}_1(E_{[0,\infty)}) \cong \mathrm{K}_0(B \otimes \mathrm{C}_0([0,\infty)))\,.$$

PROOF. **??** □

`Alternatively: use explicit` KK`-equivalences in` $\mathbb{E}(\mathbb{C}, \mathrm{C}_0(\mathbb{R})_{(1)})$ `and` $\mathbb{E}(\mathrm{C}_0(\mathbb{R})_{(1)}, \mathbb{C})$

PROOF OF PROPOSITION 8.4.5. : Since there are exact sequence

$$0 \to F(\mathbb{R}_+) \to \mathrm{K}_0(H_{[0,\infty)}(D)' \cap E_{[0,\infty)}) \to \mathrm{K}_1(B \otimes \mathrm{C}_0(\mathbb{R}_+)),$$
$$0 \to F(\mathbb{R}_-) \to \mathrm{K}_0(H_{[0,-\infty)}(D)' \cap E_{[0,-\infty)}) \to \mathrm{K}_1(B \otimes \mathrm{C}_0(\mathbb{R}_-)),$$

the monomorphism $F(Y) := G(H_Y;\, D, E_Y) \to \mathrm{K}_0(H_Y(D)' \cap E_Y)$ and the morphisms $\mathrm{K}_*(H_Y(D)' \cap E_Y) \to \mathrm{K}_*(E_Y)$ and the isomorphisms $\mathrm{K}_0(E_Y) \cong \mathrm{K}_1(B \otimes \mathrm{C}_0(Y))$ behave in a natural way equivariant with respect to homeomorphisms σ of Y, because $\widehat{\sigma} \circ H_Y = H_Y$. If X is a closed subset of Y such that for every $f \in \mathrm{C_b}(Y)_+$ with $f|X \in \mathrm{C}_0(X)$ there exists $g \in \mathrm{C_b}(Y)_+$ with $gf \in \mathrm{C}_0(Y)$ and $gf|X = f|X$, then the natural map $H_Y(D)' \cap E_Y \to H_X(D)' \cap E_X$ is an *epimorphism* (that is induced by the natural epimorphism $E_Y \to E_X$).

The algebra $H_\mathbb{R}(D)' \cap E_\mathbb{R}$ is the pullback of $H_{(-\infty,0])}(D)' \cap E_{(-\infty,0])}$ and $H_{[0,\infty)}(D)' \cap E_{[0,\infty)}$ by its natural morphism *onto* $H_{[0,\infty)}((D)' \cap E_{\{0\}}$. Since $\mathrm{K}_*(\mathrm{C}_0(\mathbb{R}_+, B)) = 0$ and $\mathrm{K}_*(H_{[0,\infty)}(D)' \cap E_{[0,\infty)}) = 0$ (by the first part), we get that $\mathrm{K}_1(\pi_B(H_0(D))' \cap \mathrm{Q^s}(B)) \cong \mathrm{K}_0(H_\mathbb{R}(D)' \cap E_\mathbb{R})$.

The diagram (and thus the isomorphism) is invariant under $\widehat{\sigma}$ if σ is an order preserving homeomorphism of $\mathbb{R}$ and $\sigma(0) = 0$. Then the morphisms of the Mayer–Vietoris sequence are also $\widehat{\sigma}$-equivariant. Since $\mathrm{K}_1(\widehat{\sigma}|\{0\}) = \mathrm{K}_1(\mathrm{id})$ on $\mathrm{K}_1(H_0(D)' \cap E_0)$ it follows $\widehat{\sigma} = \mathrm{id}$ on $\mathrm{K}_0(H_\mathbb{R}(D)' \cap E_\mathbb{R})$

More??? **??**

□

One can check that the kernel of $\mathrm{K}_1(H_0(D)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B)) \cong \mathrm{K}_0(B)$ is nothing else $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ if H_0 corresponds to $\mathrm{CP}_{\mathrm{nuc}}(A, B)$ (and similarly in the residually nuclear case $\mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$ or in the case of $\mathrm{KK}(\mathcal{C};\, A, B)$ with general non-degenerate m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$). Notice here that A and B are trivially graded.

`Protocol (until 16.9.2009):`
`Some TO DO list for chp.8, applied in Chapter 11:`

(0.1) In all points (*.*) and (*) below:

We should work with a suitable G-equivariant bi-functor $(A, B) \mapsto \mathcal{C}_{A,B} \subseteq \mathrm{CP}(A, B)$ on suitable categories of G-algebras with G-equivariant morphisms, where G is a Polish group.

(0.2) The only additional requirement for (G, X)-equivariant Kasparov modules is that A and B are graded G algebras. That ϕ, and the gradings are G-equivariant, and that the bi-module is G-modular, and that $(g(F) - F)\phi(A) \in \mathbb{K}(E)$ for all $g \in G$, and $g \to g(F)$ is norm-continuous.

(1) Each element of $\mathrm{KK}^G(X;\, A, B)$ can be represented by a non-degenerate $(G, X)-$ Kasparov A–B-module (E, ϕ, F) (i.e., $\phi(A)E = E$).

(It gives the σ-additivity of $\mathrm{KK}^G(X\, ;\, \cdot\, , B)$???

But it is probably only for a special class of groups G possible, that contains ??????? compact ????? groups (?).

For them, one can reduce all to the stable case and then use the Brown stability or Kasparov trivialization in a G-equivariant manner ????.)

(2) $\mathrm{KK}^G(X;\, \cdot\, , B)$ is half exact on (G, X)-semi-split sequences (and additive on (G, X)-split-exact sequences). Needed (!!!), because of the KK^G–proof in chapter 11.

(3) Desire: Give an explicit module for the KK-equivalence of split sequences.

(4) Morphisms:
If $\psi\colon A \to B$ satisfies $\psi \circ \mathcal{C}_1 \subseteq \mathcal{C}_2$, (that is symbolic $\psi \in \mathcal{C}_2\mathcal{C}_1^{-1} \subseteq \mathrm{CP}(A, B)$), then $\psi_*\colon \mathrm{KK}(\mathcal{C}_1\, ;\, D, A) \to \mathrm{KK}(\mathcal{C}_2\, ;\, D, B)$ is well-defined in the usual way.

If there are G-actions δ, α and β on D,A,B with $\alpha(g) \circ \mathcal{C}_1 \circ \delta(g^{-1}) \subseteq \mathcal{C}_1$ and $\beta(g) \circ \mathcal{C}_2 \circ \delta(g^{-1}) \subseteq \mathcal{C}_2$ for all $g \in G$, and if $\psi \circ \alpha(g) = \beta(g) \circ \psi$, then $\psi_*\colon \mathrm{KK}^G(\mathcal{C}_1\, ;\, D, A) \to \mathrm{KK}^G(\mathcal{C}_2\, ;\, D, B)$ is well-defined.

Suppose that G acts on a T_0-space X and that D, A, B are (G, X)-algebras. If $\psi\colon A \to B$ is compatible with the (G, X)-actions on A and B, then $\psi_*\colon \mathrm{KK}^G(X;\, D, A) \to \mathrm{KK}^G(X;\, D, B)$ *and* $\psi_*\colon {\mathrm{KK}_{\mathrm{nuc}}}^G(X;\, D, A) \to {\mathrm{KK}_{\mathrm{nuc}}}^G(X;\, D, B)$ are well-defined. (This can be deduced from the case of cone-depending KK, by taking $\mathcal{C}_1 := \mathcal{C}(X;\, D, A)$ and $\mathcal{C}_2 := \mathcal{C}(X;\, D, B)$, respectively $\mathcal{C}_1 := \mathcal{C}_{\mathrm{rn}}(X;\, D, A)$ and $\mathcal{C}_2 := \mathcal{C}_{\mathrm{rn}}(X;\, D, B)$.)

Similar (obvious "opposite") sufficient conditions can be given for the existence of $\psi^*\colon \mathrm{KK}^G(\mathcal{C}_4\, ;\, B, E) \to \mathrm{KK}^G(\mathcal{C}_3\, ;\, A, E)\colon$ ψ should be G-equivariant and $\mathcal{C}_4 \circ \psi \subseteq \mathcal{C}_3$

We get that $\psi^*\colon \mathrm{KK}^G(X;\, B, E) \to \mathrm{KK}^G(X;\, A, E)$ is well-defined if ψ is (G, X)-equivariant.

(5) The cones $\mathcal{C}$ needed for Mayer-Vietoris sequence:

(5a) Pull-back of epic morphisms $\varphi_k\colon A_k \to C$:
In the case of the pull-back $P := A_1 \oplus_{\varphi_1, \varphi_2} A_2$ of $\varphi_k\colon A_k \to C$ (with $\varphi_k \circ \mathcal{C}_{D,A_k} \subseteq \mathcal{C}_{D,C}$) one has to take the "pull-back" of the cones $\mathcal{C}_{D,A_k}$ as follows:

Let $\mathcal{C}_{D,P} \subseteq \mathrm{CP}(D,P)$ denote the set of all maps $d \mapsto V_1(d) \oplus V_2(d)$ with $V_1 \in \mathcal{C}_{D,A_1}$, $V_2 \in \mathcal{C}_{D,A_2}$, and $\varphi_1(V_1(d)) = \varphi_2(V_2(d))$ for all $d \in D$.

Notice that $\pi \circ \mathcal{C}_{D,P} \subseteq \mathcal{C}D, C$ for $\pi\colon a_1 \oplus a_2 \in P \to (1/2)(\varphi_1(a_1) + \varphi_2(a_2)) \in C$ and is G-equivariant, because the c.p. map $W\colon a_1 \oplus a_2 \in A_1 \oplus A_2 \mapsto (1/2)(\varphi_1(a_1) + \varphi_2(a_2))$ is G-equivariant and $W \circ (\mathcal{C}_{D,A_1} \oplus \mathcal{C}_{D,A_2}) \subseteq \mathcal{C}_{D,C}$.

The *co-homological case* has to consider given $\mathcal{C}_{A_k,E} \subseteq \mathrm{CP}(A_k, E)$ and $\mathcal{C}_{C,E} \subseteq \mathrm{CP}(C, E)$. The assumptions need the G–invariance of the cones and maps, e.g. $\eta(g) \circ \mathcal{C}_{C,E} \circ \gamma(g^{-1}) \subseteq \mathcal{C}_{C,E}$, etc.

(5b) How to construct the $(G, \mathcal{C})$-lift from lifts of φ_k?

In case that the φ_k are epimorphisms, the required lifts $T_k\colon C \to A_k$ (of φ_k for homology) should satisfy $T_k \circ \mathcal{C}_{D,C} \subseteq \mathcal{C}_{D,A_k}$ (and should be G-equivariant). In the co-homological situation, one has to require that $\mathcal{C}_{A_k,E} \circ T_k \subseteq \mathcal{C}_{C,E}$ etc.

Then $T\colon c \mapsto T_1(c) \oplus T_2(c)$ is a G-equivariant c.p. split of the epimorphism $\pi\colon P \to C$ with $\pi \circ T = \mathrm{id}_C$, and $T \circ \mathcal{C}_{D,C} \subseteq \mathcal{C}_{D,P}$.

(6) Mapping cones, Puppe sequence, Bott periodicity. (The modifications for the $\mathcal{C}$-version!):

$\psi\colon A \to B$ should be $(G, \mathcal{C})$ equivariant (in the relevant sense, depending from the homological or co-homological situation). The mapping cone C_ψ of ψ is the pull-back of $f \in CB = B \otimes \mathrm{C}_0((0,1]) \mapsto f(1) \in B$ and of $\psi\colon A \to B$. $\mathcal{C} =: \mathcal{C}_{D,B} \subseteq \mathrm{CP}(D, B)$ has to be extended to $\mathcal{C}_{D,CB} := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0((0,1])) =: \mathcal{C}(0,1] \subseteq \mathrm{CP}(D, CB)$.

This implies that $\mathcal{C}_{D,C_\psi} \subseteq \mathrm{CP}(D, C_\psi)$ has to be the the set of all those c.p. maps $V_1 \oplus V_2\colon D \to A \oplus CB$ with $V_1 \in \mathcal{C}_{D,A}$, $V_2 \in \mathcal{C}_{D,CB}$ and $\psi(V_1(d)) = V_2(d)(1)$ for all $d \in D$.

In the co-homological situation one has given $\mathcal{C}_{A,E}$ and $\mathcal{C}B, E$ and has to define $\mathcal{C}CB, E \subseteq \mathrm{CP}(CB, E)$ and $\mathcal{C}C_\psi, E \subseteq \mathrm{CP}(C_\psi, E)$: $\mathcal{C}CB, E := \mathcal{C}B, E \otimes \mathrm{CP}(\mathrm{C}_0((0,1]), \mathbb{C})$ $\mathcal{C}C_\psi, E$ as in the above given pull-back construction.

(7) The cones (and conditions in G-equivariant case) needed for the half-exactness on $(G, \mathcal{C})$-semi-split exact sequences.

(On (G, X)-semi-split exact sequences it should be nothing in addition to require.)

Let G a second countable l.c. group, that acts on A and D via $\alpha(g)$ and $\delta(g)$, and let $J \lhd A$ a G-invariant ideal with quotient map $\pi\colon A \to A/J$ and inclusion $\iota\colon J \to A$. The action on J and A/J should be the induced one.

Further, $\mathcal{C}_{D,A} \subseteq \mathrm{CP}(D, A)$, $\mathcal{C}_{D,J} \subseteq \mathrm{CP}(D, J)$, and $\mathcal{C}_{D,A/J} \subseteq \mathrm{CP}(D, A/J)$ are given. We require that $\pi \circ \mathcal{C}_{D,A} \subseteq \mathcal{C}_{D,A/J}$ and $\iota \circ \mathcal{C}_{D,J} \subseteq \mathcal{C}_{D,A}$, moreover, that $\alpha(g)\mathcal{C}_{D,A}\delta(g^{-1}) \subseteq \mathcal{C}_{D,A}$, $\alpha(g)\mathcal{C}_{D,J}\delta(g^{-1}) \subseteq \mathcal{C}_{D,J}$, and $[\alpha](g)\mathcal{C}_{D,A/J}\delta(g^{-1}) \subseteq \mathcal{C}_{D,A/J}$.

DEFINITION 8.4.7. We say that $V\colon A/J \to A$ is a $(G,\mathcal{C})$**-equivariant lift**, if V is completely positive, G-equivariant, and $V \circ \mathcal{C}_{D,A/J} \subseteq \mathcal{C}_{D,A}$. The sequence $0 \to J \to A \to A/J \to 0$ is called **homological** $(G,\mathcal{C})$**-semi-split**, if such V exists.

Similar, one has to require in the *co-homological* case that e.g. $\mathcal{C}_{A,D} \circ V \subseteq \mathcal{C}A/J, D$, etc.

The proof of the 6-term sequences of usual KK-theory applies now. because, the $(G,\mathcal{C})$-analogs of [**73**, lem. 19.5.3, thm. 19.5.5] work.

PROPOSITION 8.4.8. *If the sequence* $0 \to J \to A \to A/J \to 0$ *is* homological $(G,\mathcal{C})$*-semi-split, then the six-term exact sequence of the "homology"* $X \mapsto \mathrm{KK}^G(\mathcal{C}_{D,X}\,;\, D, X)$ *holds (cf.* [**73**, thm. 19.5.7]*).*

If the sequence $0 \to J \to A \to A/J \to 0$ *is* co-homological $(G,\mathcal{C})$*-semi-split, then the six-term exact sequence of the "cohomology"* $X \mapsto \mathrm{KK}^G(\mathcal{C}_{X,D}\,;\, X, D)$ *holds.*

(8) Comments on the applications of $KK^G(\mathcal{C};\,\cdot,\cdot)$ in Chapters 9-12.

(8a) The σ-additivity of $\mathrm{KK}^G(X;\,\cdot\,, D)$ and its consequences:
The σ-additivity immediately follows from the fact that each element can be represented by a *non-degenerate* Kasparov module. One of the consequences is the following lemma:

LEMMA 8.4.9. *Let* B_∞ *denote the inductive limit of a sequence of* (G, X)*-invariant injective morphisms* $\psi_n\colon B_n \to B_{n+1}$*, and suppose that the morphisms* ψ_n *satisfy the following conditions* (i) *and* (ii) *for every* $n \in \mathbb{N}$.

(i) *The element* $[(B_{n+1}, \psi_n, 0)]$ *of* $\mathrm{KK}^G(X;\, B_n, B_{n+1})$ *has an inverses element in* $\mathrm{KK}^G(X;\, B_{n+1}, B_n)$.
(ii) *For every finite subset* $F \subseteq B_n$ *and each* $\varepsilon > 0$*, there are* $m_2 > m_1 \geq n$ *and a* (G, X)*-equivariant c.p. contraction* $V\colon B_{m_2} \to B_{m_1}$ *such that* $\|\psi_n^{m_1}(x) - V(\psi_n^{m_2}(x))\| < \varepsilon$ *for all* $x \in F$.

Then $\psi_1^\infty\colon B_1 \to B_\infty := \mathrm{indlim}(\psi_n\colon B_n \to B_{n+1})$ *defines a* $\mathrm{KK}^G(X;\,\cdot,\cdot)$*–equivalence of* B_1 *and* B_∞*, i.e.,* $\mathrm{KK}(\psi_1^\infty) \in \mathrm{KK}^G(X;\, B_1, B_\infty)$ *admits an inverse in* $\mathrm{KK}^G(X;\, B_\infty, B_1)$.

Notice that (ii) is trivially satisfied for compact G and nuclear B_n (if the action of X on B_n is continuous).

PROOF. The condition (ii) ensures that there is a sequence $n_1 < n_2 < \cdots$ and (G, X)-equivariant completely positive maps $T_k\colon B_\infty \to B_{n_k}$ such that

$$S_k := psi_k^\infty \circ T_k$$

converges point-wise to id_{B_∞}. It follows that the epimorphism from the ((G, X)-contractible) mapping telescope T of the maps $(\psi_1, \psi_2, \ldots)$ onto the inductive limit B_∞ has an (G, X)-equivariant c.p. lift.

Then B_∞ is $\mathrm{KK}^G(X;\cdot,\cdot)$-equivalent to the kernel of $T \to B_\infty$. After realizing that, one is able to apply the standard arguments of cohomology theory to $h(\cdot) := \mathrm{KK}^G(X;\cdot,D)$ (for fixed D), because of homotopy invariance and half-exactness on (G,X)-split-exact sequences. Then one uses that $\mathrm{proj}-\lim^{(1)}(\psi_1^*,\psi_2^*,\ldots) = 0$, and gets that $\psi_1^\infty\colon B_1 \to B_\infty$ satisfies that $(\psi_1^\infty)^*\colon \mathrm{KK}^G(X;B_\infty,D) \to \mathrm{KK}^G(X;B_1,D)$ defines an isomorphism for $\mathrm{KK}^G(X;B_\infty,D)$ onto $\mathrm{KK}^G(X;B_1,D)$ for each separable (G,X)-algebra D.

In particular, there is $y \in \mathrm{KK}^G(X;B_\infty,B_1)$ with $(\psi_1^\infty)^*(y) = [\mathrm{id}_{B_1}]$. Since ψ_1^∞ is (G,X)-equivariant, it defines an element $z := [\psi_1^\infty] := [(B_\infty,\psi_1^\infty,0)]$ of $\mathrm{KK}^G(X;B_1,B_\infty)$ with $z\otimes_{B_\infty} y = (\psi_1^\infty)^*(y) = [\mathrm{id}_{B_1}]$. Then

$$(\psi_1^\infty)^*([\mathrm{id}_{B_\infty}] - (y\otimes_{B_1} z)) = z - (z\otimes_{B_\infty} y)\otimes_{B_1} z = 0\,,$$

hence, $y\otimes_{B_1} z = [\mathrm{id}_{B_\infty}]$ by injectivity of $(\psi_1^\infty)^*$. □

(8b) If A and B are trivially graded, then $[(B,\psi,0)] \in \mathrm{KK}^G(\mathcal{C};A,B)$ corresponds to the element of $\mathrm{Ext}^G(\mathcal{C}\otimes\mathrm{CP}(\mathbb{C},\mathrm{C}_0(\mathbb{R}));A,\mathrm{C}_0(\mathbb{R},B))$ that is defined by the extension of A by $\mathrm{C}_0((0,1),B)$ given by the mapping cone.

(8c) Notice that the mapping cone C_ψ of an (G,X)-equivariant morphism $\psi\colon A\to B$ is in a natural way a (G,X)-algebra (because defined by passage to the cone of B and a pull-back construction), and the lift $a\mapsto a\oplus(a\otimes f_0)\in C_\psi$ is a (G,X)-equivariant. The canonical 6-term exact sequence for the (G,X)-semi-split sequence $0\to SB\to C_\psi\to A\to 0$ has as its connecting maps δ are just the maps $(S\psi)^*\,\mathrm{KK}^G(X;SB,D)\to\mathrm{KK}^G(X,SA,D)$ and $\psi^*\colon \mathrm{KK}^G(X;B,D)\to \mathrm{KK}^G;A,D)$ (for each separable (G,X)-algebra D). Thus, we get:

Lemma 8.4.10. *The (G,X)-algebra C_ψ is $\mathrm{KK}^G(X;\cdot,\cdot)$-trivial, if and only if, $\psi\colon A\to B$ is a $\mathrm{KK}^G(X;\cdot,\cdot)$-equivalence.* □

This has a nice application in chapter 11.

Remark 8.4.11. Suppose that G is a Polish l.c. group, and A is a separable C^*-algebra with two actions α and β of G on A. Let $\mathcal{C} = \mathcal{C}_{in} \subseteq \mathrm{CP}(A,A)$ the cone of approximately inner c.p. maps of A.

If α and β are exterior *equivalent then there is an invertible element in $\mathrm{KK}^G(\mathcal{C};(A,\alpha),(A,\beta))$.*

Here α and β are **exterior equivalent** if there is a strictly continuous map $g\mapsto U(g)$ from G into the the unitary group $\mathcal{U}(\mathcal{M}(A))$, such that $U(gh) = U(g)\alpha(g)(U(h))$, and $\beta(g)(\cdot) = U(g)\alpha(g)(\cdot)U(g^{-1})$ for $g,h\in G$. (Here we extend $\alpha(g)$ naturally to $\mathcal{M}(A)$. It is the same as our notation $\mathcal{M}(\alpha(g))$ in other chapters.)

Proof. The arguments in the *proof* of [**698**, prop. 3.1] apply to our situation, because the cone of approximately inner c.p. maps V is invariant under conjugations, i.e., $\beta(g)\circ V\circ\alpha(g^{-1})$ if α and β are exterior equivalent. □

5. Verifying Condition (DC) of the basic Theorem 4.4.6

The verification of Condition (DC) of Theorem 4.4.6 for our axiomatic picture of $\mathrm{KK}(\mathcal{C}; A, B) \cong \mathrm{Ext}(\mathcal{C}(\mathbb{R}); A, SB)$ is completely contained in the following Proposition 8.5.1. Let us first collect some informations that allow to give a proof at the end of this Section.

In the following we make the overall assumptions that A and B are stable C^*-algebras (real or complex), A is separable, B is σ-unital, and E is any unital C^*-algebra.

We consider in the following Lemmata and in Proposition 8.5.4 also the (trivially graded) real case. Clearly in the complex case the skew-adjoint operators $a^* = -a$ are of the form $a = ib$ with $b^* = b$.

Say/Recall what is here a "non-degenerate" morphism.
Need to check first following facts:

Let A stable and separable, B stable and σ-unital, $CB := \mathrm{C}_0([0, \infty), B)$, $H \colon A \to \mathcal{M}(B)$ a non-degenerate C^*-morphism that is unitarily equivalent to $\delta_\infty \circ H$.

Consider the natural unital injection $\mathcal{M}(B) \subset \mathcal{M}(CB)$ of $\mathcal{M}(B)$ in $\mathcal{M}(CB)$. Then (by Kasparov ...?), $\mathcal{U}(\pi_{CB}(H(A))' \cap (\mathcal{M}(CB)/CB))$ is equal to
$\mathcal{U}_0(\pi_{CB}(H(A))' \cap (\mathcal{M}(CB)/CB))$
. ????

Proposition 8.5.1. *Let A and B stable C^*-algebras, where A is separable and B is σ-unital, and $H \colon A \to \mathcal{M}(B)$ a non-degenerate C^*-morphism in "general position" – in the sense of Definition 3.3.1, i.e., $H(A)B = B$ and H is unitarily homotopic to its infinite repeat $\delta_\infty \circ H$, cf. Definition 5.0.1 and Remark 5.1.1(8).*

Decide ! :
Is it equal to $H(A)B = B$ and that H is approximately unitary equivalent to $\delta_\infty \circ H$ in Norm or strict topology? ([10]).

Let $S, T \in H(A)' \cap \mathcal{M}(B)$ isometries with $SS^* + TT^* = 1$.

Suppose that there exists a strictly continuous map $t \in \mathbb{R} \mapsto U(t) \in \mathcal{M}(B)$ from $(-\infty, +\infty)$ into the unitaries of $\mathcal{M}(B)$ such that

(α) *$[U(t), H(a)] := U(t)H(a) - H(a)U(t) \in B$ for each $a \in A$,*
(β) *the map $\mathbb{R} \ni t \mapsto [U(t), H(a)]$ from $\mathbb{R}$ to B is continuous for each $a \in A$, and*
(γ) *$\lim_{t \to -\infty} \|U(t)H(a) - H(a)U(t)\| = 0$ for each $a \in A$.*

Then there exist strictly continuous maps $t \in \mathbb{R} \mapsto V(t) \in \mathcal{U}(\mathcal{M}(B))$ and $t \in \mathbb{R} \mapsto W(t) \in \mathcal{U}(\mathcal{M}(B))$ that have the following properties for each *$a \in A$:*

(i) *$U(t) \oplus_{S,T} 1 = V(t)W(t)$ for all $t \in \mathbb{R}$,*

[10] Then $\delta_\infty \circ H$ is approximately unitary equivalent to H in $\mathcal{M}(B)$ with respect to the strict topology on $\mathcal{M}(B)$ in sense of Remarks 5.1.1.

(ii) $t \mapsto [V(t), H(a)]$ *is a continuous map from* $\mathbb{R}$ *into* B,
(iii) $\lim_{t\to-\infty} \|[V(t), H(a)]\| = 0$ *and* $\lim_{t\to+\infty} \|[V(t), H(a)]\| = 0$,
(iv) $W(t) = 1$ *for all* $t \leq -1$ *and* $W(t) \in \mathcal{U}(B + \mathbb{C} \cdot 1) \subset \mathcal{U}(\mathcal{M}(B))$ *for all* $t \in \mathbb{R}$,
(v) $t \mapsto [W(t), H(a)] \in B$ *is a continuous map.*

Proposition 8.5.1 says with other words: If $U \in \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ is a unitary with $UH(a) - H(a)U \in \mathrm{C_b}(\mathbb{R}, B)$ and and $\lim_{t\to-\infty} \|U(t)H(a) - H(a)U(t)\| = 0$ for each $a \in A$, then $U \oplus_{S,T} 1$ decomposes as $U \oplus_{S,T} 1 = VW$, $V, W \in \mathcal{U}(\mathcal{M}(\mathrm{C}_0(\mathbb{R}, B)))$ with $W \in 1 + \mathrm{C_b}(\mathbb{R}, B)$ – which implies obviously $WH(a) - H(a)W \in \mathrm{C_b}(\mathbb{R}, B)$ –, with $W(t) = 1$ for $t \leq -1$ and $VH(a) - H(a)V \in \mathrm{C}_0(\mathbb{R}, B)$ for all $a \in A$.

Notice that the assumptions of Proposition the map $t \in \mathbb{R}_+ \to U(t) \in \mathcal{M}(B)$ corresponding to U is only strictly continuous.

We remind first some simple facts and (easy to see) rough estimates concerning skew-adjoint contactions $-c^* = c, c_1, \ldots, c_n \in E$, $\|c\| \leq 1$ in some real or complex unital C^*-algebra E:

(1) $\exp(c) \in \mathcal{U}_0(E)$ if $c^* = -c$.
(If E is complex then $c = ih$ for the self-adjoint element $h = -ic \in E$.)
(2) $\| [\exp(c), T] \| \leq 3 \| [c, T] \|$ for all $T \in E$.
(Use that $\|[c^n, T]\| \leq (n\|c\|^{n-1}) \cdot \|[c, T]\|$, $\|c\| \leq 1$ and $e \leq 3$.)
(3) If $c_1, \ldots, c_n \in E$ are skew-adjoint contraction and $T \in E$, then

$$\| [\exp(c_1) \cdot \exp(c_2) \cdot \ldots \cdot \exp(c_n), T] \| \leq 3n \max \{ \|[c_k, T]\| ; 1 \leq k \leq n \} .$$

(Use part (2) and that $\|\exp(c)\| = 1$ if $c = -c^*$.)

In the following Lemma let B denote a σ-unital (real or complex) C^*-algebra and let $D \subseteq \mathcal{M}(B)$ a (real or complex) separable C^*-subalgebra.

Recall that $\mathcal{M}(B)$ means the multiplier algebra of B and that $\mathcal{M}(B) \subseteq \mathcal{M}(CB)$ unitally by the natural non-degenerate embedding $B \subseteq \mathcal{M}(CB)$, where we define here $CB := \mathrm{C}_0((0, 1], B) \cong \mathrm{C}_0((\alpha, \beta], B)$ for each $\alpha < \beta \in \mathbb{R}$. In this way also $D \subseteq \mathcal{M}(B) \subseteq \mathcal{M}(CB)$ becomes a C^*-subalgebra of $\mathcal{M}(CB)$ by natural identification cf. prove of following Lemma 8.5.2 for details.

Lemma 8.5.2. *Let* B *a* σ*-unital (real or complex)* C^**-algebra,* $D \subseteq \mathcal{M}(B)$ *a separable* C^**-subalgebra of the multiplier algebra of* B, $b_1, \ldots, b_n \in \mathcal{M}(CB)$ skew-adjoint *contractions with the property that* $[b_k, d] \in CB$ *for all* $d \in D$ *and for each* $k = 1, \ldots, n$.

Then, for each given fixed contractions $d_1, \ldots, d_m \in D$ *and* $\varepsilon > 0$*, there exist a positive contraction* $e \in CB$ *and a normal element* $g \in CB$ *such that – with*

$$c_k := (1 - e^2)^{1/2} b_k (1 - e^2)^{1/2}$$

for $k \in \{1, \ldots, n\}$ *– and for* $\ell \in \{1, \ldots, m\}$ *holds:*

(i) $[c_k, d] \in CB$ *for all* $d \in D$,

(ii) $\|[c_k, d_\ell]\| < \varepsilon/(6n+7)$,
(iii) $1+g \in \mathcal{U}_0(\mathbb{C}\cdot 1 + CB)$, *– respectively* $1+g \in \mathcal{U}_0(\mathbb{R}\cdot 1 + CB)$ *–, and*
(iv) $V\cdot(1+g) = \exp(b_1)\cdot\exp(b_2)\cdot\ldots\cdot\exp(b_n)$
for the unitary $V := \exp(c_1)\cdot\exp(c_2)\cdot\ldots\cdot\exp(c_n) \in \mathcal{U}_0(\mathcal{M}(CB))$.

It implies $\|\,[V, d_\ell]\,\| < \varepsilon/2$ *for* $\ell = 1,\ldots,m$, $[V,d] \in CB$ *and* $[g,d], gd, dg \in CB$ *for all* $d \in D$.

There exist $\delta > 0$ *and* $-f^* = f \in CB$ *such that, for* $\ell = 1,\ldots,m$,

$$\|f\| \leq 1\,,\ \exp(-f)(1+g)|(0,\delta] \;=\; 1|(0,\delta] \ \textit{and}\ \|[f,d_\ell]\| < \varepsilon/(3(n+1))\,.$$

It implies that $h =: \exp(-f)(1+g) - 1 \in CB$ *satisfies* $h(t) = 0$ *for* $t \in (0,\delta]$, $V\cdot(1+g) = (V\cdot\exp(f))\cdot(1+h)$ *and*

$$\|\,[V\cdot\exp(f), d_\ell]\,\| < \varepsilon \quad \textit{for}\ \ell = 1,\ldots,m\,.$$

Moreover $1+h = \exp(-f)(1+g) \in \mathcal{U}_0(\mathbb{C}1 + CB)$ *(respectively* $1+h \in \mathcal{U}_0(\mathbb{R}1+CB)$ *in case of a real* C^**-algebra* B*).*

`The later used point is to manage that the`
`product of exponentials and`
`the unitary in` $\mathcal{U}(\mathrm{C}_0((-1,0], B+\mathbb{C}1))$
`becomes equal to 1 -- say at` $(-1,-1/2]$ `if defined on` $(-1,+\infty)$`.`

PROOF. The natural C^*-morphism from B into $\mathcal{M}(CB)$ is non-degenerate, i.e., the span of $B\cdot CB$ is dense in CB. Thus, there is a natural strictly continuous unital *-monomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(CB)$. We identify the elements of $\mathcal{M}(B)$ with there images in $\mathcal{M}(CB)$, i.e., $\mathcal{M}(B) \subset \mathcal{M}(CB)$ in a natural manner. Let $f_0 \in \mathrm{C}_0(0,1]$ the function with $f_0(t) := t$ for $t \in [0,1]$ and let $e_0 \in B_+$ a strictly positive contraction in B. Then $f_0 \otimes e_0 \in CB$ is a strictly positive contraction in CB and the intersection E of CB with the separable C^*-subalgebra $A := C^*(D \cup \{b_1,\ldots,b_n, f_0\otimes e_0\})$ of $\mathcal{M}(CB)$ is a separable C^*-subalgebra of CB and contains the strictly positive element $f_0 \otimes e_0$ of CB.

Thus, the natural inclusion of E in CB defines a unital monomorphism $\phi\colon \mathcal{M}(E) \to \mathcal{M}(CB)$ with the property that $A \subseteq \phi(\mathcal{M}(E)) \subseteq \mathcal{M}(CB)$.

It follows that $C^*(f_0 \otimes e_0)_+ \subseteq CB$ contains a quasi-central approximate unit $0 \leq e_1 \leq e_2 \leq \ldots$ for the elements in $C^*(f_0\otimes e_0)_+$, cf. [**616**, thm. 3.12.14, cor. 3.12.15]. It is important that we can find the e_n with the additional property $e_k e_\ell = e_k$ for $1 < k < \ell$.

This allows to apply the "mirror method". (The new application to the mirrored problem can be done without harming the before done steps.) The result is a decomposition of U into unitaries $U = VW$ with $W \in 1 + J$ and $[V, A] \subseteq \mathrm{C}_0(\mathbb{R}, B)$.

In particular, for each finite subsets $X \subset A$, $Z \subset (A \cap CB)$ and each $\delta > 0$ there exists a positive contraction $e \in CB_+$ with $\|ex - xe\| \,<\, \delta$ and $\|z - ze\| + \|z - ez\| \,<\, \delta$

for $x \in X$ and $z \in Z$. (It suffices to take here $Z := \{f_0 \otimes e_0\}$ and then the $\delta > 0$ sufficiently small.)

?? See for this also Remark 5.1.1(1) and its proof in Section 1 and the Section 3 of Chapter 5 for more details.??

By Proposition 5.3.1, for all $a \geq 0$, every *contraction* x and each $\beta \in [1, \infty)$,

$$\|[x, a^{1/\beta}]\| \leq 3\|[x, a]\|^{1/\beta}.$$

Apply this to $\beta := 2$, $a := 1 - e^2$ and x with $\|x\| \leq 1$ and get

$$\|[x, (1-e^2)^{1/2}]\| \leq 3\|[x, e^2]\|^{1/2} \leq 5\|[x, e]\|^{1/2}.$$

(i): Let $e \in (CB)_+$ a positive contraction. Then $b - (1-e^2)^{1/2} b (1-e^2)^{1/2} \in CB$ for all $b \in \mathcal{M}(CB)$ because $\pi_{CB}(1) = \pi_{CB}((1-e^2)^{1/2})$ for the quotient map $\pi_{CB} \colon \mathcal{M}(CB) \to \mathcal{M}(CB)/CB$.

If we apply π_{CB} to $c := (1-e^2)^{1/2} b (1-e^2)^{1/2} \in \mathcal{M}(CB)$ for $b \in \mathcal{M}(CB)$ with the property $[b, d] \in CB$ for all $d \in D \subseteq \mathcal{M}(CB)$, we get that $\pi_{CB}(c) = \pi_{CB}(b)$ and $\pi_{CB}([c, d]) = \pi_{CB}([b, d])$. Thus $[c, d] \in CB$ for all $d \in D$ if $[b, d] \in CB$ for all $d \in D$.

(ii): $\|[c_k, d_\ell]\| \leq \|[b_k, d_\ell]\| + 2\|[(1-e^2)^{1/2}, d_\ell]\|$ ...

But we have only $[b_k, d_\ell] \in CB$, need that $\|[c_k, d_\ell]\|$ is small.

Since $b_k \in \mathcal{M}(CB)$, $d_\ell \in \mathcal{M}(B) \subseteq \mathcal{M}(CB)$ and $[b_k, d_\ell] \in CB$ for $k = 1, \ldots, n$ $\ell = 1, \ldots, m$ – by assumptions –, we find for given $\delta > 0$ a positive contraction $e \in C * (f_0 \otimes e_0)_+ \subseteq CB$ with $\|[e, b_k]\| < \delta$, $\|[e, d_\ell]\| < \delta$ and $\|(1-e)[b_k, d_\ell]\| + \|[b_k, d_\ell](1-e)\| < \delta$ $f := (1-e^2)^{1/2}$, $1 \leq k \leq n$ and $1 \leq \ell \leq m$.

This ????????????

Let $b := b_k$ and $d := d_\ell$. All are positive contractions.

$(fbfd - dfbf) = fb(fd - df) - (fd - df)bf + f(bd - db)f$ can be estimated by $2\|b\|\|fd - df\| + \|f(bd - db)f\|$. Since $bd - db \in CB$

Let $\gamma := \varepsilon/(6n + 7)$. Since

$$G := C^*(D, b_1, \ldots, b_n) \subseteq \phi(\mathcal{M}(E)) \subseteq \mathcal{M}(CB)$$

is separable we find a quasi-central approximate unit in the positive contractions of CB for the elements of G by [**616**, thm. 3.12.14, cor. 3.12.15], i.e., we find a positive contraction $e \in CB$ with $\|[e, x]\| < \gamma/2$ for $x \in \{d_1, \ldots, d_m, b_1, \ldots, b_n\}$.

and $\|(1-e)[?, ?]\| + \|[?, ?](1-e)\| <$????? (small) for ????

Find contraction $e \in CB_+$ with $\|[(1-e^2)^{1/2} b_k (1-e^2)^{1/2}, d_\ell]\| < \gamma$ ...

(iii,iv): Define g by

$$g := \left(V^{-1} \cdot \exp(b_1) \cdot \exp(b_2) \cdot \ldots \cdot \exp(b_n)\right) - 1$$

for $V := \exp(c_1) \cdot \exp(c_2) \cdot \ldots \cdot \exp(c_n)$.

Since $\pi_{CB}(c_k) = \pi_{CB}(b_k)$ it follows that $\pi_{CB}(V) = \pi_{CB}\big(\exp(b_1)\cdot\exp(b_2)\cdot\ldots\cdot\exp(b_n)\big)$ and therefore $\pi_{CB}\big(V^{-1}\cdot\exp(b_1)\cdot\ldots\cdot\exp(b_n)\big) = \pi_{CB}(1)$, i.e., $g \in CB$.

Clearly $1+g \in \mathcal{U}(\mathbb{C}\cdot 1 + CB)$, respectively $1+g \in \mathcal{U}(\mathbb{R}\cdot 1 + CB)\cap(1+CB)$ in real case. In complex case we have always $\mathcal{U}(\mathbb{C}\cdot 1+CB) = \mathcal{U}_0(\mathbb{C}\cdot 1+CB)$ and in case of real C^*-algebras B holds $\mathcal{U}(\mathbb{R}\cdot 1+CB)\cap(1+CB) \subseteq \mathcal{U}_0(\mathbb{R}\cdot 1+CB)$.

`To be filled in??`

It follows that $\|\,[V, d_\ell]\,\| < \varepsilon/2$ for $\ell = 1,\ldots,m$, $[V,d] \in CB$ and $[g,d], gd, dg \in CB$ for all $d \in D$.

There exist $\delta > 0$ and $-f^* = f \in CB$ such that $\|f\| \leq 1$, $\exp(-f)(1+g)|(0,\delta] = 1|(0,\delta]$ and $\|[f, d_\ell]\| < \varepsilon/(3(n+1))$ for $\ell = 1,\ldots,m$.

It implies that $h =: \exp(-f)(1+g)-1 \in CB$, $h(t) = 0$ for $t \in (0,\delta]$, $V\cdot(1+g) = (V\cdot\exp(f))\cdot(1+h)$ and

$$\|\,[V\cdot\exp(f), d_\ell]\,\| < \varepsilon \quad \text{for } \ell = 1,\ldots,m\,.$$

□

`The following Lemma remains to be checked:`

In the following let A and B denote stable (real or complex) C^*-algebras. We require that A is separable and that B is σ-unital. We fix a a non-degenerate C^*-morphism $H\colon A \to \mathcal{M}(B)$ in general position, i.e., $H(A)B = B$ and H is in $\mathcal{M}(B)$ approximately unitary equivalent to its infinite repeat $\delta_\infty \circ H$.

Lemma 8.5.3. *Let A, B stable C^*-algebras, A separable B σ-unital, $H\colon A \to \mathcal{M}(B)$ non-degenerate faithful and in "general position", i.e., H and $\delta_\infty\circ H$ unitary homotopic in $\mathcal{M}(B)$.*

Given $U \in \mathcal{U}(\mathcal{M}(B[0,2]))$ with $U(t) = 1$ for $t \in [0,1)$ and $b_1,\ldots,b_n \in \mathcal{M}(B[0,2])$, $b_k^* = -b_k$ with $[b_k, H(a)] \in \mathrm{C}_0((1,2],B)$ for $k = 1,\ldots,n$, for all $a \in A$, and such that

$$U(t) = \exp(b_1(t))\cdot\ldots\cdot\exp(b_n(t))W(t)$$

with $W|[0,1] \in 1 + \mathrm{C}([0,1],B)$.

We find modifications b_k' of b_k and $W'(t)$ of $W(t)$ (the latter inside $1+\mathrm{C}([0,1],B)$) such that such that $W'(t) = 1$ for $t \in [0,1/2]$, $W'(1) = W(1)$, $b_k'(1) = b_k(1)$ and

$$U(t) = \exp(b_1'(t))\cdot\ldots\cdot\exp(b_n'(t))\cdot W'(t)$$

on $[0,1]$, and $\lim_{t\to 0} b_k'(t) = 0$ strictly.

Perhaps use an approximately central unit of B? With $b_k'(t) := (1-e(t)^2)^{1/2}b_k(t)(1-e(t)^2)$. Converges strictly to 0.

$$\pi_B(\exp(b_1'(t))\cdot\ldots\cdot\exp(b_n'(t))) = \pi_B(\exp(b_1(t))\cdot\ldots\cdot\exp(b_n(t)))$$

PROPOSITION 8.5.4. *Let A and B stable C^*-algebras (real or complex), where A is separable and B is σ-unital, and let $H\colon A \to \mathcal{M}(B)$ a non-degenerate C^*-morphism "in general position" (i.e., H is approximately unitary equivalent to $\delta_\infty \circ H$).*

in strict or norm topology ???
Compare Chp. 5

Moreover, let $S, T \in H(A)' \cap \mathcal{M}(B)$ isometries with $SS^* + TT^* = 1$.

The then following holds:

If $\alpha < \beta < \gamma \in \mathbb{R}$ and $t \in [\alpha, \gamma] \to u(t) \in \mathcal{U}(\mathcal{M}(B))$ is a strictly continuous map into the unitaries of $\mathcal{M}(B)$ with $u(t) = 1$ for $t \in [\alpha, \beta]$ and $u(t)H(a) - H(a)u(t) \in B$ for each $a \in A$ and $t \in [\alpha, \gamma]$.

Assume moreover that the maps $t \mapsto u(t)H(a) - H(a)u(t) \in B$ is continuous on $[\alpha, \gamma]$ for each $a \in A$ (i.e., $U := \{u(t)\}$ derives $H(A)$ into into $\mathrm{C}([\alpha, \gamma], B)$.

Then for each finite subset $X \subset A$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a decomposition $u(t) \oplus_{S,T} 1 = v(t) \cdot w(t)$ into a product of strictly continuous maps $t \in [\alpha, \gamma] \to v(t) \in \mathcal{U}(\mathcal{M}(B))$ and $t \in [\alpha, \gamma] \to w(t) \in \mathcal{U}(\mathcal{M}(B))$ such that $w(t) = 1 = v(t)$ for $t \in [\alpha, (\alpha + \beta)/2]$, $w(t) \in B + \mathbb{C} \cdot 1$ for $t \in [\alpha, \gamma]$

one can only deduce that it $w(t)$ is strictly continuous!!!!

with $t \in [\alpha, \gamma] \to w(t)$ norm-continuous, even with $w(t) \in 1 + B[\alpha, \gamma]$??

and

$$v(t) := \exp(b_1(t)) \cdot \exp(b_2(t)) \cdot \ldots \cdot \exp(b_n(t))$$

for suitable strictly continuous maps $t \mapsto b_k(t) = -b_k(t)^* \in \mathcal{M}(B)$ from $((\alpha + \beta)/2, \gamma]$ with $b_k(t) \in \mathrm{Der}(H(A), B)$ and $\sum_k \|[H(a), b_k(t)]\| < \varepsilon$ for all $a \in X$ and $t \in ((\alpha + \beta)/2, \gamma]$, and the maps

$$t \in ((\alpha + \beta)/2, \gamma] \mapsto [H(a), b_k(t)] \in B$$

are (norm-) continuous and satisfy

$$\lim_{t \to (\alpha+\beta)/2} \|[H(a), b_k(t)]\| = 0\,.$$

PROOF. The problem is:
Given $b_1(t), \ldots, b_n(t)$, with $b_k^* = -b_k \in \mathcal{M}(\mathrm{C}_0((-1/2, 0], B))$
strictly continuous with respect to t $\in$??? and bounded on $(-1/2, 0]$
with
$t \mapsto b_k(t)H(a) - H(a)b_k(t) \in \mathrm{C}_0((-1/2, 0], B)$
and $1 - \exp(b_1) \cdot \ldots \cdot \exp(b_n) \in \mathrm{C}_0((-1/2, 0], B)$.
Can we connect it in a controlled way inside
$[-1, 0]$ and $\mathcal{U}(B + \mathbb{C} \cdot 1)$ with 1 ?? □

Notice that we do not require that $\lim_{t \to (\alpha+\beta)/2} b_k(t) = 0$. Even not strictly.

But ?????

It can happen that distance of $w(\gamma)$ from $\mathbb{C} \cdot 1_{\mathcal{M}(B)}$ is equal to 2.

Remark 8.5.5. It seems that one can show that the number $n \in \mathbb{N}$ in Proposition 8.5.4 has an universal upper bound (clearly *not* depending on ε).

But it could be that ????????

Perhaps one get it by an indirect argument ?

For example one could consider for B the cone over $c_0(B_1, B_2, \ldots)$ from a sequence with $A = \mathbb{K} \otimes (\mathrm{C}(0,1] * \mathrm{C}(0,1])$ of $h_n \colon A \to \mathcal{M}(B_n)$, $\alpha := -1$, $\beta := 0$, $\gamma := 1$, $u_n \colon [-1,1] \to \mathcal{U}(\mathcal{M}(B))$ –

We are now in position to give the Proof of Proposition 8.5.1.

Proof of Proposition 8.5.1. `We can use a ''mirror principle'' and apply induction ...` □

CHAPTER 9

Scale-invariant maps (i+ii of Thm's B,M)

NEEDED???/CHECK???/PROBLEM:
In case A, B, stable and separable, that ???

$$\mathrm{Ext}(\mathcal{C};\, A, B) \cong \ker(\mathcal{U}(\pi_B(H_{\mathcal{C}}(A))' \cap \mathrm{Q}(B)) \to \mathcal{U}(\mathrm{Q}(B)))\,.$$

THIS should apply to $\mathrm{Ext}(\mathcal{C}(0,1]\,;\, A, \mathrm{C}_0((0,1], B)) = 0$ and gives then that each strictly continuous path of unitaries $t \in\to U(t) \in \mathcal{M}(B)$ with

$$\lim_{t\to 0} \|U(t)a - aU(t)\| = 0$$

and $U(t)a - aU(t) \in B$ for all $t \in (0,1]$

`This has NOT been proven somewhere:` is a finite product of exponentials $\exp(h(t))$ with $h(t)^* = -h(t)$ and $\lim_{t\to 0} \|h(t)a - ah(t)\| = 0$ and $h(t)a - ah(t) \in B$.

The number of this exponentials would then have moreover an universal upper bound.

The `injectivity` of

$$\vartheta\colon\ \mathrm{R}(\mathcal{C};\, A, B) \to \mathrm{KK}(\mathcal{C};\, A, B)$$

requires to use that $\mathcal{C} = \mathcal{C}(h_0)$ and we must show that the elements in the kernel of ϑ are homotopic in $\mathrm{R}(\mathcal{C};\, A, B)$ (via boundary maps from $\mathrm{R}(\mathcal{C}[0,1];\, A, B[0,1])$ at $\{0,1\}$ to $\mathrm{R}(\mathcal{C};\, A, B)$) to the class $[h_0]$.

Moreover, one has to show with the method of N.Ch. Phillips that "boundary-homotopic" elements of $\mathrm{SR}(\mathcal{C};\, A, B)$ define the same element of $\mathrm{Gr}(\mathrm{SR}(\mathcal{C};\, A, B))$.

??

In the meantime we rediscovered also our *direct* proof of the injectivity of the natural group homomorphism from $\mathrm{SR}(\mathcal{C};\, A, B)$ onto $\mathrm{Ext}(\mathcal{C};\, A, SB) \cong \mathrm{KK}(\mathcal{C};\, A, B)$ is injective. The proof of the injectivity uses a mirror-principle and triviality of $\mathrm{KK}(\mathcal{C};\, A, \mathcal{C}(\mathbb{R}_-, B))$, which follows from the homotopy invariance of KK via $\mathrm{KK}_c = \mathrm{KK}_{oh} = \mathrm{KK}_h$ (an observations of G. Kasparov).

We prove the parts (i) and (ii) of Theorem B and of Theorem M in this Chapter. We *assume the existence* of the non-degenerate nuclear C^*-morphism $h_0\colon A \to B$ as described in Chapter 1 before Theorem M. The existence of h_0 comes from Theorem A (in the case of Theorem B), and, — if B is separable, stable, strongly purely infinite and contains an Abelian regular C^*-subalgebra —, comes from Corollary 6.3.2.

The general existence of h_0 comes later from Theorem K (in the case of Theorem M). Theorem K will be proven in Chapter 12 with help of Corollary 6.3.2 and Corollary 9.1.7.

In Section 1 we derive also some useful applications of asymptotic maps that are asymptotically scale invariant up to approximate unitary equivalence.

The proof, as given in the second part of this Chapter, is (in conjunction with Theorem 4.4.6) a more detailed reformulation of the proof in the appendix of our preprint from December 1994 (3rd draft), more precisely: We show that the natural morphism from $\mathrm{R}(A, B)$ into $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ (resp. from $\mathrm{R}(X, A, B)$ into $\mathrm{KK}_{\mathrm{nuc}}(X; A, B)$ for $X := \mathrm{Prim}(B)$) is an isomorphism, cf. Corollary 9.4.2. Here we start with the *-monomorphism h_0 described below Theorem A (resp. before Theorem M) which defines the zero element of the R- and $\mathrm{KK}_{\mathrm{nuc}}$-groups and reduce the proofs of the parts (i) and (ii) of Theorems B and M to Theorem 4.4.6. This reduction requires results of Chapters 5 and 7, and Lemma 8.3.2.

Note that the material of Chapter 8 are modifications of well-known results of Kasparov, Skandalis and others, and can also be deduced from textbooks as e.g. [**73**], [**389**], at least in the situation of nuclear C^*-algebras, and only so far as it is needed for Corollary C, if the reader is only interested in the classification result Corollary C.

Then we use the isomorphism $\mathrm{R}(A, B) \cong \mathrm{KK}_{\mathrm{nuc}}(A, B)$ (resp. $\mathrm{R}(X;\, A, B) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$, $\mathrm{R}(\mathcal{C};\, A, B) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, A, B)$) to conclude with help of Corollary 8.3.4,(ii) (respectively with help of the generalization of Corollary 8.3.4 for $\mathrm{Ext}_{\mathrm{nuc}}(X;\, .,.)$ and $\mathrm{Ext}(\mathcal{C};\, .,.)$) that the elements of $\mathrm{R}(A, B)$ (respectively of $\mathrm{R}(X;\, A, B)$) are invariant under scaling in the sense of the below given Definition 9.1.1. We conclude that every element of $\mathrm{R}(A, B)$ (respectively of $\mathrm{R}(X;\, A, B)$, $\mathrm{R}(\mathcal{C};\, A, B)$) is given by a nuclear (respectively by a morphism $\varphi\colon A \hookrightarrow B$ with $\varphi \in \mathcal{C}$, e.g. for $\mathcal{C} = \mathrm{CP}_{\mathrm{rn}}(X;\, A, B)$). But this proves parts (i) and (ii) of Theorems B and M.

More generally, we show that
*if A is separable, B is σ-unital, A and B are stable, $h_0 = k(\cdot \otimes 1)$ for some non-degenerate *-monomorphism $k\colon A \otimes \mathcal{O}_2 \hookrightarrow B$, and if $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ denotes the point-norm closed matrix operator-convex cone generated by h_0, then the natural morphism*

$$\mathrm{R}(\mathcal{C};\, A, B) \to \mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, A, B \otimes \mathrm{C}_0(\mathbb{R}))$$

is an isomorphism.
The homotopy invariance of $\mathrm{KK}(\mathcal{C};\, A, B)$ and the natural isomorphism

$$\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, A, B \otimes \mathrm{C}_0(\mathbb{R})) \cong \mathrm{KK}(\mathcal{C};\, A, B)$$

(shown in Chapter 8) then imply that every element of $\mathrm{KK}(\mathcal{C};\, A, B)$ can be represented by a morphism h from A into B that is asymptotically dominated by h_0 (i.e., by $h \in \mathrm{Hom}(A, B)$ with $[h] \in \mathrm{SR}(\mathcal{C};\, A, B)$). Then the definition of $\mathrm{R}(\mathcal{C};\, A, B) = [h_0] + \mathrm{SR}(\mathcal{C};\, A, B) \cong \mathrm{Gr}(\mathrm{SR}(\mathcal{C};\, A, B))$ and the relations in $\mathrm{SR}(\mathcal{C};\, A, B)$ yield that

$[h] = [k]$ in KK($\mathcal{C}$; A, B) if and only if $h \oplus h_0$ and $k \oplus h_0$ are unitarily equivalent. (The latter can be deduced from the identity and isomorphism

$$G(h_0;\, A, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))) = \mathrm{R}(\mathcal{C};\, A, B) \cong KK(\mathcal{C};\, A, B)$$

and the considerations of Chapter 4 on this type of groups.)

1. Scale-invariant elements of SR(X; A, B)

We introduce a notion of generalized invariance under scaling for asymptotic maps:

Consider a topological isomorphism σ from $\mathbb{R}_+$ onto $\mathbb{R}_+$. That is, σ is a bijective continuous function of $\mathbb{R}_+$ onto $\mathbb{R}_+$. Note that $\sigma(0) = 0$ and that σ has to be strictly increasing.

Now let F be a C^*-algebra. The homeomorphism σ induces an automorphism $\widehat{\sigma}$ of $\mathrm{Q}(\mathbb{R}_+, F)$ in a natural way by mapping $\pi(f)$ to $\pi(f \circ \sigma)$, for $f \in \mathrm{C_b}(\mathbb{R}_+, F)$, where $\pi\colon \mathrm{C_b}(\mathbb{R}_+, F) \to \mathrm{Q}(\mathbb{R}_+, F)$ denotes the quotient map. Recall from Chapter 7 that $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ is a unital C^*-subalgebra of the multiplier algebra $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, F))$ of $\mathrm{Q}(\mathbb{R}_+, F)$.

Definition 9.1.1. Let X be a set and $h\colon X \to \mathrm{Q}(\mathbb{R}_+, F)$ a map. The map h will be called **invariant under scaling (up to unitary equivalence)** , if, for every topological isomorphism σ from $\mathbb{R}_+$ onto $\mathbb{R}_+$, $\widehat{\sigma} \circ h$ is unitarily equivalent to h by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F)) \subseteq \mathcal{M}(\mathrm{Q}(\mathbb{R}_+, F))$.

We say that h is **approximately scale-invariant** , if, for every topological isomorphism σ from $\mathbb{R}_+$ onto $\mathbb{R}_+$, $\widehat{\sigma} \circ h$ is approximately unitarily equivalent to h by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$.

An class $[h] \in [\mathrm{Hom}(A, \mathrm{Q}(\mathbb{R}_+, F))]$ will be called **invariant under scaling** if $[h] = [\widehat{\sigma} \circ h]$ for every topological isomorphism σ from $\mathbb{R}_+$ onto $\mathbb{R}_+$ ([1]).

Recall form Chapters 5 and 7 that, for a C^*-algebra F,

$$\mathrm{Q}(\mathbb{R}_+, F) := \mathrm{C_b}(\mathbb{R}_+, F)/\, \mathrm{C_0}(\mathbb{R}_+, F)\,,$$

that $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ is in a natural way a C^*-subalgebra of the multiplier algebra $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+.F))$ of $\mathrm{Q}(\mathbb{R}_+, F)$, and that, for every separable subset Y of $\mathrm{Q}(\mathbb{R}_+, F)$ and unitary v in the multiplier algebra of $\mathrm{Q}(\mathbb{R}_+, F)$ there is a unitary $u \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ with $u^*bu = v^*bv$ for every $b \in Y$, *so far as F is stable or is unital*, cf. Proposition 7.4.1. Thus, *one can replace* $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ *by* $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, F))$ *in Definition 9.1.1 if F is unital or F is stable.*

It is obvious from Definition 9.1.1 that, for every set X and every map $k\colon X \to F$ into the "constant" elements of $\mathrm{Q}(\mathbb{R}_+, F)$ and for every unitary $u \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$, the map $h(x) := u^*k(x)u$ is, up to unitary equivalence, invariant under scaling. The following Proposition 9.1.2 and Corollary 9.1.3 show that

[1] That means, by our definition of the classes $[\,\cdot\,]$ in Chapters 4 and 7, that $\widehat{\sigma} \circ h$ is unitarily equivalent to h in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$, i.e., h is invariant under scaling up to unitary equivalence.

for separable $h(X)$ this is the only example of a map which is scaling invariant up unitary equivalence.

PROPOSITION 9.1.2. *Suppose that X is a compact metric space, F is a σ-unital C^*-algebra, and that $h\colon X \to \mathrm{Q}(\mathbb{R}_+, F)$ is a continuous map.*

If h is approximately scale-invariant, then there exists a continuous map h_1 from X into F, such that h_1 is unitarily equivalent to h by a unitary U in the C^*-subalgebra $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$, of the multiplier algebra of $\mathrm{Q}(\mathbb{R}_+, F)$.

The following proof shows that we can find the map $h_1\colon X \to F$ and the unitary U such that $U = W + \mathrm{C}_0(\mathbb{R}_+, \mathcal{M}(F))$ with a unitary $W \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(F))$ such that $W(1) = 1_{\mathcal{M}(F)}$. It implies for the above example $h := u^* k(\cdot) u$, that there exists a unitary $W_0 \in \mathcal{M}(F) \subset \mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(F))$ such that $h_1 := W_0^* k(\cdot) W_0$ and $U = W_0^* u$ are as desired.

PROOF. Since $h \in \mathrm{C}(X, \mathrm{Q}(\mathbb{R}_+, F))$ and since $\mathrm{Q}(\mathbb{R}_+, F)$ is a quotient C^*-algebra of $\mathrm{C_b}(\mathbb{R}_+, F)$, there exists $T \in \mathrm{C}(X, \mathrm{C_b}(\mathbb{R}_+, F))$ such that $h(b) = T(b) + \mathrm{C}_0(\mathbb{R}_+, F)$ for every $b \in X$, i.e., $T\colon X \to \mathrm{C_b}(\mathbb{R}_+, F)$ is a topological lift of h.

If σ is a topological isomorphism of $\mathbb{R}_+$, then the bounded continuous function $t \mapsto T(b)(\sigma(t)) \in F$ is a lift of $\widehat{\sigma}(h(b))$ to $\mathrm{C_b}(\mathbb{R}_+, F)$.

Let u be a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$ such that, for all $b \in X$,

$$\|u^* h(b) u - \widehat{\sigma}(h(b))\| < \varepsilon/8\,.$$

There exists a unitary w in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(F))$ such that $u = w + \mathrm{C}_0(\mathbb{R}_+, \mathcal{M}(F))$. w is given by a norm continuous map $w\colon t \in \mathbb{R}_+ \to w(t) \in \mathcal{U}(\mathcal{M}(F))$ ([2]). It follows that

$$\gamma(b,t) := \|T(b)(\sigma(t)) - w(t)^* T(b)(t) w(t)\|$$

satisfies, for every $b \in X$,

$$\lim_{t\to\infty} \gamma(b,t) = \|u^* h(b) u - \widehat{\sigma}(h(b))\| < \varepsilon/8\,.$$

On the other hand, by the triangle inequality, for $t \in \mathbb{R}$ and $a, b \in X$,

$$|\gamma(a,t) - \gamma(b,t)| \,\leq\, 2\|T(a) - T(b)\|$$

(where the last norm comes from $\mathrm{C}(X, \mathrm{C_b}(\mathbb{R}_+, F))$), i.e., $\gamma(b,t)$ is uniformly continuous on the compact space X and, therefore,

$$\lim_{t\to\infty} (\sup_{b\in X} \gamma(b,t)) < \varepsilon/8\,.$$

Let $y \geq 0$. For $x > y$ and $\varepsilon > 0$, consider the set $M(x, \varepsilon)$ of the $s > x$ with the property that there exists a norm continuous map

$$v\colon t \in [x, s] \to v(t) \in \mathcal{M}(F)$$

[2]Note that the choice of w depends from u and thus from σ.

from $[x, s]$ into the unitary group $\mathcal{U}(\mathcal{M}(F))$ of the multiplier algebra of F with $v(x) = 1$ and

$$\|v(t)^* T(b)(t)v(t) - T(b)(x)\| < \varepsilon \tag{1.1}$$

for $b \in X$ and $t \in [x, s]$ ([3]). From the definition of $M(x, \varepsilon)$ we see that $M(x, \varepsilon/2) \subseteq M(x, \varepsilon)$ and $t \in M(x, \varepsilon)$ if $x < t < s$ and $s \in M(x, \varepsilon)$.

We define $m(x, \varepsilon) := \sup M(x, \varepsilon) \in [x, +\infty]$. The real number $2m(x, \varepsilon)$ is not in $M(x, \varepsilon)$ if $m(x, \varepsilon) < \infty$, because $x > 0$.

The properties of $M(x, \varepsilon)$ and

???? what else? **??**

lead to

$$m(x, \varepsilon/2) \leq m(x, \varepsilon)$$

for $\varepsilon > 0$. Moreover, $x < m(x, \varepsilon/2)$ for every $\varepsilon > 0$, because T is (uniformly) continuous.

Intuition says that for every $y \in \mathbb{R}_+$ and every $\varepsilon > 0$ there exists at least one $x > y$ with $m(x, \varepsilon) = \infty$. But we give a very detailed proof of this almost obvious fact:

Let $\varepsilon > 0$ and $y \in \mathbb{R}_+$ be fixed. We show below that the assumption that

$$m(x, \varepsilon) < \infty \quad \text{for every} \quad x > y \tag{1.2}$$

contradicts that h is of approximate scaling invariant.

If $m(x, \varepsilon) < \infty$ for every $x > y$, then we can find a sequence (x_n) in (y, ∞) such that, for every n,

$$n < x_n < m(x_n, \varepsilon/2) < 2m(x_n, \varepsilon) < x_{n+1} < \infty\,. \tag{1.3}$$

This allows us to define a topological isomorphism of $\mathbb{R}_+$ by taking a strictly increasing continuous piecewise linear map $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$, satisfying

$$\sigma(x_n) = x_n \qquad \text{and} \qquad \sigma(2m(x_n, \varepsilon)) = x_n + (m(x_n, \varepsilon/2) - x_n)/2 \qquad \text{for every} \quad n\,.$$

As we have seen above, by the assumption of approximate invariance of h under scaling (applied to σ), there exists a norm continuous map $t \mapsto w(t) \in \mathcal{U}(\mathcal{M}(F))$ with

$$\lim_{t\to\infty} \sup_{b\in X} \|T(b)(\sigma(t)) - w(t)^* T(b)(t)w(t)\| \leq \varepsilon/8.$$

Thus there exists x_n in our sequence such that

$$\|w(t)^* T(b)(t)w(t) - T(b)(\sigma(t))\| < \varepsilon/4 \tag{1.4}$$

for $t \geq x_n$ and $b \in X$.

On the other hand, by the definition of $m(x, \varepsilon/2)$, we can find a norm continuous map $t \mapsto z(t) \in \mathcal{U}(\mathcal{M}(F))$ such that $z(x_n) = 1$ and

$$\|z(t)^* T(b)(t)z(t) - T(b)(x_n)\| < \varepsilon/2 \tag{1.5}$$

[3] Note here that v can depend on s and ε.

for $t \in [x_n, x_n + (m(x_n, \varepsilon/2) - x_n)/2]$ and $b \in X$.

Since σ maps $[x_n, 2m(x_n, \varepsilon)]$ onto $[x_n, x_n + (m(x_n, \varepsilon/2) - x_n)/2]$ and since $z(t)$ is unitary, we get from equations (1.4) and (1.5) that, for $t \in [x_n, 2m(x_n, \varepsilon)]$ and $b \in X$,

$$\|w(x_n)T(b)(x_n)w(x_n)^* - T(b)(x_n)\| < \varepsilon/4,$$
$$\|z(\sigma(t))^* w(t)^* T(b)(t) w(t) z(\sigma(t)) - z(\sigma(t))^* T(b)(\sigma(t)) z(\sigma(t))\| < \varepsilon/4,$$
$$\|z(\sigma(t))^* T(b)(\sigma(t)) z(\sigma(t)) - T(b)(x_n)\| < \varepsilon/2 .$$

Let $v(t) := w(t)z(\sigma(t))w(x_n)^*$ for $t \in [x_n, 2m(x_n, \varepsilon)]$. Then the equation $v(x_n) = 1$ and applications of the triangle inequality and of the (isometric) inner automorphisms $w(x_n)(\cdot)w(x_n)^*$ show (together) that

$$\|v(t)^* T(b)(t) v(t) - T(b)(x_n)\| < \varepsilon$$

for every $t \in [x_n, 2m(x_n, \varepsilon)]$ and $b \in X$. It gives $2m(x_n, \varepsilon) \in M(x_n, \varepsilon)$. This leads to the (desired) *contradiction* by inequalities (1.3) because $m(x_n, \varepsilon) < \infty$ (by assumption).

We have seen above that, *for every $y \in \mathbb{R}_+$ and $\varepsilon > 0$, there exist $x > y$ with $m(x, \varepsilon) = \infty$.* Hence, we find a sequence $0 < x_1 < x_2 < \ldots \in \mathbb{R}$ such that $x_{n+1} > \max(x_n, n)$ and $m(x_n, 2^{-n}) = \infty$.

By the definition of $M(x_n, 2^{-n})$ and of $m(x_n, 2^{-n})$ there are norm continuous maps v_n from $[x_n, x_{n+1}]$ into $\mathcal{U}(\mathcal{M}(F))$ such that $v_n(x_n) = 1$ and

$$\|v_n(t)^* T(b)(t) v_n(t) - T(b)(x_n)\| < 2^{-n}$$

for every $t \in [x_n, x_{n+1}]$ and every $b \in X$.

By induction we define a norm continuous map W from $\mathbb{R}_+$ into the unitaries of $\mathcal{M}(F)$ by $W(t) := 1$ for $t \in [0, x_1]$ and $W(t) := v_n(t)V(x_n)$ for $t \in [x_n, x_{n+1}]$.

Since $v_n(x_n) = 1$ and v_n is norm continuous, the map $W : t \mapsto W(t)$ is well-defined and is continuous in norm. Then we have

$$\|W(t)^* T(b)(t) W(t) - W(s)^* T(b)(s) W(s)\| < 2^{(1-n)}$$

for $t, s \in [x_n, x_{n+1}]$ and $b \in X$. Iterated use of the triangle equation gives

$$\|W(t)^* T(b)(t) W(t) - W(s)^* T(b)(s) W(s)\| < 2^{(2-n)} \quad \text{for all } s, t \geq x_n .$$

Thus,

$$h_1(b) := \lim_{t \to \infty} W(t)^* T(b)(t) W(t)$$

exists in F for $b \in X$. If we consider h_1 as a map from X to $F \subseteq \mathrm{Q}(\mathbb{R}_+, F)$ then h_1 is unitarily equivalent to h by the unitary $U = W + \mathrm{C}_0(\mathbb{R}_+, \mathcal{M}(F))$ in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$. Hence h_1 is again continuous. □

COROLLARY 9.1.3. *Let D be a separable C^*-algebra, F a stable or a unital C^*-algebra and $h : D \to \mathrm{Q}(\mathbb{R}_+, F)$ a C^*-morphism.*

If h is approximately scale-invariant, then there exists a ("constant") C^*-morphism k from D into $F \subseteq \mathrm{Q}(\mathbb{R}_+, F)$ such that h is unitarily equivalent to k by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$.

In particular, for separable stable exact A and σ-unital stable B, an element $[h]$ of $\mathrm{SR}(X, A, B)$ is invariant under scaling if, and only if, $[k] = [h]$ for a "constant" C^*-morphism k from A into $B \subseteq \mathrm{Q}(\mathbb{R}_+, B)$.

PROOF. Let $d_1, d_2, \ldots$ be a dense sequence in the unit ball of D and let $X := \{0\} \cup \{2^{-n} d_n : n = 1, 2, \ldots\}$. Then D is the closed span of the compact set X, and the restriction of h to X satisfies the requirements of Proposition 9.1.2.

Thus, there is continuous map $h_1 : X \to F$ and a unitary u in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$, which is considered as a C^*-subalgebra of the multiplier algebra of $\mathrm{Q}(\mathbb{R}_+, F)$, such that $u^* h(d) u = h_1(d)$ for $d \in X$.

The map $k : d \in D \mapsto u^* h(d) u$ is a C^*-morphism from D into $\mathrm{Q}(\mathbb{R}_+, F)$ which extends h_1 to a C^*-morphism k from D into $\mathrm{Q}(\mathbb{R}_+, F)$, and k is unitarily equivalent to h by a the unitary u in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$.

$k(D) \subseteq F$, because the linear span of X is dense in D and $k|X = h_1$, i.e., k is "constant".

If $D := A$ and $F := B$, then, by Proposition 7.4.1 the various versions of unitary equivalence coincide and we may consider the unitary equivalence by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(F))$. Then the scale-invariance of $[h]$ means that h is invariant under scaling up to unitary equivalence. Hence h is approximately scale-invariant. □

We get the following useful Corollaries 9.1.4 and 9.1.6, which we apply in Chapters 10 and 12.

COROLLARY 9.1.4. *Suppose that A and B are C^*-algebras, where A is stable and separable. Let $h_0 : A \to B$ be a non-degenerate *-monomorphism.*

*Then h_0 is unitarily homotopic to $h_0 \oplus h_0$, if and only if, there exists a *-monomorphism $k : A \otimes \mathcal{O}_2 \to B$, such that $k_0(a) := k(a \otimes 1)$ is unitarily homotopic to h_0.*

The monomorphism k can be chosen to be non-degenerate, and then $k_0(A)$ commutes with a copy of $\mathcal{O}_2$ which is unitally contained in $\mathcal{M}(B)$.

PROOF. B is stable and σ-unital, because A is stable and separable and h_0 is non-degenerate, i.e., $h_0(A)B$ is dense in B.

Suppose that $k : A \otimes \mathcal{O}_2 \to B$ is such that $k_0(a) := k(a \otimes 1)$ is unitarily homotopic to h_0. Then $k(A \otimes \mathcal{O}_2)$ is a stable C^*-subalgebra that generates B as a closed ideal, because h_0 is non-degenerate. By Corollary 5.5.6(iv), k is unitarily homotopic to a non-degenerate *-monomorphism from $A \otimes \mathcal{O}_2$ into B, because B is stable and σ-unital. Therefore, we can assume that k is non-degenerate.

The extension $\mathcal{M}(k)$ of k to a unital *-monomorphism from $\mathcal{M}(A \otimes \mathcal{O}_2)$ to $\mathcal{M}(B)$ maps $1 \otimes \mathcal{O}_2$ unitally into the commutant of $k_0(A)$. Thus, k_0 is unitarily equivalent to $k_0 \oplus k_0$. It follows that h_0 is unitarily homotopic to $h_0 \oplus h_0$.

Now suppose, that h_0 is unitarily homotopic to $h_0 \oplus h_0$.

By Proposition 7.4.2, there exists, up to unitary equivalence by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$, a unique *-monomorphism $k_1 \colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, B)$ with $k_1(a \otimes 1) = h_0(a)$ such that $k_1 \oplus k_1$ is unitarily equivalent to k_1 in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

If we compose k_1 with $\widehat{\sigma}$ for a scaling σ of $\mathbb{R}_+$, then $\widehat{\sigma} k_1$ has the same properties as k_1 with respect to h_0. Thus, $\widehat{\sigma} k_1$ is unitarily equivalent to k_1 by Proposition 7.4.2.

By Corollary 9.1.3, k_1 is unitarily equivalent to a *-monomorphism k from $A \otimes \mathcal{O}_2$ into B by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$. But then $k_0 = k((\cdot) \otimes 1)$ is unitarily equivalent to h_0 in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$, and this means that h_0 and k_0 are unitarily homotopic. □

REMARK 9.1.5. In view of the following assumptions on asymptotic embeddings of a separable C^*-algebra A it seems to be useful to remember the following basic observations at the beginning of Chapter 3:

Let A a C^*-algebra. Then the following are equivalent:

(i) A is an exact C^*-algebra.
(ii) There exists a faithful nuclear *-representation $\rho \colon A \to \mathcal{L}(\mathcal{H})$ over some Hilbert space $\mathcal{H}$.
(iii) Each C^*-morphism $h \colon A \to B$ into a weakly injective C^*-algebra B is nuclear.
(iv) There exists a nuclear faithful C^*-morphism $h \colon A \to C$ into some C^*-algebra C.
(v) There exists a nuclear completely isometric embedding of A into some C^*-algebra D.

PROOF. The equivalence of (i) and (ii) is discussed in Remarks **??** ????? Section 1 of Chapter 3. The equivalence of (ii)-(v) follow essentially from the Arveson extension theorem [**42**], i.e., that each of the conditions (iii)-(v) implies that the faithful *-representations of A on a Hilbert space $\mathcal{H}$ are nuclear, cf. Chapter 3 for more details. □

COROLLARY 9.1.6. *Suppose that A and B are stable C^*-algebras, where A is separable and exact, and that k is a nuclear *-monomorphism from $A \otimes \mathcal{O}_2$ into* $\mathrm{Q}(\mathbb{R}_+, B)$.

Let $k_0(a) := k(a \otimes 1)$ for $a \in A$.

If $k_0(A) \cap J_1 = k_0(A) \cap J_2$ for all pairs of closed ideals J_1 and J_2 of $\mathrm{Q}(\mathbb{R}_+, B)$ *with $J_1 \cap B = J_2 \cap B$, then there exists a nuclear *-monomorphism h from $A \otimes \mathcal{O}_2$ into B, such that the *-monomorphisms $h_0 := h((\cdot) \otimes 1)$ and k_0 from A into* $\mathrm{Q}(\mathbb{R}_+, B)$ *are unitarily equivalent by a unitary in* $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

If, moreover, B is σ-unital, and B is contained in the closed ideal of $\mathrm{Q}(\mathbb{R}_+, B)$ *that is generated by the image of k, then h can be found such that, moreover, h_0 is non-degenerate and unitarily equivalent to $h_0 \oplus h_0$ by a unitary in $\mathcal{M}(B)$.*

PROOF. If σ is a homeomorphism of $\mathbb{R}_+$, $a \in A$, $I \in \mathcal{I}(\mathrm{Q}(\mathbb{R}_+, B))$, then $\widehat{\sigma} k_0(a) \in I$, if and only if, $k_0(a) \in \widehat{\sigma}^{-1}(I)$, if and only if, $k_0(a) \in I$, because $B \cap I = B \cap \widehat{\sigma}^{-1}(I)$.

Thus, k_0 and $k_1 := \widehat{\sigma} \circ k_0$ are nuclear *-monomorphisms, such that k_0 and k_1 extend to C^*-morphisms k and $\widehat{\sigma} k$ from $A \otimes \mathcal{O}_2$ into $\mathrm{Q}(\mathbb{R}_+, B)$, and, for every $I \in \mathcal{I}(\mathrm{Q}(\mathbb{R}_+, B))$,

$$k_0^{-1}(k_0(A) \cap I) = k_1^{-1}(k_1(A) \cap I)\,.$$

By Corollary 7.4.3, this implies that k_0 and k_1 are approximately unitarily equivalent.

Therefore k_0 is approximately scale-invariant. By Corollary 9.1.3, k_0 is unitarily equivalent to a C^*-morphism h_1 from A into B by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$. By Corollary 7.4.3, k_0 and $k_0 \oplus k_0$ are unitarily equivalent in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$. Thus h_1 is unitarily homotopic to $h_1 \oplus h_1$. Now we can proceed as in the proof of Corollary 9.1.4, and get a *-monomorphism $h\colon A \otimes \mathcal{O}_2 \to B$, such that $h_0 := h((\cdot) \otimes 1)$ is unitarily homotopic to h_1. Thus, h_0 and k_0 are unitarily equivalent in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$.

It follows that h_0 is a nuclear map, if we consider it as a map from A into $\mathrm{Q}(\mathbb{R}_+, B)$. But B is relatively weakly injective in $\mathrm{Q}(\mathbb{R}_+, B)$, i.e., there is a normal conditional expectation from the second conjugate of $\mathrm{Q}(\mathbb{R}_+, B)$ onto the second conjugate of B ([4]). Thus h_0 is also a nuclear map from A into B.

We use the nuclearity criteria (i) in Remark 3.1.2, to show that h is nuclear if h_0 is nuclear: Let C be a C^*-algebra. Consider the natural C^*-morphism

$$g := \mathrm{id}_C \otimes^{max} h\colon C \otimes^{max} (A \otimes \mathcal{O}_2) \to C \otimes^{max} B\,.$$

The restriction of g to $C \otimes^{max} (A \otimes 1)$ annihilates the kernel of the natural epimorphism from $C \otimes^{max} (A \otimes 1)$ onto $C \otimes (A \otimes 1)$, because h_0 is nuclear. Since, by nuclearity of $\mathcal{O}_2$, on the algebraic tensor product $(C \otimes A) \odot \mathcal{O}_2$ there is a unique C^*-norm, we get that g annihilates the kernel of the natural epimorphism from $C \otimes^{max} (A \otimes \mathcal{O}_2)$ onto $C \otimes (A \otimes \mathcal{O}_2)$. Thus h is nuclear.

Now we suppose, in addition, that the closed ideal I_0 of $\mathrm{Q}(\mathbb{R}_+, B)$, which is generated by $k_0(A)$, contains B, and that B is σ-unital.

Let J denote the closed ideal of B, which is generated by $h_0(A)$. The ideal $\mathrm{Q}(\mathbb{R}_+, J)$ of $\mathrm{Q}(\mathbb{R}_+, B)$ contains $k_0(A)$ and I_0, because $h_0(A)$ and $k_0(A)$ generate the same ideal of $\mathrm{Q}(\mathbb{R}_+, B)$. Since $B \cap \mathrm{Q}(\mathbb{R}_+, J) = J$ and $B \cap I_0 = B$, we get $B = J$. Thus, $h(A \otimes O_2)$ is a stable C^*-subalgebra of B and the closed ideal generated by $h(A \otimes O_2)$ is the σ-unital stable C^*-algebra B. By Corollary 5.5.6(iv), h is unitarily homotopic to a non-degenerate *-monomorphism h' from A to B. The

[4]Use normalizations of suitable c.p. maps $\mathrm{Q}(\mathbb{R}_+, B) \to \ell_\infty(B^{**})/c_0(B^{**}) \to B^{**}$.

non-degenerate *-monomorphism h_0' is again unitary equivalent to k_0 by a unitary in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$. The image of h_0' commutes element-wise with the unital copy $\mathcal{M}(h')(1_{\mathcal{M}(A)} \otimes \mathcal{O}_2)$ of $\mathcal{O}_2$, if $\mathcal{M}(h')$ is the unital extension of h' to the multiplier algebras. □

We can *rephrase* Corollary 9.1.6 in the language of actions of $\mathrm{Prim}(B)$ on A as follows:

Corollary 9.1.7. *Suppose that A and B are stable separable C^*-algebras, and that $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, such that $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$.*

*If $k\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, B)$ is a nuclear *-monomorphism, such that, for every closed ideal I of $\mathrm{Q}(\mathbb{R}_+, B)$ and for $k_0 := k((\cdot) \otimes 1)$,*

$$k_0(\Psi(I \cap B)) = k_0(A) \cap I,$$

then there exists a non-degenerate nuclear C^*-morphism $h\colon A \otimes \mathcal{O}_2 \to B$, such that $h_0(\Psi(J)) = h_0(A) \cap J$ for every closed ideal J of B, where $h_0(a) := h(a \otimes 1)$.

*The morphism h_0 is unitarily equivalent to $h_0 \oplus h_0$. Every nuclear *-monomorphism $h_1\colon A \to B$ which satisfies $h_1(\Psi(J)) = h_1(A) \cap J$ for $J \in \mathcal{I}(B)$, and is unitarily homotopic to $h_1 \oplus h_1$, is unitarily homotopic to h_0.*

Proof. Since the *-monomorphism $k\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}(\mathbb{R}_+, B)$ is nuclear, the algebra A is exact.

By assumption, the intersection of the image of k_0 with a closed ideal of $\mathrm{Q}(\mathbb{R}_+, B)$ depends only from the intersection of the ideal with B.

The closed ideal I_0 of $\mathrm{Q}(\mathbb{R}_+, B)$ which is generated by the image of $k_0 := k((\cdot)\otimes 1)$ satisfies $k_0(\Psi(B \cap I)) = k_0(A)$. Since $\Psi^{-1}(A) = \{B\}$, we have $B \cap I = B$. Thus Corollary 9.1.6 applies to k. It gives the existence of the desired non-degenerated nuclear h with h_0 unitarily equivalent to $h_0 \oplus h_0$ by a unitary in $\mathcal{M}(B)$.

Since h_0 is unitarily homotopic to k_0, for $a \in A$ and closed ideals I of $\mathrm{Q}(\mathbb{R}_+, B)$, $h_0(a) \in I$ if and only if $k_0(a) \in I$. This applies to the ideals $I := \mathrm{Q}(\mathbb{R}_+, J)$, where $J \in \mathcal{I}(B)$. We have $J = I \cap B$. Thus, $h_0(\Psi(J)) = h_0(A) \cap J$ for $J \in \mathcal{I}(B)$.

If $h_1\colon A \to B$ is a nuclear *-monomorphism, such that h_1 is unitarily homotopic to $h_1 \oplus h_1$, and such that, for $J \in \mathcal{I}(B)$,

$$h_1^{-1}(h_1(A) \cap J) = h_0^{-1}(h_0(A) \cap J),$$

then h_0 and h_1 are unitarily homotopic by Corollary 7.4.3. □

2. Mapping-cone construction defines isomorphism

Let D a stable separable C^*-algebra. We start with any non-degenerate *-monomorphism $k\colon D\otimes\mathcal{O}_2 \to B$, i.e., we suppose that D, k and $\mathcal{C}$ with the following properties (a)–(c) are given:

(a) D is a separable and stable C^*-algebra,
(b) $k\colon D\otimes\mathcal{O}_2\to B$ is a non-degenerate *-monomorphism, and
(c) $\mathcal{C}\subseteq \mathrm{CP}(D,B)$ is the point-norm closed matrix operator-convex cone that is generated by the *-monomorphism $k_0 := k((\cdot)\otimes 1)\in \mathrm{Hom}(A,B)$.

The p.i. algebra $k(A\otimes\mathcal{O}_2)$ is a non-degenerate C^*-subalgebra of B by (b), i.e., $k(A\otimes\mathcal{O}_2)B$ is dense in B and k extends uniquely to a strictly continuous unital monomorphism $\mathcal{M}(k)$ from $\mathcal{M}(A\otimes\mathcal{O}_2)$ into $\mathcal{M}(B)$. It follows that

(d) B is σ-unital and stable, cf. Remark 5.1.1(9),
(e) the morphism $k_0 := k((\cdot)\otimes 1)$ is is unitarily equivalent to $k_0\oplus k_0$ by a unitary in $\mathcal{M}(B)$ by Proposition 4.3.5(iii), because $k_0(A)$ commutes with the unital copy $\mathcal{M}(k)(1_{\mathcal{M}(A)}\otimes\mathcal{O}_2)$ of $\mathcal{O}_2$ in $\mathcal{M}(B)$), and
(f) $k_0\colon D\to B$ is non-degenerate.

We know from Corollary 7.4.19, that

$$\mathrm{SR}(\mathcal{C};\, D,B) = S(k_0;\, D, \mathrm{Q}(\mathbb{R}_+,\mathcal{M}(B)))\,,$$

and that a non-degenerate *-morphism $k_1\in\mathrm{Hom}(A,B)\cap\mathcal{C}$ is necessarily unitarily homotopic to k_0 if k_1 generates $\mathcal{C}$ and if k_1 is unitarily homotopic to $k_1\oplus k_1$. In so far k_0 and $\mathcal{C}$ determine each other uniquely.
We recall some definitions and results of Chapters 5, 7 and 8:

Remark 9.2.1. For locally compact Y, e.g. if Y equals one of $\mathbb{R}$, $\mathbb{R}_+=[0,\infty)$ or $\mathbb{R}_- := (-\infty,0]$, we have defined in Chapter 7 the asymptotic coronas $\mathrm{Q}(Y,B)$:

$$\mathrm{Q}(Y,B) := \mathrm{C_b}(Y,B)/\,\mathrm{C_0}(Y,B).$$

Recall that there is a natural isomorphism

$$\mathrm{Q}(\mathbb{R},B)\cong \mathrm{Q}(\mathbb{R}_-,B)\oplus\mathrm{Q}(\mathbb{R}_+,B),$$

and that $\mathrm{Q}(\mathbb{R},B)$ is an ideal in $E_{\mathbb{R}}=\mathrm{Q}^s(SB)$, where we write

$$SB := \mathrm{C_0}(\mathbb{R},B)\qquad\text{and}\qquad E_{\mathbb{R}} := \mathcal{M}(SB)/SB\cong \mathrm{Q}^s(SB).$$

We denote the canonical epimorphism from $\mathcal{M}(SB)$ onto $E_{\mathbb{R}}$ by π_{SB}.

Thus $\mathrm{Q}(\mathbb{R}_+,B)$ contains B naturally and $\mathrm{Q}(\mathbb{R}_+,B)$ is naturally isomorphic to an ideal of $E_{\mathbb{R}}$. We let J denote the closed ideal of $E_{\mathbb{R}}$ that is naturally isomorphic to $\mathrm{Q}(\mathbb{R}_+,B)$ and let $I_1\colon\, \mathrm{Q}(\mathbb{R}_+,B)\to J$ denote the natural isomorphism from $\mathrm{Q}(\mathbb{R}_+,B)$ onto $J\subseteq E_{\mathbb{R}}$. We define a monomorphism $h_0\colon D\to J$, by

$$h_0 := I_1\circ k_0\,. \tag{2.1}$$

To make I_1 and all related calculations of later identities explicit, we can take the continuous function $\lambda(t):=\min(1,\max(0,t))$ (for $t\in\mathbb{R}$) and build an element $\widetilde{f}(t):=\lambda(t)f(t)$ in $\widetilde{f}\in\mathrm{C_b}(\mathbb{R},B)$ by $\widetilde{f}(t):=\lambda(t)\cdot f(t)$ ($t\in[0,\infty)$) and $\widetilde{f}(t):=0$ for $t<0$, where $f\in\mathrm{C_b}(\mathbb{R}_+,B)$ is a given representative of an element $f+\mathrm{C_0}(\mathbb{R}_+,B)$ in $\mathrm{Q}(\mathbb{R}_+,B)$. Then

$$I_1\bigl(f+\mathrm{C_0}(\mathbb{R}_+,B)\bigr)=\widetilde{f}+\mathrm{C_0}(\mathbb{R},B)\,.$$

All notions of unitary equivalence in $\mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))$ coincide with the equivalence induced by unitaries of $E_{\mathbb{R}} := \mathcal{M}(SB)/SB \cong \mathrm{Q}^s(SB)$, or by unitaries coming from $\mathrm{C_b}(\mathbb{R}, \mathcal{M}(B))$, because B is stable, cf. Proposition 7.4.1. Therefore, the inclusion map $I_1\colon \mathrm{Q}(\mathbb{R}_+, B) \hookrightarrow E_{\mathbb{R}} = \mathrm{Q}^s(SB)$ defines a natural semigroup monomorphism ψ from the semigroup $[\mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))]$ of classes of unitarily equivalent morphisms in $\mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))$ (by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)) \subseteq \mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$) to $[\mathrm{Hom}(D, E_{\mathbb{R}})]$ those of $\mathrm{Hom}(D, E_{\mathbb{R}})$. Note that *the additive map ψ generalizes the usual mapping cone construction* in a natural way.

The natural inclusions and isomorphism

$$\mathcal{M}(B) \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B)) \subseteq \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$$

of Lemma 7.4.5 induce natural inclusions

$$\mathcal{M}(B) \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B))/\,\mathrm{C_0}(\mathbb{R}, B) \subseteq E_{\mathbb{R}}\,.$$

We denote by I_2 is the unital inclusion map of $\mathcal{M}(B)$ into $E_{\mathbb{R}}$.

We define a *-monomorphism

$$H_0' := \delta_\infty k_0\colon D \to \mathcal{M}(B)\,.$$

Let $H_0 := H_{\mathbb{R}} := I_2 \circ H_0'\colon D \to E_{\mathbb{R}}$. Then H_0 for H_0' is related to $\mathcal{C} := \mathcal{C}(k_0)$ in the sense of Corollary 5.4.4.

Let $\widetilde{\beta}$ be the *-automorphism of $\mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ which is defined by $\widetilde{\beta}(f)(t) = f(-t)$ for $f \in \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$. $\widetilde{\beta}$ maps $\mathrm{C_0}(\mathbb{R}, B)$ onto $\mathrm{C_0}(\mathbb{R}, B)$ and defines therefore an automorphism β of $E_{\mathbb{R}}$ with $\beta^2 = \mathrm{id}$.

Certainly, $\beta(b) = b$ for $b \in I_2(\mathcal{M}(B)) \subseteq E_{\mathbb{R}}$, and $\beta(\mathrm{Q}(\mathbb{R}_+, B)) = \mathrm{Q}(\mathbb{R}_-, B)$. Thus $\beta(J) \cap J = 0$, $\beta H_0 = H_0$ and $J + \beta(J) = \mathrm{Q}(\mathbb{R}, B) \subseteq E_{\mathbb{R}}$. Furthermore $h_0 + \beta h_0 = I_2 \circ k_0$, because $I_2(b) = I_1(b) + \beta I_1(b)$ for $b \in B$.

Since H_0 dominates h_0, the map ψ induces a semigroup homomorphism

$$\vartheta\colon [h] \in S(h_0;\, D, E_{\mathbb{R}}) \to [h] + [H_0] \in G(H_0;\, D, E_{\mathbb{R}})\,.$$

Then $\vartheta | G(h_0, D, E_{\mathbb{R}})$ is a group homomorphism and $\vartheta([h] + [h_0]) = [h] + [H_0]$ by Proposition 4.4.2(ii).

We have seen in Chapters 5 and 8 that

$$\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C_0}(\mathbb{R}));\, D, SB) = G(H_0, D, E_{\mathbb{R}})$$

(cf. Corollary 5.9.23) and

$$\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C_0}(\mathbb{R}));\, D, SB) \cong \mathrm{KK}(\mathcal{C};\, D, B)$$

(cf. Lemma 8.4.2).

The semigroup $S(h_0;\, D, E_{\mathbb{R}})$ is a sub-semi-group of $S(H_0;\, D, E_{\mathbb{R}})$ because H_0 dominates h_0 (almost obviously). Thus, $[h] \mapsto [h \oplus H_0] = [h] + [H_0]$ defines a natural group morphism ϑ from $\mathrm{R}(\mathcal{C};\, D, B) = G(h_0;\, D, E_{\mathbb{R}})$ into $\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C_0}(\mathbb{R}));\, D, SB) = G(H_0;\, D, E_{\mathbb{R}})$.

By Chapters 4 and 7, we have

$$G(k_0;\, D, Q(\mathbb{R}_+, B)) = \mathrm{R}(\mathcal{C};\, D, B) \cong G(h_0;\, D, E_{\mathbb{R}}) = [h_0] + S(h_0;\, D, E_{\mathbb{R}})\,.$$

Proof of Prop. 9.2.4 and Cor. 9.2.5 requires to prove first homotopy invariance of $\mathrm{R}(\mathcal{C}\,;\, D, B)$,
i.e., that

$$\pi_0, \pi_1 \colon\, \mathrm{R}(\mathcal{C}[0,1]\,;\, D, B[0,1]) \to \mathrm{R}(\mathcal{C}\,;\, D, B)$$

satisfy $\pi_0 = \pi_1$.

Is the homotopy invariance of $\mathrm{R}(\mathcal{C}\,;\, D, B)$ equivalent to the to the injectivity of

$$\mathrm{R}(\mathcal{C}\,;\, D, B) \to \mathrm{KK}(\mathcal{C}\,;\, D, B)\,.$$

???

But there is also a a direct proof of the criteria in Theorem 4.4.6 for the injectivity of $G(h_0;\, D, E_{\mathbb{R}}) \;\to\; G(H_0;\, D, E_{\mathbb{R}})$, given by an inductive decomposition procedure (see below).

Lemma 9.2.2. *Suppose that D and B are stable, D is separable, B is σ-unital, $H\colon D \to \mathcal{M}(B)$ is a non-degenerate C^*-morphism, and a_1, a_2 are self-adjoint generators of D, i.e., $D = C^*(a_1, a_2)$.*

Then there exists exists a constant $\Omega(H) \in \mathbb{N}$ with the following property:

For every unitary $U \in \mathcal{M}\big(\mathrm{C}_0((-\infty, 1], B)\big)$, – given by a strictly continuous map $U\colon (-\infty, 1] \ni t \mapsto U(t) \in \mathcal{U}(B)$ – with $\gamma_{U,k} \in \mathrm{C}_0((-\infty, 1], B)$ for $\gamma_{U,k}(t) := H(a_k)U(t) - U(t)H(a_k)$ and every $\delta > 0$ there exist $n \le \Omega(D, B, H)$ and contractions $T_1, \ldots, T_n \in \mathcal{M}\big(\mathrm{C}_0((-\infty, 1], B)\big)$ with $T_j^* = -T_j$ and a unitary $V \in 1 + \mathrm{C}_0((-\infty, 1], B)$, such that $[T_j, H(a_k)] \in \mathrm{C}_0((-\infty, 1], B)$, $\|[T_j, H(a_k)]\| < \delta$ for $j = 1, \ldots, n$, $k = 1, 2$, and

$$U \oplus_{s_1,s_2} 1 = V \cdot \exp(T_1) \cdot \ldots \cdot \exp(T_n)\,.$$

In particular,

$$\|[V^*(U \oplus 1), H(a_k)]\| \le n \cdot \delta \le \Omega(D, B, H) \cdot \delta\,.$$

Proof. Is not very likely that a proof exist!!!

To be filled in ?? □

Next lemma describes a correction method for paths of unitaries by mirroring the interval $[\alpha, \beta)$ onto $(-\infty, \alpha]$ via the bijective map

$$[\alpha, \beta) \ni t \mapsto \alpha - (t - \alpha)/(\beta - t) \in (-\infty, \alpha]\,.$$

Lemma 9.2.3. *Let D and B stable C^*-algebras, D separable and B σ-unital, $H\colon D \to \mathcal{M}(B)$ a non-degenerate C^*-morphism in general position (i.e., $\delta_\infty \circ H$ unitarily homotopic to H), $0 \le \alpha < \beta$ real numbers, $a_1, a_2 \in D$ self-adjoint contractions,*

$$(-\infty, \beta] \ni t \mapsto U(t) \in \mathcal{U}(\mathcal{M}(B))$$

a strictly continuous map.

Let $\lambda(t) := U(t)H(a) - H(a)U(t)$ *in* $\mathcal{M}(B)$ *for* $t \in (-\infty, \beta]$ *and suppose that* $\lambda \in \mathrm{C}((-\infty, \beta], B)$ *and* $\lim_\infty \|\lambda(t)\| = 0$.

Let $n, m \in \mathbb{N}$ *with* $\max(m, n) \le \Omega(D, B, H)$, *and* $0 < \delta \le \varepsilon$. *Suppose that the following are given (e.g. coming from Lemma 9.2.2):*

(i) $S_1, \ldots, S_m \in \mathrm{C}_{\mathrm{b,st}}((-\infty, \alpha], \mathcal{M}(B))$ *with* $S_j^* = -S_j$, $\|S_j\| \le 1$, $[S_j, a_k] \in \mathrm{C}_0((-\infty, \alpha], B)$, $\|[S_j, H(a_k)]\| < \varepsilon$ *and*

$$V_1^*(U|(-\infty, \alpha]) \in 1 + \mathrm{C}_0(-\infty, \alpha], B)\,,$$

for $V_1 := \exp(S_1) \cdot \ldots \cdot \exp(S_m)$.

(ii) $T_1, \ldots, T_n \in \mathrm{C}_{\mathrm{b,st}}((-\infty, \alpha], \mathcal{M}(B))$ *with* $T_j^* = -T_j$, $\|T_j\| \le 1$, $[T_j, H(a_k)] \in \mathrm{C}_0((-\infty, \beta], B)$, $\|[T_j, H(a_k)]\| < \varepsilon$ *and*

$$V_2^* U \in 1 + \mathrm{C}_0(-\infty, \beta], B)\,,$$

for $V_2 := \exp(T_1) \cdot \ldots \cdot \exp(T_n)$.

Then there exists a unitary in $W \in 1 + \mathrm{C}([\alpha, \beta], B)$ *with* $W(\beta) = 1$, $W(\alpha) = V_2(\alpha)^* V_1(\alpha)$ *and, for* $t \in [\alpha, \beta]$, *?????*

$\|[W(t), H(a_k)]\| \le \|[V_1, H(a_k)]\| + \|[(V_2|(-\infty, \alpha]), H(a_k)]\| \le n\delta + m\varepsilon \le 2\varepsilon\Omega(?????)$.

Then $W(\alpha)^* V_2(\alpha)^* U(\alpha) = V_1(\alpha)^* U(\alpha)$, $V_2(\alpha)W(\alpha) = V_1(\alpha)$, $V_2(\beta)W(\beta) = V_2(\beta)$.

Moreover $V_3(t) := V_1(t)$ *for* $t \in (-\infty, \alpha]$ *and* $V_3(t) := V_2(t)W(t)$ *for* $t \in (\alpha, \beta]$ *is strictly continuous on* $(-\infty, \beta]$, *with* $V_3^* U \in 1 + \mathrm{C}_0((-\infty, \beta], B)$,

$$\|[V_3(t), H(a_k)]\| \le m\varepsilon + 2n\delta \le 2\varepsilon\Omega(D, B, H)$$

and $\|[V_3(\beta), H(a_k)]\| = \|[V_2(\beta), H(a_k)]\| \le n\delta \le \delta\Omega(D, B, H)$.

PROOF. `To be filled in` ?? □

PROPOSITION 9.2.4. *The semigroup homomorphism* ψ *defines a surjective group homomorphism* $\vartheta\colon [h] + [h_0] \mapsto [h] + [H_0]$ *from* $G(h_0; D, E_{\mathbb{R}})$ *onto* $G(H_0; D, E_{\mathbb{R}})$, *if* $H_0\colon D \to E_{\mathbb{R}} := \mathcal{M}(B[\mathbb{R}])/B[\mathbb{R}]$ *is as above defined.*

The epimorphism ϑ *is an* isomorphism *from* $G(h_0; D, E_{\mathbb{R}})$ *onto* $G(H_0; D, E_{\mathbb{R}})$, *if* $Y \mapsto \mathrm{R}(\mathcal{C}[Y]; D, B[Y])$ *(*Y *compact metric space) is homotopy invariant, in the sense that the evaluation maps*

$$\pi_t\colon\ \mathrm{R}\big(\mathcal{C}[Y \times I]; D, B[Y \times I]\big) \to \mathrm{R}\big(\mathcal{C}[Y]; D, B[Y]\big)$$

have the same images for $t \in \{0, 1\} \subset I := [0, 1]$.

Recall that we denote by $\mathcal{C}[\mathbb{R}]$ (or likewise $\mathcal{C}_{\mathbb{R}}$, $S\mathcal{C}$) respectively by the matrix operator convex cone $\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))$, that is the tensor product of $\mathcal{C}$ and $\mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))$ inside the *category of m.o.c. cones*, and we urge the reader to remind that the tensor product in the category of m.o.c. cones has not *not* much to do with a completion of some tensorproduct in the ordinary sense, because of the many different "canonical" matrix-cones on tensor products, compare Chapter 3.

Remember that $B[\mathbb{R}]$ (or likewise $B_{\mathbb{R}}$, SB) are other notations for the C^*-algebra $\mathrm{C}_0(\mathbb{R}, B) \cong B \otimes \mathrm{C}_0(\mathbb{R})$. The above listed results of Chapters 5, 7 and 8 immediately yield the following.

It suffices to consider the case $Y :=$ point in Proposition 9.2.4 by replacing B by $B[Y] = \mathrm{C_b}(Y, B)$.

Corollary 9.2.5. *The semigroup homomorphism ψ induces a natural group epimorphism ϑ of $\mathrm{R}(\mathcal{C};\, D, B)$ onto $\mathrm{Ext}(\mathcal{C}[\mathbb{R}];\, D, B[\mathbb{R}]) \cong \mathrm{KK}(\mathcal{C};\, D, B)$. The epimorphism ϑ is an isomorphism if the evaluation maps*

$$\pi_0, \pi_1 \colon\ \mathrm{R}(\mathcal{C}[0,1];\, D, B[0,1]) \to \mathrm{R}(\mathcal{C};\, D, B)$$

have the same image. □

We need several elementary observations for the proofs of Proposition 9.2.4 and Corollary 9.2.5, e.g.

Lemma 9.2.6. *Suppose that D is a stable and σ-unital C^*-algebra and that k is a non-degenerate *-monomorphism from $D \otimes \mathcal{O}_2$ into B.*

Let $r_1, r_2, \ldots$ a sequence of isometries in $\mathcal{M}(B)$ such that $\sum r_n (r_n)^*$ converges strictly to 1, and let $\delta_\infty := \sum r_n(\cdot)(r_n)^*$ the corresponding infinite repeat endomorphism of $\mathcal{M}(B)$. Then:

- (i) *There exists a unital *-morphism $\epsilon \colon \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}) \hookrightarrow \mathcal{M}(B)$ such that*
  - (0) *ϵ is a strictly continuous monomorphism,*
  - (1) *$\epsilon(\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}))$ commutes element-wise with $\delta_\infty(k(D \otimes 1))$,*
  - (2) *$\epsilon(\mathcal{O}_2 \otimes 1_{\mathcal{M}(\mathbb{K})})$ commutes element-wise with $k(D \otimes 1)$, and*
  - (3) *$r_1 r_1^* = \epsilon(1_{\mathcal{O}_2} \otimes p_{11})$, and*
  - (4) *$\delta_\infty(k(D \otimes 1)) \cdot \epsilon(\mathcal{O}_2 \otimes \mathbb{K}) \subseteq B$.*
- (ii) *If $\epsilon \colon \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}) \to \mathcal{M}(B)$ is any unital *-morphism that has the properties (0)–(4) of Part (i), then*

$$d \otimes f \mapsto \delta_\infty(k(d \otimes 1))\epsilon(f \otimes p_{11})$$

  *extends to a *-monomorphism $k_1 \colon D \otimes \mathcal{O}_2 \to B$, such that $d \in D \mapsto k_1(d \otimes 1)$ is unitarily homotopic to $d \in D \mapsto k(d \otimes 1)$.*
- (iii) *$\epsilon(\mathcal{O}_2 \otimes \mathbb{K})B$ is dense in B, and there exists an isometry $t_0' \in \epsilon(\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}))$ such that $t_0' t_0'^* = 1 - r_1 r_1^*$.*

Proof. (i): First we consider the case $B = D \otimes \mathcal{O}_2$ and $k := \mathrm{id}$.

Let $\lambda \colon \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ be an isomorphism. Then there exist sequences (I_n), (s_n) of isometries in $\mathcal{M}(\mathbb{K})$, such that $\sum I_n I_n^*$ and $\sum s_n s_n^*$ strictly converge to 1, $\lambda(a \otimes p_{nn}) = I_n a I_n^*$ and $\lambda(p_{nn} \otimes a) = s_n a s_n^*$ for $a \in \mathbb{K}$.

In particular $\mathcal{M}(\lambda)(1 \otimes p_{11}) = I_n I_n^*$.

Then the infinite repeats $\delta^{(1)} := \sum I_n(\cdot)I_n^*$ and $\delta^{(2)} := \sum s_n(\cdot)s_n^*$ have element-wise commuting images and $\delta^{(1)}(a)\delta^{(2)}(b) = \lambda(a \otimes b)$ for $a, b \in \mathbb{K}$. Indeed, $\delta^{(1)} = \mathcal{M}(\lambda)((\cdot) \otimes 1)$ and $\delta^{(2)} = \mathcal{M}(\lambda)(1 \otimes (\cdot))$.

Let $r'_n := 1 \otimes I_n \otimes 1$, $s'_n := 1 \otimes s_n \otimes 1$, and $\epsilon'(f) := \sum s'_n(1 \otimes \mathcal{M}(\kappa)(f))s'_n$ for $f \in \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$, where κ denotes here the flip-isomorphism from $\mathcal{O}_2 \otimes \mathbb{K}$ onto $\mathbb{K} \otimes \mathcal{O}_2$. Then ϵ' has the quoted properties with respect to $r'_1, r'_2, \ldots$ and id (in place of $\epsilon, r_1, r_2, \ldots, k$):

For $a \in A$ and $b \in \mathbb{K}$, where $D := A \otimes \mathbb{K}$

`Is` A `sufficiently well-defined? ??`

let ??????

$$\delta_\infty(a \otimes b \otimes 1) = a \otimes \delta^{(1)}(b) \otimes 1,$$

and, for $e \in \mathbb{K}$ and $f \in \mathcal{O}_2$,

$$\epsilon'(e \otimes f) = 1 \otimes \delta^{(2)}(e) \otimes f.$$

In particular,

$$\epsilon'(1 \otimes p_{11}) = 1 \otimes \mathcal{M}(\lambda)(1 \otimes p_{11}) \otimes 1 = r'_1(r'_1)^* .$$

$D \otimes 1$ and $\epsilon'(\mathcal{O}_2 \otimes 1)$ commute, because, for $a \in A$, $b, c \in \mathbb{K}$ and $f \in \mathcal{O}_2$,

$$(a \otimes \lambda(b \otimes c)) \cdot \epsilon'(f \otimes 1) = a \otimes \lambda(b \otimes c) \otimes f .$$

In the *general case*, let $\mathcal{M}(k) \colon \mathcal{M}(D \otimes \mathcal{O}_2) \hookrightarrow \mathcal{M}(B)$ the strictly continuous unital extension of k. By Lemma 5.1.2(i) there is a unitary $u \in \mathcal{M}(B)$ such that $r_n = u\mathcal{M}(k)(r'_n)$. Let $\epsilon := u\mathcal{M}(k)(\epsilon'(\cdot))u^*$, then ϵ satisfies (0)-(4).

(ii): By the properties listed in (i), there is a C^*-morphism g from the algebraic tensor product $D \odot \mathcal{O}_2$ into B such that, for $d \in D$, $f \in \mathcal{O}_2$,

$$g(d \otimes f) = \delta_\infty(k(d \otimes 1))\epsilon(f \otimes p_{11}) ,$$

moreover g extends to a C^*-morphism k_1 from $D \otimes \mathcal{O}_2$ into B, because $\mathcal{O}_2$ is nuclear and simple.

Let $k_2 := k_1((\cdot) \otimes 1)$ and $k_0 := k((\cdot) \otimes 1)$. Since $r_1(r_1)^* = \epsilon(1 \otimes p_{11})$, $k_2 = r_1 k_0(\cdot) r_1^*$. Thus $k_0 = r_1^* k_2(\cdot) r_1$. By Corollary 7.4.4, k_2 and k_0 are unitarily homotopic, because B is stable, D is separable and $r_1 \in \mathcal{M}(B)$.

(iii): $\epsilon(\mathcal{O}_2 \otimes \mathbb{K})B$ is dense in B, because ϵ is strictly continuous and unital.

By (i), $\epsilon(1 \otimes (1 - p_{11})) = 1 - r_1 r_1^*$.

Let T_0 an isometry in $\mathcal{M}(\mathbb{K}) \cong \mathcal{L}(H)$ with $T_0(T_0)^* = 1 - p_{11}$, and let $t'_0 := \epsilon(1 \otimes T_0)$. Then t'_0 is an isometry in $\epsilon(\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}))$ such that $t'_0 t'^*_0 = 1 - r_1 r_1^*$. □

Proof of Proposition 9.2.4. We use the notations and observations of Remark 9.2.1 and reduce the prove to Theorem 4.4.6. It means, that we are going to find $F \subseteq H_0(D)' \cap E_{\mathbb{R}}$ and elements s, t, p_0, s_0, t_0, u_1 in $E_{\mathbb{R}}$, such that the above defined D, $E_{\mathbb{R}}$, J, h_0, H_0, β, together with with the below defined F, s, t, p_0, s_0, t_0 and u_1 satisfy the assumptions (i)–(vi) of Theorem 4.4.6. Then Theorem 4.4.6 implies that the generalized mapping cone construction $\psi\colon S(h_0;\, E, E_{\mathbb{R}}) \to G(H_0;\, D, E_{\mathbb{R}})$ defines a group *epimorphism* ϑ from $G(h_0;\, D, E_{\mathbb{R}})$ onto $G(H_0;\, D, E_{\mathbb{R}})$. It becomes an *isomorphism* if we later can show in addition that that $\Gamma(H_0(D), J, E) = 0$.

We are now going to define the missing ingredients for this list. Parallel, we check that the conditions (i)-(vi) of Theorem 4.4.6 are fulfilled for them:

Let $r_1, r_2, \ldots$ be a sequence of isometries in $\mathcal{M}(B)$, such that $\sum r_n(r_n)^* = 1$, and let $\epsilon\colon \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}) \to \mathcal{M}(B)$ the strictly continuous unital *-monomorphism that satisfies (0)-(4) of Lemma 9.2.6(i) with respect to $k\colon D \otimes \mathcal{O}_2 \to B$ and $(r_1, r_2, \ldots)$.

By Lemma 9.2.6(iii), there is an isometry t'_0 in $\epsilon(\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}))$, such that $t'_0(t'_0) = 1 - r_1(r_1)^*$. Thus t'_0 commutes element-wise with $H'_0(D)$.

Therefore $\delta_\infty(k_0(D))$ and $k_0(D)$ both commute element-wise with the unital copy $\eta(\mathcal{O}_2) := \epsilon(\mathcal{O}_2 \otimes 1)$ of $\mathcal{O}_2$ in $\mathcal{M}(B)$, where

$$\eta(\cdot) := \epsilon((\cdot) \otimes 1_{\mathcal{M}(\mathbb{K})})\,.$$

Let s_1, s_2 generators of $\mathcal{O}_2$ and let $s := I_2(\eta(s_1))$, $t := I_2(\eta(s_2))$. Then both are isometries in $I_2(\mathcal{M}(B)) \subseteq E_{\mathbb{R}}$ and generate a unital copy of $\mathcal{O}_2$. Since β fixes the elements of $I_2(\mathcal{M}(B))$, we have $\beta(s) = s$ and $\beta(t) = t$.

By Lemma 9.2.6(i), $\eta(\mathcal{O}_2) \subseteq H'_0(D)' \cap k_0(D)'$. If we apply I_2, we get that s and t are in $H_0(D)' \cap (h_0 + \beta h_0)(D)' \cap E_{\mathbb{R}}$.

Recall that $k_0\colon D \to B$ is non-degenerate.

By Lemma 7.4.17, there exists a contraction $g_1 \in \mathrm{C_b}(\mathbb{R}, \epsilon(1 \otimes \mathbb{K}))_+$ such that $g_1(x) = 0$ for $x \leq 0$ and $\lim_{x\to+\infty} g_1(x)k_0(a) = k_0(a)$ for every $a \in D$, because $k_0(D)$ is separable. Let $g := g_1 + SB \in E_{\mathbb{R}}$. Then $(h_0 + \beta h_0)(a)g = h_0(a)$. The positive contraction g commutes with s and t, because $\epsilon(1 \otimes \mathbb{K})$ commutes with $\epsilon(\mathcal{O}_2 \otimes 1_{\mathcal{M}(\mathbb{K})})$. Thus s and t also commute element-wise with $h_0(D)$.

Commutation of s, t with $h_0(D)$
follows also from commutation of s, t with
$(h_0 + \beta h_0)(D)$ and $J \cap \beta J = 0$. ??

Hence, *$s, t \in E_{\mathbb{R}}$ satisfy condition (i) of Theorem 4.4.6.*

Our non-degenerate *-monomorphism $H_0\colon D \to E_{\mathbb{R}}$ satisfies $\mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, D, SB) = G(H_0, D, E_{\mathbb{R}})$ by Corollary 5.9.22(i), as it was explained above.

Therefore, by Lemma 8.4.3, for every homomorphism $k\colon D \to E_{\mathbb{R}}$ that is dominated by H_0, there exists a unitary $u \in E_{\mathbb{R}}$ such that $u^*(k \oplus H_0)(a)u - H_0(a) \in J$ for each $a \in D$.
Thus *condition (ii) of Theorem 4.4.6 is valid.*

Now we want to show that conditions (iii), (iv) and (v) of Theorem 4.4.6 are satisfied:
For $P := \epsilon(\mathcal{O}_2 \otimes \mathbb{K}) \subseteq \mathcal{M}(B)$ with $\epsilon\colon \mathcal{O}_2 \otimes \mathbb{K} \to \mathcal{M}(B)$ we have that $P \cong \mathcal{O}_2 \otimes \mathbb{K}$, and that $P \cdot B$ is dense in B (by Lemma 9.2.6). Moreover $P \cap B = \{0\}$ because

P is simple and P is not contained in B by condition (3) of Lemma 9.2.6(i). Let $F_1 := \mathrm{C_b}(\mathbb{R}, P) \subseteq \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(SB)$. Certainly $\widetilde{\beta}(F_1) \subseteq F_1$, and $F_1 \cap \mathrm{C_0}(\mathbb{R}, B) = \{0\}$, because $\epsilon(g(t)) = f(t)$ for $g(t) \in \mathcal{O}_2 \otimes \mathbb{K}$ and because $f(t) \in B$ implies $f(t) = 0$. Therefore $F := \pi_{SB}(F_1) \subseteq E_{\mathbb{R}}$ is a C^*-subalgebra of $E_{\mathbb{R}}$ that is isomorphic to $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$ and satisfies $\beta(F) = F$. Since $H_0'(D)P \subseteq B$, we get $F_1 H_0'(D) \subseteq \mathrm{C_b}(\mathbb{R}, B)$ and $F H_0(D) \subseteq J + \beta J$.

By Proposition 7.4.18, for every projection $q \in F_1$, and every element $f_1 \in \mathrm{C_b}(\mathbb{R}, B)$ there exists a projection p in F_1 such that $p f_1 p - f_1 \in \mathrm{C_0}(\mathbb{R}, B)$, $q \leq p$ and $q \neq p$.

If we apply π_{SB}, we get that *condition (iii) of Theorem 4.4.6 is satisfied.*

By Lemma 9.2.6(iii), there are isometries r_1, t_0' in $\mathcal{M}(B)$ such that $t_0'(t_0')^* = 1 - p_0'$, $r_1(r_1)^* = p_0' \in P$, $r_1^* H_0'(\cdot) r_1 = k_0$, and $t_0' \in \epsilon(\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K}))$ commutes with the elements of $H_0'(D)$. If considered as elements of $\mathcal{M}(SB)$ they are fixed by $\widetilde{\beta}$. Let $s_0 := I_2(r_1)$, $t_0 := I_2(t_0')$ and $p_0 := I_2(p_0')$ the corresponding elements in $E_{\mathbb{R}}$. Then s_0 and t_0 are fixed by β, $p_0 = s_0 s_0^* \in F$, and t_0 commutes with the elements of $H_0(D) = I_2(H_0'(D))$. Since $r_1^* H_0'(\cdot) r_1 = k_0$, we get $s_0^* H_0(\cdot) s_0 = I_2 k_0 = h_0 + \beta h_0$. Thus, *condition (iv) of Theorem 4.4.6 is satisfied.*

By [**172**] any two nonzero projections in $\mathcal{O}_2 \otimes \mathbb{K}$ are unitarily equivalent by a unitary in the *-semigroup $1 + \mathcal{O}_2 \otimes \mathbb{K}$ in $\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$. Therefore, *condition (v) of Theorem 4.4.6 is satisfied* by Lemma 7.4.13, because p_0 is a nonzero projection of F and F is isomorphic to $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K})$.

BEGIN (Old vi)

```
The old assumption (OLD vi) has been
proven in new Lemma 4.4.7
from the other assumptions (i)-(v) and (OLD vii).
Thus (OLD vi) is superfluous and Part (OLD vii)
is now the new Part (vi).
```

Check of condition (OLD vi) of Theorem 4.4.6:
Let $h\colon D \to J$. Suppose that H_0 dominates $h + \beta h_0$, i.e., there exists an isometry $T \in E_{\mathbb{R}}$ with $T^* H_0(\cdot) T = h + \beta h_0$. By Lemma 7.4.22(i), there exists a contraction $y \in (J + \beta(J))_+ = \mathrm{Q}(\mathbb{R}, B)_+$ such that $y(h + \beta h_0)(\cdot) y = h$. It follows that $h = y T^* H_0(\cdot) T y$ is dominated by H_0. Since $z := Ty$ is a contraction in the ideal $\mathrm{Q}(\mathbb{R}, B)$ of $E_{\mathbb{R}} = \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B)) / \mathrm{C_0}(\mathbb{R}, B))$ and, since

$$H_0(D) \subseteq \pi_{SB}(\mathcal{M}(B)) \subseteq \pi_{SB}(\mathrm{Q}(\mathbb{R}, \mathcal{M}(B))) \subseteq E_{\mathbb{R}},$$

by the natural embeddings

$$\mathcal{M}(B) \subseteq \mathrm{C_b}(\mathbb{R}, \mathcal{M}(B)) \subseteq \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) = \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B)),$$

there is a contraction $w \in \mathrm{C_b}(\mathbb{R}, B)$ with $w + \mathrm{C_0}(\mathbb{R}, B)) = z$

$$h = w^*(\delta_\infty \circ k_0)(\cdot) w + \mathrm{C_0}(\mathbb{R}, B).$$

By 7.4.22(iv), there is a contraction $v \in \mathrm{C_b}(\mathbb{R}, B)$ with $v(t) = 0$ for $t < 0$ and $v^* k_0(\cdot) v + \mathrm{C}_0(\mathbb{R}, B) = h$. Thus, $d^* h_0(\cdot) d = d^*(h_0 + \beta h_0)(\cdot) d = h$ for the contraction $d := v + \mathrm{C}_0(\mathbb{R}, B) \in J \cong \mathrm{Q}(\mathbb{R}_+, B)$. Since B is stable and $h_0(D)$ is separable, there is an isometry $S \in \mathrm{C_b}(\mathbb{R}, M(B)) \subseteq \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ such that $s := S + \mathrm{C}_0(\mathbb{R}, B) \in E_{\mathbb{R}}$ satisfies $s^* h_0(\cdot) s = d^* h_0(\cdot) d = h$ (cf. Lemma 7.4.22(v)). Thus *assumption (OLD vi) of Theorem 4.4.6 is satisfied.*

END (OLD vi)

`Check again assumption (vi) ?? ??`

Now we verify assumption (vi) of Theorem 4.4.6:

First we define a continuous map $\xi \in \mathbb{R} \to v(\xi)$ into the unitaries of $O_2 = C^*(s_1, s_2)$ for $\xi \in [0, 1]$ by

$$v(\xi) := \xi(s_1 s_1^* + s_2 s_2^*) + (1 - \xi^2)^{1/2}(s_2 s_1^* - s_1 s_2^*)\,,$$

and by $v(\xi) := 1$ for $\xi \geq 1$, $v(\xi) := v(0) = s_2 s_1^* - s_1 s_2^*$ for $\xi \leq 0$.

Let $w(\xi) := \eta(v(\xi))$, then $w \in \mathrm{C_b}(\mathbb{R}, \eta(\mathcal{O}_2))$ is a unitary in $\mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(SB)$. Finally, let $u_1 := w + SB = \pi_{SB}(w)$. Then $u_1 \in J' \cap E_{\mathbb{R}}$, because every $a \in J \cong \mathrm{Q}(\mathbb{R}_+, B \otimes \mathbb{K})$ has a representative $b \in \mathrm{C_b}(\mathbb{R}, B)$ with $b(\xi) = 0$ for $\xi \leq 1$, and, for those b, $wb = b = bw$.

Since β fixes $C^*(s, t) = I_2(\eta(\mathcal{O}_2)) \subseteq I_2(\mathcal{M}(B))$ and s, t commutes element-wise with $h_0(D)$, we get that s and t commute element-wise with $\beta h_0(D)$. The elements of βJ can be represented by elements b in $\mathrm{C_b}(\mathbb{R}, B)$ such that $b(\xi) = 0$ for $\xi \geq 0$. We get $wb = \eta(s_2 s_1^* - s_1 s_2^*) b$ and $bw = b\eta(s_2 s_1^* - s_1 s_2^*)$. It implies that the equation

$$u_1 \beta h_0(\cdot) s s^* = t t^* \beta h_0(\cdot) u_1$$

is equivalent to

$$(ts^* - st^*) s s^* \beta h_0(\cdot) = \beta h_0(\cdot) t t^* (ts^* - st^*)\,,$$

i.e., is equivalent to the above shown element-wise commutation of ts^* with $\beta h_0(D)$.

Therefore *condition (vi)* of Theorem 4.4.6 is *satisfied.*

Now we have seen that all conditions of Theorem 4.4.6 are satisfied by D, $E_{\mathbb{R}}$, H_0, h_0, J and β. Therefore

$$\vartheta \colon [k \oplus h_0] \to [k \oplus H_0]$$

is a group epimorphism from $G(h_0;\, D, \mathrm{Q}^s(SB)) \cong \mathrm{R}(\mathcal{C};\, D, B)$ onto

$$G(H_0;\, D, \mathrm{Q}^s(SB)) = \mathrm{Ext}(\mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}));\, D, SB) \cong \mathrm{KK}(\mathcal{C};\, D, B)\,.$$

□

The following Corollary 9.2.7 and its proof uses the notations and conventions of Remark 9.2.1.

COROLLARY 9.2.7. *Suppose that B and D are stable (and trivially graded), D is separable, B σ-unital, and that $h_0\colon D \to B$ is non-degenerate with $[h_0 \oplus h_0] = [h_0]$. Let $\mathcal{C} := \mathcal{C}(h_0)$ and define* $\mathrm{Hom}(\mathcal{C};\, D, B) := \mathrm{Hom}(D, B) \cap \mathcal{C}$ *. Then:*

(i) $S(h_0\,;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))) = \mathrm{SR}(\mathcal{C};\, D, B)$.
If $[k] \in S(h_0\,;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ *then* $h := k \oplus h_0$ *is in* $[h_0] + \mathrm{SR}(\mathcal{C};\, D, B) \cong \mathrm{R}(\mathcal{C};\, D, B)$, *and* $[\widehat{\sigma} \circ h] = [h]$ *in* $\mathrm{SR}(\mathcal{C};\, D, B)$, *for every homeomorphism σ of $\mathbb{R}_+$,*
i.e., h is invariant under scaling *up to unitary equivalence.*

(ii) *The difference construction* $[h - 0] := [(B, h, 0)] \in \mathrm{KK}(\mathcal{C};\, D, B)$ *for* $h \in \mathrm{Hom}(\mathcal{C};\, D, B)$ *defines a semigroup* epi*morphism*

$$\alpha\colon [\mathrm{Hom}(\mathcal{C};\, D, B)] \to \mathrm{KK}(\mathcal{C};\, D, B)$$

from the "constant" sub-semigroup $[\mathrm{Hom}(\mathcal{C}\,;\, D, B)]$ *of* $\mathrm{R}(\mathcal{C}\,;\, D, B)$ onto $\mathrm{KK}(\mathcal{C}; D, B)$.
Elements $[h_1], [h_2] \in [\mathrm{Hom}(\mathcal{C};\, D, B)]$ *define the same element* $[h_1 - 0] = [h_2 - 0]$ *of* $\mathrm{KK}(\mathcal{C};\, D, B)$*, if and only if, $h_1 \oplus h_0$ and $h_2 \oplus h_0$ are unitarily homotopic (cf. Definition 5.0.1).*

`Next example OK ???`
`More information ? ??`

Note that, in "contrast" to Corollary 9.2.7, the natural group homomorphism $G(h_0\,;\, D, B) \to G(h_0\,;\, D, \mathrm{Q}(\mathbb{R}_+, B))$ is neither surjective nor injective in general, e.g. for $D = \mathcal{O}_2 \otimes \mathbb{K}$, $B = \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and $h_0(\cdot) := 1 \otimes (\cdot)$.

`Not clear, e.g.:`
`Then` $G(h_0\,;\, D, \mathrm{Q}(\mathbb{R}_+, B))$ `contains only one element ?`
`What about this` $G(h_0\,;\, D, B)$`?`

PROOF. The homotopy invariance of $\mathrm{R}(\mathcal{C};\, D, B)$ follows from different more involved considerations ??

(i): Let $E_{\mathbb{R}} := \mathcal{M}(SB)/SB \cong \mathrm{Q}^s(SB)$, let $I_1\colon \mathrm{Q}(\mathbb{R}_+, B) \cong J \hookrightarrow E_{\mathbb{R}}$ be the natural inclusion, and let $I_2\colon \mathcal{M}(B) \to E_{\mathbb{R}}$ denotes the natural unital monomorphism induced by $\mathcal{M}(B) \subseteq \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B)) = \mathcal{M}(SB)$.

By Proposition 7.4.21(i), a morphism $k \in \mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))$ has unitary equivalence class $[I_1 \circ k]$ in $S(h_0\,;\, D, E_{\mathbb{R}})$, if an only if $[k] \in \mathrm{SR}(\mathcal{C};\, D, B)$. (The unitary equivalence classes coincide, i.e., $[k] = [k']$ in $[\mathrm{Hom}(D, \mathrm{Q}(\mathbb{R}_+, B))]$ if and only if $[I_1 \circ k] = [I_1 \circ k']$ in $[\mathrm{Hom}(D, E_{\mathbb{R}})]$, cf. Lemma 7.4.20.)

The class $[h] = [(I_1 \circ k) \oplus h_0] = [(I_1 \circ k)] + [h_0]$ is in $G(h_0;\, D, E_{\mathbb{R}}) = [h_0] + S(h_0\,;\, D, E_{\mathbb{R}}) \cong \mathrm{R}(\mathcal{C};\, D, B)$, by Proposition 7.4.21(ii) (and by Proposition 4.4.3).

If σ is a topological isomorphism of $\mathbb{R}_+$ then $\sigma(0) = 0$ and therefore σ extends naturally to an orientation preserving topological isomorphism of $\mathbb{R}$ by letting $\sigma(t) = t$ for $t \in \mathbb{R}_-$. We denote this extension again by σ. It induces in a natural way an automorphism $\widehat{\sigma}$ of $E_{\mathbb{R}} = \mathcal{M}(SB)/(SB)$ such that its restriction to $J \cong \mathrm{Q}(\mathbb{R}_+, B)$ is just the above considered induced automorphism of $\mathrm{Q}(\mathbb{R}_+, B)$, which we also denote by $\widehat{\sigma}$, because all is naturally related by the natural inclusion and restriction maps.

The generalized mapping cone construction $k \mapsto I_1 \circ k$ satisfies $\widehat{\sigma} \circ (I_1 \circ k) = I_1 \circ (\widehat{\sigma} \circ k)$.

On the other hand the isomorphism ϑ of Proposition 9.2.4 satisfies, by definition of ϑ,

$$\vartheta([\widehat{\sigma} \circ (k \oplus h_0)]) = [I_1 \circ (\widehat{\sigma} \circ (k \oplus h_0))] + [H_0] = [\widehat{\sigma} \circ (I_1 \circ (k \oplus h_0))] + [H_0]\,.$$

On the other hand $\widehat{\sigma} \circ H_0 = H_0$ and $\widehat{\sigma} \circ (H \oplus H_0) = (\widehat{\sigma} \circ H) \oplus H_0$, because $\widehat{\sigma}$ fixes $I_2(\mathcal{M}(B))$, $H_0(D)$ is in $I_2(\mathcal{M}(B))$ and $\oplus = \oplus_{s,t}$ (up to unitary equivalence) with canonical generators s, t of $\mathcal{O}_2$ in $I_2(\mathcal{M}(B))$. Thus

$$[\widehat{\sigma} \circ (I_1 \circ (k \oplus h_0))] + [H_0] = [\widehat{\sigma} \circ ((I_1 \circ (k \oplus h_0)) \oplus H_0)]\,.$$

Since $[I_1 \circ (k \oplus h_0)] + [H_0] \in G(H_0;\, D, E_{\mathbb{R}})$, we get from Corollary 8.3.4(ii) that

$$[\widehat{\sigma} \circ (I_1 \circ (k \oplus h_0) \oplus H_0)] = [I_1 \circ (k \oplus h_0) \oplus H_0] = \vartheta([k \oplus h_0]).$$

Since $[k \oplus h_0]$ is in $\mathrm{R}(\mathcal{C};\, D, B)$ and ϑ is faithful on $\mathrm{R}(\mathcal{C};\, D, B)$ by Proposition 9.2.4, this shows the invariance of $[k \oplus h_0]$ under scaling:

$$[k \oplus h_0] = [\widehat{\sigma} \circ (k \oplus h_0)].$$

By Corollary 7.4.19(i),

$$S(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))) = \mathrm{SR}(\mathcal{C};\, D, B)\,.$$

The Definition of $S(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ yields that $h = k \oplus h_0$ is scaling invariant up to unitary equivalence (by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$. By Corollary 7.4.19(iii),

$$\mathrm{R}(\mathcal{C};\, D, B) \cong \mathrm{SR}(\mathcal{C};\, D, B) + [h_0] \subseteq \mathrm{SR}(\mathcal{C};\, D, B)\,.$$

(ii): Since h_0 is unitarily equivalent to $h_0 \oplus h_0$ we get from Lemma 7.4.24 that

$$\mathrm{Hom}(D, B) \cap \mathcal{C} = \{h \in \mathrm{Hom}(A, B)\,;\ [h] \in \mathrm{SR}(\mathcal{C};\, D, B)\,\}\,.$$

By part (i) and by Corollary 9.1.3, for every $[h] \in G(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))) \cong \mathrm{R}(\mathcal{C};\, D, B)$, there exists a unitary $u \in \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$ and a *-morphism $k\colon D \to B$ such that $u^* k(\cdot) u = h$. It follows that $k \in \mathrm{R}(\mathcal{C};\, D, B)$ and $[k \oplus h_0] = [k] + [h_0] = [h] + [h_0] = [h]$. Thus, $k \oplus h_0$ is unitarily homotopic to k, and $k \in \mathrm{Hom}(\mathcal{C};\, D, B)$. We can replace k by $k \oplus h_0$, and may assume that k is a monomorphism that dominates h_0 (in $\mathcal{M}(B)$ itself). It follows that $k \in \mathrm{Hom}(\mathcal{C};\, D, B) \to [k] \in \mathrm{R}(\mathcal{C};\, D, B)$ is surjective.

Morphisms $h_1, h_2 \in \mathrm{Hom}(\mathcal{C};\, D, B)$ define the same element $[h_1] = [h_2]$ in $\mathrm{R}(\mathcal{C};\, D, B)$ if and only if $h_1 \oplus h_0$ and $h_2 \oplus h_0$ are unitarily homotopic by definition of $\mathrm{R}(\mathcal{C};\, D, B) = G(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$.

It remains to check that the isomorphism $\mathrm{R}(\mathcal{C};\, D, B) \cong \mathrm{KK}(\mathcal{C};\, D, B)$ given by Proposition 9.2.4 and of Corollary 9.2.5 maps $[h] \in \mathrm{R}(\mathcal{C};\, D, B)$ to the difference construction $[h - 0] \in \mathrm{KK}(\mathcal{C};\, D, B)$ if $h \in \mathrm{Hom}(\mathcal{C};\, D, B)$.

Let $\mathcal{C}(\mathbb{R}) := \mathcal{C} \otimes \mathrm{CP}(\mathbb{C}, \mathrm{C}_0(\mathbb{R}))$. By Corollary 5.9.23 and Corollary 8.3.3 there are natural isomorphisms

$$\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB) \cong G(H_0;\, D, E_{\mathbb{R}})\,,$$

and

$$\mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB) \cong \mathrm{KK}(\mathcal{C};\, D, B)$$

that sends $[H_0]$ to zero and the mapping cones $C_h \in \mathrm{SExt}(\mathcal{C}(\mathbb{R});\, A, SB)$ (corresponding to the Busby invariant $[I_1 \circ h] \in S(H_0;\, D, E_{\mathbb{R}})$) of a morphism $h \in \mathrm{Hom}(\mathcal{C};\, D, B) = \mathrm{Hom}(D, B) \cap \mathcal{C}$ to the difference construction $[h-0] = [(B, h, 0)] \in \mathrm{KK}(\mathcal{C};\, D, B)$.

If we combine this isomorphism with the isomorphism $[h] \mapsto [h] \oplus [H_0]$ from $\mathrm{R}(\mathcal{C};\, D, B) \cong G(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$ onto $G(H_0;\, D, E_{\mathbb{R}}) \cong \mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, SB)$, then we get an *epimorphism* α from the Abelian semigroup $[\mathrm{Hom}(\mathcal{C};\, D, B)] \subseteq [\mathrm{Hom}(D, B)]$ of unitary equivalence classes [h] of morphisms $h \in \mathrm{Hom}(\mathcal{C};\, D, B)$ *onto* $\mathrm{KK}(\mathcal{C};\, D, B)$.

The construction of α shows (by inspection of the values of the composed maps) that $\alpha([h]) = [(B, h, 0)] =: [h - 0] \in \mathrm{KK}(\mathcal{C};\, D, B)$. Since the last two morphisms of the compositions

$$[\mathrm{Hom}(\mathcal{C};\, D, B)] \to \mathrm{R}(\mathcal{C};\, D, B) \to \mathrm{Ext}(\mathcal{C}(\mathbb{R});\, D, B) \to \mathrm{KK}(\mathcal{C};\, D, B)$$

are isomorphisms, we get that $[h_1 - 0] = [h_2 - 0]$ in $\mathrm{KK}(\mathcal{C};\, D, B)$ if and only if h_1 and h_1 have the same class in $G(h_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B)))$. The latter means that $h_1 \oplus h_0$ and $h_2 \oplus h_0$ are unitarily homotopic. □

3. Unsuspended stable E-theory versus Ext-groups

We consider the case where A is separable and stable, i.e., that $D := A \otimes \mathbb{K} \cong A$, that B is σ-unital and stable and that $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is a countably generated point-norm closed m.o.c. cone. Moreover, we assume that $\mathcal{C}$ is "faithful" and "non-degenerate" in the sense that if $a \in A_+$ $V(a) = 0$ for all $V \in \mathcal{C}$ implies $a = 0$, and that $\{V(a)\,;\ V \in \mathcal{C}, a \in A\}$ generates B as a closed two sided ideal.

[With other words: There does not exist closed ideals $I \neq \{0\}$ of A or $J \neq B$ of B such that $V(I) = \{0\}$ or $V(A) \subseteq J$ for all $V \in \mathcal{C}$. We exclude this, because otherwise we can in the below considerations A replace by A/I and B by J.]

Recall that in case where A is amenable and B is separable, the m.o.c. cone $\mathcal{C}$ is the same as the m.o.c. cone of all $\Psi_{\mathcal{C}}$-equivariant nuclear c.p.-maps from A to B, where $\Psi_{\mathcal{C}}$ is the lower semi-continuous action of $\mathrm{Prim}(B)$ on A defined by $\mathcal{C}$.

Respectively, $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is the m.o.c. cone of all Ψ-equivariant c.p.-maps for a given non-degenerate action $\Psi\colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathcal{I}(A)$ of $\mathrm{Prim}(B)$ on A that is lower semi-continuous.

More specially we can take – in the situation where A is exact and the action Ψ is lower s.c. and monotone upper s.c. – the m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}_{\mathrm{nuc}}(A, B)$ of all

Ψ-equivariant nuclear c.p.-maps $V \colon A \to B$. The exactness of A yields that those V are automatically Ψ-residual nuclear.

This equivalences show that working with an m.o.c. cone $\mathcal{C}$ covers all possible cases of "ideal-equivariant" KK- and Ext-groups.

What is optimal notation?
h, H, H_0 or H_0? Mostly used?
Unify all notation?
What about $h_0 \colon A \to B$ if $\mathcal{C} := \mathcal{C}(h_0)$?

Recall that there is a universal non-degenerate *-monomorphism $H \colon A \to \mathcal{M}(B)$ in "general position", i.e., H is unitarily equivalent to its infinite repeat, that is in 1-1-correspondence to $\mathcal{C}$, cf. Chapters 3 and 5 Precise Refs ?? in Chapters 3 and 5.

Let $H_1 := \pi_B \circ H \colon\ A \to \mathrm{Q}(B) := \mathcal{M}(B)/B$.

Basic observations:

Let A, B, $\mathcal{C}$, $H \colon A \to \mathcal{M}(B)$, $H_1 := \pi_B \circ H$, as above.

Partly obvious generalizations of work of G. Kasparov, Connes/Higson and J. Cuntz proves homotopy invariance of $\mathrm{KK}(\mathcal{C};\, A, \cdot)$, that

$$\mathrm{KK}(\mathcal{C}_{(1)}\,;\, A, B_{(1)}) \cong \mathrm{Ext}(\mathcal{C}\,;\, A, B)$$

and that

$$\mathrm{KK}(\mathcal{C};\, A, B) = \mathrm{Ext}(\mathcal{C};\, A, SB)\,.$$

Moreover, if the m.o.c. cone $\mathcal{C} \subset \mathrm{CP}(A, B)$ is countably generated and non-degenerate, this results allow to give the following "reduction" to a description by "ordinary" K_*-theory (using also a result of Cuntz and Higson):

(1)

$$\mathrm{KK}\big(\mathcal{C}(-\infty, 1]\,;\ A, \mathrm{C}_0((-\infty, 1], B)\big) = 0\,.$$

(2)

$$\mathrm{KK}(\mathcal{C};\, A, B) \ \cong\ \ \text{kernel of}\ \ \mathrm{K}_1(H_1(A)' \cap \mathrm{Q}^s(B)) \to \mathrm{K}_1(\mathrm{Q}^s(B)) = \mathrm{K}_0(B)\,.$$

(3) $\mathrm{Q}^s(B)$ is K_1-bijective.

(4) Homotopy invariance of KK: Each evaluation map

$$B[0,1] = \mathrm{C}([0,1], B) \ni f \mapsto f(t) \in B$$

defines the same natural isomorphism

$$\mathrm{KK}(\mathcal{C}[0,1]\ \ A, B[0,1]) \cong \mathrm{KK}(\mathcal{C}\,;\ A, B)\,.$$

(5) In particular, with $CB := \mathrm{C}_0((0,1], B)$,

$$\mathrm{KK}(\mathcal{C}(0,1];\, A, CB) = 0\,.$$

We need the following natural isomorphisms and descriptions of $\mathrm{KK}(\mathcal{C}\,;\, A, B)$ and and of representatives of its zero elements:

PROPOSITION 9.3.1. *Suppose that A is separable and stable, B is σ-unital and stable, and if the m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ is non-degenerate and countably generated, and let then*

$$\mathrm{KK}(\mathcal{C}\,;\,A,B) \;\cong\; \textit{kernel of}\;\; \mathrm{K}_1(H(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B))\,,$$

and

$$\mathrm{Ext}(\mathcal{C}\,;\,A,B) \;\cong\; \textit{kernel of}\;\; \mathrm{K}_0(H(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_0(\mathrm{Q}(B))\,.$$

(i) *The kernel of $\mathrm{K}_1(H_1(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B))$ is the image of the map*

$$u \mapsto [\pi_B(u)] \in \mathrm{K}_1(H_1(A)' \cap \mathrm{Q}(B))$$

where $u \in \mathcal{M}(B)$ is in the group of those unitary elements of $\mathcal{M}(B)$ that satisfy $uh(a) - h(a)u \in B$ for all $a \in A$.

(ii) *The defining equivalence relation $u_1 \sim u_2$ – given by $[\pi_B(u_1)] = [\pi_B(u_2)]$ in $\mathrm{K}_1(H(A)' \cap \mathrm{Q}(B))$ – can be equivalently expressed by the property that there exist $T_k \in \mathcal{M}(B)$ $(k = 1, \ldots, n)$ with $T_k^* = -T_k$ and $T_k h(a) - h(a)T_k \in B$, for $a \in A$, $k \in \{1, \ldots, n\}$, such that*

$$(u_1^* u_2) \oplus_{s,t} 1 = \exp(iT_1) \cdot \ldots \cdot \exp(iT_n)\,.$$

Here $s, t \in h(A)' \cap \mathcal{M}(B)$ are isometries with $ss^* + tt^* = 1$.

THIS formulation of Parts (i) and (ii) use the K_1-bijectivity of $\mathrm{Q}^s(B)$ for stable σ-unital B.
See Parts (c) and (iii) of Proposition 4.2.15.
It implies that all unitaries $v \in \mathrm{Q}^s(B)$ with $[v] = 0$ in $\mathrm{K}_1(\mathrm{Q}^s(B))$ are products of exponentials. Thus, there is a unitary $u \in \mathcal{M}(B)$ with $\pi(u) = v$ if and only if $[v] = 0$ in $\mathrm{K}_1(\mathrm{Q}^s(B))$.

PROOF. The formulas for $\mathrm{KK}(\mathcal{C}\,;\,A,B)$ and $\mathrm{Ext}(\mathcal{C}\,;\,A,B)$ have been mentioned/proven somewhere above!

(i): The unit 1 of the C^*-algebra $H_1(A)' \cap \mathrm{Q}(B)$ is properly infinite.

Thus, each element z of $\mathrm{K}_1(H_1(A)' \cap \mathrm{Q}(B))$ has a representative $[v] = z$ by a unitary in $v \in H(A)' \cap \mathrm{Q}(B)$. and $v \oplus 1$ defines the same class in $\mathrm{K}_1(H(A)' \cap \mathrm{Q}(B))$ as v.

If $v = \pi_B(u) \in H(A)' \cap \mathrm{Q}(B)$ for some unitary in $u \in \mathcal{M}(B)$ then $[v] = 0$ in $\mathrm{K}_1(\mathrm{Q}(B))$ because $\mathrm{K}_1(\mathcal{M}(B)) = 0$ by stability of B. Clearly, $[u, h(a)] \in B$ for all $a \in A$ if and only if $\pi_B(u) \in H(A)' \cap \mathrm{Q}(B)$.

Conversely, if v is unitary in $\mathrm{Q}(B)$ and $[v] = 0$ in $\mathrm{K}_1(\mathrm{Q}(B))$, then $v \oplus 1 \in \mathcal{U}_0(\mathrm{Q}(B))$.

If $[u \oplus 1] = 0$ in $\mathrm{K}_1(\mathrm{Q}(B))$, then $u \oplus 1$ is a product of exponentials $\exp(X_1) \cdot \ldots \cdot \exp(X_m)$ with $X_\ell^* = -X_\ell$ and $X_\ell \in \mathrm{Q}(B)$, cf. Lemma 4.2.6(v,2). And this implies that there exists a unitary $u \in \mathcal{M}(B)$ with $\pi_B(u) = v \oplus 1$.

(ii): If $[u] = 0$ in $\mathrm{K}_1(H(A)' \cap \mathrm{Q}(B))$, then – moreover – there exists $Y_1, \ldots, Y_n \in H(A)' \cap \mathrm{Q}(B)$ with $Y_k^* = -Y_k$ such that $u \oplus 1 = \exp(Y_1) \cdot \ldots \cdot \exp(Y_n)$. See Lemma 4.2.6(v,2).

Given any finite subset M of A and $\varepsilon > 0$ then there exist $T_k \in \mathcal{M}(B)$ ($k = 1, \ldots, n$) with $\pi_B(T_k) = Y_k$ and $T_k^* = -T_k$ and $[T_k, h(a)] \in B$ for all $a \in A$ and $\|[T_k, h(a)]\| < \varepsilon$ for all $a \in M$. The latter property can be obtained by lifting the Y_k to $S_k \in \mathcal{M}(B)$ with $S_k^* = -S_k$ and $\pi_B(S_k) = Y_k$ and then one defines $T_k := (1-e)S_k(1-e)$ for a suitable positive contraction $e \in B_+$ in a approximate unit of B that is quasi-central for $h(A) \subseteq \mathcal{M}(B)$. □

The following conjecture

CONJECTURE 9.3.2. *The "logarithmic length" of the unitary* $\pi_{CB}((u_1^* u_2) \oplus_{s,t} 1)$ *with unitaries of* $u_1, u_2 \in \mathcal{M}(CB)$ *such that* $[u_1, h(a)] - [u_2, h(a)] \in CB$ *is bounded by a universal constant, where above considered* B *is replaced here by* $CB := \mathrm{C}_0\big((-\infty, 1], B\big)$.

One can correct this by passing to $u_k \oplus 1$ ($k = 1, 2$) in $\mathcal{U}(\pi_B^{-1}(H(A)' \cap \mathrm{Q}(B)))$.

This is possible, because A is separable and stable, B is σ-unital and stable, and $h \approx_u \delta_\infty h$ for $h \colon A \to \mathcal{M}(B)$ with $H = \pi_B \circ h$, the map

$$u \oplus 1 \mapsto \pi_B(u \oplus 1) \in \mathcal{U}(H(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1$$

is surjective map onto the kernel of

$$\mathrm{K}_1(H(A)' \cap \mathrm{Q}(B)) \to \mathrm{K}_1(\mathrm{Q}(B))\,.$$

LEMMA 9.3.3. *An unitary* $u \in \mathcal{M}(B)$ *represents the zero element of* $\mathrm{KK}(\mathcal{C}\,; A, B)$*, if and only if,* $\pi_B(u)$ *represents the zero of* $\mathrm{K}_1(H(A)' \cap \mathrm{Q}(B))$*, if and only if, there exist elements* $T_1, \ldots, T_n \in \mathcal{M}(B)$ *such that* $T_j^* = -T_j$*,*

$$\exp(T_1) \cdot \ldots \cdot \exp(T_n) - (u \oplus 1) \in B$$

and $T_k h(a) - h(a) T_k \in B$ *for all* $a \in A$ *and* $k = 1, \ldots, n$.

Given $\varepsilon > 0$ *and a finite subset* $M \subset A$*, then the* T_k *in this relations can be chosen such that moreover* $\|T_k h(a) - h(a) T_k\| < \varepsilon$ *for* $a \in M$.

Here is the same problem with the possibly missing K_1-injectivity of $H(A)' \cap \mathrm{Q}(B)$ if one hopes to replace $u \oplus 1$ by u itself in this relations.

Is $H(A)' \cap \mathrm{Q}(B)$ K_1-injective if A or B is $\mathcal{O}_\infty$-absorbing?

Is it equivalent to the question about K_1-injectivity of all properly infinite unital E?

The answer is positive if B is s.p.i. and H is nuclear.

Let $B[0,1] := \mathrm{C}([0,1], B) \cong B \otimes \mathrm{C}([0,1])$. Define $\mathcal{C}[0,1] \subseteq \mathrm{CP}(A, B[0,1])$ as the set of c.p. maps $V \colon A \to B[0,1]$ with $V(\cdot)(t) \in \mathcal{C}$ for each $t \in [0,1]$. The corresponding realization of $\mathcal{C}[0,1]$ is induced by the natural embedding $\eta\colon b \mapsto b \otimes 1 \in B[0,1]$ of B into $B[0,1]$ in the sense that $\mathcal{M}(\eta) \circ h\colon A \to \mathcal{M}(B[0,1])$ is the corresponding universal *-monomorphism from A into $\mathcal{M}(B[0,1])$ that defines $\mathcal{C}[0,1]$.

Thus, the kernel of

$$\mathrm{K}_1((\pi_{B[0,1]} \circ \mathcal{M}(\eta) \circ h)(A)' \cap Q(B[0,1])) \to \mathrm{K}_1(Q(B[0,1]))$$

is naturally isomorphic to $\mathrm{KK}(\mathcal{C}[0,1]\,;\, A, B[0,1])$.

A natural generalization of Kasparov's result of homotopy invariance of KK- and Ext-groups is the following:
For (given fixed) $t \in [0,1]$, the evaluation map $f \in B[0,1] \mapsto f(t) \in B$ defines a natural *isomorphism* from $\mathrm{KK}(\mathcal{C}[0,1]\,;\, A, B[0,1])$ onto $\mathrm{KK}(\mathcal{C}\,;\, A, B)$. ("Homotopy invariance" of $\mathrm{KK}(\mathcal{C}\,;\, A, B)$.)

On the set of "representing" unitaries $u \in \mathcal{M}(B)$, respectively of $\{u(t)\} = U \in \mathcal{M}(B[0,1])$ (with $t \mapsto u(t) \in \mathcal{M}(B)$ strictly continuous), this isomorphism corresponds to the evaluation map $U = \{u(t)\} \mapsto u(t) \in \mathcal{M}(B)$.

Consequently:

Corollary 9.3.4. *Suppose that A and B are stable, A separable and B σ-unital, $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ a countably generated non-degenerate m.o.c. cone.*

Let $a_0 \in A$ such that $A = C^*(a_0)$ and let $h\colon A \to \mathcal{M}(B)$ a non-degenerate C^*-monomorphism that is unitary homotopic to is infinite repeat $\delta_\infty \circ h$, satisfies $V_b := \langle h(\cdot)b, b\rangle \in \mathcal{C}$ for all $b \in B$ and that $\mathcal{C}$ is generated by $\{V_b\,;\; b \in B\}$.

Let $B[0,1] := \mathrm{C}([0,1], B)$ and $\eta\colon \mathcal{M}(B) \to \mathrm{C}_{\mathrm{b,st}}([0,1], \mathcal{M}(B)) \cong \mathcal{M}(B[0,1])$ the natural embedding.

If $U \in \mathcal{M}(B[0,1])$ is a representing unitary for an element of

$$\mathrm{KK}(\mathcal{C}[0,1]\,;\, A, B[0,1])\,,$$

i.e., and if $U(0) \oplus 1$ can be written modulo B as finite product of $\exp(T_k)$, $k = 1, \ldots, n$, with $-T_k^* = T_k \in \mathcal{M}(B[0,1])$ and $T_k h(a) - h(a)T_k \in B[0,1]$ for $a \in A$, $k = 1, \ldots, n$, then $U \oplus 1$ can be written modulo $B[0,1]$ as a finite product of $\exp(S_\ell)$, $\ell = 1, \ldots, m$, $-S_\ell^* = S_\ell \in \mathcal{M}(B[0,1])$ and $S_\ell(\mathcal{M}(\eta) \circ h)(a) - (\mathcal{M}(\eta) \circ h)(a) S_\ell \in B[0,1]$ for $a \in A$, i.e.,

$$-S_\ell^* = S_\ell \in \mathrm{Der}(\mathcal{M}(\eta)(h(A)), B[0,1]) \subseteq \mathcal{M}(B[0,1])$$

and `Commutator norms ????`

??????????????????????????

Here $\mathcal{M}(\eta)\colon \mathcal{M}(B) \to \mathcal{M}(B[0,1])$ is the natural inclusion map.

Proof. `To be filled in` ?? ?? □

COROLLARY 9.3.5. *Let* A, B, $\mathcal{C} \subseteq \mathrm{CP}(A,B)$ *and* $h\colon A \to \mathcal{M}(B)$ *as above,* $\mathcal{M}(\eta)\colon \mathcal{M}(B) \to \mathrm{C}_{\mathrm{b,st}}((-\infty,1],\mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C}_0((-\infty,1],B))$ *the natural monomorphism.*

Let $u \in \mathcal{M}(\mathrm{C}_0((-\infty,1],B))$ *a unitary given by the strictly continuous map* $(-\infty,1] \ni t \mapsto u(t) \in \mathcal{U}(\mathcal{M}(B))$ *from* $(-\infty,1]$ *into the unitaries in* $\mathcal{M}(B)$.

Suppose that $[u(t),h(a)] \in B$ *for all* $a \in A$ *and* $t \in (-\infty,1]$ *and that* $t \mapsto [u(t),h(a)]$ *is continuous, and*

$$\lim_{t\to-\infty} \|[u(t),h(a)]\| = 0 \quad \forall\, a \in A\,,$$

i.e., $[u,h(A)] \subseteq \mathrm{C}_0((-\infty,1],B)$.

Then there exist $S_1,\ldots,S_n \in \mathcal{M}(\mathrm{C}_0((-\infty,1],B))$ *with* $S_k^* = -S_k$, $[s_k,h(a)] \in B$ *for all* $a \in A$ *and*

$$U = \exp(S_1)\cdot\ldots\cdot\exp(S_n)\cdot W$$

for some $W \in \mathcal{U}(\mathrm{C}_0((-\infty,1],B)+\mathbb{C}\cdot 1)$ *with* $1 - W \in \mathrm{C}_0((-\infty,1],B)$.

Such W *is a product of exponentials* $W = \exp(T_1)\cdot\ldots\cdot\exp(T_\ell)$ *with* $T_j = -T_j^* \in \mathrm{C}_0((-\infty,1],B)$.

In particular, $W(t) \in \mathcal{U}_0(B+\mathbb{C}\cdot 1)\cap(1+B)$ *for each* $t \in (-\infty,1]$.

QUESTION 9.3.6. Suppose that $T_1,\ldots,T_m \in \mathrm{Der}(h(A),B) \subseteq \mathcal{M}(B)$ and

$$S_1,\ldots,S_n \in \mathrm{Der}(h(A),B[0,1]) \subseteq \mathcal{M}(B[0,1])$$

with $T_j^* = -T_j$, $S_k^* = -S_k$, $V_1,V_2 \in U_0(B+\mathbb{C}\cdot 1)\cap(1+B)$ and $a \in A$ are given with

$$\exp(T_m)\cdot\ldots\cdot\exp(T_1)V_1 = W(0)V_2\,,$$

for $W(t) := \exp(S_1(t))\cdot\ldots\cdot\exp(S_n(t))$.

Does there exist for each $\gamma > 0$ and $\varepsilon \in (0,\gamma)$ a unitary

$$U \in \mathcal{U}_0(B[0,1]+\mathbb{C}\cdot 1)\cap(1+B[0,1])$$

with $U(0) = 1$,

$$\|[W(t)V_2U(t),h(a)]\| \le \gamma + \|[W(0)V_2,h(a)]\|$$

for $t \in [0,1]$ and

$$\|[W(1)V_2U(1),h(a)]\| \le \varepsilon \quad ?$$

LEMMA 9.3.7. *Suppose that* M *is a* C^**-algebra,* $B \subseteq M$ *a closed ideal,* $X = \{a_1,\ldots,a_n\} \subseteq M$ *a finite subset, and let*

$$\nu(a;\, S_1,\ldots,S_n) := \exp(S_1)\cdot\ldots\cdot\exp(S_n)a - a\exp(S_1)\cdot\ldots\cdot\exp(S_n)$$

for $a, S_1,\ldots,S_n \in M$, *with* $S_k = -S_k^*$.

Suppose that $T_1,\ldots,T_n \in M$ *satisfy* $T_k = -T_k^*$ *and* $T_ka - aT_k \in B$ *for all* $a \in X$ *and* $k = 1,\ldots,n$.

Then for each $\varepsilon > 0$ there exists a positive contraction $b = b(\varepsilon) \in B_+$ such that

$$\big\| \nu\big(a;\, (1-b)T_1(1-b), \dots, (1-b)T_n(1-b)\big)\big\| \leq \varepsilon \quad \forall\, a \in X\,.$$

`But the QUESTION is ???:`
`Let` $B := \mathrm{C}([0,1], C)$, `with` C σ`-unital and stable,`
$T_1, \dots, T_n \in \mathcal{M}(B)$ `self-adjoint contractions`
`with` $T_k h(a) - h(a)T_k \in B$ `for all` $a \in A$ `and` $k = 1, \dots, n$`, and, in addition,`

$$\exp(T_1(0)) \cdot \ldots \cdot \exp(T_n(0)) = 1\,.$$

`Can we find the` $b \in B_+$ `in this case such that` $b = \{c(t)\}$ `satisfies`
`in addition the following inequality ?`

$$\|1 - \exp\big(i(1-c(0))T_1(0)(1-c(0))\big) \cdot \ldots \cdot \exp\big(i(1-c(0))T_1(0)(1-c(0))\big)\| < \varepsilon\,.$$

Proof. Without loss of generality, one can suppose that M is separable and $a \neq 0$, because we can replace M and B in the proof by by $C^*(X \cup \{T_1, \dots, T_n\})$ and B by $B \cap C^*(X \cup \{T_1, \dots, T_n\})$. Then B contains a strictly positive contraction $e \in B_+$. If we define by functional calculus elements $f(e) \in B_+$ with $f \in \mathrm{C}[0,1]_+$, $\|f\| \leq 1$ and $f([0,\delta]) = 0$ for some $\delta \in (0,1)$, then we can select from this family of positive contractions a commutative approximate unit $e_1, e_2, \dots$ of B that is an approximately central sequence for M and satisfies $e_{m+1}e_m = e_m$. It implies $(1-e_m)(1-e_{m+k})^\ell = (1-e_{m+k})^\ell$ for $k, \ell \geq 1$.

Given $\varepsilon > 0$ we let $\gamma := \max(\|T_1\|, \dots, \|T_n\|)$ and

$$\delta := \varepsilon / n(1+2\gamma) \cdot \exp(\gamma)\,.$$

Since $[a, T_k] := aT_k - T_k a \in B$ for $a \in X$, and since (e_m) is an approximate unit of B that is approximately central for the elements of M, there exists $m_0 \in \mathbb{N}$ with $\|e_m a - a e_m\| < \delta$ and $\|[a, T_k](1-e_m)\| < \delta$ for all $m \geq m_0$ and $a \in X$.

Recall that $[a, x] := ax - xa$ satisfies the Leibnitz rule $[a, xy] = [a, x]y + x[a, y]$.

We obtain with this rule that

$$\|[a, (1-e_m)T_k(1-e_m)]\| \leq \|[a, T_k](1-e_m)\| + 2\|[a, e_m]\|\|T_k\| < \delta \cdot (1 + 2\|T_k\|)\,.$$

It follows that

$$\|[a, x_1 \cdot \ldots \cdot x_n]\| \leq \gamma^{n-1}(\|[a, x_1]\| + \ldots + \|a, x_n\|)$$

for $\gamma = \max\{\|x_1\|, \dots, \|x_n\|\}$, and, for self-adjoint $S, S_1, \dots, S_n \in M_{s.a.}$ that

$$\big\|\nu(a;\, S_1, \dots, S_n)\big\| \leq \sum_{k=1}^{n} \|[a, \exp(iS_k)]\|\,.$$

Moreover, $\|[a, \exp(iS)]\| \leq \|[a, S]\| \cdot (\exp \|S\|)$.

With $\gamma := \max\{\|S_1\|, \ldots, \|S_n\|\}$ and $S_k \in M_{s.a.}$, we get

$$\big\|\nu(a;\, S_1, \ldots, S_n)\big\| \leq \exp(\gamma) \cdot \sum_{k=1}^{n} \|[a, S_k]\| \,.$$

If $T_k \in M_{s.a.}$ with $T_k a - aT_k \in B$ for all $a \in X$ then we can find $m_0 \in \mathbb{N}$ such that $\|[a, (1-e_m)T_k(1-e_m)]\| < \delta \cdot (1+2\gamma)$ for all $m \geq m_0$. Thus, for $m \geq m_0$,

$$\big\|\nu(a;\, (1-e_m)T_1(1-e_m), \ldots, (1-e_m)T_n(1-e_m))\big\| < n\delta \cdot (1+2\gamma) \cdot \exp(\gamma) = \varepsilon \,.$$

□

We denote by $\xi\colon \mathcal{M}(B) \to \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ the natural unital embedding given by the strictly continuous extension of

$$B \ni b \to 1 \otimes b \in \mathrm{C_b}(\mathbb{R}, B) \subseteq \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B)) \,.$$

The next proposition is the base for the final proof the injectivity of

$$\mathrm{R}(\mathcal{C};\, A, B) = G(h_0, A, E) \to G(H_0, A, E) = \mathrm{KK}(\mathcal{C};\, A, B) \,.$$

Proposition 9.3.8. *Suppose that A and B are stable C^*-algebra, A is separable and B is σ-unital, and let $h\colon A \to \mathcal{M}(B)$ a non-degenerate C^*-morphism such that $\delta_\infty \circ h$ is unitary equivalent to h. Denote by $\xi\colon \mathcal{M}(B) \hookrightarrow \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$ the natural inclusion.*

If $U \in \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B)) \cong \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$ a unitary such that, for all $t \in \mathbb{R}$,

$$U(t)h(a) - h(a)U(t) \in B$$

and

$$\lim_{t\to-\infty} \|U(t)h(a) - h(a)U(t)\| = 0$$

for each $a \in A$, i.e.,

$$U \cdot (\xi \circ h)(a) - (\xi \circ h)(a) \cdot U \in J_{-\infty}$$

for $J_{-\infty} := \{c \in \mathrm{C_b}(\mathbb{R}, B)\,;\ \lim_{t\to-\infty} c(t) = 0\,\}$.

Then there exist unitaries $V, W \in \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ such that $U \oplus 1 = VW$,

$$W(\xi \circ h)(a) - (\xi \circ h)(a)W \in \mathrm{C}_0(\mathbb{R}, B) \quad \forall\, a \in A \,,$$

$V \in 1 + \mathrm{C_b}(\mathbb{R}, B)$ and $V(t) = 1$ for $t \leq -1$.

Proof. Let $a_0 \in A$ a contraction that generates A as C^*-algebra.

First we define $W_0 \in \mathcal{M}(SB)$ by $W_0(t) := W(t)$ for $t \leq 0$ and $W_0(t) := W(-t)$ for $t > 0$. Then $W_0(t)h(a) - h(a)W_0(t) \in B$ for each $t \in \mathbb{R}$ and $a \in A$ and $U_0(t) := W_0(t)^*U(t)$ satisfies $U_0(t) = 1$ for $t \leq 0$ and $[U_0(t), h(a)] \in B$ for all $t \geq 0$.

It follows that, for each (positive) $n \in \mathbb{N}$, there exists a decomposition of $U_n := U_0|(-\infty, 2n]$ of the form $U_n = V_n \cdot W_n$ where $W_n := \exp(T_{n,1}) \cdot \ldots \cdot \exp(T_{n,k_n})$ with $-T_{n,\ell}^* = T_{n,\ell} \in \mathcal{M}(\mathrm{C}_0(-\infty, 2n])$ $[T_{n,\ell}(t), h(a)] \in B$, $V_n(t) \in B+1$, $\lim_{t\to-1} V_n(t) = 1$, $\|[T_{n,\ell}(t), h(a_0)]\| < 4^{-n}$

Connect then by going back to -1 by both decompositions for U_n and U_{n+1} and by shifting and rescaling the "mirrors" at -1.

It gives a path in $[2n, 2n+2]$ of elements in $\mathcal{U}(B + \mathbb{C} \cdot 1)$ that corrects W_n on $[2n, 2n+1]$ in controlled way.

□

We could use the above mentioned results on intervals $[n, n+1]$ (instead of $[0,1]$) for a proof if we can find at the end-points a "fitting-together" procedure for the above described approximate decomposition. Or if we can find similar approximate decompositions with help of approximately central units in the ideal $\mathrm{C_b}([n,\infty), B)$ of $\mathcal{M}(\mathrm{C_0}([n,\infty), B))$. (But suitable combined).

It has to do with the question if unsuspended m.o.c. cone-related E-theory for strongly purely infinite separable C^*-algebras is homotopy-invariant, i.e., if for the Rørdam groups $\mathrm{R}(\mathcal{C};\, A, B)$ two asymptotic homomorphisms $h(0), h(1) \colon A \to \mathrm{Q}(\mathbb{R}_+, B)$ are asymptotically stably unitary equivalent in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{M}(B))/\,\mathrm{C_0}(\mathbb{R}_+, B)$ if they are homotopic, i.e., are boundaries of $h \colon A \to \mathrm{Q}(\mathbb{R}_+, \mathrm{C}([0,1], B)$ with $[h] \in \mathrm{R}(\mathcal{C};\, A, B[0,1])$. One can directly prove this kind of homotopy invariance for the Rørdam groups $\mathrm{R}(\mathcal{C};\, A, B)$.

We discuss now the natural relation of this homotopy invariance and the decomposition condition (DC) of Theorem 4.4.6.

Suppose that A and B are stable, A is separable and B is σ-unital. Let $h_0 \colon A \to B$ a non-degenerate *-monomorphism such that h_0 extends to a *-monomorphism $k_0 \colon A \otimes \mathcal{O}_2 \to B$ with $h_0 = k_0((\cdot) \otimes 1)$.

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ the corresponding m.o.c. cone. $H := \delta_\infty \circ h_0 \colon A \to \mathcal{M}(B)$ denote the infinite repeat of h_0. Consider $\mathcal{M}(B)$ naturally as a unital C^*-subalgebra of constant elements of $\mathcal{M}(\mathrm{C_0}(\mathbb{R}, B)) \cong \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$. Let consider also $\mathrm{C_b}(\mathbb{R}, B)$ as an ideal of $\mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$.

There exists a *-monomorphism $k_1 \colon \mathcal{O}_2 \otimes \mathbb{K} \to H(A)' \cap \mathcal{M}(B)$ such that $k_1(\mathcal{O}_2 \otimes \mathbb{K})B = B$, such that $H(a)k_1(c) \in B$ for all $a \in A$ and $c \in k_1(\mathcal{O}_2 \otimes \mathbb{K})$, and that $p_0 := k_1(1 \otimes p_{11})$ is a full in $\mathcal{M}(B)$ and $1 - p_0$ is a full and properly infinite projection in $H(A)' \cap \mathcal{M}(B)$. Since p_0 is properly infinite in $k_1(\mathcal{O}_2 \otimes \mathbb{K})$ with $[p_0] = 0 \in \mathrm{K_0}(k_1(\mathcal{O}_2 \otimes \mathbb{K}))$, there exists an isometry in $s_0 \in M(B)$ with $s_0 s_0^* = p_0$ and an isometry $t_0 \in H(A)' \cap \mathcal{M}(B)$ with $t_0 t_0^* = 1 - p_0$.

Let F denote the image of $\mathrm{C_b}(\mathbb{R}, \mathcal{O}_2 \otimes \mathbb{K}) \supset \mathcal{O}_2 \otimes \mathbb{K}$. We consider F naturally as a non-degenerate C^*-subalgebra of $\mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B)) \cong \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ using the *-monomorphism $f \mapsto \{k_1(f(t))\} \in \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$. Then $p_0 \in F$, $1 - p_0 \in 1 + F$. It is easy to see that each non-zero projection $p \in F$ is equivalent to p_0 by a unitary $u \in 1 + F$, because this is true for $\mathrm{C}([0,1], \mathcal{O}_2 \otimes \mathbb{K})$ (in place of F).

Remarks 9.3.9. For each positive element $g \in \mathrm{C_b}(\mathbb{R}, B)$ there is a projection $p \in F$ with $g - pg \in \mathrm{C_0}(\mathbb{R}, B)$, $p \geq p_0$.

$F \subseteq H(A)' \cap \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$.

If $V = \{V(t)\} \in \mathcal{M}(\mathrm{C}_0(\mathbb{R},B)) \cong \mathrm{C}_{\mathrm{b,st}}(\mathbb{R},\mathcal{M}(B))$ is a unitary with $\lim_{t\to-\infty}\|V(t)H(a) - H(a)V(t)\| = 0$ and $VH(a) - H(a)V \in \mathrm{C}_{\mathrm{b}}(\mathbb{R},B)$ for all $a \in A$ (i.e., $V(t)H(a) - H(a)V(t) \in B$ for all $t \in \mathbb{R}$ and $a \in A$). Then $W \in \mathcal{M}(\mathrm{C}_0(\mathbb{R},B))$ with $W(t) := V(t)^*$ for $t \leq 0$ and $W(t) := V(-t)^*$ for $t \geq 0$ is a unitary that satisfies $WH(a) - H(a)W \in \mathrm{C}_0(\mathbb{R},B)$ for all $a \in A$. If $U := VW$ then $U(t) = 1$ for $t \leq 0$ and $UH(a) - H(a)U \in \mathrm{C}_{\mathrm{b}}(\mathbb{R},B)$ for all $a \in A$.

Let $U \in \mathcal{M}(\mathrm{C}_0(\mathbb{R},B))$ a unitary with $U(t) = 1$ for $t \leq 1$ and $UH(a)-H(a)U \in \mathrm{C}_{\mathrm{b}}(\mathbb{R},B)$ for all $a \in A$. Define $U_e(t,s) := U(t - s^{-1} + 1)$ for $t \in \mathbb{R}$ and $s \in (0,1]$ and $U_e(t,0) := 1$ for all $t \in \mathbb{R}$. Then U_e is a unitary in

$$\mathcal{M}(\mathrm{C}_0(\mathbb{R}\times[0,1],B)) \cong \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}\times[0,1],\mathcal{M}(B))$$

with $U_e(t,s) = 1$ if $t \leq 1$ and $U_eH(a) - H(a)U_e \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}\times[0,1],B)$ for all $a \in A$.

Notice $\mathrm{C}_{\mathrm{b}}(\mathbb{R},\mathrm{C}([0,1],B)) \cong \mathrm{C}_{\mathrm{b}}(\mathbb{R}\times[0,1],B)$. Same for C_0 in place of C_{b}, and for $(\mathrm{C}_{\mathrm{b,st}},\mathcal{M}(\mathrm{C}([0,1],B)),\mathcal{M}(B))$ in place of $(\mathrm{C}_{\mathrm{b}},\mathrm{C}([0,1],B),B)$.

For each $s \in [0,1]$ define the unitary $U_s(t) := U_e(t,s)$. The U_s are unitaries in $\mathcal{M}(\mathrm{C}_0(\mathbb{R},B))$ that satisfy $U_0 = 1$ and $U_s(t) = 1$ for $t \leq s^{-1} - 1$ if $s \in (0,1]$. In particular, $U_s(t) = 1$ for all $t \leq 0$ and $s \in [0,1]$.

There exists a projection $p \in F$ such that

$$(1-p)(U_eH(a) - H(a)U_e) \in \mathrm{C}_0(\mathbb{R}\times[0,1],B) \quad \text{for all } a \in A\,.$$

Let $v \in 1 + F$ a unitary with $vpv^* = p_0$. Then

$$(1-p_0)(vU_sH(a) - H(a)vU_s) \in \mathrm{C}_0(\mathbb{R},B)$$

for all $a \in A$.

The projections $(1-p_0)$ and p_0 commute with $H(a)$ and commute modulo $\mathrm{C}_0(\mathbb{R},B)$ with $(vU_s)H(a)(vU_s)^*$ for each $a \in A$. We define for $s \in [0,1]$ the completely positive map $T_s\colon A \to \mathrm{C}_{\mathrm{b}}(\mathbb{R},B)$ by

$$T_s(a) := p_0(vU_s)H(a)(vU_s)^*p_0 \quad \text{for } a \in A$$

Then

$$T_s(a) - p_0(vU_s)H(a)(vU_s)^* \in \mathrm{C}_0(\mathbb{R},B)$$

and

$$T_s(a^*a) - T_s(a)^*T_s(a) \in \mathrm{C}_0(\mathbb{R},B)\,.$$

Recall that $T_s(a)(t) = p_0v(t)U(s,t)H(a)(v(t)U(s,t))^*p_0$ for $(t,s) \in (-\infty,\infty)\times[0,1]$. In particular, $T_0(a) = p_0H(a)$, because $U(0,t) = 1$ and $v(t)$ commutes with $H(a)$. For $t \in \mathbb{R}$ holds $T_1(a)(t) = p_0v(t)U(t)H(a)U(t)^*v(t)^*p_0$ where $U := \{U(t)\}$ is the above considered unitary $U \in \mathcal{M}(\mathrm{C}_0(\mathbb{R},B))$ with $U(t) = 1$ for $t \leq 0$.

The restriction to $(t,s) \in [0,\infty)\times[0,1]$ of T_s defines a homotopy in $\mathrm{SR}(\mathcal{C};\, A,B)$ from $T_0 = p_0H(\cdot)$ to $T_1(\cdot)|[0,\infty)$.

It holds $T_s(a)(t) = p_0H(a)$ for all $t \leq 0$ and $a \in A$, and $a \in A \mapsto T_s(a) \in \mathrm{C}_{\mathrm{b}}(\mathbb{R},B)$ is a c.p. contraction, with $T_s(a^*a) - T_s(a)^*T_s(a) \in \mathrm{C}_0(\mathbb{R},B)$ for $a \in A$.

PROOF. to be filled in ?? □

PROPOSITION 9.3.10. *Let* $U = \{U(t)\} \in \mathrm{C_{b,st}}(\mathbb{R}, B) = \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ *and* $v = \{v(t)\} \in 1 + F \in H(A)' \cap \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ *the above considered unitary operators.*

Then $(1-p_0)(vU)H(a)(vU)^* - (1-p_0)H(a) \in \mathrm{C_0}(\mathbb{R}, A)$, $p_0(vU)H(a)(vU)^*p_0 \in \mathrm{C_b}(\mathbb{R}, A)$ *and* $p_0(vU)H(a)(vU)^*(1-p_0) \in \mathrm{C_0}(\mathbb{R}, A)$ *for* $a \in A$.

Let $k(a) := s_0^* p_0(vU)H(a)(vU)^* p_0 s_0$ *and* $h_0(a) := s_0^* p_0 H(a) s_0$ *for* $a \in A$.

Suppose that there exists a unitary $V \in \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$ *such that*

$$V^* h_0(a) V - k(a) \oplus_{s_0,t_0} h_0(a) \in \mathrm{C_0}(\mathbb{R}, B) \quad \textit{for all} \;\; a \in A\,.$$

Then $U = U_1 U_2$ *for suitable unitary operators* $U_1, U_2 \in \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ *with*

$$H(a)(U_1 - 1),\, U_2 H(a) - H(a) U_2 \;\in \mathrm{C_0}(\mathbb{R}, B) \quad \textit{for all} \;\; a \in A\,.$$

PROOF. We consider first the special case where the c.p. map $k\colon A \to \mathrm{C_b}(\mathbb{R}, B)$ given by

$$k(a)(t) := \; s_0^*(v(t)U(t))H(a)(v(t)U(t))^* s_0$$

is itself unitary equivalent modulo $\mathrm{C_0}(\mathbb{R}, B)$ to h_0, i.e., we suppose in this special case that there exists a unitary $W = \{W(t)\} \in \mathrm{C_{b,st}}(\mathbb{R}, \mathcal{M}(B))$ with

$$W^* h_0(a) W - k(a) \in \mathrm{C_0}(\mathbb{R}, B) \quad \text{for all} \;\; a \in A\,.$$

By definition of $k(\cdot)$ and Cuntz addition $\oplus_{s_0,t_0}$,

$$s_0 k(a) s_0^* + (1-p_0)H(a) = k(a) \oplus_{s_0,t_0} H(a)\,,$$

where t_0 is an isometry in $H(A)' \cap \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ with $t_0 t_0^* = 1 - p_0$, and

$$s_0 k(a) s_0^* = p_0(vU)H(a)(vU)^* p_0\,.$$

Since $(1-p_0)H(a) - (1-p_0)(vU)H(a)(vU)^* \in \mathrm{C_0}(\mathbb{R}, B)$ and

$$p_0 vUH(a)(vU)^* - p_0(vU)H(a)(vU)^* p_0 \in \mathrm{C_0}(\mathbb{R}, B)\,,$$

we get that

$$\big(k(a) \oplus_{s_0,t_0} H(a)\big) - (vU)H(a)(vU)^* \in \mathrm{C_0}(\mathbb{R}, B) \quad \text{for all} \;\; a \in A\,.$$

Let $V_1 := s_0 W s_0^* + (1-p_0)$. Then

$$V_1^*(h_0(a) \oplus_{s_0,t_0} H(a))V_1 - (k(a) \oplus_{s_0,t_0} H(a)) \in \mathrm{C_0}(\mathbb{R}, B) \quad \text{for all} \;\; a \in A\,.$$

Since $h_0(a) \oplus_{s_0,t_0} H(a) = H(a)$ for all $a \in A$, it follows that $V_1^* H(a) V_1 - (vU)H(a)(vU)^* \in \mathrm{C_0}(\mathbb{R}, B)$ for all $a \in A$. Thus, $V_2 := V_1 vU$ commutes with $H(a)$ modulo $\mathrm{C_0}(\mathbb{R}, B)$ for $a \in A$. Notice $U = U_1 U_2$ with $U_1 := v^* V_1^* v$ and $U_2 := v^* V_2$. Since $v \in 1 + F \subseteq H(A)' \cap \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$ we get that U_2 satisfies $U_2 H(a) - H(a) U_2 \in \mathrm{C_0}(\mathbb{R}, B)$. The operator $V_1 - 1 = s_0 W s_0 - p_0$ is contained in $p_0 \mathcal{M}(\mathrm{C_0}(\mathbb{R}, B)) p_0$. Since $p_0 H(a) = H(a) p_0 \in B$ for $a \in A$ it follows that $H(A)(V_1 - 1) \subseteq \mathrm{C_b}(\mathbb{R}, B)$. Since v commutes with $H(a)$ and $\mathrm{C_b}(\mathbb{R}, B)$ is an ideal of $\mathcal{M}(\mathrm{C_0}(\mathbb{R}, B))$, we get that $H(A)(U_1 - 1) \subseteq \mathrm{C_b}(\mathbb{R}, B)$.

Now we reduce the general case to the above considered special case:

Recall:

$p_0 := k_1(1 \otimes p_{11}) \in k_1(\mathcal{O}_2 \otimes \mathbb{K}) \supset F$

$k(a) := s_0^* p_0 (vU) H(a) (vU)^* p_0 s_0$

$h_0(a) := s_0^* p_0 H(a) s_0$ for $a \in A$.

It exits unitary $V \in \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B))$ with

$$V^* h_0(a) V - k(a) \oplus_{s_0,t_0} h_0(a) \in \mathrm{C}_0(\mathbb{R}, B) \quad \text{for all} \quad a \in A\,.$$

Want to show:

There exists $p_1 \geq p_0$ in and a unitary $W \in p_1 \mathcal{M}(B) p_1 + (1 - p_1)$ such that $WH(a)W^* - (vU)H(a)(vU)^* \in \mathrm{C}_0(A, B)$.

By properties of F, – or directly by simple properties of $\mathcal{O}_2 \otimes \mathbb{K}$ –, there exist unitary $w_1, w_2 \in 1 + F$ such that $w_1^* p_0 w_1 \leq 1 - p_0$ and $w_2^* p_1 w_2 = p_0$ for $p_1 := p_0 + w_1^* p_0 w_1$. Notice

$$w_1 (p_1 - p_0) w_1^* = p_0 \leq w_1 (1 - p_0) w_1^*\,.$$

For $a \in A$ it follows that

$$p_1 (vU) H(a) (vU)^* - (s_0 k(a) s_0^* + (p_1 - p_0) H(a)) = p_0 (vU) H(a) (vU)^* (1 - p_0)$$

is in $\mathrm{C}_0(\mathbb{R}, A)$.

Let $h_1(a) := t_0^* (p_1 - p_0) H(a) t_0 = t_0^* (p_1 - p_0) t_0 H(a)$ and $z := t_0^* w_1^* s_0\,.$

The mapping h_1 is a C^*-morphism because $p_1 - p_0 \leq (1 - p_0) = t_0 t_0^*$ and $p_1 - p_0$ commutes with $H(A)$. Then

$$s_0 k(a) s_0^* + t_0 h_1(a) t_0^* = s_0 k(a) s_0^* + (p_1 - p_0) H(a)\,.$$

We show below that h_1 is unitary equivalent to h_0 modulo $\mathrm{C}_0(\mathbb{R}, B)$.

Let $V_1 \in \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ a unitary with $V_1^* h_0(a) V_1 - h_1(a) \in \mathrm{C}_0(\mathbb{R}, B)$ and let $V_2 := s_0 s_0^* + t_0 V_1 t_0^*$. Then

$$(s_0 k(a) s_0^* + t_0 h_1(a) t_0^*) - V_2^* (s_0 k(a) s_0^* + t_0 h_0(a) t_0^*) V_2 \in \mathrm{C}_0(\mathbb{R}, B)\,.$$

On the other hand, there ?????

It follows that ?????

We have

$$p_1 (vU) H(a) (vU)^* - (s_0 k(a) s_0^* + t_0 h_1(a) t_0^*) \in \mathrm{C}_0(\mathbb{R}, B)$$

Next ????????????

In the same way we get that $p_1 H(a) = s_0 h_0(a) s_0^* + t_0 h_1(a) t_0^*$

$$(s_0 h_0(a) s_0^* + t_0 h_0(a) t_0^*) - V_3^* (s_0 h_0(a) s_0^* + t_0 h_1(a) t_0^*) V_3 \in \mathrm{C}_0(\mathbb{R}, B)\,,$$

and ?????

We want to show that ?????

Recall $h_0(a) = s_0^* p_0 H(a) s_0$.

We get from $t_0 t_0^* = 1 - p_0$ that

$$z^* z = s_0^* w_1 (1 - p_0) w_1^* s_0 = 1\,,$$

and

$$zz^* = t_0^* w_1^* s_0 s_0^* w_1 t_0 = t_0^* w_1^* p_0 w_1 t_0 == t_0^* (p_1 - p_0) t_0\,.$$

Notice that w_1, p_1, p_0, t_0 commute with $H(a)$. Thus $(w_1(p_1 - p_0)t_0)(t_0^*(p_1 - p_0)) = w_1(p_1 - p_0)$ commutes with $H(a)$. Since

$$w_1 t_0 h_1(a) t_0^* w_1^* = s_0^* w_1 (p_1 - p_0) H(a) w_1^* = w_1 (p_1 - p_0) w_1^* H(a)$$

it follows

$$z^* h_1(a) z = s_0^* w_1 (p_1 - p_0) w_1^* H(a) s_0 = s_0^* p_0 H(a) s_0 = h_0(a)\,,$$

and

$$\begin{aligned} zh_0(a)z^* = zs_0^* H(a) s_0 z^* &= t_0^* w_1^* p_0 H(a) w_1 t_0 = \\ t_0^* w_1^* p_0 w_1 H(a) t_0 &= t_0^* (p_1 - p_0) H(a) t_0 = h_1(a)\,. \end{aligned}$$

Since h_1 dominates h_0 and h_0 dominates 0 modulo $\mathrm{C}_0(\mathbb{R}, B)$. It follows that h_1 dominates zero modulo $\mathrm{C}_0(\mathbb{R}, B)$, i.e.,

$$[\pi_{\mathrm{C}_0(\mathbb{R},B)} \circ h_1] + [0] = [\pi_{\mathrm{C}_0(\mathbb{R},B)} \circ h_1]$$

in $\mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))/\mathrm{C}_0(\mathbb{R}, B)$. It follows that there exists isometries $S, T \in \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$ with $Sh_1(a) - h_1(a)S \in \mathrm{C}_0(\mathbb{R}, B)$, $SS^* + TT^* = 1$ and $T^* h_1(a) T \in \mathrm{C}_0(\mathbb{R}, B)$.

???????? □

We extend an m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ naturally to $\mathcal{C}[Y] \subseteq \mathrm{CP}(A, B[Y])$ by letting $B[Y] := \mathrm{C}_0(Y, B)$ and $V \in \mathrm{CP}(A, B[Y])$ is an element of $\mathcal{C}[Y]$ if and only if there is a point-norm continuous map $y \in Y \mapsto V_y \in \mathcal{C}$ with $y \mapsto \|V_y(a)\|$ is in $\mathrm{C}_0(Y)$, such that $V(a)(y) = V_y(a)$ for $a \in A$.

Corollary 9.3.11. *Suppose that A and B are stable C^*-algebras, A is separable, B is σ-unital, the m.o.c. cone $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ is generated by $h_0 \in \mathcal{C}$, where $h_0(a) := k(a \otimes 1)$ for some a non-degenerate *-monomorphism $k \colon A \otimes \mathcal{O}_2 \to B$.*

Then $\mathrm{R}(\mathcal{C};\, A, B[Y]) \to \mathrm{KK}(\mathcal{C};\, A, B[Y])$ is injective, if and only if, $Y \mapsto \mathrm{R}(\mathcal{C};\, A, B[Y])$ is homotopy invariant on locally compact metric spaces Y, – in the sense that the canonical evaluation group morphisms

$$\mathrm{R}(\mathcal{C};\, A, B[Y \times [0, 1]]) \to \mathrm{R}(\mathcal{C};\, A, B[Y])$$

are isomorphisms for $t \in [0, 1]$.

Proof. `to be filled in` ?? □

4. Proofs of Parts (i) and (ii) of Theorems B and M

Now we carry out the program for the proof of Parts (i) and (ii) of Theorems B and M that we have described in the beginning of this Chapter 9. First we explain why Parts (i) and (ii) of Theorem B are a special cases of Parts (i) and (ii) of Theorem M (from the point of view of our program), and why they are both special cases of Corollary 9.2.7.

Remark 9.4.1. Let A a separable exact C^*-algebra. we use in the remaining part of this section the definition $D := A \otimes \mathbb{K}$. Note that, by Corollary 5.5.6, the *-monomorphism $a \in A \mapsto a \otimes p_{11}$ is is unitarily homotopic to an isomorphism from A onto D, if A is stable.

For notational simplicity we assume that in the sequel B denotes a stable σ-unital C^*-algebra. Then the assumptions of Theorem B can be reformulated as follows: *The C^*-algebra A is unital, separable and exact, and B is a stable C^*-algebra that contains a copy N of $\mathcal{O}_2 \otimes \mathbb{K}$ such that NB is dense in B.*

We modify the definition of the *-monomorphism $h_0 \colon D \to N \subseteq B \otimes \mathbb{K}$ (given in Chapter 1 after Theorem A):

By Theorem A, we find a unital *-monomorphism $h_1 \colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Then we define a unital *-monomorphism $k_1 \colon D \to \mathcal{O}_2 \otimes \mathbb{K}$ by $k_1 := k_1^u \otimes \mathrm{id}_{\mathbb{K}}$, where $k_1^u(a) := h_1(a \otimes 1)$.

Since B contains the non-degenerate copy N of $\mathcal{O}_2 \otimes \mathbb{K}$, the monomorphism k_1 defines a unital nuclear *-monomorphism k_0 from D into B by $k_0 := \vartheta k_1$, where ϑ is an isomorphism from $\mathcal{O}_2 \otimes \mathbb{K}$ onto N.

k_0 is nuclear and non-degenerate, and there is a non-degenerate *-monomorphism k from $D \otimes \mathcal{O}_2$ into $N \subseteq B$ such that $k_0 = k((\cdot) \otimes 1)$. Thus k_0 and $k_0 \oplus k_0$ are unitarily equivalent.

The *-monomorphism k can be defined by $\vartheta(h_1 \otimes id_{\mathbb{K}})\gamma$, where γ means here the natural isomorphism from $(A \otimes \mathbb{K}) \otimes \mathcal{O}_2$ onto $(A \otimes \mathcal{O}_2) \otimes \mathbb{K}$.

Later it follows, as a corollary of Theorem B, that any copy of $\mathcal{O}_2 \otimes \mathbb{K}$ in B, that generates a full hereditary subalgebra of B, could be used in place of our particular fixed copy N of $\mathcal{O}_2 \otimes \mathbb{K}$ in B. Also later the reader can deduce from Corollary 9.4.2 that the above defined k_0 could be replaced in the sequel by any *-monomorphism from $D = A \otimes \mathbb{K}$ into $\mathcal{O}_2 \otimes \mathbb{K}$, because of $\mathrm{R}(D, \mathcal{O}_2 \otimes \mathbb{K}) \cong \mathrm{KK}(A, \mathcal{O}_2) = 0$ and of the absorption result (iii) of Theorem B, proved in Chapter 7, cf. Corollary 7.4.16.

Suppose, now that $N \subseteq B$ is a (non-simple) strongly purely infinite C^*-subalgebra of B with NB dense in B, and that $k \colon D \otimes \mathcal{O}_2 \to N$ is a non-degenerate *nuclear* *-monomorphism. Let $k_0 := k((\cdot) \otimes 1) \in \mathrm{Hom}_{\mathrm{nuc}}(D, B)$. With the definitions and elementary observations of Chapter 1 in hand, the reader can see, that $X := \mathrm{Prim}(N)$ acts on B upper semi-continuously by Ψ_B where $\Psi_B(J)$ is the closed ideal of B generated by the closed ideal J of N. The T_0 space X acts lower semi-continuously on D by Ψ_D, and, moreover, the action Ψ_D is *monotone upper* *semi-continuous*, i.e., satisfies also condition (ii) of Definition 1.2.6. Here $\Psi_D(J)$ for $J \in \mathcal{I}(N)$ is defined by

$$\Psi_D(J) := k_0^{-1}(k_0(D) \cap J).$$

The later in Chapter 12 given proof of Theorem K shows that $k_0 \colon D \to B$ (and $k \colon D \otimes \mathcal{O}_2 \to B$) can be constructed up to unitary homotopy from the actions Ψ_D and Ψ_B of X on D respectively B. In fact, by Corollary 7.4.23 holds $\mathrm{CP}_{\mathrm{rn}}(\Psi) = \mathcal{C}(k_0)$ and

$$\mathrm{SR}(X;\, D, B) = \mathrm{SR}(\mathrm{CP}_{\mathrm{nuc}}(\Psi);\, D, B) = S(k_0;\, D, \mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))\,.$$

Let $X := \mathrm{Prim}(N)$, and use the natural lattice isomorphism between the open subsets of of X and the closed ideals of N, cf. Section 2. Then $\Psi_D(J) := (k_0)^{-1}(k_0(D) \cap J)$ and $\Psi_B(J) := \Psi_0^{N,B}(J) = \overline{\mathrm{span}(BJB)}$ are actions of X on D and B, respectively. Here Ψ_D is lower semi-continuous and Ψ_B is upper semi-continuous.

We define actions of $\mathcal{I}(N)$ on $E_{\mathbb{R}} = \mathrm{Q}(SB)$ and on $\mathrm{Q}(\mathbb{R}_+, B) \cong J$ by $\Psi_{E_{\mathbb{R}}}(I) := \pi_{SB}(\mathcal{M}(SB, S\Psi_B(I)))$, $\Psi_{\mathrm{Q}(\mathbb{R}_+,B)}(I) := \mathrm{Q}(\mathbb{R}_+, \Psi_B(I)) = \Psi_{E_{\mathbb{R}}}(I) \cap J$ for $I \in \mathcal{I}(N)$.

Def.of action OK? ??

From the exactness of D it follows that h is Ψ-residually nuclear, if and only if, h is nuclear and is Ψ-residually equivariant (cf. Chapter 3).

Thus H_0 is *nuclear* and has a lift $H_0' := \delta_\infty \circ k_0$ which maps D into the multiplier algebra

$$\mathcal{M}(N) \subseteq \mathcal{M}(B) \subseteq \mathrm{C}_{\mathrm{b,st}}(\mathbb{R}, \mathcal{M}(B)) = \mathcal{M}(\mathrm{C}_0(\mathbb{R}, B))$$

of the purely infinite C^*-algebra $N \subseteq B$, but $H_0'(D)$ intersects N exactly in zero.

Therefore, by Corollary 5.9.23,

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB) = G(H_0;\, D, E_{\mathbb{R}}).$$

If D is exact, we have moreover $G(H_0, D, E_{\mathbb{R}}) = [H_0] + [\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, E_{\mathbb{R}})]$, but we don't need this (for the proofs in section 9.2).

In Chapters 5 and 8 we have seen that there are natural isomorphisms

$$\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)\,.$$

Since $h_0 \colon D \to E_{\mathbb{R}}$ is dominated by H_0, every element in $S(h_0\,;\, D, E_{\mathbb{R}})$ is the unitary equivalence class $[h]$ with nuclear h and defines an element of $\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB)$ by the unitary equivalence class $[h \oplus H_0]$, cf. Corollary 5.9.22(??).

By Proposition 7.4.15, $S(h_0\,;\, D, E_{\mathbb{R}})$ is nothing else the semigroup of unitary equivalence classes Ψ-residually nuclear C^*-morphisms h from D into $\mathrm{Q}(\mathbb{R}_+, B)$, where $\Psi_A(J)$ and $\Psi_{\mathrm{Q}(\mathbb{R}_+,B)}$ are defined as above.

In Chapter 7 we have seen that

$$\mathrm{R}(X;\, D, B) = G(h_0;\, D, E_{\mathbb{R}}) = [h_0] + [\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, J)]\,.$$

(If h_0 resp. k exists.)

COROLLARY 9.4.2. *If D is exact and $k\colon D\otimes\mathcal{O}_2\to B$ is a non-degenerate nuclear *-morphism that induces the above considered action of $X := \mathrm{Prim}(B)$ on D, then ψ induces an isomorphism α of $\mathrm{R}(X;\, D, B)$ onto $\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB)\cong \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)$.*

PROOF. `to be filled in` ?? □

Following Lemma uses above conventions and explanations in Remark 9.4.1.

LEMMA 9.4.3. *If A is exact and h_0 is nuclear, let Ψ_A denote the action of $\mathrm{Prim}(B)$ induced by h_0.*

For every nuclear Ψ-residually equivariant C^*-morphism $k\colon D\to \mathrm{Q}(\mathbb{R}_+, B)$, i.e., for $[k]\in \mathrm{SR}(X;\, D, B)$, the morphism $h := k\oplus h_0$ defines an element $[h]$ in $\mathrm{R}(X;\, D, B)$ that is approximately scale-invariant.

PROOF. Use Corollary 9.2.7 with $\mathcal{C} := \mathcal{C}(h_0) = \mathcal{C}_{\mathrm{rn}}(X; D, B)$.

`Give new proof based on Corollary 9.2.7` ??

Let $E_{\mathbb{R}} := \mathrm{Q}^s(SB)$. By Proposition 7.4.15(??), **??** for separable stable and exact D, $k\in \mathrm{Hom}(X;\, D, \mathrm{Q}(\mathbb{R}_+, B))$ has unitary equivalence class $[k]$ in $S(h_0\,;\, D, E_{\mathbb{R}})$, if an only if k is nuclear.

$[h] = [k\oplus h_0] = [k] + [h_0]$ is in $\mathrm{R}(X;\, D, B)$, because $\mathrm{R}(X;\, D, B) = [h_0] + S(h_0\,;\, D, E_{\mathbb{R}})$ by Proposition 7.4.15(??) and Proposition 4.4.3.

If σ is a topological isomorphism of $\mathbb{R}_+$ then $\sigma(0) = 0$ and therefore σ extends naturally to an orientation preserving topological isomorphism of $\mathbb{R}$ by letting $\sigma(t) = t$ for $t\in\mathbb{R}_-$. We denote this extension again by σ. It induces in a natural way an automorphism $\widehat{\sigma}$ of $\mathrm{Q}^s(SB)$ such that its restriction to $J\cong \mathrm{Q}(\mathbb{R}_+, B)$ is just the above considered induced automorphism of $\mathrm{Q}(\mathbb{R}_+, B)$, which we also denote by $\widehat{\sigma}$, because all is trivially related by restriction maps.

Let $I_1\colon\ \mathrm{Q}(\mathbb{R}_+, B)\cong J\hookrightarrow \mathrm{Q}^s(SB)$ the natural inclusion. Then for the generalized mapping cone construction $k\mapsto I_1\circ k$ we have, obviously, $\widehat{\sigma}\circ(I_1\circ k) = I_1\circ(\widehat{\sigma}\circ k)$.

On the other hand the isomorphism α of Proposition 9.2.4 satisfies, by definition of α,

$$\alpha([\widehat{\sigma}\circ(k\oplus h_0)]) = [I_1\circ(\widehat{\sigma}\circ(k\oplus h_0))] = [\widehat{\sigma}\circ(I_1\circ(k\oplus h_0))].$$

By Corollary 8.3.4(ii), we have

$$[\widehat{\sigma}\circ(I_1\circ(k\oplus h_0))] = [I_1\circ(k\oplus h_0)] = \alpha([k\oplus h_0]).$$

Since $[k\oplus h_0]$ is in $\mathrm{R}(X;\, D, B)$ and α is faithful on $\mathrm{R}(X;\, D, B)$ by Proposition 9.2.4, this shows the invariance of $[k\oplus h_0]$ under scaling:

$$[k\oplus h_0] = [\widehat{\sigma}\circ(k\oplus h_0)]\,.$$

□

Proof of (i) and (ii) of Theorems B and M.. Give new proof based on Corollary 9.2.7!!! ??

In case of Theorem B we pass to the stabilizations $D := A \otimes \mathbb{K}$ and replace B by $B \otimes \mathbb{K}$.

In the case of Theorem M we assume the existence of h_0 (i.e., we prove (i) and (ii) modulo the result of Theorem K, see our above made conventions).

By Corollary 9.4.2, the natural map $h \mapsto I_1 \circ h$ from $\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, \mathrm{Q}(\mathbb{R}_+, B))$ into $\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, \mathrm{Q}^s(SB))$ defines an isomorphism

$$\alpha\colon\ \mathrm{R}(X;\, D, B) \xrightarrow{\sim} \mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB) \cong \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B).$$

By Corollary 8.3.3(iii), for a nuclear C^*-morphism h from D to B the mapping cone construction defines an element $\alpha[h] = [C_h]$ of $\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB)$ which is mapped to the element $[h - 0] \in \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)$ under the natural isomorphism from $\mathrm{Ext}_{\mathrm{nuc}}(X;\, D, SB)$ onto $\mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)$.

Thus the relations considered in Theorems B(ii) and M(ii) are just the relations which are induced by the semigroup homomorphism from the unitary equivalence classes of elements of $\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, B)$ into $\mathrm{R}(X;\, D, B)$. I.e., if h and k are in $\mathrm{Hom}_{\mathrm{nuc}}(X;\, D, B)$ then $[h - 0] = [k - 0] \in \mathrm{KK}_{\mathrm{nuc}}(X;\, D, B)$ if and only if they define the same element of $\mathrm{R}(X;\, D, B)$. But this means that there is a strongly continuous map $t \mapsto U(t)$ from $\mathbb{R}_+$ into the unitary group of $\mathcal{M}(B)$ such that $h \oplus h_0 = \lim_{t\to\infty} U(t)^*(k \oplus h_0)U(t)$. This proves the second part of Theorems B and M.

To complete the proof of the first part of Theorem B, we have to show that every element of $\mathrm{R}(X;\, D, B)$ can be represented by a Ψ-residually nuclear C^*-morphism $h\colon D \to B$. But this follows from Lemma 9.4.3 and Corollary 9.1.3, because $\mathrm{R}(X;\, D, B) = \mathrm{SR}(X;\, D, B) + [h_0] \subseteq \mathrm{SR}(X;\, D, B)$ and $[h_0]$ is the neutral element of $\mathrm{R}(X;\, D, B)$, cf. Chapter 4. □

Remark 9.4.4. Let us shortly mention alternative approaches as e.g. by N. C. Phillips [**627**] in the case where X is a point:
If one finds another way to show the homotopy invariance of $\mathrm{R}(A, B)$ with respect to the second variable B, then one can directly apply Corollary 9.1.3. And one gets that the representatives of $\mathrm{R}(A, B)$ can be chosen as nuclear C^*-morphisms from $A \otimes \mathbb{K}$ into $B \otimes \mathbb{K}$. Then it follows that there is a composition that replaces the Kasparov products. But then it still remains to show that the natural morphism from $\mathrm{R}(A, B)$ to $\mathrm{KK}_{\mathrm{nuc}}(A, B) \cong E_{\mathrm{nuc}}(A, B)$ is both faithful and surjective ([5]). The zero functor satisfies all requirements of category theory descriptions of Kasparov or E-theory type functors, *except of the universality.* A less trivial example, which is also a two-sided ideal of KK-theory (and even of E-theory) is the subgroup of the elements of $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ that factorize through a commutative C^*-algebra (it is a

[5] In principle, it could happen that h_0 is – up to asymptotic unitary homotopy – the only element of $\mathrm{R}(A, B)$.

group by Bott periodicity). Therefore, formal category theoretic (algebraic) arguments that prove the coincidence of $\mathrm{R}(A, B)$ and $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ *without assuming UCT* for A or B require some care.

The *-epimorphism $b \mapsto b(t)$, for $b \in \mathrm{C}([0,1], B)$, extends naturally to a *-epimorphism $\pi_t \colon \mathrm{C_b}(\mathbb{R}_+ \times [0,1], B) \to \mathrm{C_b}(\mathbb{R}_+, B)$ by

$$\pi_t((b_s)_{s\in\mathbb{R}_+}) := (b_s(t))_{s\in\mathbb{R}_+} .$$

The π_t define evaluation semi-group morphisms

$$(\pi_t)_* \colon \mathrm{SR}(\mathcal{C}[0,1];\, A, B[0,1]) \to \mathrm{SR}(\mathcal{C};\, A, B)$$

that naturally define group morphisms

$$(\pi_t)_* \colon \mathrm{R}(\mathcal{C}[0,1];\, A, B[0,1]) \to \mathrm{R}(\mathcal{C};\, A, B) .$$

CHAPTER 10

Unitarily homotopic algebras and Theorem B(iv)

We give in this chapter the proof of part (iv) of Theorem B and obtain related results. This finishes the proof of Theorem B, because parts (iii), (i)+(ii) have been shown in Chapters 7 and 9 respectively.

We consider here also *non-simple* σ-unital C^*-algebras.

1. Unitary homotopies and Proof of Theorem B(iv)

We recall that two morphisms $\varphi, \psi \colon A \to B$ are **unitarily homotopic** if there is a norm-continuous map $t \in \mathbb{R}_+ \mapsto U(t) \in \mathcal{M}(B)$ into the unitary operators in the multiplier algebra $\mathcal{M}(B)$ of B such that $\varphi(a) = \lim_{t\to\infty} U(t)^*\psi(a)U(t)$ for all $a \in A$, cf. Definition 5.0.1 ([1]).

We say that *-homomorphism $\varphi \colon A \to B$ *defines* a **unitary homotopy** between A and B, if there exists a *-homomorphism $\psi \colon B \to A$ such that $\psi \circ \varphi$ is unitarily homotopic to id_A and $\varphi \circ \psi$ is unitarily homotopic to id_B.

A simple calculation shows that unitarily homotopic morphisms have the same kernel. Therefore, *a unitary homotopy $\varphi \colon A \to B$ always is a monomorphism.*

If $\varphi \colon A \to B$ is a unitary homotopy and $\psi \colon B \to A$ is a *-homomorphism as in the above definition, then ψ defines again a unitary homotopy between A and B.

In the following we denote by $\overline{BXB}$ the closed ideal of B which is generated by a subset X of the multiplier algebra $\mathcal{M}(B)$ of B, i.e., $\overline{BXB}$ means the *closure of the linear span* of BXB.

A unitary homotopy $\varphi \colon A \to B$ always satisfies that $\varphi(A)$ is not contained in a non-trivial closed ideal of B, i.e. $\overline{B\varphi(A)B} = B$:

PROOF. $J := \overline{B\varphi(A)B}$ is a closed ideal of B and $\varphi\psi(B) \subset \varphi(A) \subset J$. Since $\varphi\psi$ is unitarily homotopic to id_B, this implies that every element of B is approximately unitarily equivalent to an element of the ideal J. Thus $B = J$. □

[1]It suffices to suppose that $t \mapsto U(t) \in$ is strongly continuous. Then we say that φ and ψ are "weakly unitarily homotopic". Since the unitary operators are a topological group with strict topology (and strong and strict topology coincides on the unitaries), the "weakly unitary homotopy" of maps φ and ψ is an equivalence relation. If A and B are σ-unital and B is stable then weakly unitarily homotopic φ and ψ are unitarily homotopic (by an other norm-contiuous path $t \mapsto \widetilde{U}(t)$), cf. Corollary 7.1.5.

A similar argument shows that B *and* φ *must be unital (resp.* B *is* σ*-unital) if* A *is unital (resp. is* σ*-unital) and* $\varphi\colon A \to B$ *is a unitary homotopy.*

In order to prove Theorem B(iv), we use the following observations (I)-(III):

(I) *Suppose that* A *and* B *are* σ*-unital and stable, and* $h\colon A \to B$ *is a monomorphism, such that* $\overline{Bh(A)B} = B$.
Then h *is unitarily homotopic to a monomorphism* $k\colon A \to B$ *with* $k(A)B = B$ ([2]).

(II) *If* $h_1\colon A \to B$ *and* $h_2\colon B \to C$ *are unitarily homotopic to* $k_1\colon A \to B$ *respectively* $k_2\colon B \to C$*, and if* $k_2(B)C$ *is dense in* C*, then* $h_2 \circ h_1$ *is unitarily homotopic to* $k_2 \circ k_1$.

(III) *Suppose that* A *and* B *are separable, that* $\varphi\colon A \to B$ *defines an unitary homotopy of* A *and* B*, and that* $\varphi(A)$ *contains a strictly positive element of* B*. Then* φ *is unitarily homotopic to an isomorphism* χ *from* A *onto* B.

Observation (I) is a special case of Corollary 7.4.6. The observations (I), (II) and (III) together imply that unitarily homotopic separable C^*-algebras are isomorphic if they are σ-unital and stable or if they are unital, i.e., (I), (II) and (III) imply Theorem B(iv).

Proof of the implication { (I),(II),(III) } $\Rightarrow$ (iv) of Theorem B:. Let $\varphi\colon A \to B$ a unitary homotopy and $\psi\colon B \to A$ a *-homomorphism such that $\varphi \circ \psi$ is unitarily homotopic to id_B and $\psi \circ \varphi$ is unitarily homotopic to id_A. Above we have shown that φ and ψ must be monomorphisms such that $B\varphi(A)B$ is dense in B and $A\psi(B)A$ is dense in A. By (I) there exists monomorphisms $\varphi_1\colon A \to B$ and $\psi_1\colon B \to A$ which are unitarily homotopic to φ and ψ respectively, such that $\varphi_1(A)B$ and $\psi_1(B)A$ are dense in B and A respectively.

By (II), the φ_1 and ψ_1 again define a unitary homotopy between A and B.

Now (III) applies, and we get an isomorphism χ from A onto B such that χ is unitarily homotopic to φ_1. But then χ is unitarily homotopic to φ by (II).

We have that B, φ and ψ are unital if A unital. Thus (III) applies to φ in the unital case. □

Proof of (II): . The definition of unitary homotopy shows immediately that k_2h_1 is unitarily homotopic to h_2h_1. Since $k_2(B)C$ is dense in C there is a unital strictly continuous *-homomorphism ψ from the multiplier algebra $\mathcal{M}(B)$ of B into the multiplier algebra $\mathcal{M}(C)$ of C such that $\psi|B = k_2$. If $t \in \mathbb{R}_+ \mapsto u(t) \in \mathcal{M}(B)$ is a strictly continuous map into the unitaries of $\mathcal{M}(B)$ such that $\lim \|u(t)^*h_1(a)u(t) - k_1(a)\| = 0$ for $a \in A$, then $t \in \mathbb{R}_+ \to v(t) := \psi(u(t)) \in \mathcal{M}(C)$ is strictly continuous and $\lim \|v(t)^*k_2(h_1(a))v(t) - k_2(k_1(a))\| = 0$ for $a \in A$. Thus k_2h_1 and k_2k_1 are

[2] The norm-continuous path $t \in \mathbb{R}_+ \mapsto U(t) \in \mathcal{M}(B)$ with $\lim_{t\to\infty} \|k(a) - U(t)^*h(a)U(t)\| = 0$ (for all $a \in A$) can be taken in the unitization $\widetilde{B} = B + \mathbb{C}\cdot 1 \subset \mathcal{M}(B)$.

unitarily homotopic. Since unitary homotopy is an equivalence relation, k_2k_1 is unitarily homotopic to h_2h_1. □

2. Proof of property (III)

This Section is concerned with the proof of (III), that finishes the proof of Theorem B. The proof of (III) is given above Corollary 10.3.1. Then we get some other Corollaries of our technical Proposition 10.2.1.

We start with a very general Proposition 10.2.1 that is not exactly stated in [**438**] as here, but is proved in [**438**], cf. proofs of [**438**, cor. 2.4(II), prop. 2.3].

Let $X_n \subset Y$ be a sequence of Banach spaces and closed subspaces, and let $V_n \colon X_n \to X_{n+1}$ be a sequence of contractions. Then the inductive limit $\operatorname{indlim}(V_n \colon X_n \to X_{n+1})$ is well-defined and naturally contained in $\ell_\infty(Y)/c_0(Y)$ (or in a norm ultrapower Y_ω) as a closed subspace. By

$$V_n^\infty \colon X_n \to \operatorname{indlim}(V_n \colon X_n \to X_{n+1})$$

we denote the natural contractions which are given in the definition of inductive limits by universal diagrams (and satisfy $V_{n+1}^\infty \circ V_n = V_n^\infty$). See [**438**, sec. 2] for definitions and explicit formulas (in the case of operator systems).

In the following Proposition 10.2.1 let X be a separable Banach space and Y a closed subspace of X, $\mathcal{S} \subset \mathcal{L}(X)$ a semigroup of linear contractions on X, G be a topological *semigroup* $\pi \colon G \to \mathcal{S}$ a strongly continuous semigroup epimorphism.

To simplify notation we write V, V_n, $V(t)$ and $V_n(t)$ instead of $\pi(g)$, $\pi(g_n)$, $\pi(g(t))$, $\pi(g_n(t))$ etc.

Proposition 10.2.1. *Let X, Y, $\mathcal{S}$, G and π be as above.*

(i) *Assume that for every finite dimensional subspace Z_1 of X and for every finite dimensional subspace Z_2 of Y and every $\varepsilon > 0$ there exist $V \in \mathcal{S}$ with*

$$\|z - Vz\| \le \varepsilon \|z\| \quad \forall z \in Z_2 \quad \text{and} \quad \operatorname{dist}(Vz, Y) \le \varepsilon \|z\| \quad \forall z \in Z_1.$$

Then there exist a sequence $V_1, V_2, \ldots$ in $\mathcal{S}$ and an isometric isomorphism

$$\varphi \colon Y \xrightarrow{\sim} \operatorname{indlim}(V_n \colon X \to X)$$

from Y onto $\operatorname{indlim}(V_n \colon X \to X)$*, such that, for $y \in Y$,*

$$\varphi(y) = \lim_{n\to\infty} V_n^\infty(y)\,.$$

If, moreover, G is a group and $\pi(1_G) = \mathrm{id}_X$, then $(V_1^\infty)^{-1}$ is an isometric isomorphism from $\operatorname{indlim}(V_n \colon X \to X)$ *onto X and there is a sequence (of isometries) $T_n \in \mathcal{S}$ such that, for $y \in Y$,*

$$(V_1^\infty)^{-1}(\varphi(y)) = \lim_{n\to\infty} T_n(y).$$

$(V_1^\infty)^{-1} \circ \varphi$ is an isomorphism from Y onto X.

(ii) *Suppose that, moreover,* $\mathrm{id}_X \in \mathcal{S}$, *G is unital,* $\pi(1_G) = \mathrm{id}_X$, *and for every pair of subspaces* $Z_1 \subset X$, $Z_2 \subset Y$ *of finite dimension and for every* $\varepsilon > 0$, *there exists a continuous map* $s \in [0,1] \mapsto g(s) \in G$ *with* $g(0) = 1_G$, *such that, with* $V(s) := \pi(g(s))$, *for all* $s \in [0,1]$,

$$\|z - V(s)z\| \leq \varepsilon \|z\| \quad \forall z \in Z_2 \quad \textit{and} \quad \mathrm{dist}(V(s)z, Y) \leq \varepsilon \|z\| \quad \forall z \in Z_1 .$$

Then, for every dense sequence $y_1, y_2, \ldots \in Y$ *the sequence* $V_1, V_2, \ldots \in \mathcal{S}$ *in (i) and the sequence* $g_1, g_2, \cdots \in G$ *for which* $\pi(g_n) = V_n$ *in (i) can be selected such that moreover for every* $n \in \mathbb{N}$ *there is a continuous map* $t \in [0,1] \mapsto g_n(t) \in G$ *with* $\|V_n(t)y_k - y_k\| < 2^{-n}$ *for every* $k \leq n$, $g_n(0) = 1_G$, $V_n(1) = V_n$, $V_0(t) \equiv 1$.

Furthermore, $W(t) := V_{n+1}^{\infty} \circ V_n(n+1-t)$ *(for* $n \leq t \leq n+1$*) has the property that*

$$t \in \mathbb{R}_+ \mapsto W(t)y \in \mathrm{indlim}(V_n \colon X \to X)$$

is continuous with $W(0) = V_1^{\infty}$, *and*

$$\lim_{t \to \infty} W(t)y = \varphi(y) .$$

If, moreover, G is a group and $\pi(1_G) = \mathrm{id}_X$, *then* $(V_1^{\infty})^{-1}$ *is an isometric isomorphism from* $\mathrm{indlim}(V_n \colon X \to X)$ *onto X.*

The map $t \in \mathbb{R}_+ \mapsto g(t) \in G$, *which is defined by*

$$g(t) := g_1^{-1} \cdot \ldots \cdot g_n^{-1} g_{n+1}(t-n)^{-1} \quad \textit{for} \;\; n \leq t \leq n+1 ,$$

is continuous and satisfies $\pi(g(t)) = (V_1^{\infty})^{-1} W(t)$.

The inclusion map $Y \hookrightarrow X$ *is homotopic to the isometric isomorphism* $(V_1^{\infty})^{-1} \circ \varphi$ *from Y onto X by the strongly continuous map* $t \in \mathbb{R}_+ \mapsto V(t) := \pi(g(t))$.

Proof. (i): The proof [**438**, cor. 2.4(II)] can be used here (word by word in the same way), but the inductive existence proof of the $V_n \in \mathcal{S}$ with the desired properties does not need in the induction step a Hahn-Banach separation argument, as in [**438**], the existence is just our assumption in (i) that $\mathcal{S}$ is a semigroup and has (!) the the needed approximation property.

If G is a group and $\pi(1_G) = \mathrm{id}_X$, then the V_n are isometries from X onto X and thus V_1^{∞} is an isometric isomorphism from X onto $\mathrm{indlim}(V_n \colon X \to X)$. The desired T_n can be found as iterates (products) of the V_n.

(ii): It is in principle the same proof as of [**438**, cor. 2.4(II)], but one has to construct the finite-dimensional filtrations $X_n \subset X$, $Y_n \subset Y$ and the $V_n = V_n(1)$ in [**438**, prop. 2.3] such that the ball of $V(t)V_m^n(X_n)$ is 8^{-n}-contained in X_{n+1} for $t \in [0,1]$ and $V_n(t) := \pi(g_n(t))$ with $g_n(0) = 1$, $V(1) = V_n$. That can be done by our assumptions in (ii).

Now assume that moreover G is a group and $\pi(1) = \mathrm{id}_X$. Than V_1^{∞} is an isometric isomorphism as one can easily see from the natural model for indlim in

$l_\infty(X)/c_0(X)$, cf. e.g. beginning of [**438**, sec. 2]. The rest can be verified by straight forward calculations. □

Note that on the unitary operators in $\mathcal{M}(B)$ the strong and the strict topology coincide. We get the applications of Proposition 10.2.1 if we considered $\mathcal{M}(B)$ as a closed subalgebra $\{T \in \mathcal{L}(B) \colon \exists (S \in \mathcal{L}(B)), (Sb)^*a = b^*Ta, \forall a, b \in B\}$ of the bounded operators $\mathcal{L}(B)$ on B. The strict topology is the *-strong topology on $\mathcal{L}(B)$ given by the seminorms $T \mapsto \|T(b)\| + \|T(b^*)\|$, $b \in B$.

The natural group epimorphism $u \in \mathcal{M}(B) \mapsto \iota_u \in \mathrm{Int}(B) \subset \mathcal{L}(B)$, with $\iota_u(a) := uau^*$ for $a \in B$, is continuous with respect to the strong topologies (on both sides).

Corollary 10.2.2. *Let $\varphi \colon A \to B$ be a *-homomorphism and suppose that A and B are separable. Then φ is unitarily homotopic to an isomorphism ψ from A onto B, if and only if, φ satisfies the following property (δ):*

φ is a monomorphism and, for every $\varepsilon > 0$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in B$ there exist a strongly continuous map $t \in [0,1] \mapsto v(t) \in \mathcal{M}(B)$ from $[0,1]$ into the unitary operators in $\mathcal{M}(B)$ such that

(i) *$v(0) = 1$,*
(ii) *$\|[v(t), \varphi(a_i)]\| < \varepsilon$ for $i = 1, \ldots, n$, $t \in [0,1]$,*
(iii) *$dist(v(1)^* b_j v(1), \varphi(A)) < \varepsilon$ for $j = 1, \ldots, m$.*

Proof. Assume $\varphi \colon A \to B$ satisfies the property (δ), then with $X := B$, $Y := \varphi(A)$, $G = U(\mathcal{M}(B)) :=$ unitary group of $\mathcal{M}(B)$ with strict topology, $\pi(u) := \iota_u$ and $\mathcal{S} := \{\iota_u \colon u \in U(\mathcal{M}(B))\}$ the assumptions of Proposition 10.2.1(ii) are satisfied. Thus the inclusion map $\varphi(A) \hookrightarrow B$ is unitarily homotopic to an isomorphism from $\varphi(A)$ onto B.

Since φ is a monomorphism, it follows that φ is unitarily homotopic to an isomorphism from A onto B.

Conversely, suppose that φ is unitarily homotopic to an isomorphism ψ from A onto B. Along a unitary homotopy the norm of an element remains constant. Thus φ is a monomorphism.

Let $t \in \mathbb{R}_+ \mapsto u(t) \in \mathcal{M}(B)$ be the strongly continuous map into the unitaries corresponding to the unitary homotopy. Then there exists a t_0 such that, for every $t \geq t_0$,

$$\|\varphi(a_i) - u(t)\psi(a_i)u(t)^*\| < \varepsilon/2.$$

Let $f_j := \psi^{-1}(u(t_0)^* b_j u(t_0)) \in A$.

We find $t_1 > t_0$ with $\|\varphi(f_j) - u(t)\psi(f_j)u(t)^*\| < \varepsilon$ for every $t \geq t_1$.

Let $s(t) := t_0 + t(t_1 - t_0)$ and $v(t) := u(t_0)u(s(t))^*$ for $t \in [0,1]$.

Then $v(0) = 1$, $v(t)$ is unitary, $t \mapsto v(t)$ is strongly continuous and, by the triangle inequality, for $t \in [0,1]$, $\|v(t)^*\varphi(a_i)v(t) - \varphi(a_i)\| < \varepsilon$.

Furthermore, $\mathrm{dist}(v(1)^*b_jv(1),\varphi(A)) \leq \|v(1)^*b_jv(1) - \varphi(f_j)\| \leq \varepsilon$, because $v(1)^*b_jv(1) = u(t_1)\psi(f_j)u(t_1)^*$. □

Corollary 10.2.3. *Suppose that $\varphi_n\colon A_n \to A_{n+1}$, $n = 1, 2, \ldots$, are *-homomorphisms, where A_n is separable, $\varphi_n(A_n)$ contains a strictly positive element of A_{n+1}, and there exists an isomorphism ψ_n from A_n onto A_{n+1} such that ψ_n is unitarily homotopic to φ_n.*

Then $\varphi_1^\infty\colon A_1 \to \mathrm{indlim}(\varphi_n\colon A_n \to A_{n+1})$ is unitarily homotopic to an isomorphism from A_1 onto $\mathrm{indlim}(\varphi_n\colon A_n \to A_{n+1})$.

Proof. We have to check the criteria (δ) of Corollary 10.2.2. By Corollary 10.2.2 every φ_n is a monomorphism.

$\bigcup \varphi_n^\infty(A_n)$ is dense in $B := \mathrm{indlim}(\varphi_n\colon A_n \to A_{n+1})$.

The assumptions imply that $\varphi_n^\infty(A_n)$ contains, for every n, a strictly positive element of B, because $\varphi_n(A_n)A_{n+1}$ is dense in A_{n+1}, and then by induction:

$$\overline{\varphi_{n+k} \circ \ldots \circ \varphi_n(A_n)A_{n+k+1}} = A_{n+k+1}, \qquad \overline{\varphi_n^\infty(A_n)B} = B\,.$$

Thus $\mathcal{M}(\varphi_n^\infty(A_n)) \cong \mathcal{M}(A_n)$ is unitally contained in $\mathcal{M}(B)$ and the strict topology on $\mathcal{M}(\varphi_n^\infty(A_n))$ is the same as the topology which is induced by the strict topology of $\mathcal{M}(B)$. Therefore it suffices to check that $\varphi_1^\infty\colon A_1 \to \varphi_n^\infty(A_n)$ satisfies the criteria (δ) of Corollary 10.2.2. But this follows from Corollary 10.2.2 and the following observation:

By induction and (II), $(\varphi_n^\infty)^{-1} \circ \varphi_1^\infty = \varphi_{n-1} \circ \ldots \circ \varphi_1$ is unitarily homotopic to the isomorphism $\psi_{n-1} \circ \cdots \circ \psi_1$ from A_1 onto A_n, where $\psi_k\colon A_k \to A_{k+1}$ are isomorphisms which are unitarily homotopic to $\varphi_k\colon A_k \to A_{k+1}$ ($k = 1, \ldots, n-1$). □

Now we are ready for the proof of the above statement (III), and *finish here the proof of Theorem B*:

check next proof again!!

Proof of (III). Suppose that A and B are separable.

Let $h\colon A \to B$ and $k\colon B \to A$ monomorphisms, such that $\overline{h(A)B} = B$ and $\overline{k(B)A} = A$, i.e., $h(A)$ contains a strictly positive element of B, and $h(B)$ contains a strictly positive element of A. It follows

$$\overline{hk(B)B} = \overline{hk(B)h(A)B} = \overline{h(k(B)A)B} = \overline{h(A)B} = B.$$

In the same way $\overline{kh(A)A} = A$. If, moreover, kh is unitarily homotopic to id_A, then, by Corollary 10.2.3, there exists an isomorphism α from A onto the inductive limit A_∞ of the sequence of morphisms $\chi_n\colon A \to A$ with $\chi_n := kh$, such that α is unitarily homotopic to $\chi_1^\infty\colon A \to A_\infty$.

If hk is unitarily homotopic to id_B, then there exists an isomorphism β from B onto the inductive limit B_∞ of $\psi_n \colon B \to B$ with $\psi_n := hk$, such that β is unitarily homotopic to $\psi_1^\infty \colon B \to B_\infty$.

If we consider the natural inclusions

$$A_\infty \subset l_\infty(A)/c_0(A), \qquad B_\infty \subset l_\infty(B)/c_0(B),$$

we see, that

$$(a_1, a_2, \dots) \mapsto (h(a_1), h(a_2), \dots) \qquad \text{and} \qquad (b_1, b_2, \dots) \mapsto (k(b_1), k(b_2), \dots)$$

define (modulo zero-sequences) monomorphisms μ from A_∞ into B_∞ and ν from B_∞ into A_∞ such that with $\nu\mu = id$ and $\mu\nu = id$. Therefore, μ is an isomorphism.

Thus $\varphi := \beta^{-1} \circ \mu \circ \alpha : A \to B$ is an isomorphism from A onto B. A simple calculation shows that $\mu \circ \chi_1^\infty = \psi_1^\infty \circ h$.

$\beta^{-1} \circ \mu \circ \alpha$ is unitarily homotopic to $\beta^{-1} \circ \mu \circ \chi_1^\infty = \beta^{-1} \circ \psi_1^\infty \circ h$.

Since ψ_1^∞ is unitarily homotopic to β, we get that $\psi_1^\infty \circ h$ is unitarily homotopic to $\beta \circ h$.

Thus $\varphi = \beta^{-1} \circ \mu \circ \alpha$ is unitarily homotopic to h. □

3. Some corollaries

We add some related corollaries of the first part of Proposition 10.2.1, which can be proved by almost the same arguments as above.

COROLLARY 10.3.1. *Let $\varphi \colon A \to B$ be a *-homomorphism, where A and B are separable. Then the following are equivalent:*

(i) *φ is approximately unitarily equivalent to an isomorphism ψ from A onto B.*
(ii) *φ is a monomorphism and for every $\varepsilon > 0$, $a_1, \dots, a_n \in A$, $b_1, \dots, b_m \in B$ there exists a unitary $v \in \mathcal{M}(B)$ with $\|v^*\varphi(a_i)v - \varphi(a_i)\| < \varepsilon$ for $i = 1, \dots, n$ and $\operatorname{dist}(v^*b_jv, \varphi(A)) < \varepsilon$ for $j = 1, \dots, m$.*
(iii) *φ is a monomorphism, and in the relative commutant $\varphi(A)' \cap \mathcal{M}(B)_\omega$ of $\varphi(A)$ in the ultrapower of $\mathcal{M}(B)$ there exists, for every finite sequence $b_1, \dots, b_m \in B$ and every $\varepsilon > 0$, a unitary $v \in \varphi(A)' \cap \mathcal{M}(B)_\omega$ such that v^*b_jv has distance $< \varepsilon$ from $\varphi_\omega(A_\omega)$ for $j = 1, \dots, m$.*

REMARK 10.3.2. Part (iii) of Corollary 10.3.1 is just an equivalent formulation of (ii) in the more rigorous language of ultrapowers. There is a more elegant and far reaching version in the ultrapower context.

One concludes Corollary 10.3.1 from Proposition 10.2.1(i) essentially in the same way as we have deduced Corollary 10.2.2 from Proposition 10.2.1(ii). And one can use the same arguments for the implications "Corollary 10.3.1 ⇒ Corollary 10.3.3 ⇒ Corollary 10.3.4" below as for the implications "Corollary 10.2.2 ⇒ Corollary 10.2.3 ⇒ (III)" above.

Corollaries 10.3.4 and 10.3.5 overlap with (partly not published) results of Elliott.

Corollary 10.3.3. *Suppose that, for $n = 1, 2, \ldots$, A_n is separable, $\varphi_n \colon A_n \to A_{n+1}$ are *-homomorphisms, $\varphi_n(A_n)$ contains a strictly positive element of A_{n+1} and φ_n is approximately unitarily equivalent to an isomorphism from A_n onto A_{n+1}.*

Then $\varphi_1^\infty \colon A_1 \to \operatorname{indlim}(\varphi_n \colon A_n \to A_{n+1})$ is approximately unitarily equivalent to an isomorphism from A_1 onto $\operatorname{indlim}(\varphi_n \colon A_n \to A_{n+1})$.

Corollary 10.3.4. *If A and B are separable C^*-algebras, $h \colon A \to B$ and $k \colon B \to A$ are C^*-algebra morphisms such that $\overline{h(A)B} = B$, $\overline{k(B)A} = A$, kh and hk are approximately unitarily equivalent to id_A and id_B respectively, then h is approximately unitarily equivalent to an isomorphism from A onto B.*

Corollary 10.3.5. *Suppose that A is a separable C^*-algebra, which is stable or unital and that B is a separable unital C^*-subalgebra of the multiplier algebra $\mathcal{M}(A_\omega)$ of A_ω, such that B commutes element-wise with A, and such that $b \in B \mapsto b \otimes 1 \in B \otimes B$ and $b \in B \mapsto 1 \otimes b \in B \otimes B$ are approximately unitarily equivalent (to each other in $B \otimes B$), then*

(i) *B is simple and nuclear.*
(ii) *$A \cong A \otimes B \otimes B \otimes \ldots$ with an isomorphism $\psi \colon A \to A \otimes B \otimes B \otimes \ldots$ which is approximately unitarily equivalent to $a \in A \mapsto a \otimes 1 \otimes 1 \otimes \ldots$*

Proof. (i): Let J be a closed ideal of B and $J \neq B$, ν a pure state of B with $\nu(J) = 0$. Since B is separable, we find a sequence of unitaries $u_n \in B \otimes B$ and 2^{-n}-approximations a_n of u_n in the algebraic tensor product $B \odot B$, such that $b \otimes 1 = \lim_{n\to\infty} u_n^*(1 \otimes b)u_n$. Thus $b = \lim(\mathrm{id} \otimes \nu)(a_n^*(1 \otimes b)a_n)$ for every $b \in B$. The right hand side is a sequence of completely positive finite rank maps with kernels which contain J. Thus $J = 0$ and B is nuclear. Every closed ideal of B is trivial, i.e. B is simple.

(ii): Since B is nuclear and simple, there is a copy of $A \otimes (B \otimes B) \cong (A \otimes B) \otimes B$ in the ultrapower $(A \otimes B)_\omega$ of $A \otimes B$, such that $A \otimes 1 \otimes B$ corresponds naturally to the canonical embedding of $A \otimes B$ in $(A \otimes B)_\omega$ by constant sequences, and $A \otimes B \otimes 1$ is in $C^*(A \cdot (A' \cap \mathcal{M}(A_\omega))) \subset A_\omega \subset (A \otimes B)_\omega$, where A_ω is naturally identified with $(A \otimes 1_B)_\omega$.

Let $v \in \mathcal{M}(A \otimes B)_\omega$ denote the unitary, which is represented by $1_{\mathcal{M}(A)} \otimes u_n$, where u_n is as in part (i). We see that the criteria (iii) of Corollary 10.3.1 is satisfied for $\varphi \colon a \in A \mapsto a \otimes 1_B \in A \otimes B$. Thus φ is approximately unitarily equivalent to to an isomorphism ψ_1 from A onto $A \otimes B$.

It follows by induction that $\varphi_n \colon A_n \to A_{n+1}$ is approximately unitarily equivalent to an isomorphism from A_n onto A_{n+1}, where inductively $A_1 = A$, $A_{n+1} = A_n \otimes B$ and $\varphi_n(a) = a \otimes 1_B$ for $a \in A_n$. But then Corollary 10.3.3 gives that the monomorphism $a \mapsto a \otimes 1_B \otimes 1_B \otimes \ldots$ from A into $A \otimes B \otimes B \otimes \ldots =$

indlim($\varphi_n\colon A_n \to A_{n+1}$) is approximately unitarily equivalent to an isomorphism from A onto $A \otimes B \otimes B \otimes \ldots$. □

Remark 10.3.6. Since we have finished the proof of Theorem B with the above given proof of (III), now we can use the results A-F and H as stated in Chapter 1.

One easily sees from Theorem B(ii)+(iii) and Corollaries C(ii) and F that $B = \mathcal{O}_2$ and $B = \mathcal{O}_\infty$ satisfy the requirements of Corollary 10.3.5. Therefore, also Corollary G now follows from Corollary F and Corollary 10.3.5, if we do not use the original paper [**678**].

We recall from Definition 3.10.1 that a *-homomorphism $h_1\colon A \to B$ *approximately dominates* a completely positive contraction $h_2\colon A \to B$, if and only if, there exists a sequence of contractions $d_n \in B$ such that $\lim \|h_2(a) - d_n^* h_1(a) d_n\| = 0$ for $a \in A$. Clearly, we can choose the d_n's as isometries if A, B, h_1 and h_2 are unital.

If A and B are stable and σ-unital, then, by Proposition 7.4.7, we find isometries $t_n \in \mathcal{M}(B)$ such that $\lim \|h_2(a) - t_n^* h_1(a) t_n\| = 0$ for $a \in A$.

Corollary 10.3.7. *Suppose that A is a separable C*-algebra which is stable or is unital and contains a copy of $\mathcal{O}_2$ unitally.*

If id_A approximately dominates $\mathrm{id}_A \oplus \mathrm{id}_A$, then $A \cong \mathcal{O}_\infty \otimes A$.

If id_A is approximately unitarily equivalent to $\mathrm{id}_A \oplus \mathrm{id}_A$ by a sequence of unitaries in the multiplier algebra of A, then $A \cong \mathcal{O}_\infty \otimes A \cong \mathcal{O}_2 \otimes A$.

Proof. By the above Remark 10.3.6, the assumptions mean that there is an isometry $T \in \mathcal{M}(A)_\omega$ such that $T^*aT = sas^* + tat^*$ for $a \in A$ and suitable canonical generators of $\mathcal{O}_2 \subset \mathcal{M}(A)$. It follows that $t_1 := Tt$ and $s_1 := Ts$ are isometries in $\mathcal{M}(A)_\omega$ such that $t_1^* s_1 = 0$ and the elements t_1 and s_1 commute elementwise with $A \subseteq A_\omega$. $C^*(s_1, t_1)$ contains a copy of $B := \mathcal{O}_\infty$ unitally.

The natural unital *-homomorphism from $\mathcal{M}(A)_\omega$ into $\mathcal{M}(A_\omega)$ fixes $A \subset A_\omega$ and maps, therefore, B unitally and, by simplicity of B, faithfully into $A' \cap \mathcal{M}(A_\omega)$. By Corollary 10.3.5 and Remark 10.3.6, $A \cong \mathcal{O}_\infty \otimes A$.

If id_A is approximately unitarily equivalent to $\mathrm{id}_A \oplus \mathrm{id}_A$, then the above considered T is unitary and $t_1 t_1^* + s_1 s_1^* = 1$, i.e., $C^*(s_1, t_1)$ is naturally isomorphic to $\mathcal{O}_2$. By Corollary 10.3.5 and Remark 10.3.6, $A \cong \mathcal{O}_2 \otimes A$. □

Corollary 10.3.8. *Suppose that A is a separable nuclear C*-algebra. Then A satisfies the WvN-property of Definition 1.2.3, if and only if, $A \otimes \mathbb{K} \cong A \otimes \mathbb{K} \otimes \mathcal{O}_\infty$.*

Proof. We may suppose that A is stable, because A satisfies the WvN-property if and only if $A \otimes \mathbb{K}$ satisfies the WvN-property, as one can see immediately from Definition 1.2.3.

Since $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$, we get from Proposition 3.10.15 that A is strongly p.i. and has the WvN-property if $A \cong A \otimes \mathcal{O}_\infty$.

If A is nuclear, then $\mathrm{id} \oplus \mathrm{id}$ is residually nuclear. Thus, if A satisfies the WvN-property, then $\mathrm{id} \oplus \mathrm{id}$ is approximately dominated by id. Hence, $A \cong A \otimes \mathcal{O}_\infty$ by Corollary 10.3.7. □

`Revise next. Give ref.`

REMARK 10.3.9. In the paper with M. Rørdam [**463**] we show that all strongly purely infinite C^*-algebras have the WvN-property.

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Mention the corrections that have to be done
in the paper with Rørdam [463]
(as in Info for Ando)!!!
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It is an open question if every purely infinite separable nuclear C^*-algebra has the WvN-property.

`Mention here:`

"regular Abelian C^*-subalgebra"

implies

"Abelian factorization",

implies existence of

"ideal-separating residually nuclear map"

(= "residually nuclear separation")

The progress is partly mentioned in Chp. 12.

COROLLARY 10.3.10. *Suppose that A and B are stable separable nuclear C^*-algebras, and that $h\colon A \to B$, $k\colon B \to A$ and $\psi\colon A \otimes \mathcal{O}_2 \to A \otimes \mathcal{O}_2$ are *-monomorphisms.*

(i) *If, for every primitive ideal $J \subset A$,*

$$\psi(J \otimes \mathcal{O}_2) = \psi(A \otimes \mathcal{O}_2) \cap (J \otimes \mathcal{O}_2),$$

then ψ is unitarily homotopic to the identity map of $A \otimes \mathcal{O}_2$.

(ii) *$h \otimes \mathrm{id}_{\mathcal{O}_2}$ is unitarily homotopic to an isomorphism from $A \otimes \mathcal{O}_2$ onto $B \otimes \mathcal{O}_2$ if $kh(J) = kh(A) \cap J$ and $hk(K) = hk(B) \cap K$ for every primitive ideal $J \subset A$ and every primitive ideal $K \subset B$.*

PROOF. (i): By assumption on ψ, ψ and id are both residually nuclear *-monomorphisms and induce on $A \otimes \mathcal{O}_2$ the same action of the primitive ideal space of A. Thus, by Corollary 7.4.3, ψ and id are unitarily homotopic.

(ii): Since A and B are stable, we find by (I) and (II) *-monomorphisms $h_1\colon A \to B$ and $k_1\colon B \to A$, such that h_1 is unitarily homotopic to h, k_1 is unitarily homotopic to k, $h_1(A)B$ is dense in B, $k_1(B)A$ is dense in A, and hk is unitarily homotopic to h_1k_1, kh is unitarily homotopic to k_1h_1. Thus h_1k_1 and k_1h_1 again take the system of closed ideals invariant. This carries over to $h_1k_1 \otimes \mathrm{id}_{\mathcal{O}_2}$ and

$k_1h_1 \otimes \mathrm{id}_{\mathcal{O}_2}$. Thus, part(i) and the above proven (III) imply, that $h_1 \otimes \mathrm{id}_{\mathcal{O}_2}$, and, therefore, $h \otimes \mathrm{id}_{\mathcal{O}_2}$ are unitarily homotopic to an isomorphism. □

We do not know if it is necessary to assume in the following Corollary 10.3.11 that the algebras B_n are stable.

Corollary 10.3.11. *Suppose that (B_n) is a sequence of separable stable C^*-algebras, and that $h_n \colon B_n \to B_{n+1}$ are *-homomorphisms.*

Let $B := \mathrm{indlim}(h_n : B_n \to B_{n+1})$.

If $B_n \cong B_n \otimes \mathcal{O}_\infty$ (resp. $B_n \cong B_n \otimes \mathcal{O}_2$) for $n = 1, 2, \ldots$, then $B \cong B \otimes \mathcal{O}_\infty$ (resp. $B \cong B \otimes \mathcal{O}_2$).

Proof. Let $h_n^\infty \colon B_n \to B$ the natural morphism and let $C_n := h_n^\infty(B_n)$. Since $\mathbb{K}$, $\mathcal{O}_\infty$ and $\mathcal{O}_2$ are simple and nuclear, C_n is stable and $C_n \cong C_n \otimes \mathcal{O}_\infty$ (resp. $C_n \cong C_n \otimes \mathcal{O}_2$).

The union of the stable subalgebras C_n is dense in B. By [**373**], cf. also Corollary 5.5.3, B is stable, i.e. $B \cong B \otimes \mathbb{K}$.

Therefore, we can assume, that $\mathcal{M}(B)$ contains isometries s, t with $ss^* + tt^* = 1$, such that $dC_n + C_n d \subset C_n$ for every $d \in C^*(s,t) \cong \mathcal{O}_2$ and $n \in \mathbb{N}$. Otherwise, we replace B by $B \otimes \mathbb{K}$, C_n by $C_n \otimes \mathbb{K}$, and h_n by $h_n \otimes \mathrm{id}_{\mathbb{K}}$.

By Corollary 10.3.7, $\mathrm{id}_B |C_n$ approximately dominates $(\mathrm{id}_B \oplus \mathrm{id}_B)|C_n$. Therefore, by Corollary 10.3.7, $B \cong B \otimes \mathcal{O}_\infty$.

If $C_n \cong C_n \otimes \mathcal{O}_2$, then, by Corollary 10.3.7, $\mathrm{id}_B |C_n$ is approximately unitarily equivalent to $(\mathrm{id}_B \oplus \mathrm{id}_B)|C_n$ by unitary elements of the multiplier algebra of C_n.

By [**180**], cf. [**816**, chp. 16], the unitary group of $\mathcal{M}(C)$ is contractible if C is σ-unital and stable, cf. our version **??** of a proof. It follows, that the unitary operators of $\mathcal{M}(C)$ can be approximated by unitary operators in the unitization of C in the strict (= strong*) topology.

Therefore, $\mathrm{id}_B |C_n$ is also approximately unitarily equivalent to $(\mathrm{id}_B \oplus \mathrm{id}_B)|C_n$ by unitaries in the unitization of B. Thus, the morphisms id_B and $\mathrm{id}_B \oplus \mathrm{id}_B$ are approximately unitarily equivalent. Thus, by Corollary 10.3.7, $B \cong B \otimes \mathcal{O}_2$. □

Corollary 10.3.12. *The class of separable nuclear C^*-algebras with WvN-property is closed under inductive limits.*

Proof. There is a more general result for inductive limits of strongly p.i. algebras (cf. [**463**]), but here we use Corollary 10.3.7.

Let $h_n : A_n \to A_{n+1}$, $n = 1, 2, \ldots$, a sequence of *-homomorphisms, where A_n is separable, nuclear and has the WvN-property. Let A denote its inductive limit, and let $B := A \otimes \mathbb{K}$.

By Definition 1.2.3, the class of C^*-algebras with WvN-property is closed under stabilization and passage to hereditary C^*-subalgebras, i.e., it suffices to show that B satisfies the WvN-property.

By Corollary 10.3.8, $B_n := A_n \otimes \mathbb{K}$, satisfies $B_n \cong B_n \otimes \mathcal{O}_\infty$ and B is the inductive limit of the sequence $h_n \otimes \mathrm{id}_\mathbb{K} \colon B_n \to B_{n+1}$.

Therefore, by Corollary 10.3.11, $B \cong B \otimes \mathcal{O}_\infty$.

B is separable, stable and nuclear. Thus B has the WvN-property by Corollary 10.3.8. □

CHAPTER 11

PI-SUN algebras represent the nuclear KK-classes

In this Chapter we prove Theorem I of Chapter 1 and some additional results.

There are different methods to construct purely infinite algebras that are $\mathrm{KK}^G(X;\cdot,\cdot)$-equivalent to a given separable (G,X)-algebra, and have simple fibers with respect to the action of X.

(1) inductive limit construction in the case of exact algebras and exact discrete groups, 1a) direct inverses in Ext. 1b) using σ-additivity of KK w.r.t. first variable.

(2) construction of a suitable Toeplitz-Pimsner algebra (such that it is equal to the corresponding Cuntz-Pimsner algebra).

`Citation from Chapter 1, changes to bundle/equvariant case: ??`

In fact, we prove in Chapter 11 a more general result for continuous fields $(A_x)_{x\in X}$ of exact C^*-algebras A_x on a compact Hausdorff space X with the additional property that the algebra $\mathcal{A}$ of continuous sections is exact (which is e.g. the case if all fibers A_x are nuclear or if X is finite):

One has to take again $\mathcal{A}_t := \mathcal{A}\otimes\mathcal{O}_\infty{}^{\mathrm{st}}$ if $\mathcal{A}$ is unital, and define $\mathcal{A}_t$ as the natural unital split extension of

$$\mathrm{C}(X,\mathcal{O}_2)\cong[\mathrm{C}(X)1_{\mathcal{M}(\mathcal{A})}]\otimes\mathcal{O}_2$$

by $\mathcal{A}\otimes\mathcal{O}_\infty{}^{\mathrm{st}}$ if $\mathcal{A}$ is not unital. Then one can use the sub-trivialization theorem of E. Blanchard [**89**] (that can be shown using Theorem A and is a special case of Theorem K), to get a unital $\mathrm{C}(X)$-module *-monomorphism $h_0^u\colon\mathcal{A}_t\to\mathrm{C}(X,\mathcal{O}_2)\subset\mathcal{A}_t$. We define again $h:=\mathrm{id}_{\mathcal{A}_t}\oplus h_0^u$ and let $\mathcal{P}(\mathcal{A}):=\mathrm{indlim}(h\colon\mathcal{A}_t\to\mathcal{A}_t)$. All the operations are consistent with the above defined constructions of $(A_x)_t$ and $P(A_x)$ for the fibers A_x of $(A_x)_{x\in X}$. It is not difficult to see that $(P(A_x))_{x\in X}$ is a continuous field of C^*–algebras, and that $\mathcal{P}(\mathcal{A})$ is the algebra of continuous sections of this field. By construction, $\mathcal{P}(\mathcal{A})$ is exact. Further, the *-monomorphisms $\eta_x\colon A_x\to P(A_x)$ define a $\mathrm{C}(X)$-module monomorphism η from $\mathcal{A}$ into $\mathcal{P}(\mathcal{A})$.

We show in Chapter 11 the more general result:
The monomorphism η defines a $\mathrm{KK}(X;\cdot,\cdot)$-equivalence between $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$. ([1]).

At the end of Chapter 11 we outline an other construction that leads to $\mathrm{KK}(X;\cdot,\cdot)$-equivalent exact (respectively nuclear) algebras with primitive ideal space isomorphic to X (cf. Theorems 11.4.1 and O):

[1]The $\mathrm{KK}(X;\mathcal{A},\mathcal{B})$ is for compact metric spaces X just the Kasparov functor $\mathcal{R}\,\mathrm{KK}^G(X;.,.)$ for the base space X, G is the trivial group).

One starts with a suitable Hilbert bimodule and builds the corresponding Cuntz–Krieger–Pimsner algebra. The construction has the advantage that G-actions of locally compact groups G on $(A_x)_{x\in X}$ lead in a natural way to G-actions on $(P(A_x)\otimes \mathbb{K})_{x\in X}$ ([2]).

Furthermore, we modify in Chapter 11 the above described construction of $A \Rightarrow P(A)$ in a way that we get (cf. Theorem 11.0.7):

Suppose that A is a unital separable exact C^*-algebra, G is a countable (discrete) exact *group, and that $\alpha\colon G\to \mathrm{Aut}(A)$ is a group-morphism.*

Then there exist a purely infinite exact unital C^*-algebra B, a group-morphism $\beta\colon G\to \mathrm{Aut}(B)$ and a G-equivariant unital monomorphism *$\eta\colon A\to B$ that defines a KK^G-equivalence $[\eta]\in \mathrm{KK}^G(A,B)$.*

It can be managed that $\eta(A)$ is the range $E(B)$ of a $\beta(G)$-equivariant conditional expectation E on B.

If — in addition — A is nuclear, then one can find B, β and E, such that B is nuclear — in addition —.

`some new approach:`

Definition: ????????????

To get some more general and farer going results on KK-equivalent embedding into purely infinite algebras, we have to make some definitions, concerning maps that changes the identity maps not to much in the sense of KK-theory.

Definition 11.0.1. Suppose that D is unital, separable and exact. A unital monomorphism $h\colon D\to E$ will be called **elementary** if there are

(i) a unital *-monomorphism $h_0\colon D\to \mathcal{O}_2$,
(ii) a *-monomorphism $\gamma\colon \mathcal{O}_2\to\mathcal{O}_\infty$, and
(iii) an isometry $T\in\mathcal{O}_\infty$
(iv) an isomorphism $\iota\colon D\otimes\mathcal{O}_2\to E$ such that

$$h(d)=\iota((d\otimes TT^*)+(1\otimes\gamma(h_0(d))))\quad \text{for all}\ \ d\in D\,.$$

We call h **plain** if $E=D\otimes\mathcal{O}_\infty$ and $\iota=\mathrm{id}$.

`More generally:`

Suppose $\mathcal{O}_\infty\subset\mathcal{M}(D)$, $k_0\colon D\otimes\mathcal{O}_2\to D\otimes\mathcal{O}_\infty$, $h_0(d):=k_0(d\otimes 1)$, “sufficiently” degenerate. consider similar sum-construction variants

$h(d):=TdT^*+h_0(d)$ etc.

Consider the cases of bundles, to see what is the right definition.

$h_0\colon D\to \mathrm{C}_0(X,\mathcal{O}_2)$

[2] At least in the case where A_x is is nuclear, because then, by Theorem M(iii), there are $\mathrm{C}(X)$-module isomorphisms of $\mathcal{P}(\mathcal{A})\otimes\mathbb{K}$ onto any other purely infinite stable separable nuclear $\mathrm{C}(X)$-algebra $\mathcal{B}$ (with simple fibers B_x) that is $\mathrm{KK}(X;\cdot,\cdot)$-equivalent to $\mathcal{A}$.

generally:

Some continuous action of a lattice $\mathbb{O}$ on D should be given.

$$k_0 \colon D \otimes \mathcal{O}_2 \to D \otimes \mathcal{O}_\infty$$

universally for that action (if k_0 goes into $D \otimes \mathcal{O}_\infty$???)

Suppose that $D_1 \to D_2 \to \cdots \to D_k$ are given by elementary morphisms $h_n \colon D_n \to D_{n+1}$ and suppose that $D_1 \cong D_1 \otimes \mathcal{O}_\infty$.

By definition there are isomorphisms $\iota_n \colon D_n \otimes \mathcal{O}_\infty \to D_{n+1}$ related to h_n. Consider

$$\lambda_{j,k} := \iota_j \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \colon D_j \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty{}^{\otimes(k-j)} \to D_{j+1} \otimes \mathcal{O}_\infty{}^{\otimes(k-j)}$$

$\eta_k := \lambda_k \otimes \cdots \otimes \lambda_1 \colon D_1 \otimes \mathcal{O}_\infty{}^{\otimes k} \to D_{k+1}$.

The system is called homotopy-commutative, if $\eta_k \otimes \mathrm{id}$ is unitarily homotopic to η_{k+1} and if every unitary homotopy on the boundary of a triangle extends to a homotopy in the interior.

`more precise !!!  ???`

Technical Lemma:

LEMMA 11.0.2. *Iterations of elementary morphisms h are elementary.*

Elementary morphisms $h_1, h_2 \colon D \to E$ are unitarily homotopic if

$$\iota_1, \iota_2 \colon D \otimes \mathcal{O}_\infty \to E$$

are unitarily homotopic.

Elementary h is a KK-*equivalence.*

There is a conditional expectation $P_h \colon E \to h(D)$.

Others???

PROPOSITION 11.0.3. *Suppose that D_1 is separable and exact, $h_n \colon D_n \to D_{n+1}$, $n \in \mathbb{N}$, is a sequence of elementary morphisms and $D_1 \cong D_1 \otimes \mathcal{O}_\infty$. Let*

$$P := \mathrm{indlim}(h_n \colon D_n \to D_{n+1})\,.$$

(i) *P is simple, purely infinite, separable, unital and exact (respectively is pi-sun if the D_n are nuclear).*
(ii) *There is an approximately inner conditional expectation $E \colon P \to P$ onto $h_{1,\infty}(D_1)$, $h_{1,\infty}$ is injective.*
(iii) *The unital monomorphism $h_{1,\infty} \colon D_1 \to P$ defines a* KK-*equivalence between D_1 and P, if the ι_n-diagrams (to/for $D_1 \otimes \mathcal{O}_\infty$) are homotopy commutative.*

The idea is, that one considers the m.o.c. cones $\mathcal{C}_{k,\ell} \subset \mathrm{CP}(D_k, D_\ell)$, $\mathcal{C}_{k,\infty} \subset \mathrm{CP}(D_k, P)$ and $\mathcal{C}_{\infty,k} \subset \mathrm{CP}(P, D_k)$ that are generated by the $h_{k,\ell}$, $h_{k,\infty}$ and conditional expectations from P onto $h_{1,k}(D_1) \subseteq D_k$ etc.

The consider (perhaps $\mathcal{C}$ and "action" - equivariant) asymptotic morphisms:

I.e. consider/construct/define a "natural" asymptotic morphism $\phi\colon P \to \mathrm{Q}(\mathbb{R}_+, D_1)$, such that $\mathrm{Q}(\mathbb{R}_+, h_{1,\infty})\colon \mathrm{Q}(\mathbb{R}_+, D_1) \to \mathrm{Q}(\mathbb{R}_+, P)$ has the property that $[\mathrm{Q}(\mathbb{R}_+, h_{1,\infty}) \circ \phi]\colon P \to \mathrm{Q}(\mathbb{R}_+, P)$ is homotopic to $\mathrm{id}_P\colon P \to \mathrm{Q}(\mathbb{R}_+, P)$ and $\phi \circ h_{1,\infty}\colon D_1 \to \mathrm{Q}(\mathbb{R}_+, D_1)$ is homotopic to $\mathrm{id}_{D_1}\colon D_1 \to \mathrm{Q}(\mathbb{R}_+, D_1)$.

But all this inside the given m.o.c. cones.

The proof will be postponed to Section **??**. (Sec. 11.2 ??)

Instead we first deduce from Proposition 11.0.3 the proof of Theorem I, the proof of its $C(X)$-modular generalization Corollary 11.0.5, and the proof of Theorem 11.0.7, that gives a G-equivariant generalization of Theorem I.

Theorem

THEOREM 11.0.4. *Suppose that A is a stable separable C^*-algebra.*

*Let X a T_0 space and $\Psi\colon \mathbb{O}(X) \to \mathcal{I}(A)$ an action of X on A such that there exists a non-degenerate nuclear *-monomorphism $k_0\colon A \otimes \mathcal{O}_2 \to A \otimes \mathcal{O}_\infty$ that is maximal with respect to the Ψ-residually nuclear maps* and reproduces Ψ*, i.e., $J \in \mathcal{I}(A)$ is in the image of Ψ, if and only if, $k_0(J \otimes \mathcal{O}_2) = k_0(A \otimes \mathcal{O}_2) \cap (J \otimes \mathcal{O}_\infty)$.*

????????????????

Let $h_0 := k_0((\cdot) \otimes 1)$ and $h := \mathrm{id} \oplus (h_0 \otimes \mathrm{id})$. Then $P := \mathrm{indlim}(h\colon A \otimes \mathcal{O}_\infty \to A \otimes \mathcal{O}_\infty)$ is exact (respectively nuclear if A is nuclear), strongly purely infinite (more precisely: $P \cong P \otimes \mathcal{O}_\infty$), $\mathrm{Prim}(P) \cong X$, and $\eta\colon a \in A \mapsto h_{1,\infty}(a \otimes 1) \in P$ defines a is a $\mathrm{KK}(X; \cdot, \cdot)$-equivalence $[\eta] \in \mathrm{KK}(X;\, A, P)$.

COROLLARY 11.0.5. *Let X a locally compact Polish space. Suppose that $\mathcal{A}$ is a separable exact $\mathrm{C}_0(X)$-C^*-algebra, such that $x \in X \to \|a_x\|$ is continuous for every $a \in \mathcal{A}$ (i.e., $\mathcal{A}$ is the algebra of continuous sections vanishing at infinity of a continuous field of C^*-algebras A_x). And let $\mathcal{P}(\mathcal{A})$ be defined as above.*

Then $\mathcal{P}(\mathcal{A})$ is a strongly purely infinite exact $\mathrm{C}_0(X)$-C^*-algebra with simple purely infinite fibers $P(A_x)$, and $\mathcal{P}(\mathcal{A}) \cong \mathcal{P}(\mathcal{A}) \otimes \mathcal{O}_\infty$.

The natural map $\eta\colon A \to P(A)$ defines a $\mathrm{KK}(X; \cdot, \cdot)$-equivalence.

LEMMA 11.0.6. *If $D \cong D \otimes \mathcal{O}_\infty$ and if discrete exact G acts on D by $\alpha\colon G \to \mathrm{Aut}(D)$, then there exists a unitary representation $U\colon G \to \mathcal{U}(\mathcal{O}_\infty)$ and an elementary morphism $h\colon D \to D \otimes \mathcal{O}_\infty$ with ι unitarily homotopic to* id*, such that $h \circ \alpha(g) = (\alpha(g) \otimes \mathrm{Ad}(U(g))) \circ h$ for $g \in G$.*

PROOF. Use an unital embedding η of the exact unital algebra $D \rtimes_{r,\alpha} G$ into $\mathcal{O}_2$. One may suppose that $\mathcal{O}_2$ is unitally contained in $(1 - s_1 s_1^*)\mathcal{O}_\infty(1 - s_1 s_1^*)$ for $\mathcal{O}_\infty = C^*(s_1, s_2, \ldots)$. Then $U(g) = \eta(g) + TT^*$ has the property for $T = s_1$. □

THEOREM 11.0.7. *Suppose that A is an separable unital nuclear C^*-algebra, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a countable discrete exact group G on A.*

Then there exist a pi-sun algebra B with an action $\beta\colon G \to \mathrm{Aut}(B)$ of G on B and a G-equivariant unital monomorphism *$\eta\colon A \to B$ that defines a* KK*-equivalence $[\eta] \in \mathrm{KK}(A, B)$.*

In particular, $\mathrm{K}_*(\beta(g))\circ \mathrm{K}_*(\eta) = \mathrm{K}_*(\eta)\circ \mathrm{K}_*(\alpha(g))$ *and* $\mathrm{K}_*(\eta)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ *is invertible.*

It seems that (B, β) and η can be chosen, such that η defines even a KK^G-equivalence $[\eta] \in \mathrm{KK}^G(A, B)$. (only "seems" ???)

Proof. Let G act on $D_1 := A \otimes \mathcal{O}_2$ by $\alpha_1(g) := \alpha(g) \otimes \mathrm{id}$ for $g \in G$. Clearly the map $k\colon a \to a \otimes 1$ is a G-equivariant monomorphism from A into D_1, and $[k] \in \mathrm{KK}^G(A, D_1)$ is a KK^G-equivalence, in particular $[k] \in \mathrm{KK}(A, D_1)$ is a KK-equivalence. Let $D_n := A \otimes ({\mathcal{O}_\infty}^{\otimes n})$. By Lemma 11.0.6 there exists actions $\alpha_n\colon G \to \mathrm{Aut}(D_n)$ and G-equivariant plain elementary morphisms $h_n\colon D_n \to D_{n+1}$.

Now let $B := \mathrm{indlim}\, D_n$ and $\eta := h_{1,\infty} \circ k$. Since the h_n are G-equivariant, there is a group morphism $\beta\colon G \to \mathrm{Aut}(B)$ with $h_{1,\infty} \circ \alpha_1(g) = \beta(g) \circ h_{1,\infty}$ for $g \in G$. By Proposition 11.0.3, B is pi-sun and the morphism η defines a KK-equivalence $[\eta] \in \mathrm{KK}(A, B)$. □

Remark 11.0.8. We call a group G ***C*-liftable*** if, for every pi-sun algebra B in the UCT-class with finitely generated $\mathrm{K}_*(B)$ and every grading-preserving action $\gamma\colon G \to \mathrm{Aut}(\mathrm{K}_*(B))$ of G on $\mathrm{K}_*(B)$ that fixes the class $[1] \in \mathrm{K}_0(B)$, there exists an action $\beta\colon G \to \mathrm{Aut}(B)$ such that for the natural group morphism $\mu\colon \mathrm{Aut}(B) \to \mathrm{Aut}(\mathrm{K}_*(B))$ holds $\mu \circ \beta(g) = \gamma \circ \mu$.

The Theorem 11.0.7 implies that the problem of the determination of all C^*-liftable groups G is equivalent to the question under which circumstances for a finitely generated Abelian group H and an action $\gamma_0\colon G \to \mathrm{Aut}(H)$ with G-fixed element $x_0 \in H$ there is a unital nuclear C^*-algebra A in the UCT-class and an action $\alpha\colon G \to \mathrm{Aut}(A)$ such that $\mathrm{K}_1(A) = 0$ and there is an isomorphism $\gamma\colon H \to \mathrm{K}_0(A)$

???????????

Then the case $\mathrm{K}_0(A) = 0$ and $\mathrm{K}_1(A) = H$ can then be studied by replacing A by $A \otimes \mathcal{P}_\infty$, where $\mathcal{P}_\infty$ is the unique pi-sun algebra in the UCT class with $\mathrm{K}_0(\mathcal{P}_\infty) = 0$ and $\mathrm{K}_1(\mathcal{P}_\infty) = \mathbb{Z}$ ([3]).

The general case follows from the consideration of $A = A_1 \oplus (A_2 \otimes \mathcal{P}_\infty)$ in Theorem 11.0.7, where $\mathrm{K}_1(A_j) = 0$ for $j = 1, 2$.

It is evident that every free group is in this class of groups.

`It is not clear if` $\mathrm{Aut}(B) \to \mathrm{KK}(B, B)^{-1}$ `is surjective for pi-sun` B `in UCT-class !!! ??` By the natural epimorphisms (???) $\mathrm{Aut}(B) \to \mathrm{KK}(B, B)^{-1} \to \mathrm{Aut}(\mathrm{K}_*(B))$, (given by the UCT and Theorem B).

---

[3]One can easily show that $\mathcal{P}_\infty = {\mathcal{O}_\infty}^{st} \rtimes \mathbb{Z}_2$ for the flip action μ of $\mathbb{Z}_2$ on the standard form ${\mathcal{O}_\infty}^{st} \cong \mathcal{P}_\infty \otimes \mathcal{P}_\infty$ with $\mathrm{K}_0(\mu)(x) = -x$ — it is simply the flip on the tensor product, cf. [**448**, Rem.4.3(2)] and observe that, for the coordinate flip on $\mathbb{R}^2$, the crossed product $\mathrm{C}_0(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ is isomorphic to $\mathrm{C}_0(\mathbb{R}) \otimes C$, with $C := \{f \in \mathrm{C}_0(\mathbb{R}_0, M_2)\,;\ f(0) = \text{diagonal}\}$, i.e., $\mathrm{K}_*(C) = (\mathbb{Z}, 0)$.

Contributions of J. Spielberg [**732**] and (farer going) results of T. Kasura ([**416**], [**417**]) show that in particular all cyclic groups are C^*-liftable.

Begin of old text:

Note that we can construct in the case of exact algebras only KK-equivalences, but not $\mathrm{KK}_{\mathrm{nuc}}$-equivalences because thre is no useful non-elementary criteria for which exact separable A the identity map of A defines a class $[\mathrm{id}_A]$ in $\mathrm{KK}_{\mathrm{nuc}}(A,A)$. (But there is a natural map from $\mathrm{KK}_{\mathrm{nuc}}(A,A)$ onto an ideal of the ring $\mathrm{KK}(A,A)$.)

Recall that ${\mathcal{O}_\infty}^{st} := (1-s_1s_1^*)\mathcal{O}_\infty(1-s_1s_1^*) \subset \mathcal{O}_\infty$. It contains a copy of $\mathcal{O}_2$ unitally.

If A is exact, separable and unital let $A_t := A \otimes {\mathcal{O}_\infty}^{st}$. If A is exact, separable and non-unital, then let A_t denote the unital C^*-subalgebra of $\mathcal{M}(A)\otimes {\mathcal{O}_\infty}^{st}$ given by

$$A_t := A \otimes {\mathcal{O}_\infty}^{st} + 1 \otimes \mathcal{O}_2 \subseteq \mathcal{M}(A) \otimes {\mathcal{O}_\infty}^{st} \,,$$

a split extension of $\mathcal{O}_2$ by $A \otimes {\mathcal{O}_\infty}^{st}$.

One can variate the construction if A is stable to get a stable version of A_t by considering first a copy of $\mathbb{K} \subset \mathcal{M}(A)$ with $\mathbb{K} \cap A = \{0\}$ and $\mathbb{K}\cdot A = A$, existing by Remark 5.1.1(8), and then let $A_t := A \otimes {\mathcal{O}_\infty}^{st} + \mathbb{K}\otimes \mathcal{O}_2$, contained in $\mathcal{M}(A)\otimes {\mathcal{O}_\infty}^{st}$.

Let S, T the generators of $\mathcal{O}_2 = C^*(S,T;\, SS^* + TT^* = 1, S^*S = 1 = T^*T)$.

The algebra A_t is exact because A is exact and ${\mathcal{O}_\infty}^{st}$ and $\mathcal{O}_2$ are nuclear (cf. [**432**]).

To avoid later problems with notations, now we change the notation for A_t: Let $B := A_t$ in the sequel. Note that the exact C*-algebra B is unital and contains $\mathcal{O}_2$ unitally.

Define $h := \mathrm{id}_B \oplus h_0^u$ where the unital monomorphism $h_0^u \colon B \to \mathcal{O}_2 \subset B$ is as described in Chapter 1 before Theorem B. (Respectively, if A is stable, let $h := \mathrm{id}_B \oplus h_0$ for the more general non-degenerate "maximal" ideal-system preserving nuclear map h_0 from $A \otimes \mathcal{O}_\infty$ into $C \subset A \otimes \mathcal{O}_\infty$, where C is a nuclear and stable, $\mathcal{O}_2$ absorbing C^*-subalgebra of $A \otimes \mathcal{O}_2$ with the ideal system of A)

$P(A) := \mathrm{indlim}(h\colon B \to B)$ is purely infinite and simple by the criteria in

check if (iv) or (v) is used Proposition 2.2.5(iv)

for $h_n = h$, $B_n = B$, because for $\varepsilon > 0$, a, b in B_+ with $\|a\| = \|b\| = 1$ we can find an isometry v in $\mathcal{O}_2$ such that $\|1 - v^* h_0(a) v\| < \varepsilon$. Thus

$$\|h(b) - h(b)^{1/2} v^* t^* h(a) t v h(b)^{1/2}\| < \varepsilon \,.$$

Since exactness is closed under inductive limits ([**432**]), $P(A)$ is exact. $P(A)$ is nuclear if moreover A (and hence B) is nuclear.

The ideal system is the same as that from A because h is ideal system preserving.

For a proof of Theorem I it remains to show:

(i) *There is a conditional expectation from the hereditary C^*-subalgebra D of $P(A)$ generated by $A \otimes p$ onto $A \otimes p \cong A$, and*

(ii) *the inclusion map* $A \cong A \otimes p \subset B \cong h_1^\infty(B) \subset P(A)$ *defines a* KK-*equivalence between* A *and* $P(A)$.

The projection $p = s_2 s_2^*$ is in ${\mathcal{O}_\infty}^{st} = (1 - s_1 s_1^*)\mathcal{O}_\infty(1 - s_1 s_1^*) \subset \mathcal{O}_\infty$. $A \cong A \otimes p \to A \otimes {\mathcal{O}_\infty}^{st}$ is a KK-equivalence, because $\mathbb{C}p \to {\mathcal{O}_\infty}^{st}$ defines a KK-equivalence between $\mathbb{C}$ and ${\mathcal{O}_\infty}^{st}$ and because one can tensor KK-equivalences (by nuclear algebras). It can be seen from the the 6-term exact sequences and from $\mathrm{KK}(., \mathcal{O}_2) = \mathrm{KK}(\mathcal{O}_2, .) = 0$ that $A \otimes {\mathcal{O}_\infty}^{st} \to B$ defines a KK-equivalence. Thus $A \cong A \otimes p \to B$ defines a KK-equivalence between A and B. The hereditary subalgebra of B generated by $A \otimes p$ is $A_0 := A \otimes p\mathcal{O}_\infty p \subset A \otimes {\mathcal{O}_\infty}^{st} \subset B$. Thus $A \otimes p$ is the range of a conditional expectation from the hereditary C*-subalgebra A_0 onto $A \otimes p$.

Above we have seen that the proof of Theorem I, (i.e. of (i) and (ii) above) reduces to the proof of the following:

(i*) $h_1^\infty(B)$ *is the range of a conditional expectation on* $P(A)$, *and*
(ii*) $h_{1,\infty}\colon B \to P(A)$ *defines a* KK-*equivalence* $[h_{1,\infty}] \in \mathrm{KK}(B, P(A))$.

We show $(i*)$ by an induction procedure and then use homotopy invariance of unsuspended E-theory $\mathrm{R}(B, P(A))$ and $\mathrm{R}(P(A), B)$ to prove (ii^*). More precisely we construct a unital asymptotic morphism $g\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+, B)$ such that $g \circ h_{1,\infty}$ is homotopic to id_B and $h_{1,\infty} \circ g$ is homotopic to $\mathrm{id}_{P(A)}$. By the homotopy invariance of Rørdam groups it follows that $h_{1,\infty}$ is a KK-equivalence.

Cuntz addition of morphisms is the same as the "direct" sum of morphisms (after stabilization) and thus induces the sum for the corresponding KK-elements:

$[\mathrm{id}_B] + [h_0] = [\mathrm{id}_B \oplus h_0] = [h]$ and $2[h_0] = [h_0 \oplus h_0] = [h_0]$. It follows that $[h_0] = 0$ and $[h] = [\mathrm{id}]$ in $\mathrm{KK}(B, B)$.

Thus it suffices to show that the following statements (α) and (β) are true in order to prove (i*) and (ii*):

(α) *There is a family of conditional expectations* E_n^m *from* $h_m^\infty(B)$ *onto* $h_n^\infty(B)$ *for* $n < m$ *such that* $E_n^{m+k} h_m^\infty = E_n^m h_m^\infty$. *and* $E_n^k E_m^k h_k^\infty = E_n^k h_k^\infty$ *for* $n < m < k$.

They define conditional expectations $E_n\colon P(A) \to h_n^\infty(B) \subset P(A)$ *with* $E_n(P(A)) = h_n^\infty(B)$ *and* $E_n E_m = E_{min(n,m)}$.

(β) *There exists a completely positive unital map* $V\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+, B)$ *such that* $V(b^*b) - V(b)^*V(b) \in \mathrm{C_0}(\mathbb{R}_+, B)$ *for every* $b \in P(A)$, *and* $[(\pi \circ V) \circ h_1^\infty] = [h] \in \mathrm{KK}(B, B)$, $[h_1^\infty \circ (\pi \circ V)] = [\mathrm{id}_{P(A)}] \in \mathrm{KK}(P(A), P(A))$.

The completely positive maps V, $V \circ h_1^\infty$ and $h_1^\infty \circ V$ in statement (β) define elements of $\mathrm{Ext}^{-1}(P(A), SB)$, $\mathrm{Ext}^{-1}(B, SB)$ and $\mathrm{Ext}^{-1}(P(A), SP(A))$ and [.] denote the corresponding elements of $\mathrm{KK}(., .) \cong \mathrm{Ext}^{-1}(., S(.))$.

1. Proof of (α)

We start with the proof of (α). The proof of (β) is conceptional quite simple but requires some preparation.

PROOF OF (α): By induction we find $(s^n)^*h^n(b)s^n = b$ for $b \in B$ and $n = 1, 2, \ldots$:

$$b = s^*(sbs^* + th_0^u(b)t^*)s = s^*h(b)s = s^*(s^n)^*h^n(h(b))(s^n)s = (s^{n+1})^*h^{n+1}(b)s^{n+1}.$$

Let T_n be given by $T_n(b) := h^n((s^n)^*bs^n)$ and $T_0 = \mathrm{id}_B$. Then T_n is a conditional expectation from B onto $h^n(B)$, because $T_n(B) \subset h^n(B)$ and $T_n(b) = b$ for b in $h^n(B)$, because $(s^n)^*h^n(b)s^n = b$.

The conditional expectations T_n satisfy $T_{n+k}h^k = h^kT_n$ and $T_nT_m = T_{\min(n,m)}$:

The first equation follows from $h^{n+k}((s^n)^*(s^k)^*h^k(b)s^ks^n) = h^k(h^n((s^n)^*bs^n))$.

Certainly $T_nT_m = T_m$ if $n \leq m$ if $n \leq m$ and for $m < n$ we have

$$T_nT_m(b) = h^n((s^n)^*h^m((s^m)^*bs^m)s^n) = h^n((s^{n-m})^*(s^m)^*bs^ms^{n-m}) = T_n.$$

Now we define conditional expectations E_n^m ($n < m$) from $h_m^\infty(B)$ onto $h_n^\infty(B) = h_m^\infty(h^{m-n}(B))$ by

$$E_n^m(b) := h_m^\infty T_{m-n}(h_m^\infty)^{-1}(b)$$

for b in $h_m^\infty(B)$, i.e. by $E_n^m h_m^\infty = h_m^\infty T_{m-n}$.

Then $E_n^m h_k^\infty = E_n^k h_k^\infty$ for $n < m$ and $n < k \leq m$ because

$$E_n^m h_k^\infty = E_n^m h_m^\infty h^{m-k} = h_m^\infty T_{m-n} h^{m-k} = h_m^\infty h^{m-k} T_{k-n} = h_k^\infty T_{k-n} = E_n^k h_k^\infty.$$

If $n < m < k$ we have $E_n^k E_m^k = E_n^k$ because

$$E_n^k E_m^k h_k^\infty = E_n^k h_k^\infty T_{k-m} = h_k^\infty T_{k-n} T_{k-m} = h_k^\infty T_{k-n} = E_n^k h_k^\infty.$$

The consistency of the E_n^m implies the existence and uniqueness of conditional expectations $E_n \colon P(A) \to h_n^\infty(B)$ from $P(A)$ onto $h_n^\infty(B)$ such that $E_n h_k^\infty = E_n^m h_k^\infty$ for $n \leq k \leq m$. Then $E_n E_m = E_m$ if $m \leq n$.

Let $n < m < k \in \mathbb{N}$ then

$$E_n E_m h_k^\infty = E_n E_m^k h_k^\infty = E_n^k E_m^k h_k^\infty = E_n^k h_k^\infty = E_n h_k^\infty\,.$$

Thus $E_n(P(A)) = h_n^\infty(B)$, $E_n E_m = E_{min(n,m)}$ and (α) is shown. □

2. Preliminaries for the proof of (β)

We need some observations for the proof of (β). They are based on elementary properties of $\mathcal{O}_2$ and of Proposition 11.2.2 (which is related to Theorem B).

LEMMA 11.2.1. *Let A be a unital C*-algebra, then the unitary group of $A \otimes \mathcal{O}_2$ is simply connected.*

In particular, any continuous map $u \colon \partial([0,1]\times[0,1]) \to U(\mathcal{O}_2)$ from the boundary of the square $[0,1]\times[0,1]$ into the unitaries of $\mathcal{O}_2$ extends to a continuous map from the square into the unitaries of $\mathcal{O}_2$.

PROOF. Let $F := A \otimes \mathcal{O}_2$. We consider $\mathcal{O}_2 \cong 1 \otimes \mathcal{O}_2$ as a unital subalgebra of F with generators $\{s,t\}$ (to simplify notation). The unital endomorphism $\delta(a) = sas^* + tat^* = a \oplus a$, is point-norm homotopic inside the endomorphisms of $\mathcal{O}_2$ to $\mathrm{id}_{\mathcal{O}_2}$, by [**172**, prop.2.2], $\mathcal{O}_2$ is K_1-injective by [**172**, thm.1.9], and $\mathrm{K}_1(\mathcal{O}_2) = 0$ by [**172**, thm.3.8].

Thus, $\mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$, and it follows that $\mathrm{id}_A \otimes \delta$ is homotopic to id_F and $u \sim_h u \oplus u$ in $\mathcal{U}(F)$.

Notice that $v \oplus 1$ is unitarily equivalent to $1 \oplus v$ by the selfadjoint unitary $U_c := ts^* + st^*$ in $M_2 \subset \mathcal{O}_2$ and $(u_1 \oplus u_2) \oplus u_3$ is unitarily equivalent to $u_1 \oplus (u_2 \oplus u_3)$ by the unitary $U_d \in \mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$, defined by equation (2.4), see Lemma 4.2.6(o) or Proposition 4.3.2.

Since the unitary group of $\mathcal{O}_2$ is connected by [**172**, thm.1.9, thm.3.8], we get that the unitary $U_d \in \mathcal{O}_2$ defined by equation (2.4) is homotopic to 1 in E. Together we get the following homotopies in the unitary group of F:

$u \sim u \oplus u = (u \oplus 1)(1 \oplus u) \sim u^2 \oplus 1$, and with $v = u^2$

$$1 = (v \oplus 1)(v^* \oplus 1) \sim v \oplus v^* \sim (v \oplus v) \oplus v^* \sim v \oplus (v \oplus v^*) \sim v \oplus 1 = u^2 \oplus 1.$$

Thus $u \sim 1$.

For $T := \partial([0,1]\times[0,1])$ and $D := [0,1]\times[0,1]$ this implies that the epimorphism $\mathrm{C}(D, \mathcal{O}_2) \to \mathrm{C}(T, \mathcal{O}_2)$ maps the unitary group of $\mathrm{C}(D, \mathcal{O}_2)$ onto the unitary group of $\mathrm{C}(T, \mathcal{O}_2) \cong \mathrm{C}(T) \otimes \mathcal{O}_2$. □

The following Proposition 11.2.2 generalizes some aspects of Theorem B.

The unitary equivalence modulo $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$ implies that k and h define the same element of $\mathrm{Ext}_{\mathrm{nuc}}(A, S\,\mathrm{C}(X,D))$.

Thus $[h] = [k]$ in $\mathrm{KK}_{\mathrm{nuc}}(A, \mathrm{C}(X,D))$.

Parts (i) and (ii) of Proposition 11.2.2 simply mean that $h \oplus h_0^u$, h and k are unitarily homotopic in $\mathrm{C}(X,D)$ if $h(A)$ and $k(A)$ are contained in $\mathrm{C}(X,D) \subset \mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathrm{C}(X,D))$.

By $\pi_y \colon \mathrm{C}_\mathrm{b}(Y,D) \to D$ we denote the evaluation at the point $y \in Y = \mathbb{R}_+ \times X$.

PROPOSITION 11.2.2. *Suppose that X is a compact space, D is a unital simple purely infinite C*-algebra which contains a copy of $\mathcal{O}_2$ unitally, that A a unital separable exact C*-algebra, and that $h, k \colon A \to \mathrm{C}_\mathrm{b}(\mathbb{R}_+ \times X, D) \cong \mathrm{C}_\mathrm{b}(\mathbb{R}_+, \mathrm{C}(X,D))$ are unital C*-morphisms with the property that, for every $y \in \mathbb{R}_+ \times X$, $\pi_y h$ and $\pi_y k$ are monomorphisms. Then:*

(i) *There is a unitary* $U_0 \in \mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D))$ *such that* $(h \oplus h_0^u)(a) - U_0^* h(a) U_0$ *is in* $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$ *for every* $a \in A$.

(ii) *If, moreover,* h *and* k *are nuclear and* $[h] = [k]$ *in* $\mathrm{KK_{nuc}}(A, \mathrm{C}(X,D))$, *then there exist a unitary* $U \in \mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D))$ *such that* $h(a) - U^* k(a) U$ *is in* $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$ *for every* $a \in A$.

(iii) *For* $\varphi \colon A \otimes \mathbb{K} \to \mathrm{C}(X,D) \otimes \mathbb{K}$ *with* $[\varphi(1 \otimes p_{11})] = 0$ *in* $\mathrm{K}_0(\mathrm{C}(X,D))$ *there exits a unital* C^**-morphism* $h \colon A \to \mathrm{C}(X,D)$ *such that* $\pi_y h$ *is a monomorphism for every* $y \in X$ *and* $h \otimes \mathrm{id}_{\mathbb{K}}$ *and* $\varphi \oplus h_0$ *(where* $h_0 = h_0^u \otimes \mathrm{id}_{\mathbb{K}}$*) are unitarily homotopic.* h *is nuclear if* φ *is nuclear. In particular, every* $z \in \mathrm{KK_{nuc}}(A, \mathrm{C}(X,D))$ *with* $[1_A] \otimes_A z = [1_{\mathrm{C}(X,D)}] = 0$ *is of the form* $z = [h]$ *for a nuclear* h *of the type above.*

Proof. Ad(i): Let $\mathrm{Q}(\mathbb{R}_+, \mathrm{C}(X,D)) := \mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D)) / \mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$ and let π denotes the quotient map $\pi \colon \mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D)) \to \mathrm{Q}(\mathbb{R}_+, \mathrm{C}(X,D))$.

$\mathrm{C_b}(\mathbb{R}_+ \times X, D) \cong \mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D))$ and $\mathrm{C}_0(\mathbb{R}_+ \times X, D) \cong \mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$ by a natural isomorphism.

Consider $\mathrm{C}(X,D)$ as a unital C*-subalgebra of $\mathrm{Q}(\mathbb{R}_+, \mathrm{C}(X,D))$. Then the relative commutant $h_0^u(A)' \cap \mathrm{Q}(\mathbb{R}_+, \mathrm{C}(X,D))$ contains a copy of $\mathcal{O}_2$ unitally (by the definition of h_0^u in Chapter 1 before Theorem B).

Thus by Proposition 4.3.5 it suffices to show that $\pi h \colon A \to \mathrm{Q}(\mathbb{R}_+ \times X, D)$ dominates $h_0^u \colon A \to \mathcal{O}_2 \subset \mathrm{Q}(\mathbb{R}_+ \times X, D)$. But this follows from Corollary 7.4.6, because the lift $h_0^u \colon A \to \mathcal{O}_2 \subset \mathrm{C_b}(\mathbb{R}_+ \times X, D)$ is nuclear and the evaluations $\pi_y h$ at $y \in \mathbb{R}_+ \times X$ of the lift $h \colon A \to \mathrm{C_b}(\mathbb{R}_+ \times X, D)$ of πh are unital monomorphisms.

Ad(ii): By the proof of Theorem B(i) and (ii) in Chapter 9 we have for $h_0 := h_0^u \otimes \mathrm{id}_{\mathbb{K}}$ that the morphisms $h \otimes \mathrm{id}_{\mathbb{K}} \oplus h_0$ and $k \otimes \mathrm{id}_{\mathbb{K}} \oplus h_0$ from $A \otimes \mathbb{K}$ into $\mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D) \otimes \mathbb{K})$ are unitarily equivalent modulo $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D)) \otimes \mathbb{K}$ Since $h \otimes \mathrm{id}_{\mathbb{K}} \oplus h_0 = (h \oplus h_0^u) \otimes \mathrm{id}_{\mathbb{K}}$ the argument before Corollary C in Chapter 1 shows that this implies that $h \oplus h_0^u$ and $k \oplus h_0^u$ are unitarily equivalent in $\mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D))$ modulo $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$. By the assumptions on the $\pi_y h$ and $\pi_y k$ we get from part(i) that moreover h and k are unitarily equivalent in $\mathrm{C_b}(\mathbb{R}_+, \mathrm{C}(X,D))$ modulo $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X,D))$.

Ad(iii): By Theorem B(i) there exists for $z \in \mathrm{KK_{nuc}}(A, \mathrm{C}(X,D))$ a nuclear C^*-monomorphism $\varphi \colon A \otimes \mathbb{K} \to \mathrm{C}(X,D) \otimes \mathbb{K}$ with $[\varphi] = z$, because $\mathrm{C}(X,D)$ contains a unital copy of $\mathcal{O}_2$.

Then $[\varphi(1 \otimes p_{11})] = z \otimes_A [1_A] = 0$ in $\mathrm{K}_0(\mathrm{C}(X,D))$.

$[(\varphi \oplus h_0)] = [\varphi] + [h_0] = [\varphi]$ in $\mathrm{KK}(A, \mathrm{C}(X,D))$ because $2[h_0] = [h_0]$.

Since $[\varphi(1 \otimes p_{11})] = 0$ in $\mathrm{K}_0(\mathrm{C}(X,D))$, there is a unitary V in $M(\mathrm{C}(X,D) \otimes \mathbb{K})$ such that

$$1 \otimes p_{11} = V^*(\varphi(1 \otimes p_{11}) \oplus (1 \otimes p_{11}))V = V^*(\varphi \oplus h_0)(1 \otimes p_{11})V.$$

Thus there is a unital C^*-morphism $h\colon A \to \mathrm{C}(X, B)$ such that, for $a \in A$

$$h(a) \otimes p_{11} = V^*(\varphi \oplus h_0)(a \otimes p_{11})V.$$

$A \otimes p_{11}$ is a full corner of $A \otimes \mathbb{K}$. Therefore, $h \otimes \mathrm{id}_{\mathbb{K}}$ is unitarily homotopic to $V^*(\varphi \oplus h_0)V$ and thus to $\varphi \oplus h_0$. If we apply the evaluation maps $\pi_y \otimes \mathrm{id}_{\mathbb{K}}$ to this unitary homotopy, then we see that $\pi_y h$ is a monomorphism for every $y \in X$.

From the definition of a unitary homotopy it follows that a morphism is nuclear, if it is unitarily homotopic to a nuclear morphism. □

Let X be a compact space and D a unital C^*-algebra. Then cone(X) means the one-point compactification of of $\mathbb{R}_+ \times X$.

By the isomorphism $[0, 1) \cong \mathbb{R}_+$, the C^*-algebra $\mathrm{C}(cone(X), D)$ is the unital subalgebra of $\mathrm{C}([0, 1] \times X, D) \cong \mathrm{C}([0, 1], \mathrm{C}(X, D))$ which is naturally isomorphic to the unital split extension of $\mathrm{C}_0(\mathbb{R}_+, \mathrm{C}(X, D))$ by $D \subset \mathrm{C}(X, D)$.

Now we consider the case $D = \mathcal{O}_2$.

Corollary 11.2.3. *Suppose that X is a compact space, B is a unital separable exact C^*-algebra, and $h\colon B \to \mathrm{C}(X, \mathcal{O}_2)$ is a unital C^*-morphism such that $\pi_y h$ is a monomorphism for every $y \in X$.*

Then there exists $k\colon B \to \mathrm{C}(cone(X), \mathcal{O}_2)$ with $\pi_y k = \pi_y h$ for every $y \in X \cong X \times \{0\}$ and $\pi_y k$ is a monomorphism for every $y \in cone(X)$.

Proof. Since $\mathrm{KK}(A, \mathrm{C}(X) \otimes \mathcal{O}_2) = 0$, h is unitarily homotopic to h_0^u by Proposition 11.2.2(ii). This homotopy yields a strongly continuous family of morphisms $k_1(\tau, y)\colon B \to \mathcal{O}_2$ ($\tau \in [0, 1/2]$) such that $k_1(0, y) = h(y) := \pi_y h$ for $y \in X$. And there is a continuous map $u\colon (0, 1/2] \times X \to U(\mathcal{O}_2)$ such that $k_1(\tau, y) = u(\tau, y)^* h_0^u u(\tau, y)$ for $y \in X$ and $\tau \in (0, 1/2]$. The unitary $v \in \mathrm{C}(X, \mathcal{O}_2)$ with $v(y) := u(1/2, y)$ is homotopic to 1 in the unitaries of $\mathrm{C}(X, \mathcal{O}_2) \cong \mathrm{C}(X) \otimes \mathcal{O}_2$ by Lemma 11.2.1.

Thus there is an extension w of u to $(0, 1] \times X$ with $w(1, y) = 1$ for $y \in X$ and $w(\tau, y) = u(\tau, y)$ for $y \in X$, $\tau \in [0, 1/2]$, $k(\tau, y) := k_1(\tau, y)$ for $y \in X$, $\tau \in [0, 1/2]$, $k(\tau, y) := w(\tau, y)^* h_0(.) w(\tau, y)$ for $y \in X$, $\tau \in (1/2, 1]$ is the desired extension of h to $cone(X)$. □

Lemma 11.2.4.
(i) *If $\varphi\colon B \to \mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_2)$ is a unital C^*-morphism such that $\pi_\tau \varphi$ is a monomorphism for every $\tau \in \mathbb{R}_+$ then there exists a unitary $U \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_2)$ such that $\varphi(a) - U^* h_0^u(a) U \in \mathrm{C}_0(\mathbb{R}_+, \mathcal{O}_2)$ for every $a \in B$, where $h_0^u\colon B \to \mathcal{O}_2$ is as defined before Theorem B in Chapter 1.*

    *In particular, every unital *-monomorphism $k\colon B \to \mathcal{O}_2$ is unitarily homotopic to h_0^u.*

(ii) *Suppose that $h := \mathrm{id}_B \oplus_{s,t} h_0^u$, and that u_k are unitaries in $\mathcal{O}_2$ with $u_k s = s^{k+1}$, $u_0 = 1$.*

    Then $h^{k+1} = u_k(\mathrm{id}_B \oplus_{s,t} h_k) u_k^*$, where $h_k\colon B \to \mathcal{O}_2$ is a unital C^*-morphism.

In particular, h^2 is unitarily homotopic to h by unitaries $\tau \in [0,1) \mapsto u(\tau) \in \mathcal{O}_2$ such that $u(1/2) = u_1$, $u(0) = 1$, $u(1/2+\tau) = u_1(1 \oplus_{s,t} v(2\tau))$ for $\tau \in [0,1/2)$, where $h_1(b) = \lim_{\tau\to 1} v(\tau)h_0(b)v(\tau)^*$ is a unitary homotopy between h_1 and h_0 with $v(x) \in \mathcal{O}_2$ and $v(0) = 1$.

(iii) *For $k \in \mathbb{N}$ there exist unitaries $v_k \in \mathcal{O}_2$ with $v_k(s^k) = s$.*

*If $k \geq 2$ and v_k is a unitary in $\mathcal{O}_2$ with $v_k(s^k) = s$, then there is exactly one unital *-monomorphism $\psi_k \colon B \to \mathcal{O}_2$ such that $h^k = v_k^*(\mathrm{id}_B \oplus \psi_k)v_k$, where Cuntz addition is taken with $\{s,t\}$.*

(iv) *Let $u(t)$ as in (ii), and define a unital C*-morphism φ from B into $\mathrm{C_b}(\mathbb{R}_+, B)$ by $\varphi(b)(t) := h(b)$ for $t \in [0,1]$ $\varphi(b)(n) := h^n(b)$ for $n = 1,2,\ldots$ and $\varphi(b)(n+t) := u(t)\varphi(b)(n)u(t)^*$ for $n = 1,2,\ldots$, $t \in [0,1)$.*

Suppose $n+k \geq 1$ and that u_{k+n+2} are unitaries in $\mathcal{O}_2$ with

$$u_{k+n+1} \cdot s^{k+n+1} = s \quad \textit{and} \quad u_{k+n+2} \cdot s^{k+n+2} = s.$$

Then there are a continuous map $u \colon [0,1] \to \mathcal{O}_2$ into the unitaries of $\mathcal{O}_2$ and a strongly continuous map $\psi \colon [0,1] \to \mathrm{Hom}(B, \mathcal{O}_2) \subset \mathcal{L}(B)$ into the unital C-morphisms from B into $\mathcal{O}_2$ such that $u(0) = u_{k+n+1}$, $u(1) = u_{k+n+2}$ and*

$$h^k\varphi(1+\tau)h^n = u(\tau)^*(\mathrm{id}_B \oplus (h_0 \oplus \psi(\tau)))u(\tau).$$

Proof. Ad(i): Since $\mathrm{KK_{nuc}}(B, \mathcal{O}_2) = \mathrm{KK}(B, O_2) = 0$, (i) follows as a special case from Proposition 11.2.2(ii) where X= point, $k = h_0^u$ and $D = \mathcal{O}_2$.

Ad(ii): Above we have seen that $(s^k)^*h^k(b)s^k = b$, cf. construction of E_n.

Suppose that $h_{k-1} \colon B \to \mathcal{O}_2$ and $u_{k-1} \in \mathcal{O}_2$ with $h^k = u_{k-1}(\mathrm{id} \oplus_{s,t} h_{k-1})u_{k-1}^*$ are given.

Let $h'_k := h_0 \oplus_{s,t} (h_{k-1} \circ h)$. Then h'_k is a unital morphism from B into O_2. We have

$$h^{k+1}(b) = u_{k-1}(s^2b(s^*)^2 + sth_0(b)t^*s^* + t(h_{k-1}\circ h)(b)t^*)u_{k-1}^*.$$

Let w be the unitary in $\mathcal{O}_2$ with $ws^2 = s$, $wst = ts$, $wt = t^2$.

Then $(\mathrm{id} \oplus_{s,t} h_{k-1})h = h \oplus_{s,t} (h_{k-1} \circ h) = (\mathrm{id}_B \oplus_{s,t} h_0) \oplus_{s,t} (h_{k-1} \circ h) = w^*(\mathrm{id}_B \oplus_{s,t} (h_0 \oplus_{s,t} h_{k-1} \circ h))w = w^*(\mathrm{id}_B \oplus_{s,t} h'_k)w$.

It follows $h^{k+1} = u_{k-1}w^*(\mathrm{id}_B \oplus_{s,t} h'_k)w(u_{k-1})^*$.

On the other hand, $u_{k-1}w^*s = s^{k+1} = u_k s$ and $u_{k-1}, u_k, w \in \mathcal{O}_2$.

Thus $u_k = u_{k-1}w^*(1 \oplus_{s,t} v)$ for some unitary $v \in \mathcal{O}_2$. Now let $h_k := v^*h'_k v$.

Ad(iii): There exist unitaries v_k in $\mathcal{O}_2$ with $v_k(s^k) = s$, because $s^k(s^k)^*$ and $1 - s^k(s^k)^*$ are all Murray–von Neumann equivalent for $k \in \mathbb{N}$, by [**172**].

The existence of ψ_k follows from (ii): Let $\psi_k := h_{k-1}$ for $u_{k-1} := (v_k)^*$.

ψ_k is uniquely determined by $\psi_k(b) = t^*v_kh^k(b)(v_k)^*t$.

Ad(iv): By definition of φ, we have $\varphi(1+\tau)(b) = u(\tau)^*(b \oplus h(\tau)(b))u(\tau)$, where $u \colon [0,1] \to \mathcal{O}_2$ is a continuous map into the unitaries of O_2 with $u(0) = 1$,

$u(1)s^2 = s$, and $h\colon [0,1] \to \mathrm{Hom}(B, \mathcal{O}_2)$ is a strongly continuous map into the unital *-monomorphisms from B into $\mathcal{O}_2$ with $h(0) = h_0$.

If u_1, u_2 in $\mathcal{O}_2$ are unitaries with $u_0^* s = s^k$, $u_1^* s = s^{k+1}$ such that

$$\lambda(\tau)(b) = u(\tau)^*(b \oplus \psi(\tau)(b))u(\tau) \tag{*}$$

where $\psi\colon \tau \mapsto \psi(\tau) \in \mathrm{Hom}(B, \mathcal{O}_2)$, and $u\colon \tau \mapsto u(\tau) \in U(\mathcal{O}_2)$ are chosen with $u(0)^* s = s^k$, $u(1)^* s = s^{k+1}$ then $u(0) = (1 \oplus v_0)u_0$, $u(1) = (1 \oplus v_1)u_1$, where v_0 and v_1 are unitaries in $\mathcal{O}_2$.

Let $v\colon [0,1] \to U(\mathcal{O}_2)$ be continuous with $v(0) = v_0$, $v(1) = v_1$, then $\chi(\tau)(b) = w(\tau)^*(\oplus\chi(\tau)(b))w(\tau)$ where $\chi(\tau)(b) := v(\tau)^*\psi(\tau)(b)v(\tau)$, $w(\tau) := (1 \oplus v(\tau))^* u(\tau)$.

Thus in all cases in question it suffices to find a decomposition (*) for at least one pair $(\psi(\tau), u(\tau))$ for every $\tau \in O_2$, such that $\psi(\tau)$ is a unital monomorphism and $u(0)s^k = s$, $u(1)s^{k+1} = s$. This is the case for $\lambda(\tau) := \varphi(1+\tau)$ by definition of φ. If $\lambda(\tau)$ has the decomposition (*) of the desired type, then (by a simple computation) $h\lambda(\tau) = u_1(\tau)^*(b \oplus \psi_1(\tau)(b))u_1(\tau)$ and $\lambda(\tau) = u_2(\tau)^*(b \oplus \psi_2(\tau)(b))u_2(\tau)$ where $u_1(\tau) := w(u(\tau) \oplus 1)$, $u_2(\tau) := wu(\tau)$, $\psi_1(\tau) := \psi(\tau)(b) \oplus h_0(\lambda(\tau)(b))$, $\psi_2(\tau) := h_0(b) \oplus \psi(\tau)((b))$ and $w := s(t^*)^2 + tst^*s^* + t^2 s^*$. Thus by induction we get (iv). □

Now we define a unital C^*-morphism $\varphi\colon B \to \mathrm{C_b}(\mathbb{R}_+, B)$ by $\varphi(b)(t) := h(b)$ for $t \in [0,1]$ $\varphi(b)(n) := h^n(b)$ for $n = 1, 2, \ldots$ and $\varphi(b)(n+t) := u(t)\varphi(b)(n)u(t)^*$ for $n = 1, 2, \ldots$, $t \in [0,1)$, where $u(t)$ is as in Lemma 11.2.4(ii). The definition fits because $u(0) = 1$.

Then by the above Lemma 11.2.4(iv) we have $\varphi(b) = w(b \oplus_{s,t} \psi(b))w^*$, where $w \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_2)$ is defined inductively by $w(t) = 1$ for $t \in [0,1]$, $w(n+t) = u(t)w(n)$ for $n = 1, 2, \ldots$, $t \in [0,1)$ (note that $u(0) = 1$), $w(n+t) = u(1/2)w(n)$ for $n = 1, 2, \ldots$, $t \in (1/2, 1]$ and where $\psi(b) := t^* w^* \varphi(b) w t$ is a unital C^*-morphism from B into $\mathcal{O}_2 \subset B$.

Thus by Lemma 11.2.4(i) there exists a unitary $v \in \mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_2)$ such that $\psi(b) - v h_0^u(b) v \in \mathrm{C_0}(\mathbb{R}_+, \mathcal{O}_2)$ for every $b \in B$ and thus $\varphi(b) - (w(1 \oplus v))h(b)(w(1 \oplus v))^*$ is in $\mathrm{C_0}(\mathbb{R}_+, \mathcal{O}_2)$ for every b in B, and $w(1 \oplus v)$ is a unitary in $\mathrm{C_b}(R_+, O_2)$.

Lemma 11.2.5. *There exist unital C^*-morphisms*

$$\theta_n\colon B \to \mathrm{C}([0,1] \times [0,1], B)$$

that satisfy the following boundary conditions for $\sigma \in [0,1]$, $\tau \in [0,1]$:

$\theta_n(0, \sigma) = h\varphi(1 + n\sigma)$, $\theta_n(1, \sigma) = h\varphi(1 + (n+1)\sigma)$,

$\theta_n(\tau, 0) = h^2$, and $\theta_n(\tau, 1) = h^{n+1}\varphi(1 + \tau)$.

Proof. By Lemma 11.2.1 and Corollary 11.2.3 it is enough to show that there is a continuous map $u\colon \partial([0,1] \times [0,1]) \to \mathcal{O}_2$ into the unitaries of $\mathcal{O}_2$ and a strongly continuous map $h\colon \partial([0,1] \times [0,1]) \to \mathrm{Hom}(B, \mathcal{O}_2) \subset \mathcal{L}(B, \mathcal{O}_2)$ into the unital *-monomorphisms from B into $\mathcal{O}_2$, such that the boundary conditions on θ_n are given by $\psi\colon \partial([0,1] \times [0,1]) \to \mathrm{Hom}(B, B)$ with $\psi(b)(\tau, s) = u(\tau, s)^*(b \oplus h(\tau, s)(b))u(\tau, s)$

for $(t,s) \in \partial([0,1]\times[0,1])$ because then $\theta_n(b)(\tau,s) := w(\tau,s)^*(b\oplus k(\tau,s)(b))w(\tau,s)$, $(\tau,s) \in [0,1]\times[0,1]$ is as desired, where $w\colon [0,1]\times[0,1] \to U(\mathcal{O}_2)$ is a continuous extension of u and $k\colon [0,1]\times[0,1] \to \mathrm{Hom}(B,\mathcal{O}_2)$ is a strongly continuous map from $[0,1]\times[0,1] \cong cone(\partial([0,1]\times[0,1]))$ into the unital *-monomorphisms from B into $\mathcal{O}_2$ such that $h = k|\partial([0,1]\times[0,1])$.

On the sides of the square $[0,1]\times[0,1]$, the boundary conditions are given by strongly continuous maps into $\mathrm{Hom}(B,B)$, which coincide on the vertices: h^2 in $(0,0)$ and $(1,0)$, h^{n+2} in $(0,1)$ and h^{n+3} in $(1,1)$. By the construction of φ we have $\varphi(1+k+\xi) = \varphi(1+\xi)h^k$ for $\xi \in [0,1]$. Thus we find $m = 2n+3$ points $z_1, z_2, \ldots, z_m$ on $\partial([0,1]\times[0,1])$ (containing the vertices $z_1 = (0,0), z_2 = (1,0)$) such that $\lambda_k(\tau) := \theta_n(z_{k+1} + \tau(z_{k+1} - z_k))$ is one of the following functions: $\lambda_1(\tau) \equiv h^2, \lambda_2(\tau) = \varphi(1+\tau)h,\ \lambda_3(\tau) = \varphi(1+\tau)h^2,\ \ldots\ ,\ \lambda_{n+2}(\tau) = \varphi(1+\tau)h^{n+1}$, $\lambda_{n+3}(\tau) = h^{n+1}\varphi(2-\tau),\ \lambda_{n+4}(\tau) = h\varphi((n+1)-\tau) = h\varphi(2-\tau)h^{n+1},\ \ldots\ ,$ $\lambda_{2n+3}(\tau) = h\varphi(2-\tau)$.

Now apply Lemma 11.2.4(iv) to this function to get the above desired particular form.

□

3. Proof of (β)

We construct in some sense a "completely positive liftable unsuspended E-equivalence" as an element of $\mathrm{R}(P(A),B)$, given by by a path of u.c.p. maps $V_t\colon P(A) \to B$ with the property that $V_t \circ h_{1,\infty}\colon B \to B$ is equivalent in $\mathrm{R}(B,B)$ to $h\colon B \to B$. for the maps $h_{1,\infty}\colon B \to P(A)$ and $h\colon B \to B$:

Since $h\colon B \to B$ is a KK-equivalence, (and is equivalent to id_B in $\mathrm{R}(B,B)$), it is sufficient to define a unital completely positive map $V\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+,B)$ with the following properties:

(i) $V(b^*b) - V(b)^*V(b) \in \mathrm{C_0}(\mathbb{R}_+,B)$ for every $b \in P(A)$, i.e., V defines a unital and completely positively liftable homomorphism from $P(A)$ into $\mathrm{Q}(\mathbb{R}_+,B)$.

(ii) There are a *- homomorphism $\varphi_1\colon B \to \mathrm{C_b}(\mathbb{R}_+,B)$ and a unitary $U \in \mathrm{C_b}(\mathbb{R}_+,\mathcal{O}_2)$ such that $\varphi_1(a) - U^*h(a)U \in \mathrm{C_0}(\mathbb{R}_+,B)$

(Is not necessary! "Stable" homotopy of φ_1 with $h\colon B \to B$ is enough.)

and

$\{\varphi_1(a)(t) - V(h_{1,\infty}(a))(t)\} \in \mathrm{C_0}(\mathbb{R}_+,B)$ for $a \in B$, $\tau \in \mathbb{R}_+$.

(It gives – by $\mathrm{R}(C,D) = \mathrm{KK}(C,D)$ for unitally $\mathcal{O}_\infty$-containing C,D – $[V]\circ[h_{1,\infty}] = [h] = [\mathrm{id}_B]$ in $\mathrm{KK}(B,B)$)

This is equivalent to $V \circ h_{1,\infty}$ homotopic to $U^*h(\cdot)U$ for suitable path $U(t)$. Thus $[h] = [V]\circ[h_{1,\infty}]$ in $\mathrm{KK}(B,B)$

(iii) Need a proof that $h_{1,\infty}(V(\cdot)(t))$ is homotopic to $[\mathrm{id}_{P(A)}]$ in $\mathrm{R}(P(A),P(A))$.

(A kind of "reconstruction".)

(iv) Another way to describe it would be the following:

There is unital completely positive $W\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+ \times [0,1], P(A))$, such that $\lim_{\tau\to\infty} \|W(b^*b)(\tau,t) - W(b)(\tau,t)^* W(b)(\tau,t)\| = 0$

and

$\lim_{\tau\to\infty} \|W(b)(\tau,1) - b\| = 0$

and

$\lim_{\tau\to\infty} \|W(b)(\tau,0) - h_{1,\infty}(V(b)(\tau))\| = 0.$

This means that $[\pi \circ V] \in \mathrm{Ext}^{-1}(P(A), SB) \cong \mathrm{KK}(P(A), B)$ has the property that

$$[h_1^\infty] \otimes_B [\pi \circ V] = [\mathrm{id}_{P(A)}] \in \mathrm{KK}(P(A), P(A))$$

and

$$[\pi \circ V] \otimes_{P(A)} [h_1^\infty] = [h] \cong [\mathrm{id}_B] \in \mathrm{KK}(B, B).$$

Thus Theorem I is proved if we have shown the existence of V, φ_1, U and W with the above listed properties (i)-(iii).

We define the unital completely positive map $V\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+, B)$ by the following constructions.

For $n = 1, 2, \ldots$ let $\varphi_n(b)(\tau) := \varphi(b)(\tau - n)$ for $\tau \geq n$ and $\varphi_n(b)(\tau) := h(b)$ for $0 \leq \tau < n$. Then $\varphi_n(n + k) = h^k(b)$, $\varphi_n(b)(n + \tau) = h(b)$ for $k = 1, 2, \ldots$, $\tau \in [0, 1]$ and $\varphi_{n+k}(h^k(b))(\tau) = \varphi_n(b)(\tau)$ for $\tau \geq n + k$. We let $V_n := \varphi_n(h_n^\infty)^{-1} E_n$.

Then $V_n h_n^\infty = \varphi_n$, and we have $V_m(h_n^\infty(b))(\tau) = V_n(h_n^\infty(b))(\tau)$ for $m > n$, $\tau \geq m$:

$$V_m \circ h_n^\infty = V_m \circ h_m^\infty h^{m-n} = \varphi_m h^{m-n}.$$

For the parameter $\tau \geq m$ we get $\varphi_m(h^{m-n}(b))(\tau) = \varphi_n(b)(\tau) = V_n(h_n^\infty(b))(\tau)$. Let

$$V(d)(\tau) := (n + 2 - \tau) V_n(d)(\tau) + (\tau - n - 1) V_{n+1}(d)(\tau)$$

for $n + 1 \leq \tau \leq n + 2$ and $V(d)(\tau) := V_1(d)(\tau)$ for $0 \leq \tau \leq 2$.

The definition is fits, because on the boundary points of the intervals in question we have $V(d)(n+1) = V_n(d)(n+1)$ and $V(d)(n+2) = V_{n+1}(d)(n+2)$ for $n = 1, 2, \ldots$.

Then $V\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+, B)$ is a unital completely positive map and

$$V(h_n^\infty(b))(\tau) = \varphi_n(b)(\tau)$$

for $\tau \geq n+1$, $n \geq 1$, because if $\tau \geq n+1$ then there is an $m \in \mathbb{N}$ with $n+1 \leq m+1 \leq \tau \leq m+2$ and for this m we get $V(h_n^\infty(b))(\tau) = (m+2-\tau) V_m(h_n^\infty(b))(\tau) + (\tau - m - 1) V_{m+1}(h_n^\infty(b))(\tau)$ and the right hand side is equal to $V_n(h_n^\infty(n))(\tau) = \varphi_n(b)(\tau)$.

In particular, since $P(A)$ is the closure of $\bigcup h_n^\infty(B)$, we get that V is asymptotically multiplicative, i.e., $V(d^*d) - V(d)^*V(d) \in \mathrm{C_0}(\mathbb{R}_+, B)$ for $d \in P(A)$.

Thus V is an asymptotic morphism (which is not nuclear if B is not nuclear) from $P(A)$ in B with $V(h_1^\infty(b)) - \varphi_1(b) \in \mathrm{C_0}(\mathbb{R}_+, B)$. As we have seen above, φ_1 (as a scaling of φ) is unitarily equivalent to $h\colon B \to B \subset \mathrm{C_b}(\mathbb{R}_+, B)$ by a unitary U in $\mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_2)$. Thus

$$[V] \otimes_{P(A)} [h_1^\infty] = [h] = [\mathrm{id}_B]$$

in $\mathrm{KK}(B, B)$.

We construct an asymptotic morphism

$$W\colon P(A) \to \mathrm{C_b}(\mathbb{R}_+ \times [0,1], P(A))$$

with

$$W(d)(\tau, 0) - h_1^\infty(V(d)(\tau)) \in \mathrm{C}_0(\mathbb{R}_+, P(A)) \quad \text{and} \quad W(d)(\tau, 1) - d \in \mathrm{C}_0(\mathbb{R}_+, P(A)).$$

We use the unital C^*-morphisms $\theta_n\colon B \to \mathrm{C}([0,1] \times [0,1], B)$ of Lemma 11.2.5.

Let us list some equations for V, h_n^∞, φ and θ_n that we have shown above or are obtained by straight forward calculations from the definitions:

Just from the definition of φ, φ_n, V and from $V(h_n(b))(n+1+\sigma) = \varphi_n(n+1+\sigma)$ we get $\varphi(1+\sigma)(b) = V(h_n^\infty(b))(n+1+\sigma))$ for $\sigma \in [0,1]$.

Then we need moreover the equations $h_{m+k}^\infty(h^k(b)) = h_m(b)$ and the boundary conditions listed in Lemma 11.2.5.

From the boundary conditions it follows that $h(\theta_n(b)(1,\tau)) = \theta_{n+1}(h(b))(1,\tau)$.

At first we define unital, completely positive maps W_n.

Let $W_n\colon P(A) \to \mathrm{C}([n+1, n+2] \times [0,1], P(A))$ be defined for $\tau, \sigma \in [0,1]$ by

$$W_n(d)(n+1+\sigma, \tau) := h_{n+2}^\infty(\theta_n((h_n^\infty)^{-1} E_n(d))(\sigma,\tau)).$$

It implies

$$W_n(h_n^\infty(b))(n+1+\sigma,\tau) = h_{n+2}^\infty(\theta_(b)(\sigma,\tau)),$$

and from $E_n(P(A)) = h_n^\infty(B)$ and the above listed equations (and the boundary conditions of θ_n in Lemma 11.2.5) the reader easily get by straight forward calculations (by replacing $E_n(d)$ by $h_n^\infty(b)$) the following three equations (where $n \in \mathbb{N}$, $d \in P(A)$, and $\sigma, \tau \in [0,1]$):

$$W_n(E_n(d))(n+1+\sigma, 0) = E_n(d),$$

$$W_n(E_n(d))(n+1+\sigma, 1) = h_1^\infty(V(E_n(d))(n+1+\sigma))$$

and

$$W_n(E_n(d))(n+2,\tau) = W_{n+1}(E_n(d))(n+2,\tau).$$

The latter equation shows that

$$W(d)(n+1+\sigma,\tau) := W_n((1-\sigma)E_{n-1}(d) + \sigma E_n(d))(n+1+\sigma,\tau)$$

for $n = 2, 3, \ldots$ and $\sigma, \tau \in [0,1]$ and $W(d)(\eta,\sigma) := W_2(E_1(d))(3,\sigma)$ for $\tau, \sigma \in [0,1]$, $0 \le \eta \le 3$ is a well-defined completely positive and unital map from $P(A)$ into $\mathrm{C_b}(\mathbb{R}_+, P(A))$.

Since the union of the $E_n(P(A))$ is dense in $P(A)$ we see from first two the above listed three equations for the W_n's that

$$\lim_{\sigma\to\infty} \|W(d)(\sigma,0) - d\| = 0$$

and

$$\lim_{\sigma \to \infty} \| W(d)(\tau, 1) - h_1^{\infty}(V(d)(\tau)) \| = 0$$

for every $d \in P(A)$.

This means $[h_1^{\infty} \circ (\pi \circ V)] = [\mathrm{id}]$ in $\mathrm{KK}(P(A), P(A))$. On the other hand $[h_1^{\infty} \circ (\pi \circ V)] = [h_1^{\infty}] \otimes_B [V]$.

This completes the proof of (β) and ends the proof of Theorem I.

4. Examples via Cuntz-Pimsner algebras

The following (X, G)-equivariant construction of generalized Fock–Toeplitz algebra (respectively of a Cuntz–Krieger–Pimsner algebra) using a suitable Hilbert bi-module.

THEOREM 11.4.1. *Text from lectures on equivariant reconstruction.*

PROOF. □

Cite from Chapter 1: One starts with a suitable "universal" Hilbert bimodule and builds the corresponding Cuntz–Krieger–Pimsner algebra.

The universality of the construction has the advantage that G-actions of locally compact groups G on $(A_x)_{x\in X}$ lead in a natural way to G-actions on $(P(A_x)\otimes \mathbb{K})_{x\in X}$

Next taken from Toronto 2014 lectures

Definition (in Chp.1??) of regular subalgebras:
Let $C \subset A$ a C^*-subalgebra. C is regular for A if

(i) C separates the ideals J of A: $J_1 \cap C = J_2 \cap C$ implies $J_1 = J_2$.
(ii) $C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)$ for all $J_1, J_2 \in \mathcal{I}(A)$.

THEOREM 11.4.2 (Realization of Ψ, H.H.,E.K.). *Suppose that B is separable and stable. Let $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ a non-degenerate lower s.c. action of* Prim(B) *on A.*

*If $B \otimes \mathcal{O}_2$ contains a regular abelian C^*-subalgebra C then $\Psi = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathcal{C}_\Psi$. In particular, Ψ comes from a non-degenerate *-monomorphism $h \colon A \otimes \mathbb{K} \to \mathcal{M}(B)$, that is unique up to unitary homotopy of its infinite repeats.*

COROLLARY 11.4.3 (Reconstruction Theorem, H.H.,E.K.). *Suppose that A is a nuclear and stable, that Ω is a* sup–inf *closed sub-lattice of $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ with* $\mathrm{Prim}(A), \emptyset \in \Omega$. *Then there is a non-degenerate *-monomorphism $H_0 \colon A \to \mathcal{M}(A)$ with following properties:*

(i) *The infinite repeat $\delta_\infty \circ H_0$ is unitarily equivalent to H_0.*
(ii) *For every $U \in \mathbb{O}(\mathrm{Prim}(A))$ holds $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$ where $V \in \Omega$ is given by $V = \bigcup\{W \in \Omega\,;\ W \subset U\}$.*

The H_0 is uniquely determined up to unitary homotopy.

The uniques up to unitary homotopy means by Definition 5.0.1:

If $H_1 \colon A \to \mathcal{M}(A)$ also satisfies the conditions (i) and (ii) then there is a norm-continuous path $t \in \mathbb{R}_+ \to U(t) \in \mathcal{U}(\mathcal{M}(A))$ such that $U(t)^* H_2(a) U(t) - H_0(a) \in A$ for all $a \in A$ and $t \in \mathbb{R}_+$ and $\lim_{t\to\infty} U(t)^* H_2(a) U(t) = H_0(a)$.

COROLLARY 11.4.4 (Continuation of Reconstruction Theorem). *The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ of the Hilbert A-A-module $\mathcal{H} := (A, H_0)$ is stable and strongly purely infinite; and it is the same as the C^*-Fock algebra $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$.*

The natural embedding of A into $\mathcal{O}_{\mathcal{H}}$ defines a lattice isomorphism from Ω onto $\mathbb{O}(\mathrm{Prim}(\mathcal{O}_{\mathcal{H}}))$ and is a $\mathrm{KK}(X;\cdot,\cdot)$-equivalence for $X := \mathrm{prime}(\Omega) \cong \mathrm{Prim}(\mathcal{O}_{\mathcal{H}})$.

`Discussion of some general ways:`

The algebra A will be changed to nice Cuntz "standard" form $B := A_t$ up to KK, such that $\mathcal{O}_2 \subseteq \mathcal{M}(B)$, $\alpha\colon A \to B$ is $\mathrm{KK}(\mathcal{C}\,;\,A,B)$ a KK-equivalence (!!!!), e.g. given by tensoring with $\mathcal{O}_2 \otimes \mathbb{K}$ and "adding" $\mathcal{O}_2$ or $\mathcal{O}_2 \otimes \mathbb{K}$, depending on $\mathcal{C}$.

Better would be: Can it be chosen stably homotopic to direct sum $A \oplus_{s_1,s_2} \mathcal{O}_2$ sitting in $B = A_t$?

`What is the inverse? Can stabilize ? Use` $\mathcal{C}\widehat{\otimes}\,\mathrm{CP}(\mathcal{O}_\infty)$`?`

– after stabilization also an E-equivalence?? –

and the endomorphism $\gamma := \mathrm{id}_B \oplus \gamma_0$, γ_0, $\mathcal{C}$-compatible "inner homotopic": γ_0^2 and $\gamma_0 \oplus \gamma_0$ "controlled" (inner?) homotopic to γ_0, γ should have has in $\mathrm{KK}(\mathcal{C};\,B,B)$ the same class as id_B.

$C := \mathrm{indlim}_n(\gamma_n\colon B \to B)$ with $\gamma_n = \gamma$, $\tau\colon C \to \mathrm{Q}([0,\infty),B)$ by intertwining γ_n and γ_{n+1} in suitable way and using conditional expectations onto ????,

given by moving the conditional expectations from C onto $\gamma_{n,\infty}(B)$ to that of with $\gamma_{1,\infty}\colon B \to C$, satisfies $\tau \circ \gamma_{1,\infty}\ {}_h\epsilon_B$ in a controlled sense ("inner" sense ?), with $\epsilon_B(b) = b + B[0,\infty)$.

An idea would be convex combinations $T_s := (n+1-s)T_n + (s-n)T_{n+1}$ for $s \in [n,n+1]$ as maps from C into $B[n,n+1]$, where $T_n := \gamma_{1,n}^{-1} \circ P_n\colon C \to B$, $P_n\colon C \to \gamma_{1,n}(B) \subset C$.

Thus, $[\epsilon_B] = [\mathrm{id}_B]$ in $E(B,B) \cong \mathrm{KK}(\mathcal{C};B,B)$, $[\tau] \circ [\gamma_{1,\infty}] = [\epsilon_B] = [\mathrm{id}_B]$, $\gamma_{1,\infty}\colon\ \mathrm{Q}([0,\infty),B)$

`Does it explain the KK-equivalence ??`

It gives classes X =???? $\in \mathrm{KK}(\mathcal{C}\,;\,C,B)$, $Y \in \mathrm{KK}(\mathcal{C}\,;\,B,C)$ inclusion, with $Y = [\gamma_{1,\infty}]$, $X = [\tau]$ corresponds to asymptotic morphism from C to B.

that is $\mathrm{KK}(\mathcal{C};\cdot,\cdot)$ equivalent to A.

B has equivariant endomorphism that is $\mathrm{KK}(\mathcal{C};\,\cdot,\cdot)$ equivalent to id_B, ...

Use the "fact" that certain kinds of ??? and a non-commutative variant of a classical result of J. Milnor ???? given by L.G. Brown [**?**]. give simply ref to Blackadar Rosenberg [1987, 1.12]. not found????

Countable additivity w.r.t. first variable: Blackadar's K-theory book, [**73**, thm. 19.7.1]

Continuity in first variable [**73**, thm. 21.5.2] with Milnor $\lim^1$-sequence.

CHAPTER 12

Non-commutative Selection and Proof of Thm. K

In this chapter we use the in Chapters 4 -11 confirmed results for a complete proof of Theorems K (e.g. using Corollary 6.3.2) and Theorem O in Chapter 1 (e.g. using Theorem **??**). As it was explained in the introductory Chapter 1, this completes the proofs of *all* the results stated in Chapter 1, because we have already proved Theorem M in Chapter 9 under the *additional assumption*, that *there exists* $h_0: A \hookrightarrow B$ with the properties that h_0 is injective, non-degenerate, nuclear, is unitarily homotopic to $h_0 \oplus h_0$, and realizes the given non-degenerate action $\Psi: \mathcal{I}(B) \to \mathcal{I}(A)$ via $\Psi(J) = h_0^{-1}(J \cap h_0(A))$. The existence of h_0 is now the main point of Theorem K, because we have seen the uniqueness of such h_0 up to unitary homotopy in Chapter 9. Moreover we have shown there, that, for σ-unital stable B, there is — up to unitary equivalence (by unitaries in $\mathrm{Q}(\mathbb{R}_+, \mathcal{M}(B))$) — at most one nuclear *-monomorphism $k_0: A \to \mathrm{Q}(\mathbb{R}_+, B)$ with k_0 approximately unitarily equivalent (by unitaries in the bigger algebra $\mathcal{M}(\mathrm{Q}(\mathbb{R}_+, B))$) to $k_0 \oplus k_0$, such that k_0 realizes the given action Ψ of $\mathrm{Prim}(B)$ on A via $\Psi(J \cap B) = k_0^{-1}(J \cap h_0(A))$ for any closed ideal J of $\mathrm{Q}(\mathbb{R}_+, B)$. We have seen in Chapter 9, that this properties of k_0 imply that such k_0 must be unitarily equivalent to some $h_0: A \to B$ of the desired kind ([1]).

The Theorem 6.3.1 implies the existence of $h_0: A \to B$ (realizing Ψ) under the *additional* assumptions that B is separable and contains a regular abelian C^*-subalgebra in the sense of Definition 1.2.9.

Thus, to complete the proof of Theorem K, we must show only the following (but get more general) results:

(i) B can be replaced by a suitable separable subalgebra B_0 of B and Ψ by a suitable action Ψ_0 of $\mathrm{Prim}(B_0)$ on B_0, i.e., there exists a *separable* C^*-subalgebra B_0 of B such that
  (a) B_0 is stable and contains a strictly positive element of B.
  (b) Each ideal I of B_0 is the intersection $I = B_0 \cap J$ of an ideal J of B with B_0.
  (c) $\Psi(J_1) = \Psi(J_2)$ if $J_1 \cap B_0 = J_2 \cap B_0$, $J_1, J_2 \in \mathcal{I}(B)$.
  (d) B_0 is strongly purely infinite.

[1] Combine Corollaries 9.1.6 and 9.1.4 – with the B in 9.1.4 replaced by our $k_0(A)\,\mathrm{Q}(\mathbb{R}_+, B) k_0(A)$.

(2) If B is separable ([2]), then we show the existence of a separable C^*-subalgebra $B_1 \subset \mathrm{Q}(\mathbb{R}, B)$ such that
  (e) $B \subset B_1$ and B contains a strictly positive element of B_1 (in particular, B_1 is stable).
  (f) Each ideal I of B_1 is the intersection $I = B_1 \cap J$ of an ideal J of $\mathrm{Q}(\mathbb{R}_+, B)$ with B_1.
  (g) B_1 contains a regular abelian C^*-algebra $C \subset B_1$.
  (h) B_1 is strongly purely infinite.

If we consider B_1 (in place of B) and the action $\Psi_1(I) := \Psi(I \cap B)$ of $\mathrm{Prim}(B_1)$ on A, then it turns out that the action Ψ_1 is non-degenerate, lower semi-continuous and monotone upper semi-continuous ([3]). Thus, Theorem 6.3.1 (respectively Corollary 6.3.2 — a special case of Theorem K) applies to (A, B_1, Ψ_1) and gives that there is nuclear

$$k_0 \colon A \hookrightarrow B_1 \subset \mathrm{Q}(\mathbb{R}_+, B_0) \subset \mathrm{Q}(\mathbb{R}_+, B)$$

with k_0 unitarily equivalent to $k_0 \oplus k_0$, and $k_0(A) \cap J = k_0(\Psi_1(B_1 \cap J)) = k_0(\Psi(J \cap B))$ for all closed ideals J of $\mathrm{Q}(\mathbb{R}_+, B)$. Then Corollaries 9.1.6 and 9.1.4 imply that there is non-degenerate nuclear $h \colon A \otimes \mathcal{O}_2 \to B$ such that $h_0 := h((\cdot) \oplus 1)$ is unitarily equivalent to k_0. Thus, h_0 is as desired, and Theorem K follows from the existence of (B_0, Ψ_0) and (B_1, Ψ_1) with the above listed (1a)–(2h).

The technical preparations for the existence of (B_0, Ψ_0) and (B_1, Ψ_1) lead also to the asymptotic non-commutative 'selection' Theorem 12.1.8.

The below given characterization of the Dini functions on $\mathrm{Prim}(A)$, Proposition 12.2.6, the reduction result in Lemma 12.2.14, together with the results in the last sections of Chapters 5 and 6 yield a proof of Proposition 12.2.15.

For notations we refer the reader to Chapter 1, Definitions 1.2.1–1.2.8.

1. Non-commutative asymptotic Selection

First we need some lemmata for our proof of the asymptotic non-commutative Selection Theorem 12.1.8.

Lemma 12.1.1. *Suppose that $H \colon A \to \mathcal{M}(B)$ is a C^*-morphism, where A and B are σ-unital. For $J \in \mathcal{I}(B)$, let*

$$\Psi(J) := H^{-1}(\Psi^{\mathrm{up}}_{B,H(A)}(J)) := H^{-1}(H(A) \cap \mathcal{M}(B, J)).$$

(i) *Ψ is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, i.e., it satisfies parts (iii) and (iv) of Definition 1.2.6.*

[2]The separability of B can be supposed, because we can replace B by B_0, and Ψ by $\Psi_0(J) := \Psi(K)$ for arbitrary $K \triangleright B$ with $J = B_0 \cap K$. The properties (b) and (c) imply that Ψ_0 is well-defined and is lower s.-c. and monotone upper s.-c., because $\Phi(J) := \overline{\mathrm{span}(BJB)}$ is upper semi-continuous, satisfies $B_0 \cap \Phi(J) = J$, and $B_0 \cap \bigcap_\tau \Phi(J_\tau) = \bigcap_\tau J_\tau$, and Ψ is lower s.-c. and monotone upper s.-c.

[3]The l. s.-continuity and monotone u. s.-continuity of Ψ_1 follows from the that continuity properties of Ψ and of $I \mapsto I \cap B$.

(ii) $\Psi(0) = \ker(H)$, *and* $\Psi^{-1}(A)$ *is the set of* $J \in \mathcal{I}(B)$ *with* $BH(A)B \subset J$.

(iii) Ψ *satisfies (ii) of Definition 1.2.6 if* $H(A) \subset B$.

(iv) *If* C *is a* σ*-unital* C^**-subalgebra of* $\mathcal{M}(B)$*, such that* CB *is dense in* B*,* $H(A) \subset \mathcal{M}(C)$ *(i.e.* $H(A)C \subset C$*), and, for* $I \in \mathcal{I}(C)$*,*

$$\Phi(I) := H^{-1}(\Psi^{\mathrm{up}}_{C,H(A)}(I)) \;=\; H^{-1}(H(A) \cap \mathcal{M}(C, I))\,,$$

then, for $J \in \mathcal{I}(B)$*,*

$$\Psi(J) = \Phi(\Psi^{\mathrm{up}}_{B,C}(J))\,.$$

(v) *The morphism* $H \colon A \to \mathcal{M}(B)$ *is weakly* Ψ*-residually nuclear if* H *satisfies at least one of the following properties (a)–(c):*

(a) $H \colon A \to \mathcal{M}(C)$ *is* Φ*-residually nuclear.*

(b) A *is exact and* $H \colon A \to \mathcal{M}(C)$ *is weakly nuclear.*

(c) C *is nuclear.*

Proof. (i)-(iv) follow straight from the definitions. The details are left to the reader. E.g., (iv) follows from the easily poved identity

$$\mathcal{M}(C) \cap \mathcal{M}(B, J) = \mathcal{M}(C, C \cap \mathcal{M}(B, J))$$

for *non-degenerate* C^*-subalgebras C of $\mathcal{M}(B)$.

Note that (iv) implies (v,a), because, for $J \in \mathcal{I}(B)$, $I := \Psi^{\mathrm{up}}_{B,C}(J)$, we have the natural inclusions

$$[H] \colon A/\Phi(I) \hookrightarrow \mathcal{M}(C/I) \subset \mathcal{M}(B/J),$$

the map $a + \Phi(I) \mapsto b^*(d^*(H(a) + \mathcal{M}(C, I))d)b$ is a nuclear map from $A/\Phi(I)$ into B/J for $b \in B/J$, $d \in C/I \subset \mathcal{M}(B/J)$, and C/I is a non-degenerate C^*-subalgebra of $\mathcal{M}(B/J)$.

It is obvious that (v,c) implies (v,a). By Proposition **??**, $H \colon A \to \mathcal{M}(C)$ is (norm-) nuclear if A is exact and H is weakly nuclear. Then all the quotient maps $[H] \colon A/\Phi(I) \to \mathcal{M}(C)/\mathcal{M}(C, I) \subset \mathcal{M}(C/I)$ are nuclear by exactness of A (cf. Remark **??**). □

Lemma 12.1.2. *Suppose that* A *and* B *are stable and separable* C^**-algebras, and that* $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ *is a lower semi-continuous action of* $\mathrm{Prim}(B)$ *on* A*, such that* $\Psi(0) = 0$ *and* $\Psi^{-1}(A) = \{B\}$*.*

Then the following are equivalent:

(i) *There exists a non-degenerate weakly* Ψ*-residually nuclear *-monomorphism* $H_0 \colon A \to \mathcal{M}(B)$*, such that* $\delta_\infty H_0$ *is unitarily equivalent to* H_0*, and, for* $J \in \mathcal{I}(B)$*,*

$$\Psi(J) = H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J)).$$

(ii) *For every* $J \in \mathrm{Prim}(B)$*,* $a \in A \setminus \Psi(J)$ *there exist a* Ψ*-residually nuclear completely positive map* $V \colon A \to B$ *such that* $V(a) \notin J$*.*

If H_0 *with (i) exists, then* H_0 *is unique up to unitary homotopy.*

Proof. (i)⇒(ii): If $a \in A$ and $H_0(a) \notin \mathcal{M}(B, J)$, then there exists $b \in B_+$ with $bH_0(a)b \notin J$. $V = bH_0(.)b$ is a Ψ-residually nuclear map from A to B with $V(a) \notin J$.

(i)⇒(ii): If I is a closed ideal of B, and if $a \in A$ is not in $\Psi(I)$, then there exists a primitive ideal $J \in \mathrm{Prim}(B)$, such that $I \subset J$ and $a \notin \Psi(J)$, because Ψ is lower semi-continuous and I is the intersection of the primitive ideals J which contain I.

Thus, there is a Ψ-residually nuclear map $V \colon A \to B$ with $V(a) \notin I$.

The set K of Ψ-residually nuclear completely positive contractions V from A to B is a convex set. It is a separable metric space as a subset of $\mathrm{CP}(A, B) \subset \mathcal{L}(A, B)$ with strong topology on $\mathcal{L}(A, B)$ (i.e., topology of point-wise convergence in norm, given by the system of semi-metrics $\rho_a(V_1, V_2) := \|V_1(a) - V_2(a)\|$).

Let $V_1, V_2, \ldots$ be a dense sequence in K. We get weakly Ψ-residually nuclear C^*-morphisms d_n from A to $\mathcal{L}(\mathcal{H}_B)$, if we apply Stinespring-Kasparov dilation to V_n, $n = 1, 2, \ldots$. Here $\mathcal{H}_B$ denotes the (right) Hilbert-B-module which is given by the sequences $(b_1, b_2, \ldots)$ with norm convergent series $\sum b_n^* b_n$ in B. Since B is stable and σ-unital, we get that there is a natural isomorphism $\mathcal{H}_B \cong B$ as (right) Hilbert-B-modules and $\mathcal{L}(\mathcal{H}_B) \cong \mathcal{M}(B)$ as C^*-algebras under this isomorphism. If we use this isomorphism, we get weakly Ψ-residually nuclear C^*-morphisms $h_n \colon A \to B$, such that V_n is the point-norm limit of Ψ-residually nuclear completely positive contractions of form $W = b^* h_n(.)b$, with $b \in B$ a contraction.

Let $h \colon A \to \mathcal{M}(B)$ be the direct sum of $h_1, h_2, \ldots$, i.e., $h(a) := \sum_n s_n h_n(a) s_n^*$ for $a \in A$, where $s_1, s_2, \ldots$ is a sequence of isometries in $\mathcal{M}(B)$ with $\sum s_n (s_n)^*$ strictly convergent to 1 in $\mathcal{M}(B)$.

Then h is weakly Ψ-residually nuclear and every Ψ-residually nuclear completely positive contraction $V \colon A \to B$ is the point-norm limit of maps $b^* h(.)b$ with contractions $b \in B$.

It follows, that $h(\Psi(I)) \subset \mathcal{M}(B, I)$ and $h(a) \notin \mathcal{M}(B, I)$ for $I \in \mathcal{I}(B)$, $a \notin \Psi(I)$.

Thus $h \colon A \to \mathcal{M}(B)$ is a weakly Ψ-residually nuclear *-monomorphism, such that, for $I \in \mathcal{I}(B)$,

$$h(\Psi(I)) = h(A) \cap \mathcal{M}(B, I).$$

The hereditary C^*-subalgebra D of B, which is generated by $h(A)Bh(A)$, is stable, because A is stable.

The existence of an approximate unit in A shows that $h(A)D \subset D$ and that $h(a)Bh(a^*)$ is contained in the closure of $h(a)Dh(a^*)$ for $a \in A$.

Let $k(a) := h(a)|D$ for $a \in A$. It follows, that $k \colon A \to \mathcal{M}(D)$ is a non-degenerate *-monomorphism from A into $\mathcal{M}(D)$, such that, for $J \in \mathcal{I}(B)$ and $a \in A_+$, $k(a) \in \mathcal{M}(D, D \cap J)$ if and only if $h(a) \in \mathcal{M}(B, J)$. Since every closed ideal of D is the intersection of a closed ideal of B with D, we get that k is weakly Φ-residually nuclear for $\Phi := \Psi^{D,B}_{\mathrm{down}}$, and that $\Phi(D \cap J) = \Psi(J)$ for $J \in \mathcal{I}(B)$. In particular, $\Psi(J_1) = \Psi(J_2)$ if $J_1 \cap D = J_2 \cap D$.

The ideal I which is generated by D, satisfies $h(A) \subset \mathcal{M}(B, I)$. Thus $\Psi(I) = A$ and, by assumption, $I = B$. By Corollary 5.5.6, there is a *-isomorphism γ from B onto D, which is approximately inner as a *-monomorphism from B to B and satisfies $\gamma(I) = D \cap I$ for $I \in \mathcal{I}(B)$.

Let H_0 denote the infinite repeat of $\mathcal{M}(\gamma)^{-1}k$. Then H_0 is a non-degenerated weakly $(\Phi \circ \gamma)$-residually nuclear *-monomorphism from A to B with $H_0(a) \in \mathcal{M}(B, I)$, if and only if, $k(a) \in \mathcal{M}(D, \gamma(I))$. Since δ^2 is unitarily equivalent to δ, $\gamma(I) = D \cap I$, and $\Psi = \Phi\gamma$, we get that H_0 has the desired properties.

By Corollary 5.9.15, H_0 with (i) is unique up to unitary homotopy. □

LEMMA 12.1.3. *Suppose that C is a separable commutative C*-algebra, that A is a separable C*-algebra, and that*

$$\Psi \colon \mathcal{I}(C) \cong \mathbb{O}(\mathrm{Prim}(C)) \to \mathcal{I}(A)$$

is a lower semi-continuous action of $X := \mathrm{Prim}(C)$ on A with $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{C\}$.

*Then, there is a *-monomorphism H from A into $\mathcal{M}(C \otimes \mathbb{K})$ such that, for every closed ideal J of C,*

$$H(\Psi(J)) \; = \; H(A) \cap \mathcal{M}(C \otimes \mathbb{K}, J \otimes \mathbb{K})\,.$$

*If A is stable, then H can be taken as a non-degenerate *-monomorphism, i.e., such that $H(A)(C \otimes \mathbb{K})$ is dense in $C \otimes \mathbb{K}$.*

PROOF. Since $J \mapsto J \otimes \mathbb{K}$ defines a lattice isomorphism from $\mathcal{I}(A)$ onto $\mathcal{I}(A \otimes \mathbb{K})$, it suffices to consider the case where A is stable. Moreover, it suffices to show that condition (ii) of Lemma 12.1.2 is satisfied for $B := C \otimes \mathbb{K}$ under the above natural identification $J \otimes \mathbb{K} \leftrightarrow J$ of $\mathcal{I}(B)$ and $\mathcal{I}(C)$.

Note that here $X := \mathrm{Prim}(C)$ is the maximal ideal space of C, and that X itself or its one point compactification $Y := X \cup \{C\}$ is a metrizable compact space.

We define a map g from Y into the set of closed split faces of the quasi-state space $S_q(A) := \{f \in A^* \colon 0 \le f, \|f\| \le 1\}$ of A as follows: For $J \in Y$ let

$$g(J) := \{f \in S_q(A) \colon f(\Psi(J)) = 0\}.$$

Then the lower semicontinuity of Ψ implies that the map $(J, f) \mapsto J$ is an open map from $\{(J, f) \colon J \in Y, f \in g(J)\} \subset Y \times S_q(A)$ onto Y.

Let $a \in A$ and I a closed ideal of C. The lower semicontinuity of Ψ implies that, if $a \notin \Psi(I)$, then there is a maximal ideal J_0 of C such that $I \subset J_0$ and $a \notin \Psi(J_0)$. Thus there is a quasi-state $f_0 \in g(J_0)$ with $f_0(a) = \|a + \Psi(J_0)\|$. Since g satisfies the requirements of the Michael selection principle [**553**], we find a continuous map $J \in Y \mapsto f(J) \in S_q(A)$ such that $f(J) \in g(J)$ and $f(J_0) = f_0$. In particular, $f(C) = 0$. The map

$$V \colon b \in A \mapsto \{f(J)(b)\}_{J \in X} \in \mathrm{C}_0(X) \cong C \otimes p_{11}$$

is a Ψ-equivariant completely positive contraction from A into B with $V(a) \notin I \otimes \mathbb{K}$. □

Lemma 12.1.4. *Suppose that B, C and D are separable C^*-subalgebras of a C^*-algebra G, such that*

(i) *B and D are stable,*
(ii) *D contains B and $\overline{DBD} = D$, and,*
(iii) *C is a commutative C^*-subalgebra of D with the following properties (α) and (β):*
  (α) *For every closed ideals J_1 and J_2 of D, $J_1 \cap C = J_2 \cap C$ implies $J_1 \cap B = J_2 \cap B$.*
  (β) *There exists a sequence of (in D) approximately inner completely positive contractions $V_n \colon D \to C$ with $\lim_{n\to\infty} V_n(a) = a$ for every $a \in C$.*

Suppose moreover that A is a separable stable C^*-algebra and that $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A with $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$.

Let $E := \overline{B \cdot G \cdot B} \subseteq G$ and $\Phi(J) := \Psi(B \cap J)$ for $J \in \mathcal{I}(E)$.

*Then there exists a non-degenerate weakly Φ-residually nuclear *-monomorphism $H_1\colon A \hookrightarrow \mathcal{M}(E)$, such that $\delta_\infty H_1$ is unitarily equivalent to H_1, and, for every closed ideal J of E,*

$$H_1(\Phi(J)) = H_1(\Psi(B \cap J)) = H_1(A) \cap \mathcal{M}(E, J).$$

H_1 with this property is unique up to unitary homotopy.

In particular, for every automorphism φ of E with $\varphi|B = \mathrm{id}_B$, the composition $\mathcal{M}(\varphi)H_1 = \varphi H_1 \varphi^{-1}$ is unitarily homotopic to H_1.

Condition (iii,β) implies that $(J_1 \cap C) + (J_2 \cap C) = (J_1 + J_2) \cap C$ for all closed ideals J_1 and J_2 of D. If a commutative C^*-subalgebra $C \subseteq D$ satisfies this condition and has property (iii,α) then C is called **regular Abelian C^*-subalgebra** of D.

The map $I \lhd C \mapsto J(I) \lhd D$ considered in the *proof* of Lemma 12.1.4 is lower semi-continuous. Thus, the family of c.p. maps $W\colon D \to C$ with $W(J(I)) \subset I$ is closed under point-norm convergence, satisfies $W(J) \subset J \cap C$ for all $J \lhd D$ by $J(J \cap C) \supseteq J$ (hence W is approximately inner in D). Since C is nuclear, it follows that W is residually nuclear.

Can we find, for $c_1, \ldots, c_n \in C_+$ and $\varepsilon > 0$, a contraction $V \in \mathrm{CP_{in}}(A, A)$ with $V(C) \subset C$ and $\|V(c_k) - c_k\| < \varepsilon$?

Proof. The hereditary C^*-subalgebra E of G is σ-unital and stable because B is σ-unital and stable. Therefore, the uniqueness of H_1 follows from Lemma 12.1.2. The φ-invariance up to unitary homotopy follows from the uniqueness.

Let $J_1, J_2 \subset D$ closed ideals, then $C \cap J_i \subset C \cap (J_1 + J_2)$ for $i = 1, 2$. Now let $a \in (J_1 + J_2) \cap C$. There exist $b \in J_1$ and $c \in J_2$ with $a = b + c$. By assumption (iii, β), for $\varepsilon > 0$ there is an approximately inner completely positive contraction $V \colon D \to C$ with $\|V(a) - a\| < \varepsilon$. Then $V(b) \in J_1 \cap C$, $V(c) \in J_2 \cap C$ and $V(a) \in (C \cap J_1) + (C \cap J_2)$. But $(C \cap J_1) + (C \cap J_2)$ is a closed ideal of C.

Thus, for closed ideals J_1 and J_2 of D,

$$C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)\,.$$

Therefore, for every closed ideal I of C, the set of closed ideals J of D with $C \cap J \subset I$ is upward directed. It is easy to see, that this set is also upward monotonous closed. Hence, for every closed ideal I of C, there is a biggest ideal $J(I)$ in the set of closed ideals J of D with $J \cap C \subset I$.

Let $\Psi_D(I) := J(I)$ for $I \in \mathcal{I}(C)$.

The definition of Ψ_D shows that $\Psi_D \colon \mathcal{I}(C) \to \mathcal{I}(D)$ defines a lower semi-continuous action of $\mathrm{Prim}(C)$ on D.

$\mathrm{Prim}(D)$ acts lower semi-continuous on B by $J \in \mathcal{I}(D) \mapsto B \cap J \in \mathcal{I}(B)$.

It follows that the composition with the given action Ψ of $\mathrm{Prim}(B)$ on A,

$$\Psi_A \colon J \in \mathcal{I}(C) \mapsto \Psi(B \cap \Psi_D(J))$$

is a lower semi-continuous action of $\mathrm{Prim}(C)$ on A.

Let $J \in \mathcal{I}(D)$, then, by definition of Ψ_D, $J \subset \Psi_D(C \cap J)$ and $C \cap \Psi_D(C \cap J) \subset C \cap J$. By assumption (iii,$\alpha$), this implies $B \cap J = B \cap \Psi_D(C \cap J)$. Thus, for every $J \in \mathcal{I}(D)$,

$$\Psi(B \cap J) = \Psi_A(C \cap J).$$

In particular, $\Psi_A(0) = \Psi(0) = 0$ and $\Psi_A(C) = \Psi(B) = A$.

Let $I \in \mathcal{I}(C)$ and suppose that $\Psi_A(I) = A$, i.e., $\Psi(B \cap \Psi_D(I)) = A$. Since, by assumption on Ψ, $\Psi^{-1}(A) = \{B\}$, this implies $B \subset \Psi_D(I)$. By (ii) and definition of Ψ_D, it follows $\Psi_D(I) = D$, $C = \Psi_D(I) \cap C \subset I$, and therefore, $(\Psi_A)^{-1}(A) = \{C\}$.

Thus, Ψ_A is a lower semi-continuous action of $\mathrm{Prim}(C)$ onto A with $\Psi_A(0) = 0$ and $(\Psi_A)^{-1}(A) = \{C\}$.

By Lemma 12.1.3, there exists a non-degenerate *-monomorphism H from A into $\mathcal{M}(C \otimes \mathbb{K})$ such that, for $I \in \mathcal{I}(C)$,

$$H(\Psi_A(I)) = H(A) \cap \mathcal{M}(C \otimes \mathbb{K}, I \otimes \mathbb{K}).$$

We have $D \cap E = \overline{BDB}$, because both are hereditary C^*-subalgebras of D which are generated by B. Therefore, $D \cap E$ is σ-unital and stable. By (ii), D is the smallest closed ideal of D which contains B. Let $J := \overline{DCD}$ the closed ideal of D which is generated by C. Then $C \cap J = C = C \cap D$, thus $J \cap B = D \cap B = B$ and $J = D$, by (iii,α) and (ii). It follows, that $(D \cap E) \otimes p_{11}$ and $C \otimes \mathbb{K}$ are stable σ-unital C^*-subalgebras of $D \otimes \mathbb{K}$ which both generate the same ideal $D \otimes \mathbb{K}$.

By Corollary 5.5.6, there is a *-monomorphism

$$k\colon C\otimes\mathbb{K}\hookrightarrow D\cap E$$

such that $k(C\otimes\mathbb{K})$ contains a strictly positive element of $D\cap E$, and that, for every closed ideal J of D,

$$k(C\otimes\mathbb{K})\cap J=k((C\cap J)\otimes\mathbb{K}).$$

Since k is non-degenerate, $\mathcal{M}(k(C\otimes\mathbb{K}))\subset\mathcal{M}(E)$ and, for $I\in\mathcal{I}(G)$,

$$\mathcal{M}(k(C\otimes\mathbb{K}))\cap\mathcal{M}(E,I\cap E)=\mathcal{M}(k(C\otimes\mathbb{K}),k((C\cap I)\otimes\mathbb{K})).$$

Here we used that $C\cap I=C\cap(I\cap D)$.

Let $H_1:=\mathcal{M}(k)\delta_\infty H=k\delta_\infty(H(.))k^{-1}$. Then H_1 is a non-degenerate *-monomorphism, $H_1(A)$ is contained in $\mathcal{M}(k(C\otimes\mathbb{K}))$, and, for every closed J of E, and

$$H_1(\Phi(J))\;=\;H_1(\Psi(B\cap J))\;=\;H_1(A)\cap\mathcal{M}(E,J)\,.$$

Note here that $B\cap J=B\cap(I\cap D)$, and $C\cap I=C\cap(I\cap D)$ for the unique ideal I of G with $J=I\cap E$.

Since $H_1(A)$ is contained in $\mathcal{M}(k(C\otimes\mathbb{K}))$, we get from Lemma 12.1.1(v) that H_1 is weakly Φ-residually nuclear. $\delta_\infty H_1$ is unitarily equivalent to H_1, because $\delta_\infty\mathcal{M}(k)\delta_\infty$ is unitarily equivalent to $\mathcal{M}(k)\delta_\infty$ by Lemma 5.1.2. □

We denote by $[a,b]$ the commutator $ab-ba$ in the following.

LEMMA 12.1.5. *For every $n\in\mathbb{N}$, there is a continuous function f_n with $f_n(0)=0$ which has the following universal property:*

If A is a C^*-algebra with $A\cong A\otimes\mathcal{O}_\infty\otimes\mathcal{O}_\infty\otimes\ldots$, $a_1,\ldots,a_n,b\in A_+$ are contractions, and $\varepsilon>0$, with $\|[a_j,a_k]\|<\varepsilon$ for $j,k=1,\ldots,n$, then, there exists a contraction $c\in A$ with

$$\|[a_j,c]\|<f_n(\varepsilon)\,,\quad \|(1-c^*c)a_j\|<f_n(\varepsilon)\quad\textit{and}\quad \|[a_j,c^*bc]\|<f_n(\varepsilon)\,.$$

PROOF. Let C be a commutative C^*-subalgebra of a C^*-algebra D. Then the set $K(C)$ of completely positive maps $V(a):=\sum(c_i)^*ac_i$ with $c_1,\ldots,c_m\in C$ and $\|V(1)\|\le 1$ is convex. Its point weak closure on D^{**} is $K(C^{**})$, where we identify C^{**} naturally with the weak closure of C in D^{**}. For $b\in D$, $a_1,\ldots,a_n\in C$, and $\delta>0$, there exist $V\in K(C^{**})$ with

$$\|(1-V(1))a_j\|<\delta\qquad\text{and}\qquad\|[a_j,V(b)]\|\;<\;\delta. \tag{*}$$

To see this, approximate $a_1,\ldots,a_n$ by elements in the span of mutually orthogonal projections $p_1,\ldots,p_k\in C^{**}$ and consider $\sum_j p_jbp_j$. A Hahn-Banach separation argument shows that we can find V with (*) even in $K(C)$.

Suppose that there is a C^*-algebra B and a *-monomorphism φ from $B\otimes\mathcal{O}_\infty$ into D, such that $C\subset\varphi(B\otimes 1)$ and $b\in\varphi(B\otimes 1)$. Let $c:=\sum_{1\le i\le m}\varphi(d_i\otimes s_i)$, where $\varphi(d_i\otimes 1)=c_i$ and s_i denote the canonical generators of $\mathcal{O}_\infty$. Then c satisfies $c^*bc=V(b)$, $V(1)=c^*c$, and $[a_j,c]=0$. Therefore, $\|(1-c^*c)a_j\|<\delta$, $\|[a_j,c^*bc\|<\delta$, and c is a contraction in D commuting with a_j for $j=1,\ldots,n$.

Now we define numbers $g(a,b) := \|[a,b]\|$, where $[a,b] = ab - ba$,

$$\eta(a_1,\ldots,a_n) := \max\{g(a_j,a_k)\colon j,k = 1,\ldots,n\},$$

$$\mu(a;b;c) := \max(g(a,c), g(a,c^*bc), \|(1-c^*c)a\|),$$

and, for C^*-algebras A and contractions $a_1,\ldots,a_n,b,c \in A_+$,

$$v(a_1,\ldots,a_n;b;A) := \inf_c\{\max(\mu(a_1;b;c),\ldots,\mu(a_n;b;c))\},$$

where the infimum runs over all contractions $c \in A$.

For every $t \in \mathbb{R}_+$, and for every C^*-subalgebra A, we denote by $Y(t,A)$ the set of sequences $(a_1,\ldots,a_n,b)$ of contractions in A_+ such that $\eta(a_1,\ldots,a_n) \le t$. Now we define increasing functions $F(t,A)$ and $F(t)$ as follows:

$$F(t,A) := \sup\{v(a_1,\ldots,a_n;b;A)\colon\ (a_1,\ldots,a_n,b) \in Y(t,A)\},$$

and $F(t) := \sup_A\{F(t,A)\}$, where the supremum runs over all separable C^*-subalgebras A of $\mathcal{L}(\ell_2)$ with $A \cong A \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$. $F(t,A)$ and $F(t)$ are increasing on $\mathbb{R}_+$.

It is not hard to see, that f_n with the desired properties exists if and only if $\lim_{t\to 0} F(t) = 0$. Since $F(t)$ is increasing, we have $\lim_{t\to 0} F(t) = \inf\{F(1/k)\colon k \in \mathbb{N}\}$.

Suppose that $F(1/k) \ge 3\gamma > 0$ for $k \in \mathbb{N}$. Then we can find, for $k = 1,2,\ldots$,

(i) separable C^*-algebras A_k with $A_k \cong A_k \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$, and
(ii) $(a_1^{(k)},\ldots,a_n^{(k)},b^{(k)}) \in Y(1/k,A_k)$ such that $v(a_1^{(k)},\ldots,a_n^{(k)};b^{(k)};A_k) > 2\gamma$.

Now we use ultrapowers: Let $D := \prod_\omega(A_k)$, and $a_1,\ldots,a_n,b \in D$ with representatives $(a_j^{(1)},a_j^{(2)},\ldots)$ of a_j, and $(b^{(1)},b^{(2)},\ldots)$ of b. Let C and B be the C^*-subalgebras of D that are generated by $\{a_1,\ldots,a_n\}$ and $\{b,a_1,\ldots,a_n\}$ respectively. Then C is commutative and B is separable. By Proposition 7.4.11, there exists a *-monomorphism φ from $B \otimes \mathcal{O}_\infty$ into D, such that $\varphi(d \otimes 1) = d$ for $d \in B$. As we have seen above, there is a contraction $c \in D$ such that

$$\max(\mu(a_1;b;c),\ldots,\mu(a_n;b;c)) \,<\, \gamma\,.$$

This contradicts the above property (ii). □

LEMMA 12.1.6. *Suppose that $B \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$. Let $G := \mathrm{Q}(\mathbb{R}_+,B)$, the corona of B, or let $G := B_\omega$, the ultrapower of B.*

If D is a separable C^*-subalgebra of G and C a separable commutative C^*-subalgebra of D, then there exist a contraction $d \in C' \cap G$ and a commutative C^*-subalgebra A of G such that $C \subset A$, $d^*da = a$ for $a \in C$, and $d^*Dd \subset A$.

PROOF. We consider the case $G = \mathrm{Q}(\mathbb{R}_+,B)$. The proof for the case $G = B_\omega$ is similar.

Let $e_1,e_2,\ldots$ be a dense sequence in the positive contractions of D.

By Corollary 7.4.10, the approximately inner completely positive contractions $T\colon D \to G$ are one-step inner, i.e., there is a contraction $d \in G$ such that $T(b) =$

d^*bd. If $T|C = \mathrm{id}\,|C$, then $[a,d] = (1-dd^*)ad$ and $\|(1-dd^*)^{1/2}ad\|^2 = \|T(a^*a) - T(a^*)T(a)\| = 0$ for $a \in C$. It follows $d \in C' \cap G$ and and $d^*da = d^*ad = a$ for $a \in C$.

Thus it suffices to construct a sequence of approximately inner completely positive maps $T_n\colon G \to G$, such that $T_n|C = \mathrm{id}\,|C$, $\lim_n [T_n(e_k), T_n(e_j)] = 0$ and that $\lim_n T_n(e_k)$ exists for $k, j \in \mathbb{N}$.

The approximately inner completely positive contractions $T\colon G \to G$ with $T|C = \mathrm{id}\,|C$ form a semigroup for every C^*-subalgebra $C \subset G$. Therefore, we attempt to construct a sequence of approximately inner completely positive contractions $T_n\colon G \to G$ in the special way $T_n := S_{n+1}S_n \ldots S_1$. Here $S_n\colon G \to G$ should be approximately inner completely positive contractions which satisfy $S_n|C_n = \mathrm{id}\,|C_n$ and $S_n(g_n)$ commutes with C_n, where $C_1 := C$, $g_1 := e_1$, and, by induction, $g_n := S_{n-1}\ldots S_1(e_n)$, $C_{n+1} := C^*(C_n, S_n(g_n))$.

Then the sequence of restrictions $T_n|D$ has the desired properties, and $A := \mathrm{indlim}\, C_n$, $T(b) := \lim T_n(b)$ for $b \in D$ have the properties as stated in Lemma 12.1.6.

Thus it remains to show that for every separable commutative C^*-subalgebra $C \subset G$ and every contraction $g \in G_+$ there is an approximately inner completely positive contraction $S\colon G \to G$ with $S|C = \mathrm{id}\,|C$ such that $S(g)$ commutes elementwise with C.

Let $a_1, a_2, \ldots$ a sequence of positive contractions in $\mathrm{C_b}(\mathbb{R}_+, B)$, such that the sequence of the $c_n := a_n + \mathrm{C_0}(\mathbb{R}_+, B)$ is dense in the positive contractions in C. Let $b \in \mathrm{C_b}(\mathbb{R}_+, B)_+$ a contraction which represents g. With the notation of Lemma 12.1.5, we find a sequence $s_1 > s_2 > \ldots$ in $(0,1]$ such that $f_m(\varepsilon) < 1/n$ for $\varepsilon \in (0, s_n]$, $m \le n$. This is possible by Lemma 12.1.5.

Since $[c_j, c_k] = 0$, there are positive real numbers x_n with $x_{n+1} > \max(x_n, n)$, such that $\|[a_i(y), a_j(y)]\| < s_n$ for $y \ge x_n$ and $1 \le i, j \le n$.

Let $I_n := [x_n, x_{n+2}]$. Since $\mathrm{C}(I_n, B)$ absorbs $\mathcal{O}_\infty$ tensorial, we can apply Lemma 12.1.5 to $\mathrm{C}(I_n, B)$, $a_j|I_n$, $j = 1, \ldots, n$, and $b|I_n$, and find contractions $d_n \in \mathrm{C}(I_n, B)$ such that, for $j = 1, \ldots, n$, $\|(1 - d_n^* d_n)(a_j|I_n)\|$, $\|(a_j|I_n) - d_n^*(a_j|I_n)d_n\|$ and $\|[(a_j|I_n), d_n^*(b|I_n)d_n]\|$ are smaller then $1/n$.

Now we choose a continuous function $\mu_1(y)$ on $\mathbb{R}_+$ with $0 \le \mu_1(y) \le 1$, $\mu_1(x_{2n}) = 0$ and $\mu_1(x_{2n-1}) = 0$, for $n \in \mathbb{N}$. Let $\mu_2(y) := 1 - \mu_1(y)$.

We define elements $h_1, h_2 \in \mathrm{C_b}(\mathbb{R}_+, B)$ as follows:

Let $h_k(y) = 0$ for $y \in [0, x_k]$, $k = 1, 2$, and, for $y \in [x_{2n-k}, x_{2n+2-k}]$, $n \in \mathbb{N}$,

$$h_k(y) := (\mu_k(y))^{1/2} d_{2n-k}.$$

Then the corresponding elements $d_1, d_2 \in G$ satisfy $\|d_1^*d_1 + d_2^*d_2\| \le 1$ and define an inner completely positive contraction $S(a) := d_1^* a d_1 + d_2^* a d_2$ on G with $S|C = \mathrm{id}\,|C$, such that $S(g)$ commutes with C. □

LEMMA 12.1.7. *Suppose that B is separable and stable, and that B is isomorphic to $B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \dots$.*

Let $G := \mathrm{Q}(\mathbb{R}_+, B)$. Then there exist separable C^*-subalgebras C and D such that B, C, D and G satisfy the assumptions (i), (ii) and (iii) of Lemma 12.1.4.

PROOF. Below, we show the existence of sequences of separable C^*-subalgebras $C_n \subset D_n \subset F_n \subset D_{n+1}$ of G, of contractions $d_n \in G$, and of *-monomorphisms h_n from $F_n \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ into G, such that

(i) $D_1 := B$,
(ii) F_n is the C^*-subalgebra of G generated by D_n and d_n,
(iii) $h_n(f \otimes 1 \otimes p_{11}) = f$ for every $f \in F_n$
(iv) D_{n+1} is the image $h_n(F_n \otimes \mathcal{O}_\infty \otimes \mathbb{K})$ of h_n,
(v) C_n is a commutative C^*-subalgebra of D_n such that, for every closed ideal J of D_n, J is the smallest closed ideal of D_n which contains $C_n \cap J$, and
(vi) d_n commutes elementwise with C_n, $d_n^* d_n a = a$ for every $a \in C_n$, and $d_n^* D_n d_n$ generates a commutative C^*-subalgebra A_n of F_n with $C_n \subset A_n$,
(vii) $A_n \subset C_{n+1}$.

In particular, we have $B \subset D_n \subset D_{n+1}$ and $C_n \subset A_n \subset C_{n+1} \subset D_{n+1}$.

Let D and C be the closures of the unions of the sequences (D_n) and (C_n) respectively. Then $C \subset D$ and $B \subset D$. D is stable by (iv) and [**373**]. B is stable by assumption. By (ii),(iii) and (iv), D_{n+1} is the closed span of $D_{n+1} D_n D_{n+1}$. Thus, by (i), the span of DBD is dense in D. C is commutative.

Let J_1 and J_2 be closed ideals of D with $J_1 \cap C = J_2 \cap C$. Then $J_1 \cap C_n = J_2 \cap C_n$ and, by (v), $J_1 \cap D_n = J_2 \cap D_n$. But J_k is the inductive limit of $J_k \cap D_n$ for $n = 1, 2, \dots$ and $k = 1, 2$.

We can define approximately inner completely positive contractions $T_n \colon D \to C$ with $T_n | C_n = \mathrm{id} | C_n$, as follows: Let $e_{nk} := d_n d_{n+1} \dots d_{n+k}$ and $T_n(a) := \lim_{k\to\infty} (e_{nk})^* a e_{nk}$ Note here that e_{nm} is in D_{n+m+1} and that, therefore, by (vi) and (vii), $(e_{nk})^* a (e_{nk})$ is in C_{n+m+2} and $(e_{nk})^* a(e_{nk}) = (e_{nl})^* a(e_{nl})$ for $a \in D_{n+m+1}$, and $k, l > m$.

Thus, *B, C, D and G satisfy the assumptions (i), (ii) and (iii) of Lemma 12.1.4.*

Now we show the existence of C_n, D_n, d_n, A_n, F_n and h_n by induction:

We start at $n = 0$. Let $C_0 := 0$, $d_0 := 0$, $A_0 := 0$ and let φ denote an isomorphism from $B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ onto $B \subset G$. φ exists by our assumptions on B.

We define $F_0 \subset G$ as $\varphi(B \otimes 1 \otimes p_{11})$. Then there is a unique *-monomorphism h_0 from $F_0 \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ onto B, such that, for $b \in B$, $e \in \mathcal{O}_\infty$ and $k \in \mathbb{K}$,

$$h_0(\varphi(b \otimes 1 \otimes p_{11}) \otimes e \otimes k) = \varphi(b \otimes e \otimes k).$$

In particular, $h_0(f \otimes 1 \otimes p_{11}) = f$ for $f \in F_0$.

In other words, $h_0(f \otimes e \otimes k) = \varphi(\psi(f) \otimes e \otimes k)$, where ψ denotes the inverse of the isomorphism $b \mapsto \varphi(b \otimes 1 \otimes p_{11})$ from B onto F_0.

By (ii), for $n = 1, 2, \ldots$, $F_n := C^*(D_n, d_n)$ is defined by D_n and d_n.

If we have found F_n, we find a *-monomorphism h_n from $F_n \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ into G with property (iii) by Corollary 7.4.9.

Since D_{n+1} is the image of h_n, we have $B = D_1$, and it is enough to to define d_n and C_n with properties (v), (vi) and (vii). If $C_n \subset D_n$ are given, then the existence of d_n and $A_n := C^*(d_n^* D_n d_n)$ with (vi) follows from Lemma 12.1.6.

So we may assume that $A_n \subset F_n$ and h_n are given. It remains to construct C_{n+1} with properties (vii) and (v), where n has to be replaced by $n+1$ in (v).

Let $(f_k) \subset F_n$ a dense sequence in the set of positive contractions in F_n, andlet X_n denote the commutative C^*-subalgebra of $F_n \otimes \mathbb{K}$ which is generated by $A_n \otimes p_{11}$ and the set $\{f_k \otimes p_{kk} \colon k = 2, 3, \ldots\}$.

We identify $F_n \otimes \mathbb{K}$ with the subalgebra $F_n \otimes 1 \otimes \mathbb{K}$ of $K_n := F_n \otimes \mathcal{O}_\infty \otimes \mathbb{K}$. For every closed ideal J of K_n, we have that $X_n \cap J$ generates J as a closed ideal of K_n. Since $h_n(K_n) = D_{n+1}$, we can define C_{n+1} as $h_n(X_n)$. Now properties (v) and (vii) follow from property (iii) of h_n. □

THEOREM 12.1.8 (Asymptotic non-commutative Selection). *Suppose that A and B are separable and stable C^*-algebras, where B is isomorphic to $B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$, and that*

$$\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$$

defines a lower semi-continuous action of $\operatorname{Prim}(B)$ *on A in the sense of Definition 1.2.6 with $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$.*

Let E denote the hereditary C^*-subalgebra of $\mathrm{Q}(\mathbb{R}_+, B)$ *which is generated by B, and let $\Phi(I) := \Psi(I \cap B)$ for $I \in \mathcal{I}(E)$.*

*Then, there exists a non-degenerate weakly Φ-residually nuclear *-monomorphism H_0 from A into the multiplier algebra $\mathcal{M}(E)$ of E such that $\delta_\infty H_0$ is unitarily equivalent to H_0, and, for $I \in \mathcal{I}(E)$,*

$$\Psi(I \cap B) = \Phi(I) = H_0^{-1}(H_0(A) \cap \mathcal{M}(E, I)).$$

H_0 with this property is unique up to unitary homotopy.

Conversely H_0 determines Ψ in the sense of Lemma 12.1.1, i.e., for $J \in \mathcal{I}(B)$,

$$\Psi(J) := H_0^{-1}(H_0(A) \cap \mathcal{M}(E, E \cap \mathrm{Q}(\mathbb{R}_+, J)))\,.$$

In particular, H_0 and $\widehat{\sigma} \circ H_0$ are unitarily homotopic, for every automorphism $\widehat{\sigma}$ of $\mathcal{M}(E)$, which is induced by a scaling homeomorphism σ of $\mathbb{R}_+$.

PROOF. The existence of H_0 is the logical sum of Lemma 12.1.4 and Lemma 12.1.7, because B is separable, stable and is isomorphic to $B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$.

The uniqueness up to unitary homotopy of a non-degenerate weakly Φ-residually nuclear *-monomorphism $H_0\colon A \to \mathcal{M}(E)$ with

$$\Phi(I) = H_0^{-1}(H_0(A) \cap \mathcal{M}(E, I)).$$

and $\delta_\infty H_0$ unitarily equivalent to H_0 follows from Lemma 12.1.2.

For $J \in \mathcal{I}(B)$, $\mathrm{Q}(\mathbb{R}_+, J)$ is an ideal of $\mathrm{Q}(\mathbb{R}_+, B)$ and $J = B \cap \mathrm{Q}(\mathbb{R}_+, J)$.

Note that $\widehat{\sigma}$ fixes the elements of B and that $B \cap \widehat{\sigma}(I) = B \cap I$ for every $I \in \mathcal{I}(E)$ and every homeomorphism σ of $\mathbb{R}_+$. Thus $\Phi\widehat{\sigma} = \Phi$, and, therefore the uniqueness result applies to $\widehat{\sigma}H_0$.

Since $\Psi(J) = \Phi(E \cap \mathrm{Q}(\mathbb{R}_+, J))$ for $J \in \mathcal{I}(B)$, H_0 determines Ψ in the sense of Lemma 12.1.1. □

2. Selections and Ψ-equivariant embedding

We give now some definitions and lemmata that are needed for the generalization of Theorem A to the case of non-simple strongly purely infinite algebras as target algebras instead of $\mathcal{O}_2$ (cf. Theorem K).

Definition 12.2.1. Let $\mathcal{I}(A)$ denote the set of closed ideals of A with the below described topology and let $\mathrm{Prim}(A)$ denote the space of primitive ideals of A with the hull-kernel topology, i.e. $J \in \mathcal{I}(A)$ is in $\mathrm{Prim}(A)$ if it is the kernel of a (non-zero) irreducible representation of A, and the closed subsets of $\mathrm{Prim}(A)$ are given by the sets of "hulls" $h(I) := \{J \in \mathrm{Prim}(A)\colon I \subset J\}$. Thus the closure operation $X \subseteq \mathrm{Prim}(A) \mapsto \overline{X} \subset \mathrm{Prim}(A)$ is given by $\overline{X} := h(k(X))$, where the "kernel" ideal $k(X)$ of X is defined as $k(X) := \bigcap_{J\in X} J$.

We define a sort of **generalized Gelfand transform** $b \in A \mapsto \widehat{b}$ from A into the bounded functions on $\mathcal{I}(A)$, where $\widehat{b}\colon \mathcal{I}(A) \to \mathbb{R}_+$, for $b \in A$, is defined by

$$\widehat{b}(J) := \|b + J\| := \inf\{\|b + c\|\colon c \in J\}.$$

Recall that the ideal space $\mathcal{I}(A)$ of a C^*-algebra A is a compact Hausdorff space if we take the coarsest topology under which every generalized Gelfand transform is continuous. The latter is the *Fell topology* on $\mathcal{I}(A)$, cf. [**559**].

The compactness follows simply from the fact that every character on the commutative C^*-algebra $C^*(\{\widehat{b};\ b \in A\})$ defines a C^*-seminorm on A. Here $C^*(\{\widehat{b};\ b \in A\})$ means the C^*-subalgebra of $\ell_\infty(\mathcal{I}(A)_{\mathrm{discrete}})$ that is generated by the Gelfand transforms $\widehat{b}$ $(b \in A)$. All Gelfand transforms vanish on the point $J = A$, i.e. $\widehat{b}(A) = 0$. Thus, for every $b \in A$, $\widehat{b}$ is in $\mathrm{C}_0(\mathcal{I}(A) \setminus \{A\})$.

We shall use also the notation $\widehat{b}$ for the restriction $\widehat{b}\,|\,\mathrm{Prim}(A)$ of $\widehat{b}$ to $\mathrm{Prim}(A) \subset \mathcal{I}(A)$. This is justified, because $\widehat{b}$ is determined by its restriction to $\mathrm{Prim}(A)$, for $I \in \mathcal{I}(A)$, i.e.,

$$\widehat{b}(I) = \sup\{\,\widehat{b}(J)\,;\ J \in h(I)\,\}.$$

If A is commutative, then $\mathrm{Prim}(A)$ is the space of maximal ideals of A and $\widehat{b}$ is the ordinary Gelfand transform of the absolute value $|b| = \sqrt{b^*b}$ of $b \in A$.

REMARK 12.2.2. If A is separable, then, with the homeomorphism which we have introduced in Chapter 1, $\mathcal{I}(A)$ becomes a closed subspace of the Hilbert cube $[0,1]^\infty$, which contains $(0,0,\ldots)$ and is closed under component-wise maximum $\alpha\vee\beta$ for $\alpha,\beta\in\mathcal{I}(A)$. Since $\mathcal{I}(A)$ is compact, it follows that, for every subset $K\subset\mathcal{I}(A)$, $\bigvee K\in\mathcal{I}(A)$, and, for every decreasing sequence $\alpha_1\geq\alpha_2\geq\ldots$, $\bigwedge_{n=1}^\infty\alpha_n$ is in $\mathcal{I}(A)$.

Note here, that the natural map from the ideals of A into the Hilbert cube reverses the natural order, i.e., the elements α in the corresponding subset of the Hilbert cube correspond better to the closed subsets of the primitive ideal space of A. If $A\cong\mathrm{C}(X)$ for a compact metric space X, the image is just the classical compact space of closed subsets of X, the topology is then given by the Hausdorff distance metric.

The reader should note that, as explained in Chapter 1, $\mathrm{Prim}(A)$ has *not* the relative topology induced by $\mathcal{I}(A)$:

$\mathrm{Prim}(A)$ is a subset of $\mathcal{I}(A)$, but is not a topological subspace. The topology of $\mathrm{Prim}(A)$ is given by the above cited hull-kernel topology (Jacobson topology). It is easy to see, that the system $\mathbb{O}(\mathrm{Prim}(A))$ of open sets is the coarsest T_0-topology on $\mathrm{Prim}(A)$ under which every generalized Gelfand transform $\widehat{a}$ of an element a in A is *lower semi-continuous*. Thus, $\mathrm{Prim}(A)$ has the topology that is induced by the T_0 topology on the Hilbert cube $[0,1]^\infty$ given by the family of open subsets in Hausdorff topology, that are upward directed (with respect to the coordinate-wise order on elements of $[0,1]^\infty$).

If A is separable, one can re-discover $\mathrm{Prim}(A)$ as the $\vee$-prime elements of $\mathcal{I}(A)$ with system of closed sets induced by the intersection of $\mathcal{I}(A)$ with the set $\{\beta\colon\beta\vee\alpha=\alpha\}$ of points below $\alpha\in\mathcal{I}(A)$.

The reader, who is not familiar with this topics, could learn from considering the following simple example:

Let X a point-complete T_0-space with a countable base of its topology (e.g. $X\cong\mathrm{Prim}(A)$ with separable A), and let $f\colon X\to\mathbb{R}_+$ be a bounded non-negative function on X with $\|f\|_\infty:=\sup f(X)=1$. We can define an action of $\mathbb{O}(X)$ on $\mathrm{C}_0((0,1])$ by $\Psi_f(Z):=\mathrm{C}_0((\sup f(X\setminus Z),1])$ for open subsets Z of X. Here we define $\sup(\emptyset):=0$, because we consider only subsets and elements of $[0,1]$.

The reader easily checks that the action Ψ_f is lower semi-continuous if and only if f is lower semi-continuous.

It is interesting to note, that Ψ_f satisfies property (ii) of Definition 1.2.6, if and only if, for every decreasing sequence $K_1\supset K_2\supset\ldots$ of closed subsets K_n of X,

$$\sup f(\bigcap K_n)=\inf\{\sup f(K_n)\colon n=1,2,\ldots\}.$$

Now let $X:=\mathrm{Prim}(A)$ for a separable C^*-algebra A. As we shall see below, Ψ_f satisfies properties (ii), (iii) and (iv) of Definition 1.2.6, if and only if, $J\mapsto f(h(J))$ is a continuous function on $\mathcal{I}(A)$, and this is the case, if and only if, $f=\widehat{b}$ for an element $b\in A\otimes\mathcal{O}_2$.

The property (i) of Definition 1.2.6 is satisfied for Ψ_f only in the rare and exceptional case where f nsatisfies $\sup f(K_1 \cap K_2) = \min(\sup f(K_1), \sup f(K_2))$ for every closed subsets $K_1, K_2 \subset \mathrm{Prim}(A)$. E.g., this happens for commutative A, if and only if, f is 0 except on one point of $\mathrm{Prim}(A)$.

Let $s, t \in \mathcal{M}(A)$ isometries, $a, b, c, d \in A$ elements and denote by $\beta(a, b; t)$ the maximum of $\|t^*bt - a\|$, $\|t^*b^*bt - a^*a\|$ and $\|t^*bb^*t - aa^*\|$. Clearly $t \in \mathcal{M}(A) \mapsto \beta(a, b; t)$ is strictly continuous and $\beta(c, d; t) \le \beta(a, b; t) + \|a - c\| + \|b - d\|$.

In the following Lemma 12.2.3, let

$$D(a, b; s, t) \;:=\; \max(\beta(b, a, s), \beta(a, b, t))\,.$$

The definition causes $D(a, b; s, t) \le D(c, d; s, t) + \|a - c\| + \|b - d\|$. A similar argument shows that $D(a, b; s, t)$ is strictly continuous in $s, t \in \mathcal{M}(A)$. We define a sort of "distance" ρ_1 of contractions $a, b \in A$ by

$$\rho_1(a, b) := \rho_1(a, b, A) := \inf\{D(a, b; s, t) \colon\; s, t \in \mathcal{M}(A),\; s^*s = t^*t = 1\,\}\,.$$

Lemma 12.2.3. *For every C^*-algebra A with $A \cong A \otimes \mathcal{D}_2$, and for every contractions $a, b \in A$ with $\rho_1(a, b, A) < \varepsilon$, there exists a unitary $U \in \mathcal{M}(A)$ such that $\|U^*aU - b\| < 8\sqrt{\varepsilon}$.*

Proof. It suffices to consider separable C^*-algebras A (up to isomorphisms). The properties of $D(a, b; s, t)$ imply $\rho_1(a, b) \le \rho_1(c, d) + \|a - c\| + \|b - d\|$. Obviously $\|U^*aU - b\| \le \|U^*cU - d\| + \|a - c\| + \|b - d\|$ for $a, b, c, d \in A$ by triangle inequality for unitary $U \in \mathcal{M}(A)$.

Recall that $\mathcal{D}_2 := \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$. By the above inequalities we can suppose for our estimates that $a = c \otimes 1, b = d \otimes 1 \in A \otimes 1$ and that there are isometries $s, t \in \mathcal{M}(A)$ with $D(a, b; s \otimes 1, t \otimes 1) = D(c, d; s, t) < \varepsilon$, by approximation of contractions $a, b \in A \cong A \otimes \mathcal{D}_2$ by those in $A \otimes 1$ and by strict approximation of the isometries $s, t \in \mathcal{M}(A)$ by those in $\mathcal{M}(A) \otimes 1 \subset \mathcal{M}(A)$. Then we have to find $U \in \mathcal{M}(A) \otimes \mathcal{O}_2 \subset \mathcal{M}(A)$ with $\|U^*(c \otimes 1)U - d \otimes 1\| \le 8 \cdot D(c, d; s, t)^{1/2}$.

Let $B := C^*(x)$ denote the universal C^*-algebra generated by a single contraction $x \in B$ and define representations $h, k \colon B \to E := \mathcal{M}(A)$ by $h(x) := c$, $k(x) := d$, then we find a unitary $U \in E \otimes \mathcal{O}_2$ that satisfies the inequality

$$\|U^*(h(x) \otimes 1)U - k(x) \otimes 1\| \;<\; 8 \cdot \mu(s, t;\, h, k;\, x)^{1/2}\,,$$

by Corollary 4.5.2, where $\mu(s, t;\, h, k;\, x)$ denotes the maximum of the 6 values

$$\big\{\|s^*h(y)s - k(y)\|\,,\; \|t^*k(y)t - h(y)\|\,;\; y \in \{\, x,\, x^*x,\, xx^*\,\}\big\}\,.$$

This definition of $\mu(s, t;\, h, k;\, x)$ shows that $D(c, d; s, t) = \mu(s, t;\, h, k; x)$, i.e., U is as desired. □

Lemma 12.2.4. *Suppose that A is a separable C^*-algebra, and that F a simple and exact C^*-algebra (e.g. $F = \mathcal{O}_2$ or $F = \mathbb{K}$). Let $B := A \otimes \mathcal{O}_2$ and $X := \mathrm{Prim}(A)$. Then:*

(i) *The map* $J \mapsto J \otimes F$ *defines a natural topological isomorphism* $X \cong \mathrm{Prim}(A \otimes F)$.
(ii) $\{\widehat{a} :\ a \in A \otimes \mathbb{K}\}$ *is closed under point-wise maximum.*
(iii) *Every lower semi-continuous non-negative function* f *on* X *is the point-wise l.u.b. of a sequence* $(\widehat{a_n})$ *with* $a_n \in A_+$.
(iv) $\{\widehat{a}\colon a \in B\} \subset \ell_\infty(X_{\mathrm{disk}})$ *is a maximum-closed and* $\|.\|_\infty$ *norm-closed subset of* $\ell_\infty(X_{\mathrm{disk}})$,
(v) *The* $\|\,.\,\|_\infty$*- closure of* $\{\widehat{a}\colon a \in A \otimes \mathbb{K}\}$ *is* $\{\widehat{a}\colon a \in B\}$.

Proof. Ad(i): By Proposition B.4.2, $J \mapsto J \otimes F$ defines topological isomorphism from $\mathrm{Prim}(A)$ onto $\mathrm{Prim}(A \otimes F)$, for every simple separable exact C^*-algebra F, e.g. $F = \mathcal{O}_2$ or $F = \mathbb{K}$.

Ad(ii): Let s, t be canonical generators of a unital copy of $\mathcal{O}_2$ in $\mathcal{M}(A \otimes \mathbb{K})$. The Cuntz sum $a \oplus b := sas^* + tbt^*$ satisfies

$$\|(a \oplus b) + J\| = \max(\|sas^* + J\|, \|tbt^* + J\|) = \max(\|a + J\|, \|b + J\|)\,,$$

i.e., $\widehat{a \oplus b} = \max(\widehat{a}\,, \widehat{b}\,)$.

Ad(iii): Let $f\colon X \to \mathbb{R}_+$ be a lower semi-continuous function on X. For $c \in \mathbb{R}_+$, let $Y(c) := f^{-1}((c, \infty))$, and let $\chi(c)$ denote the characteristic function of $Y(c)$. Then $c\chi(c) \le f$. If $z \in X$ and $0 < c < f(z)$, then $c = c\chi(c)(z)$. Thus f is the point-wise the l.u.b. of the at most countable set $\{c\chi(c) : c = m/n, m, n \in \mathbb{N}\}$. Since f is lower semi-continuous, $Y(c)$ is an open subset of X and, therefore $\chi(c)$ is lower semi-continuous. Moreover, by the definition of the topology on X, there is a closed ideal $I(c)$ of A such that $Y(c) = X \setminus h(I(c))$.

Thus it suffices to show that, for every closed ideal I of A and $c \in \mathbb{R}_+$, there exist $a \in A_+$, such that the characteristic function χ of $X \setminus h(I)$ is the l.u.b. of $g_n := \widehat{b_n} = \widehat{a}^{\,1/n}$ where $b_n = a^{1/n}$. Since I is separable, we find a strictly positive contraction a in I_+. But this means: $0 < \|a + J\| \le 1$ if $J \in X$ is a primitive ideal of A which is not in $h(I)$, and $\|a + J\| = 0$ for $J \in h(I)$, i.e., $\widehat{a} \le \chi$ and $\widehat{a}(J) > 0$ if $\chi(J) > 0$. Thus, $\sup \widehat{a}(J)^{1/n}$ is the characteristic function χ of $X \setminus h(I)$.

Ad(iv): Let $b \in B$, $d := (b^*b)^{1/2}$ and $c \in (\mathcal{O}_2)_+$ with $Spec(c) = [0, 1]$, then

$$\|b + J\| = \|(d \otimes c) + J \otimes \mathcal{O}_2\|.$$

There is an isomorphism φ from $B \otimes \mathcal{O}_2$ onto B, such that $g \mapsto \varphi(g) \otimes 1$ is an approximately inner endomorphism of $B \otimes \mathcal{O}_2$. In particular $\varphi(J \otimes \mathcal{O}_2) = J$, and, for $a := \varphi(d \otimes c)$,

$$\|b + J\| = \|a + J\|.$$

This, together with the logical sum of Corollary 3.10.11 and Lemma 12.2.3, says that, for every uniformly convergent sequence $f_n = \widehat{b_n}$ with $b_n \in B$, there is a sequence $(a_n) \subset B_+$, such that (a_n) converges in B and $\widehat{b_n} = \widehat{a_n}$. $a \mapsto \widehat{a}$ is a norm-continuous map from B into $\ell_\infty(\mathrm{Prim}(B)_{\mathrm{discrete}})$. Thus $\lim f_n = \widehat{b}$ for $b = \lim a_n$.

Part (v) follows from (iv), because $\mathcal{O}_2$ is nuclear. □

DEFINITION 12.2.5. Suppose X is a T_0-space and that $f\colon X \to \mathbb{R}_+ := [0,\infty)$ is a non-negative function on X. We say that f is a **Dini function** on X (or say f **is Dini**) if f is lower semi-continuous and, for every increasing sequence $g_1 \leq g_2 \leq \ldots$ of lower semi-continuous functions $g_n\colon X \to \mathbb{R}_+$, (g_n) converges uniformly to f in $\ell_\infty(X)$, i.e., $\lim_{n\to\infty} \|f - g_n\|_\infty = 0$, if f is point-wise the supremum of the sequence (g_n), i.e., if $f(x) = \sup\{g_n(x)\colon n \in N\}$.

Every Dini function on X is automatically bounded, because we can consider $g_n := \min(n, f)$.

If X is a metrizable, locally compact and σ-compact Hausdorff space, then a function f is a non-negative Dini function in the sense of the above given Definition, if and only if, $f \in \mathrm{C}_0(X)$. This follows from the classical Lemma of Dini.

The following is a weak version of a stronger result in [**447**], where $B = A \otimes \mathcal{O}_2$ in (ii) is replaced by A itself.

PROPOSITION 12.2.6. *Suppose that A is a separable C^*-algebra and let $B := A \otimes \mathcal{O}_2$, $X := \mathrm{Prim}(A)$ and $h(I) := \{J \in X\colon I \subseteq J\}$ for $I \in \mathcal{I}(A)$.*

Then for a non-negative function f on X the following properties are equivalent:

(i) *f is a Dini function.*

(ii) *There exists $b \in B$ with $\widehat{b} = f$, i.e., $f(J) = \widehat{b}(J \otimes \mathcal{O}_2)$ for $J \in X$.*

(iii) *f is lower semi-continuous, bounded, and, for every increasing sequence $I_1 \subset I_2 \subset \ldots$ of closed ideals I_n of A,*

$$\sup f(h(I)) = \inf\{\sup f(h(I_n))\colon n = 1, 2, \ldots\},$$

where I is the closure of $\bigcup_{n=1}^\infty I_n$.

PROOF. (i)⇒(ii): Suppose that $f\colon X \to \mathbb{R}_+$ is a Dini function. By Lemma 12.2.4(iii), there is a sequence (a_n) in A such that f is the point-wise supremum of the sequence $(\widehat{a_n})$. Let $g_n(J) := \max(\widehat{a_1}(J), \ldots, \widehat{a_n}(J))$ for $J \in X$. Then g_n is an increasing sequence of non-negative lower semi-continuous functions on X with $f = \sup\{g_n\colon n \in \mathbb{N}\}$ point-wise. Since f is Dini, (g_n) converges uniformly to f.

Let $b_n := (a_1 \otimes 1) \oplus \ldots \oplus (a_n \oplus 1)$ in B, then $g_n(J) = \|b_n + (J \otimes \mathcal{O}_2)\|$.

Thus $g_n \in \{\widehat{b}\colon\ b \in B\}$. By Lemma 12.2.4(iv), $f = \widehat{b}$ for some $b \in B$.

(ii)⇒(iii): Suppose that there is $b \in B$ such that $f(J) = \widehat{b}(J \otimes \mathcal{O}_2)$ for $J \in X$. Let $g(I) := \|b + (I \otimes \mathcal{O}_2)\|$ for every closed ideal I of B. Then, for every subset $Z \subset \mathcal{I}(A)$, $g(I) = \sup\{g(J)\colon J \in Z\}$, where $I := \bigcap\{J\colon J \in Z\}$. If K is the closure of an increasing sequence $I_1 \subset I_2 \subset \ldots$ of closed ideals I_n in A, $g(K) = \inf\{g(I_n)\colon n \in \mathbb{N}\}$.

Since $g|X = f$, it follows that f satisfies $\sup f(Z) = g(k(Z))$ for every subset Z of X, where $k(Z) = \bigcap\{J\colon J \in Z\}$ is the kernel-operation on $\mathcal{I}(A)$.

But $hk(Z)$ is the closure of Z in X and $kh(I) = I$ for every closed ideal I of A. We get that f is lower semi-continuous, and that $\sup f(h(I)) = g(I)$ for every closed ideal I of A. Now the upward monotone continuity of g implies, for every increasing sequence $I_1, I_2, \ldots$ of closed ideals I_n of A, that, for the closure I of $\bigcup\{I_1, I_2, \ldots\}$,

$$\sup f(h(I)) = \inf\{\sup f(h(I_n)) \colon n = 1, 2, \ldots\}.$$

(iii)⇒(i): We define a map T from the bounded functions f on X into the functions F on $\mathcal{I}(A)$, by $T(f)(I) := \sup f(h(I))$ for $I \in \mathcal{I}(A)$. Then $T(g) \leq T(f)$ if $g \leq f$. If $I_1, I_2 \in \mathcal{I}(A)$ then $T(f)(I_1 \cap I_2) = \max(T(f)(I_1), T(f)(I_2))$, because $h(I_1 \cap I_2) = h(I_1) \cup h(I_2)$. In particular, $T(f)$ is monotone, i.e., $T(f)(I) \leq T(f)(J)$ if $J \subset I$.

If f is lower semi-continuous on X, then $T(f)(J) = f(J)$ for every primitive ideal J of A:

The set $\{y \in X \colon f(y) \leq f(J)\}$ is closed and contains J. The closure of the one-point set $\{J\}$ is $h(J)$. Thus $f(J) \leq T(f)(J) = \sup f(h(J)) \leq f(J)$.

If f is lower semi-continuous on X, then $T(f)$ is also a lower semi-continuous function on $\mathcal{I}(A)$:

Let M be a subset of $\mathcal{I}(A)$, and let $J(M)$ denote the intersection of the ideals $I \in M$. Then $h(J(M))$ is the closure in X of of the union of the subsets $h(I) \subset X$ for $I \in M$. Since f is lower semi-continuous,

$$\sup f(h(J(M))) = \sup f(\bigcup\{h(I) \colon I \in M\}) = \sup\{\sup f(h(I)) \colon I \in M\}.$$

The right hand equality comes from $f(\bigcup\{h(I) \colon I \in M\}) = \bigcup\{f(h(I)) \colon I \in M\}$.

Let $c \in \mathbb{R}_+$ and $P \in \mathcal{I}(A)$ a point in the closure of the set $\mathcal{M}(c) := \{I \in \mathcal{I}(A) \colon T(f)(I) \leq c\}$. Then $J(\mathcal{M}(c)) \subset P$ and, therefore, $T(f)(P) \leq T(f)(J(\mathcal{M}(c)))$. By the above formula for f, $T(f)(J(\mathcal{M}(c))) \leq \sup\{T(f)(I) \colon I \in \mathcal{M}(c)\} \leq c$. Thus $T(f)$ is lower semi-continuous on $\mathcal{I}(A)$.

Now suppose that f satisfies the assumptions of (iii).

We show that $T(f)$ is then also an upper semi-continuous function, i.e., $T(f)$ is a continuous function on the metrizable compact Hausdorff space $\mathcal{I}(A)$:

Let $c \in \mathbb{R}_+$, and let (P_n) be a convergent sequence in $\mathcal{I}(A)$ with limit $P \in \mathcal{I}(A)$ such that $c \leq T(f)(P_n)$ for $n = 1, 2, \ldots$. Let $I_n = \bigcap_{k \geq n} P_k$. Then $I_n \subset I_{n+1}$, $I_{n+1} \subset P_{n+1}$ and, therefore, $c \leq T(f)(P_{n+1}) \leq T(f)(I_{n+1}) \leq T(f)(I_n)$.

It follows $c \leq T(f)(I)$ for the closure I of $\bigcup\{I_1, I_2, \ldots\}$.

Let $a \in P$, then $\widehat{a}(P) = 0$, $\widehat{a}(I_n) = \sup\{\widehat{a}(P_k) \colon k \geq n\}$, and

$$\widehat{a}(P) = \lim \widehat{a}(P_n) = \lim_n \widehat{a}(I_n) = \widehat{a}(I).$$

Thus, $P \subset I$ and $c \leq T(f)(I) \leq T(f)(P)$. Which proves the upper semicontinuity of $T(f)$.

Let $0 \leq g_1 \leq g_2 \leq \ldots$ be an increasing sequence of lower semi-continuous functions on X, such that $f(y) = \sup\{g_n(y) \colon n \in \mathbb{N}\}$ for every $y \in X$. Let $I \in \mathcal{I}(A)$ and $\varepsilon > 0$. We find $y \in h(I) \subset X$ such that $f(y) + \varepsilon > \sup f(h(I))$. There is $n \in \mathbb{N}$ with $g_n(y) + \varepsilon > \sup f(h(I))$. Thus $T(g_1) \leq T(g_2) \leq \ldots$ is an increasing sequence of lower semi-continuous functions on $\mathcal{I}(A)$, which converges point-wise to the continuous function $T(f)$.

By the lemma of Dini, the sequence $T(g_n)$ converges uniformly to $T(f)$ on $\mathcal{I}(A)$. This happens also with $g_n = T(g_n)|X$ and $f = T(f)|X$.

Hence, f is a Dini function. □

Corollary 12.2.7. *Suppose that A and B are separable C^*-algebras, and that γ is a topological isomorphism from* Prim(A) *onto* Prim(B). *Let $E := A \otimes \mathcal{O}_2$ and $F := B \otimes \mathcal{O}_2$. Then $f \mapsto f \circ \gamma$ defines an isomorphism from $\{\widehat{b} \colon b \in F\}$ onto $\{\widehat{a} \colon a \in E\}$, where we naturally identify* Prim(A) *with* Prim(E) *and* Prim(B) *with* Prim(F).

Proof. $f \mapsto f \circ \gamma$ defines an isomorphism from the Dini functions on Prim(B) onto the Dini functions on Prim(A). Now apply Proposition 12.2.6. □

Corollary 12.2.8. *Suppose that A and B are separable C^*-algebras, and that Ψ is a map from $\mathcal{I}(B)$ into $\mathcal{I}(A)$ with $\Psi(J_1) \subset \Psi(J_2)$ for $J_1 \subset J_2$, $J_1, J_2 \in \mathcal{I}(B)$. Then:*

(i) *Ψ is lower semi-continuous, if and only if, for every $a \in A$, $J \mapsto \widehat{a}(\Psi(J))$ is a lower semi-continuous function on* Prim(B), *and for every $I \in \mathcal{I}(B)$ and $a \in A$ with $\widehat{a}(\Psi(I)) > 0$ there exists $J \in$* Prim(B) *with $\widehat{a}(\Psi(J)) > 0$ and $I \subset J$.*

(ii) *Ψ is, moreover, countably monotone upper semi-continuous, i.e., satisfies (ii) of Definition 1.2.6, if and only if, for every $a \in A$, $J \mapsto \widehat{a}(\Psi(J))$ is a Dini function on* Prim(B).

Proof. Let $X :=$ Prim(B) and $f_a := \widehat{a}\Psi|X$ for $a \in A$.

Suppose that Ψ is lower semi-continuous. Then, for every subset Z of X,

$$\widehat{a}(\Psi(\bigcap J \in Z) = \widehat{a}(\bigcap_{J \in Z} \Psi(J)) = \sup\{\widehat{a}(J) \colon J \in Z\}.$$

In particular, $\sup f_a(Z) = \sup f_a(hk(Z))$ and $\widehat{a}(\Psi(I)) = \sup f_a(h(I))$, because $\bigcap\{J \in Z\} = \bigcap\{J \in hk(Z)\}$ and $I = kh(I) = \bigcap\{J \in h(I)\}$. But this means that f_a is lower semi-continuous on X and that $\widehat{a}(\Psi(I)) > 0$ implies the existence of $J \in h(I)$ with $f_a(J) > 0$.

Conversely, suppose that f_a is lower semi-continuous for every $a \in A$ and that for every $I \in \mathcal{I}(B)$ and $a \in A$ with $\widehat{a}(\Psi(I)) > 0$ there exists $J \in h(I)$ with $f_a(J) > 0$. This says that $f_a(Z) = \{0\}$ implies $f_a(hk(Z)) = \{0\}$ and that $\widehat{a}(\Psi(I)) = 0$ if and only if $f_a(h(I)) = \{0\}$.

Let $I := \bigcap_\alpha I_\alpha$, $K := \bigcap_\alpha \Psi(I_\alpha)$ and $Z := \bigcup_\alpha h(I_\alpha)$. By monotony of Ψ, we have $\Psi(I) \subset K$. If $a \in K$, then $\widehat{a}(\Psi(I_\alpha)) = 0$ for every α in the index set. Thus $\{0\} = f_a(Z) = f_a(hk(Z))$. But $k(Z) = I$. Therefore, $\widehat{a}(\Psi(I)) = 0$, i.e., $a \in \Psi(I)$. Hence $K = \Psi(I)$ and Ψ is lower semi-continuous.

Ad(ii): We have seen in the proof of the first part, that $\widehat{a}(\Psi(I)) = \sup f_a(h(I))$ for $I \in \mathcal{I}(B)$ and that f_a is lower semi-continuous, if Ψ is lower semi-continuous. f_a is bounded by $\|a\|$.

Let $I_1 \subset I_2 \subset \ldots$ an increasing sequence of closed ideals of B, and let I and K denote the closures of $\bigcup I_n$ and $\bigcup \Psi(I_n)$, respectively. Then

$$\widehat{a}(K) = \inf\{\sup f_a(h(I_n)) \colon n = 1, 2, \ldots\}.$$

By Proposition 12.2.6(iii), it follows that $\widehat{a}(\Psi(I)) = \widehat{a}(K)$ if f_a is a Dini function. Thus, Ψ satisfies (ii) of Definition 1.2.6 if f_a is a Dini function on X for every $a \in A$.

If Ψ satisfies (ii) of Definition 1.2.6 then $\Psi(I) = K$ and, therefore,

$$\sup f_a(h(I)) = \inf\{\sup f_a(h(I_n)) \colon n = 1, 2, \ldots\}.$$

Thus, by Proposition 12.2.6(iii), f_a is a Dini function on X for every $a \in A$, if Ψ satisfies property (ii) of Definition 1.2.6. □

REMARK 12.2.9. Recall from Remark 3.11.3 that for a C^*-algebra B the following Properties (i)–(iv) are equivalent to each other:

(i) *B strongly purely infinite in the sense of Definition 1.2.2.*
(ii) *For every $a, b \in B_+$ and $\varepsilon > 0$ there exist contractions $s, t \in B$ such that $\|a^2 - s^*a^2s\|$, $\|b^2 - t^*b^2t\|$ and $\|s^*abt\|$ are all less then ε.*
(iii) *The ultrapower B_ω of B is strongly purely infinite.*
(iv) *The asymptotic corona $\mathrm{Q}(\mathbb{R}_+, B)$ of B is strongly purely infinite.*
(v) *$\mathrm{Q}(\mathbb{R}_+, B)$ has the WvN-property of Definition 1.2.3.*
(vi) *B admits no non-zero character, and for every separable C^*-subalgebra C of $\mathrm{Q}(\mathbb{R}_+, B)$ the following holds:*

If $t \in \mathbb{R}_+ \mapsto V(t)$ is a strongly continuous map from $\mathbb{R}_+$ into the approximately inner completely positive contractions from B into B, T is the completely positive contraction from $\mathrm{Q}(\mathbb{R}_+, B)$ into $\mathrm{Q}(\mathbb{R}_+, B)$ given by

$$T(a + \mathrm{C}_0(\mathbb{R}_+, B)) := V(a) + \mathrm{C}_0(\mathbb{R}_+, B),$$

where $a \in \mathrm{C_b}(\mathbb{R}_+, B)$ and $V(a)(t) := V(t)(a(t))$, and if $T|C$ is residually nuclear, then, there exists a contraction $d \in \mathrm{Q}(\mathbb{R}_+, B)$ such that $T(b) = d^*bd$ for $b \in C$.

From (ii) it follows that *the class of strongly purely infinite algebras is closed under inductive limits.*

It is obvious that quotient algebras and hereditary C^*-subalgebras of strongly purely infinite C^*-algebras are strongly purely infinite.

REMARK 12.2.10. In [**431**, sec. 3] we have discussed several equivalent properties of C^*-subalgebras A of a C^*-algebra B, among them the property, that there is a normal conditional expectation from the second conjugate B^{**} of B onto the second conjugate A^{**} of A, where we have naturally identified A^{**} with the weak closure of A in B^{**}.

Among others, in [**431**] it was shown that this property is equivalent to the **relative weak injectivity** (in short: rwi) of A in B, which means that the natural C^*-morphism from the maximal C^*-algebra tensor product $A \otimes^{max} C$ to $B \otimes^{max} C$ is injective, and thus isometric, for every C^*-algebra C, cf. [**431**, prop. 3.1]. Here it suffices even to take $C = C^*(F_\infty)$, the full group C^*-algebra of the free group on countably many generators.

It follows immediately that the closure A of the union of an increasing sequence $A_1 \subset A_2 \subset \ldots$ of relatively weakly injective subalgebras of B is again relatively weakly injective in B.

Unfortunately, relative weak injectivity does not pass to quotients in general:

There exist B, $A \subset B$, closed ideals J of A, and $I_1 \subset I_2$ of B such that

(i) $J := I_1 \cap A = I_2 \cap A$ and A is relatively injective in B,
(ii) A/J is relatively injective in B/I_1, but
(iii) A/J is *not* relatively weakly injective in B/I_2.

We define a stronger property as the relative weak injectivity such that it plays together with some particular quotient maps in a natural manner:

We say that $A \subset B$ is **residually relatively weakly injective** in B, if, for every $J \in \mathcal{I}(A)$, the natural monomorphism maps $A/(A \cap \Psi^{A,B}_{\text{down}}(J))$ onto a relatively weakly injective C^*-subalgebra of $B/\Psi^{A,B}_{\text{down}}(J)$ ([4]).

The above mentioned results from [**431**] and the short exactness of the maximal C^*-algebra tensor product imply immediately the following equivalent formulation:

$A \subset B$ *is residually relatively weakly injective in* B*, if and only if,*

$$A \otimes^{max} C^*(F_\infty) \subset B \otimes^{max} C^*(F_\infty)$$

and, for every $J \in \mathcal{I}(A)$*,*

$$(A \cap \Psi^{A,B}_{\text{down}}(J)) \otimes^{max} C^*(F_\infty) = (A \otimes^{max} C^*(F_\infty)) \cap (\Psi^{A,B}_{\text{down}}(J) \otimes^{max} C^*(F_\infty)).$$

If A is relatively weakly injective in B, and if I is a closed ideal of $B \otimes^{max} C$ such that

$$I \cap (A \otimes^{max} C) = (A \cap \Psi^{A,B}_{\text{down}}(J)) \otimes^{max} C,$$

then I contains $\Psi^{A,B}_{\text{down}}(J) \otimes^{max} C$.

It follows, that $A \subset B$ *is residually relatively weakly injective in* B *if, e.g.,* A *is relatively weakly injective in* B *and every closed ideal of* $A \otimes^{max} C^*(F_\infty)$ *is the intersection of* $A \otimes^{max} C^*(F_\infty)$ *with a closed ideal of* $B \otimes^{max} C^*(F_\infty)$*.*

[4]Recall that $\Psi^{A,B}_{\text{down}}(J)$ is the smallest closed ideal of B which contains $J \in \mathcal{I}(A)$.

Lemma 12.2.11. *Suppose that $A_1 \subset A_2 \subset \dots$ is a sequence of C^*-subalgebras of B which are residually relatively weakly injective in B. Then the closure A of its union is residually relatively weakly injective in B.*

Proof. First, A is again relatively weakly injective in B, i.e., $A \otimes^{max} C$ and $A_n \otimes^{max} C$ are naturally C^*-subalgebras of $B \otimes^{max} C$ for every C^*-algebra C, because the functor $(.) \otimes^{max} C$ is continuous with respect to inductive limits.

Let $J \in \mathcal{I}(A)$. Then $(A \cap \Psi^{A,B}_{\text{down}}(J)) \otimes^{max} C$ is contained in $\Psi^{A,B}_{\text{down}}(J) \otimes^{max} C$, and $\Psi^{A,B}_{\text{down}}(J)$ is the closure of the union of the increasing sequence of closed ideals $\Psi^{A_n,B}_{\text{down}}(A_n \cap J)$.

For $n < m$, the intersection of $\Psi^{A_n,B}_{\text{down}}(A_n \cap J) \otimes^{max} C$ with $A_m \otimes^{max} C$ is contained in $\big(A \cap \Psi^{A_m,B}_{\text{down}}(A_m \cap J)\big) \otimes^{max} C$.

Thus, the intersection of $\Psi^{A_n,B}_{\text{down}}(A_n \cap J) \otimes^{max} C$ with $A \otimes^{max} C$ is contained in $(A \cap \Psi^{A,B}_{\text{down}}(J)) \otimes^{max} C$.

By continuity of $(.) \otimes^{max} C$, we get that $(A \cap \Psi^{A,B}_{\text{down}}(J)) \otimes^{max} C$ is the intersection of $A \otimes^{max} C$ with $\Psi^{A,B}_{\text{down}}(J) \otimes^{max} C$. □

Lemma 12.2.12. *Suppose that $A \subset B$ is non-degenerate and residually relatively weakly injective in B, and that $A \cap \Psi^{A,B}_{\text{down}}(J) = J$ for every $J \in \mathcal{I}(A)$.*

Let C be a C^*-subalgebra of $\mathcal{M}(A)$ and $V \colon C \to \mathcal{M}(A)$ a completely positive map.

Then V is weakly residually nuclear as a map into $\mathcal{M}(A)$, if and only if, it is weakly residually nuclear as a map into $\mathcal{M}(B)$.

Proof. By Lemma 12.1.1(iv), since $\mathcal{M}(A) \subset \mathcal{M}(B)$ unitally, we have $\mathcal{M}(A) \cap \mathcal{M}(B, I) = \mathcal{M}(A, A \cap I)$ for $I \in \mathcal{I}(B)$. It follows that $\mathcal{M}(A, J) = \mathcal{M}(A) \cap \mathcal{M}(B, \Psi^{A,B}_{\text{down}}(J))$. Therefore, $V(C \cap \mathcal{M}(B, I)) \subset \mathcal{M}(B, I)$ for every $I \in \mathcal{I}(B)$, if and only if, $V(C \cap \mathcal{M}(A, J)) \subset \mathcal{M}(A, J)$ for every $J \in \mathcal{I}(A)$.

The argument for the proof of Lemma 12.1.1(v) shows, that V is also weakly residually nuclear as a map from $C \subset \mathcal{M}(B)$ into $\mathcal{M}(B)$, if V is weakly residually nuclear as a map from C into $\mathcal{M}(A)$.

Now suppose that V is weakly residually nuclear as a map from C into $\mathcal{M}(B)$, and let $J \in \mathcal{I}(A)$ and $I := \Psi^{A,B}_{\text{down}}(J)$. Then $J = A \cap I$, and $C \cap \mathcal{M}(B, I) = C \cap \mathcal{M}(A, J)$, because $C \subset \mathcal{M}(A)$. $C/(C \cap \mathcal{M}(A, J))$ is in a natural way a C^*-subalgebra of $\mathcal{M}(A/J) \subset \mathcal{M}(B/I)$. The natural map $[V]_I \colon C/(C \cap \mathcal{M}(B, I)) \to \mathcal{M}(B/I)$ is weakly nuclear as a map from $C/(C \cap \mathcal{M}(B, I)) \subset \mathcal{M}(B/I)$ into $(B/I)^{**}$, and $[V]_I(C/(C \cap \mathcal{M}(B, I)))$ is contained in $\mathcal{M}(A/J)$. By assumption, $A/J \subset B/I$ is a non-degenerate and relatively weakly injective C^*-subalgebra of B/I, i.e., there is a normal conditional expectation P from $(B/E)^{**}$ onto the weak closure $(A/J)^{**}$ of A/J in $(B/E)^{**}$. The natural map $[V]_J$ from $C/(C \cap \mathcal{M}(A, J))$ into $(A/J)^{**}$ equals $P[V]_I$. Thus $[V]_J$ is weakly nuclear.

Thus $V \colon C \to \mathcal{M}(A)$ is weakly residually nuclear. □

REMARK 12.2.13. Recall that, by Definition 1.2.3, a C^*-algebra B has **residually nuclear separation** if, for every separable C^*-subalgebra $C \subset B$, every $a \in C_+$ and every $\varepsilon > 0$, there exists a residually nuclear completely positive contraction $V: C \to B$ such that $\|V(a) - a\| < \varepsilon$. Obviously, residually nuclear separation passes to hereditary C^*-subalgebras and to quotient algebras. An application of Lemma 12.1.2 to $A := B$ and $\Psi := \Psi_B$ shows that

*a separable stable C^*-algebra B has residually nuclear separation, if and only if, there is a non-degenerate weakly residually nuclear *-monomorphism $H_0: B \to \mathcal{M}(B)$, such that $\delta_\infty H_0$ is unitarily equivalent to H_0, and, for $J \in \mathcal{I}(B)$,*

$$H_0(J) = H_0(B) \cap \mathcal{M}(B, J).$$

On can conlude from [**463**] and [**443**] that

a C^*-algebra B with residually nuclear separation has the WvN-property if and only if it is strongly purely infinite.

LEMMA 12.2.14. *Suppose that B is strongly purely infinite, and that A is a separable C^*-subalgebra of $\mathcal{M}(B)$.*

Then, for every separable C^*-subalgebra C of B, there exists a separable C^*-subalgebra D of B, such that

(i) *$C \subset D$,*
(ii) *D is strongly purely infinite,*
(iii) *for every closed ideal J of D, there exists a closed ideal I of B such that $J = D \cap I$, i.e., $J = D \cap \Psi^{D,B}_{\text{down}}(J)$,*
(iv) *there is a weakly continuous conditional expectation from $(B/\Psi^{D,B}_{\text{down}}(J))^{**}$ onto $(D/J)^{**}$, i.e., D is residually relatively weakly injective in B, and*
(v) *$AD \subset D$ and $aD = \{0\}$ implies $a = 0$ for $a \in A$, i.e., the natural C^*-morphism $L: a \in A \to L_a|D \in \mathcal{M}(D)$ from A to $\mathcal{M}(D)$ is a monomorphism, where $L_a(b) := ab$.*

The subalgebras D of B with the properties (i)-(v) are closed under inductive limits, i.e., closures D of increasing sequences $D_1 \subset D_2 \subset \dots$ of C^*-subalgebras of B with properties (i)-(v) satisfy again (i)-(v).

If B is stable, then D can be chosen as stable subalgebra of B.

*If B is stable and has residually nuclear separation, then D can be chosen such that D is stable and there exists a non-degenerate residually nuclear *-monomorphism $H_0: D \to \mathcal{M}(D)$, such that $\delta_\infty H_0$ and H_0 are unitarily equivalent, and $H_0(J) = H_0(D) \cap \mathcal{M}(D, J)$ for every closed ideal J of D.*

If B is σ-unital (with strictly positive element $e \in B_+$) then D with (i)-(iv) for C replaced by $C^*(C, e)$ satisfies $IL(A \cap \mathcal{M}(B, I)) = L(A) \cap \mathcal{M}(D, D \cap I)$ for every $I \in \mathcal{I}(B)$.

If $\mathrm{id}|A: A \hookrightarrow \mathcal{M}(B)$ is weakly residually nuclear, then $L: A \to \mathcal{M}(D)$ is weakly residually nuclear.

PROOF. If $a_1, a_2, \ldots$ is a dense sequence in A, then we find contractions $b_n \in B_+$ with $\|a_n b_n\| + 1/n > \|a_n\|$. The smallest C^*-subalgebra C_1 of B with $AC_1 \subset C_1$, $C \subset C_1$ and $\{b_1, b_2, \ldots\} \subset C_1$ is separable, and $L \colon a \mapsto L_a | C_1$ is a *-monomorphism from A to $\mathcal{M}(C_1)$.

Let C_1 be a separable C^*-subalgebra of B, and let M be a countable dense subset of the positive cone of C_1.

By Remark 12.2.9 we can choose, for $a, b \in M$ and $k \in \mathbb{N}$, contractions $c, d \in B$, such that $\|a^2 - c^* a^2 c\|$, $\|b^2 - d^* b^2 d\|$ and $\|c^* abd\|$ are less then $1/k$. The chosen elements are in a countable subset S of B. Let C_2 be the separable C^*-subalgebra of B which is generated by C_1 and S.

Then for positive elements $a, b \in C_1$ and $\delta > 0$ there exist contractions $c, d \in C_2$ such that $\|a^2 - c^* a^2 c\|$, $\|b^2 - d^* b^2 d\|$ and $\|c^* abd\|$ are less then δ.

In the same way we construct an increasing sequence $C_1 \subset C_2 \subset \ldots$ of separable C^*-subalgebras C_n of B such that, for positive elements $a, b \in C_n$ and $\delta > 0$ there exist contractions $c, d \in C_{n+1}$ such that $\|a^2 - c^* a^2 c\|$, $\|b^2 - d^* b^2 d\|$ and $\|c^* abd\|$ are less then δ.

Let D_1 be the closure of the union of the C_n. Then, for positive elements $a, b \in D_1$ and $\delta > 0$ there exist contractions $c, d \in D_1$ such that $\|a^2 - c^* a^2 c\|$, $\|b^2 - d^* b^2 d\|$ and $\|c^* abd\|$ are less then δ. Thus D_1 is a separable strongly purely infinite C^*-subalgebra of B and contains C_1.

Now let G be a C^*-algebra. For every C^*-subalgebra C of G and $a, b \in G$, $k \in \mathbb{N}$, we denote by $\rho(a, b, k, C)$ the infimum of the numbers $\|a - \sum c_j^* b c_j\|$, where $c_j \in C$, $1 \leq j \leq k$, with $\|\sum c_j^* c_j\| \leq 1$. If $C_1 \subset C_2$ then $\rho(a, b, k, C_2) \leq \rho(a, b, k, C_1)$. For fixed C and k, $\rho(a, b, k, C)$ is uniformly continuous in a and b.

Let $C \subset G$ and $a, b \in C$ positive contractions. Then a is in the closed ideal generated by b if and only if the infimum of $\{\rho(a, b^{1/n}, k, C) \colon n, k \in N\}$ is zero.

Thus, the equations $\rho(a, b, k, C) = \rho(a, b, k, G)$ for every $a, b \in C_+$, $k \in \mathbb{N}$, imply that, for every $J \in \mathcal{I}(C)$,

$$\Psi_{\text{down}}^{C,G}(J) \cap C = J.$$

Now let E_1 be a separable C^*-subalgebra of G and let M denote a dense countable subset of $(E_1)_+$.

By definition of ρ, we can choose, for $a, b \in M$ and $k, m \in \mathbb{N}$, contractions $c_1, \ldots, c_k \in G$ with $\|\sum_{j=1}^k c_j^* c_j\| \leq 1$ such that $\|a - \sum_{j=1}^k c_j^* b c_j\|$ is less then $\rho(a, b, k, G) + 1/m$. The chosen elements are in a countable subset S of G. Let E_2 be the separable C^*-subalgebra of G which is generated by E_1 and S.

Then $\rho(a, b, k, E_2) = \rho(a, b, k, G)$ for positive elements $a, b \in E_1$ and $k \in \mathbb{N}$.

In the same way we construct an increasing sequence $E_1 \subset E_2 \subset \ldots$ of separable C^*-subalgebras E_n of G such that $\rho(a, b, k, E_{n+1}) = \rho(a, b, k, G)$ for positive elements $a, b \in E_n$ and $k \in \mathbb{N}$.

Let E be the closure of the union of the E_n. Then E is separable, $E_1 \subset E$, and $\rho(a, b, k, E) = \rho(a, b, k, G)$ for positive elements $a, b \in E$ and $k \in \mathbb{N}$. Here we used the uniform continuity of ρ in a and b.

Thus, for every separable C^*-subalgebra E_1 of a C^*-algebra G, there exist a separable C^*-subalgebra E of G, such that $E_1 \subset E$ and every closed ideal J of E is the intersection of E with a closed ideal of G, i.e., $E \cap \Psi^{E,G}_{\text{down}}(J) = J$.

By [**431**, lem. 3.4], for every separable C^*-subalgebra D_1 of B, there is a separable relatively weakly injective C^*-subalgebra D_2 of B with $D_1 \subset D_2$.

Let $F := C^*(F_\infty)$. Thus $E_1 := D_2 \otimes^{max} F$ is in natural way a separable C^*-subalgebra of $G := B \otimes^{max} F$.

We find a separable C^*-subalgebra E_2 of G which contains E_1 and has the property that every closed ideal of E_2 is the intersection of E_2 with a closed ideal of G. Since the algebraic tensor product $B \odot F$ is dense in G, there is a separable C^*-subalgebra D_3 of B such that $D_2 \subset D_3$, and the closure in G of the algebraic tensor product $D_3 \odot F$ contains E_2.

By [**431**, lem. 3.4], we find a separable relatively weakly injective C^*-subalgebra D_4 of B with $D_3 \subset D_4$. Then $E_2 \subset D_4 \otimes^{max} F$.

In this way we get sequences $D_1 \subset D_2 \subset \ldots$ and $E_1 \subset E_2 \subset \ldots$ of separable C^*-subalgebras of B and G, respectively, such that D_{2n} is relatively weakly injective in B, $D_{2n} \otimes^{max} F = E_{2n-1}$, and every closed ideal of E_{2n} is the intersection of E_{2n} with a closed ideal of G.

Let $D \subset B$ denote the closure of the union of algebras D_n. Then D is relatively weakly injective in B, and $D \otimes^{max} F$ is just the closure E of the union of the algebras E_n.

Let $J \in \mathcal{I}(E)$. The closure I of the union of $I_n := \Psi^{E_{2n},G}_{\text{down}}(E_{2n} \cap J)$ contains J, because the union of $E_{2n} \cap J$ is dense in J.

$I_n \cap E$ is contained in J, because, for $n \leq m$, $I_n \subset I_m$, and

$$E_{2m} \cap I_n \subset E_{2m} \cap I_m = E_{2m} \cap J,$$

and the union of $E_{2m} \cap I_n$ is dense in $I_n \cap E$. Thus $J = I \cap E$. By the last observation in Remark 12.2.10, D is residually relatively weakly injective in B.

Now let $J \in \mathcal{I}(D)$. Then $\Psi^{D,B}_{\text{down}}(J) \otimes^{max} F$ is the closed ideal of $B \otimes^{max} F$ which is generated by the closed ideal $I := J \otimes^{max} F$ of $E = D \otimes^{max} F$, i.e., is $\Psi^{E,G}_{\text{down}}(I)$. It contains $(D \cap \Psi^{D,B}_{\text{down}}(J)) \otimes^{max} F$, and its intersection with $E = D$ is I. Thus $J \otimes^{max} F$ contains $(D \cap \Psi^{D,B}_{\text{down}}(J)) \otimes^{max} F$. If we apply a character χ of F to this, we get that $J = D \cap \Psi^{D,B}_{\text{down}}(J))$.

Now we change the notation and iterate the constructions:

We find inductively sequences (C_n), (D_n) and (E_n) of separable C^*-subalgebras of B, such that $C \subset C_1$, $C_n \subset D_n \subset E_n$, and $E_n \subset C_{n+1}$, $AC_n \subset C_n$,

$\|L_a|C_1\| = \|a\|$ for $a \in A$, D_n is strongly p.i., E_n is residually relatively weakly injective in B, and $\Psi_{\text{down}}^{E_n,B}(J) \cap E_n = J$ for every $J \in \mathcal{I}(E_n)$.

Let D be the closure of the union of this sequence. D contains C. By Remark 12.2.9, D is strongly p.i. and, by Remark 12.2.10, D is residually relatively weakly injective in B. It is easy to see that D satisfies condition (v).

Let $J \in \mathcal{I}(D)$ and $J_n := E_n \cap J$. Then $\Psi_{\text{down}}^{D,B}(J)$ is the closure of the increasing sequence of closed ideals $I_n := \Psi_{\text{down}}^{E_n,B}(J_n)$, and $I_n \cap E_m \subset I_m \cap E_m = J_m$ for $n < m$. Thus $I_n \cap D \subset J$ and, therefore $\Psi_{\text{down}}^{D,B}(J) \cap D = J$. Thus D satisfies (i)-(v).

Now let $D_1 \subset D_2 \subset \ldots$ be an increasing sequence of separable C^*-algebras D_n of B, and suppose that D_n satisfies (i)-(v).

Let D the closure of $\bigcup D_n$. By Remarks 12.2.9 and 12.2.10, D has also properties (ii) and (iv). Above we have seen that (iii) is preserved under inductive limits, i.e. D satisfies (iii). It is easy to check that D satisfies (v).

Now suppose that B is stable. By Remark 5.1.1(8), this means that there exists a sequence $t_1, t_2, \ldots$ of isometries in $\mathcal{M}(B)$, such that $\sum t_n(t_n)^*$ strictly converges to 1. We replace A by the separable C^*-subalgebra A_1 of $\mathcal{M}(B)$ which is generated by A and $\{t_1, t_2, \ldots\}$. Then a separable C^*-subalgebra D of B is stable, if D satisfies (i)-(v) for A_1 in place of A:

$s_n := L_{t_n}|D$ are isometries in $\mathcal{M}(D)$ such that $\sum s_n(s_n)^*$ strictly converges to 1 in $\mathcal{M}(D)$.

Suppose that B is stable and has residually nuclear separation.

Let $D_1 := D$ be above found separable stable C^*-subalgebra D of B with (i)-(v) for A_1, let B_1 denote the closure of DBD, and let M_1 be a countable dense subset of the positive cone of D_1.

Residually nuclear separation passes to hereditary C^*-subalgebras. Therefore we can find a countable set S_1 of residually nuclear completely positive contractions from D_1 into B_1 such that, for $a \in M_1$ and $n \in \mathbb{N}$, there is a $V \in S_1$ with $\|a - V(a)\| < 1/n$.

The union the images of the $V \in S_1$ is a separable subset of B_1. Thus it is contained in a separable C^*-subalgebra C_2 of B_1. Necessarily, C_2 contains M_1 and, therefore, also D_1.

We find a separable C^*-subalgebra D_2 of B_1 with (i)-(v) for A_1, C_2 and B_1 in place of A, C and B, because B_1 is again strongly purely infinite, $C_2 \subset B_1$, and $A_1 \cong L(A_1)|B_1$ is contained in $\mathcal{M}(B_1)$. If M_2 is a countable dense subset of the positive cone of D_2, then we can again find a countable set S_2 of residually nuclear completely positive contractions from D_2 into B_1 such that, for $a \in M_2$ and $n \in \mathbb{N}$, there is a $V \in S_2$ with $\|a - V(a)\| < 1/n$.

If we go so on, we get sequences $D_1 \subset D_2 \subset \ldots$, $S_1, S_2, \ldots$ of separable C^*-subalgebras D_n of B_1 and of countable sets S_n of residually nuclear completely positive contractions from D_n to B_1 such that, for $n \in \mathbb{N}$:

(i) $C \subset D_1$, D_1 is stable and contains a strictly positive element of D_n
(ii) D_n is strongly purely infinite,
(iii) for every closed ideal J of D_n, $J = D_n \cap \Psi_{\mathrm{down}}^{D_n,B}(J)$, and,
(iv) D_n is residually relatively weakly injective in B,
(v) $A_1 D_n \subset D_n$ and $\|a\| = \|L_a|D_1\|$ for $a \in A$.
(vi) $V(D_n) \subset D_{n+1}$ for $V \in S_n$, and,
(vii) for every $a \in (D_n)_+$ and every $\delta > 0$ there is $V \in S_n$ with $\|V(a) - a\| < \delta$.

Let D denote the closure of the union of the C^*-algebras D_n. Then D satisfies conditions (i)-(v) of Lemma 12.2.14 and D is stable. By Lemma 12.2.12, the completely positive maps $V \in S_n$ are also residually nuclear if we consider them as maps from D_n to D.

Now we argue as in the proof of the implication (ii)⇒(i) in the proof of Lemma 12.1.2:

Since, by Kasparov stabilization theorem, $\mathcal{H}_D \cong D$ as (left) Hilbert D-module, we can take Kasparov-Stinespring dilations of the V in S_n, infinite repeats and direct sums. We get a sequence of non-degenerate weakly residually nuclear *-monomorphisms $h_n \colon D_n \to \mathcal{M}(D)$, such that $\delta_\infty h_n$ is unitarily equivalent to h_n, and $h_n(J \cap D_n) = h_n(D_n) \cap \mathcal{M}(D, J)$ for $J \in \mathcal{I}(D)$. By Lemma 12.1.2, it follows that $h_n|D_m$ is unitarily homotopic to h_m for $m < n$.

This allows to replace the h_n by unitarily equivalent *-monomorphisms k_n such that the sequence k_n is convergent in point-norm on every D_m.

The limit is a *-monomorphism h from D into $\mathcal{M}(D)$ with $h(J) = h(D) \cap \mathcal{M}(D, J)$ for $J \in D$, and $h|D_n$ is weakly residually nuclear for every $n \in \mathbb{N}$. It follows that h is residually nuclear, and that D is the closure of $Dh(D)D$.

By Lemma 12.1.2, we can construct from h with the above properties a non-degenerate residually nuclear *-monomorphism H_0 from D into $\mathcal{M}(D)$ with $H_0(J) = H_0(D) \cap \mathcal{M}(D, J)$ for $J \in \mathcal{I}(D)$ such that H_0 is unitarily equivalent to its infinite repeat $\delta_\infty H_0$.

Now suppose that B is σ-unital and let $e \in B_+$ strictly positive.

Then we can find D with the properties (i)-(v) and $e \in D$ (simply by replacing C by $C^*(C, e)$). It holds $L(A \cap \mathcal{M}(B, I)) = L(A) \cap \mathcal{M}(D, D \cap I)$ for every $I \in \mathcal{I}(B)$, because, for $a \in A$

$$a \in \mathcal{M}(B, I) \quad \Leftrightarrow \quad ae \in I \quad \Leftrightarrow \quad aD \subset D \cap I \quad \Leftrightarrow \quad a \in \mathcal{M}(D, D \cap I)\,.$$

If $\mathrm{id}\,|A \colon A \hookrightarrow \mathcal{M}(B)$ is weakly residually nuclear, then $L \colon A \to \mathcal{M}(D)$ is weakly residually nuclear, by (iv) and Lemma 12.2.12, because we replace B by its hereditary C^*–subalgebra DBD (with strictly positive $e \in D_+$). □

The following Proposition 12.2.15 generalizes the first part of Theorem A. Here the target algebra B replaces $\mathcal{O}_2$ in Theorem A, and B is in general not simple.

Theorem A(i) is implied by Proposition 12.2.15, because a separable exact algebra A always has a nuclear *-monomorphism into $\mathcal{L}(\ell_2) \subset \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$.

PROPOSITION 12.2.15 (Trivialization and Embedding).

Suppose that B is strongly purely infinite, stable, σ-unital and has residually nuclear separation, and that A is stable, separable and exact.

*Let H be a non-degenerate *-monomorphism from A into $\mathcal{M}(B)$, such that $a \in A \mapsto b^*H(a)b \in B$ is nuclear for every $b \in B$.*

The following properties (i) and (ii) are equivalent:

(i) *There exists a non-degenerate nuclear *-monomorphism $h\colon A \otimes \mathcal{O}_2 \hookrightarrow B$ such that $\delta_\infty h_0$ and $\delta_\infty H$ are unitarily homotopic, where $h_0(a) := h(a \otimes 1)$ for $a \in A$, and $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ is the infinite repeat, as defined in Remark 5.1.1(8).*

(ii) *The lower semi-continuous action*

$$\Psi\colon J \in \mathcal{I}(B) \mapsto H^{-1}(H(A) \cap \mathcal{M}(B, J)) \in \mathcal{I}(A),$$

of Prim(B) *on A is also monotone upper semi-continuous (i.e., satisfies condition (ii) of Definition 1.2.6).*

Note that $h_0 = h_0 \oplus_{s,t} h_0$ if we use the isometries $s, t \in \mathcal{M}(B)$ which are given by $s := \mathcal{M}(h)(1 \otimes r_1)$ and $t := \mathcal{M}(h)(1 \otimes r_2)$ if r_1 and r_2 denote the canonical generators of $\mathcal{O}_2$.

PROOF. (i)⇒(ii): For $a \in A$, $\delta_\infty(H(a)) \in \mathcal{M}(B, J)$ if and only if $\delta_\infty(h_0(a)) \in \mathcal{M}(B, J)$, because they are unitarily homotopic in $\mathcal{M}(B)$. For stable σ-unital B, we have that

$$\delta_\infty(\mathcal{M}(B)) \cap \mathcal{M}(B, J) = \delta_\infty(\mathcal{M}(B, J)),$$

i.e., $b \in \mathcal{M}(B, J)$ if and only if $\delta_\infty(b) \in \mathcal{M}(B, J)$ for $b \in \mathcal{M}(B)$. Since $B \cap \mathcal{M}(B, J) = J$, it follows, that $h_0(A) \cap J = h_0(\Psi(J))$. By Lemma 12.1.1(iii), $J \mapsto h_0^{-1}(h_0(A) \cap J)$ satisfies condition (ii) of Definition 1.2.6.

(ii)⇒(i): We consider at first the case that B is separable.

By Remark 5.1.1(8), there is a strictly continuous unital C^*-morphism from $\mathcal{M}(\mathbb{K})$ into the relative commutant R of $\delta_\infty(H(A))$ in $\mathcal{M}(B)$. It follows that R contains a unital copy of $\mathcal{O}_2$ and a sequence of isometries $r_1, r_2, \ldots$, such that the isometries r_n commute with the elements of $\mathcal{O}_2$, and that $\sum r_n(r_n)^*$ strictly converges to 1 in $\mathcal{M}(B)$.

There is a non-degenerate *-monomorphism $K\colon A \otimes \mathcal{O}_2 \to \mathcal{M}(B)$ with $K(a \otimes b) = \delta_\infty(H(a))b$ for $a \in A$ and $b \in \mathcal{O}_2$. The isometries r_n commute with the image of K. Thus, by Lemma 5.1.2(i), $\delta_\infty K$ is unitarily equivalent to K.

If we can find a non-degenerate nuclear *-monomorphism h from $A \otimes \mathcal{O}_2$ into B, such that $\delta_\infty h$ is unitarily homotopic to $\delta_\infty K$, then the infinite repeat of $h_0 := h((.) \otimes 1)$ is unitarily homotopic to $(\delta_\infty)^2 H$. But δ_∞^2 is unitarily equivalent to δ_∞ by Lemma 5.1.2(i).

Thus, by Theorem 6.3.1, it suffices to show that, for $a_1, \ldots, a_n \in K(A \otimes \mathcal{O}_2)$ and for $\varepsilon > 0$, there exist completely positive contractions $V \colon \mathcal{M}(B) \to B$ and $W \colon B \to \mathcal{M}(B)$ such that

(a) $\|W \circ V(a_j) - a_j\| < \varepsilon$, for $j = 1, \ldots, n$.

(b) V is strictly continuous and is residually equivariant, i.e., $\lim_n \|V(b_n) - V(b)\| = 0$ if $b_n \to b$ in $\mathcal{M}(B)$ strictly and $V(J) \subset J \cap B$ for $J \in \mathcal{I}(\mathcal{M}(B))$.

`Old version was only:` V is weakly residually nuclear.

(c) $W \colon B \to \mathcal{M}(B)$ is weakly residually nuclear, i.e., $W(J)B \subset J$ for $J \in \mathcal{I}(B)$, and the c.p. maps $W_a \colon b \in B \mapsto a^*W(b)a \in B$ are residually nuclear for all $b \in B$. I.e. the maps $[W]_J \colon B/J \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B,J)$ have the property that $([W]_J)_d \colon b \in B/J \mapsto d^*[W]_J(b)d \in B/J$ is a nuclear map for all $d \in B/J$. Here $[W]_J(a + J) := W(a) + \mathcal{M}(B, J)$, i.e., $d^*[W]_J(b)d = \pi_J(f^*W(a)f)$ for $b = a + J$ and $d = f + J$.

`Old version was only:` W is weakly residually equivariant, i.e. $W(J) \subset \mathcal{M}(B, J)$ for $J \in \mathcal{I}(B)$.

`change the old version for Thm. 6.3.1, as executed below:` ??

`NEXT comes from some beamer presentation:`

(1) $H_0(A)$ is non-degenerate, i.e., $H_0(A)B$ is dense in B, and is in "general position" (i.e., there exists a unitary $U \in \mathcal{M}(B)$ $U^*H_0(\cdot)U = \delta_\infty \circ H_0$).

(2) The given lower s.c. action $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is realized by $\Psi(J) := H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J))$ and Ψ is monotone upper semi-continuous, i.e., $\bigcup_n \Psi(J_n)$ is dense in $\Psi(J)$ for $J := \overline{\bigcup_n J_n}$, if the sequence $J_n \in \mathcal{I}(B)$ is increasing: $J_1 \subset J_2 \subset \cdots$.

(3) For every $b \in B$, the Ψ_A-compatible map $A \ni a \mapsto b^*H_0(a)b \in B$ is nuclear, and can be approximated by compositions $V_2 \circ V_1$ of the residually nuclear maps $V_1 \colon a \in A \mapsto b_1^*H_1(a)b_1 \in B$ and $V_2 \colon b \in B \mapsto b_2^*\lambda(b)b_2$. I.e. V_2 satisfies $V_2(J) \subset J$ for all $J \in \mathcal{I}(B)$ and that $[V_2]_J \colon B/J \to B/J$ is nuclear for all $J \in \mathcal{I}(B)$.

Step 2:
If A satisfies (1)–(3), then, – with $a \in A$ identified with $H_0(a) \in H_0(A) \subset \mathcal{M}(B)$ – for every $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, there exist completely positive contractions $V \colon \mathcal{M}(B) \to B$ and $W \colon B \to \mathcal{M}(B)$ that satisfy the above listed conditions (a), (b) and (c).

More on Step 2:

Consider the set of maps $V := V_c \colon \mathcal{M}(B) \to B$ given by $V_c \colon b \in \mathcal{M}(B) \mapsto c^*\mathcal{M}(\lambda)(b)c$. The point-norm closure is an m.o.c. cone $\mathcal{C}_1$.

`What is` $\lambda \colon B \to \mathcal{M}(B)$??

Perhaps $\lambda \colon B \to \mathcal{M}(B)$ a residually nuclear separation?

Do the same with the maps $W := W_T \colon B \to \mathcal{M}(B)$ given by $W_T(b) := T^*\lambda(b)T$, for $T \in \mathcal{M}(B)$, and we denote this m.o.c. cone by $\mathcal{C}_2$.

Both cones are singly generated (as m.o.c.c.), e.g. $\mathcal{C}_1$ by $e\mathcal{M}(\lambda)(\cdot)e$ where $e \in B_+$ is strictly positive, and $\mathcal{C}_2$ by λ (as u.c.p. map).

The above listed properties (b) and (c) for suitable $V = V_c$ and $W = W_T$ follow from the properties of λ and the fact that $\mathcal{M}(\lambda)$ is the unique strictly continuous extensions of λ.

Continuation 1: More on Step 2:

Notice that $\lambda(b)$ and $\delta_\infty(b)$ for each self-adjoint $b \in B$ must be unitarily homotopic in $\mathcal{M}(B)$ by the generalized W-vN theorem, because both define *-mono-morphisms from $C^*(b)$ into $\mathcal{M}(B)$ that are in "general position" and define the same action of $\mathrm{Prim}(B)$ on $C^*(b)$ by definition of λ, namely the action $J \in \mathcal{I}(B) \mapsto \mathcal{M}(B, J) \cap C^*(b)$.

It follows that each $W \in \mathcal{C}_2$ maps B into the closure I_0 of $\mathcal{M}(B)\delta_\infty(B)\mathcal{M}(B)$, the closed ideal of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$.

Moreover, $b \in \mathcal{M}(B)_+$ will be mapped by all $P \in \mathcal{C}_2 \circ \mathcal{C}_1$ into the ideal of $\mathcal{M}(B)$ that is the norm-closure of the union of the ideals $\mathcal{M}(B, J((ebe - 1/n)_+))$, with $J(b) \in \mathcal{I}(B)$ as defined above, $e \in B_+$ strictly positive contractions.

`The following Lemma supports the conjectures in ''Continuation 1'' and ''Continuation 2''.`

LEMMA 12.2.16. *Let B a σ-unital stable C^*-algebra, $e \in B_+$ a strictly positive contraction, and let $a \in \mathcal{M}(B)_+$ such that the natural l.s.c. action of* $\mathrm{Prim}(B)$ *on $C^*(a) \cong \mathrm{C}_0(\mathrm{Spec}(a) \setminus \{0\})$ is also monotone upper semi-continuous.*

Then a is in the closed ideal of $\mathcal{M}(B)$ that is generated by $\delta_\infty(eae)$.

*Suppose that – more generally – $\lambda\colon B \to \mathcal{M}(B)$ is a non-degenerate *-monomorphism such that $\delta_\infty \circ \lambda$ is approximately unitarily equivalent to λ and that* $\overline{\mathrm{span}(BbB)} = \overline{\mathrm{span}(B\lambda(b)B)}$ *for each $b \in B_+$.*

Then a is in the closed ideal of $\mathcal{M}(B)$ that is generated by $\lambda(eae)$.

PROOF. Let $I(b)$ denote the closed ideal of $\mathcal{M}(B)$ generated by $b \in \mathcal{M}(B)_+$ and let $J(b)$ the closed ideal of B generated by BbB. Note $J(\delta_\infty(b)) = J(b) = J(ebe) = J(b^{1/2}e^2b^{1/2}) = J(beb)$.

For $b \in \mathcal{M}(B)_+$ and $\gamma > 0$ holds, by Lemma 5.9.10, that

$$\delta_\infty((b-\gamma)_+) \in \mathcal{M}(B, J((b-\gamma)_+)) \subseteq I(\delta_\infty(b))\,.$$

Let $\varepsilon > 0$. Since, the natural action of $\mathrm{Prim}(B)$ on $C^*(a) \cong \mathrm{C}_0(\mathrm{Spec}(a) \setminus \{0\})$ is monotone upper semi-continuous, and since $J(eae)$ is the closure of $\bigcup_n J((eae - 1/n)_+)$, we get that a is in the closed union of the ideals $C^*(a) \cap \mathcal{M}(B, J((eae-\gamma)_+))$ for $\gamma \in (0,1)$.

It follows that $(a-\varepsilon)_+ \in \mathcal{M}(B, J((eae-\gamma)_+))$ for some suitable $\gamma > 0$. Then $(a-\varepsilon)_+ \in I(\delta_\infty(eae))$. Since $I(\delta_\infty(eae))$ is closed it follows $a \in I(\delta_\infty(eae))$.

`Add the missing cross references:`

The conditions on λ imply that, for $b \in B_+$, the upper s.c. action of $\mathrm{Prim}(C^*(b))$ on B defined by $C^*(b) \subset \mathcal{M}(B)$ and $\lambda\colon C^*(b) \to \mathcal{M}(B)$ are the same. This implies that the l.s.c. actions of $\mathrm{Prim}(B)$ on $C^*(b)$ defined by $\mathrm{id}\,|C^*(b)$ or $\lambda|C^*(b)$ are the same, because they are both the Galois adjoint of the same upper s.c. action, cf. Proposition **??**.

It follows by Corollary **??** that $\delta_\infty(\lambda(b))$ and $\delta_\infty(b)$ are approximately unitarily equivalent in $\mathcal{M}(B)$. This implies $I(\delta_\infty(\lambda(b))) = I(\delta_\infty(b))$. Since $\lambda(b)$ is approximately unitary equivalent to $\delta_\infty(\lambda(b))$ it follows $I(\lambda(b)) = I(\delta_\infty(b))$.

This applies in particular to $b := eae$.

`What about the` a `and` e `here ??`

It implies that $a \in I(\delta_\infty(eae)) = I(\lambda(eae))$. □

Continuation 2: More on Step 2:

This ideal is contained in the closed ideal of $\mathcal{M}(B)$ generated by $\delta_\infty(ebe)$ for a strictly positive element $e \in B_+$ of B.

?????????????????????

$b \in I(\delta_\infty(ebe))$ if and only if $b \in I(\delta_\infty(e)) =$ the closed ideal of $\mathcal{M}(B)$ generated by $\delta_\infty(B)$.

?????????????????????

`Conjecture:` There exists a sequence $P_n \in \mathcal{C}_2 \circ \mathcal{C}_1$ with $\|P_n(b) - b\| \to 0$ if and only if $b \in I(\delta_\infty(e))$.

¡– Seems to require further assumptions.

Continuation 3: More on Step 2:

The condition (a) is equivalent to $H_0 \in \mathcal{C}_3 := \mathcal{C}_2 \circ \mathcal{C}_1 \circ \mathcal{C}_{H_0}$. The m.o.c. cone $\mathcal{C}_3$ is contained in the cone of (norm-) nuclear c.p. maps from A into $\mathcal{M}(B)$, and the elements of $\mathcal{C}_3$ map A into the norm-closed ideal $I(\delta_\infty(B))$ of $\mathcal{M}(B)$ that is generated by $\delta_\infty(B)$.

More precisely, the elements of $\mathcal{C}_3$ map $a \in A_+$ into the closed ideal of $I(\delta_\infty(B)) \subset \mathcal{M}(B)$ generated by the element $\delta_\infty(e)\delta_\infty^2(e)\delta_\infty^3(a)\delta_\infty^2(e)\delta_\infty(e)$.

If there is $b \in B_+$ such that $\delta_\infty(b)$ and a generate the same closed ideal of $\mathcal{M}(B)$, then there are $T_n \in \mathcal{C}_2 \circ \mathcal{C}_1$ such that $\|T_n(a) - a\| \to 0$.

`END of TEXT from old beamer presentation`

`Idea: take unital weakly residual nuclear monomorphism` H `of` $\mathcal{M}(B)$ `in general position and use that` $H|C$ `and` id_C `are approximately unitarily equivalent by unitaries` $u_1, u_2, \ldots$.

`Then replace` W `by suitable` $u_n^* H(W(\cdot))u_n$.

Let $C := K(A \otimes \mathcal{O}_2)$, let $a_1, a_2, \ldots$ be a dense sequence in the unit ball of C, and let X_n denote the span of $\{a_1^*, a_1, \ldots, a_n^*, a_n\}$.

$\mathrm{id}_{\mathcal{M}(B)}|C$ is weakly residually nuclear, because C is exact and $\delta_\infty|A = K((\,\cdot\,)\otimes 1)$ is nuclear by the exactness of A. In fact, $\mathrm{id}\,|C\colon C \hookrightarrow \mathcal{M}(B)$ is even nuclear, see Chapters 3 and 5, where we have discussed those topics in every detail.

Since K is unitarily equivalent to δ_∞, $\mathrm{id}\,|C$ is unitarily equivalent to $\delta_\infty|C$.

By Remark 12.2.13, there exists a non-degenerate weakly residually nuclear *-monomorphism $d\colon B \to \mathcal{M}(B)$ such that $\delta_\infty d$ is unitarily homotopic to d, and, for $J \in B$,

$$d(J) = d(B) \cap \mathcal{M}(B, J).$$

Thus $\mathcal{M}(d)\,(\mathcal{M}(B,J)) = \mathcal{M}(d)(\mathcal{M}(B)) \cap \mathcal{M}(B,J)$ for $J \in \mathcal{I}(B)$.

It follows, that the *-monomorphisms $\mathcal{M}(d)|C$ and $\mathrm{id}\,|C$ from C to $\mathcal{M}(B)$ are both nuclear *-monomorphisms and induce the same action of $\mathrm{Prim}(B)$ on C, i.e., for $J \in \mathcal{I}(B)$,

$$\mathcal{M}(d)(C \cap \mathcal{M}(B,J)) = \mathcal{M}(d)(C) \cap \mathcal{M}(B,J).$$

Therefore, by Corollary 5.9.16, $\delta_\infty\mathcal{M}(d)|C$ and $\delta_\infty|C$ are unitarily homotopic. This implies, together with the unitary equivalence of $\delta_\infty|C$ and $\mathrm{id}\,|C$, the existence of a sequence $U_1, U_2, \ldots$ of unitaries in $\mathcal{M}(B)$, such that, for $n = 1, 2, \ldots$ and $b \in X_n$,

$$\|U_n^*\delta_\infty\mathcal{M}(d)(b)U_n - b\| \;<\; 4^{-n}\|b\|\,. \tag{1}$$

Let $e \in B_+$ a strictly positive contraction in B of norm one. By Remark 5.1.1(3), we find a sequence of functions $g_n \in \mathrm{C}_0((0,1])_+$, such that $g_n g_{n+1} = g_n$, $\|g_n\| = 1$, $g_n \to 1$ point-wise, and, for $b \in X_n$ and $m > n$,

$$\|[g_n(e), b]\| + \|[(g_m(e) - g_n(e))^{1/2}, b]\| < 4^{-n}\|b\|.$$

We define real numbers $\gamma(n,k,m,T)$, $\gamma(n,k,m)$ and $\gamma(n,k)$ for $k,m,n \in \mathbb{N}$ and for inner completely positive contractions $T\colon B \to B$ as follows:

$$\begin{aligned}
\gamma(n,k,m,T) &:= \sup\{\|T(g_k(e)ag_k(e)) - g_m(e)ag_m(e)\|\;;\; a \in X_n, \|a\| \le 1\}\\
\gamma(n,k,m) &:= \inf_T \gamma(n,k,m,T)\\
\gamma(n,k) &:= \sup_m \gamma(n,k,m).
\end{aligned}$$

Then $\gamma(n,k,m) = 0$ for $m \le k$, $\gamma(n,k,m) \le \gamma(n+1,k,m)$, $\gamma(n,k,m) \le \gamma(n,k+1,m)$ and $\gamma(n,k,m) \le \gamma(n,k,m+1)$.

Thus, $\gamma(n,k+1) \le dz\gamma(n,k) \le \gamma(n+1,k)$.

Now we show that $\lim_{k\to\infty}\gamma(n,k) = 0$.

Since $k \mapsto \gamma(n,k)$ is decreasing and $m \mapsto \gamma(n,k,m)$ is increasing, it is enough to show that $\lim_{j\to\infty}\gamma(n,k_j,k_{j+1}) = 0$ for every sequence $k_1 < k_2 < \ldots$.

Let $p_j := g_{k_j}(e)$ for $j \in \mathbb{N}$, and $P\colon C \to \ell_\infty(B)$, $Q\colon C \to \ell_\infty(B)$ the completely positive maps which are defined by $P(c) := (p_1cp_1, p_2cp_2, \ldots)$ and $Q(c) := (p_2cp_2, p_3cp_3, \ldots)$.

To see $\lim_{j\to\infty} \gamma(n, k_j, k_{j+1}) = 0$, it suffices to show that, for every $\varepsilon > 0$, there exists an inner completely positive contraction T on $\ell_\infty(B)$, such that, for $a \in X_n$,

$$\operatorname{dist}(T(P(a)) - Q(a), c_o(B)) < \|a\|\varepsilon.$$

Let h_1 denote the C^*-morphism from $C \otimes \mathrm{C}_0((0,1]) \cong \mathrm{C}_0((0,1], C)$ to $\ell_\infty(B)/c_0(B)$, with $h_1(c\otimes f) = \pi((cf(p_1), cf(p_2), \ldots))$ for $c \in C$ and $f \in \mathrm{C}_0((0,1])$, where π means the natural quotient map from $\ell_\infty(B)$ onto $\ell_\infty(B)/c_0(B)$.

Further let h_2 the *-epimorphism from $C \otimes \mathrm{C}_0((0,1])$ onto C, with $h_2(c\otimes f) = f(1)c$.

Note that $h_1(c \otimes f_0^2) = \pi(P(c))$, where $f_0(t) = t$. We show that h_1 approximately inner dominates $V := \pi Q h_2$. Then the proof of $\lim_k \gamma(n,k) = 0$ is ready, because it follows that, for $\varepsilon > 0$, there is an inner completely positive contraction T' on $\ell_\infty(B)/c_0(B)$, such that, for $a \in X_n$,

$$\|T'(\pi(P(a))) - \pi(Q(a))\| < \|a\|\varepsilon/2.$$

(because C is exact).

Since every completely positive contraction $c \mapsto p_j c p_j$, for $j > 1$, is a nuclear map, its direct sum Q is a nuclear map from C into $\ell_\infty(B) \subset \mathcal{M}(c_0(B))$. By Corollary 5.6.3, Q is nuclear, because C is exact. Thus V is nuclear.

Now we show that $V(a \otimes f) = f(1)\pi(Q(a))$ is in the closed ideal generated by $h_1(a \otimes f)$ for $a \in C_+$, $f \in \mathrm{C}_0((0,1])_+$:

$V(a \otimes f) = 0$, if $f(1) = 0$. But if $f(1) > 0$, then there is $\delta \in (0,1)$, such that $f(t) > f(1)/2$ for $t \in (\delta, 1]$. Define $\lambda \in \mathrm{C}_0((0,1])$ by $\lambda(t) := 0$ on $[0,\delta]$, $\lambda(t) :=], \min(1, 2(t-\delta)/(1-\delta))$ for $t \in (\delta, 1]$. Then $f(1)(a \otimes \lambda) \le a \otimes f$, and it suffices to show that $\pi Q(a)$ is in the closed ideal generated by $\pi(b)$, where

$$b := (a^{1/2}\lambda(p_1)a^{1/2}, a^{1/2}\lambda(p_2)a^{1/2}, \ldots).$$

By Corollary 3.10.12 and Proposition 2.12.8(iii),

what is needed: prop:2.24 ??? ??

this follows if we can show that the sequence of real numbers

$$\lambda_n = \max(0, \sup\{\|p_{n+1}ap_{n+1} + J\| - \|a^{1/2}\lambda(p_n)a^{1/2} + J\| \colon J \in \operatorname{Prim}(B)\})$$

converges to zero. We have

$$\|p_{n+1}ap_{n+1} + J\| = \|a^{1/2}p_{n+1}^2 a^{1/2} + \mathcal{M}(B,J)\| \le \|a + \mathcal{M}(B,J)\|.$$

By definition of λ, $\lambda(p_n)\lambda(p_{n+1}) = \lambda(p_n)$, $\|\lambda(p_n)\| = 1$ and $\lim \|\lambda(p_n)e - e\| = 0$.

Since $a^{1/2}\lambda(p_n)a^{1/2} \le a$ and $\lambda(p_n)$ is an increasing approximate unit of B, we get that $\xi_n(J) := \|a^{1/2}\lambda(p_n)a^{1/2} + J\|$ converges to $\|a + \mathcal{M}(B,J)\|$ for every $J \in \operatorname{Prim}(B)$.

The natural lower semi-continuous action $\Psi^{\mathrm{up}}_{B,C}$ from $\operatorname{Prim}(B)$ on C satisfies, for $J \in \mathcal{I}(B)$,

$$\delta_\infty(H(\Psi(J))) = \delta_\infty(H(A)) \cap \Psi^{\mathrm{up}}_{B,C}(J)\,.$$

Here we used that $\delta_\infty(\mathcal{M}(B,J))$ is the same as $\delta_\infty(\mathcal{M}(B)) \cap \mathcal{M}(B,J)$. Since $\mathcal{O}_2$ is simple and nuclear, and since $K(A \otimes 1) = \delta_\infty(H(A))$, we get that, for $J \in \mathcal{I}(B)$,

$$\Psi^{\mathrm{up}}_{B,C}(J) = K(\Psi(J) \otimes \mathcal{O}_2)\,.$$

Thus $\Psi^{\mathrm{up}}_{B,C}$ satisfies the condition (ii) of Definition 1.2.6, because Ψ satisfies this condition by assumption.

By Corollary 12.2.8(ii), the property (ii) of Definition 1.2.6 for $\Psi^{\mathrm{up}}_{B,C}$ implies that, for every $a \in C_+$, the real function

$$J \mapsto \widehat{a}(\Psi^{\mathrm{up}}_{B,C}(J) = \|a + \mathcal{M}(B,J)\|$$

is a Dini function on $\mathrm{Prim}(B)$.

Therefore, the functions ξ_n converge on $\mathrm{Prim}(B)$ uniformly to the function $J \mapsto \|a + \mathcal{M}(B,J)\|$, because ξ_n is an increasing sequence of lower semi-continuous functions which converges point-wise to $J \mapsto \|a + \mathcal{M}(B,J)\|$ on $\mathrm{Prim}(B)$.

Let $\theta_n := \sup\{\|a + \mathcal{M}(B,J)\| - \xi_n(J)\,;\ J \in \mathrm{Prim}(B)\}$.

Then $0 \le \lim \lambda_n \le \lim \theta_n = 0$. Thus $V(a \otimes f)$ is in the closed ideal generated by $h_1(a \otimes f)$ for $a \in C_+$, $f \in \mathrm{C}_0((0,1])_+$.

By Proposition B.4.2(ii), it follows now that, for every closed ideal J of $C \otimes \mathrm{C}_0((0,1])$, $V(J)$ is in the closed ideal of $\ell_\infty(B)/c_0(B)$ that is generated by $h_1(J)$. Thus, the kernel of h_1 is contained in the kernel of V, and the natural completely positive map S from the image of h_1 into $\ell_\infty(B)/c_0(B)$ is residually equivariant.

Since $C \otimes \mathrm{C}_0((0,1])$ is exact, by Remark 3.1.2(iv), $h_1(C \otimes \mathrm{C}_0((0,1]))$ is exact and S is nuclear.

By Corollary 3.10.7, S is approximately inner.

This completes the proof of $\lim_{k\to\infty} \gamma(n,k) = 0$.

We can take a sequence $1 < k_1 < k_2 < \ldots$ of positive integers such that $\gamma(n,k_n) < 4^{-n}$, and get an approximate unit $e_n := g_{k_n}(e) \in B_+$, of B with $e_{n+1}e_n = e_n$, $\|e_n\| = 1$, and, for $b \in X_n$,

$$\|[e_n, b]\| + \|[(e_{n+1} - e_n)^{1/2}, b]\| < 4^{-n}\|b\|. \tag{2}$$

Since $\gamma(n,k_n) < 4^{-n}$, we find inner completely positive contractions $T_n \colon B \to B$, such that, for $b \in X_n$

$$\|T_n(e_n b e_n) - e_{n+1} b e_{n+1}\| < 4^{-n}\|b\|. \tag{3}$$

`check next construction with new (a),(b),(c) of 6.3.1` ??

We define the desired weakly residually equivariant completely positive contractions $W_n \colon B \to \mathcal{M}(B)$ and $V_n \colon \mathcal{M}(B) \to B$, such that V_n is weakly residually nuclear, and $(W_n \circ V_n)|C$ tends to $\mathrm{id}\,|C$ in point-norm topology:

Let $n \in \mathbb{N}$ fixed and $k > n$. The inner completely positive contractions

$$L_k := T_{k+1} \circ T_k \circ \ldots \circ T_{n+1} T_n$$

of B are given by $L_k(b) := \sum_{j=1}^{m_k}(c_{k,j})^* b c_{k,j}$ for $b \in B$, where $c_{k,j} \in B$, $j = 1, \ldots, m_k$, $\sum_{j=1}^{m_k}(c_{k,j})^* c_{k,j}$ is a contraction. Note that, by induction, (3) implies

$$\|L_k(e_n b e_n) - e_{k+2} b e_{k+2}\| < 4^{1-n}/3, \tag{4}$$

for $b \in X_n$ and $k > n$.

Let $t_1, t_2, \ldots$ denote the sequence of isometries in $\mathcal{M}(B)$ with $\sum t_k(t_k)^*$ strictly convergent to 1, such that our infinite repeat δ_∞ is defined by $\sum t_k(\cdot)t_k^*$, cf. Remark 5.1.1(8).

Let $f_n := e_n^{1/2}$, and $f_k := (e_{k+1} - e_k)^{1/2}$ for $k > n$. We define, by induction, integers $p(n+1) := 1$ and, $p(k+1) = p(k) + m_k$ for $k > n$, and elements of $\mathcal{M}(B)$ by $d_n := 1$, $d_k := \sum_{j=1}^{m_k}(t_{p(k)+j})c_{k,j}$. Note d_k is a contraction, because $(d_k)^* d_l = \delta_{k,l} \sum_j (c_{k,j})^* c_{k,j}$.

Then $\sum_{k>n} d_k f_k$ is strictly convergent in $\mathcal{M}(B)$ and

$$S_n := \sum_{k \geq n} d_k f_k$$

is a contraction in $\mathcal{M}(B)$, cf. Remark 5.1.1(???????). **??**

Let, for $b \in \mathcal{M}(B)$,

$$W_n(b) := S_n^* \delta_\infty(b) S_n,$$

and

$$V_n(b) := e_n U_n^* \delta_\infty(\mathcal{M}(d)(b)) U_n e_n.$$

Then, $W_n|B$ is a weakly residually equivariant contraction from B into $\mathcal{M}(B)$, V_n is a weakly residually nuclear contraction from $\mathcal{M}(B)$ to B, and, for $b \in X_n$,

$$\|W_n V_n(b) - b\| \, < \, 4^{1-n} 3 \|b\|.$$

This estimate summarizes the estimate (1) and the estimates for $\|\Gamma(x-y)\|$ and $\|\Gamma(y) - b\|$ otained by summing the inequalities (4) and (2). Here Γ denotes the completely positive contraction from $\ell_\infty(B)$ into $\mathcal{M}(B)$, which is given by

$$\Gamma(b_1, b_2, \ldots) \; := \; (e_n)^{1/2} b_n (e_n)^{1/2} + \sum_{k>n} f_k b_k f_k,$$

and x, y denote the elements of $\ell_\infty(B)$, which are defined by $x_k := L_k(e_n b e_n)$ and $y_k := e_{k+2} b e_{k+2}$ for $k = 1, 2, \ldots$.

Note that $\Gamma(x) = W_n(e_n x e_n)$ and $\Gamma(y) = (e_n)^{1/2} b (e_n)^{1/2} + \sum_{k>n} f_k b f_k$. (See Remark 5.1.1(?????) **??** for the calculation of the needed estimates from those of (2) and (4).)

Thus Theorem 6.3.1 applies in the separable case.

`Here end of use of 6.3.1`

If B is *not* separable, then we can consider instead of B a separable C^*-subalgebra D of B, which contains a strictly positive element of B, is invariant by right and left multiplication by elements of $H(A)$, and satisfies all the properties listed in Lemma 12.2.14 with respect to $H(A)$, B and D. In particular,

D is residually weakly injective in B, $H(A)D \subset D$, and $D \cap \Psi_{\text{down}}^{D,B}(I) = I$ for $I \in \mathcal{I}(D)$. It follows $H(A) \subset \mathcal{M}(D) \subset \mathcal{M}(B)$ and, therefore, by Lemma 12.1.1(iv), $H(A) \cap \mathcal{M}(B, J) = H(A) \cap \mathcal{M}(D, D \cap J)$ for $J \in \mathcal{I}(B)$. Thus, $\Psi(J) = \Phi(D \cap J)$ for $J \in \mathcal{I}(B)$, where $\Phi(I) := H^{-1}(H(A) \cap \mathcal{M}(D, I))$ for $I \in \mathcal{I}(D)$.

By Lemma 12.2.12, it follows that $H\colon A \to \mathcal{M}(D)$ is is a weakly Φ-residually nuclear map from A to $\mathcal{M}(D)$.

Since $\Psi_{\text{down}}^{D,B}$ is upper semi-continuous, and $\Phi(I) = \Psi(\Psi_{\text{down}}^{D,B}(I))$ for $I \in \mathcal{I}(D)$, we get from (ii), that the lower semi-continuous action Φ also satisfies condition (ii) of Definition 1.2.6.

By construction, D is again strongly purely infinite and has residually nuclear separation.

Thus, by the above considered separable case, there exists a non-degenerated nuclear *-monomorphism $h\colon A \otimes \mathcal{O}_2 \to D \subset B$, such that the infinite repeat of $h_0 := h((\cdot) \otimes 1)$ is unitarily homotopic to the infinite repeat of H in $\mathcal{M}(D)$. Since D is a stable and non-degenerate C^*-subalgebra of B, this implies that $\delta_\infty h_0$ is unitarily homotopic to $\delta_\infty H$ in $\mathcal{M}(B)$. □

Remark 12.2.17. For every C^*-algebra B, every separable C^*-subalgebra D of $\mathrm{Q}(\mathbb{R}_+, B)$ and every commutative separable C^*-subalgebra C of D, there exists a strongly continuous map $t \mapsto V(t)$ from $\mathbb{R}_+$ into the approximately inner completely positive contractions on B, such that the completely positive contraction T on $\mathrm{Q}(\mathbb{R}_+, B)$ defined by V fixes the elements of C and $T(D)$ is contained in a commutative C^*-algebra of $\mathrm{Q}(\mathbb{R}_+, B)$.

This can be seen as follows: If g is a contraction in $\mathrm{C_b}(\mathbb{R}_+, B \otimes \mathcal{O}_\infty)$ and f a pure state on $\mathcal{O}_\infty$ then

$$V(t)(b) := (\mathrm{id} \otimes f)(g(t)^*(b \otimes 1)g(t))$$

defines an approximately inner completely positive contraction $V(t)$ of B such that $t \mapsto V(t)$ is strongly continuous. For the corresponding completely positive contraction T on $\mathrm{Q}(\mathbb{R}_+, B)$ and the contraction $d := g + \mathrm{C}_0(\mathbb{R}_+, B)$ in $\mathrm{Q}(\mathbb{R}_+, B \otimes \mathcal{O}_\infty)$ holds $T(a) = P(d^*(a \otimes 1)d)$ for $a \in \mathrm{Q}(\mathbb{R}_+, B)$. Here the completely positive contraction P from $\mathrm{Q}(\mathbb{R}_+, B \otimes \mathcal{O}_\infty)$ onto $\mathrm{Q}(\mathbb{R}_+, B)$ is given on representatives $e \in \mathrm{C_b}(\mathbb{R}_+, B \otimes \mathcal{O}_2)$ by $e(t) \mapsto (\mathrm{id} \otimes f)(e(t))$ for $t \in \mathbb{R}_+$, and $\mathrm{Q}(\mathbb{R}_+, B) \otimes \mathcal{O}_\infty$ is naturally embedded in $\mathrm{Q}(\mathbb{R}_+, B \otimes \mathcal{O}_\infty)$. Now choose d for $D \otimes 1$ and $C \otimes 1$ as in Lemma 12.1.6 to get the desired result.

If B is purely infinite then, for every positive element a in $\mathrm{Q}(\mathbb{R}_+, B)$, $T(a)$ is in the closed ideal generated by a: For $\delta > 0$, $(T(a) - \delta)_+ = e^*ae + f^*af$ for some $e, f \in \mathrm{Q}(\mathbb{R}_+, B)$. It follows that then $T|D$ is residually nuclear. This shows:

For every purely infinite C^*-algebra B the asymptotic corona $\mathrm{Q}(\mathbb{R}_+, B)$ has residually nuclear separation in the sense of Definition 1.2.3 (cf. Remark 12.2.10).

If, moreover, B is strongly purely infinite then there exists a contraction d in $\mathrm{Q}(\mathbb{R}_+, B)$ with $T(b) = d^*bd$ for $b \in D$. d commutes elementwise with C, as we have shown on the beginning of the proof of Lemma 12.1.6.

LEMMA 12.2.18. *Suppose that B is stable and σ-unital. Let $B_1 := B \otimes F$, where $F := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$. Then there is a non-degenerate *-monomorphism $d\colon B_1 \hookrightarrow \mathcal{M}(B)$, such that*

(i) *$\delta_\infty d$ is unitarily equivalent to d, and*
(ii) *$d(B_1) \cap \mathcal{M}(B, J) = d(J \otimes F)$ for $J \in \mathcal{I}(B)$.*

PROOF. Since B is stable, we can apply the infinite repeat δ_∞ to $\mathcal{M}(B)$. There is a unital copy of $\mathcal{L}(\ell_2)$ in the relative commutant of $\delta_\infty(\mathcal{M}(B))$ in $\mathcal{M}(B)$. Therefore, a unital copy of $F \subset \mathcal{M}(B)$ commutes with $\delta_\infty(B)$.

Since F is nuclear and simple, it follows that there is a unique C^*-morphism $h\colon B_1 \to \mathcal{M}(B)$ with $h(b \otimes f) = \delta_\infty(b)f$ for $b \in B$, $f \in F$. h is a non-degenerate *-monomorphism, because $\delta_\infty|B := h((.)\otimes 1)$ is a non-degenerate *-monomorphism and F is simple.

Let $d := \delta_\infty h$. Then d is non-degenerate *-monomorphism, because $d(B\otimes 1) = \delta_\infty^2(B)$ is a non-degenerate C^*-subalgebra of $\mathcal{M}(B)$.

$\delta_\infty d$ is unitarily equivalent to d, because δ_∞^2 is unitarily equivalent to δ_∞.

Let $J \in \mathcal{I}(B)$. Then $I := d^{-1}(d(B_1) \cap \mathcal{M}(B, J))$ is a closed ideal of B_1. By Proposition B.4.2(iii), there is a closed ideal K of B such that $K \otimes F = \Psi_1(J)$. We get that $b \in K$ if and only if $\delta_\infty^2(b)B \subset J$. Since $b \in B$, this is the case if and only if $b \in J$. Thus $K = J$. □

PROOF OF THEOREM K:. Let $\Psi(J) := \Psi_A(Z_J)$ for $J \in \mathcal{I}(B)$. Here $Z_J \subset \mathrm{Prim}(B)$ denotes the hull of J.

The uniqueness up to unitary homotopy of the nuclear *-monomorphism $h_0 := h((.) \otimes 1)$ follows from $h_0(\Psi(J)) = h_0(A) \cap J$ for $J \in \mathcal{I}(B)$ and from the unitary homotopy of h_0 with $h_0 \oplus h_0$, cf. Corollary 9.1.4.

We reduce the general case to the case where B is separable and Ψ is non-degenerate.

If B is separable, it suffices to consider, instead of of B, the smallest ideal I of B with $\Psi(I) = A$, to prove the existence of h. The existence of I follows from the lower semicontinuity of Ψ.

Thus, we may assume, in addition, that $\Psi^{-1}(A) = \{B\}$.

Let $F := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$, $B_1 := B \otimes F$, and let E_1 and E denote the hereditary C^*-subalgebras of $\mathrm{Q}(\mathbb{R}_+, B_1)$ and $\mathrm{Q}(\mathbb{R}_+, B)$ which are generated by B_1 and B, respectively. By Remarks 12.2.9 and 12.2.17, E_1 and E are strongly purely infinite and have residually nuclear separation.

Since F is simple and nuclear, we get from Proposition B.4.2(iii), that

$$J \leftrightarrow J \otimes F$$

defines a natural topological isomorphism from $\mathrm{Prim}(B)$ onto $\mathrm{Prim}(B_1)$ and, therefore, a lattice isomorphism from $\mathcal{I}(B)$ onto $\mathcal{I}(B_1)$. We denote the action of $\mathrm{Prim}(B_1)$ on A by Ψ_1, i.e., $\Psi_1(J \otimes F) = \Psi(J)$. Then Ψ_1 is again lower semi-continuous, satisfies property (ii) of Definition 1.2.6, $\Psi_1(0) = 0$ and $\Psi_1^{-1}(A) = \{B_1\}$.

The action $\Phi(J) := \Psi_1(B_1 \cap J)$ for $J \in \mathcal{I}(E_1)$ is a lower semi-continuous action of $\mathrm{Prim}(E_1)$ on A, and satisfies property (ii) of Definition 1.2.6, $\Phi(0) = 0$ and $\Phi^{-1}(A) = \{E_1\}$. This is the case, because Ψ_1 and $\Psi^{\mathrm{up}}_{E_1,B_1} \colon J \mapsto B_1 \cap J$ satisfy conditions (ii)-(iv) of Definition 1.2.6, and $B_1 \subset J$ implies $J = E_1$.

By Theorem 12.1.8, there is a non-degenerate weakly residually nuclear monomorphism G_0 from A to $\mathcal{M}(E_1)$ such that G_0 is unitarily equivalent to $\delta_\infty G_0$, and, for $J \in \mathcal{I}(E_1)$,

$$G_0(\Phi(J)) = G_0(A) \cap \mathcal{M}(E_1, J).$$

By Proposition 12.2.15, there is a non-degenerate nuclear *-monomorphism $k \colon A \otimes \mathcal{O}_2 \to E_1$ such that, for $k_0 := k((.) \otimes 1)$, $\delta_\infty k_0$ is unitarily homotopic to G_0 in $\mathcal{M}(E_1)$, and k_0 is unitarily equivalent to $k_0 \oplus k_0$.

In particular, for $J \in \mathcal{I}(E_1)$,

$$\delta_\infty k_0(\Phi(J)) = \delta_\infty(k_0(A)) \cap \mathcal{M}(E_1, J).$$

This implies $k_0(\Psi_1(B_1 \cap I)) = k_0(A) \cap I$ for every closed ideal I of $\mathrm{Q}(\mathbb{R}_+, B_1)$.

E_1 is a hereditary C^*-subalgebra of $\mathrm{Q}(\mathbb{R}_+, B_1)$. Therefore, k is also nuclear as a *-monomorphism from $A \otimes \mathcal{O}_2$ into $\mathrm{Q}(\mathbb{R}_+, B_1)$.

Since $\Psi_1^{-1}(A) = \{B_1\}$, it follows from Corollary 9.1.7, that there exists a nuclear *-monomorphism h from $A \otimes \mathcal{O}_2$ into B_1, such that $h_0 := h((.) \otimes 1)$ is non-degenerate and satisfies $h_0(\Psi_1(I)) = h_0(A) \cap I$ for $I \in \mathcal{I}(B)$. Thus, for $J \in \mathcal{I}(B)$,

$$h_0(\Psi(J)) = h_0(A) \cap (J \otimes F).$$

Now, it follows from Lemma 12.2.18, that $H_0 = dh_0$ is a non-degenerate nuclear *-monomorphism from from A into $\mathcal{M}(B)$, such that $\delta_\infty H_0$ is unitarily equivalent to H_0, and, for $J \in \mathcal{I}(B)$,

$$H_0(\Psi(J)) = H_0(A) \cap \mathcal{M}(B, J).$$

Since B contains a strictly positive element of E, we get from Lemma 12.1.1 that $\mathcal{M}(B) \cap \mathcal{M}(E, I) = \mathcal{M}(B, B \cap I)$ for $I \in \mathcal{I}(E)$, and we can consider H_0 as a non-degenerate nuclear *-monomorphism from A into $\mathcal{M}(E)$. Thus, for $I \in \mathcal{I}(E)$,

$$H_0(\Psi(B \cap I)) = H_0(A) \cap \mathcal{M}(E, I).$$

Now we can repeat the above applications of Proposition 12.2.15 and Corollary 9.1.7, where we have to replace B_1 by B, E_1 by E, and G_0 by $H_0 \colon A \to \mathcal{M}(E)$.

We get the desired non-degenerate nuclear *-monomorphism h from $A \otimes \mathcal{O}_2$ into B with $h_0(\Psi(J)) = h_0(A) \cap J$ for $J \in \mathcal{I}(B)$ and $h_0 = h((.) \otimes 1)$. □

COROLLARY 12.2.19. *Suppose that B is strongly purely infinite, separable and stable.*

Then $\{\widehat{b} : \ b \in B\}$ is the set of all Dini functions on $\mathrm{Prim}(B)$.

PROOF. Let $B_1 := B \otimes \mathcal{O}_2$, and let f be a Dini function on X.

By Proposition 12.2.6, there exists $a \in (B_1)_+$ with $f(J) = \widehat{a}(J \otimes \mathcal{O}_2)$ for $J \in \mathrm{Prim}(B)$. Let C denote the commutative C^*-subalgebra of B_1 which is generated by a, and let $A := C \otimes \mathbb{K}$. Then A is a nuclear C^*-subalgebra of $B_1 \otimes \mathbb{K}$.

Let $\Phi(J) := \ J \otimes \mathcal{O}_2 \otimes \mathbb{K}$.

By Lemma 12.2.4, Φ is a lattice isomorphism from $\mathcal{I}(B)$ onto $\mathcal{I}(B_1 \otimes \mathbb{K})$.

It follows that $\Psi(J) := A \cap \Phi(J)$ for $J \in \mathcal{I}(B)$ defines a lower semi-continuous action of $\mathrm{Prim}(B)$ on A. It satisfies $\Psi(0) = 0$, $\Psi(B) = A$, and the condition (ii) of Definition 1.2.6.

By Theorem K, there is a *-monomorphism $h_0\colon A \hookrightarrow B$, such that, for $J \in \mathcal{I}(B)$,

$$h_0(\Psi(J)) = h_0(D) \cap J.$$

Let $b := h_0(a \otimes p_{11})$. Then, for $J \in \mathrm{Prim}(B)$,

$$\|b + J\| = \|(a \otimes p_{11}) + \Phi(J)\| = \widehat{a}(J \otimes \mathcal{O}_2).$$

This means $\widehat{b} = f$. □

The following Corollary 12.2.20 is an equivalent reformulation of Corollary L.

COROLLARY 12.2.20. *Suppose that A and B are separable, stable and nuclear C^*-algebras, and that there is a topological isomorphism γ from $X :=$ $\mathrm{Prim}(A)$ onto* $\mathrm{Prim}(B)$.

Then there exists an isomorphism φ from $A \otimes \mathcal{O}_2$ onto $B \otimes \mathcal{O}_2$, such that φ induces γ, i.e., for $J \in$ $\mathrm{Prim}(A)$,

$$\varphi(J \otimes \mathcal{O}_2) = \gamma(J) \otimes \mathcal{O}_2\,.$$

φ with this properties is unique up to unitary homotopy.

In particular, the natural group morphism from $\mathrm{Aut}(A \otimes \mathcal{O}_2)$ *to the homeomorphisms of* $\mathrm{Prim}(A)$ *is an epimorphism with kernel equal to the group of automorphisms of $A \otimes \mathcal{O}_2$ that are unitarily homotopic to* id.

In particular, every approximately inner automorphism of $A \otimes \mathcal{O}_2$ is unitarily homotopic to id.

PROOF OF COROLLARY L AND OF COROLLARY 12.2.20. By Theorem K, there exist non-degenerate *-monomorphisms $h\colon A \otimes \mathcal{O}_2 \hookrightarrow B \otimes \mathcal{O}_2$ and $k\colon B \otimes \mathcal{O}_2 \hookrightarrow A \otimes \mathcal{O}_2$ such that h and k induce the corresponding actions of

X on B, i.e., for $J \in \operatorname{Prim}(A)$, $h(J \otimes \mathcal{O}_2)$ is the intersection of $h(A \otimes \mathcal{O}_2)$ with $\gamma(J) \otimes \mathcal{O}_2$, and similar for k. By Corollary 10.3.10(ii), h is unitarily homotopic to an *-isomorphism φ from $A \otimes \mathcal{O}_2$ onto $B \otimes \mathcal{O}_2$ with the desired property. Here we have used that every unital *-endomorphism of $\mathcal{O}_2$ is unitarily homotopic to id, and that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$, cf. Corollary F(iii) and Corollary H(ii).

The uniqueness of φ follows from Corollary 10.3.10(i). □

3. Interjection: Non-commutative Selection Conjecture

COROLLARY 12.3.1. *Suppose that A is a separable, stable and exact C^*-algebra. Then there are*

(i) *a separable stable nuclear C^*-algebra B such that $B \cong B \otimes \mathcal{O}_2$, and*
(ii) *a non-degenerate *-monomorphism $\varphi \colon A \hookrightarrow B$ such that φ induces an isomorphism λ from $\mathcal{I}(B)$ onto $\mathcal{I}(A)$ by $\lambda(I) := \varphi^{-1}(I \cap \varphi(A))$, i.e., for closed ideals $I_1, I_2 \in \mathcal{I}(B)$, $I_1 \cap \varphi(A) = I_2 \cap \varphi(A)$ implies $I_1 = I_2$, and, $\lambda(I) = J$ for $J \in \mathcal{I}(A)$ and $I := \overline{B\varphi(J)B} \in \mathcal{I}(B)$.*

B is unique up to λ-equivariant isomorphisms, and φ is unique up unitary homotopy.

*If, in addition, A strongly purely infinite, then there is also a non-degenerate *-monomorphism $\psi \colon B \hookrightarrow A$ that induces a "kind of inverse" for λ. The map $\varphi \circ \psi$ is unitarily homotopic to* id_B.

The nuclear map $\psi \circ \varphi$ is unitarily homotopic to $\mathrm{id}_A \colon A \to A$, *if and only if, A is nuclear and $A \cong A \otimes \mathcal{O}_2$.*

PROOF. The uniqueness of B follows from Corollary L. The uniqueness, up to unitary homotopy, of φ follows from Corollary 7.4.8.

We can replace A by $A \otimes \mathcal{O}_2$, to prove the existence of B and φ. Here we use that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and that every unital *-endomorphism of $\mathcal{O}_2$ is unitarily homotopic to the identity map of $\mathcal{O}_2$, cf. Corollary F(ii) and Corollary H(ii).

Suppose $A \cong A \otimes \mathcal{O}_2$. Then A is strongly purely infinite. By Theorem K, there is a non-degenerate *nuclear* *-monomorphism $h \colon A \to A$, such that h induces the natural action of $\operatorname{Prim}(A)$ on A, i.e. $h(J) = h(A) \cap J$ for every closed ideal J of A.

Let B denote the inductive limit $\operatorname{indlim}(h_n \colon A \to A)$ where $h_n = h$ for $n = 1, 2, \ldots$. Since h is nuclear, B is a nuclear C^*-algebra. By Corollary 10.3.11, B is separable, stable and $B \cong B \otimes \mathcal{O}_2$.

By Corollary 7.4.8, the nuclear maps h^n and h are unitarily homotopic for $n = 2, 3, \ldots$, because $h^n(J) = h^n(A) \cap J$ (by induction). This implies that J is the closure of $Ah^n(J)A$ for every $J \in \mathcal{I}(A)$.

If we apply the canonical maps $h^\infty_{n+k} h^n = h^\infty_k$, we obtain that, for $J \in \mathcal{I}(A)$,

$$h^\infty_k(J) = h^\infty_k(A) \cap h^\infty_{n+k}(J),$$

and $h_n^\infty(J)$ is the closure of $h_n^\infty(A)h_1^\infty(J)h_n^\infty(A)$.

Let φ denote the canonical C^*-morphism h_1^∞ from A into $B := \operatorname{indlim}(h_n \colon A \to A)$, and, for $J \in \mathcal{I}(A)$, let $\mu(J)$ denote the closure of $B\varphi(J)B$. Then $\mu(J)$ is the inductive limit of $h_n^\infty(J)$, and $\varphi(A) \cap \mu(J) = \varphi(J)$.

If $I \in \mathcal{I}(B)$, then I is the inductive limit of $h_n^\infty(A) \cap I$, because $\bigcap_n h_n^\infty(A)$ is dense in A. If we apply the above observations to

$$J_n := (h_n^\infty)^{-1}(h_n^\infty(A) \cap I)$$

we get $J_n = J_m$ for $m, n \in \mathbb{N}$ and

$$h_n^\infty(\varphi^{-1}(\varphi(A) \cap I)) = h_n^\infty(A) \cap I.$$

Thus $I = \mu(\varphi^{-1}(\varphi(A) \cap I))$, and λ defines a lattice isomorphism from $\mathcal{I}(B)$ onto $\mathcal{I}(A)$ with inverse μ. (In particular, it preserves prime elements and the hk-topology on them.)

If A strongly purely infinite, then, by Theorem K there is also a non-degenerate *-monomorphism $\psi \colon B \hookrightarrow A$ that defines the "inverse" $\mu = \lambda^{-1}$ of $\lambda \colon \mathcal{I}(B) \to \mathcal{I}(A)$, i.e., $J \cap \psi(B) = \psi(\mu(J))$.

Then $\varphi \circ \psi$ induces the identity on $\mathcal{I}(B)$, and $\varphi \circ \psi$ is unitarily homotopic to id_B by Theorem K.

If the nuclear map $\psi \circ \varphi$ is *unitarily* homotopic to $\mathrm{id}_A \colon A \to A$ then A is nuclear and id_A is approximately unitarily equivalent to $\mathrm{id}_A \oplus \mathrm{id}_A$, because ψ and φ can be taken non-degenerate (which implies that $\psi \circ (\mathrm{id}_B \oplus \mathrm{id}_B) \circ \varphi$ is unitarily equivalent to $(\psi \circ \varphi) \oplus (\psi \circ \varphi)$), and id_B is unitarily homotopic to $\mathrm{id}_B \oplus \mathrm{id}_B$.

If A is nuclear and $A \cong A \otimes \mathcal{O}_2$, then Theorem K applies also to $\psi \circ \varphi$, and is unitarily homotopic to $\mathrm{id}_A \colon A \to A$. □

CONJECTURE 12.3.2. *Suppose that B is a separable stable C^*-algebra with $B \cong B \otimes \mathcal{O}_\infty$. Let E denote the closure of $B\,\mathrm{Q}(\mathbb{R}_+, B)B$, let $H \colon A \to \mathcal{M}(E)$ a non-degenerate C^*-morphism from a separable stable C^*-algebra A into the multiplier algebra $\mathcal{M}(E)$ of E, and let Ψ_1 denote the lower semi-continuous actions of* $\mathrm{Prim}(E)$ *on A which is defined, for $I \in \mathcal{I}(E)$, by*

$$\Psi_1(I) = H^{-1}(H(A) \cap \mathcal{M}(E, I)).$$

Furthermore, suppose that

(i) *H is unitarily equivalent to its infinite repeat $\delta_\infty H$,*
(ii) *H and $\widehat{\sigma}H$ are unitarily homotopic, for every automorphism $\widehat{\sigma}$ of $\mathcal{M}(E)$, which is induced by a scaling homeomorphism σ of $\mathbb{R}_+$, and*
(iii) *H is weakly Ψ_1-residually nuclear.*

The conjecture *is:*
Then there exists a C^*-morphism H_0 from A into $\mathcal{M}(B)$, which is in $\mathcal{M}(E)$ unitarily homotopic to H.

REMARK 12.3.3. If the Conjecture 12.3.2 is true, then (i)-(iii) imply the additional condition

(iv) $\Psi_1(I_1) = \Psi_1(I_2)$ if $I_1 \cap B = I_2 \cap B$.

This is because $\mathcal{M}(B) \cap \mathcal{M}(E, I) = \mathcal{M}(B, I \cap B)$. The latter formula comes from $B \subset BEB = E$.

One could believe that the proof of Conjecture 12.3.2 can be given with ideas from Section 1. But there is a difficult point:
Let $b \in B_+$ a strictly positive contraction of B. Let X be a compact subset of A_+ with linear span dense in A. For T in $\mathcal{M}(E)$ let $\|T\|_\# := \|Tb\| + \|bT\|$.

The difficulty consists in the proof the following additional condition:

(v) For every linearly generating compact subset X of A, there exists a sequence of functions $f_n \in \mathrm{C}_0((0,1])_+$ with

$$\|f_n\| = 1, \quad f_n f_{n+1} = f_n \quad \text{and} \quad \|[f_n(b), H(a)]\|_\# < 2^{-n}$$

for $n \in \mathbb{N}$ and $a \in X$, such that, in addition, for every $\varepsilon > 0$ and every homeomorphism σ of $\mathbb{R}_+$, there is a unitary $U \in \mathcal{M}(E)$ with the following properties:
For every $a \in X$ and $n \in \mathbb{N}$, $\|[f_n(b), U]\|_\# < 2^{-n}$, and

$$\|U^* H(a) U - \widehat{\sigma} H(a)\|_\# + \|H(a) - U \widehat{\sigma} H(a) U^*\|_\# < \varepsilon.$$

Note that the existence of approximately commutative approximate units implies that, for every countable set S of homeomorphisms σ of $\mathbb{R}_+$, such a sequence f_n exists (uniformly for all $\varepsilon > 0$ and $\sigma \in S$).

If, conversely, H is approximately unitarily equivalent to a "constant" H_0, then the same argument shows the existence of the desired sequence (f_n) and the unitary $U \in \mathcal{M}(E)$. I.e., the sequence (f_n) exists, if Conjecture 12.3.2 is true.

If we make the additional assumption of the above described "uniform" existence of the sequence f_n, then the proof of the Conjecture 12.3.2 is similar to the proofs of Proposition 9.1.2 and its Corollary 9.1.3 together. But the operator norms, that we have considered there, must be replaced by the above strictly continuous norms $\|\,.\,\|_\#$. This norm is invariant under scaling automorphisms $\widehat{\sigma}$, but it is only quasi-invariant with respect to inner automorphisms.

REMARK 12.3.4. Let A, B, E and $H \colon A \to \mathcal{M}(E)$ as in Conjecture 12.3.2. Suppose that there exists a non-degenerate C^*-morphism H_0 from A into $\mathcal{M}(B)$, such that H_0 is unitarily homotopic to H in $\mathcal{M}(E)$.

Let Ψ_2 denote the lower semi-continuous actions of $\mathrm{Prim}(B)$ on A defined by

$$\Psi_2(J) := H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J)).$$

Then $\Psi_1(J) = \Psi_2(J \cap B)$ for $J \in \mathcal{I}(E)$ and H_0 is weakly Ψ_2-residually nuclear.

Indeed:
Since H_0 and H are unitarily homotopic in $\mathcal{M}(E)$, $H_0(a)$ is in $\mathcal{M}(B) \cap \mathcal{M}(E, J)$

if and only if $H(a)$ is in $\mathcal{M}(E, J)$. But it is easy to see that $\mathcal{M}(B, J \cap B) = \mathcal{M}(B) \cap \mathcal{M}(E, J)$. Thus $\Psi_1(J) = \Psi_2(J \cap B)$ for $J \in \mathcal{I}(E)$.

Let $I \in \mathcal{I}(B)$. Then $J := \mathrm{Q}(\mathbb{R}_+, I) \cap E$ is a closed ideal of E, such that $J \cap B = I$. B/I is relatively weakly injective in E/J, because E/J is naturally isomorphic to the hereditary C^*-subalgebra of $\mathrm{Q}(\mathbb{R}_+, B/I)$, that is generated by B/I.

It follows, that $I = B \cap \Psi^{B,E}_{\mathrm{down}}(I)$, and that B/I is relatively weakly injective in $E/\Psi^{B,E}_{\mathrm{down}}(I)$. Therefore, B is residually relatively weakly injective in E.

Thus H_0 is Ψ_2-residually nuclear by Lemma 12.2.12, where we have to replace (C, A, B) by $(H_0(A), B, E)$.

Definition 12.3.5. A separable C^*-algebra B has the (non-commutative) **selection property** if, for every separable C^*-algebra A and every lower semi-continuous action

$$\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$$

of $\mathrm{Prim}(B)$ on A in the sense of Definition 1.2.6 with $\Psi(0) = 0$ and $\Psi^{-1}(A) = \{B\}$, there exists a non-degenerate weakly Ψ-residually nuclear *-monomorphism H_0 from A into the multiplier algebra $\mathcal{M}(B)$ of B such that $\delta_\infty H_0$ is unitarily equivalent to H_0, and, for $J \in \mathcal{I}(B)$,

$$\Psi(J) = H_0^{-1}(H_0(A) \cap \mathcal{M}(B, J))\,.$$

(By Corollary 5.9.16, H_0 is unique up to unitary homotopy.)

Lemma 12.1.2(ii) allows to give an equivalent definition which also works for non-stable B.

In other words (in the terminology of Section **??**):
For every separable A and for every non-degenerate lower semi-continuous action Ψ of $\mathrm{Prim}(B)$ on A, $\mathrm{CP}_{\mathrm{rn}}(\Psi; A, B)$ *realizes* the action Ψ in the sense that $\Psi = \Psi_{\mathcal{C}}$ for $\mathcal{C} := \mathrm{CP}_{\mathrm{rn}}(\Psi; A, B)$.

The "selection" procedure is the following:
Let be given $a_0 \in A_+$ and a pure state λ_0 on B with $\lambda_0(\Psi(\overline{\mathrm{span}(Aa_0A)}) \neq \{0\}$. Then we have to construct a continuous map f from the Polish space $P(B)$ of pure states on B into the compact metric space $S(A^*)$ of positive linear functionals ρ on A with $\|\rho\| \leq 1$ with properties:

(i) $f(\lambda)(\Psi(J_\lambda)) = 0$ for all $\lambda \in S(A^*)$, where J_λ is the kernel of the irreducible representation d_λ corresponding to λ.

(ii) For each $a \in A_+$ there is $b \in B$ with $\lambda(b) = f(\lambda)(a)$ for all $\lambda \in S(A^*)$.

The condition (ii) is not covered by usual selection theorems.

Conjecture 12.3.6 (Non-commutative Michael selection principle). *Every separable and stable C^*-algebra B has the selection property of Definition 12.3.5.*

`Now there exists a proof of the NC Micheal selection principle.`
??

It uses that coherent Dini spaces are primitive ideal spaces of nuclear separable C^*-algebras and that every lower s.c. action of an Abelian separable C^*-algebra C on a separable and stable C^*-algebra B comes from a $H_0 \colon C \to \mathcal{M}(B)$.

Remark 12.3.7.

(1) It is known that every *exact* separable stable C^*-algebra B *has the selection property*, because $B \otimes \mathcal{O}_2$ contains a "regular" commutative C^*-subalgebra $C \subset B \otimes \mathcal{O}_2$ if B is exact and separable (cf. Remark 12.3.8).

(2) Since B has the selection property, if and only if, $B \otimes \mathcal{O}_2$ has the selection property, we may suppose that $B \cong B \otimes \mathcal{O}_2$ and let $D \subset B$ a non-degenerate copy of $C \otimes \mathbb{K}$ that is also regular in B. Then the non-commutative Michael selection principle, the observation that $\mathcal{M}(B, B \cap J)$ is equal to the set of $T \in \mathcal{M}(B)$ with $TD \subset J$, Lemma 12.1.2 and Corollary 5.9.15 imply together a proof of Conjecture 12.3.2 for all stable separable B with the property that $B \otimes \mathcal{O}_2$ contains an Abelian regular C^*-subalgebra.

(3) If the Conjecture 12.3.2 is true then also the non-commutative Michael selection principle of Conjecture 12.3.6 holds:

Let $B_1 := B \otimes \mathcal{O}_\infty$. By Theorem 12.1.8 and Conjecture 12.3.2, there is a weakly Ψ-residually nuclear C^*-morphism $H_1 \colon A \to \mathcal{M}(B_1)$ such that, for $J \in \mathcal{I}(B)$,

$$\Psi(J) = H_1^{-1}(H_1(A) \cap \mathcal{M}(B_1, J \otimes \mathcal{O}_\infty)).$$

Let $J \in \mathcal{I}(B)$ and $a \in A \setminus \Psi(J)$. Then there is $f \in B_1$, such that $f^* H_1(a) f$ is not in $J \otimes \mathcal{O}_\infty$.

Let $d \colon B_1 \to \mathcal{M}(B)$ denote the non-degenerate *-monomorphism of Lemma 12.2.18.

By Lemma 12.2.18(ii), $d(f^* H_1(a) f)$ is not in $\mathcal{M}(B, J)$, i.e., there is $e \in B$, such that $V(a)$ is not in J, where we let $V(c) := e^* d(f^* H_1(c) f) e$ for $c \in A$.

V is Ψ-residually nuclear for $f \in B_1$ and $e \in B$, because $f^* H_1(.) f$ is Ψ-residually nuclear, and $e^* \delta_\infty(d(.)) e$ is equivariant on $\mathcal{I}(B) \cong \mathcal{I}(B_1)$.

By Lemma 12.1.2, there exists the desired Ψ-residually nuclear non-degenerate *-monomorphism H_0 from A to $\mathcal{M}(B)$.

Thus a proof of Conjecture 12.3.2 would imply a proof of the non-commutative Michael selection principle.

Remark 12.3.8. If one limits to those separable C^*-algebras B such that $\mathrm{Prim}(B)$ is homeomorphic to $\mathrm{Prim}(C)$ for a separable exact C^*-algebra C, then, by Corollary 12.3.1, Theorem K, Corollary L and Lemma 12.2.18, it suffices to prove the selection property for the unique stable separable nuclear C^*-algebra A with $\mathrm{Prim}(A)$ homeomorphic to $\mathrm{Prim}(B)$, and with $A \cong A \otimes \mathcal{O}_2$. Thus, in this class of C^*-algebras B, the selection property is essentially a property of the T_0-space $\mathrm{Prim}(B)$.

Lemma 12.1.3 implies, that every separable stable C^*-algebra B with Hausdorff primitive ideal space has the selection property.

Lemma 12.1.4 and the below given Lemma 12.3.10 show that *stable separable B has the selection property if B contains a commutative C^*-subalgebra C which is regular in B in the sense of Definition B.4.1.*

This implies, e.g., that B has the selection property if its primitive ideal space is isomorphic to the primitive ideal space of an inductive limit of C^*-algebras with Hausdorff primitive ideal spaces. But the second example considered after Corollary N in Chapter 1, is not such an inductive limit but contains a commutative C^*-subalgebra which is regular in the algebra.

Examples of those B where considered by Mortensen [**559**]. We generalize his construction as follows:

Let S be a compact metric semi-group with jointly continuous multiplication, and let $C := \mathrm{C}(S)$. The multiplication defines a unital C^*-morphism $d\colon C \to C \otimes C$, which is given by $d(f)(x, y) = f(xy)$. Let $A := C \otimes \mathcal{O}_2$. We choose a unital *-monomorphism h from C into $\mathcal{O}_2$, and a *-isomorphism k from $\mathcal{O}_2 \otimes \mathcal{O}_2$ onto $\mathcal{O}_2$. Let g denote the unital *-endomorphism of A, which is given by

$$g := (\mathrm{id}_C \otimes k)((\mathrm{id}_C \otimes h)d) \otimes \mathrm{id}_{\mathcal{O}_2}).$$

Now let $B := \mathrm{indlim}(h_n\colon A \to A)$, with $h_n = h$ for $n = 1, 2, \ldots$.

Then $\mathcal{I}(B)$ is the projective limit of the iterates of the map from $\mathbb{O}(S)$ into $\mathbb{O}(S)$, which is given, for open subsets Z of S, by

$$Z \mapsto S \setminus ((S \setminus Z)S).$$

If S is a linearly ordered space, which is separable and compact in its interval topology, then S is topologically and order isomorphic to a closed subset of the interval $[0, 1]$ (cf. [**360**]). S carries semigroup structures, e.g. the multiplication $(x, y) \mapsto \min(x, y)$. The corresponding map on $\mathbb{O}(S)$ is given by

$$Z \mapsto S \cap (\max(S \setminus Z), 1],$$

and the T_0-space of the corresponding is S with S, $\emptyset$ and $S \cap (t, 1]$, $t \in [0, 1)$ as open subsets. (Together with Corollary L, this implies the main result of [**559**].)

REMARK 12.3.9. The realization $H_0\colon A \to \mathcal{M}(E)$ of a *lower* semi-continuous actions $\Psi_B\colon \mathcal{I}(A) \to \mathcal{I}(B)$ of $\mathrm{Prim}(A)$ on B (as given Theorem 12.1.8 with $E := B\,\mathrm{Q}(\mathbb{R}_+, B)B$) can be used to obtain realizations of *upper* semi-continuous actions $\Psi_A\colon \mathcal{I}(B) \to \mathcal{I}(A)$ of $\mathrm{Prim}(B)$ on A, if the action Ψ_A is 'residually non-singular' in the following sense:

We call an action $\Psi_A\colon \mathcal{I}(B) \to \mathcal{I}(A)$ of $\mathrm{Prim}(B)$ on A **residually non-singular**, if $\Psi_A(J_1) = \Psi_A(J_2)$ and $J_1 \subset J_2$ imply $J_1 = J_2$.

If Ψ_A satisfies the condition (i) or the condition (iii) of Definition 1.2.6, then Ψ_A is residually non-singular if and only if Ψ_A is an injective map from $\mathcal{I}(B)$ into $\mathcal{I}(A)$.

Let B and A stable and separable, and let $X := \operatorname{Prim}(B)$. Since in the following considerations only the lattices of ideals are considered, we assume — for simplicity — that $B \otimes F \cong B$ and $A \otimes F \cong A$, where $F \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \ldots$.

Suppose that $\Psi_A \colon \mathcal{I}(B) \cong \mathbb{O}(X) \to \mathcal{I}(A)$ satisfies the conditions (i) and (ii) of Definition 1.2.6, and $\Psi_A(0) = 0$, $\Psi_A^{-1}(A) = \{B\}$.

We define, for $I \in \mathcal{I}(A)$, a subset of $\mathcal{I}(B)$ by

$$\gamma(I) := \{J \in \mathcal{I}(B) \colon \Psi_A(J) \subset I\}\,.$$

The set $\gamma(I)$ is closed under sums $J_1 + J_2$ of closed ideals, and $J \in \gamma(I)$ if J is the closure of an increasing sequence in $\gamma(I)$. Therefore $\gamma(I)$ contains a unique maximal element $\Psi_B(I)$. $\Psi_B(I)$ is the closure of the sum of all elements in $\gamma(I)$.

It is easy to see that $\Psi_B \colon \mathcal{I}(A) \to \mathcal{I}(B)$ is a lower semi-continuous action of $\operatorname{Prim}(A)$ on B, that $\Psi_B^{-1}(B) = \{A\}$, and that, for $J \in \mathcal{I}(B)$, $I \in \mathcal{I}(A)$, $J \subset \Psi_B(I)$ if and only if $\Psi_A(J) \subset I$. In particular, $J \subset \Psi_B(\Psi_A(J))$. It follows that, for $J \in \mathcal{I}(B)$,

$$\Psi_A(J) = \bigcap\{I \in \mathcal{I}(A) \colon J \subset \Psi_B(I)\}.$$

If Ψ_A is residually non-singular, it implies that $\Psi_B(0) = 0$, and that Ψ_B is a left inverse of Ψ_A.

Let E denote the closure $A\,\mathrm{Q}(\mathbb{R}_+, A)A$, and let $H_0 \colon B \to \mathcal{M}(E)$ be the Ψ_B-residually nuclear non-degenerated *-monomorphism of Theorem 12.1.8 for Ψ_B, then an easy calculation shows that, for $J \in \mathcal{I}(B)$,

$$\Psi_A(J) = A \cap \Psi_{\mathrm{down}}^{H_0(B),E}(H_0(J))\,.$$

If, moreover, A has the selection property, then even H_0 can be found such that $H_0(B) \subset \mathcal{M}(A)$, and we get, for $J \in \mathcal{I}(B)$,

$$\Psi_A(J) = A \cap \Psi_{\mathrm{down}}^{H_0(B),A}(H_0(J))\,.$$

Lemma 12.3.10. *Suppose that A is a nuclear stable C*-subalgebra of a separable stable C*-algebra B, and that A has the selection property of Definition 12.3.5.*

Then the following are equivalent:

(i) *A is a regular subalgebra of B in the sense of Definition B.4.1.*
(ii) *$\Psi_{B,A}^{\mathrm{up}} \colon J \in \mathcal{I}(B) \mapsto A \cap J \in \mathcal{I}(A)$ is a residually non-singular action, which is continuous in the sense of Definition 1.2.5, (i)-(iv).*
(iii) *A separates the ideals of B, and, for every $I \in \mathcal{I}(A)$, the set $\gamma(I) := \{J \in \mathcal{I}(B) \colon A \cap J \subset I\}$ contains a unique maximal element.*
(iv) *A separates the ideals of B, and there exists a sequence (T_n) of residually equivariant maps from B into B, such that $T_n(B) \subset A$ for $n \in \mathbb{N}$, and $T_n|A$ converges in point-norm topology to $\mathrm{id}\,|A$.*

PROOF. If $J_1, J_2 \in \mathcal{I}(B)$, $I := (A \cap J_1) + (A \cap J_2)$, and $\gamma(I)$ of (iii) contains a maximal element K, then $J_1 \subset K$ and $J_2 \subset K$. Therefore $J_1 + J_2 \subset K$, which means $I = A \cap (J_1 + J_2)$.

This observation shows that (iii) and (ii) are obvious equivalent formulations of (i).

(iv)⇒(i): Since A separates $\mathcal{I}(B)$, it remains to show that $A \cap (J_1 + J_2)$ is contained in $(A \cap J_1) + (A \cap J_2)$, for $J_1, J_2 \in \mathcal{I}(B)$. The argument in the second part of the proof of Lemma 12.1.4 can be easily adapted, to get this.

(iii)⇒(iv): $\Psi_B(I) := \max \gamma(I)$ is the lower semi-continuous action of $\mathrm{Prim}(A)$ on B, which corresponds to the upper semi-continuous action $\Psi_A := \Psi^{\mathrm{up}}_{B,A}$ in the sense of Remark 12.3.9.

$\Psi_B(0) = 0$ and $\Psi_B^{-1}(B) = \{A\}$, because A separates the ideals of B.

Since A is separable, stable, and has the selection property, there is a non-degenerate weakly Ψ_B-residually nuclear *-monomorphism H_0 from B into $\mathcal{M}(A)$, such that, for $I \in \mathcal{I}(A)$

$$H_0(\Psi_B(I)) = H_0(B) \cap \mathcal{M}(A, I).$$

It follows $A \cap \Psi_B(I) \subset I$, and

$$H_0^{-1}(H_0(A) \cap \mathcal{M}(A, I)) = A \cap \Psi_B(I)\,.$$

Thus, by Corollary 3.10.6(II), $H_0|A$ approximately dominates $V := \mathrm{id}_A$, because A is also nuclear.

The hereditary C^*-subalgebra $D \subset B$ generated by A is stable and generates B as an ideal, because A is stable and separates $\mathcal{I}(B)$.

If we consider $H_0|D$ as a *-monomorphism from D into $\mathcal{M}(D)$, then $H_0|D$ is weakly residually nuclear, because A is nuclear, $H_0(D) \subset \mathcal{M}(A)$, and, for $J \in \mathcal{I}(B)$, $J = \Psi_B(A \cap J)$ and

$$H_0(D) \cap \mathcal{M}(D, D \cap J) = H_0(\Psi_B(A \cap J))\,.$$

Thus $T := a^* H_0(b^*(.)b)a$ is a residually nuclear map from B into B with image in A, if $a \in A$ and $d \in D$.

The convex combinations of restrictions $T|A$ approximate $\mathrm{id}\,|A$ in the point-norm topology, because, the map $H_0|A$ approximately dominates $V := \mathrm{id}_A$. □

The following Proposition 12.3.11 generalizes part (ii) of Theorem A.

PROPOSITION 12.3.11. *Suppose that A, B are stable and separable, that A is nuclear and has the selection property, B is strongly purely infinite, and that $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a residually non-singular lower semi-continuous action of* $\mathrm{Prim}(B)$ *on A with $\Psi(0) = 0$ and $\Psi(B) = A$.*

Then the following are equivalent.

(i) *Ψ is a continuous action in the sense of Definition 1.2.6, (i)-(iv).*

(ii) *There is a non-degenerate *-monomorphism* $h_0 \colon A \hookrightarrow B$ *with* $h_0(\Psi(J)) = h_0(A) \cap J$ *and* $[h_0] = [h_0] + [h_0]$ *and a residually equivariant conditional expectation* P *from* B *onto* $h_0(A)$*.*

(iii) *Every non-degenerate *-monomorphism* h *from* A *into* B*, with* $h(\Psi(J)) = h(A) \cap J$ *and* $[h] = [h] + [h]$*, maps* A *onto a regular subalgebra of* B*.*

PROOF. Combine Theorem K, Lemma 12.3.10, and the last part of Theorem 6.3.1. □

The next observations allows to extend the main results of this section considerably (see Corollary 12.3.18).

`Alternative version from Dini3 :`

Suppose that A is a stable separable C^*-algebra, and that $g \colon \operatorname{Prim}(A) \to [0, \infty)$ is a bounded lower semi-continuous function,such that the set $X := \{\sup g(F)\,;\ F \in \mathbb{F}(\operatorname{Prim}(A))\}$ is *closed* in $[0, \sup g(X)]$.

Then there exists $a \in \mathcal{M}(A)_+$ with the following properties (i)-(iii).

(i) $\operatorname{Spec}(a) = X$,
(ii) $g(J) = \|a + \mathcal{M}(A, J)\|$ for all $J \in \operatorname{Prim}(A)$, and
(iII) $\operatorname{Spec}(a + \mathcal{M}(A, I)) = X \cap [0, \sup g(\operatorname{hull}(I))]$ for every closed ideal $I \lhd A$.

If $a \in \mathcal{M}(A)_+$ satifies (i)-(iii) then its infinite repeat $\delta_\infty(a)$ also satisfies (i)-(iii). If $b \in \mathcal{M}(A)_+$ satisfies (i)-(iii) (with b in place of a), then there is norm-continuous map $t \in [0, \infty) \mapsto U(t) \in \mathcal{U}(\mathcal{M}(A))$ such that $U(0) = 1$ and $\lim_{t\to\infty} U(t)\delta_\infty(a)U(t)^* = \delta_\infty(b)$.

PROPOSITION 12.3.12. *Suppose that* B *is stable and separable,* $1 \in K \subset [0, 1]$ *a closed set and that* $f \colon \operatorname{Prim}(B) \to K$ *is a lower semi-continuous function on* $\operatorname{Prim}(B)$ *with supremum* $\sup f(\operatorname{Prim}(B)) = 1$*. Then there is* $a \in \mathcal{M}(B)_+$ *with following properties:*

(i) $\operatorname{Spec}(a) = K$
(ii) $\|a + \mathcal{M}(B, J)\| = f(J)$ *for all* $J \in \operatorname{Prim}(B)$*,*
(iii) a *is unitarily equivalent to* $\delta_\infty(a)$*.*
(iv) *The action of* $\operatorname{Prim}(B)$ *on* $K = \operatorname{Spec}(a) \cong \operatorname{Prim}(C^*(a, 1))$ *is given by* $\Psi(I) = (\|a + \mathcal{M}(B, I)\|, 1] \cap K$ *for all closed ideals* I *of* B*.*

The $a \in \mathcal{M}(B)_+$ *with (i)–(iv) is unique up to unitary homotopy.*

PROOF. We find isometries $r, t, s_1, s_2, \ldots \in \mathcal{M}(B)$ with $tt^* + rr^* = 1$ and with $\sum_n s_n(s_n)^*$ strictly converges to 1. We define $\Delta \colon \ell_\infty(\mathcal{M}(B)) \to \mathcal{M}(B)$ by $\Delta(b_1, b_2, \cdots) := \sum s_n b_n (s_n)^*$. If we identify $\ell_\infty(\mathcal{M}(B))$ with $\mathcal{M}(c_0 \otimes B)$ then we can see that Δ is a unital non-degenerate strictly continuous *-monomorphism.

If J is closed ideal of B, then J is stable, and $tBt^* + sJs^*$ is stable and generates a full hereditary C^*–subalgebra D of B. Then D is stable, and, by Corollary 5.5.6 (which is the Ψ-equivariant version of the Brown stable isomorphism theorem), there exists an approximately inner isomorphism ψ from B onto D. Thus, there is an projection $p \in \mathcal{M}(B)$ such that pBp and $(1-p)B(1-p)$ are stable, pBp generates J and $(1-p)B(1-p)$ is full in B. It follows that there is an isometry $v \in \mathcal{M}(B)$ with $vv^* = 1-p$, and that the strictly closed ideal of $\mathcal{M}(B)$ generated by p is just $\mathcal{M}(B, J)$.

We define a lower semi-continuous increasing function $\gamma\colon K \to K$ by $\gamma(0) := 0$ and $\gamma(s) := \sup(K \cap [0, s))$ for $s > 0$. The function γ satisfies $t \le \gamma(s) \le s$ for all $t < s \in K$, and $\gamma(s) < s$ holds if and only if $(\gamma(s), s) \cap K = \emptyset$.

It holds $\inf\{\gamma(s)\,;\ s \in (t, 1] \cap K\} = t$ for all $t \in K$. Indeed, if $t = \inf K \cap (t, 1]$, then there is a sequence $s_1 > s_2 > \cdots$ in K with $\lim s_n = t$, then $s_{n+1} \le \gamma(s_n) \le s_n$, thus $\lim \gamma(s_n) = t$. For each $s \in (t, 1] \cap K$ there is $n \in \mathbb{N}$ with $s > s_n$. Hence $\gamma(s) \ge \gamma(s_n) \ge t$ and $\inf\{\gamma(s)\,;\ s \in (t, 1] \cap K\} = t$. If $t < s_0 := \inf K \cap (t, 1]$, then $s_0 \in K$ (by compactness of K) and $(t, s_0) \cap K = \emptyset$. It follows $(t, 1] \cap K = [s_0, 1] \cap K$, $[0, s_0) \cap K = [0, t] \cap K$ and $\inf\{\gamma(s)\,;\ s \in (t, 1] \cap K\} = g(s_0) = t$.

Now let $(x_1, x_2, \ldots)$ a dense sequence of (pairwise different) points in $K \setminus \{0\}$ that contains all boundary points of K (in $(0, 1]$).

Let J_n denote the ideal of B corresponding to the open subset $f^{-1}(\gamma(x_n), \infty)$ of $\mathrm{Prim}(B)$. For each $n \in \mathbb{N}$, we find a projection $p_n \in \mathcal{M}(B, J_n)$ such that $p_n B p_n$ is stable and generates J_n.

Define $a := \Delta(x_1 p_1, x_2 p_2, \ldots)$. We show that $a \in \mathcal{M}(B)$ satisfies (i) and (ii), and we modify a later to get also (iii) and (iv).

(i): Let $p_0 := 1 - \Delta(p_1, p_2, \ldots)$, and define a unital *-morphism $\lambda\colon \mathrm{C}(K) \to \mathcal{M}(B)$, for $g \in \mathrm{C}(K)$, by

$$\lambda(g) := g(0)p_0 + \Delta(g(x_1)p_1, g(x_2)p_2, \ldots)\,.$$

Then λ is faithful, because $\{0, x_1, x_2, \ldots\}$ is dense in K. Clearly $a = \lambda(f_0)$ for $f_0(t) := t$, $t \in K$. Thus $\mathrm{Spec}(a) = \mathrm{Spec}(f_0) = K$ by spectral permanence.

(ii): Notice that $(s_n)^* B s_n = B$, $(s_n)^* J s_n = J$ and $p_n B$ generates J_n. Thus $p_n (s_n)^* B s_n \subset (s_n)^* J s_n$ is equivalent to $J_n \subset J$. If J is primitive, then the inclusion $J_n \subset J$ means that $f(J) \le \gamma(x_n)$ (by definition of J_n).

Let $J \in \mathrm{Prim}(B)$. Then $\|a + \mathcal{M}(B, J)\| \le t$, if and only if, $\|(a-t)_+ + \mathcal{M}(B, J)\| = 0$, if and only if, $(a-t)_+ B \subset J$, if and only if, $\min(x_n - t, 0) p_n (s_n)^* B s_n \subset (s_n)^* J s_n$ for all $n \in \mathbb{N}$, if and only if, $f(J) \le \gamma(x_n)$ for all n with $t < x_n$.

Thus, $\|a + \mathcal{M}(B, J)\| \le t$, if and only if, $f(J) \le t$, because

$$\inf\{\gamma(x_n)\,;\ t < x_n\} = \inf\{\gamma(s)\,;\ s \in (t, \infty) \cap K\} = t\,.$$

It follows (ii).

(iii): Replace a by $\delta_\infty(a)$ (if necessary) and use that $\delta_\infty \circ \delta_\infty$ is unitarily equivalent to δ_∞ by Lemma 5.1.2(i).

(iv): Suppose that $a' \in \mathcal{M}(B)$ satisfies (i)–(iii) (in place of a). By property (iii) and by Lemma 5.1.2(ii), one can find a unital *-monomorphism $\varphi\colon \mathrm{C}(K) \otimes \mathcal{O}_2 \to \mathcal{M}(B)$ with $\varphi(f_0 \otimes 1)$ unitarily equivalent to a' with property (i)–(iii), where $f_0(t) = t$ for $t \in K$. Take a unital monomorphism $\rho\colon \mathrm{C}(K) \to \mathcal{O}_2$ (e.g. using a continuous epimorphism of the Cantor set onto K), and let $g \in \mathrm{C}(K \times K) \cong \mathrm{C}(K) \otimes \mathrm{C}(K)$ denote the function $g(s,t) := \min(s,t)$.

Then $\mathrm{Spec}(g) = g(K \times K) = K$, and, for every closed subset F of K, one has $g(F \times K) = [0, \max F] \cap K$. It follows that $b := (\mathrm{id} \otimes \rho)(g) \in \mathrm{C}(K) \to \mathcal{O}_2$ has spectrum $\mathrm{Spec}(b) = F$ and that a natural action of $\mathrm{Prim}(\mathrm{C}(K) \otimes \mathcal{O}_2) \cong K$ on $\mathrm{Prim}(C^*(b,1)) \cong K$ is given by $F \mapsto K \cap [0, \max F]$ (all expressed by the maps for the closed subsets F). In particular, if I is a closed ideal of $\mathrm{C}(K) \otimes \mathcal{O}_2$ (corresponding to the open subset U of K), then $I \cap (\mathrm{C}(K) \otimes 1) = \mathrm{C}_0(U) \otimes 1$ and $I \cap C^*(b,1) = \mu(\mathrm{C}_0(K \cap (\max(K \setminus U), 1]))$, for the unital *-monomorphism $\mu\colon h \in \mathrm{C}(K) \to h(b) = (\mathrm{id} \otimes \rho)(h \circ g) \in \mathrm{C}(K) \to \mathcal{O}_2$. It implies $\|b + I\| = \|f_0 \otimes 1 + I\|$ and $I \cap C^*(b,1) = \mu(\mathrm{C}_0(K \cap (\|b + I\|, 1])$, for all closed ideals of $\mathrm{C}(K) \to \mathcal{O}_2$.

Let $a := \delta_\infty(\varphi(b))$. Then a satisfies (i)–(iv), because the l.s.c. action of $\mathrm{Prim}(B)$ on $C^*(a,1)$ (respectively on $C^*(a')$) factorizes over the action on of $\mathrm{Prim}(\mathrm{C}(K) \to \mathcal{O}_2) \cong K$ on $\mathrm{C}(K) \otimes 1$ (respectively on $\mathrm{C}(K) \otimes 1$). □

LEMMA 12.3.13. *Suppose that B is a σ-unital C^*–algebra, and that $J, K \in \mathcal{I}(B)$.*

(i) $(J + K)_+ = J_+ + K_+$, *and*
(ii) $\mathcal{M}(B, J + K) = \mathcal{M}(B, J) + \mathcal{M}(B, K)$.

PROOF. (i): See [**401**, exercise 4.6.64]. Alternatively: It is easy to see that the set $J + K$ is a closed ideal. Let $c \in (J + K)_+$, and $d := c^{1/2}$. There are selfadjoint $a \in J$ and $b_0 \in K$ with $a + b_0 = d$. Then $c \leq a^2 + b^2$ for $b := (b_0^2 + (ab_0 + b_0 a)_-)^{1/2}$ and $b \in K_+$. By asymmetric Riesz decomposition [**616**, prop. 1.4.10], there are $e, f \in B$ with $ee^* \leq a^2$, $ff^* \leq b^2$ and $c = e^*e + f^*f$.

(ii): Since J, K and $J + K$ are closed ideals of B, the $\mathcal{M}(B, J)$, $\mathcal{M}(B, K)$, $\mathcal{M}(B, J) + \mathcal{M}(B, K)$ and $\mathcal{M}(B, J + K)$ are closed ideals of $\mathcal{M}(B)$. The inclusion

$$\mathcal{M}(B, J) + \mathcal{M}(B, K) \subset \mathcal{M}(B, J + K)$$

comes immediately from the Definition of $\mathcal{M}(B, \cdot)$. Let $T \in \mathcal{M}(B, J + K)_+$ and $\varepsilon > 0$. Since B is σ-unital, there is an approximate unit $e_0 = 0 \leq e_1 \leq e_2 \leq \cdots \leq 1$ in B with $e_n e_{n+1} = e_n$ for $n \in \mathbb{N}$, such that $\sum_n (e_{n+1} - e_n)$ and $\sum_{n \geq 0} (e_{n+1} - e_n)^{1/2} T (e_{n+1} - e_n)^{1/2}$ are strictly convergent with sums 1 and T_1 and $\|T_1 - T\| < \varepsilon$. Let $e_{-1} := 0$ and $f_n := e_{n+2} - e_{n-1}$. By part (i), there are $a_n \in J_+$ and $b_n \in K_+$ with $a_n + b_n = f_n T f_n$. Then $S_1 := \sum_{n \geq 0} (e_{n+1} - e_n)^{1/2} a_n (e_{n+1} - e_n)^{1/2}$ and $S_2 := \sum_{n \geq 0} (e_{n+1} - e_n)^{1/2} b_n (e_{n+1} - e_n)^{1/2}$ are strictly convergent, $S_1 \in \mathcal{M}(B, J)_+$, $S_2 \in \mathcal{M}(B, K)_+$, $T_1 = S_1 + S_2$. Thus, T has distance $< \varepsilon$ from $\mathcal{M}(B, J) + \mathcal{M}(B, K)$ (for each $\varepsilon > 0$). □

REMARK 12.3.14. The (algebraic) definition of $\mathcal{M}(B,J)$ immediately shows that $\bigcap_\gamma \mathcal{M}(B,J_\gamma) = \mathcal{M}(B,\bigcap_\gamma J_\gamma)$ for for every family $\{J_\gamma\} \subset \mathcal{I}(B)$, i.e., that $\mathcal{M}(B,\cdot)$ is *lower* semi-continuous.

The monotone map $\mathcal{M}(B,\cdot)$ is *not* monotone upper semi-continuous, e.g. if $B := c_0 = \mathrm{C}_0(\mathbb{N})$ and $J_n := \mathrm{C}(\{1,2,\ldots,n\})$ (respectively $B := \mathrm{C}_0(\mathbb{N},\mathcal{O}_2\otimes\mathbb{K})$ and $J_n := \mathrm{C}(\{1,2,\ldots,n\},\mathcal{O}_2\otimes\mathbb{K})$), then $B = \overline{\bigcup_n J_n}$, but $\bigcup_n \mathcal{M}(B,J_n)$ is not dense in $\mathcal{M}(B) = \mathcal{M}(B,B)$.

`Next need Def.'s of Dini function and Dini space`

LEMMA 12.3.15. *Suppose that B is separable and stable, that X is a* Dini space, *and that $\Psi\colon \mathcal{I}(B)\cong \mathbb{O}(\mathrm{Prim}(B))\to\mathbb{O}(X)$ is a lower semi-continuous action.*

Let $J\in\mathcal{I}(\mathcal{M}(B))$ and let $\Psi^*(J) :=$ the biggest open subset U of X such that, for every Dini function $g\colon X\to[0,\infty)$ on X with support in U and $\sup g(X)=1$, $a_f\in J$ for the element $a_f\in\mathcal{M}(B)_+$ of Proposition 12.3.12 for the l.s.c. function $f(J) := \sup g(X\setminus\Psi(J))$ for every Dini function $g\colon X\to[0,\infty)$ on X with support in U and $\sup g(X)=1$.

It holds

(o) *$\Psi^*\colon \mathcal{I}(\mathcal{M}(B))\to\mathbb{O}(X)$ is well-defined.*
(i) *$U\subset\Psi^*(J)$, if and only if, $\mathcal{M}(B,\Phi(V))\subset J$ for all $V << U$ and the upper s.c. adjoint Φ of Ψ.*
(ii) *The map $J\to\Psi^*(J)$ is l.s.c. and monotone upper s.c. on the ideals J with $J\cap\delta_\infty(\mathcal{M}(B))$ generates J.*

PROOF. `Important, to be filled in ??,` □

PROPOSITION 12.3.16. *Suppose that A is separable and contains a regular exact C^*-subalgebra C. Then A has the Abelian factorization property.*

THEOREM 12.3.17. *Suppose that A and B are separable and stable, and that A has the Abelian factorization property.*

Then, for each non-degenerate lower semi-continuous action $\Psi\colon\mathcal{I}(B)\to\mathcal{I}(A)$, there is a non-degenerate faithful weakly Ψ-residually nuclear C^*-morphism $H\colon A\to\mathcal{M}(B)$ (with H unitarily equivalent to δ_∞) such that

$$\Psi(J) = H^{-1}(H(A)\cap\mathcal{M}(B,J)) \quad \text{for all } \ J\in\mathcal{I}(B)\,.$$

PROOF. `to be filled in ??,` □

COROLLARY 12.3.18. *Suppose that A and B are separable C^*-algebras, A is exact and B is stable. If $\Psi\colon\mathcal{I}(B)\to\mathcal{I}(A)$ is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A with $\Psi(\{0\})=\{0\}$ and $\Psi^{-1}(A)=\{B\}$, then there is a non-degenerate (norm-) nuclear C^*-morphism $H\colon A\to\mathcal{M}(B)$ with $\delta_\infty\circ H$ unitarily equivalent to H and $\Psi(J)=H^{-1}(H(A)\cap\mathcal{M}(B,J))$ for all $J\in\mathcal{I}(B)$.*

The nuclear morphism H is unique up to unitary homotopy.

PROOF. `to be filled in` ?? □

4. Proof of (old !!!) Theorem O

We conclude this chapter with the proof of the old !!!! Theorem O.

`beginning from here, old text has to be replaced` ??

The proof is fairly elementary and allows the reader to check his insight in the above developed methods for Ψ-residually nuclear maps.

We show that any C^*-algebra with primitive ideal space homeomorphic to $[0,1]$ and fibres $\cong \mathcal{O}_2$ respectively $\cong \mathcal{O}_2 \otimes \mathbb{K}$ are isomorphic to $\mathrm{C}([0,1], \mathcal{O}_2)$ respectively to $\mathrm{C}([0,1], \mathcal{O}_2 \otimes \mathbb{K})$.

??

`Next there is only a proof of the old Theorem O`

Suppose that A is a unital separable C^*-algebra with $\mathrm{Prim}(A) \cong [0,1]$, the interval $[0,1]$ with ordinary topology, and that the fibers, i.e. the simple quotients, are isomorphic to $\mathcal{O}_2$. Thus, A is as a C^*-algebra bundle with base space $[0,1]$ and fibers isomorphic to $\mathcal{O}_2$. This implies that A nuclear and is strongly p.i., cf. ends of Chapters 2 and 3. The K-theory of A is trivial, see below.

For nuclear C^*-algebra bundles A, B over a locally compact σ-compact metrizable spaces X, we have that $\mathrm{KK}_{\mathrm{nuc}}(X; A, B) = \mathcal{R}\,\mathrm{KK}^{trivial}(X; A, B)$. By Corollary N, the bundle is stably isomorphic to $\mathrm{C}([0,1], \mathcal{O}_2)$ if and only if $\mathcal{R}\,\mathrm{KK}^{trivial}([0,1], A, A) = 0$.

This allows to use the tautological epimorphism and the six-term exact sequence to check

`the ''following proposition'', that has been erased,`

which reduces the triviality question to the ordinary KK-theory.

We show the non-trivial direction of the statement of the old version of Theorem O, i.e., that A *is isomorphic to* $\mathrm{C}([0,1], \mathcal{O}_2)$*, if* A *is a separable unital* C^**-algebra with* $\mathrm{Prim}(A) \cong [0,1]$*, if every primitive quotient of* A *is isomorphic to* $\mathcal{O}_2$*, and if* A *satisfies the UCT for the ordinary* KK*-theory of* C^**-algebras.*

All non-zero projections in $\mathrm{C}([0,1], \mathcal{O}_2 \otimes \mathbb{K})$ are equivalent. Therefore, it suffices to show that $A \otimes \mathbb{K}$ and $\mathrm{C}([0,1], \mathcal{O}_2 \otimes \mathbb{K})$ are isomorphic.

Every primitive quotient of A is nuclear. Since A is separable, this implies that every factor representation of A generates an injective factor. Thus, A is a nuclear C^*-algebra.

By Corollary L, $A \otimes \mathcal{O}_2 \cong \mathrm{C}([0,1], \mathcal{O}_2)$ with an isomorphism that implements the isomorphism χ from $[0,1]$ onto $\mathrm{Prim}(A)$.

The isomorphism and the natural embedding of $A \otimes 1$ into $A \otimes \mathcal{O}_2$ maps $\chi(t)$ onto the intersection of $A \otimes 1$ with the kernel of the epimorphism $g \mapsto g(t)$ from

$C([0,1],\mathcal{O}_2)$ onto $\mathcal{O}_2$, and it maps the center of A onto $C([0,1])$. This gives a unital isomorphism λ from $C([0,1])$ onto the center of A such that

$$\chi(t) = \lambda(C_0([0,1]\setminus\{t\}))A \quad \text{and} \quad \pi_t(\lambda(f)a) = f(t)\pi_t(a).$$

Thus A is a (continuous) C^*-bundle with quotient maps $\pi_t\colon A \to A/\chi(t)$ in the sense of [**92**], [**471**]. Since $[0,1]$ has dimension one, and since the fibers $A/\chi(t)$ are simple and purely infinite, by Remarks 2.15.12 and 3.11.6(ii), the algebra A is strongly purely infinite, cf. [**93**]. (The latter can be seen also directly and elementary with a modification of the idea in step one, below.)

It follows that Theorem M applies to A:

$A\otimes\mathbb{K}$ is isomorphic to $C([0,1],\mathcal{O}_2\otimes\mathbb{K})$, if and only if, $KK([0,1];A,A)=0$.

We show the equation $KK([0,1];A,A)=0$ in the following five steps.

Let $A|X$ denote the *restriction of the bundle A to X*: Let Ψ denote the action of $[0,1]$ on A, which is defined by a homeomorphism χ from $[0,1]$ onto $\mathrm{Prim}(A)$. Then $\Psi(Z)=\lambda(C_0(Z))A$ for open subsets Z of $[0,1]$ and the primitive ideal $\chi(t)$ is the ideal $\Psi([0,1]\setminus\{t\})$. If X is a relatively closed subset of an open subset Z of $[0,1]$, we define $A|X$ as $\Psi(Z)/\Psi(Z\setminus X)$. Then $\pi_t\colon A\to A|\{t\}$ is the natural epimorphism on the fiber at t. Let $0<x<y\le 1$, then the natural epimorphisms $A|[0,y]\to A|[0,x]$ and $A|[0,y]\to A|[x,y]$ define just the pull back of the natural epimorphisms $A|[0,x]\to A|\{x\}$ and $A|[x,y]\to A|\{x\}$, i.e., we can compose elements of $A|[0,x]$ and $A|[x,y]$ to an element of $A|[0,y]$ if they coincide at x.

First, A contains a copy of $\mathcal{O}_2$ unitally:

Since $A|\{x_0\}$ is isomorphic to $\mathcal{O}_2$ there exist contractions a and b in A such that $a^*b=0$ and that the continuous functions $x\mapsto\|\pi_x(c)\|$ are zero at $x=x_0$ for c in $G:=\{1-a^*a, 1-b^*b, 1-aa^*-bb^*\}$.

It follows that there exists $\delta>0$ such that $\|\pi_Y(c)\|<1/8$ for $c\in G$, where $Y:=[\max(0,x_0-\delta),\min(x_0+\delta,1)]$ and π_Y denotes the natural epimorphism from A onto $A|Y$. Then $s:=\pi_Y(a)\pi_Y(a^*a)^{-1/2}$ and $t:=\pi_Y(b)\pi_Y(b^*b)^{-1/2}$ are canonical generators of a unital copy of $\mathcal{O}_2$ in $A|Y$.

We find $x_0=0<x_1<\ldots<x_n=1$ such that $A|[x_j,x_{j+1}]$ contains isometries s_j and t_j with $s_js_j^*+t_jt_j^*=1$ for $j=0,\ldots,n-1$, because $[0,1]$ is compact.

We modify them by induction, fit them together and get a unital copy of $\mathcal{O}_2$ in A:

Suppose that $k<n$ and that s,t are canonical generators of $\mathcal{O}_2$ in $A|[0,x_k]$. Let π_+ and π_- denote the natural epimorphisms $A|[0,x_k]\to A|\{x_k\}$ and $A|[x_k,x_{k+1}]\to A|\{x_k\}$ respectively.

Then $u:=\pi_+(s)\pi_-(s_k^*)+\pi_+(t)\pi_-(t_k^*)$ is a unitary in $A|\{x_k\}\cong\mathcal{O}_2$. Since the unitary group of $\mathcal{O}_2$ is connected, we find a unitary v in $A|[x_k,x_{k+1}]$ such that $\pi_-(v)=u$. Let $\tilde{s}:=vs_k$ and $\tilde{t}:=vt_k$. Then $\pi_-(\tilde{s})=\pi_+(s)$ and $\pi_-(\tilde{t})=\pi_+(t)$.

Thus there are S, T in $A|[0, x_{k+1}]$ with restrictions to $[0, x_k]$ and $[x_k, x_{k+1}]$ equal to s, t and $\tilde{s}, \tilde{t}$ respectively. Thus $S^*S = 1 = T^*T$ and $SS^* + TT^* = 1$.

Second, $\mathrm{K}_0(A) = 0$: Since A contains copy of $\mathcal{O}_2$ unitally, it suffices to show that, for every non-zero projection p in A, pAp contains a copy of $\mathcal{O}_2$ unitally. $t \in [0,1] \mapsto \|\pi_t(p)\|$ is continuous, non-zero and takes 1 or 0 as its values. Thus this function is one on $[0,1]$, and, therefore, $\mathrm{Prim}(pAp) = \mathrm{Prim}(A) = [0,1]$, and the primitive quotients are isomorphic to $\mathcal{O}_2$. The first step shows that the unital algebra pAp contains a copy of $\mathcal{O}_2$ unitally.

Third, $\mathrm{K}_1(A) = 0$: Let $B := A \otimes P_\infty$, where P_∞ is the unique unital pi-sun algebra in the UCT-class with $K_0(P_\infty) = 0$ and $K_1(P_\infty) = \mathbb{Z}$, cf. Corollary H(ii). Then $\mathrm{K}_1(A) = \mathrm{K}_0(B)$ by Künneth theorem. But $\mathrm{Prim}(B) = [0,1]$ and every simple quotient of B is isomorphic to $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes P_\infty$. Thus from the second step we see that $\mathrm{K}_0(B) = 0$.

Fourth, separable A with $\mathrm{K}_*(A) = 0$ satisfies the UCT if and only if $\mathrm{KK}(A, B) = 0$ for all separable C^*-algebras B.

The six term exact sequence implies that $\mathrm{KK}(A|I, A|I) = 0$ for every half-open interval $I = [0, t)$ or $I = (t, 0]$, because there is a natural semi-split short exact sequence

$$0 \to A|[0,t) \oplus A|(t,1] \to A \to \mathcal{O}_2 \to 0,$$

and KK is additive in each variable with respect to direct sums. It follows that $\mathrm{KK}(A|I, A|I) = 0$ for every closed subinterval $I = [s,t] \subset [0,1]$ of $[0,1]$, because $A|I = A/(A|[0,s) + A|(t,1])$. For open or half-open subintervals I of $[0,1]$ we have short exact sequences $0 \to A|I \to A|K \to B \to 0$ where K is the closure of I and B is $\mathcal{O}_2$ or $\mathcal{O}_2 \oplus \mathcal{O}_2$.

Fifth: To calculate $\mathrm{KK}([0,1]; A, A)$, we use the six-term exact sequence with respect to the second variable and the following $\mathrm{C}([0,1])$-equivariant semi-split short exact sequence:

$$0 \to J \to \mathrm{C}([0,1]) \otimes A \to A \to 0,$$

where the action on $\mathrm{C}([0,1]) \otimes A$ is induced by

$$d\colon f \in \mathrm{C}([0,1]) \mapsto f \otimes 1 \in \mathrm{C}([0,1]) \otimes A$$

and the map $\mathrm{C}([0,1]) \otimes A \to A$ is the tautological epimorphism given by

$$f \otimes a \in \mathrm{C}([0,1]) \otimes A \mapsto fa \in A\,.$$

The Ψ-equivariant semi-splitness of the exact sequence can be seen as follows: Tensor with $\mathcal{O}_2$, use that $A \otimes \mathcal{O}_2 \cong \mathrm{C}([0,1], \mathcal{O}_2)$, take the obvious lift and cut it down to A and $\mathrm{C}([0,1], A)$ with help of a state on $\mathcal{O}_2$.

J is the closed sum of ideals $\mathrm{C}_0(K) \otimes A|I$ with certain open subintervals K and I of $[0,1]$, cf. Proposition B.4.2(ii).

We consider now the action of $[0,1]$ on $\mathrm{C}_0(K) \otimes A|I$ which is induced by d.

For open or closed sub-intervals I and K of $[0,1]$,

$$\mathrm{KK}([0,1]; \mathrm{C}_0(K) \otimes A|I, \mathrm{C}_0(K) \otimes A|I) = 0,$$

because $A|I$ is KK-equivalent to $\mathcal{O}_2$ and we can tensor the identity map of $\mathrm{C}_0(K)$ with this KK-equivalence to obtain a $\mathrm{KK}([0,1]; .,.)$-equivalence of $\mathrm{C}_0(K) \otimes A|I$ and $\mathrm{C}_0(K) \otimes \mathcal{O}_2$.

The Mayer-Vietoris sequence gives that also $\mathrm{KK}([0,1]; B, C) = 0$ if $B \subset J$ and B is finite sum of ideals $\mathrm{C}_0(K) \otimes A|I$ of $\mathrm{C}([0,1]) \otimes A$.

$\mathrm{KK}([0,1]; .,.)$ is in the first variable continuous with respect to Ψ-equivariant monomorphic inductive limits of nuclear C^*-algebras J_n which are $\mathrm{KK}([0,1]; .,.)$-equivalent to $\mathrm{C}([0,1], \mathcal{O}_2)$. It follows $\mathrm{KK}([0,1]; J, \cdot) = 0$, because the corresponding Milnor $\lim^{(1)}$-sequence vanishes.

Since also $\mathrm{KK}([0,1]; \mathrm{C}([0,1]) \otimes A, \mathrm{C}([0,1]) \otimes A) = 0$, the six term exact sequence shows that $\mathrm{KK}([0,1]; A, A) = 0$.

APPENDIX A

Cuntz equivalence, multiplicative domains, ?related topics?

?? TO DO list of topics, to add:

$0 \leq b \leq n\,\mathrm{diag}(b)$ for $b \in M_n(A)_+$ etc.
(<-- this is partly done below)
Applications of $n[a] \leq m[b]$ for $a, b \in A_+$.

Remarks on Glimm halving in ultra-powers. !!!!!!! That is:
A_ω has Glimm halving property,
if and and only if,
A has *uniform* Glimm halving property.
Is Glimm halving and uniform Glimm halving really different?
See also Remarks in Chapter 2 on Condition (ii) of Definition ??.
See also Lemma A.6.8(iii).
Conjecture?:
Let A a separable stable C*-algebra.
For separable C*-subalgebras B of $\mathrm{Q}^{\mathrm{s}}(A) \cong \mathcal{M}(A)/A$
and unital separable C*-subalgebras C of
$F(A) := (A' \cap A_\omega)/\,\mathrm{Ann}(A, A_\omega)$
there exists a unital C*-morphism
of the ''winding around'' (or ''joining'') algebra $\mathcal{E}(C, C)$ into $B' \cap \mathrm{Q}^{\mathrm{s}}(A)$.
OK: It is only a very weak hope ... !!!

Generalities:
Flip of self-absorbing approximately inner ?
$\mathcal{U}(A) = \mathcal{U}_0(A)$ for all $A := \mathcal{O}_n$, $A := \mathcal{E}_n$?
$\mathcal{U}(A) = \mathcal{U}_0(A)$ for all tensorial self-absorbing A?
Jiang-Su?
Each tensorial self-absorbing A absorbs Jiang-Su? Result of Winter?
Are K_1-injective?
Has stable rank one, by result of Rørdam

What happens with the commutators of unitaries in $\mathrm{C}(S^1) \otimes \mathcal{E}(M_m, M_n)$

if $m \neq n$?
Is it K_1-injective?
In particular, is each commutator of two unitaries in $\mathcal{U}(\mathcal{E}(M_m, M_n))$,
or -- more generally -- those contained in the connected component
$\mathcal{U}_0(\mathcal{E}(M_m, M_n))$ of 1 in the unitaries of it?

1. (*) The algebras $\mathcal{O}_n$, $\mathcal{E}_n$ and $\mathcal{O}_\infty$

We outline an approach to get the used basic facts on $\mathcal{E}_n$ and $\mathcal{O}_n$, $n \in \{2, 3, \ldots, \infty\}$. All is based on elementary observations (or can be replaced by such observation in our particular cases). We work with formal algebras generated by relations, because it allows to see what carries over to general non-commutative rings... and what does not so ... E.g. the existence of suitable very important C^*-subalgebras in the central sequence algebra requires the analytic theory of some rather special types of operator algebras, defined by their defining relations.

(1) ***C*-algebras defined by relations*** .
Let A denote an algebraic *-algebra over $\mathbb{C}$ (or likewise over $\mathbb{R}$) that has finitely, countable or uncountably many generators $a_1, a_2, \ldots$ and defining relations $R_1, R_2, \ldots$ that are given explicitly by non-commutative polynomials $P_n(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*)$ with coefficients in $\mathbb{C}$ (or $\mathbb{R}$ in case of real C^*-algebras) and the relation are given by requiring the equations $P_n = 0$, A is the quotient of the free *-algebra by the ideal generated by the polynomials P_n. Sometimes the relations are written more efficient as matrix conditions and then the P_n are matrix polynomials. But this can be described also by ordinary n.c. polynomials by writing the conditions for the entries separately.

If we require bounds, e.g. that certain expressions $P(a_1, \ldots, a_n^*)$ should become a contractions in any representation then we can do this by introducing new elements, say x, y, and then $Q_1(a_1, \ldots, a_n^*, x) := P(a_1, \ldots, a_n^*) - x$ and $Q_2(x, x^*, y, y^*) = 1 - x^*x - y^*y$, but consider $C^*(a_1, \ldots, a_n; R)$ with system of relations R containing P as a C^*-subalgebra of the unital C^*-algebra $C^*(a_1, \ldots, a_n, x, y, 1; R \cup \{Q_1, Q_2\})$. We go here not into details, because we use only very special types of relations that are easily to understand and are usually "bounded" in the following sense:

The algebraic *-algebra A is called "**bounded**" if, for each generator a_n, there exist $c_1, \cdots, c_m \in \widetilde{A} := A + \mathbb{C}1$, $0 \leq \gamma_n < \infty$ such that $a_n^* a_n + c_1^* c_1 + \cdots + c_m^* c_m = \gamma_n^2 1$. If A is bounded then every *-representation d of A on a pre-Hilbert space $\mathcal{H}$ is bounded in the sense that $\|d(a_n)\| \leq \gamma_n$ for all generating elements. (Compare [**426**, lem. 5.3(iii)] that was inspired by ideas of T. Tannaka [**768**] and its abstract version introduced for the study of block-algebras of M.G. Krein [**493**]). (Today: dual of compact "quantum groups"?)

In particular, the supremum $\|a\| := \sup_{\{d\}} \|d(a)\|$ of all C^*-semi-norms on A exists and is finite for *all* elements on A. The set $J_0 := \{a \in A\,;\, \|a\| = 0\}$ is a

*-ideal of A, and the C^*-completion of A/J_0 with respect to the induced (maximal) C^*-norm is a C^*-algebra, that we denote by $C^*(a_1, a_2, \ldots ; R_1, R_2, \ldots)$.

One can check that the *below considered*(!) special examples A all have the property that $J_0 = \{0\}$ (exercises for the reader). – But in more general cases and other types of relations it can happen that $J_0 \neq \{0\}$. –

To make this definitions more flexible, one can here the "scalar coefficients" replace by elements in some other ring/algebra.

(2) **On the defining relations for $\mathcal{E}_n$ and $\mathcal{O}_{\mathrm{n}}$**

One can easily show – on a fairly algebraic level – that the *universal* C^*-algebras

$$\mathcal{O}_{\mathrm{n}} := C^*(s_1, \ldots, s_n \,;\; s_j^* s_j = 1 = s_1 s_1^* + \cdots + s_n s_n^*)$$

with $n > 1$ is "naturally" isomorphic to the *universal* unital C^*-algebra

$$C^*(B_n \cup \{S\} \,;\; S^*S = 1 \in B_n,\, SaS^* = e_{1,1} \otimes a,\, a \in B_n) \tag{1.1}$$

that contains $B_n := M_{n^\infty}$.

Notice that this formula is only a formal "reminder" if we hold up our restricted view that we, if possible, are willing to describe the basic model algebras as quotients of "free" algebras by ideals generated by "relations" (as above described in an informal manner), then B_n or a suitable dense *-subalgebra itself has to be described as quotient of free *-algebra. It means here that we first have to define B_n as the competition of the inductive limit of $A_{k,n} := M_{n^k}$ with $a \in A_{k,n} \mapsto A_{k+1,n}$ given by $a \mapsto a \otimes 1_n$ and S is an "additional variable" that satisfies the relation polynomials $P_1(S) := S^*S - 1 = 0$ with $1 = \mathrm{indlim}_{k\to\infty} 1_{n^k}$, and $S(a_n \otimes 1_n)S^* = e_{11} \otimes a_n \in A_{k+1,n}$ for all $a_n \in A_{k,n}$. It looks childish, but careful consistence would require a detailed list of generators and how the generators of one algebra can be expressed by the generators of the other algebra and vice versa. All this should be done on the algebraic level. With other words: to compare this relation-defined algebras we should her replace – for calculation with (countably many) relations on the pure algebraic level – the algebra B_n by its algebraic inductive limit of the sequence $M_{n^k} \to M_{n^{k+1}}$ given by the maps $a \mapsto a \otimes (e_{11} + \cdot + e_{nn}) = a \otimes 1_n$ and the proceed as e.g. in [**686**, proof thm. 4.2.2] or in proof of [**169**, thm. 1.13]. The isomorphism of $\mathcal{O}_{\mathrm{n}}$ with the corner cross-product in Equation (1.1) becomes true for the C^*-algebras *after completion* with the maximal C^*-norm on them. In more detail it goes as follows:

Let $I_k := \{1, \ldots, n\}^k$ and define

$$W(i_1, \ldots, i_k) := W_p := s_{i_1} \cdot \ldots \cdot s_{i_k}$$

for the "word" $p := (i_1, \ldots, i_k) \in I_k$. Then the linear span of the elements 1, W_p, $(W_p)^*$ and $W_p(W_q)^*$ $(p \in I_k,\, q \in I_\ell)$ is a dense *-subalgebra of $\mathcal{O}_{\mathrm{n}}$. Take $S := s_1$ and identify M_{n^k} with the linear span of the elements $W_p W_q^*$, $p, q \in I_k$ $(k = 1, 2, \ldots)$.

It shows that $\mathcal{O}_{\mathrm{n}} \cong B_n \rtimes_\psi \mathbb{N}$ for the corner endomorphism $\psi\colon B_n \ni a \mapsto e_{1,1} \otimes a \in B_n$. This algebra is stably isomorphic to the crossed product of a trace-scaling automorphism α of $B_n \otimes \mathbb{K}$. Trace-scaling automorphisms α of simple C^*-algebras have always the property that α^k is strictly outer for $k \in \mathbb{N}$, and therefore, $(B_n \otimes \mathbb{K}) \rtimes_\alpha \mathbb{Z}$ is simple (use here the more general results of Kishimoto [**478**] or of Olesen and Pedersen [**578**]). Crossed products of nuclear C^*-algebras by Abelian groups are always nuclear ????? give citation ?????????? (more generally crossed products of nuclear C^*-algebras by amenable groups are nuclear).

Missing citation/reference to place?

A proof would be:

Let G locally compact group, that acts point-norm continuous ([1]) on a C^*-algebra A by $\rho\colon G \to \mathrm{Aut}(A)$, and let B any C^*-algebra. There are natural isomorphisms of full crossed products

$$C^*(G, A; \rho) \otimes^{\max} B \cong C^*(G, A \otimes^{\max} B; \rho(\cdot) \otimes^{\max} \mathrm{id}_B)$$

and of reduced crossed products

$$C^*_{red}(G, A;\, \rho) \otimes^{\min} B \cong C^*_{red}(G, A \otimes^{\min} B;\, \rho(\cdot) \otimes^{\min} \mathrm{id}_B)$$

and an obvious canonical epimorphism from the first to the second that is compatible with the canonical epimorphism $(\cdot) \otimes^{\max} B \to (\cdot) \otimes^{\min} B$.

Thus, if A nuclear and G amenable, then $C^*(G, A, \rho) = C^*_{red}(G, A, \rho)$ is again nuclear.

`Here is what we need:`

But in our case we use the following general observation:

Suppose that G is a *compact* group that acts point-norm continuous on a C^*-algebra A and that the fix-point algebra $A^{\rho(G)}$ of the action ρ of G on A is nuclear. Then A nuclear.

Indeed: The kernel J of $A \otimes^{\max} B \to A \otimes^{\min} B$ is $\rho(g) \otimes^{\max}$ id-invariant. And the integral over G defines a faithful conditional expectation from J to the kernel of $A^{\rho(G)} \otimes^{\max} B \to A^{\rho(G)} \otimes^{\min} B$ that is trivial by nuclearity of $A^{\rho(G)}$. Thus, $J = \{0\}$ and A must be nuclear.

We have used here the following variant of the Definition of Nuclearity of C*-algebras (compare cite :
Choi–Effros , [**141**], [**142**], Nuclear C*-algebras and injectivity: the general case. (Shows: A is nuclear , if and only if, A^{**} is an injective W*-algebra.) [**145**]

A C^*-algebra A is nuclear, if and only if, the natural *-epimorphisms $A \otimes^{\max} B \to A \otimes^{\min} B$ are faithful (= are isomorphisms) for every C^*-algebra B.

In fact, it suffices to take here as B the algebras $C^*(F_\infty)$, $\mathcal{L}(\ell_2)$ and $\mathcal{L}(\ell_2)/\mathbb{K}(\ell_2)$. (¡ – is this from Kirchberg, E. [**426**], or from one of the upper citations?)

[1] Means that $g \in G \mapsto \rho(g)(a)$ is a continuous map from G into A for each $a \in A$

It follows that $\mathcal{O}_n$ is *simple and nuclear.* By definition, $\mathcal{O}_n$ ($2 \leq n < \infty$) is finitely generated and $\mathcal{O}_\infty$ is countably generated, in particular they all are *separable.*

`We have this ‘‘exercise in footnote’’ carried out somewhere! Find and Cite the place of it!!`

([2]).

The simplicity of the corner-endomorphism crossed product $B_n \rtimes_\psi \mathbb{N}$ and the property that
for each $a, b \in (B_n)_+$ with $\|b\| = 1 \geq \|a\|$ and for each $\varepsilon > 0$ there are $k \in \mathbb{N}$ and a contraction $d \in B_n$ with $\|\psi^k(a) - d^*bd\| < \varepsilon$
imply that $B_n \rtimes_\psi \mathbb{N}$ is purely infinite (cf. Corollary 2.18.3). Hence, $\mathcal{O}_n$ *is a pi-sun algebra in the UCT-class.*

(Urgent Question:
Where it is shown that the $\mathcal{O}_n$ are in the UCT-class, and what has to be shown to recognize the "pi-sun" algebras are in the UCT-class? For the $\mathcal{O}_n$ one has "only" to show first that all AF-algebras are in the UCT-class, and what imply the products of UCT-class algebras by circle actions, action by an endomorphism or automorphisms.)

The above mentioned property can be seen directly by the following argument of J. Cuntz:
Let $b \in B_n \subseteq \mathcal{O}_n$ with $b \geq 0$ and $\|b\| = 1$. Then we find $k \in \mathbb{N}$ and $c \in (M_{n^k})_+$ with $\|b - c \otimes 1\| < \varepsilon$ and $\|c\| = 1$. Since the k-fold tensor product e_k of $e_{1,1} \in M_n$ is a minimal idempotent in M_{n^k} there is a unitary $u \in M_{n^k}$ with $e_k u^* c u e_k = e_k$. Let $d := (u \otimes 1)\psi^k(a^{1/2})$, then $\|d\| \leq 1$ and $d^*(c \otimes 1)d = \psi^k(a)$. Thus $\|d^*bd - \psi^k(a)\| < \varepsilon$.

The Pimsner–Voiculescu exact sequence for crossed products (more precisely: its variant for crossed products by $\mathbb{N}$, [**636**]) gives "immediately" the exact sequence

$$0 \to \mathrm{K}_1(\mathcal{O}_n) \to \mathbb{Z}[\frac{1}{n}] \overset{\times(n-1)}{\longrightarrow} \mathbb{Z}[\frac{1}{n}] \to \mathrm{K}_0(\mathcal{O}_n) \to 0\,,$$

?? to be checked again !!!

where $1 \in \mathbb{Z}[\frac{1}{n}]$ maps to $[1] \in \mathrm{K}_0(\mathcal{O}_n)$. It shows that $\mathrm{K}_1(\mathcal{O}_n) = 0$ and $\mathrm{K}_0(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$ (in a way that the class of $[1]$ generates $\mathbb{Z}_{n-1}$). In particular, $\mathcal{O}_2$ is K_*-trivial.

(3) **The non-simple algebras $\mathcal{E}_n$.**

We consider the (non-simple) universal C^*-algebras

$$\mathcal{E}_n := C^*(s_1, \ldots, s_n\,;\; s_i^* s_j = \delta_{i,j} 1)\,,$$

and its (simple) natural inductive limit

$$\mathcal{O}_\infty := C^*(s_1, s_2, \ldots\,;\; s_i^* s_j = \delta_{i,j} 1)\,.$$

[2] In fact, it is easy to see (as an exercise for the readers?) that every separable C^*-algebra A with properly infinite multiplier algebra $\mathcal{M}(A)$, i.e., with $1_{\mathcal{M}(A)} \in \mathcal{E}_2 \subseteq \mathcal{M}(A)$, is single generated. This applies to all separable pi-sun algebras, in particular to $\mathcal{E}_n$, $\mathcal{O}_n$ and $\mathcal{O}_\infty$.

It turns out ([3]), that $e := e_n := 1 - (s_1 s_1^* + \cdots + s_n s_n^*)$

The element e_n is called ''p'' on some other places.

is a non-zero projection in $\mathcal{E}_n$ with $e\mathcal{E}_n e = \mathbb{C}e$, that the closed ideal $I(e)$ (generated by e) is naturally isomorphic to $\mathbb{K}$, and that the left multiplication of $\mathcal{E}_n$ on $I(e)$ is faithful (as one can see with help of the gauge action $z \in S^1 \mapsto \sigma_z \in \mathrm{Aut}(\mathcal{E}_n)$ with $\sigma_z(s_n) = zs_n$ that fixes e).

Indeed:
$X \cdot \mathcal{E}_n \cdot e = \{0\}$ implies $X^*X \cdot \mathcal{E}_n \cdot e = \{0\}$. Since $\sigma_z(\mathcal{E}_n \cdot e) = \mathcal{E}_n \cdot e$ for all $z \in S^1$, we get also $\sigma_z(X^*X) \cdot \mathcal{E}_n \cdot e = \{0\}$ for all $z \in S^1$. Thus $P(X^*X) \cdot \mathcal{E}_n \cdot e = \{0\}$. Here $P(a) := \int_{S^1} \sigma_z(a)dz$ for $a \in \mathcal{E}_n$.

It is not difficult to see: The fix-point algebra of any (continuous) circle action on the compact operators $\mathbb{K}$ of a separable Hilbert space has a non-degenerate fix-point C^*-subalgebra of $\mathbb{K}$. (Because $p \in \overline{P(p) \cdot \mathbb{K} \cdot P(p)}$ for all continuous circle actions $\sigma(z)$ on $\mathbb{K}$.)

It follows that $P(\mathcal{E}_n)$ acts faithful on the ideal $I(e)$ by left-multiplication if and only if $P(\mathcal{E}_n)$ acts faithful on $I(e) \cap P(\mathcal{E}_n)$.

It is not difficult to see that the fix-point algebra $P(\mathcal{E}_n)$ of the circle action on $\mathcal{E}_n$ is an AF-algebra that contains p. If $P(\mathcal{E}_n)$ acts not faithful on $I(e)$ then there must exist a non-zero projection in the kernel of this action.

Such a projection must be equivalent to a projection in the canonical filtration of $P(\mathcal{E}_n)$ by finite AF-algebras and each “smaller” projection is also on the kernel. It causes that for some $W := s_{k_1} \cdot \ldots \cdot s_{k_m}$ holds $WW^* \cdot I(e) = \{0\}$, in particular $WW^*(We) = 0$. But this is impossible, because W is an isometry, i.e., $W^*W = 1$ and $W^*(We) = e$.

Thus, $P(\mathcal{E}_n)$ acts faithful on $\mathcal{E}_n \cdot e$ and on $I(e)$. By above considerations it implies that $P(X^*X) = 0$ if $X\mathcal{E}_n \cdot e = \{0\}$. Since P is a faithful conditional expectation (for any continuous circle action) it follows that $X = 0$ if $X \in \mathcal{E}_n$ and $X \cdot \mathcal{E}_n \cdot e = \{0\}$. Thus, $I(e)$ is a non-degenerate ideal of $\mathcal{E}_n$, i.e., there exists no ideal orthogonal to $I(e)$.

(This arguments show that every $\sigma(z)$-invariant non-zero ideal of $\mathcal{E}_n$ has non-zero intersection with $P(\mathcal{E}_n)$.)

Therefore, $I(e)$ is an *essential* ideal of $\mathcal{E}_n$.

Clearly, the quotient $\mathcal{E}_n/I(e)$ is naturally isomorphic to the – simple – C^*-algebra $\mathcal{O}_{\mathrm{n}}$.

(This way of arguments requires to prove *before !!!* the simplicity of $\mathcal{O}_{\mathrm{n}}$, to obtain above conclusion.)

[3]Use that $\mathcal{E}_n$ is the span of elements $W_r W_s^*$ of words, where W_r and W_s are $= 1$ or are products of some of the generators $s_1, \ldots, s_n$.

Above we have seen, that there is a short-exact sequence

$$0 \to \mathbb{K} \to \mathcal{E}_n \to \mathcal{O}_{\rm n} \to 0\,.$$

In particular, $\mathcal{E}_n$ must be nuclear (if $\mathcal{O}_{\rm n}$ is proven to be nuclear). The corresponding six-term exact sequence in K_*-theory shows that (necessarily) $\mathrm{K}_1(\mathcal{E}_n) = 0$ and that there is an exact sequence

$$0 \to \mathbb{Z} \to \mathrm{K}_0(\mathcal{E}_n) \to \mathbb{Z}_{n-1} \to 0\,,$$

such that the class $[1] \in \mathrm{K}_0(\mathcal{E}_n)$ is mapped onto the $1 \in \mathbb{Z}_{n-1}$ and $1 \in \mathbb{Z}$ is mapped into $[e] \in \mathrm{K}_0(\mathcal{E}_n)$. The definition of e shows that $[e] = (n-1)[1]$ in $\mathrm{K}_0(\mathcal{E}_n)$.

The only remaining possibility is now $\mathrm{K}_0(\mathcal{E}_n) \cong \mathbb{Z}$ with generator $[1]$ of $\mathbb{Z}$.

One can see now (e.g. from $[e] = (n-1)[1]$) that the extension $\mathcal{E}_n$ of $\mathcal{O}_{\rm n}$ by $\mathbb{K}$ is not a split extension.

One can show also directly that the natural C^*-morphisms $\mathcal{E}_{n-1} \to \mathcal{E}_n \to \mathcal{O}_{\rm n}$ for $n \geq 2$ define a faithful embedding from $\mathcal{E}_{n-1}$ into $\mathcal{O}_{\rm n}$, – and therefore also a faithful embedding from $\mathcal{E}_{n-1}$ into $\mathcal{E}_n$ that does not intersect the closed ideal of $\mathcal{E}_n$ generated by $e_n := 1 - \sum_{k=1}^{n} s_k s_k^*$.

Indeed: Let $\rho\colon \mathcal{E}_{n-1} \to \mathcal{L}(\mathcal{H})$ a faithful unital *-representation of $\mathcal{E}_{n-1}$. We can here take a separable Hilbert space $\mathcal{H}$, because $\mathcal{E}_{n-1}$ is separable. The $\mathcal{H}$ has infinite dimension because $\mathcal{L}(\mathcal{H})$ contains a non-unitary isometry. Thus $\mathcal{H} \cong \ell_2(\mathbb{N})$. If we let $L := \rho(e_{n-1})\mathcal{H}$, then there exists an isometry $T \in \mathcal{L}(\mathcal{H} \otimes_2 \mathcal{H})$ with $TT^* = \rho(e_{n-1}) \otimes \mathrm{id}_{\mathcal{H}}$. The isometries $t_k := \rho(s_k) \otimes \mathrm{id}_{\mathcal{H}}$ for $k = 1, \ldots, n-1$ and $t_n := T$ satisfy the relation $\sum_{k=1}^{n} t_k t_k^* = 1$. Thus $C^*(t_1, \ldots, t_n)$ is a natural image of $\mathcal{O}_{\rm n}$ and the C^*-representation $a \mapsto \rho(a) \otimes \mathrm{id}_{\mathcal{H}}$ is a faithful unital C^*-morphism from $\mathcal{E}_{n-1}$ into $C^*(t_1, \ldots, t_n)$ that factorizes through the canonical unital C^*-morphisms $\mathcal{E}_{n-1} \to \mathcal{E}_n \to \mathcal{O}_{\rm n} \to C^*(t_1, \ldots, t_n)$. It shows that the natural unital C^*-morphism $\mathcal{E}_{n-1} \to \mathcal{E}_n$ is injective and does not intersect the ideal of $\mathcal{E}_n$ generated by e_n.

It follows that the natural unital *-morphisms from $\mathcal{E}_n$ into $\mathcal{O}_\infty$ are injective (because they factorize over the natural C^*-morphism $\mathcal{E}_n \to \mathcal{E}_{n+1}$, and that is injective) and that, therefore, $\mathrm{K}_*(\mathcal{O}_\infty) = \mathrm{indlim}_n\, \mathrm{K}_*(\mathcal{E}_n)$ by continuity of $\mathrm{K}_*(\cdot)$ with respect to inductive limits ...

(Unfortunately, it is bit more complicate ...!!!)

We get that $\mathcal{O}_\infty$ is separable, unital and nuclear, that $\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ with generator $[1] \in \mathrm{K}_0(\mathcal{O}_\infty)$ and that $\mathrm{K}_1(\mathcal{O}_\infty) = 0$. The unital C^*-morphisms $\eta_n\colon \mathcal{E}_n \to \mathcal{E}_{n+1}$ satisfy the sufficient criteria in part (iv) of Proposition 2.2.5, because $I(e) \cong \mathbb{K}$ is an essential ideal.

Indeed: It implies that, for each $a \in (\mathcal{E}_n)_+$ with $\|a\| = 1$, there is a contraction $c \in I(e)$ with $\|c^* a c - e\| < \varepsilon$, thus $\|d^* \eta_n(a) d - 1\| < \varepsilon$ for $d := \eta_n(c) s_{n+1}$.

Thus $\mathcal{O}_\infty$ is simple and *purely infinite* by Proposition 2.2.5(iv), i.e., if we all above put together the we see that $\mathcal{O}_\infty$ is a (simple) pi-sun algebra with

$\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ with generator [1] and $\mathrm{K}_1(\mathcal{O}_\infty) = \{0\}$. The pure infiniteness implies that $\mathrm{K}_1(\mathcal{O}_\infty) = 0$ is equivalent to $\mathcal{U}_0(\mathcal{O}_\infty) = \mathcal{U}(\mathcal{O}_\infty)$.

(4) δ_n **is homotopic to** id **in** $\mathrm{End}(\mathcal{O}_\mathrm{n})$

δ_2 is unitarily homotopic to id in the space of unital *-endomorphism of $\mathcal{O}_2$ (with point-norm topology).

Indeed: Notice that $X := \{s^2, st, ts, t^2\}$ are four isometries with range sum $= 1$ (X defines a unital embedding of $\mathcal{O}_4$ into $\mathcal{O}_2$). Thus $B := \mathrm{span}\{ab^*\,;\ a, b \in X\}$ is naturally isomorphic to M_4. Let $u_1 := \delta_2(s)s^* + \delta_2(t)t^* = s^2(s^*)^2 + ts(st)^* + st(ts)^* + t^2(t^*)^2$. Then u_1 is a selfadjoint unitary in B, $\delta_2(s) = u_1 s$ and $\delta_2(t) = u_1 t$. It follows that there is a (norm-)continuous path $[0, 1] \ni \tau \mapsto u(\tau) \in \mathcal{U}(B)$ with $u(0) = 1$ and $u(1) = u_1$. The unital endomorphisms h_τ of $\mathcal{O}_2$ with $h_\tau(s) = u(\tau)s$ and $h_\tau(t) = u(\tau)t$ define a point-norm continuous path in the unital endomorphisms of $\mathcal{O}_2$ with $h_0 = \mathrm{id}$ and $h_1 = \delta_2$.

Similarly, one can see that δ_n *is homotopic to* id *in the unital *-endomorphisms of* $\mathcal{O}_\mathrm{n}$ *for* $3 \le n < \infty$*,* by showing that

$$u_1 := \sum_{k=1}^{n} \delta_n(s_k)s_k^* = \sum_{k,\ell=1}^{n} s_k s_\ell (s_\ell s_k)^*$$

is a selfadjoint unitary in the natural image of $M_{n^2}(\mathcal{Z})$ spanned by

$$\{s_k s_\ell s_j^* s_i^*\,;\ k, \ell, j, i \in \{1, \dots n\}\}\ .$$

(5) **The unitary groups are contractible** $\mathcal{U}(\mathcal{O}_\mathrm{n}) = \mathcal{U}_0(\mathcal{O}_\mathrm{n})$

For the case of the *real* C^*-algebras generated by $s_1, \dots, s_n$, one can consider the unitary $u := \sum_{k=1}^{n} \delta_n(s_k)s_k^*$ in the bigger C^*-subalgebra M_{n^3} of $\mathcal{O}_\mathrm{n}$ spanned by $W_p(W_q)^*$ ($p, q \in \{1, \dots, n\}^3$).

The determinant of $u \otimes 1_n$ in $O(n^3)$ is always ± 1, but if $n = 2m$ then again $u \in 1_n \in SO(n^3)$ and $SO(k)$ is connected for all $k \in \mathbb{N}$.

Thus, for even $n \in 2\mathbb{N}$, we have again that δ_n is homotopic to id inside the unital endomorphisms of the real version $(\mathcal{O}_\mathrm{n})_\mathbb{R}$ of $\mathcal{O}_\mathrm{n}$.

`check next again?  was used!  Where??` The sign of the determinant of $u \otimes 1 \in M_{2^2} \otimes 1_2 \subseteq M_{2^3}$ is positive. Thus $u \otimes 1 \in SO(2^3)$, ... ????? But elements of $SO(2^3)$ are connected to 1 in $SO(2^3)$. This implies also the K_*-triviality of ${\mathcal{O}_2}^\mathbb{R}$. (Moreover $F({\mathcal{O}_2}^\mathbb{R}) = 0$ for every homotopy invariant “additive” functor from the category of “real” C^*-algebras into “additive” categories of groups.)

For the other “real” ${\mathcal{O}_\mathrm{n}}^\mathbb{R}$ the situation is more complicate ... Compare e.g. [**718**] in conjunction with [**173**].

(6) **The groups** $\mathcal{U}(B \otimes \mathcal{O}_2)$ **are contractible.**

$\mathcal{U}(B \otimes \mathcal{O}_2)$ is connected for all unital separable C^*-algebras: The application of the path $\mathrm{id}_B \otimes h_\tau$ from id to $\mathrm{id}_B \otimes \delta_2$ to U defines a path from U to $\delta_2(U)$ in $\mathcal{U}(B \otimes \mathcal{O}_2)$.

Now proceed as in Chapter 11. `more general? ??`.

Notice that every continuous map $\lambda\colon S^n \to \mathcal{U}(B\otimes\mathcal{O}_2)$ is an element of $\mathcal{U}(C(S^n,B)\otimes\mathcal{O}_2)$. Thus, $\mathcal{U}(B\otimes\mathcal{O}_2)$ has trivial homotopy groups. Since $\mathcal{U}(B\otimes\mathcal{O}_2)$ is a Polish space, that is locally homeomorphic to the self-adjoint part of $B\otimes\mathcal{O}_2$, it follows that $\mathcal{U}(B\otimes\mathcal{O}_2)$ is *contractible* (By a classical result of J. Milnor [**556**]). In particular, *the unitary group of* $\mathrm{C}(S^1,\mathcal{O}_2)$ *is simply connected.*

(7) **A proof that $\mathcal{O}_{\mathrm{n}}$ is pi-sun**

There is an alternative direct proof of the simplicity, nuclearity and pure infiniteness of $\mathcal{O}_{\mathrm{n}}$, considered as the *universal* C^*-algebras

$$C^*(s_1,\ldots,s_n\,;\; s_j^*s_j = 1 = s_1s_1^* + \cdots + s_ns_n^*)\,,$$

that does *not imply* (immediately) the above derived additional informations that $\mathcal{O}_{\mathrm{n}}$ is in the UCT class and $\mathrm{K}_*(\mathcal{O}_{\mathrm{n}}) \cong \mathbb{Z}_{n-1}\oplus 0$ with generator [1].

If $z\in S^1\subset\mathbb{C}$ is a complex number with $|z|=1$ then, by universality of $\mathcal{O}_{\mathrm{n}}$, there is a unique automorphism σ_z of $\mathcal{O}_{\mathrm{n}}$ with $\sigma_z(s_j)=zs_j$. It is easy to check on elementary products of the s_j and s_k^* that $z\to\sigma_z$ is a (point-norm continuous) circle action on $\mathcal{O}_{\mathrm{n}}$, and that the fixed point algebra of σ is just B_n , i.e., $P(\mathcal{O}_{\mathrm{n}}) = B_n$ for the faithful conditional expectation

$$P(x) := \frac{1}{2\pi}\int_0^{2\pi}\sigma_{e^{it}}(x)\,\mathrm{d}t\,.$$

Let F any C^*-algebra and $h\geq 0$ in the kernel J of $\mathcal{O}_{\mathrm{n}}\otimes^{\max}F\to\mathcal{O}_{\mathrm{n}}\otimes^{\min}F$. Since the kernel is invariant under the action $\sigma_z\otimes^{\max}\mathrm{id}_F$ by the functoriality of the maximal and minimal C^*-tensor products, we can integrate and get that $\frac{1}{2\pi}\int_0^{2\pi}(\sigma_{e^{it}}\otimes\mathrm{id})(h)\,\mathrm{d}t$ is in J. The conditional expectation defined by this integral can be calculated on elementary tensors, and is just $P\otimes^{\max}\mathrm{id}_F$. It is necessarily a faithful conditional expectation from $\mathcal{O}_{\mathrm{n}}\otimes^{\max}F$ onto $B_n\otimes^{\max}F = B_n\otimes^{\min}F$ (because B_n is nuclear). It follows that $J\cap(B_n\otimes^{\max}F) = P\otimes^{\max}\mathrm{id}(J) = 0$ and, thus $P\otimes^{\max}\mathrm{id}_F(h)=0$ and $h=0$, i.e., $J=\{0\}$ and $\mathcal{O}_{\mathrm{n}}$ *is nuclear.*

The proof of the simplicity and pure infiniteness uses that the faithful conditional expectation P is one-step approximately inner (by a sequence of isometries $D_1, D_2,\ldots\in\mathcal{O}_{\mathrm{n}}$). Refer to the place where the D_n are defined.

`What is here` D_k`?  How is it defined?  See further below.`

An argument of J. Cuntz in [**172**] implies that $\mathcal{O}_{\mathrm{n}}$ is both simple and purely infinite:
Let $\varepsilon>0$ and $a\in\mathcal{O}_{\mathrm{n}}$ with $0\leq a$ and $\|a\|=1$, then $P(a)\neq 0$ (because P is faithful on $(\mathcal{O}_{\mathrm{n}})_+$ by continuity of the circle action σ_z), and we find $k\in\mathbb{N}$ and a *minimal* projection $p\in M_{n^k}$ with $\|D_k^*aD_k - P(a)\| < (\varepsilon\|P(a)\|)/2$ and $\|pP(a)p - \|P(a)\|p\| < (\varepsilon\|P(a)\|)/2$.

There is an isometry T in $\mathcal{O}_{\mathrm{n}}$ with $T(T^*)=p$. We get $\|\,d^*ad-1\,\|<\varepsilon$ for $d := \|P(b)\|^{-1/2}D_kT$. Then calculation shows that $\|T^*D_k^*aD_kT - 1\|P(a)\|\| < (2/3)\varepsilon\|P(a)\|$.

(8) **Approximate Innerness of $P\colon\mathcal{O}_{\mathrm{n}}\to M_{n^\infty}$**

An algebraically defined explicit sequence of isometries $D_1, D_2, \ldots \in \mathcal{O}_n$, first described by De Schreye and Van Daele in [**209**, prop. 6], with property

$$\lim_k \| D_k^* x D_k - P(x) \| = 0 \quad \text{for all } x \in \mathcal{O}_n$$

can be given by the following modification of definitions by De Schreye and Van Daele in [**209**, prop. 5] (it uses an argument of J. Cuntz in [**169**, lem. 1.8]):
We let $I_k := \{1, \ldots, n\}^k$ and define isometries $W_p := s_{i_1} s_{i_2} \cdot \ldots \cdot s_{i_k}$ for $p = (i_1, \ldots, i_k) \in I_k$. The linear span of the elements $W_r W_s^*$ is a dense *-subalgebra of $\mathcal{O}_n$, and each element $W_r W_s^*$ $(r \in I_\ell,\ s \in I_m)$ with $\ell \le m \le k$ is in the linear span of elements $W_p W_q^*$ with $q \in I_k$ and $p \in I_{k-(m-\ell)}$. Notice that M_{n^k} is naturally isomorphic to the linear span of $W_r W_s^*$ with $r, s \in I_k$. Define $Y_k := s_1 (s_2)^k$ and

$$D_k := \sum_{p \in I_k} W_p Y_k W_p^* .$$

The W_p are isometries with mutually orthogonal ranges that sum up to 1. Therefore D_k is an isometry, and satisfies $D_k^* W_r W_s^* D_k = P(W_r W_s^*)$ if $r \in I_\ell$, $s \in I_m$ and $\max(\ell, m) \le k$.
Indeed:
It suffices to consider the case $m = k \ge \ell$, then $W_s^* D_k = Y_k W_s^*$, which implies that $D_k^* b D_k = b$ for all $b \in M_{n^k}$. In case $\ell < k$, one has $D_k^* W_r = \sum_{i \in I_j} W_r W_i Y_k^* W_i^*$ for $j := k - \ell$. Thus, the remaining case $m = k > \ell$ leads (with $j := k - \ell > 0$) to

$$D_k^* W_r W_s^* D_k = D_k^* W_r Y_k W_s^* = \sum_{i \in I_j} W_r (W_i Y_k^* W_i^* Y_k) W_s^*$$

which is zero: If $W_i Y_k^* \neq 0$ then necessarily $W_i = s_2^j$, which gives $W_i Y_k^* W_i^* Y_k = s_2^j (s_2^k)^* s_1^* (s_2^j)^* s_1 (s_2)^k = 0$, because $(s_2^j)^* s_1 = 0$.

`Change to more consistent notation!!`

(9) **Question if flip has algebraic approximation**

An interesting question for application to flip maps is the following:

It holds that $\lim_k (D_k \otimes D_k)^* (x \otimes y)(D_k \otimes D_k) = P(x) \otimes P(y)$ for all $x, y \in \mathcal{O}_n$ using suitable contractions $D_1, D_2, \ldots \in \mathcal{O}_n$ with the property

$$\lim_k \| D_k^* x D_k - P(x) \| = 0 \quad \text{for} \quad x \in \mathcal{O}_n .$$

(It would be desirable to have them also for $\mathcal{E}_n$ in place of $\mathcal{O}_n$ to get it also for $\mathcal{O}_\infty$.)

If this is the case then the question about the simplicity of the fixed point algebra of the flip map on $\mathcal{O}_n \otimes \mathcal{O}_n$ becomes a question about the simplicity of the fix-point algebra F_{n^∞} of the flip on $M_{n^\infty} \otimes M_{n^\infty}$. This is because F_{n^∞} is the range of the restriction of $P \otimes P$ to the fix-point algebra of the flip map on $\mathcal{O}_n \otimes \mathcal{O}_n$: Each (possibly existing) non-trivial ideal J of the fix-point algebra of the flip map on $\mathcal{O}_n \otimes \mathcal{O}_n$ has a nontrivial intersection with the F_{n^∞} with help of the in the flip-algebra contained $D_k \otimes D_k$.

??? Is there an opposite application ???: To show simplicity (!) of $\mathcal{O}_n$, could we show directly that the flip of $\mathcal{O}_n \otimes \mathcal{O}_n$ is approximately inner as c.p. map?

???

The unitary $u = \sum_n s_n^* \otimes s_n$ that satisfies $u^*u = 1 = uu^*$, $u(s_k \otimes 1) = 1 \otimes s_k$, and $u^*(1 \otimes s_k) = s_k \otimes 1$.

$\sum_{n,m} s_n^* s_m \otimes s_n s_m^* = \sum_n 1 \otimes s_n s_n^* = 1$ and $\sum_{n,m} s_n s_m^* \otimes s_n^* s_m = 1$ and $\sum_n \ldots = 1$

(applies also to the ultrapower ?).

$u(s_k \otimes 1) = (1 \otimes s_k)u$??? $\sum_n s_n^* s_k \otimes s_n$?? =?? $\sum_n s_n^* \otimes (s_k s_n)$ No !!!!

`Can not work??`
`because extremal ideals need not be maximal.`
`Could only work with` *extremal* S_1`-invariant ideals,`
`and would show that` $\mathbb{K} \subseteq \mathcal{E}_n$
`is the only non-trivial` S_1`-invariant ideal.`

Attempt: Go the following way:

The kernel ideal of $\mathcal{E}_n \to \mathcal{O}_n$ is generated by $p := 1 - s_1 s_1^* - \ldots - s_n s_n^*$, and the projection p satisfies $pW_i W_j^* p = 0$ and $pW_i = 0$ for $W_i \in I_k$ and $W_j \in I_\ell$. It follows that p is a minimal projection, i.e., satisfies $p(\mathcal{E}_n)p = \mathbb{C} \cdot p$ and the ideal generated by p is isomorphic to $\mathbb{K}(\ell_2(\mathbb{N}))$.

(Question: In the real case one has to take ± 1 actions (by $\mathbb{Z}_2$ or $\mathbb{Z}_2^n$) ??)

Moreover, the circle action fixes the projection p. Thus, the ideal generated by p is invariant under circle action.

`next is nonsense, because ''extremal'' is not ''maximal''.`
`Check if this used on other places!!!`

Since $\mathcal{O}_n$ and $\mathcal{E}_n$ are unital, they

???? have a maximal (? only extremal !! ?) nontrivial closed ideal (say J) that contains the ideal generated by p.

Without knowing first the existence or non-existence of an ideal we have to proof with other means that the conditional expectation P is approximately inner. Just this is established with $D_1, D_2, \ldots$.

The fixed points of the circle action on "this maximal ideal" must be necessarily an ideal of the fixed point algebra.

(10) **On simplicity of algebraic $\mathcal{O}_n$ version**

The reader can check, that the above arguments show also that the universal algebraic *-algebra (over $\mathbb{R}$, $\mathbb{C}$ or over the rational numbers) with the above suitably generalized defining relations of $\mathcal{O}_n$ is a simple *-algebra.

In fact, the above arguments also show that the (algebraic) algebra generated by elements $s_1, \ldots, s_n, t_1, \ldots, t_n$ with defining relations $t_j s_k = \delta_{jk} 1$ and $\sum_k s_k t_k = 1$ is simple.

(With suitably modified $D_1, D_2, \ldots$.)

One can easily see that there is a unique anti-isomorphism $x \mapsto x^*$ of order 2 with $s_k^* = t_k$. (The involution * has to be defined as conjugate linear if we consider the algebraic algebra over $\mathbb{C}$ defined by this relations.)

(11) **List** $\mathrm{K}_*(\mathcal{O}_\mathrm{n})$

$(\mathrm{K}_0(\mathcal{O}_\mathrm{n}), [1], \mathrm{K}_1(\mathcal{O}_\mathrm{n})) = (\mathbb{Z}_{n-1}, 1, 0)$ $(n = 3, \ldots)$, $(\mathrm{K}_0(\mathcal{O}_2), \mathrm{K}_1(\mathcal{O}_2)) = (0, 0)$ and $(\mathrm{K}_0(\mathcal{O}_\infty), [1], \mathrm{K}_1(\mathcal{O}_\infty)) = (\mathbb{Z}, 1, 0)$.

Has been discussed above in Old(0.1) ?= New (3)? .

(12) **Unital endomorphisms of $\mathcal{O}_\mathrm{n}$ are homotopic.**

Every unital endomorphism of $\mathcal{O}_\mathrm{n}$ is homotopic to id (in particular δ_n is homotopic to id).

[And thus, $(n-1)\,\mathrm{K}_*(\mathcal{O}_\mathrm{n}) = 0$]

(See Old (0.2) ?= ? New (4) or (A1.4, it follows also from UCT and classification for $n = 3, 4, \ldots$).

But we need before that $\mathcal{O}_\infty$ and $\mathcal{O}_2$ are pi-sun with $(\mathrm{K}_0(\mathcal{O}_2), \mathrm{K}_1(\mathcal{O}_2)) = (0, 0)$

(It follows from: δ_2 unitarily homotopic to id) and $(\mathrm{K}_0(\mathcal{O}_\infty), [1], \mathrm{K}_1(\mathcal{O}_\infty)) = (\mathbb{Z}, 1, 0)$ (should come from $\mathrm{K}_*(\mathcal{E}_n) = (\mathbb{Z}, [1_{\mathcal{E}_n}] = 1, 0)$ and $\mathcal{O}_\infty = \mathrm{indlim}_n\, \mathcal{E}_n$ with unital $\mathcal{E}_n \to \mathcal{E}_{n+1}$).

$\mathcal{E}_n \to \mathcal{E}_2 \to \mathcal{E}_n$ homotopic to id, and $\mathcal{E}_2 \to \mathcal{E}_n \to \mathcal{E}_2$ homotopic to id, where $\mathcal{E}_m \to \mathcal{E}_n$ for $n \geq m$ natural $s_k \mapsto s_k$, and $\mathcal{O}_\infty \to \mathcal{E}_2$ given by $s_1 \mapsto s_1$ and $s_{n+1} \mapsto s_2^n s_1$. Notice $s_1^*(s_2^m)^* s_2^n s_1 = 0$ for $m = 0, 1, 2, \ldots$, $m < n$.

The proof of homotopy follows from the K_1-injectivity implied by the squeezing property (sq) of the $\mathcal{E}_n$.

We must show before that $\mathcal{O}_2 \cong \mathcal{D}_2 := \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ and $\mathcal{O}_\infty \cong \mathcal{D}_\infty := \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$???.

(How to get that shortly, elementary, sufficient ???)

(13) **General $F(\mathcal{O}_2) = 0$ for homotopy functors.**

For every

(enough: "with respect to unitary homotopy" ??)

homotopy-invariant additive functor $F \colon C^* \to G$ into an Abelian (or only additive) category G holds $F(\mathcal{O}_2) = 0$ where C is a subcategory of the category of C^*-algebras that contains $\mathcal{O}_2$ (as object) and all unital endomorphisms of $\mathcal{O}_2$ (as morphisms).

Also every functor F that takes on approximately unitary equivalent C^*-morphisms into the same element/morphism in G.

(14) $\mathcal{O}_n$ **as Cuntz-Pimsner algebras.**

There is another way to study $\mathcal{O}_n$ (and $\mathcal{E}_n$) – including all the former results on nuclearity, simplicity, pure infiniteness, UCT and the calculation of the K-theories.

This algebras are very special cases of Cuntz-Pimsner algebras (respectively generalized Fock–Toeplitz algebras) as considered by M. Pimsner in [**633**], because $\mathcal{O}_n \cong \mathcal{O}(\mathcal{H})$ and $\mathcal{E}_n \cong \mathcal{T}(\mathcal{H})$ for the Hilbert $\mathbb{C}$ –$\mathbb{C}$- bimodule $\mathcal{H} := \ell_2(n)$, $\mathcal{O}_\infty \cong \mathcal{O}(\mathcal{H}) = \mathcal{T}(\mathcal{H})$ for the $\mathbb{C}$ –$\mathbb{C}$-bimodule $\mathcal{H} := \ell_2(\mathbb{N})$. The paper of M. Pimsner unifies and generalizes some parts of related work of J. Cuntz.

1.1. Remarks on $\mathcal{O}_n$ and $\mathcal{O}_\infty$ (2). Where is the homotopy $a \otimes 1 \sim 1 \otimes a$ in $\mathcal{O}_\infty$, $\mathcal{O}_2$?

(and $\mathcal{O}_n$?? depends on the class in $K_1(\mathcal{O}_n \otimes \mathcal{O}_n)$ of ??? a possible flipping unitaries v_n, discussed somewhere below ?? with $\lim_n v_n^*(s_k \otimes 1)v_n = 1 \otimes s_k$, $\lim_n v_n^*(1 \otimes s_k)v_n = s_k \otimes 1$??? or $V := n^{-1/2} \sum_{k,\ell} s_k^* \otimes s_\ell$??? with $V^* = V$)

constructed respectively described?

For $\mathcal{O}_2 \otimes A$ we have always $K_*(\mathcal{O}_2 \otimes A) = 0$. Indeed, $\delta_2 \otimes \mathrm{id}_A$ is (unitary) homotopic to id on $\mathcal{O}_2 \otimes A$, because δ_2 on $\mathcal{O}_2$ has this property.

If A is unital, then this implies $\mathcal{U}(\mathcal{O}_2 \otimes A) = \mathcal{U}_0(\mathcal{O}_2 \otimes A)$ by Proposition 4.2.15 because $\mathcal{O}_2$ satisfies as a quotient of $\mathcal{E}_2$ the "squeezing" Property (sq) of Definition 4.2.14 and $\mathcal{O}_2 \otimes A$ inherits it from $\mathcal{O}_2$.

It follows moreover that $\mathcal{U}(\mathcal{O}_2 \otimes A)$ is contractible, because $\mathcal{O}_2 \otimes (A \otimes C(S_n))$ has also contractible $C(S_n, \mathcal{U}(\mathcal{O}_2 \otimes A)) \cong \mathcal{U}(\mathcal{O}_2 \otimes (A \otimes C(S_n)))$.

The case of $\mathcal{O}_\infty$ requires to show that $K_1(\mathcal{O}_\infty) = 0$ and $K_1(\mathcal{O}_\infty \otimes \mathcal{O}_\infty) = 0$. It was shown in Section **??** of Chapter 2 that $\mathcal{O}_\infty$ is p.i. and simple.

< -- NO! That $\mathcal{O}_n$ is simple has to be shown -- before we can apply this arguments !!

The definitions of $\mathcal{O}_\infty$, $\mathcal{O}_n$ and $\mathcal{E}_n$ show that all have an properly infinite unit element, thus are K_1-surjective by Lemma 4.2.6(v).

By Proposition 4.2.15, $\mathcal{E}_n$, $\mathcal{O}_\infty$ and $\mathcal{O}_n$ have the squeezing property (sq) of Definition 4.2.14, because the $\mathcal{E}_n$ have it and $\mathcal{O}_\infty$ respectively $\mathcal{O}_n$ inherits it as inductive limit of all $\mathcal{E}_n$ respectively quotients of $\mathcal{E}_n$.

Those C^*-algebras A that absorb $\mathcal{O}_2$ or $\mathcal{O}_\infty$ tensorial are also *strongly* purely infinite (and are simple if A is simple), and then are K_1-injective by Proposition 4.2.15 or ?????????????

Since $\mathcal{O}_\infty$ is in the "bootstrap" category (as inductive limit of the $\mathcal{E}_n$), the Künneth theorem for tensor products gives that $K_1(\mathcal{O}_\infty \otimes \mathcal{O}_\infty) = 0$ and that $K_0(\mathcal{O}_\infty \otimes \mathcal{O}_\infty) = \mathbb{Z}$ with $[1 \otimes 1] = 1$. This argument can be repeated and yields $K_1(\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots) = 0$ and $K_0(\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots) = \mathbb{Z}$ with $[1] = 1$ by induction.

Use v_n get what??

It needs to consider the partial isometries $v_n := \sum_{k=1}^{n} s_k \otimes s_k^*$ in $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ (respectively the unitaries in $\mathcal{O}_{\rm n} \otimes \mathcal{O}_{\rm n}$). They satisfy $v_n^* v_n = 1 \oplus (\sum_{k=1}^{n} s_k s_k^*)$ and $v_n v_n^* = (\sum_{k=1}^{n} s_k s_k^*) \oplus 1$.

Are the isometries $s_k \otimes s_\ell$ generators of $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$? (Certainly the $s_k \otimes 1$ and $1 \otimes s_\ell$ are generators.)

If one uses K_1-injectivity, then one has to use also *Künneth* theorem to calculate $\mathrm{K}_1(\mathcal{O}_\infty \otimes \mathcal{O}_\infty)$.

Needs K_1-injectivity of all s.p.i. algebras:

Some has to be shown (more practical?) in Section **??** of Chapter 2.

The notion of K_1-injectivity is defined and studied in Section 2 of Chapter 4.

Gives that *all unital *-morphisms $\mathcal{O}_\infty \to E$ are unitarily homotopic if and only if E is K_1-injective*:

Let $S_1, S_2, \ldots \in E$ generators of $\mathcal{O}_\infty$. $s_n := h_1(S_n)$, $t_n := h_2(S_n)$.

$s_1, s_2, \ldots$ homotopic to $s_1, s_1 s_2^n$ and $t_1, t_1 t_2^n$, then to $t_1, t_2, \ldots$.

Gives embeddings $\mathcal{E}_2 \hookrightarrow \mathcal{E}_3 \hookrightarrow E$.

All unital *-monomorphisms $\mathcal{E}_n \hookrightarrow \mathcal{E}_{n+k}$ $(n, k \geq 1)$ are homotopic in $\mathcal{E}_{n+k}$ and are – moreover – homotopic in E – if E is K_1-injective.

If E unital and has properly infinite unit, then this means $\mathcal{U}(E)/\mathcal{U}_0(E) \cong \mathrm{K}_1(E)$ if E is K_1-bijective.

Uses $\mathcal{U}_0(\mathcal{E}_n) = \mathcal{U}(\mathcal{E}_n)$ and that $\mathcal{E}_n \cap \mathbb{K} = \{0\}$ in $\mathcal{E}_{n+1}$, because $\mathbb{K} \subseteq \mathcal{E}_m$ is an essential ideal of $\mathcal{E}_m$ for each $m \geq 2$ and otherwise the unital map $\mathcal{E}_n \to \mathcal{O}_{n+1}$ would define a unital C^*-morphism from $\mathcal{O}_{\rm n}$ into $\mathcal{O}_{n+1}$.

It follows that $\mathcal{D}_\infty$ has a flip automorphism. that is homotopic to id.

SOME COLLECTION OF RESULTS USED IN CHAPTERS 2 and 4 about $\mathcal{O}_{\rm n}$:

Transfer or Repeat ??
Needs to refer to "Property (sq)" !!!
Give the precise reference to definition and Prop. with results in Chp. 4 !!!

?? From Chp. 4:

(a): $\mathcal{E}_n := C^*(s_1, \ldots, s_n;\ s_i^* s_j = \delta_{ij})$, $n = 2, \ldots, \infty$, satisfy Property (sq):

Indeed: Let $w := k_1 k_2 \ldots k_p$, $v := \ell_1 \ell_2 \ldots \ell_q$, "words" of length $p, q \in \mathbb{N}$ with "letters" $k_j, \ell_i \in \{1, \ldots, n\}$ from the "alphabet" $\{1, \ldots, n\}$.

Define $T_0 := 1$ and an isometry by the product

$$T_p(w) := s_{k_1} s_{k_2} \cdot \ldots \cdot s_{k_p} .$$

It is easy to see ??? that the linear span of the element 1 and of the "elementary" products $T_q(w)$, $T_q(w)^*$ and $T_q(w)(T_r(v)^*)$ is dense in the C^*-algebra $\mathcal{E}_n$ (in both of real or complex case), because

(i) the product $T_q(v)^*T_p(w)$ is equal to 1 if $v = w$,
(ii) is equal to zero if there exists $g \leq r := \min(p, q)$ with $k_g \neq \ell_g$,
(iii) is equal to $T_{(p-q)}(u_1)$ if $q < p$ and $w = vu_1$ and
(iv) is equal to $T_{(q-p)}(u_2)^*$ if $p < q$ and $v = wu_2$.

We denote by G one of this elementary products or let $G = 1$.

Perhaps change $L(...)$ to $\lambda(...)$???

A sort of "length" $L(G)$ is defined by $L(1) := 0$, $L(T_q) := q$, $L((T_q)^*) := q$ and $L(T_q \cdot (T_r)^*) := q + r$. Clearly, $s_k^*T_p(w) = \delta_{k,k_1}T_{p-1}(v)$ for all $k = 1, \ldots, n$ with word $v := k_2k_3 \ldots k_n$ for $w = k_1k_2 \ldots k_n$.

Let $G := T_p(w)T_q(v)^*$ with $w = k_1 \ldots k_p$, $v = \ell_1 \ldots \ell_q$ $(p, q \geq 1)$, then, $L(s_k^*Gs_\ell) = L(G) - 2$ if and only if $k_1 = k$ *and* $\ell_1 = \ell$, otherwise only $s_k^*Gs_\ell = 0$ can happen.

In case $G = T_p(w)$, $w = k_1 \ldots k_p$ we get $s_k^*Gs_\ell \neq 0$ if and only if $k = k_1$ and then $L(s_k^*Gs_\ell) = L(G)$, because then $G = T_p(u)$ with $u = k_2 \ldots k_p\ell$. Since $L(G^*) = L(G)$ the same holds for $G = T_p(w)^*$ with the role of k and ℓ interchanged.

Therefore, one can find, for each linear combination $C = \alpha_0 1 + \sum_{j=1}^m \alpha_j G_j$ (with scalars α_j), an index $1 \leq j_0 \leq m$ and suitable $k, \ell \in \{1, \ldots, n\}$ such that $k \neq \ell$, $s_k^*G_{j_0}s_\ell = 0$ for at least one $j_0 \in \{1, \ldots, m\}$ and $L(s_k^*G_js_\ell) \leq L(G_j)$ for $j \neq j_0$. Thus $s_k^*Cs_\ell = \sum_{j\neq 0, j\neq j_0} \alpha_j s_k^*G_js_\ell$ with elementary products $s_k^*G_js_\ell$ (it can be equal to 1).

Iteration – say m-times – of this operation leads to words w and v in the alphabet $\{1, \ldots, n\}$ such that for the isometries $T_m(w)$ and $T_m(v)$ holds $T_m(w)^*T_m(v) = 0$ and $T_m(w)^*CT_m(v) = 0$. It says that $\mathcal{E}_n$ has Property (sq).

It follows that its quotients $\mathcal{O}_n$ also have Property (sq).

END OF (sq) PROOF for $\mathcal{E}_n$!

We can also use also the proof of J. Cuntz in [**172**, thm. 1.9, thm. 3.7] to get $\mathcal{U}_0(\mathcal{O}_\infty) = \mathcal{U}(\mathcal{O}_\infty)$.

On the real version $(\mathcal{O}_2)_\mathbb{R}$ of $\mathcal{O}_2$:

The homotopy id $\sim_h \delta_2$ exists also in the semi-group of unital *-endomorphisms of $(\mathcal{O}_2)_\mathbb{R}$. It follows that $\mathcal{U}((\mathcal{O}_2)_\mathbb{R}) = \mathcal{U}_0((\mathcal{O}_2)_\mathbb{R})$ and that all unital *-endomorphisms φ of $(\mathcal{O}_2)_\mathbb{R}$ are homotopic to the identity isomorphism of $(\mathcal{O}_2)_\mathbb{R}$.

Indeed, φ is uniquely defined by $\varphi(s_j) = u_\varphi s_j$ $(j = 1, 2)$, where

$$u_\varphi := \sum_{j=1}^{2} \varphi(s_j)s_j^* \,,$$

and, for each unitary $u \in (\mathcal{O}_2)_\mathbb{R}$ holds:

$u \oplus_{s_1,s_2} u \sim_h u$ in $\mathcal{U}((\mathcal{O}_2)_\mathbb{R})$. Since $u \oplus v \sim_h v \oplus u$ and $uv \oplus 1$ by Part (v,4) of Lemma 4.2.6 we get that

$$(u \oplus u) \oplus u^* \sim_h u \oplus u^* \;\sim_h\; 1 \,.$$

Since U_d or U_d^* first?

$$U_d^*((u \oplus u) \oplus u^*)U_d = u \oplus (u \oplus u^*)$$

it follows

$$u \oplus 1 \sim_h u \oplus (u \oplus u^*) \sim_h 1$$

in $\mathcal{U}((\mathcal{O}_2)_{\mathbb{R}})$. We get that

$$u \sim_h u \oplus u \sim_h (u \oplus 1) \cdot (1 \oplus u) \sim_h 1$$

for every $u \in \mathcal{U}((\mathcal{O}_2)_{\mathbb{R}})$, i.e., $\mathcal{U}((\mathcal{O}_2)_{\mathbb{R}}) = \mathcal{U}_0((\mathcal{O}_2)_{\mathbb{R}})$.

The self-adjoint unitary $C(2) := \sum_{1 \le i,j \le 2} s_i s_j s_i^* s_j^*$ in the real C^*-algebra

$$A_2 := C^*(s_i s_k (s_j s_\ell)^*; 1 \le i,j,k,\ell \le 2) \cong M_4(\mathbb{R})$$

has determinate $= -1$. But $\delta_2(C(2))$ is unitary equivalent to $C(2)$ in

$$M_8(\mathbb{R}) \cong A_3 := C^*(s_i s_k s_m (s_j s_\ell s_n)^*; 1 \le i,j,k,\ell \le 2)$$

by the unitary $C(3) := \sum_{1 \le i,j,k \le 2} s_i s_j s_k (s_j s_k s_i)^*$. This implies that $C(2) \in \mathcal{U}_0((\mathcal{O}_2)_{\mathbb{R}})$ and then that δ_2 is homotopic to id inside the unital *-endomorphisms of $(\mathcal{O}_2)_{\mathbb{R}}$.

The equality $\mathcal{U}(\mathcal{E}_2) = \mathcal{U}_0(\mathcal{E}_2)$, can be seen (in case of complex E) from the short-exact sequence $\mathbb{K} \to \mathcal{E}_2 \to \mathcal{O}_2$, and from the equalities $\mathcal{U}_0(\mathbb{K} + \mathbb{C} \cdot 1) = \mathcal{U}(\mathbb{K} + \mathbb{C} \cdot 1)$ and $\mathcal{U}_0(\mathcal{O}_2) = \mathcal{U}(\mathcal{O}_2)$.

Transfer or Repeat? FROM Chp. 2:

In proof of property (sq) for $\mathcal{E}_n$ in Chp. 4 is already some said about the algebraic version of $\mathcal{E}_n$.

The C^*-algebra $\mathcal{E}_n$ is the universal C^*-algebra generated by n isometries with orthogonal ranges, i.e., $\mathcal{E}_n := C^*(s_1, \ldots, s_n\, ;\ s_k^* s_\ell = \delta_{k\ell} 1)$. The C^*-algebra $\mathcal{O}_\infty$ is the inductive limit of the sequence of the natural unital C^*-morphisms $h_n \colon \mathcal{E}_n \to \mathcal{E}_{n+1}$ defined by $h_n(s_k) := s_k$ for $k = 1, \ldots, n$.

The *-monomorphisms h_n satisfy the criteria 2.2.5(iv):

The closed ideal $J(p_n)$ of $\mathcal{E}_n$ generated by $p_n := 1 - s_1 s_1^* - \cdots - s_n s_n^*$ is isomorphic to $\mathbb{K}(\ell_2(\mathbb{N}))$ and p_n is a minimal projection of $J(p_n) \cong \mathbb{K}$, because $p_n T_q(w_1) T_r(w_2)^* p_n = 0$ for w_1, w_2 words in the "alphabet" $\{1, \ldots, n\}$ with at least one of w_1 and w_2 is not the "empty" word (i.e., the $q + r > 0$ for the lengths q of w_1 and r of w_2).

It implies that $p_n \cdot \mathcal{E}_n \cdot p_n = \mathbb{C} \cdot p_n$.

The ideal $J(p_n)$ is an *essential* ideal of $\mathcal{E}_n$, because $\mathcal{O}_n \cong \mathcal{E}_n / J(p_n)$ is simple.

(Refer here to the place where simplicity of $\mathcal{O}_n$ is proven!)

The simplicity of $\mathcal{O}_n$ follows from the fact that M_{n^∞} is the fixed point algebra of the "gauge" circle action on $\mathcal{O}_n$ and that (!!!!) the conditional expectation from $\mathcal{O}_n$ onto M_{n^∞} defined by this action is an approximately inner u.c.p. map.

(Where is the appox innerness is shown here?)

(The point is, that one proves first – e.g. Cuntz – that the conditional expectation $P_n\colon \mathcal{O}_{\mathrm{n}} \to M_{n^\infty}$ is element-wise approximately inner by taking approximation be elementary .)

Ref's to simplicity (! where shown?) of $\mathcal{O}_{\mathrm{n}}$??:
$J(p_n) \cong \mathbb{K}$
$p_n \mathcal{E}_n p_n = \mathbb{C} p_n$
$\mathcal{O}_{\mathrm{n}}$ simple, implies that
$J(p_n)$ is essential ideal of $\mathcal{E}_n$.

(But the argument for "$J(p_n)$ essential" needs only that $\mathcal{O}_{\mathrm{n}}$ is G-simple for $G := S^1$ and the "gauge" circle action.)

Hence, $a \in \mathcal{E}_n \mapsto H(a) \in \mathcal{M}(J(p_n))$ with $H(a)b = ab$ for $b \in J(p_n)$ is a faithful C^*-morphism, and for $a \in (\mathcal{E}_n)_+$ with $\|a\| = 1$ there exists a partial isometry $z \in J(p_n) \cong \mathbb{K}$ with $z^*z = p_n$ and $\| p_n - z^*az \| < \varepsilon$.

Define $c := h_n(z)s_{n+1}h_n(b^{1/2})$. If we use that $h_n(p_n) = p_{n+1} + s_{n+1}s_{n+1}^* \in \mathcal{E}_{n+1}$ and $p_{n+1}s_{n+1} = 0$, then we get

$$\|c^* h_n(a)c - h_n(b)\| < \varepsilon\,.$$

New ?? concepts for Appendix of book:

Try to find a direct simple proof that the gauge action of $\mathcal{O}_{\mathrm{n}}$ is approximately inner ...???

NOT found yet !!!

Definitions of $\mathcal{O}_{\mathrm{n}}$, $\mathcal{E}_n$, $\mathcal{O}_\infty$.

Its K_*-groups:
$\mathrm{K}_*(\mathcal{E}_n) = (\mathbb{Z}, 0)$, $\mathrm{K}_*(\mathcal{O}_{\mathrm{n}}) = (\mathbb{Z}/(n-1)\mathbb{Z}, 0)$, $n = 2, 3, \ldots$, i.e., cyclic with generators [1] $\mathrm{K}_*(\mathcal{O}_\infty) = (\mathbb{Z}, 0)$.

$\mathcal{O}_\infty$ is the inductive limit of the inclusion maps $\mathcal{E}_n \to \mathcal{E}_{n+1}$, $n = 2, 3, \ldots$. It is stationary on [1]. Shows $(\mathbb{Z}, 1) \cong (\mathrm{K}_0(\mathcal{O}_\infty), [1])$.

It was shown in Chp.2, sec.2.2:
indlim $\mathcal{E}_n \to \mathcal{E}_{n+1}$ is simple and p.i.
But it did use that the ideal $I(p)$ is *essential* in $\mathcal{E}_n$,
with $p = p_0 = e_n$ defined below/above several times.
Requires that $I(p)$ is essential for $\mathcal{E}_n$.

$\mathcal{O}_{\mathrm{n}} := \mathcal{E}_n / I(p)$, where $I(p)$ is the closed ideal of $\mathcal{E}_n$ generated by the projection $p := 1 - s_1 s_1^* - \ldots - s_n s_n^*$.

Here $\mathcal{E}_n := C^*(s_1, \ldots, s_n\,;\, s_i^* s_j = \delta_{ij}1) = C^*(A_n)$ is the C^*-completion of A_n, defined as the universal algebraic *-algebra generated by $s_1, \ldots, s_n$ subject to the relations $s_i^* s_j = \delta_{i,j}1$.

The existence and well-defined-ness of $C^*(A_n)$ follows from an old observation of M.G. Krein in 1949 [**493**] mentioned in his study of Krein bloc-algebras:

Let A a unital *-ring with relations that allow to prove that for (some) given generators $a_1, a_2, \ldots ; a_1^*, a_2^*, \ldots$ and relations $R_1, R_2, \ldots$ of A there exists $b_{1,n}, \ldots, b_{k,n} \in A$ and $\gamma_n \geq 0$ (depending from a_n) such that in the unitization $\widetilde{A} := A + \mathbb{C} \cdot 1$ of A holds

$$a_n^* a_n + b_{1,n}^* b_{1,n} + \cdots + b_{k,n}^* b_{k,n} = \gamma_n 1 .$$

Then there is a well-defined maximal C^*-seminorm on A. We denote the corresponding C^*-algebra then by $C^*(a_1, a_2, \ldots ; R_1, R_2, \ldots)$. The natural C^*-morphism from the algebraic A into its C^*-algebraic version $C^*(a_1, a_2, \ldots ; R_1, R_2, \ldots)$ is not necessarily faithful.

(In non-unital case one can adjoin a unit element e and require that ??? for desired "contractions" $w \in A$ there exist $v \in A_e$ in an extended version $A_e \supseteq A$ with $w^* w + v^2 = 1$ or $w^* w + v^* v = 1$.)

We show that our A_n is the linear span of the expressions $W_\alpha W_\beta^*$ for α and β words in from the "alphabet" $\{1, 2, \ldots, n\}$, where $W_\alpha := 1$ for the empty word (of length 0) or $W_\alpha = s_{k_1} s_{k_2} \cdot \ldots \cdot s_{k_\ell}$ for the word $\alpha = k_1 k_2 \ldots k_\ell$ of length $\ell = \ell(\alpha)$ with $k_1, k_2, \ldots, k_\ell \in \{1, \ldots, n\}$.

Notice that W_α and W_β satisfy $W_\alpha^* W_\beta = \delta_{\alpha,\beta} 1$ if α and β have the same length. If $\ell(\alpha) < \ell(\beta)$, then we have to write β as $\beta = \beta' \gamma$ with $\ell(\beta') = \ell(\alpha)$ and get $W_\alpha^* W_\beta = \delta_{\alpha,\beta'} \cdot W_\gamma$. In this way only terms like 1 and $W_\alpha \cdot W_\beta^*$ appear in the sub-semigroup of A_n generated by $1, s_1, \ldots, s_n, s_1^*, \ldots, s_n^*$. Thus A_n is the linear span of this terms.

If we let $p_0 := 1 - s_1 s_1^* - \cdots - s_n s_n^*$ then $p_0 W_\alpha = 0$ if α has length ≥ 1, i.e., if $W_\alpha \neq 1$, because $p_0 s_k = 0$ for all $k \in \{1, \ldots, n\}$. It follows that $p_0 A_n p_0 = \mathbb{C} \cdot p_0$ and $p_0 \mathcal{E}_n p_0 = \mathbb{C} \cdot p_0$.

It follows that $W(\alpha) p_0 W(\alpha)^* W(\beta) p_0 W(\beta)^* = 0$, i.e., that the ideal $I(p_0)$ of A_n generated by $\{p_0\}$ contains a countable number of mutually orthogonal projections $p_\alpha := W(\alpha) p_0 W(\alpha)^*$ with 1-dimensional $p_\alpha A_n p_\beta$.

Clearly $p_0 = p_0^*$, and, for $k \neq \ell$ and $i = j$, or $i \neq j$,

$$(s_i)^k p_0 ((s_i)^k)^* (s_j)^\ell p_0 ((s_j)^\ell)^* = 0 .$$

It implies that the closed ideal $I(p_0)$ of $\mathcal{E}_n$ must be isomorphic to $\mathbb{K}$.

There is a circle action σ_z on A_n (by universality of A_n) given on generators by $\sigma_z(s_k) := z s_k$ for $z \in S^1 := \{z \in \mathbb{C} ;\ |z| = 1\}$. The action fixes p_0 and is compatible with the relations of A_n, i.e., the $t_k := \sigma_z(s_k)$ define the same relations $t_j^* t_k = \delta_{jk} 1$ and $p_0 := 1 - t_1^* t_1 - \cdots - t_n^* t_n$ as the $s_1, \ldots, s_n$. Moreover $\sigma_z(a^*) = \sigma_z(a)^*$, because this happens on generators.

By universality of A_n it follows that σ extends to a circle action on $\mathcal{E}_n$ that fixes p_0. In particular, it takes $I(p_0) \cong \mathbb{K}$ invariant.

(In fact one has moreover a natural $\mathcal{U}(M_n)$ action on $\mathcal{E}_n$ that fixes p_0.)

Thus, σ extends naturally to a circle action σ on $\mathcal{E}_n$, and then to an action on $\mathcal{O}_n$ respectively on $\mathcal{O}_\infty$, because they are the completions with the maximal possible C^*-norms on it.

Notation below differs from that of above:

The fix-point-algebra of σ on $\mathcal{O}_n$ is generated by the elements $W_\alpha W_\beta^*$ with W_α and W_β words in "letters" $s_1, \ldots, s_n$ of the same length $\ell(\alpha) = \ell(\beta)$. We have $\sum s_n s_n^* = 1$ in $\mathcal{O}_n$. It follows that $W_\alpha W_\beta^* = \sum_{j=1}^n (W_\alpha s_j)(W_\beta s_j)^*$ is contained in the span of the $W_\gamma W_\delta$ with $1 + \ell(\alpha) = \ell(\gamma) = \ell(\delta) = 1 + \ell(\beta)$.

It is easy to see that this is naturally isomorphic to the inductive limit of unital embeddings

$$M_{n^k} \to M_{n^k} \otimes 1_n \subseteq M_{n^{k+1}} .$$

Thus ${\mathcal{O}_n}^\sigma \cong M_{n^\infty}$ and

$$P_n(a) := \int_{z \in S^1} \sigma_z(a) dz$$

is a conditional expectation from $\mathcal{O}_n$ onto $M_{n^\infty} \subseteq \mathcal{O}_n$.

It turns out that P_n is 1-step approximately inner in $\mathcal{O}_n$ (cf. [**209**]), but this has been shown also by J. Cuntz itself.

The corresponding contractions T_m with $P_n(a) := \lim_m T_m^* a T_m$ are given by

$$T_m := \sum_{\alpha,\beta} V_\alpha W_\beta (V_\alpha)^* (W_\beta)^* = \sum_\beta \delta_n^m (W_\beta)(W_\beta)^* = \sum_\alpha V_\alpha \delta_n^m (V_\beta) ,$$

where α and β run through $\{1, \ldots, n\}^m$.

If leads to the observation that $\mathcal{O}_n$ is simple, p.i. (hence s.p.i.) and nuclear:

The nuclearity follows from the general observation that compares maximal and minimal tensor products and shows that a C^*-algebra with a continuous action of a compact group is nuclear if its fix-point-algebra is nuclear.

Where explained in detail? Refer to this.

Since $P_n(\mathcal{O}_n) \cong M_{n^\infty}$ is simple and P_n is faithful on the positive cone of $\mathcal{O}_n$, it follows that "one-step approximate innerness" of P_n would show also the simplicity of $\mathcal{O}_n$, cf. [**169**], [**172**] or [**209**].

Nothing found in the cited papers?!

The argument of J. Cuntz goes a follows:

(1a.) The conditional expectation $P_n \colon \mathcal{O}_n \to M_{n^\infty} \subseteq \mathcal{O}_n$ is faithful, because it is the integral of a circle action.

(1a is My or his argument?)

(1b.) The nonzero projections in $M_{n^\infty} \subseteq \mathcal{O}_n$ are all infinite in $\mathcal{O}_n$.

In particular Why it should be a special case of what???, there exists for each nonzero projection $p \in \mathcal{O}_n$ an isometry $s \in \mathcal{O}_n$ with $s^* p s = 1$.

(1c.) It implies that each nonzero positive element $a \in M_{n^\infty}$ is infinite in $\mathcal{O}_n$, and there exists $d \in \mathcal{O}_n$ with $d^* a d = (\|a\|/2) \cdot 1$.

(2a.) Let $b \in \mathcal{O}_n$ an element in $W_?$????? then there exists $d \in W_{??}$ such that $d^*b^*bd = P_n(b^*b)$ ...??

(Range of such d must almost commute with given finite subset in the fixed point algebra of the circle action but must have orthogonal ranges if composed with a finite subset of non-fixed points ...?)

(2b.) Let $a \in \mathcal{O}_n$ positive with $\|a\| = 1$ and $\varepsilon > 0$. Then there exists an element $e \in A$ with $\|e^*ae - P_n(a)\| \le \varepsilon$.

WHERE THIS IS SHOWN?

(2a.)

If we can show that $P_n(a)$ is contained in $\{d^*ad\,;\; d \in \mathcal{O}_n\}$ for each $a \in \mathcal{O}_n$ with $a \ge 0$.

Thus, if $0 \ne a \ge 0$ and P_n is faithful on $(\mathcal{O}_n)_+$ then $\mathcal{O}_n$ is simple.

Let $a \in \mathcal{O}_n$ positive and $\varepsilon > 0$, then find $n \in \mathbb{N}$ and $b^* = b \in W_n$ with $\|b^2 - a\| < \varepsilon$. Leads to $\|P_n(a) - P_n(b^2)\| < \varepsilon$.

Then find element (contraction?) $D \in W_m$ with $\|D^*b^2D - P_n(b_2)\| < \varepsilon$. ...

If one can show that each projection in $P_n(\mathcal{O}_n)$ dominates the range of an isometry in $\mathcal{O}_n$ then the one-step approximation of P_n shows moreover that $\mathcal{O}_n$ must be purely infinite.

That is, each minimal projection of the span of $\{W_\alpha W_\beta^*\,;\; \ell(\alpha) = \ell(\beta) = k\} = M_{n^k}$ is the range of an isometry in $\mathcal{O}_n$, because all non-zero minimal projections in M_{n^k} are equivalent.

Since $I(p) \cong \mathbb{K}$ (where shown? ref!) and $\mathcal{O}_n := \mathcal{E}_n/I(p)$ is nuclear, it follows that the algebras $\mathcal{E}_n$ are nuclear. This passes to its inductive limit $\mathcal{O}_\infty$, i.e., $\mathcal{O}_\infty$ is nuclear.

The defining element $u \in \mathcal{O}_2$ with $us_j = \delta_2(s_j)$ of the endomorphism $\delta_2 \colon \mathcal{O}_2 \to \mathcal{O}_2$ given by $a \in \mathcal{O}_2 \mapsto \delta_2(a) := s_1as_1^* + s_2as_2^*$ can be written down explicitly as

$$u := s_1s_1s_1^*s_1^* + s_2s_1s_2^*s_1^* + s_1s_2s_1^*s_2^* + s_2s_2s_2^*s_2^*\,.$$

It is a symmetry in the real C^*-algebra $M_4(\mathbb{R}) \cong A_2 := C^*_{\mathbb{R}}(s_is_ks_\ell^*s_j^*\,;\; i,j,k,\ell = 1,2)$ with determinate $= -1$.

Since $C^*_{\mathbb{R}}(s_is_ks_\ell^*s_j^*\,;\; i,j,k,\ell = 1,2)$ is a unital *-subalgebra of $M_8(\mathbb{R}) \cong A_3 := C^*_{\mathbb{R}}(s_is_ks_ms_n^*s_\ell^*s_j^*\,;\; i,j = 1,2)$, and $O(4) \otimes 1_2 \subseteq SO(8)$, there is a continuous path inside $SO(8)$ that connects $u \otimes 1_2$ in $SO(8)$ with 1. This happens then also in $\mathcal{O}_2$.

An explicit path can be found with help of suitable elementary transformations:

Bit more general for $n \in \mathbb{N}$, $n > 1$, we consider $\mathcal{O}_n$ and $\mathcal{E}_n$. Let $W(k) := \{1, \ldots, n\}^k$. If $w = (j_1, \ldots, j_k) \in W(k)$ is a "word" of length k, let $\lambda w := (j_k, j_1, \ldots, j_{k-1})$ the cyclic change of the word $w \in W(k)$ and $\overline{w} := (j_k, j_{k-1}, \ldots, j_2, j_1)$ the reversed word in $W(k)$. They define isometries $S(w) := s_{j_1} \cdot \ldots \cdot s_{j_k}$. Notice that the elements $Z(k) := \sum_{w \in W(k)} Z(\lambda w)Z(w)^*$

and $U(k) := \sum_{w\in W(k)} Z(\overline{w})Z(w)^*$ of $\mathcal{O}_{\rm n}$ are orthogonal operators in $A_k :=$ span$(\{S(w_1)S(w_2)^*\,;\ w_1, w_2 \in W_k\}) \cong M_{n^k}$ respectively a symmetry (i.e., $U(k)^* = U(k)$, $U(k)^2 = 1$).

Notice that $A_k \subseteq A_{k+1}$ because

$$S(w_1)S(w_2)^* = \sum_{\ell=1}^{n}(S(w_1)s_\ell)(S(w_2)s_\ell)^*$$

and $S(w)s_\ell = S((w,\ell)$ where $(w,\ell) := (j_1,\ldots,j_k,\ell) \in W(k+1)$.

$$Z(k+1)^*S(w_1)S(w_2)^*Z(k+1) \;=\; \delta_n(S(w_1)S(w_2)^*) \quad \text{for } w_1, w_2 \in W(k)\,.$$

This follows from $Z(k+1)^*S(w_1)s_\ell s_\ell^* S(w_2)^*Z(k+1) = s_\ell S(w_1)S(w_2)^*s_\ell^*$ or $s_\ell^* S(w_2)^*Z(k+1) = S(w_2)^*s_\ell^*$.

In case $n=2$ and $k=2$, one gets

$$U(2) = Z(2) = s_1^2(s_1^2)^* + s_2^2(s_2^2)^* + s_1s_2(s_2s_1)^* + s_2s_1(s_1s_2)^*$$

Moreover $U(2)s_\ell = \delta_2(s_\ell)$ for $\ell = 1,2$.

More generally, for $n \geq 2$, $U(2)s_\ell = \delta_n(s_\ell)$ for $\ell = 1,\ldots,n$. It follows that δ_n is homotopic to id on $\mathcal{O}_{\rm n}$, if and only if, $U(2)$ is homotopic to 1 in $\mathcal{U}(\mathcal{O}_{\rm n})$ (respectively in $O(\mathcal{O}_{\rm n})$ for the real version of $\mathcal{O}_{\rm n}$).

In case $n=2$ we get with $V := (s_1s_2^* - s_2s_1^*)$ that

$$Z(3)^*U(2)Z(3) \;=\; \delta_2(U(2)) \;=\; (s_1U(2)s_1^* + s_2s_2^*)V^*(s_1U(2)s_1^* + s_2s_2^*)V\,.$$

Notice that $\exp((\pi/2)V) = V + (1 - V^4)$ if $V^* = -V = V^3$. In particular, $V = \exp((\pi/2)V)$ if $V^* = -V$ and $V^4 = 1$. Let $C = [\alpha_{jk}] \in SO_2$ the $90°$ rotation matrix, i.e., $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{21} = -\alpha_{12} = 1$ and recall $[s_1,s_2]^* = [s_1^*, s_2^*]^\top \in M_{2,1}(\mathcal{O}_2) \subseteq M_2(\mathcal{O}_2)$. We define a unitary in (the real version of) $\mathcal{O}_2$ by

$$V_\theta := \exp(\theta(\pi/2)V) = [s_1,s_2]\exp(\theta\pi/2C)[s_1,s_2]^* \in A_3$$

Since $V^* = -V$, we get that V_θ is in $SO(A_3) \cong SO_3$ for $\theta \in [0,1]$.

It follows that

$$U_\theta := Z(3)(U(2)\oplus 1)V(\theta)^*(U(2)\oplus 1)V(\theta)Z(3)^*$$

is a path in $SO(A_3)$ that connects the unitary $U(2) = U_1 \in O(A_2)$ with $1 = U_0$.

It follows that the unital *-endomorphism δ_2 of $\mathcal{O}_2$ is homotopic inside the unital endomorphisms $\mathrm{End}_1(\mathcal{O}_2)$ of $\mathcal{O}_2$ to the identity isomorphism.

Similar considerations show that $\delta_n \in \mathrm{End}_1(\mathcal{O}_{\rm n})$ is homotopic inside the unital endomorphisms $\mathrm{End}_1(\mathcal{O}_{\rm n})$ of $\mathcal{O}_{\rm n}$ to the identity isomorphism id of $\mathcal{O}_{\rm n}$.

If one can find a unitary U in the ultra-power $(\mathcal{O}_2)_\omega$ of $\mathcal{O}_2$, or in $\mathrm{Q}(\mathbb{R}_+,\mathcal{O}_2) := \mathrm{C}_b(\mathbb{R}_+,\mathcal{O}_2)/\,\mathrm{C}_0(\mathbb{R}_+,\mathcal{O}_2)$, or in $(\mathcal{O}_2)_\infty := \ell_\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$ with $U\delta_2(x)U^* = x$ for $x \in \mathcal{O}_2$ then one gets for $t_j := Us_j$ that t_1, t_2 commute element-wise with $\mathcal{O}_2$, i.e., $\mathcal{O}_2 \cong C^*(t_1,t_2)$ is a copy of $\mathcal{O}_2$ that is unital contained in $\mathcal{O}_2{}' \cap (\mathcal{O}_2)_\omega$ (respectively in ${\mathcal{O}_2}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ or in ${\mathcal{O}_2}' \cap (\mathcal{O}_2)_\infty)$. Then, by repeat of this construction, we find a unital copy of $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots \subseteq \mathcal{O}_{2\omega}$ that commutes with $\mathcal{O}_2$ element-wise in this asymptotic algebras. It allows to conclude that $\mathcal{O}_2 \cong \mathcal{D}_2 := \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$.

`Next is contained in proof prop. 2.2.5?`

The following argument for pure infiniteness of $\mathcal{O}_\infty$ uses (and needs) that $\mathcal{E}_n \to \mathcal{O}_\infty$ is injective. (Somewhere above proven !)
and that $e \in \mathcal{E}_n$ satisfies $s_{n+1}s_{n+1}^* \leq e$.

Claim: The algebra $\mathcal{O}_\infty$ is purely infinite.

Idea of arguments:

"Have somewhere above shown" that the natural map $\eta \colon \mathcal{E}_n \to \mathcal{E}_{n+1}$ is injective and that $e_n := 1 - \sum_{k=1}^n s_k s_k^*$ satisfies $e_n \mathcal{E}_n e_n = \mathbb{C} \cdot e_n$.

(E.g. via looking to $\mathcal{E}_{n-1} \to \mathcal{E}_n \to \mathcal{O}_\mathrm{n}$ and extending a faithful unital representation $\rho \colon \mathcal{E}_{n-1} \to \mathcal{L}(\mathcal{H})$ with separable $\mathcal{H}$ to $\rho \otimes \mathrm{id} \colon \mathcal{E}_{n-1} \otimes \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ and finding an isometry $T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ with

$$TT^* = 1 \otimes 1 - \rho(s_1 s_1^* + \ldots + s_{n-1}s_{n-1}^*) \otimes 1 \,.$$

– Here we denote $\mathrm{id}_\mathcal{H}$ by 1.)

It follows that $I(e)$ embeds injective into $\mathcal{E}_{n+1}$ and $s_{n+1}^* \eta(e) s_{n+1} = 1$. If $c \in \mathcal{E}_n$ is positive and $g \in I(e)$ with $\gamma \cdot e = g^* c g \neq 0$, $g^* g = e$, then $s_{n+1}^* g^* c g s_{n+1} = \gamma 1$. In particular, c is full and properly infinite in $\mathcal{E}_{n+1}$.

Let $a \in \mathcal{O}_\infty$ positive with $\|a\| = 1$. Then there exists $n \in \mathbb{N}$ and a positive contraction $b \in \mathcal{E}_n$ with $\|a - b\| < 1/8$. Thus, $\|b\| \geq 7/8$ and $\|(b - 1/8)_+\| \geq 3/4$. There is a contraction $d \in \mathcal{O}_\infty$ with $d^* a d = (b - 1/8)_+$, cf. Lemma 2.1.9. It follows that $d^* a d \in \mathcal{E}_n$ and $\|d^* a d\| \geq 3/4$.

Rest of the proof: $e = 1 - \sum_k s_k s_k^*$ satisfies $I(e) \cong \mathbb{K}$ and $e \in I(e)$ minimal and $e \in \mathcal{E}_{n+1} \subseteq \mathcal{O}_\infty$

`Need for final conclusion that`
`(o)` $I(e) \cong \mathbb{K}$,
`(i)` e `is a properly infinite full projection in` $\mathcal{E}_{n+1} \subseteq \mathcal{O}_\infty$ `and`
`(i)` $I(e)$ `essential for` $\mathcal{E}_n$.
`To be shown:`

The closed ideal $I(e) \cong \mathbb{K}$ of $\mathcal{E}_n$ is an *essential* of $\mathcal{E}_n$.

Steps:

(1.) $I(e) \cong \mathbb{K}$, because:
It is a closed ideal of $\mathcal{E}_n$ generated by the projection $e := 1 - (s_1 s_1^* + \ldots + s_n s_n^*)$. It satisfies $e s_j = 0$ for $j = 1, \ldots, n$. Thus, $e \mathcal{E}_n e = \mathbb{C} \cdot e$, i.e., the projection e is a minimal idempotent in $\mathcal{E}_n$.
(2.) $I(e)$ is invariant under the canonical circle action, because e is fixed by the circle action
(3.) The set of positive elements orthogonal to $I(e)$ is (by (2.)) invariant under the

circle action and is a closed ideal J of $\mathcal{E}_n$ orthogonal to $I(e)$.
(4.) The image $\pi_{I(e)}(J)$ of J in $\mathcal{O}_n := \mathcal{E}_n/I(e)$ is a closed ideal of $\mathcal{O}_n$ that is isomorphic to J and is invariant under the circle action on $\mathcal{O}_n$ and ...
(5.) $\pi_{I(e)}(J)$ does not contain the class $1 + I(e) \in \mathcal{E}_n/I(e)$:
Otherwise it means that $(1 - Y) \cdot \mathcal{E}_n \cdot e = \{0\}$ for some $Y \in I(e)$ with $0 \leq Y \leq 1$. It implies that $Ya = a$ for all $a \in I(e)$ and that $Y - Y^2 = 0$. Thus, Y is a unit element of $I(e)$, but $I(e)$ is not unital.
(6.) The circle-action invariant ideal $\pi_{I(e)}(J)$ has – if it would be non-zero – a non-zero intersection with $P(\mathcal{O}_n)$ that is a closed ideal of $P(\mathcal{O}_n)$, that can not contain 1 by (5.).

Since $P(\mathcal{O}_n) \cong M_{n^\infty}$ it follows that $J \cong \pi_{I(e)}(J) = 0$.

This finishes the proof that $I(e)$ is an essential ideal of $\mathcal{E}_n$ (without showing before that $\mathcal{O}_n$ is simple).

`Follows also immediately from the` *`simplicity`* `of` $\mathcal{O}_n$ (– if this has been shown before together with the property of $e := 1 - s_1 s_1^* - \cdots - s_n s_n^*$ that $e \cdot \mathcal{E}_n \cdot e = \mathbb{C}e$ – but we give an other proof that does not use the simplicity of $\mathcal{O}_n$):

$eW(\alpha) = 0$ for $\ell(\alpha) > 0$, where $W(\alpha) :=$????.

Recall: $a \in \mathcal{O}_\infty$ with $\|a\| = 1$ there exists $n \in \mathbb{N}$ and a positive contraction $b \in \mathcal{E}_n$ with $\|a - b\| < 1/8$. Thus, $\|b\| \geq 7/8$ and $\|(b - 1/8)_+\| \geq 3/4$. There is a contraction $d \in \mathcal{O}_\infty$ with $d^*ad = (b - 1/8)_+$, cf. Lemma 2.1.9.

We find a minimal projection q in $I(p)$ with

$$\|q(b - 1/8)_+ q\| \geq \|(b - 1/8)_+\| - 1/4 \geq 1/2 .$$

Then $qd^*adq = \alpha q$ for some $\alpha \geq 1/2$. There is a contraction $z \in \mathcal{E}_{n+1}$ with $z^*qz = 1$, because there is a partial isometry $v \in I(p) \cong \mathbb{K}$ with $v^*qv = p$ and $s_{n+1}^* p s_{n+1} = 1$. Thus $f := \alpha^{-1/2} dqz$ satisfies $f^*af = 1$.

It follows that $\mathcal{O}_\infty$ *is simple and purely infinite* (if we have shown before the *simplicity* of $\mathcal{O}_n$ or only that $I(e)$ is an essential ideal of $\mathcal{E}_n$).

1.2. Remarks on Real $\mathcal{O}_n$ and $\mathcal{O}_\infty$ (2). The defining unitary $u \in \mathcal{O}_2$ with $us_j = \delta_2(s_j)$ for the endomorphism

$$\delta_2 \colon \mathcal{O}_2 \ni a \mapsto \delta_2(a) := s_1 a s_1^* + s_2 a s_2^* \in \mathcal{O}_2$$

can be written down explicitly as

$$u := s_1 s_1 s_1^* s_1^* + s_2 s_1 s_2^* s_1^* + s_1 s_2 s_1^* s_2^* + s_2 s_2 s_2^* s_2^* .$$

It is a symmetry with determinate $= -1$ in the real C^*-algebra $M_4(\mathbb{R}) \cong A_2 := C^*_{\mathbb{R}}(s_i s_k s_\ell^* s_j^* \,;\ i,j,k,\ell = 1,2)$.

But since $C^*_{\mathbb{R}}(s_i s_k s_\ell^* s_j^* \,;\ i,j,k,\ell = 1,2)$ is a unital *-subalgebra of

`Check formula for` A_3`:`

$$M_8(\mathbb{R}) \cong A_3 := \ C^*_{\mathbb{R}}(s_i s_k s_m s_n^* s_\ell^* s_j^* \,;\ i,j,k,\ell,m \in \{1,2\})$$

, and the natural embedding $A_2 \subset A_3$, $O(4) \otimes 1_2 \subseteq SO(8)$, there is a continuous path inside $SO(8)$ that connects $u \otimes 1_2$ in $SO(8)$ with 1. This happens then also in the real version of $\mathcal{O}_2$.

An explicit path can be found using an isomorphism from $M_8(\mathbb{R})$ onto A_3.

Bit more general (but only in complex case) for $n \in \mathbb{N}$, $n > 1$, we consider $\mathcal{O}_\mathrm{n}$ and $\mathcal{E}_n$. Let $W(k) := \{1, \dots, n\}^k$. If $w = (j_1, \dots, j_k) \in W(k)$ is a "word" of length k, let $\lambda w := (j_k, j_1 \dots, j_{k-1})$ the cyclic change of the word $w \in W(k)$ and $\overline{w} := (j_k, j_{k-1}, \dots, j_2, j_1)$ the reversed word in $W(k)$. The define isometries $S(w) := s_{j_1} \cdot \ldots \cdot s_{j_k}$. Notice that the elements $Z(k) := \sum_{w \in W(k)} Z(\lambda w) Z(w)^*$ and $U(k) := \sum_{w \in W(k)} Z(\overline{w}) Z(w)^*$ of $\mathcal{O}_\mathrm{n}$ are orthogonal operators in $A_k := \mathrm{span}(\{S(w_1)S(w_2)^*\,;\ w_1, w_2 \in W_k\}) \cong M_{n^k}$ respectively a symmetry (i.e., $U(k)^* = U(k)$, $U(k)^2 = 1$).

Notice that $A_k \subseteq A_{k+1}$ because

$$S(w_1)S(w_2)^* = \sum_{\ell=1}^{n} (S(w_1)s_\ell)(S(w_2)s_\ell)^*$$

and $S(w)s_\ell = S((w, \ell)$ where $(w, \ell) := (j_1, \dots, j_k, \ell) \in W(k+1)$.

$$Z(k+1)^* S(w_1) S(w_2)^* Z(k+1) \;=\; \delta_n(S(w_1)S(w_2)^*) \quad \text{for } w_1, w_2 \in W(k)\ .$$

This follows from $Z(k+1)^* S(w_1) s_\ell s_\ell^* S(w_2)^* Z(k+1) = s_\ell S(w_1) S(w_2)^* s_\ell^*$ or $s_\ell^* S(w_2)^* Z(k+1) = S(w_2)^* s_\ell^*$.

In case $n = 2$ and $k = 2$, one gets

$$U(2) = Z(2) = s_1^2 (s_1^2)^* + s_2^2 (s_2^2)^* + s_1 s_2 (s_2 s_1)^* + s_2 s_1 (s_1 s_2)^*$$

Moreover $U(2)s_\ell = \delta_2(s_\ell)$ for $\ell = 1, 2$.

More generally, for $n \geq 2$, $U(2)s_\ell = \delta_n(s_\ell)$ for $\ell = 1, \dots, n$. It follows that δ_n is homotopic to id on $\mathcal{O}_\mathrm{n}$, if and only if, $U(2)$ is homotopic to 1 in $\mathcal{U}(\mathcal{O}_\mathrm{n})$ (respectively in $O(\mathcal{O}_\mathrm{n})$ for the real version of $\mathcal{O}_\mathrm{n}$).

In case $n = 2$ we get with $V := (s_1 s_2^* - s_2 s_1^*)$

$$Z(3)^* U(2) Z(3) = \delta_2(U(2)) = (s_1 U(2) s_1^* + s_2 s_2^*) V^* (s_1 U(2) s_1^* + s_2 s_2^*) V\ .$$

Notice that $\exp((\pi/2)V) = V + (1 - V^4)$ if $V^* = -V = V^3$. In particular, $V = \exp((\pi/2)V)$ if $V^* = -V$ and $V^4 = 1$. Let $C = [\alpha_{jk}] \in SO_2$ the 90° rotation matrix, i.e., $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{21} = -\alpha_{12} = 1$ and recall $[s_1, s_2]^* = [s_1^*, s_2^*]^\top \in M_{2,1}(\mathcal{O}_2) \subseteq M_2(\mathcal{O}_2)$. We define a unitary in (the real version of) $\mathcal{O}_2$ by

$$V_\theta := \exp(\theta(\pi/2)V) = [s_1, s_2] \exp(\theta\pi/2C) [s_1, s_2]^* \in A_3$$

Since $V^* = -V$, we get that V_θ is in $SO(A_3) \cong SO_3$ for $\theta \in [0, 1]$.

It follows that

$$U_\theta := Z(3)(U(2) \oplus 1)V(\theta)^*(U(2) \oplus 1)V(\theta)Z(3)^*$$

is a path in $SO(A_3)$ that connects the unitary $U(2) = U_1 \in O(A_2)$ with $1 = U_0$.

It follows that the unital *-endomorphism δ_2 of $\mathcal{O}_2$ is homotopic inside the endomorphisms of $\mathcal{O}_2$ to the identity isomorphism.

This is an important information, because it allows to show that with an approximation argument one can apply the following observation:

If one can find a unitary U in the ultra-power $(\mathcal{O}_2)_\omega$ of $\mathcal{O}_2$, or in $\mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2) := \mathrm{C}_b(\mathbb{R}_+, \mathcal{O}_2)/\,\mathrm{C}_0(\mathbb{R}_+, \mathcal{O}_2)$, or in $(\mathcal{O}_2)_\infty := \ell_\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$ with $U\delta_2(x)U^* = x$ for $x \in \mathcal{O}_2$ then one gets for $t_j := Us_j$ that t_1, t_2 commute element-wise with $\mathcal{O}_2$ in the asymptotic algebra, i.e., $\mathcal{O}_2 \cong C^*(t_1, t_2)$ is unital contained in $\mathcal{O}_2{}' \cap (\mathcal{O}_2)_\omega$ (respectively in $\mathcal{O}_2{}' \cap \mathrm{Q}(\mathbb{R}_+, \mathcal{O}_2)$ or in $\mathcal{O}_2{}' \cap (\mathcal{O}_2)_\infty$). Then, by repeat of this construction, we find a unital copy of $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots \subseteq \mathcal{O}_{2\omega}$ that commutes with $\mathcal{O}_2$ element-wise in this asymptotic algebras. It allows to conclude that $\mathcal{O}_2 \cong \mathcal{D}_2 := \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$. See Chapter 11.

`Remarks on real` $\mathcal{O}_2$ `and` $\mathcal{O}_\infty$ `(3):`

The endomorphism $\delta_2(a) := s_1 a s_1^* + s_2 a s_2^*$ is homotopic to id in the endomorphisms of real Cuntz algebra $(\mathcal{O}_2)_\mathbb{R}$.

It follows $\mathcal{U}(\mathcal{O}_2) = \mathcal{U}_0(\mathcal{O}_2)$ by [**172**, thm. 1.9, thm. 3.7].

We have for the "real version" of $\mathcal{O}_\infty$ also $\mathcal{U}_0(\mathcal{O}_\infty) = \mathcal{U}(\mathcal{O}_\infty)$ by [**172**, thm's. 1.9, 3.7], (cf. arguments in Chapter 4, modify them for the real case).

$\mathcal{U}_0((\mathcal{O}_2)_\mathbb{R}) = \mathcal{U}((\mathcal{O}_2)_\mathbb{R})$ and $\mathcal{U}_0((\mathcal{O}_\infty)_\mathbb{R}) = \mathcal{U}((\mathcal{O}_\infty)_\mathbb{R})$ hold also for the "real" version $(\mathcal{O}_2)_\mathbb{R}$ of $\mathcal{O}_2$.

Here we use in the real cases $\mathcal{E} = \mathcal{E}_\mathbb{R}$, $(\mathcal{O}_2)_\mathbb{R}$ and $(\mathcal{E}_2)_\mathbb{R}$ also the "complex" notations "unitary" and $\mathcal{U}(E), \mathcal{U}_0(E)$ instead of $O(E)$, respectively $SO(E)$, for the orthogonal elements $u \in E$, respectively $u \in E$ in the connected component of 1 in $O(E)$.

The homotopy id $\sim_h \delta_2$ exists also in the semi-group of unital *-endomorphisms of $(\mathcal{O}_2)_\mathbb{R}$. It follows that $\mathcal{U}((\mathcal{O}_2)_\mathbb{R}) = \mathcal{U}_0((\mathcal{O}_2)_\mathbb{R})$ and that all unital *-endomorphisms φ of $(\mathcal{O}_2)_\mathbb{R}$ are homotopic to the identity isomorphism of $(\mathcal{O}_2)_\mathbb{R}$. Indeed, φ is uniquely defined by $\varphi(s_j) = u_\varphi s_j$ $(j = 1, 2)$ where

$$u_\varphi := \sum_{j=1}^{2} \varphi(s_j) s_j^*\,,$$

and, for each unitary $u \in (\mathcal{O}_2)_\mathbb{R}$ holds:

$u \oplus_{s_1,s_2} u \sim_h u$ in $\mathcal{U}((\mathcal{O}_2)_\mathbb{R})$. Since $u \oplus v \sim_h v \oplus u \sim uv \oplus 1$ by part (v,4) of Lemma 4.2.6 we get that

$$(u \oplus u) \oplus u^* \sim_h u \oplus u^* \;\sim_h\; 1\,.$$

Since U_d `or` U_d^* `first?`

$$U_d^*((u \oplus u) \oplus u^*)U_d = u \oplus (u \oplus u^*)$$

it follows

$$u \oplus 1 \;\sim_h\; u \oplus (u \oplus u^*) \;\sim_h\; 1$$

in $\mathcal{U}((\mathcal{O}_2)_\mathbb{R})$. We get that

$$u \sim_h u \oplus u \sim_h (u \oplus 1) \cdot (1 \oplus u) \sim_h 1$$

for every $u \in \mathcal{U}((\mathcal{O}_2)_\mathbb{R})$, i.e., $\mathcal{U}((\mathcal{O}_2)_\mathbb{R}) = \mathcal{U}_0((\mathcal{O}_2)_\mathbb{R})$.

The selfadjoint unitary $C(2) := \sum_{1\leq i,j\leq 2} s_i s_j s_i^* s_j^*$ in the real C^*-algebra $A_2 := C^*(s_i s_k (s_j s_\ell)^*; 1 \leq i,j,k,\ell \leq 2) \cong M_4(\mathbb{R})$ has determinate $= -1$.

But $\delta_2(C(2))$ is unitary equivalent to $C(2)$ in

$$M_8(\mathbb{R}) = A_3 := C^*(s_i s_k s_m (s_j s_\ell s_n)^*; 1 \leq i,j,k,\ell \leq 2)$$

by the unitary $C(3) := \sum_{1\leq i,j,k\leq 2} s_i s_j s_k (s_j s_k s_i)^*$. This implies that $C(2) \in \mathcal{U}_0((\mathcal{O}_2)_\mathbb{R})$ and then that δ_2 is homotopic to to id in the unital *-endomorphisms of $(\mathcal{O}_2)_\mathbb{R}$.

The equality $\mathcal{U}(\mathcal{E}_2) = \mathcal{U}_0(\mathcal{E}_2)$, can be seen (in case of complex E) from the short-exact sequence

$$\mathbb{K} \to \mathcal{E}_2 \to \mathcal{O}_2$$

and from the equalities

$\mathcal{U}_0(\mathbb{K} + \mathbb{C} \cdot 1) = \mathcal{U}(\mathbb{K} + \mathbb{C} \cdot 1)$

and $\mathcal{U}_0(\mathcal{O}_2) = \mathcal{U}(\mathcal{O}_2)$.

But one can also use the general property (sq) to prove K_1-injectivity.

[**209**, prop. 5]:

(Let $T_1, \ldots, T_n$ isometries with $\sum_k T_k T_k^* = 1$.) Define $R_k := T_2 T_1 \cdot \ldots \cdot T_1$, where we take k-times T_1.

??

`Is it not simply the original argument of J. Cuntz ?`

Let (U_k is not a unitary !)

$$U_k = \sum_{p\in\Gamma_k} V_p R_k V_p^*$$

Then $\phi(x) = \lim_{k\to\infty} U_k^* x U_k$ for $x \in \mathcal{O}_\mathrm{n}$.

???????

(What happens in $\mathcal{E}_n$ in place of $\mathcal{O}_\mathrm{n}$??)

Here $\phi(x) := \int_G \sigma(g)(x)\, \mathrm{d}\mu_G(g)$. It is different defined in the paper. (Above defined as $P(x)$ for the circle action.)

`How is the group` G `defined??` Likely by circle action – or by $\mathbb{Z}_2$-action: $\chi\colon \{-1, 1\}$?

Here

$$\Gamma_k := \{1, 2, \ldots, n\}^k := \{p = (p_1, \ldots, p_k);\ p_j \in \{1, 2, \ldots, n\}\}$$

and $V_p := T_{p_1} \cdot \ldots \cdot T_{p_k}$.

(Also written somewhere as $T_k(w)$ for $w = p_1 \dots p_k$, $p_j \in \{1, \dots, n\}$).

2. Semi-Projectivity of the $\mathcal{E}_n$ and $\mathcal{O}_n$

Recall that $\mathcal{E}_2 := C^*(s_1, s_2\,;\, s_j^* s_k = \delta_{jk}1)$. The C^*-algebra $\mathcal{E}_2$ satisfies a version of *weak semi-projectivity* shown in the following lemma.

LEMMA A.2.1. *If A is the closure of an increasing sequence of C^*-subalgebras $A_1 \subseteq A_2 \subseteq \cdots$ and $\psi\colon \mathcal{E}_2 \to A$ is a (not necessarily unital) C^*-morphism, then for every $\varepsilon \in (0,1)$ there exist $n = n(\varepsilon) \in \mathbb{N}$, a C^*-morphism $\psi_n\colon \mathcal{E}_2 \to A_n$ and elements $h_1, h_2 \in A$ of norm $\|h_k\| \leq \varepsilon$ $(k = 1, 2)$ such that $h_k^* = -h_k$ and $\psi_n(1) := \exp(-h_1)\psi(1)\exp(h_1)$, $\|\psi(s_j) - \psi_n(s_j)\| < \varepsilon$ and $\psi_n(s_j) = \exp(h_2)\psi(s_j)\exp(h_1)$ for $j = 1, 2$.*

Reorganize following proof:

Explain the relations between ``$\|1 - u\| < 1$ small'' and $u := \exp(h)$ for $h \in A$ with $h^* = -h$ and ``$\|h\|$ small''.

Start then (! 1st) the unital case with range projections in A_n for all n.

Do (2nd) the reduction from unital case where the range projections are not in A_n to the case where they are in all A_n

then (3rd) the reduction from most general to the unital case.

PROOF. Let $p_k := \psi(s_k s_k^*)$ for $k = 1, 2$. We consider first the C^*-morphism $\psi_0\colon \mathbb{C} \oplus \mathbb{C} \to A$ defined by $\psi_0(\xi_1, \xi_2) := \xi_1 p_1 + \xi_2 p_2$.

Let $y := p_1 - p_2$. It has spectrum $\{-1, 1\} \subseteq \mathrm{Spec}_A(y) \subseteq \{-1, 0, 1\}$. We find a sequence of selfadjoint contractions $y_n \in A_n$ such that $y = \lim y_n$. It follows that there exists n_0 such that $\|y_n - y\| < 1/8$ for all $n \geq n_0$. Thus, $\mathrm{Spec}(y_n)$ is in $[-1, -3/4] \cup [-1/4, 1/4] \cup [3/4, 1]$ for all $n \geq n_0$. Let $\gamma(t)$ the continuos pice-wise linear function with (only) break points $\{-3/4, -1/4, 1/4, 3/4\}$, values $\gamma(t) := -1$ on $[-1, -3/4]$, $\gamma(t) := 0$ on $[-1/4, 1/4]$, $\gamma(t) := 1$ on $[3/4, 1]$ and $\gamma(t)$ linear on $[-3/4, -1/4] \cup [1/4, 3/4]$.

Then $\mathrm{Spec}(\gamma(y_n)) \subseteq \{-1, 0, 1\}$ for all $n \geq n_0$, and $p_{1,n} := \gamma(y_n)_+$, $p_{2,n} := \gamma(y_n)_-$ are projections for $n \geq n_0$ and $\psi_{0,n}(\xi_1, \xi_2) := \xi_1 p_{1,n} + \xi_2 p_{2,n}$ are C^*-morphisms from $\mathbb{C} \oplus \mathbb{C}$ into A_n $(n \geq n_0)$ that satisfy $\lim_n \psi_{0,n}(\xi_1, \xi_2) = \psi_0(\xi_1, \xi_2)$.

More precisely, there exist $h_{1,n} \in A$ with $h_{1,n}^* = -h_{1,n}$ and $\|h_{1,n}\| \leq \arcsin????$, such that

$$\psi_{0,n}(1, 1) = \exp(-h_{1,n})(p_1 + p_2)\exp(h_{1,n})$$

for all $n \geq n_0$.

Let $z_j := \psi(s_j)$, $P := z_1^* z_1 = z_2^* z_2$, $p_j := z_j z_j^* \leq P$ $(j = 1, 2)$. Notice that $p_1 + p_2 \leq P$ (implies $p_1 p_2 = 0$).

Since the self-adjoint contractions in the algebraic *-algebra $\bigcup_n A_n$ are dense in the self-adjoint contractions of A, we can start with a sequence $x_n^* = x_n \in A_n$ with $\|x_n\| \leq 1$ and $\lim \|x_n - P\| = 0$. Let $\gamma_n := \|x_n - P\|$. Then the spectrum $\mathrm{Spec}(x_n)$ of x_n is contained in $[-\gamma_n, \gamma_n] \cup [1 - \gamma_n, 1]$. Thus, $P_n := \varphi(x_n) \in A_n$ is a projection with $\|x_n - P_n\| \leq \gamma_n$ for all n with $\gamma_n < 1/4$ where

$$\varphi(t) := \min(1, 2\max(0, 2t - 1)) = 4[(t - 1/2)_+ - (t - 3/4)_+] .$$

We get the existence of n_0 such that there are projections $P_n \in A_n$ with $\|P_n - P\| \leq 1/2$ for $n \geq n_0$ and that $\lim_n \|P_n - P\| = 0$.

Let $\lambda(t)$ a "suitable" continuous strictly increasing functions on $[0, 2]$ with $\lambda(0) = 0$.

By ????

`Check Ref's for changes!!`

Part (v) of Lemma 4.1.3

there exist $h_n \in A_+$ with $\|h_n\| \leq \arcsin \|P_n - P\|$ such that $e^{-ih_n} P e^{ih_n} = P_n$. Notice that here e^{ih_n} is build in $A + \mathbb{C}1$ after adjoining a unit to A and that $|e^{it} - 1| \leq |t|$ (for $t \in [-\pi, \pi]$) implies

$$\|1 - e^{ih_n}\| \leq \|h_n\| \leq \arcsin \|P_n - P\| .$$

An opposite estimate follows from $2|\sin(t/2)| = |\exp(-it/2) - \exp(it/2)| = |1 - \exp(it)|$ for $t \in [-\pi, \pi]$ and gives $\|h\| \leq 2\arcsin(\|1 - \exp(ih)\|/2)$ for $h^* = h$ with $\|h\| \leq \pi/2$.

Notice also that $\|1 - u\| < 1$ implies for unitary u that $s := \log(u) = \log(1 - (1 - u))$ exists, satisfies $s^* = -s$, $\|s\| \leq \pi/3$ and $\exp(s) = u$. (One can consider $C^*(u)$ as a quotient of $\mathrm{C}(S^1)$ and reformulate it as properties of characters of $\mathrm{C}(S^1)$ i.e., as conditions on points of $S^1 \subseteq \mathbb{C}$.) Using that $|\log(1 - t)| \leq -\log(1 - |t|) \leq |t|/(1 - |t|)$ for $|t| < 1$, we get also the estimate $\|s\| \leq -\log(1 - \|1 - u\|) \leq \|1 - u\|/(1 - \|1 - u\|)$ if u is a unitary with $\|1 - u\| < 1$. Let $h := -is$, then $u = \exp(ih)$ and

$$\|h\| \leq \|1 - u\|/(1 - \|1 - u\|) .$$

If we take some suitable $\ell > n_0$ such that $\|h_\ell\| < \lambda(\varepsilon)$, then we can replace A by $P_\ell A P_\ell$, ψ by the then unital C^*-morphism $\psi' \colon \mathcal{E}_2 \to P_\ell A P_\ell$ $b \in \mathcal{E}_2 \to A$ given by $\psi'(b) := e^{-ih_\ell} \psi(b) e^{ih_\ell}$ and consider only the $P_\ell A_n P_\ell$ with $n \geq \ell > n_0$. We store n_0, n_ℓ and $H_0 := h_\ell$ in our bookkeeping.

This reduces the considerations to unital A and a unital C^*-morphism $\psi \colon \mathcal{E}_2 \to A$, i.e., now $z_j^* z_k = \delta_{j,k} 1$. We proceed in the same way as above with the projections $p_j = z_j z_j^*$ and find self-adjoint $h_1 \in A$, $n_1 < n_2$ in $\mathbb{N}$ and self-adjoint $h_2 \in (1 - q_1)A(1 - q_1)$ both with $\|h_1\|, \|h_2\| \leq \lambda(\varepsilon)$, such that $e^{-ih_1} p_1 e^{ih_1} =: q_1 \in A_{n_1}$

and $e^{-ih_2}e^{-ih_1}p_2e^{ih_1}e^{ih_2} =: q_2 \in A_{n_2}$. Notice that also $e^{-ih_2}e^{-ih_1}p_1e^{ih_1}e^{ih_2} = q_1$, because $q_1e^{ih_2} = e^{ih_2}q_1 = q_1$. Again store $n_1 < n_2$ and $H_1 := h_1$ and $H_2 := h_2$ to our list.

In this way we have reduced the consideration to the case where we can start with the more comfortable assumptions $z_j^*z_k = \delta_{jk}1$ and $p_j = z_jz_j^* \in A_n$ for all $n \in \mathbb{N}$. Let S_n denote the closed unit-ball of A_n and S the closed unit-ball of A. Since $(p_jS_n)_n$ is an increasing sequence of closed convex subsets of p_jS and p_jS is the closure of $\bigcap_n p_jS_n$ there exists for given $\varepsilon > 0$ and $m \in \mathbb{N}$ an $n := n(\varepsilon, m) \in \mathbb{N}$ with $n > m$ and $\mathrm{dist}(z_j, p_jS_n) < \lambda(\varepsilon)$ for $j = 1, 2$. It means that we can find $w_j \in p_jS_n$ with $\|w_j - z_j\| < \lambda(\varepsilon)$ $(j = 1, 2)$. It implies that $w_jw_j^* \leq p_j$, $1-2\varepsilon \leq w_j^*w_j \leq 1$ and $\|w_jw_j^* - p_j\| < 2\lambda(\varepsilon)$. Thus, $(w_j^*w_j)^{-1/2}$ exists and satisfies

$$1 \leq (w_j^*w_j)^{-1/2} \leq (1 - 2\lambda(\varepsilon))^{-1/2} .$$

Let $u_j := w_j(w_j^*w_j)^{-1/2}$. Then u_j is an isometry with $p_ju_j = u_j$ and

$$\|u_ju_j^* - p_j\| \leq 2\lambda(\varepsilon) + [(1 - 2\lambda(\varepsilon))^{-1/2} - 1] < 1 ,$$

i.e., with $u_ju_j^* = p_j$ and

$$\|u_j - z_j\| \leq \lambda(\varepsilon) + [(1 - 2\lambda(\varepsilon))^{-1/2} - 1] .$$

The functions $\lambda(t)$ should be chosen such that $\lambda(\varepsilon) + [(1 - 2\lambda(\varepsilon))^{-1/2} - 1)] < \varepsilon/3$, and that $|e^{it} - 1| \leq 2\lambda(\varepsilon) + 2[(1 - 2\lambda(\varepsilon))^{-1/2} - 1]$ implies $|t| \leq \varepsilon/2$.

It follows that $U := u_1z_1^* + u_2z_2^* + (1 - p_1 - p_2)$ is a unitary with $Uz_j = u_j$ and

$$\|U - 1\| \leq \|u_1 - z_1\| + \|u_2 - z_2\| \leq 2\lambda(\varepsilon) + 2[(1 - 2\lambda(\varepsilon))^{-1/2} - 1] .$$

Then $u_j = Uz_j$ with unitary U such that $\|U - 1\|$ is sufficiently small.

Summing up we get that that $u_j = UV^*z_jV$ with $V := e^{iH_0}e^{iH_1}e^{iH_2}$ and $\|1 - V\|$????? and $\|1 - UV\| \leq \|1 - U\| + \|1 - V\|$.

Since $|1 - \exp(it)|^2 = 2(1 - \cos(t))$ the norm $\|H\| \leq \pi/2$ of $H^* = H$ approaches zero if $\|1 - \exp(iH)\|$ goes to zero. This allows to show that products of unitaries sufficiently near to 1 are exponentials $\exp(iH)$ with $H^* = H$ of small norm. □

3. On the Jiang-Su algebra $\mathcal{Z}$

Let $m, n \in \mathbb{N} = \{1, 2, \ldots\}$ natural numbers with g.c.d.$(n, m) = 1$, the dimension-drop C^*-algebra $\mathcal{E}(M_m, M_n)$ is defined as the C^*-subalgebra of $\mathrm{C}([0, 1], M_m \otimes M_n)$ consisting of the maps $f\colon [0, 1] \to M_m \otimes M_n$ with $f(0) \in M_m \otimes 1_n$ and $f(1) \in 1_m \otimes M_n$.

(We call it also “winding around” algebra ... build by a construction on pairs of algebras similar to the construction of a “Join” polyhedron in $\mathbb{R}^m \times \mathbb{R}^n$ from two polyhedra in $\mathbb{R}^m$ and $\mathbb{R}^n$ ...)

Need: Suitable central sequences, ...

There exists a trace-collapsing unital endomorphism from $\mathcal{E}(M_{2^\infty}\,,\ M_{3^\infty})$ into $\mathcal{E}(M_{2^\infty}\,,\ M_{3^\infty})$.

Definition A.3.1. The **Jiang-Su algebra** is isomorphic to the inductive limit of this endomorphism ...

There are several other definitions. See original Jiang-Su paper ... Needed properties of the Jiang-Su algebra...

Containment of Jiang-Su algebra in other algebras, e.g. infinite tensor products of suitable amenable unital C^*-algebras.

Need:
Special Central sequences of the JS algebra that allow to apply p.i. and s.p.i. criteria.

Each tensorial self-absorbing C^*-algebra absorbs the Jiang-Su algebra. (Winter)

Compare with my homotopy criterium ...

Something lost??

4. Fix-point algebras of compact group actions

Suppose that G is a compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ a point-norm continuous group homomorphism from G into the group $\mathrm{Aut}(A)$ of automorphisms of A. It is not difficult to see, that the map

$$a \mapsto P^G(a) := \int_G \alpha(g)(a)\,\mathrm{d}(g)$$

defines a *faithful* conditional expectation P^G from A onto the fix-point algebra A^G of G. Here "faithful" means that $P^G(a) \neq 0$ for all non-zero $a \in A_+$.

Notice that one can not conclude similar properties for A from the corresponding property of the fix-point C^*-subalgebra of a $\mathbb{Z}$-action of A (by an automorphism of A).

Proposition A.4.1. *Suppose that a compact group G acts continuous in point-norm topology on a C^*-algebra A.*
The fix-point subalgebra A^G of A is nuclear (respectively is exact, weakly injective, QWEP, has the local lifting property), if and only if, A is nuclear (respectively is exact, weakly injective, QWEP, has the local lifting property).

Moreover, if the conditional expectation P_G can be approximated point-wise by inner c.p. maps

$$V_\tau(a) := \sum_{k=1}^{n(\tau)} d_{k,\tau}^* a d_{k,\tau}\,,$$

then each non-zero closed ideal of A has non-zero intersection with A^G.

It is interesting that the question if P_G respects also *local reflexivity* – as defined for operator spaces, which is different from the well-known (and almost trivial) local reflexivity of Banach spaces – remains open even in case of circle actions!

It could be that the new operator exact sequences criteria for left-ideals of (stable) locally reflexive C^*-algebras implies now a positive answer for "local reflexivity" ... ???

But this requires to check what happens with the the fixed points under compact actions?

Question: What about "relative" properties? In particular, "relative weak injectivity" could be a candidate ...

Equivalent formulation of r.w.i. is given by: $A \subseteq B$ r.w.i. in B if and only if $A \otimes^{\max} C^*(F_\infty) \subseteq B \otimes^{\max} C^*(F_\infty)$.

If G acts on B by $\rho\colon g \in G \mapsto \rho(g) \in \mathrm{Aut}(B)$ and $\rho(g)(A) = A$ for all $g \in G$, then $B^G \cap A$ is r.w.i. in B^G?

Here B^G denotes the fix-point algebra of the G-action on B. Seems to work for compact G.

Proof. Wa adapt an idea of A. Grothendieck for the study of locally convex vector spaces with help of tensor product functors. A simple variant of his idea transferred to the study of analytic properties of C^*-algebras is to express this properties with help of tensor product functors and use the automatic G-invariance of the functors. We use this proof to say something more general about the applications of this method and its limitation.

Let (at the moment) C denote a fixed C^*-algebra. Let $\alpha\colon G \to \mathrm{Aut}(A)$ denote the G-action on A, and let

$$(B, C) \to B \otimes_\mu C \quad \text{and} \quad (B, C) \to B \otimes_\nu C$$

denote C^*-algebra tensor product "functors" that satisfy $\|\cdot\|_\mu \geq \|\cdot\|_\nu$ on the algebraic tensor product $B \odot C$ for all C^*-algebras B, i.e., the closed ideals $J_\mu(B)$ and $J_\nu(B)$ of maximal C^*-algebra tensor product $B \otimes^{\max} C$ given as kernels of the natural *-epimorphisms $B \otimes^{\max} C \to B \otimes_\mu C$ and $B \otimes^{\max} C \to B \otimes_\nu C$ satisfy $J_\mu(B) \subseteq J_\nu(B)$. (In fact, a suitable functorial selection of those "kernels" contains all necessary information on the tensor product functors!)

They are defined by a B-functorial choice of C^*-norms $\|\cdot\|_\mu \geq \|\cdot\|_\nu$ on the algebraic tensor products $B \odot C$ and are the corresponding topological tensor-products are the completions of $B \odot C$ with this norms. The functoriality means that for each C^*-morphism $h\colon A \to B$ and $x \in A \odot C$ in the algebraic tensor product holds that $\|h \otimes \mathrm{id}(x)\|_\mu \leq \|x\|_\mu$ and similar for $\|\cdot\|_\nu$. Or, equivalently, $(h \otimes^{\max} \mathrm{id})(J_\mu(A)) \subseteq J_\mu(B)$ for each C^*-morphism $h\colon A \to B$ and fixed C.

With other words, we allow only "natural" functor transformations

$$\varphi_B \colon B \otimes^\mu C \to B \otimes^\nu C$$

that satisfy

$$\varphi_B \circ (h \otimes^\mu \mathrm{id}) = (h \otimes^\nu \mathrm{id}) \circ \varphi_A$$

for $h\colon A \to B$.

In addition, we suppose, that $V\otimes_\mu$id and $W\otimes_\nu$id remain contractions, whenever V and W are c.p. contractions. It is not difficult to see, – with help of Kasparov-Stinespring dilations of V and using the outer-unital extension $\widetilde{V}\colon \widetilde{A}\to\widetilde{B}$ –, that this property is equivalent to the property of $\otimes_\mu$ (and similar of $\otimes_\nu$) that $B\otimes_\mu C\subseteq \widetilde{B}\otimes_\mu C$ and $pBp\otimes_\mu C\subset B\otimes_\mu C$ for all C, and all full projections $p\in B$. (We let it to reader to check this.)

Let ???????

Thus, A^G is nuclear (respectively, weakly injective, QWEP, locally liftable) if A is so.
Exactness always passes to C^*-subalgebras.

The conditional expectations P^G are faithful for compact G. Indeed, if $a\geq 0$ and χ is a positive functional on A with $\chi(a)>0$, then $\chi(P^G(a))>0$, because it is the integral over the non-zero, non-negative, and continuous function $g\mapsto \chi(\alpha(g)(a))$.

The maps $\eta\otimes_\mu\mathrm{id}\colon A^G\otimes_\mu C\to A\otimes_\mu C$ and $V\otimes_\mu\mathrm{id}\colon A\otimes_\mu C\to A^G\otimes_\mu C$ satisfy $(V\otimes_\mu\mathrm{id})(\eta\otimes_\mu\mathrm{id})=\mathrm{id}$. Thus, $A^G\otimes_\mu C$ is naturally isomorphic to the closure of the subspace $A^G\odot C$ in $A\otimes_\mu C$, i.e., $A^G\otimes_\mu C\subset A\otimes_\mu C$.

Now let the inclusion $A\subset B$ be G-equivariant, for some (point-norm) continuous G-action $\beta\colon G\to \mathrm{Aut}(B)$, i.e., $\alpha_g(a)=\beta_g(a)$ for $a\in A$ and $g\in G$. Then it holds:
If $A^G\otimes_\mu C\subset B^G\otimes_\nu C$, then $A\otimes_\mu C\subset B\otimes_\nu C$.

Indeed, let J denote the kernel of the natural C^*-morphism $\varphi\colon A\otimes_\mu C\to B\otimes_\nu C$. The functoriality of $\otimes_\mu$ and $\otimes_\nu$ implies

$$\varphi\circ(\alpha(g)\otimes_\mu \mathrm{id}_C)=(\beta(g)\otimes_\mu\mathrm{id}_C)\circ\varphi\,.$$

Thus, $\alpha(g)\otimes_\mu\mathrm{id}_C(J)=J$ for all $g\in G$.

One can see, with help of elementary tensors $a\otimes c$, that $g\mapsto(\alpha(g)\otimes_\mu\mathrm{id}_C)$ and $g\mapsto(\beta(g)\otimes_\nu\mathrm{id}_C)$ are point-norm continuous, and that the corresponding faithful conditional expectations $P_1:=P^G_X\colon x\mapsto\int(\alpha(g)\otimes\mathrm{id})(x)\,\mathrm{d}g$ and $P_2:=P^G_Y\colon y\mapsto\int(\alpha(g)\otimes\mathrm{id})(x)\,\mathrm{d}g$ map $X:=A\otimes_\mu C$ onto the closure of $A^G\odot C$ in $A\otimes_\mu C$, respectively, maps $Y:=B\otimes_\nu C$ onto the closure of $B^G\odot C$ in $B\otimes_\nu C$ and satisfy $\varphi\circ P_1=P_2\circ\varphi$. for $\varphi\colon A\otimes_\mu C\to B\otimes_\nu C$. It follows, that $P_1(J)\subset J\cap\overline{A^G\otimes C}$

Since $A^G\otimes_\mu C\subset A\otimes_\mu C$ and $B^G\otimes_\nu C\subset B\otimes_\nu C$ are complemented subspaces, we get that $P_G(J)$ is in the kernel of the monomorphism $A^G\otimes_\mu C\to B^G\otimes_\nu C$ (after natural identifications of $A^G\otimes_\mu C$ and $B^G\otimes_\nu C$ with its images). Thus $P_G(J)=\{0\}$, and $J=0$.

Now we apply the latter to our cases:

One obtains that A is nuclear if A^G is nuclear if one uses above $\otimes_\mu:=\otimes^{\max}$, $\otimes_\nu:=\otimes^{\min}$, $B:=A$ and C arbitrary.

It has been shown in [**431**, prop.1.1] that B is weakly injective, if an only if, $B\otimes^{\max}C^*(F)=B\otimes^{\min}C^*(F)$, where F is the free group on countably many generators. Thus, we obtain that A is weakly injective if A^G is weakly injective, if we let $\otimes_\mu := \otimes^{\max}$, $\otimes_\nu := \otimes^{\min}$, $A = B$ and $C := C^*(F)$.

By [**431**, prop.1.1], B has the local lifting property, if and only if, $B \otimes^{\max} \mathcal{L}(\ell_2) = B \otimes^{\min} \mathcal{L}(\ell_2)$. We get that A has the local lifting property if A^G has the local lifting property, if we take $\otimes_\mu := \otimes^{\max}$, $\otimes_\nu := \otimes^{\min}$, $A = B$ and $C := C^*(F)$

To see that A has the QWEP if A^G has the QWEP, one can take $\otimes_\mu := \otimes^{\max}$, $\otimes_\nu := \otimes^\ell$, $B := A$, $C := C^*(F)$, because B has QWEP, if and only if, $B \otimes^{max} C^*(F) = B \otimes^\ell C^*(F)$.

Here (generally) $B \otimes_\ell C$ – "the minimal left-exact C^*-tensor-product functor" – is defined by the completion of $B \odot C$ with respect to the C^*-norms

$$\| \sum_k b_k \otimes c_k \|_\ell := \sup_{F,\pi} N_F(\sum_k b_k \otimes c_k)\,,$$

given by (sufficiently many) C^*-algebras F and epimorphisms $\pi\colon F \to B$ and,

$$N_F(\sum_k b_k \otimes c_k) := \operatorname{dist}(\sum_k f_k \otimes c_k, I \otimes^{\min} C)\,,$$

where $\pi(f_k) = b_k$ and $I := \ker(\pi)$, and with distance taken in $F \otimes^{\min} C$.

That the *exactness* of A follows from the exactness of A^G, can be seen by putting $\otimes_\nu := \otimes^{\min}$, $\otimes_\mu := \otimes_r$, $B := A$ and consider arbitrary C.

Here we define the functor $(B, D) \mapsto B \otimes^r D$ – "the minimal right exact C^*-tensor-product functor" – from $D \otimes_\ell B$ by

$$\| \sum_k b_k \otimes d_k \|_r := \| \sum_k d_k \otimes b_k \|_\ell ?????\,.$$

??? `More` ????

If P_G can be approximated point-wise by inner c.p. maps

$$V_\tau(a) := \sum_{k=1}^{n(\tau)} d_{k,\tau}^* a d_{k,\tau}\,,$$

then $P_G(J) \subset J \cap A^G$.

Since P_G is faithful on positive elements, each non-zero closed ideal of A has non-zero intersection with A^G. □

QUESTION A.4.2. Is the class of locally reflexive C^*-algebras is invariant under crossed product by abelian *compact* groups G?

(Could this follows from Takai Duality: $A \otimes \mathbb{K} \cong (A \rtimes G) \rtimes \widehat{G}$. For discrete groups H acting on B – e.g. for $H := \widehat{G}$ and

????

holds $B \subseteq B \rtimes H$. Now take $B := A \rtimes G$ and $H := \widehat{G}$.)

The Question is:

Can we deduce from the local reflexivity of the crossed product $A \rtimes G$ (respectively of the fixed point algebra A^G) of A by a continuous action of a compact group G on A that A itself is locally reflexive?

Even in the case of circle actions $G := T$ $(= S^1)$ the answer is not known.

5. Comparison of elements in C^*-algebras

HERE is some collection of observations on $\approx$ and $\precsim$ and some easy consequences:

Notice $a \approx a^*a \sim_{MvN} aa^*$ in A.

It holds $a \otimes b \approx (a+b) \otimes 0$ in $M_2(A)$ if $a^*b = 0 = ba^*$. Indeed: It implies $b^*a = 0$, $a^*ab^*b = 0$ and Thus $(a+b) \approx (a+b)^*(a+b) = a^*a + b^*b$,

It follows that the $\approx$-classes $[\mathbb{K}(\mathcal{H})]_{approx}$ for all (non-zero) Hilbert spaces $\mathcal{H}$ is isomorphic to $\{0, 1, \ldots, \mathrm{Dim}(\mathcal{H})\}$ if $\mathcal{H}$ has finite dimension and $\{0, 1, 2, \ldots\} \cup \{+\infty\}$ if $\mathcal{H} \cong \ell_2(\mathbb{N})$.

$S, T \in \mathbb{K}(\ell_2(\mathbb{N}))$ satisfy $S \approx T$ if an only if the ranks (= dimension of image) of S^*S and T^*T are the same.

The positive operators $g(\alpha) := (\alpha, \alpha^2, \alpha^3, \ldots) \in \mathbb{K}(\ell_2(\mathbb{N}))$ for $\alpha \in (0,1)$ are properly infinite in $\mathbb{K}(\ell_2(\mathbb{N}))$ with trace $= \alpha/(1-\alpha)$, because $g(\alpha) \precsim g(\alpha^2) = (\alpha^2, \alpha^4, \alpha^6, \ldots)$, and $g(\alpha) \precsim \alpha \cdot g(\alpha^2) = (\alpha^3, \alpha^5, \alpha^7, \ldots)$ imply $g(\alpha) \oplus g(\alpha) \precsim g(\alpha)$, because $\alpha g(\alpha) \sim_{MvN} g(\alpha^2) \oplus \alpha \cdot g(\alpha^2)$ in $\mathbb{K}(\ell_2(\mathbb{N}))$ and $\sim_{MvN}$ implies $\approx$. But $\alpha g(\alpha) \approx g(\alpha)$.

Notice also that inside the diagonal operators in $\mathbb{K}(\ell_2(\mathbb{N}))$ it holds $g(\alpha)^2 = g(\alpha^2)$. It gives $g(\alpha) \oplus g(\alpha) \precsim g(\alpha)^2 \oplus \alpha g(\alpha)^2 \approx \alpha \cdot g(\alpha)$.

The $g(\alpha) \in \mathbb{K}_+$ are all properly infinite and $\lim_{n\to\infty} \|g(1/n)\| \leq 1/n$ with traces $\mathrm{Tr}(g(\alpha)) = \alpha/(1-\alpha)$.

Asymptotic: $2\alpha/(1-\alpha)$ versus $(1+\alpha)\alpha^2/(1-\alpha^2)$ on $\alpha \in (0,1)$ behave similar/different for $\alpha \nearrow 1$.

A flexible almost "topological" version of comparison has been introduced in special cases in the papers [**170, 171**] and was later generalized e.g. by M. Rørdam in [**690**] and others to the following definition.

Definition A.5.1. The **Cuntz comparison** between elements $a, b \in A$ is defined as follows:

We say that b **majorizes** a (in A and in the sense of J. Cuntz [**170, 171**]), and write it as $a \precsim b$ or $b \succsim a$), if there are sequences $c_1, c_2, \ldots; d_1, d_2, \ldots \in A$ such that $a := \lim_n c_n b d_n$.

Below it is shown that $a \precsim b$ is a transitive order relation, i.e., $a \precsim b$ and $b \precsim c$ imply $a \precsim c$. Obviously always $a \precsim a$.

Thus, we can define an equivalence relation $a \approx b$ by $a \precsim b$ and $b \precsim a$. It has the obvious property $a \approx a$ for all $a \in A$.

`move to/ or see one of the below given lemmas:`

It is evident that $a \precsim b$ is a transitive relation: $a \precsim a$ because e.g. always $a = \lim_n (aa^*)^{1/n} a (a^*a)^{1/n}$, and $a = \lim_n g_n c h_n$ for suitable choices $g_n := c_{k_n} e_{\ell_n}$ and $h_n := f_{\ell_n} d_{k_n}$ if $e_1, e_2, \ldots ; f_1, f_2, \ldots \in A$ satisfies $b = \lim_n e_n c f_n$.

Elements $a, b \in A$ are **Cuntz equivalent** (notation: $a \approx b$) if $a \precsim b$ and $b \precsim a$. Since $a \precsim a$ and $\precsim$ is transitive, $\approx$ is an equivalence relation in the usual sense. The $a \approx b$–equivalence classes of $a \in A \otimes \mathbb{K}$ in $A \otimes \mathbb{K}$ will be denoted sometimes by $[a]_\approx$ or simply by $[a]$ for elements a in the *algebraic* inductive limit

$$\operatorname{indlim}_n M_n(A) \;=\; \operatorname{indlim}_n A \otimes M_n \subseteq\; A \otimes \mathbb{K}$$

(but sometimes even for $a \in A \otimes \mathbb{K}$). The corresponding semigroup of this classes with Cuntz-addition has later been written as $W(A)$ by M. Rørdam, cf. [**690**]. We denote by $\mathrm{Cu}(A)$ the classes in $A \otimes \mathbb{K}$, and call it the large Cuntz semi-group.

Since $\mathcal{M}(A \otimes \mathbb{K})$ contains a copy of $\mathcal{E}_2 := C^*(S_1, S_2;\, S_j^* S_k = \delta_{j,k} \cdot 1)$ unitally (It contains also a copy of $\mathcal{O}_2$, but that is for the definitions here not of interest), we can define a (Cuntz-)addition of elements by $a \oplus b := S_1 a S_1^* + S_2 b S_2^*$ for $a, b \in A \otimes \mathbb{K}$, it induces an addition of the $\approx$ classes, and allow to define the "general" and the "local" Cuntz semigroups, namely the "general" for all elements of $A \otimes \mathbb{K}$ and the "local" for the Pedersen ideal $P(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$. Notice here that $P(A \otimes \mathbb{K})$ is usually different from the – naturally in $P(A \otimes \mathbb{K})$ as inductive limit contained – union (respective inductive limit) $\bigcap_{n \in \mathbb{N}} M_n(P(A))$, but from the definition of the Pedersen ideals $P(A)$ of A and $P(\mathbb{K} \otimes A)$ of $\mathbb{K} \otimes A$ as "the minimal dense algebraic ideals" it is easy to see that each element $x \in P(A \otimes \mathbb{K})_+$ is in the stronger sense of Murray–von-Neumann equivalent (see below) to some element in $y \in M_n(P(A))_+$ for some $n := n(x) \in \mathbb{N}$.

The semigroup $W(A)$ for elements in $\bigcup_n M_n(A)$ as introduced by M. Rørdam, [**690**], is in general strictly between the "large" and the "small" Cuntz semi-groups $\mathrm{Cu}(A)$ and $\mathrm{CS}(A)$.

It is often considered as the algebraic inductive limit

$$M_n \otimes A = M_n(A) \to M_{n+1}(A) = M_{n+1} \otimes A$$

via the embedding $M_n \oplus \{0\} \subseteq M_{n+1}$. But this is inessential because $\precsim$ and $\approx$ are weaker than $\leq$ and $\sim_{\mathrm{MvN}}$ combined, but this requires some care.

A stronger equivalence relation $\sim$ is the *Murray–von-Neumann equivalence*: Elements $a, b \in A_+$ are **MvN-equivalent** (denoted by $a \sim b$ or $a \sim_{\mathrm{MvN}} b$), if they are equivalent in sense of *Murray* and *von Neumann*, i.e., if there exists $d \in A$ with $d^*d = a$ and $dd^* = b$. Indeed, $a \approx b$ because one can take $d_n := d((d^*d)^{1/2} + 1/n)^{-1}$ and $c_n := d_n^*$ to get $a := \lim d_n^* b d_n$ and $b := \lim d_n a d_n^*$.

We define n**-step majorization** (denoted $b \succ_n a$) by

$$a \oplus 0_{n-1} \cong a \otimes p_{11} \precsim b \otimes 1_n\,.$$

?? Check all chapters for closed ideals: $J(b)$, J_b, $I(b)$ generated by an element or subset of B.
Decide which of them is the best and change others!!!

Definition A.5.2. Let B a C^*-algebra. A non-zero *contraction* $c \in B$ will be called a **scaling element** if $c^*cc = c$. We call a scaling element $c \in B$ "strictly scaling" if in addition $c^*c \neq cc^*$.

A positive contraction $b \in B_+$ is a **scale generator** if there exists $d \in B$ and $0 < \delta < \varepsilon < 1$ such that $d^*(b-\varepsilon)_+d = (b-\delta)_+$

In particular then, $(b-\delta)_+$ is in the closed ideal generated by $(b-\varepsilon)_+$.

A positive element $b \in B_+$ is a **compactly supported** element in B if there exist $\delta \in (0, \|b\|)$ such that b is contained in the closed ideal generated by $(b-\delta)_+$. (In particular each projection $p \in B$ is compactly supported.)

The element $b \in B$ is called **compactly n-supported** if there exists $\delta > 0$ such that $b \otimes p_{11} \precsim (b-\delta)_+ \otimes 1_n$.

Remark A.5.3. An equivalent definition of a scaling element $c \in B$ is that $e := c^*c$ is a positive contraction in B with $ecc^* = cc^* = cc^*e$.

A scaling element $c \in B$ that is not *strictly* scaling in sense of our Definition A.5.2 is simply a normal element in B with the property that $\operatorname{Spec}(c) \subseteq \{0\} \cup S^1$.

Notice that B. Blackadar and J. Cuntz call elements $c \in B$ in [**78**, def. 1.1] "scaling" only if they are *strictly* scaling elements in sense of our Definition A.5.2.

Remark A.5.4. The usual definition of compactness implies for Hausdorff spaces X that compact subsets of X are closed. But notice that for non-Hausdorff spaces this is not the case if we take the usual definition of a "compact" subset $Y \subseteq X$ that we find in every family $\mathcal{V} \subseteq \mathbb{O}(X)$ with $Y \subseteq \bigcup_{U \in \mathcal{V}} U$ a finite collection $U_1, \ldots, U_n \in \mathcal{V}$ with $Y \subseteq U_1 \cup \ldots \cup U_n$. This kind of subsets Y of X has been called in the Bourbaki monographs "quasi-compact" for non-Hausdorff spaces.

We have to consider sometimes the T_0 spaces $X := \operatorname{Prim}(A)$ for separable C^*-algebras A – or in cases where A is non-separable its "point" completions $X := \operatorname{prime}(A)$ ([4]). *We write often "compact" instead of "quasi-compact" in our considerations.*

Moreover this kind of ("quasi -") compact subsets K of X are even not "*saturated*", i.e., are not the intersections of all open subsets of X that contain K. Only the very special "coherent" spaces X defined by the property that each (quasi-) compact subset of X is saturated in this sense.

The class of saturated spaces does not contain $\operatorname{Prim}(A_0)$ for

$$A_0 := \{ f \in C([0,1], M_2) \, ; \; f(0) \in \mathbb{C} \oplus \mathbb{C} = \operatorname{diag}(M_2) \} .$$

[4] Such "adding" of all prime closed subsets of a T_0 space X to the set of points in X also called a "sobrification" of X, in a sense it is a kind of completion of X that does not change its lattice of open subsets.

The notions of Definition A.5.2 will be justified by the following Lemma:

LEMMA A.5.5. *Let $b \in B_+$, $c \in I((b-\gamma)_+)_+$ for some $\gamma > 0$.*

(i) *For each $\varepsilon > 0$ there exists $n := n(\varepsilon) \in \mathbb{N}$ such that $(b-\varepsilon)_+ \otimes p_{11} \precsim c \otimes 1_n$ in $A \otimes M_n$, if and only if, b is in the closed ideal generated by c. In particular, then $b \in I((b-\gamma)_+)$.*

(ii) *If b is contained in the closed ideal $I((b-\gamma)_+)$, then the Dini function $f_b \colon J \in \mathrm{Prim}(B) \mapsto \|\pi_J(b)\|$ on $\mathrm{Prim}(B)$ has an open and compact subset U_b of $\mathrm{Prim}(B)$ as its support, and $f_b(J) \geq \gamma$ for all $J \in U_b$. In particular, the closed ideal $I(b) = \overline{\mathrm{span}(BbB)}$ has (quasi-) compact primitive ideal space $\mathrm{Prim}(I(b))$.*

(iii) *If the closed ideal $I(b)$ of B generated by $b \in B_+$ has compact primitive ideal space, then there exists $\gamma > 0$ such that b is in the closed ideal generated by $(b-\gamma)_+$.*

In particular, for each $\varepsilon > 0$ there exists $n := n(\varepsilon) \in \mathbb{N}$ and $d_1, \ldots, d_n \in B$ such that $(b-\varepsilon)_+ = \sum_{k=1}^n d_k^*(b-\gamma)_+ d_k$.

(iv) *Let $c \in B$ a* scaling element, *i.e., $\|c\| \leq 1$ and $c^*cc = c$. The element $V(c) := c + i(1-c^*c)^{1/2}$ is an isometry in $B + \mathbb{C}\cdot 1$, and the projection*

$$p := p(c) := 1 - V(c)V(c)^* = (c^*c - cc^*) + i(c(1-c^*c)^{1/2} - (1-c^*c)^{1/2}c^*)$$

is a projection in the closed ideal $I(c)$ of B.

The ideal $I(p)$ generated by p contains a sequence of mutually orthogonal Murray–von-Neumann equivalent projections $p = p(c) =: p_1, p_2, \ldots$.

(v) *The closed ideal $I(p) \subseteq I(c)$ given by the projection $p \in I(c)$ of part (iv) is generated as a closed ideal by $c^*c - cc^* \in I(c)$.*

The ideal $I(p)$ contains $c^*c(1-c^*c)$ and each $d \in B$ that satisfies $dc = 0$ and $dc^*c = d$.

(vi) *If b and γ satisfy* (iii) *and that $(b-\gamma)_+$ is properly infinite, then for each $\varepsilon \in (0,\gamma)$ there exists $d_k := d_k(\varepsilon) \in B$ $(k = 1,2)$ such that*

$$d_j^*(b-\gamma)_+ d_k = \delta_{jk} \min(1, \varepsilon^{-1}(b-\varepsilon)_+) = \delta_{jk}\varepsilon^{-1}\big((b-\varepsilon)_+ - (b-(1+\varepsilon))_+\big).$$

In particular, the elements $c_k := (b-\gamma)_+^{1/2} d_k$ are "scaling" elements in B with

$$c_j^*c_k = \delta_{jk} c_1^* c_1\,, \;\; c_k^* c_k c_k = c_k \;\; \text{and } \|c_k\| = 1$$

that generates the closed ideal $I(b) := \overline{\mathrm{span}(BbB)}$ of B.

(vii) *The closed ideal $I(p_1)$ of B generated by $p_1 := 1 - V(c_1)V(c_1)^*$ with scaling elements $c_1, c_2 \in B$ as in Part (vi) coincides with $I(b)$. In particular, $I(b)$ is generated by a hereditary stable C^*-subalgebra D that is generated by a sequence of mutually orthogonal and mutually equivalent projections $p_1, p_2, \ldots$.*

PROOF. (i): Let $b \in B_+$, and $c \in I((b-\gamma)_+)_+$ for fixed $\gamma > 0$.

If b is contained in the closed ideal $I(c)$ generated by c then for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in B$ with $\|b - \sum_k d_k^* c d_k\| < \varepsilon/2$. By norm-continuity of $\delta \mapsto (c-\delta)_+$ and by Lemma 2.1.9, there exists a contraction $e \in B$ such that

$$(b-\varepsilon)_+ \;=\; \sum_{1\leq k\leq n} (d_k e)^*(c-\delta)_+(d_k e)\,.$$

We can this equation express by $(b-\varepsilon)_+ \otimes p_{11} = D^*((c-\delta)_+ \otimes 1_n)D$ in $B \otimes M_n$ with column $D := \sum_k (d_k e) \otimes p_{k1} \in M_{n,1}(B)$.

In particular, for each $\varepsilon > 0$ there exists $n := n(\varepsilon) \in \mathbb{N}$ such that $(b-\varepsilon)_+ \otimes p_{11} \precsim c \otimes 1_n$ in $A \otimes M_n$.

Suppose now that for each $\varepsilon > 0$ there exists $n := n(\varepsilon) \in \mathbb{N}$ such that $(b-\varepsilon)_+ \otimes p_{11} \precsim c \otimes 1_n$ in $A \otimes M_n$. It says that there exists a sequence of column matrices $D_1, D_2, \ldots \in M_{n,1}(B) \subseteq M_n(B)$ with $(b-\varepsilon)_+ \otimes p_{11} = \lim_n D_n^*(c \otimes 1_n)D_n$ in $M_n(B)$. It follows that $(b-\varepsilon)_+ \otimes p_{11} \in M_n(I(c))$ with $I(c) \lhd B$ the ideal generated by c. It follows that $(b-\varepsilon)_+ \in I(c) \subseteq I((b-\gamma)_+)$ for each $\varepsilon > 0$. Thus, b is in the closed ideal $I(c)$ generated by c, in particular, $b \in I((b-\gamma)_+)$.

(ii): Recall here that $f_b^{-1}[\gamma, \infty) \subseteq \mathrm{Prim}(A)$ is (quasi-) compact for every $\gamma > 0$, cf. [**213**, prop. 3.3.7], where we consider here the Dini function f_b on $\mathrm{Prim}(A)$ defined by $f_b(J) := \|\pi_J(b)\|$ for $b \in B$ and $J \in \mathrm{Prim}(B)$.

If b is contained in the closed ideal

$$I((b-\gamma)_+) \;=\; \bigcap \{ J \in \mathrm{Prim}(B)\,;\; \|\pi_J(b)\| \leq \gamma \}\,,$$

then the Dini function $f_b \colon J \in \mathrm{Prim}(B) \mapsto \|\pi_J(b)\| \in [0, \infty)$ on $\mathrm{Prim}(B)$ has support

$$U_b := \; f_b^{-1}(0, \infty) = \{ J \in \mathrm{Prim}(B)\,;\; f_b(J) > 0 \}\,.$$

This set U_b is identical with the (quasi-) compact subset

$$f_b^{-1}[\gamma, \infty) = \{ J \in \mathrm{Prim}(B)\,;\; f_b(J) \geq \gamma \}\,.$$

Indeed: The inequality $f_b(J) = \|\pi_J(b)\| < \gamma$ for $J \in \mathrm{Prim}(A)$ implies that there are $\varepsilon > 0$ and $\delta \in [f_b(J), \gamma)$ such that $\pi_J((b-\delta)_+) = 0$ and, – by assumption $b \in I((b-\gamma)_+)$ –, that there exist $d_1, \ldots, d_n \in B$ with $(b-\varepsilon)_+ = \sum_k d_k^*(b-\delta)_+ d_k$. If we apply to this π_J, we see that

$$(f_b(J) - \varepsilon)_+ = \|\pi_J((b-\varepsilon)_+)\| \leq \sum_k \|d_k\|^2 \|\pi_J((b-\delta)_+)\| = 0\,.$$

Thus, $f_b(J) \leq \varepsilon$ for all $\varepsilon > 0$, i.e., $f_b(J) = 0$ if $f_b(J) < \gamma$. It means $U_b := f_b^{-1}(0, \infty) = f_b^{-1}[\gamma, \infty)$.

Since the set $f_b(J) = \|\pi_J(b)\| \geq \gamma$ is a compact subset of $\mathrm{Prim}(B)$, the closed ideal $I(b) = \overline{\mathrm{span}(BbB)} = \bigcap \{ J \in \mathrm{Prim}(B)\,;\; f_b(J) = 0 \}$ has compact primitive ideal space $\mathrm{Prim}(I(b)) = f_b^{-1}[\gamma, \infty)$.

(iii): Suppose that $I(b)$ has compact primitive ideal space $\mathrm{Prim}(I(b)) \subseteq \mathrm{Prim}(B)$. The corresponding open subset of $\mathrm{Prim}(B)$ is given by $f_b^{-1}(0, \infty)$. The family of open subsets $f_b^{-1}(\delta, \infty)$ of $f_b^{-1}(0, \infty)$ with $\delta > 0$ covers the compact space $f_b^{-1}(0, \infty)$. Thus, there exists $\gamma > 0$ such that $f_b^{-1}(\gamma, \infty) = f_b^{-1}(0, \infty)$. It is easy to see that $f_b^{-1}(\gamma, \infty) = f_c^{-1}(0, \infty)$ for $c = (b-\gamma)_+$. It follows that $(b-\gamma)_+$ and b generate the same closed ideal of B.

This implies that for each $\varepsilon > 0$ there exists $n := n(\varepsilon) \in N$ and $d_1, \ldots, d_n \in B$ such that $(b-\varepsilon)_+ = \sum_{k=1}^n d_k^* \, (b-\gamma)_+ \, d_k$.

(iv,v): Let $c \in B$ a contraction with $cc^*c^*c = cc^*$. The element $V := V(c) := c + i(1-c^*c)^{1/2}$ of $B + \mathbb{C} \cdot 1$ is an isometry, because $cc^*(1-c^*c) = 0$ implies $c^*(1-c^*c) = 0$, $c^*(1-c^*c)^{1/2} = 0$ and $(1-c^*c)^{1/2}c = 0$.

The projection $p := p(c) := 1 - V(c)V(c)^*$, is in $I(c) \subseteq B$. The projections

$$p_n := V^{n-1}(V^*)^{n-1} - V^n(V^*)^n$$

are mutually orthogonal, where we let $p_1 := p(c)$. If $m < n$ then $p_m \sim_{MvN} p_n$ by $Z = V^{n-m}p_m$, i.e., $Zp_mZ^* = p_n$. In particular, $p_n \in I(p(c))$ for all $n \in \mathbb{N}$.

The equations $c^*(1-c^*c)^{1/2} = 0$ and $(1-c^*c)^{1/2}c = 0$ imply $p := 1 - V(c)V(c)^* = (c^*c - cc^*) + i(c(1-c^*c)^{1/2} - (1-c^*c)^{1/2}c^*)$ and that $(1-c^*c)^{1/2}p(1-c^*c)^{1/2} = c^*c(1-c^*c)$. Thus, $J := I(c^*c(1-c^*c)) \subseteq I(p)$. Since $c(1-c^*c) \in J$, we get $c(1-c^*c)^{1/2} \in J$ and that $\pi_J(p) = \pi_J(c^*c - cc^*)$, i.e., $p - (c^*c - cc^*) \in J$. The epimorphism π_J maps $I(p) \supseteq J$ onto $I(p)/J$ and has kernel $J \subseteq I(p)$. We get that the positive element $c^*c - cc^*$ must be in $I(p)$.

Let $d \in B$ with $d^*c = 0$ and $dc^*c = d$. Then $dd^*V(c) = 0$, because $d(1-c^*c)^{1/2} = 0$. It follows $dd^* \leq \|d\|^2 p$, which implies $d \in I(p)$.

(vi): Suppose that $b \in B_+$ and γ satisfy (iii) and that $(b-\gamma)_+$ is properly infinite.

It implies that for each $\tau > \gamma$ and $n \in \mathbb{N}$ the existence of $e_1, \ldots, e_n \in B$ such that $e_j^*(b-\gamma)_+ e_k = \delta_{ij}(b-\tau)_+$.

Indeed, for each $\mu > 0$ and $n \in \mathbb{N}$ there exists a matrix $R \in M_n(B)$ that satisfies $R^*((b-\gamma)_+ \otimes p_{11})R = ((b-\gamma)_+ - \mu)_+ \otimes 1_n$. Here we use again Lemma 2.1.9 in $M_n(B)$ to obtain equality. Define $\mu := \tau - \gamma$ and $e_1, \ldots, e_n$ by $[e_1, \ldots, e_n] := R$ with R for μ. The $e_1, \ldots, e_n$ have the desired property.

Let $\varepsilon \in (0, \gamma)$. There exists $\tau > \gamma$ and $m := m(\varepsilon, \tau) \in \mathbb{N}$ $g_\ell := g_\ell(\varepsilon, \tau) \in B$ ($\ell = 1, \ldots, m$) such that

$$\sum_{1 \leq \ell \leq m} g_\ell^*(b-\tau)_+ g_\ell = \min(1, \varepsilon^{-1}(b-\varepsilon)_+) = \varepsilon^{-1}\big((b-\varepsilon)_+ - (b-(1+\varepsilon))_+\big)\,,$$

because we find $f_\ell \in B$ with

$$\| \sum_\ell f_\ell^*(b-\gamma)_+ f_\ell - (b-\varepsilon/3)_+ \| < \varepsilon/3$$

and can apply the continuity of $\tau \mapsto (b-\tau)_+$ and the Lemma 2.1.9 to get the equation proper by multiplying the f_ℓ with a suitable contraction e from the right to reach the equality

$$\sum_{1 \leq \ell \leq m} (f_\ell e)^*(b-\tau)_+(f_\ell e) = (b - 2\varepsilon/3)_+ \,.$$

Finally we let $g_\ell := f_\ell e h(b)$ where

$$h(t) := \min(\varepsilon, (t-\varepsilon)_+)/(\varepsilon(t-2\varepsilon/3)_+)\,.$$

We take $n := 2m$ in the above given construction of the e_k and let $d_1 := \sum_{1\leq \ell\leq m} e_\ell g_\ell$ and $d_2 := \sum_{1\leq \ell\leq m} e_{m+\ell} g_\ell$. Then

$$d_j^*(b-\gamma)_+ d_k \;=\; \delta_{jk}\min(1, \varepsilon^{-1}(b-\varepsilon)_+)\,.$$

In particular, the $c_k := (b-\gamma)_+^{1/2} d_k$ are "scaling" elements in B with $c_j^* c_k = \delta_{jk} c_1^* c_1$, $c_k c_k^* c_k^* c_k = c_k c_k^*$, such that $I(c_k) = I(b)$ for the the closed ideals $I(c_k) := \overline{\mathrm{span}(Bc_kB)}$ and $I(b) := \overline{\mathrm{span}(BbB)}$ of B. □

We have sometimes to deduce estimates for the infimum $\gamma(n,a,b,\varepsilon)$ of the norms $\|\sum d_k^* d_k\|$ for $d_1,\ldots,d_n \in A$ with $\|b - \sum_k d_k^* a d_k\| < \varepsilon$. Notice that then there are *contractions* $e_1,\ldots,e_m \in A$ such that $\|b - \sum_j e_j^* a e_j\| < \varepsilon$ and $m \leq n(\gamma(n,a,b,\varepsilon)+1)$.

Lemma A.5.6. *Let $X = [b_{k\ell}] \in M_n(B)_+$, $d_1,\ldots,d_n \in B$, and let $\mathrm{diag}(X) \in M_n(B)$ denote the diagonal matrix with entries $b_{11},\ldots,b_{nn}$ in the diagonal of X.*

(i) *The linear map $X \mapsto n\,\mathrm{diag}(X) - X$ on $M_n(B)$ is completely positive. In particular, $X \leq n\,\mathrm{diag}(X)$ for all $X \in M_n(B)_+$.*
(ii) *Let $\mathrm{Tr}_n([\alpha_{jk}]) := (\alpha_{11}+\cdots+\alpha_{nn})\cdot 1_n$ for $[\alpha_{jk}] \in M_n$. The map $X \mapsto n(\mathrm{id}_B \otimes \mathrm{Tr}_n)(X) - X$ on $M_n(B)$ is completely positive.*
In particular, $X \leq n(\mathrm{id}_B\otimes \mathrm{Tr}_n)(X) = n(b_{11}+\ldots+b_{nn})\otimes 1_n$ for all $X = [b_{k,\ell}] \in M_n(B)_+$.
(iii) *$0 \leq \sum_{k,\ell=1}^n b_{k\ell} \leq n(b_{11}+\ldots+b_{nn})$ for all $X = [b_{k,\ell}] \in M_n(B)_+$.*
(iv) *For every $b \in B_+$ and $d_1,\ldots,d_n \in B$,*

$$(d_1+\cdots+d_n)^* b (d_1+\cdots+d_n) \leq n\,(d_1^* b d_1 + \cdots + d_n^* b d_n)\;.$$

Proof. (i): We identify $M_n(\mathcal{M}(B))$ naturally with $\mathcal{M}(B)\otimes M_n$, and M_n with $1\otimes M_n$. Let $z := \exp((2\pi i)/n)$ and $U := \mathrm{diag}(1, z, \ldots, z^{n-1}) \in M_n \subseteq M_n(\mathcal{M}(B))$. Then, for $a \in \mathcal{M}(B)$,

$$\sum_{k=0}^{n-1} (U^k)^*(a\otimes e_{ij})U^k = a\otimes (\sum_k z^{(j-i)k}) e_{ij} = n\delta_{ij}(a\otimes e_{ij})\,.$$

Thus, $X = (U^0)^* X U^0 \leq \sum_{k=0}^{n-1} (U^k)^* X U^k = n\,\mathrm{diag}(X)$, and the map

$$X \mapsto n\,\mathrm{diag}(X) - X = \sum_{k=1}^{n-1} (U^k)^* X U^k$$

is completely positive, because is a sum of c.p. maps.

(ii): The map $n(\mathrm{id}_B\otimes \mathrm{Tr}_n)(X) - X = (\mathrm{id}_B \otimes (n\,\mathrm{Tr}_n - \mathrm{id}))(X)$, id_B is c.p. on B and we show tat $n\,\mathrm{Tr}_n - \mathrm{id}$ is c.p. on M_n, where we define $\mathrm{Tr}([\alpha_{jk}]) := (\sum_k \alpha_{kk})\cdot 1_n$.

The complete positivity of $n\,\mathrm{Tr}_n - \mathrm{id}$ on M_n can be seen from the formula

$$n^{-1}(\sum_k \alpha_{kk})\cdot 1_n = n^{-2}\sum_{g\in G} g^{-1}[\alpha_{jk}]g\,,$$

because this implies $n(\sum_k \alpha_{kk})\cdot 1_n - [\alpha_{jk}]$ is the sum of n^2-1 inner *-automorphisms $g^{-1}[\alpha_{jk}]g$ of M_n.

Here $G := G(U,V) \subseteq \mathcal{U}(n) = \mathcal{U}(M_n)$ is the irreducible finite subgroup G of $\mathcal{U}(M_n)$ with n^2 unitary elements that is generated by the unitary $U := \mathrm{diag}(z^0, z^1, \ldots, z^{n-1})$ with $z := \exp(2\pi i)/n$ and by the cyclic permutation $V \in \mathcal{U}(M_n)$ of the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, i.e., $V(e_k) = e_{k+1}$ $(k = 1, \ldots, n-1)$ and $V(e_n) = e_1$. To see $|G| = n^2$, notice that $U^jV^k = z^{jk}V^kU^j$ for $j, k \in \{0, \ldots, n-1\}$.

`Alternative proof:`

The map

$$S\colon X \mapsto \mathrm{diag}(X) = (b_{11}, \ldots, b_{nn}) \in B^n = B \oplus \cdots \oplus B \cong \mathrm{diag}(M_n(B)) \subseteq M_n(B)$$

is a c.p. map, because it is a conditional expectation and

$$\mathrm{diag}(X) = n^{-1}(n\,\mathrm{diag}(X) - X) + n^{-1}X\,.$$

It is clear that the maps $T_j(b_1, \ldots, b_n) := \sum_{k\neq j} b_k$ are completely positive, because they are sums of c.p. maps.

The map $(b_1, \ldots, b_n) \mapsto (b_1 + \ldots + b_n) \otimes 1_n - \mathrm{diag}(b_1, \ldots, b_n) \in M_n(B)$ is completely positive in $B^n \cong \mathrm{diag}(M_n(B))$, because it is the orthogonal sum of the c.p. maps T_j.

It follows the positivity of

$$X \mapsto [n(b_{11}+\ldots+b_{nn})]\otimes 1_n - X = n((\mathrm{id}_B \otimes \mathrm{Tr}_n)(X) - \mathrm{diag}(X)) + (n\,\mathrm{diag}(X) - X)\,.$$

(iii): Let C the column $C := [1, 1, \ldots, 1]^\top \in M_{n,1}(\mathcal{M}(B))$. Apply the c.p. map $Y \mapsto C^*YC$ to $Y := n\,\mathrm{diag}(X) - X$. Since $Y \geq 0$ by part (i), we get

$$\sum_{k,\ell=1}^{n} b_{k\ell} = C^*XC \leq nC^*\,\mathrm{diag}(X)C = n(b_{11} + \ldots + b_{nn})\,.$$

(iv): Take $X := [b_{jk}]$ in part (iii) with $b_{j,k} := d_j^* b d_k$, i.e., $X = D^*D$ for the row matrix $D := [b^{1/2}d_1, \cdots, b^{1/2}d_n] \in M_{n,1}(B)$. □

Next remark considers the equivalence given by polar decomposition of a of the hereditary C^*-subalgebras generated by a^*a and by aa^*.

`Next appears also before proof of THM.E. ??`

Remark A.5.7. Let $a \in A$ non-zero and $a := v(a^*a)^{1/2}$ its polar decomposition in the W*-algebra A^{**}, let $D := \overline{a^*aAa^*a}$ and $E := \overline{aa^*Aaa^*}$ the hereditary C^*-subalgebras of A generated by a^*a, respectively by aa^*.

Then $D = \overline{a^*Aa}$, $E = \overline{aAa^*}$, and the map $\varphi\colon d \in D \to vd\,v^* \in A^{**}$ has image in E and defines a C^*-algebra isomorphism from D onto E. The inverse isomorphism φ^{-1} is given by $e \in E \mapsto v^*ev \in D$.

It holds $\varphi(D \cap I) = E \cap I$ for every closed ideal $I \triangleleft A$ of A. In particular, if D is a full hereditary C^*-subalgebra of A, i.e., if $\overline{\mathrm{span}(ADA)} = A$, then also E is full in A.

In particular, D is simple if and only if E is simple.

PROOF. Let $a \in A$ and $a = v(a^*a)^{1/2} = (aa^*)^{1/2}v$ its unique (right) polar decomposition of a in A^{**}. Then $vx \in A$ for every x in the norm-closed right-ideal $R := \overline{a^*aA} = \overline{a^*A}$ of A that is generated by $(a^*a)^{1/2}$.

The map $x \mapsto vx$ is an isometric right-module isomorphism from R onto the closed right-ideal $\overline{aA} = \overline{aa^*A}$. The inverse map from $\overline{aA}$ onto R is given by $y \mapsto v^*y$, because $a^* = v^*(aa^*)^{1/2}$ is the polar decomposition of a^* in A^{**}.

This follows from $vf(a^*a) = f(aa^*)v$ for each non-negative continuous function $f \in \mathrm{C}_0(0, \|a\|^2]$, and can be easily seen in A^{**}.

easily seen: more details?

In particular, $y := vz^\alpha \in A$ and $y^*y = z^{2\alpha}$ for all $z \in A_+$ with $z \leq a^*a$ and all $\alpha > 0$. □

REMARK A.5.8. Let D a closed hereditary C^*-subalgebra of A.

Each closed ideal J of D is the intersection $J = D \cap I$ with D of the closed ideal $I := I(J) := \overline{\mathrm{span}(AJA)}$ of A.

For $I_1, I_2 \in \mathcal{I}(A)$ holds $I_1 \cap D = I_2 \cap D$ if and only if $I_1 \cap I(D) = I_2 \cap I(D)$, where $I(D) := \overline{\mathrm{span}(ADA)}$.

If $I(D) = A$ then the map $I \in \mathcal{I}(A) \mapsto J := I \cap D \in \mathcal{I}(D)$ is a lattice isomorphism from the lattice $\mathcal{I}(A)$ of closed ideals of A onto the lattice $\mathcal{D}$. The inverse of this map is given by $J \mapsto I := \overline{\mathrm{span}(AJA)}$.

6. Basic properties of Cuntz semigroups

Compare also with W-vN equivalence ??

Let A a C^*-algebra. Recall that $a \precsim b$ (with more precise notation $a \precsim_A b$) for $a, b \in A$ is defined by the existence of sequences of elements $d_n, e_n \in A$ with $a = \lim_n d_n b e_n$.

In particular, always $a \precsim a^*a \precsim aa^* \precsim a^*$.

The relation depends from A and passes in general not to C^*-subalgebras: Let $a, b \in A_+$ non-zero with $ab = 0$ and $a \precsim_A b$ define $D \subseteq A$ as $D := \overline{aAa + bAb}$ then $a \not\precsim b$ in D.

Notice that for $A \neq 0$, ?????

It follows that $a, b \in (A \otimes \mathbb{K})_+$ then $a \precsim b$ is equivalent to the existence of $c_1, c_2, \ldots \in A \otimes \mathbb{K}$ with $a = \lim_n c_n^* b c_n$. Moreover, if $a \neq 0$, the c_n can be replaced here by $(b-\delta_n)_+^{1/k_n} c_n (a-\delta_n)_+^{1/k_n}$ with suitable choices $k_n \in \mathbb{N}, < \delta_n < \min(\|a\|, \|b\|)$ with $k_n \to \infty$ and $\delta_n \to 0$, depending on the norms $\|c_n\|$, $\|a\|$ and $\|b\|$ in an "universal" way. (The explicit formula should be an exercise.) Thus, if a, b are in the Pedersen ideal $P(A)$ of A – which is by definition the minimal dense ideal of A denoted by $P(A)$ – then $a, b \in P(A)$ and $a \approx_A b$ implies $a \approx_{P(A)} b$. Recall also that $a \approx b$ means $a \precsim b$ and $b \precsim a$.

LEMMA A.6.1. *The relation $a \precsim_A b$ has following properties:*

(i) *The relation $\precsim$ is transitive:*

Always $a \precsim a$ in $C^*(a) \subseteq A$ itself. The relations $a \precsim_A b$ and $b \precsim_A c$ imply $a \precsim_A c$.

The relation $a \approx b$, i.e., $a \precsim b$ and $b \precsim a$ together, is an equivalence relation on A, and it satisfies that $a \precsim b$, $a \approx c$ and $b \approx d$ together imply that $c \precsim d$.

(ii) *The relations $\approx$ and $\precsim$ are compatible with $\oplus$, e.g. $a \oplus c \precsim b \oplus d$ in $M_2(A)$ if $a \precsim b$ and $c \precsim d$ in A. If, moreover, $b, d \in A$ are "orthogonal" in the sense that $bd^* = 0$ and $b^*d = 0$, then $a \precsim b$ and $c \precsim d$ imply $a \oplus c \precsim (b + d) \oplus 0 \approx b \oplus d$ in $M_2(A)$.*

(iii) *The relation $\precsim$ is compatible with *-morphisms: If $\psi \colon A \to B$ is a *-morphism, and $a \precsim_A b$ in A, then $\psi(a) \precsim_B \psi(b)$ in B.*

(iv) *The set $\xi(b)$ of $a \in A$ with $a \precsim b$ is* closed *in A and satisfies $c\xi(b)d \subseteq \xi(b)$ for $c, d \in \mathcal{M}(A)$.*

More precisely, $\xi(b)$ is the closure in A of the set *of all elements $cbd \in A$ with $c, d \in \mathcal{M}(A)$.*

(v) *It holds $aa^* \approx a^*a \approx a^* \approx a$, and $g(a^*a) \precsim a$ for all continuous functions g on $\mathrm{Spec}(a^*a)$ with $g(0) = 0$.*

If, in addition, $g(t) > 0$ for $t > 0$, then $g(a^*a) \approx a$.

(vi) *If $a, b \in A_+$, then $a \precsim b$, if and only if, for every $\varepsilon > 0$, there are $\delta = \delta(\varepsilon) > 0$ and $d = d(\varepsilon) \in A_+$ such that $d^*(b - \delta)_+ d = (a - \varepsilon)_+$. In particular, $a \precsim b$ if $a \sim_{MvN} c$ for some $0 \leq c \leq b$.*

If p and q are projections in A, then $p \precsim q$, if and only if, there is a projection $r \in A$ with $p \sim r \leq q$.

(vii) *If $a \in A_+$ then $a \precsim b$ if and only if $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.*

(viii) *If D is a hereditary C^*-subalgebra of A, and $a, b \in D \subseteq A$, then $a \precsim_D b$ in D, if and only if $a \precsim_A b$ in A.*

Particular cases are:

Always $a \precsim_D b$ for $a \in D := \overline{b^*Ab}$ and $b \in A$.

$x \oplus 0_n \precsim_B y \oplus 0_n$ in $B := M_{n+1}(A)$, if and only if, $a \precsim_A y$ (i.e., in A itself).

If $a, b \in A$, then $a \precsim_A b$ in A, if and only if, $a \precsim_{\mathcal{M}(A)} b$ in the multiplier algebra $\mathcal{M}(A)$.

(ix) *$(x \oplus y) \approx (y \oplus x)$ in $M_2(A)$ and $((x \oplus y) \oplus z) \approx (x \oplus (y \oplus z))$ in $M_3(A)$ for all $x, y, z \in A$.*

(x) *$(x + y) \oplus 0 \precsim x \oplus y$ in $M_2(A)$ for all $x, y \in A$.*

(xi) *If $a, b \in A_+$, $\pi_J(a) \precsim \pi_J(b)$ for $J \lhd A$, then, there is $c \in J_+$ with $a \precsim b \oplus c$.*

In particular, there exists $c \in J_+ \cap \overline{bAb}$ such that $(b \oplus b) \precsim (b \oplus c)$ if $\pi_J(b) \oplus \pi_J(b) \precsim \pi_J(b)$.

(xii) *Suppose that $a \in A_+$ and that there is $b \in \mathrm{Ann}(a, A)_+$ such that $a \in \overline{\mathrm{span}(AbA)}$.*

Then, for each closed ideal $J \triangleleft A$, each $\delta > 0$, each $c \in J_+$, and every $\varepsilon > 0$ there is $d = d(\varepsilon) \in J_+ \cap \mathrm{Ann}((a-\delta)_+, A)$ and $n = n(\delta, \varepsilon) \in \mathbb{N}$ such that

$$(a-\delta)_+ + (c-\varepsilon)_+ \precsim a \oplus (d \otimes 1_n)\,.$$

(xiii) *Parts (xiii) and (xiv) should be separated*

Suppose that A is an AW-algebra, $p_n, q \in A$ projections with $p_1 \leq p_2 \leq \cdots$ and with $p_n \precsim q$ for each $n \in \mathbb{N}$. Then $p \precsim q$ for $p := \sup_n p_n$.*

(xiv) *If A is an AW*-algebra, and if $a, b \in A_+$ with $a \precsim b$, then $p_a \precsim p_b$ for the support projections $p_a, p_b \in A$ of a and b, and $az \precsim bz$ for each $z \in A_+$ in the center of A.*

(xv) *Suppose that $a_n, b_n \in A$, $a = \lim a_n$, $b = \lim b_n$. If there is a continuous function $\psi \in \mathrm{C}[0,1]_+$ with $\psi(\delta) > 0$ for $\delta > 0$, such that $(a_n^* a_n - \delta)_+ \precsim (b_n^* b_n - \psi(\delta))_+$ for all $n \in \mathbb{N}$ and all rational $\delta > 0$. Then $a \precsim b$.*

(xvi) *Suppose that $C \subseteq B \subseteq A$ are C*-subalgebras of A, and that $S \subseteq C$ is a set of positive contractions with the property that, for each $c_1, c_2 \in S$ and each positive rational numbers r_1, r_2 with property $(c_1 - r_1)_+ \precsim_A (c_2 - r_2)_+$ in A holds also $(c_1 - r_1)_+ \precsim_B (c_2 - r_2)_+$ in B.*

If $S \subseteq C \subseteq B$ satisfy the following condition (D), then for all $a, b \in C$,

$$a \precsim_A b \quad \textit{if and only if} \quad a \precsim_B b\,.$$

(D) *For each positive contraction $a \in C_+$ and $\delta > 0$, there exists $c \in S$ and rational $r > 0$ with $(a-\delta)_+ \precsim_B (c-r)_+ \precsim_B a$.*

Condition (D) *is satisfied, if S is dense in the set of contractions in C_+.*

(xvii) *Let $a, b \in A$ and $B := A_\omega$ or $B := \mathrm{C_b}(X, A)/\,\mathrm{C_0}(X, A)$ with any locally compact Hausdorff space X, and $A \subseteq B$ the natural embedding. Then $a \precsim_A b$, if and only if, $a \precsim_B b$.*

Proof. The properties (i) and (iii) come straight form the definition.

(ii): $a \oplus c = \lim_n (e_n \oplus g_n)(b \oplus d)(f_n \oplus h_n)$ if $a = \lim_n e_n b f_n$ and $c = \lim_n g_n d h_n$.

If (moreover) $b, d \in A$ satisfy $bd^* = 0$ and $b^*d = 0$, then $(b+d) \oplus 0 \approx b \oplus d$ in $M_2(A)$ by part (v), because $(b+d)(b+d)^* \oplus 0 = [b,d][b,d]^*$ and $[b,d]^*[b,d] = (b \oplus d)^*(b \oplus d)$ for the row matrix $[b,d] \in M_{1,2}(A)$.

(iv): If $a = \lim_n f_n b e_n$ then $cad = \lim_n (cf_n) b (e_n d)$. Thus, $cad \in \xi(b)$ if $a \in \xi(b)$ for all $c, d \in \mathcal{M}(A)$.

Let $a := \lim a_n$ with $a_n \precsim b$ for each $n \in \mathbb{N}$, and let $\varepsilon > 0$. Put $\delta := \varepsilon/2$. There are $n = n(\delta) \in \mathbb{N}$ with $\|a - a_n\| < \delta$ and elements $f, g \in A$ with $\|a_n - fbg\| < \delta$. Thus $\|a - fbg\| < \varepsilon$ for suitable $f, g \in A$. It implies that $a \precsim b$. Thus, $a \in \xi(b)$ if $a \in \overline{\xi(b)}$.

The closed set $\xi(b)$ is the closure in A of the *set* of all elements $cbd \in A$ with $c, d \in \mathcal{M}(A)$, because $cbd \precsim b$, i.e., $cbd \in \xi(b)$, for all elements $c, d \in \mathcal{M}(A)$, and if $a \in \xi(b)$, i.e., if $a \precsim b$, then the there exist sequences (c_n) (d_n) in $A \subseteq \mathcal{M}(A)$ such that $a = \lim_n c_n b d_n$.

(v): The relations $g(b) \precsim_A b$ for $b \in A_+$ and continuous $g\colon \mathrm{Spec}(b) \to \mathbb{R}_+$ with $g(0) = 0$ can be seen in $C^*(b) \cong \mathrm{C}_0(\mathrm{Spec}(b) \setminus \{0\})$, using later part (iii). In this way one can also see that $g(b) \approx b$ if $g(t) > 0$ for all $0 < t \in \mathrm{Spec}(b)$.

In particular, $b \approx b^2$, thus $a(a^*a)a^* = (aa^*)^2 \approx aa^*$. It implies $aa^* \precsim a^*a$. If we replace a by a^*, then we get $aa^* \approx aa^*$.

Hence $b \approx c$ if $b, c \in A_+$ and $b \sim_{\mathrm{MvN}} c$.

The definition of $\precsim$ shows that $a \precsim b$ implies $a^* \precsim b^*$. Thus, $a \approx b$ if and only if $a^* \approx b^*$.

We have $a^*a = \lim_n c_n a d_n$ with $c_n = a^*$ and $d_n = (a^*a)^{1/n}$. On the other hand, $a = \lim_n e_n a^*a f_n$ for $e_n = a$, $f_n := g_n(a^*a)$, where $g_n(t) = (t + 1/n)^{-1} t^{1/n}$. It follows $a \approx a^*a$, and then $a^* \approx (a^*a)^* = a^*a$.

(vi): If $a, b \in A_+$ and $a \precsim b$, then $a \approx a^{1/2} \precsim b^{1/2} \approx b$ by parts (i) and (v). Thus, there are sequences $c_n, d_n \in A$ with $\lim_n c_n b^{1/2} d_n = a^{1/2}$. Let $\varepsilon > 0$. There is $n \in \mathbb{N}$ with $\|c_n b^{1/2} d_n d_n^* b^{1/2} c_n^* - a\| < \varepsilon/3$.

Thus, $\varepsilon/3 + \|d_n\|^2 c_n b (c_n)^* \geq a$, and Lemma 2.1.9 gives the existence of a contraction $e \in A$ with $f^*bf = e^*b_1e = (a - \varepsilon/2)_+$ for $b_1 := \|d_n\|^2 c_n b (c_n)^*$ and $f := \|d_n\| c_n^* e$. It follows $\|f^*(b-\delta)_+ f - a\| < \varepsilon$ for $\delta = \varepsilon/(4(\|f\|^2 + 1)) > 0$. Now again Lemma 2.1.9 applies and gives the existence of $g \in A$ with $d^*(b-\delta)_+ d = g^*(f^*(b-\delta)_+ f)g = (a-\varepsilon)_+$ where $d := fg$.

Conversely, if there are $\delta = \delta(\varepsilon) > 0$ and $d = d_\varepsilon \in A_+$ such that $d^*(b-\delta)_+ d = (a-\varepsilon)_+$ for every $\varepsilon > 0$, then $(a-\varepsilon)_+ \precsim (b-\delta)_+$. Since $(b-\delta)_+ \precsim b$ by part (v), then $(a - 1/n)_+ \precsim b$ for all $n \in \mathbb{N}$. It implies $a \precsim b$ by part (iv).

By part (v), $a \sim c$ implies $a \approx c$. If $0 \leq c \leq b$, then there are contractions $d_n \in A$ with $d_n^* b d_n = (c - 1/n)_+$ by Lemma 2.1.9. Thus $c \approx a \precsim b$.

It follows, that the existence of $r \leq p = p^* = p^2$ with $r \sim p$ implies $p \precsim q$, if p and q are projections in A.

If $p \precsim q$ then there are $\delta > 0$ and $d \in A$ with $(1-\delta)d^*qd = d^*(q-\delta)_+ d = (p - 1/2)_+ = (1/2)p$. It follows that $z := (2(1-\delta))^{1/2} qd$ satisfies $z^*z = p$ and $r := zz^* \leq q$.

(vii): It follows from parts (v) and (iv), because

$$(a-\varepsilon)_+ \precsim a \quad \text{and} \quad a = \lim (a - (1/n))_+ \,.$$

(viii): Use an approximate unit of D.

(ix); Part (viii) allows to use suitable (unitary) permutation matrices in $M_2(\mathcal{M}(A))$ respectively $M_3(\mathcal{M}(A))$.

(x): $d^*(x \oplus y)d = (x+y) \oplus 0$ for $d \in M_2 \subseteq M_2(\mathcal{M}(A))$ with matrix entries $d_{11} = d_{21} = 1$, $d_{12} = d_{22} = 0$.

(xi): Let $a, b \in A_+$, $\pi_J(a) \precsim \pi_J(b)$ and $\varepsilon > 0$. Then Part (vi) shows that there are $\delta = \delta(\varepsilon) > 0$ and $d = d(\varepsilon) \in A$, such that $x_\varepsilon := d^*(b-\delta)_+ d - (a-\varepsilon)_+ \in J$. It follows $(a-\varepsilon)_+ \leq b \oplus |x_\varepsilon|$. If we take here $\varepsilon \in \{1/n\,;\ n \in \mathbb{N}\}$ and let $c :=$

$\sum_n 2^{-n}(1+\|x_{1/n}\|)^{-1}|x_{1/n}|$, then $c \in J_+$, $\|c\| \leq 1$, $|x_{1/n}| \precsim c$ and $(a-1/n)_+ \precsim b \oplus c$ for all $n \in \mathbb{N}$. Thus $a \precsim (b \oplus c)$ by (vii).

(xii): Let $a \in A_+$ and $b \in \mathrm{Ann}(a, A)_+ := \{c \in A\,;\ ca = 0 = ac\}$ such that $a \in \overline{\mathrm{span}(AbA)}$. Let $\delta > 0$ and let K denote the closed ideal, that is generated by $\mathrm{Ann}((a-\delta)_+, A)$. Then $b \in \mathrm{Ann}(a, A) \subset \mathrm{Ann}((a-\delta)_+, A)$. It follows that $a \in K$. On the other hand, $\pi_K(a)$ is invertible in A/K, because K contains $(1 - f_\delta(a))a$, i.e., $\mathrm{Spec}(\pi_K(a)) \subseteq [\delta, \|a\|]$. Thus $K = A$, which means that $D := \mathrm{Ann}((a-\delta)_+, A)$ is a full hereditary C^*-subalgebra of A. Then $J \cap D$ is full in J for each $J \triangleleft A$.

Thus, for every $\varepsilon > 0$ and $c \in J_+$, there are $e_1, \ldots e_n \in (J \cap D)_+$ and $f_1, \ldots, f_n \in J$ such that $(c-\varepsilon)_+ = \sum_k f_k^* e_k f_k$. Thus $(c-\varepsilon)_+ \precsim d \otimes 1_n$ for $d := \sum_k e_k$. Now $(a-\delta)_+ + c \precsim a \oplus (d \otimes 1_n)$ follows from Parts (ii), (vi) and (x).

`Put (xiii) and (xiv) on extra place` (xiii): There is a projection $y \in qAq$ such that y is central in qAq, $y = yq$ is finite (or zero) and $(q-y)$ is properly infinite (or zero). There exist a central projection z of A such that $zq = y$. Notice that the projection $p := \sup_n p_n \in A_+$ exists, because A is an AW*-algebra. If $a \in A$ is a contraction, then the left and right support projections $p_{aa^*} := \sup_n (aa^*)^{1/n}$ and $p_{a^*a} := \sup_n (a^*a)^{1/n}$ of a are Murray–von-Neumann equivalent in A by the partial isometry $v \in A$ of the polar decomposition $a = v(a^*a)^{1/2}$ of a in A, cf. [**64**].

By Part (vi), there exist $q_n \leq q$ such that $p_n \sim q_n$ for $n \in \mathbb{N}$. It implies $zp_n \sim zq_n \subseteq zq = y$ and $(1-z)p_n \sim (1-z)q_n \subseteq (1-z)q = q-y$.

If $q \neq y$ then $q - y$ is properly infinite, i.e., there are $s, t \in A$ with $s^*s = t^*t = q - y$ and $s^*t = 0$ and $ss^* + tt^* \leq q - y$. Then there exist a sequence $s_1, s_2, \ldots \in (q-y)A(q-y)$ with $s_m^* s_n = \delta_{m,n}(q-y)$. Let $w_n \in A$ with $(1-z)p_n = w_n^* w_n$ and $w_n(w_n)^* \leq q - y$. Now we define

$$a := s_1 w_1 + \sum_{n=1} 2^{-n} s_{n+1} w_{n+1} (1-z)(p_{n+1} - p_n)\,.$$

Then $\|a\| \leq 1$, $(1-z)p = \sup_n (a^*a)^{1/n}$ and $aa^* \leq (q-y)$. Thus $p_{aa^*} = \sup_n (aa^*)^{1/n} \leq (1-z)q = q - y$ and $(1-z)p = p_{a^*a} \sim p_{aa^*} \leq (1-z)q\,.$

The AW*-algebra yAy is finite. Let v_0 a partial isometry with $v_0^* v_0 = zp_1$ and $v_0(v_0)^* \leq y = zq$. By induction, we find partial isometries $v_2, v_3, \ldots \in A$ with $v_n^* v_n = z(p_{n+1} - p_n)$, $v_n(v_n)^* \leq zq = y$ and $v_n^* v_k = 0$ for $k < n$. Indeed:
The partial isometry $w_n := v_0 + \cdots + v_{n-1}$ satisfies $w_n^* w_n = zp_n$ and $w_n(w_n)^* \leq y$. By Part (vi), there is a partial isometry $x \in A$ with $x^*x = zp_{n+1}$ and $xx^* \leq y$, because $zp_{n+1} \precsim y$. Then $xzp_n x^* \sim w_n(w_n)^*$ in the finite AW*-algebra yAy. We find a unitary $u \in yAy$ with $uxzp_n x^* u^* = (w_n w_n)^*$, because Murray–von-Neumann equivalent projections in finite AW*-algebras are unitarily equivalent, cf. [**64**, chap. 6]. Let $v_n := uxz(p_{n+1} - p_n)$, then $v_n^* w_n = 0$ and $v_n^* v_n = z(p_{n+1} - p)$ and $v_n v_n^* \leq zq$.

Now let $b := \sum 2^{-n} v_n$. Then $\|b\| \leq 1$, $bb^* \leq zq$, $zp = \sup_n (b^*b)^{1/n}$. Thus $zp = p_{b^*b} \sim p_{bb^*} \leq zq$.

It follows $p = p_{a^*a} + p_{b^*b} \sim p_{aa^*} + p_{bb^*} \leq q\,.$

(xiv): If $z \in A_+$ is in the center of A, and if (d_n), $(e_n) \subseteq A$ satisfy $\lim e_n b d_n = a$, then $\lim e_n(zb)d_n = za$, i.e., $a \precsim b$ implies $za \precsim zb$.

Let $a, b \in A_+$ with $a \precsim b$, then $(1/n)p_n \leq a \precsim b \leq \|b\| q$, $p_1 \leq p_2 \leq \cdots$ and $p = \sup_n p_n$ for the support projections $p := p_a$, $p_n := p_{a_n}$ and $q := p_b$ of a, $a_n := (a - (1/n))_+$ and b respectively. In particular, $p_n \precsim q$ for all $n \in \mathbb{N}$. Thus, Part (xiii) implies $p_a \precsim p_b$.

(xv): Suppose that $a = \lim a_n$, $b = \lim b_n$, and that ψ is a continuous function $\psi(\delta) > 0$ such that $(a_n^* a_n - \delta)_+ \precsim (b_n^* b_n - \psi(\delta))_+$ for all $n \in \mathbb{N}$ and all rational $\delta > 0$. Let $\varepsilon > 0$, and $\delta \in (0, \varepsilon/2)$ rational. There is $n_0 \in \mathbb{N}$ with $\|a^*a - a_n^* a_n\| < \delta$ and $\|b^*b - b_n^* b_n\| < \psi(\delta)/2$ for all $n \geq n_0$. This implies $a^*a \leq (a_n^* a_n - \delta)_+ + 2\delta$ and $b_n^* b_n \leq b^*b + \psi(\delta)/2$. By Lemma 2.1.9, we find $d_1, d_2 \in A$ with $(a^*a - \varepsilon)_+ = d_1^*(a_n^* a_n - \delta)_+ d_1$ and $(b_n^* b_n - \psi(\delta))_+ = d_2^* b^* b d_2$. It follows $(a^*a - \varepsilon)_+ \precsim b^*b$ (for all $\varepsilon < 0$), and (finally) $a \approx a^*a \precsim b^*b \approx b$ by Parts (i), (vii) and (v).

(xvi): Let $a_1, b_1 \in C \setminus \{0\}$ with $a_1 \precsim_A b_1$, then $a \approx_C a_1$, $b \approx_C b_1$ and $a \precsim_A b$ for the contractions $a := \|a_1\|^{-2} a_1^* a_1 \in C_+$ and $b := \|b_1\|^{-2} b_1^* b_1 \in C_+$. By Part (vi) there exist $0 < \delta_n \leq \varepsilon_n < 1/4$ with $\lim_n \varepsilon_n = 0$ such that $(a - \varepsilon_n)_+ \precsim_A (b - \delta_n)_+$. For each $n \in \mathbb{N}$ there are $c_n, d_n \in C_+$ and rational $r_n, s_n > 0$ with $(a - 2\varepsilon_n)_+ \precsim_B (c_n - r_n)_+ \precsim_B (a - \varepsilon_n)_+$ and $(b - \delta_n)_+ \precsim_B (d_n - s_n)_+ \precsim_B b$.

It follows $(c_n - r_n)_+ \precsim_A (d_n - s_n)_+$. Then $(c_n - r_n)_+ \precsim_B (d_n - s_n)_+$ by assumptions on S. It yields $(a - 2\varepsilon_n)_+ \precsim_B b$ for all $n \in \mathbb{N}$, by Part (i). Thus, $a \precsim_B b$ and $a_1 \precsim_B b_1$ by Parts (vii) and (i).

Condition *(D)* is satisfied, if S is dense in the set of contractions in C_+, because $\|c - a\| < \delta/3$ implies the existence of contractions $d_1, d_2 \in C$ and rational $r \in (\delta/3, \delta/2]$ with $(a - \delta)_+ = d_1^*(c - r)_+ d_1$ and $(c - r)_+ = d_2^* a d_2$, by Lemma 2.1.9.

proofs complete ??

(xvii): Let $a, b \in A$ and $B := A_\omega$ or $B := \mathrm{C_b}(X, A)/\mathrm{C_0}(X, A)$, and consider the natural embedding $a \mapsto \pi_\omega(a, a, \ldots) \in A_\omega$, respectively $a \mapsto f_a + \mathrm{C_0}(X, A)$ for $f_a(x) := a$. Then $a \precsim_A b$ implies $a \precsim_B b$ by Part (iii). We use $a \approx a^*a$ for the opposite direction, cf. Part (v). Let $a, b \in A_+$ with $a \precsim_B b$ and $\delta > 0$, then Part (v) says that there is $d = (d_1, d_2, \ldots) \in \ell_\infty(A)$ with $\lim_n d_n^* b d_n = (a - \delta)_+$ (respectively $d \in \mathrm{C_b}(X, A)$ with $g \in \mathrm{C_0}(X, A)$ for $g(x) := d(x)^* b d(x) - (a - \delta)_+$). Thus $(a - \delta)_+ \precsim_A b$ in both cases (since X is not compact). It follows $a \precsim_A b$ by Part (vii). □

We call a C^*-subalgebra B of A **$\precsim$-preserving** if $a, b \in B$ and $a \precsim_A b$ imply that $a \precsim_B b$.

Lemma A.6.2. *Let $C \subseteq A$ a separable C^*-subalgebra and $B_1 \subseteq B_2 \subseteq \cdots \subseteq A$ a sequence of $\precsim$-preserving C^*-subalgebras of a C^*-algebra A.*

(o) *Each hereditary C^*-subalgebra $D \subseteq A$ is $\precsim$-preserving.*
(i) *The closure B of $\bigcup_n B_n$ is $\precsim$-preserving.*

(ii) *There exists a separable $\precsim$-preserving C^*-subalgebra B of A with $C \subseteq B \subseteq A$.*

(iii) *Consider a C^*-algebra E naturally as a C^*-sub-algebra of $A := E_\omega$, (respectively of $A := \mathrm{C_b}(X, E)/\mathrm{C_0}(X, E)$ for a locally compact Hausdorff space X). Then E is a $\precsim$-preserving C^*-subalgebra of A.*

PROOF. (o): See Lemma A.6.1(viii).

(i): We prove a stronger result: Suppose that $a \precsim_A b$ and $a, b \in B_n$ imply $a \precsim_{B_{n+1}} b$. Then, obviously, the relation $a \precsim_A b$ for $a, b \in \bigcup_n B_n$ implies $a \precsim_B b$. Let $C := B$ and let $S \subseteq C_+$ denote the set of positive contractions in $\bigcup_n B_n$. Then S, C, B and A satisfy the assumptions of Lemma A.6.1(xvi). Thus, for all $a, b \in B$, $a \precsim_A b$ if and only if $a \precsim_B b$.

(ii): Since C is separable, there exists a countable dense subset S in the set of positive contractions in C_+.

If $c, d \in S$ and $r, t \in [0, 1]$ are rational with $(c - r)_+ \precsim_A (d - t)_+$, then there is a sequence $d_1, d_2, \ldots \in A$ with $(c - r)_+ = \lim_n d_n^*(d - t)_+ d_n$.

The set $S \times S \times ([0, 1] \cap \mathbb{Q}) \times ([0, 1] \cap \mathbb{Q})$ is at most countable. It follows, that there is a separable C^*-subalgebra $C \subseteq B_1 \subseteq A$ such that $(c - r)_+ \precsim_A (d - t)_+$ implies $(c - r)_+ \precsim_{B_1} (d - t)_+$ for $c, d \in S$ and $r, t \in [0, 1]$. By Lemma A.6.1(xvi), for each $a, b \in C$, $a \precsim_A b$ is equivalent to $a \precsim_{B_1} b$.

We can repeat this argument for B_1 in place of C, and get a separable C^*-subalgebra $B_1 \subseteq B_2 \subseteq A$ such that, for each $a, b \in B_1$, $a \precsim_A b$ is equivalent to $a \precsim_{B_2} b$. Going on this way, we can find a sequence $B_1, B_2, \ldots$ of separable C^*-subalgebras $B_n \subseteq B_{n+1} \subseteq A$ such that $B_0 := C \subseteq B_1$ and, for each $a, b \in B_n$, $a \precsim_A b$ is equivalent to $a \precsim_{B_{n+1}} b$. Then the closure B of $\bigcup_n B_n$ is a $\precsim$-preserving separable C^*-subalgebra of A that contains C.

(iii): The cases $E \subseteq E_\omega$ and $E \subseteq \mathrm{C_b}(X, E)/\mathrm{C_0}(X, E)$ follow from Lemma A.6.1(xvii). □

REMARK A.6.3. Suppose that $B \subseteq A$ is a C^*-subalgebra. Let $a \in A_+$, $b \in B_+$ and let $J_A(b) := \overline{\mathrm{span}(AbA)}$ (respectively $J_B(b) := \overline{\mathrm{span}(BbB)}$) the closed ideal of A (respectively of B) generated by b.

Notation $I(a)$ was in conflict with absorption ideal

$$I_A(a) = \{b \in A\,;\ [b] + [a] \leq [a]\}$$

The following observation (o) is trivial, and the implications (o)⇒(i)⇒(ii) are straight-forward:

(o) The smallest number $t \geq 0$ with $(a - t)_+ \in J_A(b)$ is given by

$$\mathrm{dist}(a, J_A(b)) = \|a + J_A(b)\| = \|\pi_{J_A(b)}(a)\|\,.$$

(i) For each $\delta > 0$ there exists $n = n(\delta) \in \mathbb{N}$ such that $(a - (t+\delta))_+ \precsim b \otimes 1_n$, if and only if, $t \geq \operatorname{dist}(a, J_A(b))$.

(ii) $\operatorname{dist}(a, J_A(b)) = \operatorname{dist}(a, J_B(b))$ for all $b \in B_+$ if $B \otimes \mathbb{K} \subseteq A \otimes \mathbb{K}$ is $\precsim$-preserving.

Let $B \subseteq A$ a C^*-sub-algebra of A and $a, b \in A$ positive contractions, and let $d_1, d_2 \in A$ contractions. We define

$$\rho(a, b;\, d_1, d_2) := \|a^2 - d_1^* a^2 d_1\| + \|b^2 - d_2^* b^2 d_2\| + \|d_1^* a b d_2\|\,,$$

and denote by $\rho(a, b; B)$ the infimum of the non-negative numbers

$$\{\rho(a, b; d_1, d_2)\,;\; d_1, d_2 \in B,\, \|d_1\| \leq 1,\, \|d_2\| \leq 1,\,\}\,.$$

The following Lemma establishes the continuity of the property s.p.i. in the category of C^*-algebras with respect to inductive limits.

LEMMA A.6.4. *Let $C \subseteq A$ and $B_1 \subseteq B_2 \subseteq \ldots \subseteq A$ separable C^*-subalgebras. Let B denote the closure $\bigcup_n B_n$. Then:*

(i) *For contractions $a, b, a', b' \in A_+$ holds*

$$|\,\rho(a', b'; A) - \rho(a, b; A)\,| \leq 5\|a' - a\| \,+\, 5\|b' - b\|$$

and $\rho(a, b; C) \geq \rho(a, b; A)$ if $a, b \in C$.

(ii) *$\rho(a, b; A) = \rho(a, b; A_\omega)$ for all contractions $a, b \in A_+$.*

(iii) *If $\rho(a, b; B_{n+1}) = \rho(a, b; A)$ for all positive contractions $a, b \in B_n$ for $n = 1, 2, \ldots,$ then $\rho(a, b; B) = \rho(a, b; A)$ for all positive contractions $a, b \in B$.*

(iv) *There exists a separable C^*-subalgebra B of A with $C \subseteq B \subseteq A$, such that $\rho(a, b; B) = \rho(a, b; A)$ for all contractions $a, b \in B_+$.*

PROOF. (i): The inequality $\rho(a, b; C) \geq \rho(a, b; A)$ is immediate from the definition. Straight calculation shows

$$|\,\rho(a', b'; d_1, d_2) - \rho(a, b; d_1, d_2)\,| \leq 5\|a' - a\| \,+\, 5\|b' - b\|\,.$$

(ii): By Part (i), $\rho(a, b; A) \geq \rho(a, b; A_\omega)$. Here we have identified $a \in A$ with $\pi_\omega(a, a, \ldots) = \Delta(a) + c_\omega(A)$ in A_ω.

Let $a, b \in A_+$ contractions and $\delta > 0$. There are sequences of contraction $e_1, e_2, \ldots; f_1, f_2, \ldots \in A$ such that, for $e := \pi_\omega(e_1, e_2, \ldots)$ and $f := \pi_\omega(f_1, f_2, \ldots)$,

$$\rho(a, b; A_\omega) + \delta > \rho(a, b; e, f) = \lim_\omega \rho(a, b; e_n, f_n) \geq \inf_n \rho(a, b; e_n, f_n)\,.$$

Clearly, $\inf_n \rho(a, b; e_n, f_n) \,\geq\, \rho(a, b; A)\,.$

(iii): By assumptions and Part (i), we have $\rho(a, b; B) = \rho(a, b; A)$ for all positive contractions $a, b \in \bigcup_n B_n$. It implies $\rho(a, b; B) = \rho(a, b; A)$ for all positive contractions $a, b \in B$, because $(a, b) \mapsto \rho(a, b; B) - \rho(a, b; A)$ is continuous.

(iv): Let $X \subseteq C_+$ a countable subset of contractions, that is dense in the set of all positive contractions in C. For each $a, b \in X$ there are sequences of contractions

$e_1, e_2, \ldots ; f_1, f_2, \ldots \in A$, such that $\lim_n \rho(a,b;e_n,f_n) = \rho(a,b;A)$. Thus, there is a countable subset $X \subseteq Y \subseteq A$ of the contractions in A, such that

$$\inf\{\, \rho(a,b;d_1,d_2)\,;\; d_1,d_2 \in Y \,\} = \rho(a,b;A) \quad \forall\, a,b \in X\,.$$

Let $C \subseteq B_1 \subseteq A$ denote the separable C^*-subalgebra that is generated by Y. Then $\rho(a,b;B_1) = \rho(a,b;A)$ for all $a,b \in X$. This is true also for all positive contractions $a,b \in C$, because $(a,b) \mapsto \rho(a,b;B) - \rho(a,b;A)$ is continuous.

Now we can repeat the argument with B_1 in place of C, and get a separable C^*-subalgebra $B_1 \subseteq B_2 \subseteq A$ with $\rho(a,b;B_2) = \rho(a,b;A)$ for all $a,b \in B_1$. Thus, we find a sequence $B_1 \subseteq B_2 \subseteq \cdots \subseteq A$ that satisfies the assumptions of Part (iii). Then the closure B of $\bigcup_n B_n$ is a separable C^*-subalgebra of A with $C \subseteq B$ and $\rho(a,b;B) = \rho(a,b;A)$ for all $a,b \in B$. □

LEMMA A.6.5. *Let M a W^*-algebra, $a \in M_+$ with $\|a\| = 1$, let $P_b \in M$ denote the support projection of $b \in M_+$, and let $d_1, \ldots, d_m \in M$ with $\sum d_j^* d_j \leq 1$.*

Suppose that the support projections of $z(a-t)_+$ and of $z(1-a-t)_+(a-s)_+$ are either zero or infinite for all $t,s \in (0,1)$ and for all central projections $z \in M$. Then:

(i) *The support projections of $(a-t)_+$ and $(1-a-t)_+ \cdot (a-s)_+$ are properly infinite or zero for every $t,s \in [0,1]$.*
(ii) *For each $\delta > 0$, there exists mutually orthogonal projections $p_1, \ldots, p_n \in \{a\}' \cap M$ and $0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq 1$ such that p_k is properly infinite in M for $k = 1, \ldots, n$, $\sum_k p_k = P_a$ and $(a-\delta)_+ \leq \sum_k \alpha_k p_k \leq a$.*
(iii) *For each $\delta > 0$ and $m \in \mathbb{N}$ there exists partial isometries $v_1, \ldots v_m \in P_a M P_a$ with $v_j^* v_k = \delta_{jk} P_a$ and $\|\, v_j\, a - a v_j \,\| < \delta$ for $j,k = 1, \ldots, m$.*
(iv) *For each $\varepsilon > 0$ there exist self-adjoint $h_k \in M$ with $\|h_k\| \leq \pi$ $(1 \leq k \leq 4)$ such that $h_2 = -h_1$, $h_4 = -h_3$ and*
`next ok ?? ??`

$$\Big(\big(\sum_j d_j^* a d_j\big) - \varepsilon\Big)_+ \;\leq\; \sum_{k=1}^{4} \exp(ih_k)\, a \exp(-ih_k)$$

PROOF. `check proof of (i) again` (i): Suppose that the support projections of $z(a-t)_+$ and of $z(1-a-t)_+(a-s)_+$ are either zero or infinite for all $t \in (0,1)$ and for all central projections $z \in M$.

Let $b \in M_+$ such that the support projection $P_b \in M$ of b is infinite. Then there exists a central projection $z \in M$ such that zP_b is finite (or is zero) and that $(1-z)P_b$ is properly infinite (or is zero). The support projection of zb is given by zP_b. Thus, if the support projection of zb is not finite, then $zP_b = 0$ and $P_b = (1-z)P_b$ is properly infinite.

Thus, all support projections $P_{(a-t)_+}$ and and $P_{(1-a-t)_+(a-s)_+}$ of the elements $(a-t)_+$ respectively $(1-a-t)_+(a-s)_+$ are properly infinite ore zero in M.

If follows that $(a-t)_+$ and $(1-a-t)_+(a-s)_+$ are zero or are properly infinite elements of M for each $s,t \in [0,1]$, by Lemma 2.5.3(xv).

(ii): By Lemma A.6.2(ii), there exists a separable C^*-subalgebra $B \subseteq M$ with $a \in B$, such that $c \precsim_B d$ if and only if $c \precsim_M d$ for all $c, d \in B$. Let $M_0 \subseteq M$ the $\sigma(M, M_*)$-closure (i.e., ultra-weak closure) of B in M.

Then M_0 is a direct sum $M_0 \cong \prod_\alpha N_\alpha$ of W*-algebras N_α *with separable predual*, and the elements $(a-t)_+$ and $(1-a-t)_+(a-s)_+$ are properly infinite or zero in M_0 for all $s,t \in [0,1]$. It follows, that the image a_α of $a = \{a_\alpha\}_\alpha \in \prod_\alpha N_\alpha$ have the properties that the elements $(a_\alpha - t)_+$ and $(1-a_\alpha-t)_+(a_\alpha-s)_+$ are properly infinite or zero.

Fix some of the α. Let $b := a_\alpha \in N_\alpha =: N$. Then $0 \leq b \leq 1$ and the elements $(b-t)_+$ and $(t-b)_+(b-s)_+$ are properly infinite for all $s,t \geq 0$. Let $E := P_b \in N$ the support projection of b in N. By 2.5.3(xv), the support projections $P(t) := P_{(b-t)_+}$ of $(b-t)_+$ and the support projections $Q(t) := P_{(t-b)_+ b}$ of $(t-b)_+ b$ are properly infinite projections in N (but, clearly, they can't be properly infinite in $\{b\}'' \subseteq N$, and they are not necessarily properly infinite in $\{b\}' \cap N$). It holds $Q(t), P(t) \in \{b\}''$, $Q(t)P(t) = 0$, $Q(s) \leq Q(t)$, $P(s) \geq P(t)$ for $s < t$ $Q(t) + P(t) \leq E$. The projection $R(t) := E - Q(t) - P(t)$ satisfies $bR(t) = tR(t)$ for each $t \in [0,1]$. In particular, $R(t)R(s) = 0$ for $s \neq t$. Since N has separable pre-dual, there is a faithful normal state ρ on N. It follows $\rho(R(t)) > 0$ if $R(t) \neq 0$, and that $\{t \in [0,1]\,;\, \rho(R(t)) \geq 1/n\}$ contains at most n points. It implies, that the set $S = \{t \in [0,1]\,;\, R(t) \neq 0\}$ is countable. We have $0 \notin S$, because $P(0) = E$.

If $s < t$ and $R(t) = 0$, then $P(s) - P(t) = P(s)Q(t)$. But $P(s)Q(t)$ is the support projection of $(t-b)_+(b-s)_+$, and $(t-b)_+(b-s)_+$ is zero or properly infinite properly infinite. Now, Lemma 2.5.3(xv) implies that the support projection $P(s) - P(t) = P(s)Q(t)$ of $(t-b)_+(b-s)_+$ is properly infinite if $P(s) \neq P(t)$ and $t \notin S$.

Since $b = \sup_+ tP(t)$ (in the lattice $(\{b\}'')_+$), we can approximate b from below in norm by $\sum t_k(P(t_k) - P(t_{k+1})) = \sup_k t_k P(t_k)$ with $0 = t_1 < t_2 < \ldots < t_n < 1$, $t_{k+1} \notin S$ and where $t_k \in ((k-1)/n, k/n]$.

It follows that each $R_k = P(t_k) - P(t_{k+1})$ (with $P(t_{n+1}) := 0 = P(1)$) is a properly infinite projection in N_α or is zero, $R_k \in \{b\}''_+$, $b - 1/n \leq \sum(k-1)/nR_k \sum t_k R_k \leq b$.

If we do this for each summand N_α of M_0, then we get projections $p_k = (R_k^\alpha)_\alpha \in M_0$, $k = 1, \ldots, n$, such that the p_k commute with a, are mutually orthogonal, are properly infinite or zero, have sum $\sum p_k = P_a$, and $a - (2/n) \leq \sum(k-1)/np_k \leq a$.

(iii): Let $\delta > 0$ and $m \in \mathbb{N}$. Let $n = [2/\delta] + 1$. We find $p_k \in \{a\} \cap M$, $k = 1, \ldots, n$, with the properties in (ii). It follows, that, for each $j \in \{1, \ldots, n\}$,

$$\|p_j T p_j b - b p_j T p_j\| \leq \|b - \sum_k (k-1)/n) p_k\| \|p_j T p_j\| \leq (2/n)\|p_j T p_j\|\,.$$

Since the projections p_k are properly infinite, we find $z_{\ell,j} \in p_j M p_j$, $\ell = 1, \ldots, m$ with $z_{\ell,j}^* z_{k,j} = \delta_{\ell,k} p_j$.

Let $v_\ell = \sum_j z_{\ell,j}$. Then $v_1, \ldots v_m \in P_a M P_a$ and $v_\ell^* v_k = \delta_{\ell,k} P_a$. Since the p_j commute with a and v_ℓ, we have that $v_\ell\, a - a\, v_\ell = \sum_j p_j (z_{\ell,j}\, a - a z_{\ell,j}) p_j$. Thus, the above estimate gives $\| v_\ell\, a - a\, v_\ell \| < \delta$ for $\ell, k = 1, \ldots, m$,

end above? more?

(vi): Given $d_1, \ldots, d_m \in M$ with $\sum d_j^* d_j \leq 1$, and $\varepsilon > 0$, we let $\delta := \varepsilon/(1 + \sqrt{m})$. We find the $v_1, \ldots, v_m$ with the properties of (iii), and define a contraction $d \in M$ by $\sum_{\ell=1}^m v_\ell d_\ell$. Then $d^* a d - \sum_\ell d_\ell^* a d_\ell = CBC^*$ for the row $C = [d_1, \ldots, d_m] \in M_{1,m}(M)$ and the matrix $B = [b_{\ell,k}] \in M_m(M)$ with entries

$$b_{\ell,k} = \delta_{\ell,k} a - v_\ell^* a v_k = v_\ell^* (v_k a - a v_k)\,.$$

Thus, $B = V^* D$ for the rows $V := [v_1, \ldots, v_m]$ and $D = [v_1 a - a v_1, \ldots, v_m a - a v_m]$ in $M_{1,m}(M)$. Since $\|C\| \leq 1$, $V^* V \leq P_a \otimes 1_n$, and $\|D\| \leq \delta\sqrt{m}$, it follows $\|CBC^*\| \leq \|B\| \leq \|D\| < \varepsilon$. Hence,

$$\sum_\ell d_\ell^* a d_\ell - \varepsilon \leq d^* a d\,.$$

Let $e := (d + d^*)/2$, $f := (d - d^*)/2i$, $h_1 := \arcsin(e)$, $h_2 := -h_1$, $h_3 := \arcsin(f)$ and $h_4 := -h_3$, then $\|e\| \leq 1$, $\|f\| \leq 1$, and $\|h_k\| \leq \pi$ for $k = 1, 2, 3, 4$.

$$\begin{aligned} d^* a d \leq d^* a d + d a d^* = 2(eae + faf) &\leq \\ 2(eae + faf) + 2(\cos(h_1) a \cos(h_1) + \cos(h_3) a \cos(h_3)) &= \\ 4 \sum_k \exp(ih_k) a \exp(-ih_k)\,. & \end{aligned}$$

□

Lemma A.6.6. *For every $a \in A$ and every unitary $u \in \mathcal{M}(A)$, the element $u^* a u$ is in the norm-closed convex hull of the set $\{\exp(ih) a \exp(-ih)\,;\ h^* = h \in A, \|h\| < \pi\}$.*

In particular, if $C \subseteq A_+$ is a hereditary closed convex sub-cone of A_+, then

$$K := \{a \in A_+\,;\ \exp(ih) a \exp(-ih) \in C \quad \textit{for all} \quad h^* = h \in A \quad \textit{with} \quad \|h\| < \pi\}$$

is the positive part $I_+ := K$ of a closed ideal I of A.

The closed ideal I is the largest ideal of A with $I_+ := I \cap A_+ \subseteq C$.

Proof. Recall that $\mathcal{M}(A) \subseteq A^{**}$. The set $\{\exp(ih)\,;\ h^* = h \in A, \|h\| < \pi\}$ is *-ultra-strongly dense in the unitaries of A^{**}, because $\exp\colon A^{**} \to A^{**}$ is *-ultra-strongly continuous on bounded parts, and because every unitary V of a W*-algebra is of the form $V = \exp(ih)$ for some self-adjoint $h \in A^{**}$ with $\|h\| \leq \pi$.

Use Hahn-Banach separation from the closed convex hull of

$$\{\exp(ih) a \exp(-ih)\,;\ h^* = h \in A, \|h\| < \pi\}\,.$$

If $\exp(ih)a\exp(-ih) \in C$ for all $h^* = h \in A$ with $\|h\| < \pi$ and if C is closed and convex, then $u^*au \in C$ for all unitaries $u \in \mathcal{M}(A)$, i.e.,

$$a \in K := \bigcap_{u\in\mathcal{U}(\mathcal{M}(A))} u^*Cu\,.$$

If C is a hereditary closed convex sub-cone of A_+, then $K := \bigcap_{u\in\mathcal{U}(\mathcal{M}(A))} u^*Cu$ is a hereditary closed convex sub-cone of A_+ that is invariant under "inner" automorphisms $\mathrm{Ad}(u)$ of A.

Thus, the hereditary C^*-subalgebra $I := \overline{KAK}$ of A is a closed ideal of A with $I_+ = K$.

If $J \lhd A$ is a closed ideal with $J_+ \subseteq C$, then $u^*au \in C$ for all $a \in J_+$ and $u = \exp(-ih)$ with $h^* = h \in A$. Thus, $J \subseteq I$. □

Lemma A.6.7. *Suppose that A is a C^*-algebra with the property that every* additive *lower semi-continuous trace $\tau\colon A_+ \to [0,\infty]$ takes only the values 0 and ∞.*

Let $\mathcal{S}_{in}$ denote the set of approximately inner completely positive contractions, and let $\mathcal{S}_u$ denote the point-norm closed convex hull of the (exponential) inner automorphisms

$$a \mapsto \exp(ih)a\exp(-ih)$$

with $h^* = h$ of norm $< \pi$.

Then, for each contraction $a \in A_+$, $V \in \mathcal{S}_{in}$ and $\varepsilon > 0$, there exists $T \in \mathcal{S}_u$ with

$$V(a) - \varepsilon \leq 4T(a)\,.$$

Proof. to be filled in ?? □

Lemma A.6.8. *Let $a \in A_+$, and suppose that there is $m \in \mathbb{N}$ such that, for every* contraction *$b \in A_+$ in the (algebraic) ideal generated by $\{a\}$ there are $d_1, \ldots, d_m \in A$ (depending on b) with $\sum_{k=1}^m d_k^* a d_k = (b - 1/2)_+$.*
We denote by $m(a)$ the smallest $m \in \mathbb{N}$ with this property.

Let $I(a)$ denote the closed ideal of A generated by $a \in A_+$. Then:

(i) *For every $c \in I(a)_+$ and $\varepsilon > 0$ there are $\delta > 0$ and $d_1, \ldots, d_{m(a)} \in aAc$ with $\sum_k d_k^*(a-\delta)_+ d_k = (c-\varepsilon)_+$ and $\|\sum_k d_k^* d_k\| \leq 2\delta^{-1}\|c\|$.*
check above estimate
(ii) *If $A_1, A_2, \ldots$ is a sequence of C^*-algebras, such that there is $\mu \in \mathbb{N}$ with $m(a) \leq \mu$ for each $n \in \mathbb{N}$ and $a \in (A_n)_+$ then $m(b) \leq \mu$ for every positive $b \in \prod_n A_n$.*
(iii) *If $m(a) \leq m_0$ for all $a \in A_+$, and if J is a closed ideal of A, then $m(b) \leq m_0$ for every positive $b \in (A/J)_+$.*
(iv) *Suppose that, for each $b \in (A_\omega)_+$ and $\varepsilon > 0$, there exists $m = m(b,\varepsilon) \in \mathbb{N}$ such that $(b-\varepsilon)_+ \otimes 1_{2m} \precsim b \otimes 1_m$.*
Then there exists $n \in \mathbb{N}$ such that $a \otimes 1_{2n} \precsim a \otimes 1_n$ for all $a \in A_+$.

PROOF. (i): Let $c \in I(a)_+$ and $\varepsilon > 0$. There are $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in A$ with $\|c - \sum_k f_k^* a f_k\| < \varepsilon/2$.

Question:

Are there $\delta > 0$ and $d_1, \ldots, d_m \in aAc$ with $\sum_k d_k^*(a-\delta)_+ d_k = (c-\varepsilon)_+$ and $\|\sum d_k^* d_k\| \leq 2\delta^{-1}\|c\|$?

(If it does not work in A^{**} then not in A!)

`check above estimates`

`to be filled in` ?? □

QUESTION A.6.9. If $0 \leq a \leq b$ and $\|b\| \leq 1$, then, for every $\varepsilon > 0$ there is a contraction $d \in C^*(a,b) \subseteq A$ with $d^*bd = (a-\varepsilon)_+$ and $\|db - bd\| \leq f(\|ab - ba\|, \varepsilon)$, where

$$f(t,\varepsilon) := \sup\{\mu(a,b;\varepsilon)\,;\ \ a,b \in \mathcal{L}(\ell_2)_+\,,\ 0 \leq a \leq b \leq 1\,,\ \|ab - ba\| \leq t\,\}$$

for $t > 0$ and

$$\mu(a,b;\varepsilon) := \inf\{\|db - bd\|\,;\ d \in C^*(a,b),\ \|d\| \leq 1,\ d^*bd = (a-\varepsilon)_+\,\}\,.$$

Is $\lim_{t\to 0} f(t,\varepsilon) = 0$ for each (fixed) $\varepsilon > 0$?

That is "in the limit" (i.e., in $\mathcal{L}(\ell_2)_\omega$) the case of commuting $0 \leq a \leq b$ and

$$d := \lim_{\delta\to 0}(b+\delta)^{-1/2}(a-\varepsilon)_+^{1/2}\,.$$

The question has to do with the study of images in $\mathrm{Q}(X,B)|\omega$ of the fibers $F(X;A,B)|\omega$ of the unital $\mathrm{C}(\gamma X)$-algebra

$$F(X\,;\,A,B) := (A' \cap \mathrm{Q}(X,B))/\operatorname{Ann}(A, \mathrm{Q}(X,B))$$

for separable C^*-subalgebra $A \subseteq \mathrm{Q}(X,B)$.

QUESTION A.6.10. Is there a universal continuous function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$, such that ρ is increasing and has the following property (*)?

(*) If $a, b \in A_+$, $0 \leq \gamma < \varepsilon$ satisfy $b \leq a + \gamma$, then there is a contraction $d := d(a,b;\gamma,\varepsilon) \in A$ with $d^*ad = (b-\varepsilon)_+$ and

$$\|da - ad\| \leq \rho\big(\sup_{\gamma\leq\delta\leq\varepsilon}\|a(b-\delta)_+ - (b-\delta)_+ a\|\big)\,.$$

What about using ultrapowers here?

7. Basics on Quasi-traces

`Is only collection/list of topics now`

DEFINITION A.7.1. `To be defined:`

Dimension function d, **dimension function**

Rank function = subadditive local rank function, local rank function r, quasi-trace τ, 2-quasi-trace bounded, unbounded, l.s.c. traces = additive q-traces, ...

For every compact Hausdorff space X of dimension ≥ 2 there exists a compact Hausdorff space Y of dimension $\leq \mathrm{Dim}(X)+1$, a continuous injective map $\varphi\colon X \to Y$ (with adjoint epimorphism $\widehat{\varphi}\colon \mathrm{C}(Y) \to \mathrm{C}(X)$ given by $\widehat{\varphi}(f)(x) := f(\varphi(x))$ for $x \in X$) and a unital extension

$$0 \to \mathbb{K} \otimes c_0 \to A \to \mathrm{C}(Y) \to 0$$

(– with fixed canonical epimorphism $\pi\colon A \to \mathrm{C}(Y)$ –) such that for each quasi-measure μ on X (in the sense of Aarnes [**3**]) with $\mu(X) = 1$ the composition $\tau_\mu := \nu_\mu \circ \widehat{\varphi} \circ \pi$ is a unital quasi-trace on A_+, where $\nu_\mu\colon \mathrm{C}(X)_+ \to [0, \infty)$ is the *local* quasi-trace defined by the quasi-measure μ. It is then easy to see that τ_μ can't be 2-sub-additive on the positive part A_+ of the type-I C^*-algebra A_+ if ν_μ is not additive on $\mathrm{C}(X)_+$, because otherwise the (again) 2-sub-additive class map $[\tau_\mu]$ on $\mathrm{C}(X)_+$ coincides with ν_μ, but the quasi-state ν_μ on $\mathrm{C}(X)_+$ can't be 2-sub-additive if the quasi-measure μ is not a (then finite) Borel measure on X.

Recall here that a quasi-trace τ is 2-*sub-additive* if it has the property $\tau(a+b) \leq 2(\tau(a) + \tau(b))$ for $a, b \in A_+$. This property is equivalent to both of $\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2}$ (U. Haagerup) and the existence of a quasi-trace τ_2 on $M_2(A)_+$ with $\tau_2(a \otimes p_{11}) = \tau(a)$,

Aarnes has defined in [**3**] a quasi-state on $\mathrm{C}([0,1]^2)$ that is not additive, therefore it can not be 2-sub-additive. Thus, *there are bounded quasi-traces that are not* 2-*quasi-traces on unital type-I* C^*-*algebras.*

Haagerup [**342**] (cf. also [**348**]) has shown that every bounded 2-quasi-trace on an exact unital C^*-algebra is additive (i.e., is a bounded trace). One can technically improve the arguments of Haagerup [**342**] and Blackadar/Handelman [**79**] to get this result also for all (not necessarily bounded !) lower semi-continuous 2-quasi-traces $\tau\colon A_+ \to [0, \infty]$.

REMARK A.7.2. The lower semi-continuous dimension functions $d\colon M_\infty(A) \to [0, \infty]$, where $M_\infty(A) := \bigcup_n M_n(A)$, are in one-to-one correspondence to lower semi-continuous 2-quasi-traces $\tau\colon A_+ \to [0, \infty]$ by :

$$\tau_d(a) := \lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty d((a-t)_+)\, \mathrm{d}t$$

and $d_\tau(b)$ for $b \in M_n(A)$ and given τ – extended to $\tau_n\colon M_n(A)_+ \to [0, +\infty]$, by

$$d_\tau(b) := \limsup_m \tau_n((b^*b)^{1/m})\,.$$

Check last definition!
Is here $((b^*b)^{1/m} - 1/m)_+$ better??

The local quasi-trace τ (respectively rank function d) is on A_+ lower semi-continuous, if and only if, for every $a \in A_+$: $\tau(a) = \sup_{\delta > 0} \tau((a-\delta)_+)$ (respectively $d(a) = \sup_{\delta>0} d((a-\delta)_+)$). All on $A_+^1 := \{a \in A_+ \,;\, \|a\| \leq 1\}$ bounded local quasi-traces are lower semi-continuous.

An l.s.c. quasi-trace $\tau\colon A_+ \to [0, \infty]$ is a 2-quasi-trace, if and only if, $\tau(a+b) \leq 2(\tau(a) + \tau(b))$ for all $a, b \in A_+$, if and only if, $\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2}$ for all

$a, b \in A_+$. It is equivalent to the original definition of J. Cuntz that proposes the existence of a quasi-trace τ_2 on $M_2(A)_+$ with $\tau(a) = \tau_2(\mathrm{diag}(a, 0))$ for all $a \in A_+$.

The dimension functions $d \colon M_\infty(A) \to [0, \infty]$ are in one-to-one correspondence to the *monotone* additive maps $\lambda \colon \mathrm{CS}(A \otimes \mathbb{K}) \to \mathbb{R}_+ \cup \{+\infty\}$.

Need definitions of $\mathrm{Cu}(A)$, $\mathrm{CS}(A)$ and $M_\infty(A)$ given in Chp. 2 ! Or move them to here? ??? Or only remind them here? ???

If $d \colon M_\infty(A) \to [0, \infty]$ is a dimension function (respectively a rank function $r \colon A \to [0, \infty]$, respectively $\tau \colon A_+ \to [0, \infty]$ a quasi-trace) then $d_*(b) := \sup_{\varepsilon > 0} d((b^*b - \varepsilon)_+)$ for $b \in M_\infty(A)$ (respectively $r_*(b) := \sup_{\varepsilon>0} r((b^*b - \varepsilon)_+)$ for $b \in A$, $\tau_*(a) := \sup_{\varepsilon>0} \tau((a - \varepsilon)_+)$ for $a \in A_+$) is a lower semi-continuous dimension function on $M_\infty(A)$ (respectively, lower semi-continuous rank function, lower semi-continuous quasi-trace). r_* is sub-additive on A if r is sub-additive. (τ_* is a 2-quasi-trace if τ is a 2-quasi-trace.

It holds $\tau_* = \tau_{d_\tau}$ and $d_* = d_{\tau_d}$.

Let $a \in A_+$ and denote by S the semigroup of $x \in \mathrm{CS}(A \otimes \mathbb{K})$ with the property that there is $n \in \mathbb{N}$ with $x \leq n[a] = [a \otimes 1_n]$, i.e., $b \precsim a \otimes 1_n$ for $[b] = x$.

????

then $(S, \leq, [a])$ is a scaled abelian semigroup (with the induced preorder $\leq$ and order unit $[a]$). Every monotone and additive map $\lambda \colon S \to [0, \infty]$ extends to a monotone and additive map $\lambda_e \colon \mathrm{CS}(A \otimes \mathbb{K}) \to [0, \infty]$ by $\lambda_e(x) := \lambda(x)$ for $x \in S$ and by $\lambda_e(x) := +\infty$ for $x \notin S$.

Proof. If $x \leq y$ and $\lambda_e(y) < \infty$, then $x, y \in S$, in case that $\lambda_e(x) \leq \lambda_e(y)$. If $\lambda_e(x + y) < \infty$ then $x + y \in S$. Since $0 \leq y$ and $0 \leq x$ in $\mathrm{CS}(A \otimes \mathbb{K})$ it follows that $x = 0 + x \leq x + y \leq n[a]$.

and ?????

If $\lambda_e(x) + \lambda_e(y) < \infty$, then $\lambda_e(x) < \infty$ and $\lambda_e(y) < \infty$, i.e., $x, y \in S$. □

Proof. `to be filled in ??` □

Corollary A.7.3. *Let A denote a C^*-algebra, and $a \in A_+$ with the property that $d(a) = 0$ for every lower semi-continuous dimension function d on $M_\infty(A) := \bigcup_n M_n(A)$ with $d(a) < \infty$, then, for every $\varepsilon > 0$, there is $n(a, \varepsilon) \in \mathbb{N}$ (depending on ε) such that $(a - \varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ in $M_{2n}(A)$ for all $n \geq n(a, \varepsilon)$.*

Proof. Let I denote the algebraic ideal of A generated by a. It contains the minimal dense ideal (Pedersen ideal) of the hereditary C^*-subalgebra $\overline{aAa}$, in particular, $(a - \varepsilon)_+ \in I$ for all $\varepsilon > 0$. Consider the sub-semigroup S of classes $[c] \in \mathrm{CS}(\overline{aAa})$ with the property that there is $n \in \mathbb{N}$ with $[c] \leq n[a]$, i.e., $c \precsim a \otimes 1_n$. Then $(S, \leq, [a])$ is a "scaled" semigroup with the preorder induced from $\mathrm{CS}(\overline{aAa})$.

Since every monotone additive map $\lambda \colon S \to \mathbb{R}_+$ extends to monotone additive map $\lambda_e \colon \mathrm{CS}(A) \to [0, \infty]$, and $D(x) := \lambda_e([x])$ is a dimension function on $M_\infty(a)$, and since $D_*(x) := \sup_{\varepsilon>0} D(((x^*x)^{1/2} - \varepsilon)_+)$ is a lower semi-continuous dimension function on A, we have $D_*(a) = 0$. Thus, $\lambda((a-\varepsilon)_+) = 0$ for all $\varepsilon > 0$.

By A.13.10, for every $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ with $2n[(a-\varepsilon)_+] \leq n[a]$ in $\mathrm{CS}(\overline{aAa})$ for all $n \geq n_0$, i.e., $(a-\varepsilon)_+ \otimes 1_{2n} \precsim a \otimes 1_n$ for all $n \geq n_0$. □

Lemma A.7.4. *Suppose that $\tau \colon A_+ \to [0, \infty]$ is a lower semi-continuous quasi-trace.*

(i) *If τ is a 2-quasi-trace, and if $e \in A_+$ satisfies $\tau(e) < \infty$ and $\|e\| = 1$, then there is a bounded quasi-trace $\rho \colon M_2(\widetilde{D})_+ \to \mathbb{R}_+$ with $\rho(d \oplus 0) = \tau(d)$ for $d \in D_+$, where $D := \{d \in A\,;\ ed = de = d\}$.*

(ii) *The l.s.c. quasi-trace τ is* additive *on A_+, if $\tau|D_+$ is* sub-additive *for every hereditary C^*-subalgebra $D \subseteq A$ with the property that there is $e \in A_+$ with $\tau(e) < \infty$, $\|e\| = 1$ and $de = ed = d$ for all $d \in D$.*

Proof. (i): We have $\tau(d) \leq \|d\|\tau(e)$ for all $d \in D$. By [**79**, cor.II.2.5], the bounded 2-quasi-trace $\tau' := \tau|D_+$ on D extends to a bounded 2-quasi-trace $\tau''|\widetilde{D}_+$ on the unitization $\widetilde{D}$ of D with

$$\tau''(1) = \sup\{\tau(d)\,;\ d \in D_+,\ \|d\| \leq 1\} \ \leq\ \tau(e)\,.$$

The proof of [**79**, cor.II.2.5] is based on the technically engaged theorem [**79**, thm.I.4.1]. A *detailed and elementary* proof of Part (i) goes as follows:

It follows that $f := 2(e - 1/2)_+$ satisfies $df = fd = d$ for all $d \in D$ and $f^{1/n} \leq 2e$ for all $n \in \mathbb{N}$. In particular, f is a strictly positive contraction in the center of the C^*-subalgebra $C^*(D, f)$ of A, because $h(f)d = h(1)d$ for $h \in \mathrm{C}_0(0,1]$ and d, which implies $C^*(D, f) = D + C^*(f)$ and $DC^*(f) = D$.

The restriction of τ to $C^*(D, f)_+$ is bounded, because, for $c \in C^*(D, f)_+$,

$$\tau(c) \leq \|c\| \sup_n \tau(f^{1/n}) \leq 2\|c\|\tau(e)\,.$$

The element $f(1-f)$ is orthogonal to D and is a strictly positive element of an ideal I of $C^*(D, f)$, with $\widetilde{D} \cong C^*(D, f)/I$. The isomorphism is defined by the epimorphism $\phi \colon C^*(D, f) \to \widetilde{D}$ with kernel $C^*(f(1-f))$, given by $\phi(f) = 1$ and $\phi(d) = d$.

Then we have $M_2(D) = \{b \in M_2(A)\,;\ b(e \oplus e) = (e \oplus e)b = b\}$. Thus $M_2(C^*(D, f))$ is generated by $M_2(D)$ and $M_2(C^*(f))$, $f \oplus f = f \otimes 1_2$ is in the center of $M_2(C^*(D, f))$, and the ideal $M_2(C^*(f(1-f)))$ is the hereditary C^*-subalgebra of $M_2(C^*(D, f))$ that is generated by $g := (f(1-f)) \otimes 1_2 = (f - f^2) \oplus (f - f^2)$.

Use the identification $M_2(A) = A \otimes M_2$, then $e \oplus e = e \otimes 1_2$, $f \oplus f = f \otimes 1_2$, $M_2(C^*(D, f)) = C^*(M_2(D), M_2(C^*(f))$, $M_2(C^*(f(1-f))) = C^*(f(1-f)) \otimes M_2$, $f \oplus f$ is in the center of $M_2(C^*(D, f))$.

`check again!::`

We define on the unital C^*-algebra $M_2(\widetilde{D}) \cong M_2(C^*(D, f))/M_2(I)$ a function $\nu(a + s) := \inf_n \tau_2((1 - g^{1/n})(a + f \otimes s)(1 - g^{1/n})$ for selfadjoint $a^* = a \in M_2(D)$

and $s^* = s \in M_2$ with ???????? $\leq a + s \geq 0$. It is obvious that $\nu(d \oplus 0) = \tau(d)$ for all $a \in D_+$, and it is not hard to see, that $\nu|C_+$ is additive for every commutative C^*-subalgebra $C \subseteq M_2(\widetilde{D})$, because $C^*(C \cdot (f \otimes 1_2), g) \subseteq M_2(C^*(D, f))$ is a commutative C^*-subalgebra of $M_2(C^*(D, f))$.

Thus $\nu_1(d + t) = \nu((d + t) \oplus 0)$ (for $d^* = d \in D$ and $t \in \mathbb{R}_+$ with $d + t \geq 0$) is a bounded 2-quasi-trace on $\widetilde{D}_+$ with $\nu_1(d) = \tau(d)$ for $d \in D_+$.

(ii): Let $a, b \in A_+$. Since τ is in particular a lower s.c. 2-quasi-trace, one has $\tau(a + b) \leq 2(\tau(a) + \tau(b))$ and $\max(\tau(a), \tau(b)) \leq \tau(a + b)$. Thus $\tau(a) + \tau(b) = +\infty$, if and only if, $\tau(a + b) = +\infty$. Hence, we may suppose that $\tau(a + b) < \infty$.

Let $g_\delta(t) := \min((2/\delta)\max(t - \delta/2, 0), 1)$ for $t \in \mathbb{R}_+$ and $\delta \in (0, 1]$, and notice that $g_\delta g_\varepsilon = g_\varepsilon$ for $\delta \leq \varepsilon/2$.

If $\tau(a+b) < \infty$ and $\delta > 0$, let $e := g_\delta(a+b)$ and $D := \{x \in A;\ xe = ex = x\}$. The elements $c := g_{4\delta}(a+d)^{1/2}$ and $d := \|a+b\| g_{2\delta}(a+d)^{1/2}$ are in D_+, d commutes with all elements in cAc and $d \geq c(a+b)c$. In particular, $\tau(d-cac)+\tau(cac) = \tau(d)$, $\tau(d - cbc) + \tau(cbc) = \tau(d)$, and

$$2\tau(d) - \tau(c(a + b)c) = \tau(2d - c(a + b)c) = \tau((d - cac) + (d - cbc))$$

The sub-additivity of τ implies $\tau(c(a + b)c) \leq \tau(cac) + \tau(cbc)$, and

$$2\tau(d) - \tau(c(a + b)c) \leq \tau(d - cac) + \tau(d - cbc) = 2\tau(d) - (\tau(cac) + \tau(cbc)).$$

Hence, $\tau(c(a + b)c) = \tau(cac) + \tau(cbc)$.

The elements a, b, $a + b$ are in $E := \overline{(a + b)A(a + b)}$. Since

$$\tau(y^{1/2} g_{2\delta}(a + b) y^{1/2}) = \tau(g_{2\delta}(a + b)^{1/2} y g_{2\delta}(a + b)^{1/2})$$

and

$$\tau(y) = \lim_{\delta \searrow 0} \tau(y^{1/2} g_{2\delta}(a + b) y^{1/2})$$

for all $y \in E_+$ by monotony and lower semi-continuity of τ, it follows $\tau(a + b) = \tau(a) + \tau(b)$. □

REMARK A.7.5. Similar arguments show:
A lower s.c. local *quasi-trace* $\tau\colon A_+ \to [0, \infty]$ *is* 2-subadditive*,*
i.e., $\tau(a + b) \leq 2(\tau(a) + \tau(b))$ *for all* $a, b \in A_+$*, if and only if*
$\tau(a + b) = \infty$ *implies* $\tau(a) + \tau(b) = \infty$*, and, for every* $e \in A_+$ *with* $\tau(e) < \infty$ *and* $\|e\| = 1$*,* $\tau|D_+$ *is* 2*-subadditive, where* D *is the the hereditary* C^**-subalgebra* $D := \{a \in A;\ ae = ea = a\}$.

COROLLARY A.7.6. *Let* A *an exact* C^**-algebra. Then every lower semi-continuous* 2*-quasi-trace* $\tau\colon A_+ \to [0, \infty]$ *is an additive trace.*

PROOF. If A is exact, unital and $\tau(1) < \infty$, then this was shown by U. Haagerup in [**342**] (see also [**348**]).

It suffices to show in the non-unital case that $\tau|D_+$ is sub-additive for every hereditary C^*-subalgebra $D \subseteq A$ with the property that there is $e \in A_+$ with $\tau(e) < \infty$, $\|e\| = 1$ and $de = ed = d$ for all $d \in D$, cf. Lemma A.7.4(ii). But then there is an extension $\widetilde{\tau}\colon \widetilde{D}_+ \to [0,\infty)$ of $\tau|D_+$ by Lemma A.7.4(i). Where we use that the unitization $\widetilde{D}$ of $D \subseteq A$ is exact if A is exact. □

REMARK A.7.7. There exist unital quasi-traces that are not 2-quasi-traces on the unital C^*-algebra $A := \mathrm{C}([0,1]) * \mathrm{C}([0,1])$, i.e., on the unital free product of two copies of $\mathrm{C}([0,1])$, and on a type-I C^*-algebra A that is a unital extension

$$0 \to c_0(\mathbb{K}) \to A \to \mathrm{C}([0,1]^2) \to 0\,.$$

`Cite! place of exact definition.`

8. Projectivity of some C^*-algebras

DEFINITION A.8.1. A C^*-algebra B is **projective** if for every C^*-algebra A, closed ideal $J \triangleleft A$ of A and C^*-morphism $\psi\colon B \to A/J$ there exists a C^*-morphism $\phi\colon B \to A$ with $\psi = \pi_J \circ \phi$ (ϕ is a **lift** of ψ).

Obvious examples of projective C^*-algebras are the free algebras generated by a finite, countable or uncountable number of contractions. (The latter by using the Axiom of Choice.)

Here we display a proof of the projectivity of the cones $CF := \mathrm{C}_0((0,1],F)$ over finite-dimensional C^*-algebras F.

The following Lemma A.8.2 reformulates [**540**, thm. 3.3]. We use a variant of [**616**, prop. 1.5.10] for a proof.

LEMMA A.8.2. *Let $J \triangleleft A$, $a,b \in A_+$ and $d \in A/J$ with $d^*d \leq \pi_J(a)$ and $dd^* \leq \pi_J(b)$. Then there exists $g \in A$ with $g^*g \leq a$, $gg^* \leq b$ and $\pi_J(g) = d$.*

REMARK A.8.3. Lemma A.8.2 is a two-sided version of an order lifting theorem of Combes [**155**], cf. also [**616**, prop. 1.5.10]:

Let $\rho\colon B \to C$ a C^*-algebra epimorphism. If $x \in B_+$ and $y \in C$ satisfy $y^*y \leq \rho(x)$ then there exists some $z \in B$ with $\rho(z) = y$ and $z^*z \leq x$.

Inspection of the proof of [**616**, prop. 1.5.10] allows to see moreover the following:

For each $b \in B$ with $\rho(b) = y$ there exists $z \in B$ with $\rho(z) = y$, $z^*z \leq x$ and (in addition) $zz^* \leq bb^*$.

Indeed: Let $b \in B$ with $\rho(b) = y$ and $J := \rho^{-1}(0)$. Then $(x - b^*b)_- \in J_+$, and $c = (x - b^*b)_- + x$ satisfies $\rho(c) = \rho(x)$, $b^*b \leq c$ and $x \leq c$.

Let $z := \lim_{n\to\infty} b(n^{-1}1 + c)^{-1/2}x^{1/2}$. This limit exists because the elements $t_n := b(n^{-1}1+c)^{-1/2}x^{1/2}$ satisfy via monotony and centrality of C^*-algebra norms the inequalities

$$\|t_m - t_n\|^2 \;\leq\; \|\,\big((m^{-1}1 + c)^{-1/2} - (n^{-1}1 + c)^{-1/2}\big)c\,\|\,.$$

The equation $\pi_J(x) = \pi_J(c)$ implies $\pi_J(z) = \pi_J(b)$. The additional inequality $zz^* \leq bb^*$ comes from $\|(n^{-1}1 + c)^{-1/2}x^{1/2}\|^2 \leq 1$ via $x \leq c$.

Proof of Lemma A.8.2: Let $J \triangleleft A$, $a, b \in A_+$ and $d \in A/J$ with

$$d^*d \leq \pi_J(a) \quad \text{and} \quad dd^* \leq \pi_J(b)\,.$$

We find some $e \in A$ with $\pi_J(e) = d$ by surjectivity of π_J. There exists $f \in A$ with $f^*f \leq a$ and $\pi_J(f) = d$, by Remark A.8.3.

We can apply again Remark A.8.3, but this time with (d^*, f^*, b) in place of (y, b, x), because $\pi_J(f) = d$ and $dd^* \leq \pi_J(b)$. Get $z \in A$ with $\pi_J(z) = d^*$, $z^*z \leq b$ and $zz^* \leq f^*f$.

The element $g := z^*$ has the desired properties: $g^*g \leq f^*f \leq a$, $gg^* = z^*z \leq b$ and $\pi_J(g) = \pi_J(z)^* = d$. □

Proposition A.8.4. [**540**, cor. 3.8] *For each C^*-algebra F of finite dimension, the cone $CF := \mathrm{C}_0((0,1], F)$ of F is projective in sense of Definition A.8.1.*

Proof. The following simple observation works only for σ-unital C^*-algebras B_1 and B_2. This is necessary to observe because the (free) universal C^*-algebra generated by un-countably many contractions is projective (by using the Axiom of Choice) but is not σ-unital.

If B_1 and B_2 are projective σ-unital C^*-algebras, then $B_1 \oplus B_2$ is projective.

Indeed: Let $\psi\colon B_1 \oplus B_2 \to A/J$ a C^*-morphism, $e_1 \in B_1$ and $e_2 \in B_2$ strictly positive contractions, and let $a \in A$ with $\pi_J(a) = \psi(e_1 \oplus (-e_2))$. The elements $a_1 := (a^* + a)_+$, $a_2 := (a^* + a)_- \in A_+$ and hereditary C^*-subalgebras $D_k := \overline{a_k A a_k}$ $(k \in \{1,2\})$ satisfy $D_1 D_2 = \{0\}$, $\psi(B_1 \oplus \{0\}) \subseteq \pi_J(D_1) \cong D_1/(J \cap D_1)$ and $\psi(\{0\} \oplus B_2) \subseteq \pi_J(D_2) \cong D_2/(J \cap D_2)$. Then $\psi_1(b_1) := \psi(b_1 \oplus 0)$ $(b_1 \in B_1)$ and $\psi_2(b_1) := \psi(0 \oplus b_2)$ $(b_2 \in B_2)$ are C^*-morphisms into $D_1/(J \cap D_1)$ respectively $D_2/(J \cap D_2)$. By projectivity of B_1 and B_2 there are C^*-morphisms $\phi_k\colon B_k \to D_k$ with $\pi_J(\phi_k(b_k)) = \psi_k(b_k)$ for $b_k \in B_k$ $(k \in \{1,2\})$. The map $\phi(b_1 \oplus b_2) := \phi_1(b_1) + \phi_2(b_2)$ $(b_k \in B_k)$ is a linear lift of ψ. It is a C^*-morphism, because $D_1 D_2 = \{0\}$.

This shows that finite orthogonal direct sums of σ-unital projective C^*-algebras are projective.

Let $k_1, k_2, \ldots, k_n \in \mathbb{N}$ and $F := M_{k_1} \oplus \cdots \oplus M_{k_n}$. It is easy to see that

$$CF := \mathrm{C}_0((0,1], F) \cong \mathrm{C}_0((0,1], M_{k_1}) \oplus \cdots \oplus \mathrm{C}_0((0,1], M_{k_n})\,.$$

The additive invariance of the class of σ-unital projective C^*-algebras shows that it is enough to prove the projectivity of CF only in the cases $F := \mathbb{C}$ and of $F := M_n$ for $n > 1$.

If $F := \mathbb{C}$ then $CF \cong \mathrm{C}_0(0,1]$ and has the generator f_0 defined by $f_0(t) = t$ $(t \in [0,1])$. $\psi\colon CF \to A/J$ is determined by the positive contraction $b := \psi(f_0)$. Let $d \in A$ with $\pi_J(d) = b^{1/2}$. Then $\pi_J(f(a_0)) = \psi(f)$ for $f \in \mathrm{C}_0(0,1]$ and $a_0 := d^*d - (d^*d - 1)_+$. Thus $\phi\colon f \mapsto f(a_0)$ is a lift of ψ.

Since projectivity is invariant under direct sums we get that CF is projective for $F = \mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

If $F := M_n$ with $n > 1$ then the C^*-algebra $CF = CM_n := \mathrm{C}_0((0,1], M_n)$ is isomorphic to the universal C^*-algebra $\mathcal{A}_n := C^*(x_2, \ldots, x_n; R_n)$ with conditions and relations R_n given by $\|x_2\| \leq 1$, $x_j^* x_k = \delta_{jk} x_2^* x_2$ and $x_j x_k = 0$ for $j, k \in \{2, \ldots, n\}$.

Indeed (by using arguments in the proof [**538**, prop. 2.7]): It is easy to see that the elements $\gamma_n(x_k) := f_0 \otimes p_{k,1} \in \mathrm{C}_0((0,1], M_n)$ – with the canonical matrix units $p_{j,k} \in M_n$ – generate $\mathrm{C}_0(0,1] \otimes M_n = \mathrm{C}_0((0,1], M_n)$ and satisfy the norm-condition and relations R_n and therefore define a *-epimorphism onto $\mathrm{C}_0((0,1], M_n)$. Moreover, it is not difficult to check that the positive element

$$Z_n := (x_{1,2} x_{1,2}^* + x_{1,2}^* x_{1,2} + \ldots + x_{1,n}^* x_{1,n})^{1/2}$$

is a contraction in the center of $\mathcal{A}_n$ and $\gamma_n(Z_n) = f_0 \otimes 1_n$. Since it is in the center, it must be strictly positive in $\mathcal{A}_n$ (by the defining relations). Thus, every irreducible *-representation ρ of $\mathcal{A}_n$ maps Z_n to a scalar $\rho(z_n) = \alpha 1$ for some $\alpha \in (0,1]$ and that $\rho(\mathcal{A}_n) \cong M_n$ with $\rho(x_{1,k}) = \alpha p_{1,k}$.

Use now that $\gamma_n(Z_n) = f_0 \otimes 1_n$ to get that the natural C^*-morphism γ_n from $\mathcal{A}_n$ onto $\mathrm{C}_0((0,1], M_n)$ is an isomorphism, because any irreducible representation of A is the compositions of γ_n with the irreducible representations of $\mathrm{C}_0((0,1], M_n)$ and those exhaust all the corresponding $\alpha \in (0,1]$.

The relations R_n (or its realization in $CM_n = \mathrm{C}_0(0,1] \otimes M_n$) show the existence of canonical C^*-morphisms from $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ that agrees with the natural C^*-morphisms from CM_n into CM_{n+1} given by the inclusion $M_n \subseteq M_{n+1}$ (as left upper corner). Moreover the natural inclusion $CF \subset \mathrm{C}_0((0,1], M_n)$ for $F := \mathbb{C}^n$ defines an isomorphism from CF onto

$$C^*((x_2 x_2^*)^{1/2}, (x_2^* x_2)^{1/2}, \ldots, (x_n^* x_n)^{1/2}) \subseteq \mathcal{A}_n\,,$$

in a way such that $f_0 \otimes 1_n \in CF$ corresponds to Z_n, $f_0 \otimes p_{11}$ to $(x_2^* x_2)^{1/2}$ and the $f_0 \otimes p_{kk}$ map to $(x_k\, x_k^*)^{1/2}$ for $k = 2, \ldots, n$.

Let $\psi \colon \mathcal{A}_n \to A/J$ a C^*-morphism. The projectivity of $CF \cong \mathrm{C}_0((0,1], \mathbb{C}^n)$ for $F := \mathbb{C}^n$ shows that there is a C^*-morphism $\phi_1 \colon \ell_n(\mathrm{C}_0(0,1]) = C\mathbb{C}^n \to A$ that is always given by mutually orthogonal positive contractions $a_1, \ldots, a_n \in A_+$ with $\phi_1((x_2^* x_2)^{1/2}) = a_1$ and $\phi_1((x_k x_k^*)^{1/2}) = a_k$ for $k = 2, \ldots, n$, and satisfies the equations

$$\pi_J(a_1)^2 \;=\; \psi(x_2^* x_2) \quad \text{and} \quad \pi_J(a_k)^2 \;=\; \psi(x_k x_k^*) \quad \text{for} \quad k \geq 2\,. \tag{8.1}$$

Let $a_1, \ldots, a_n \in A$ mutually orthogonal positive contractions, and suppose that there are given C^*-morphism $\psi \colon \mathcal{A}_n \to A/J$ with $\psi(x_2^* x_2) \leq \pi_J(a_1^2)$ and $\psi(x_k\, x_k^*) \leq \pi_J(a_k^2)$ for $k \geq 2$. We show that there exists a C^*-morphism $\phi \colon \mathcal{A}_n \to A$ with $\pi_J \circ \phi = \psi$.

In addition we show that we can find the ϕ with $\phi(x_2^* x_2) \leq a_1^2$ and $\phi(x_k\, x_k^*) \leq a_k^2$ for $k \geq 2$.

We proceed by induction over $k \geq 2$ and produce from above defined $\phi_1 \colon C\mathbb{C}^n \to A$, given by mutually orthogonal positive contractions $a_1, \ldots, a_n \in A_+$, inductively C^*-morphisms $\phi_k\colon \mathcal{A}_k \to A$ with $\phi_k(x_2)^*\phi_k(x_2) \le a_1^2$, $\phi_k(x_j)\phi_k(x_j)^* \le a_j^2$ and $\pi_J(\phi_k(x_j)) = \psi(x_j)$ for $j = 2, \ldots, k$.

If $k = 2$ then this follows immediately from Lemma A.8.2 with $a := a_1^2$, $b := a_2^2$, $d := \psi(x_2)$.

Let $2 < k < n$ and suppose that we have a C^*-morphism $\phi_k\colon \mathcal{A}_k \to A$ with the above listed properties. We apply Lemma A.8.2 to $a := \phi_k(x_2)^*\phi_k(x_2) \le a_1^2$, $b := a_{k+1}^2$ and $d := \psi(x_{k+1})$, and obtain $g \in A$ with $g^*g \le \phi_k(x_2)^*\phi_k(x_2)$, $gg^* \le a_{k+1}^2$ and $\pi_J(g) = \psi(x_{k+1})$.

We use ϕ_k and g to define a C^*-morphism $\phi_{k+1}\colon \mathcal{A}_{k+1} \to A$ with $\pi_J \circ \phi_{k+1} = \psi|\mathcal{A}_{k+1}$, $\phi_{k+1}(x_2^*x_2) \le a_1^2$ and $\phi_{k+1}(x_jx_j^*) \le a_j^2$.

Recall that $\phi_k(x_i)^*\phi_k(x_j) = \delta_{ij}\phi_k(x_2)^*\phi_k(x_2)$ for $i, j \in \{2, \ldots, k\}$ and let $S := (\phi_k(x_2)^*\phi_k(x_2))^{1/2}$. Then $\phi_k(x_j) := v_jS$ is the polar decomposition of $\phi_k(x_j)$ with the partial isometries $v_j \in A^{**}$ given by $v_j = \lim_n \phi_k(x_j)(n^{-1}1 + S)^{-1}$. It follows that $v_j^*v_i = \delta_{ij}v_2^*v_2$. Since $g^*g \le S^2$, we can define $y_j := v_j(g^*g)^{1/2} = \lim_n \phi_k(x_j)(n^{-1}1 + S)^{-1}(g^*g)^{1/2}$ $(j = 2, \ldots, k)$ and $y_{k+1} := g$.

The y_j satisfy $y_jy_j^* \le \phi_k(x_j)\phi_k(x_j)^* \le a_j^2$, $y_i^*y_j = \delta_{ij}y_2^*y_2 = \delta_{ij}g^*g$. for $2 \le i, j \le k$, and $y_{k+1}^*y_j = g^*y_j = 0$ for $j \le k$ because $gg^* \le a_{k+1}^2$ and $y_jy_j^* \le a_j^2$.

Moreover, $y_iy_j = 0$ because $y_i^*y_i \le g^*g \le a_1^2$ and $y_jy_j^* \le a_j^2$ and $a_1a_j = 0$ for $j = 2, \ldots, n$. $\|y_2\| = \|g\| \le \|\phi_k(x_2)\| \le 1$

Thus, the elements $\{y_2, \ldots, y_{k+1}\} \subset A$ satisfy the condition and relations R_{k+1} of $\mathcal{A}_{k+1}$. Moreover, they satisfy $y_2^*y_2 \le a_1^2$ and $y_jy_j^* \le a_j^2$ for $j = 2, \ldots, k+1$. Let $\phi_{k+1}\colon \mathcal{A}_{k+1} \to A$ the corresponding C^*-morphism.

We check that $\pi_J \circ \phi_{k+1}$ is the restriction of $\psi\colon \mathcal{A}_n \to A/J$ to $\mathcal{A}_{k+1}$:

Recall that $\pi_J(y_{k+1}) = \pi_J(g) = \psi(x_{k+1})$ by construction of g, and $\pi_J(\phi_k(x_j)) = \psi(x_j)$ for $2 \le j \le k$ by assumption on ϕ_k. In particular $\pi_J((g^*g)^{1/2}) = (\psi(x_2)^*\psi(x_2))^{1/2}$, and $\pi_J(S) = (\psi(x_2)^*\psi(x_2))^{1/2}$ by definition of $S := (\phi_k(x_2)^*\phi_k(x_2))^{1/2}$. Let $T := (\psi(x_2)^*\psi(x_2))^{1/2}$. It satisfies $\psi(x_j)^*\psi(x_j) = T^2$. Then the definition of the y_j for $j = 2, \ldots k$ show the norm convergence

$$\pi_J(y_j) = \lim_n \psi(x_j)(n^{-1}1 + T)^{-1}T = \psi(x_j)\,.$$

It says that ϕ_{k+1}, – defined by $\phi_{k+1}(x_j) := y_j$ –, satisfies $\pi_J \circ \phi_{k+1} = \psi|\mathcal{A}_{k+1}$. □

Corollary A.8.5. *For each $n \in \mathbb{N}$ the universal C^*-algebra A generated by n contractions $t_1, \ldots, t_n$ with mutually orthogonal ranges is projective.*

Proof. Notice that A is given by universal C^*-algebra with relations

$$A := C^*(t_1, \ldots, t_n\,;\ \|t_k\| \le 1, t_k^*t_\ell = 0 \text{ for } k \ne \ell\,, k, \ell \in \{1, \ldots, n\}\,)$$

with contractions t_k. The relations imply the additional relation $t_1t_1^* + \cdots + t_nt_n^* \le 1$. Let B a C^*-algebra, $J \subseteq B$ a closed ideal of B and $\rho\colon A \to B/J$ a C^*-morphism. We let $s_k := \rho(t_k)$. All we need is to find contractions $c_1, \ldots, c_n \in B$ with $c_j^*c_k = 0$

for $j \neq k$ and $\pi_J(c_k) = s_k := \rho(t_k)$, because then there is a C^*-morphism $\psi\colon A \to B$ with $\psi(t_k) = c_k$, which implies $\pi_J \circ \psi = \rho$.

The C^*-subalgebra of A generated by $(t_k t_k^*)^{1/4}$ $(k = 1, \dots, n)$ is the image of a unique C^*-morphism $\gamma\colon CF \to A$ for $F := \mathbb{C}^n$ and

$$CF \cong C^*(y_1, \dots, y_n\, ;\ 0 \leq y_k \leq 1,\, y_k y_\ell = 0, \text{ for } k \neq \ell\,)$$

with $\gamma(y_k) = (t_k t_k^*)^{1/4}$. By the projectivity of CF there exists a C^*-morphism λ from CF into B that is a lift of $\rho \circ \gamma\colon CF \to B/J$. The C^*-morphism λ satisfies the equations

$$\pi_J(\lambda(y_k)) = \rho((t_k t_k^*)^{1/4}) = (s_k s_k^*)^{1/4}\,.$$

The polar decomposition $s_k = v_k(s_k^* s_k)^{1/2} = (s_k s_k^*)^{1/2} v_k$ of s_k in $(B/J)^{**}$ has the property that $(s_k s_k^*)^{1/4} v_k = v_k (s_k^* s_k)^{1/4} \in B/J$. Let $x_k \in B$ elements with $\pi_J(x_k) = (s_k s_k^*)^{1/4} v_k$. Then the elements $b_k := \lambda(y_k) x_k$ satisfy $b_\ell^* b_k = 0$ for $k \neq \ell$ and $\pi_J(b_k) = s_k$ for $k = 1, \dots, n$. We define elements $c_k \in B$ by $c_k := b_k f(b_k^* b_k) = \lim_{p\to\infty} b_k f(1/p + b_k^* b_k)$ for the continuous function $f(t) := \max(1,t)^{-1/2}$ on $[0,\infty)$. Use here that the element $f(b_k^* b_k)$ is a multiplier of $C^*(b_k^* b_k)$. The elements $c_1, \dots, c_n$ are contractions in B with $c_k^* c_j = 0$ for $j \neq k$ (i.e., with orthogonal ranges) and $\pi_J(c_k) = \rho(t_k)$. □

Contents of next Remark should be exist on other places! Find it! And delete one of them.

Delete following Lemma ref.lem:A.old.2.5
Replace citations as follows:
ref.lem:A.old.2.5(i,ii) by Remark A.8.6????,
ref.lem:A.old.2.5(iii) by Remark 2.1.16(ii),
Old ref.lem:A.1.25 =? ref.lem:A.old.2.5(iv) by ?????
ref.lem:A.old.2.5(v) by ?????
ref.lem:A.old.2.5(iv) = Old A.1.25?? by ??????

Begin: Old Lemma ref.lem:A.old.2.5

REMARK A.8.6. The projectivity of CF can be equivalent expressed by the following formulation:

If F is a C^*-algebra of finite linear dimension then every extension

$$0 \to J \to E \to \mathrm{C}_0((0,1], F) \to 0$$

of $\mathrm{C}_0((0,1], F)$ by a C^*-algebra J is a split extension.

9. Projectivity and c.p.c. order zero maps

Elements $a, b \in A$ of a C^*-algebra A are *orthogonal* if $(aa^* + a^*a)(bb^* + b^*b) = 0$. We denote this by $a \perp b$. It is equivalent to the 4 equations $ab = 0$, $ba = 0$, $a^*b = 0$ and $ab^* = 0$. And it says equivalently that $h_{1,j} h_{2,k} = 0$ for $j, k \in \{1, 2\}$, where $h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2} \in A$ are the selfadjoint elements with $a = h_{1,1} + i h_{1,2}$ and

$b = h_{2,1} + ih_{2,2}$. In particular, for self-adjoint $a, b \in A$ holds $a \perp b$ if and only if $ab = 0$.

DEFINITION A.9.1. Let A and B C^*-algebras and $\psi\colon A \to B$ a bounded linear map that is invariant under passage to adjoint elements, i.e., $\psi(a^*) = \psi(a)^*$ for all $a \in A$ ([5]). We call a map ψ with this property *orthogonality preserving* if ψ has the additional property that $\psi(h)\psi(k) = 0$ for selfadjoint $h, k \in A$ with $hk = 0$. ([6])

If ψ is moreover a completely positive contraction, then ψ is called an "order-zero map" by W. Winter and J. Zacharias in [**836**, def. 1.3].
(We call it here sometimes "*order-zero morphism*".)

If $\varphi\colon \mathrm{C}_0((0,1], A) \to B$ is a C^*-algebra homomorphism then $\psi_\varphi(a) := \varphi(f_0 \otimes a)$ is the contractive completely positive order-zero map defined by φ.

It turns out that there exists a natural decomposition of ψ_φ into element-wise commuting C^*-morphisms $\lambda\colon \mathrm{C}_0(0,1] \to \mathcal{M}(D)$ and $\rho\colon A \to \mathcal{M}(D)$ with $\psi_\varphi(f \otimes a) = \lambda(f)\rho(a)$ for all $f \in \mathrm{C}_0(0,1]$ and $a \in A$, where D denotes the hereditary C^*-subalgebra B generated by $\psi_\varphi(A)$ ...

We use in this book only the C^*-morphisms $\varphi\colon \mathrm{C}_0((0,1], A) \to B$ instead of order-zero morphisms, because by our

Lemmata ??, ??, ??, and Proposition ??

there is a natural bijective relation between *contractive, 2-positive and orthogonality preserving* maps $\psi\colon A \to B$ and the usual C^*-morphisms $\varphi\colon \mathrm{C}_0((0,1], A) \to B$ that is given by $\psi(a) = \varphi(f_0 \otimes a)$ for all $a \in A$. It is a bijective relation because the 2-positive contraction ψ determines φ uniquely by the following obvious equations for all $n \in \mathbb{N}$ and $a \in A_+$:

$$\varphi(f_0^n \otimes a) = \varphi(f_0 \otimes a^{1/n})^n = \psi(a^{1/n})^n .$$

In particular, this implies that all 2-positive *and* orthogonality preserving maps $\psi\colon A \to B$ are automatically completely positive!

The following Proposition A.9.2 should be moved below general observations!
One application of the projectivity of $\mathrm{C}_0((0,1], F)$ for a finite-dimensional C^*-algebra F and the characterization of 2-positive order-zero maps $\varphi\colon F \to A/J$ is the following proposition.

PROPOSITION A.9.2. *Let F a C^*-algebra of finite dimension and $\varphi\colon F \to A/J$ a 2-positive order-zero map, then there exists a completely positive order zero map $\psi\colon F \to A$ with $\pi_J \circ \psi = \varphi$.*

PROOF. For each 2-positive order-zero map φ from F into the C^*-algebra $B := A/J$ there exists a C^*-morphism $\lambda_0\colon \mathrm{C}_0((0,1], F) \to B$ and $f \in \mathrm{C}_0(0,1]_+$

[5] Such maps ψ are sometimes called "symmetric".

[6] Equivalently, $a_1 \perp a_2$ implies $\psi(a_1) \perp \psi(a_2)$, because $\psi(a^*) = \psi(a)^*$ on A. M. Wolff calls our "orthogonality preserving" maps "*disjointness preserving*" in [**838**].

with $\varphi(b) = \lambda_0(f \otimes b)$ for all $b \in B$. By Proposition A.8.4, the algebra $\mathrm{C}_0((0,1], F)$ is projective. Thus, there exists a C^*-morphism $\lambda_1 \colon \mathrm{C}_0((0,1], F) \to A$ with $\pi_J \circ \lambda_1 = \lambda_0$. Then $\psi(b) := \lambda_1(f \otimes b)$ defines a completely positive order-zero from F into A with $\pi_J \circ \psi = \varphi$. □

Here are some citations from the paper [**836**] of W. Winter and J. Zacharias.

Erase them or replace them by my own version (with citations).

Corollary 3.1 of WILHELM WINTER AND JOACHIM ZACHARIAS in COMPLETELY POSITIVE MAPS OF ORDER ZERO:

[**836**, cor. 3.1]

Let A and B be C^*-algebras, and $\varphi\colon A \to B$ a c.p.c. order zero map. Then, the map given by $\rho_\varphi(f_0 \otimes a) := \varphi(a)$ (for $a \in A$) induces a *-homomorphism $\rho_\varphi\colon \mathrm{C}_0((0,1], A) \to B$.

Conversely, any *-homomorphism $\rho\colon \mathrm{C}_0(0,1] \otimes A \to B$ induces a c.p.c. order zero map $\varphi_\rho\colon A \to B$ via $\varphi_\rho(a) := \rho(f_0 \otimes a)$. These mutual assignments yield a canonical bijection between the point-norm closed set of c.p.c. order zero maps from A to B and the *-homomorphisms from $\mathrm{C}_0(0,1] \otimes A$ to B.

My Remarks:

The positive case:

If A is abelian and $\varphi\colon A \to B$ *positive*, then φ is completely positive, because the second conjugate $\varphi^{**}\colon A^{**} \to B^{**}$ is again positive. Then φ and φ^{**} are moreover a *completely* positive map because each finite subset of A^{**} can be approximated (in norm) by the linear span of finitely many projections in A^{**}.

Thus, then there are unique (!) C^*-morphisms $\rho_C\colon \mathrm{C}(0,1] \otimes C \to B$ with $\rho_C(f_0 \otimes c) = \varphi(c)$ for each commutative $C \subseteq A$ ...

Check if it is on intersections of different abelian $C \subseteq A$ the same. (Seems to be ...)

But the PROBLEM (!) is, that general elements of $(\mathrm{C}(0,1] \otimes A)_+$ are not contained in some $\mathrm{C}(0,1] \otimes C$ ... Thus, this old idea of proof don't work.

The following theorem reformulates the theorem [**838**, thm.2.3] of M. Wolff on Jordan algebras for the more special case of C^*-algebras, where he calls the above defined "symmetric" and "orthogonality preserving" maps "*disjointness preserving*".

Theorem A.9.3. *Let A and B be C^*-algebras, with unital A, and let $\varphi\colon A \to B$ be a disjointness preserving map. Define $D := \{\varphi(1_A)\}' \cap B \subseteq B$ and*

$$C := \overline{\varphi(1_A) \cdot D}\,.$$

*Then, $\varphi(A) \subseteq C$ and there is a Jordan *-homomorphism $\psi\colon A \to \mathcal{M}(C)$ from A into the multiplier algebra of C satisfying $\varphi(a) = \varphi(1_A)\psi(a)$ for all $a \in A$.*

The ideas of his proof are related to some study of Lamperti operators by W. Arendt in [**41**].

Notice that $\varphi(1_A)$ is selfadjoint, but is here not required to be positive. The proof of Theorem A.9.3 reduces to the case where $A = \mathrm{C}[0,1]$, but is not obvious.

We use the below given Lemma **??** and consider then a slightly more general case where A is not necessarily unital, $\varphi \colon A \to B$ is a bounded linear map with the properties that φ is "symmetric" – in the sense that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$ – and is "orthogonality preserving" – in the sense that $\varphi(a_1)\varphi(a_2) = 0$ if $a_1 a_2 = 0$ for orthogonal selfadjoint elements $a_1 = a_1^*$, $a_2 = a_2^*$ (i.e., with $a_1 a_2 = 0$).

REMARK A.9.4. `First (see below explained): Banach case for` $V \colon Y^* \to X^*$.
`Then` V `is` $\sigma(Y^*, Y)$-$\sigma(X^*, X)$ `continuous`
`if and only if`
`there exists bounded linear map` $U \colon X \to Y$ `such that` $V = U^*$.
`But look to the notations!! Chaos there???`

The continuity properties for $S \colon A^{**} \to B^{**}$ and $T \colon A^{**} \to B^{**}$ with respect to $\sigma(A^{**}, A^*)$ and $\sigma(B^{**}, B^*)$ (... play role of $V := U^* \colon Y^* \to X^*$, e.g. with suitable $S_* \colon B^* \to A^*$ in the role of $U \colon X \to Y$ with continuity of V with respect to $\sigma(Y^*, Y)$ and $\sigma(X^*, X)$.)

Because then, for example, the linear functional given by $a \in A^{**} \mapsto \varphi(T(a))$ where $\varphi \in B^*$ is equal the linear functional $V(\varphi) \in A^*$. This proves the continuity properties for $T = V^*$.

$V \colon Y^* \to X^*$.
Then V is $\sigma(Y^*, Y)$-$\sigma(X^*, X)$ continuous
if and only if
there exists bounded linear map $U \colon X \to Y$ such that $V = U^*$.

Conversely, the quoted continuity properties e.g. for T on A^{**} implies that for each $\varphi \in B^*$ the map $\varphi \circ T$ is in A^* and has norm $\leq \|\varphi\| \cdot \|T\|$. Thus, $V(\varphi) := \varphi \circ T$ defines a bounded linear map $V \colon B^* \to A^*$.

Compare notational chaos with above:

From Banach space theory it follows that a continuous linear map $V \colon X \to Y$ (here $U := V^* = T_*$) for Banach spaces X and Y (here in our case $X := B^*$ and $Y := A^*$) has an adjoint $T := L^* \colon Y^* = A^{**} \to X^* = B^{**}$ that is $\sigma(A^{**}, A^*)$ – $\sigma(B^{**}, B^*)$ continuous, i.e., is continuous with respect to the topologies $\sigma(A^{**}, A^*)$ on A^{**} and $\sigma(B^{**}, B^*)$ on B^{**}, because this means that $L(\varphi) = \varphi \circ T$ is in A^* for each $\varphi \in B^*$.

In particular, then ????

Indeed, we can consider $\varphi\colon B \to \mathbb{C}$ as linear operator. Then $\varphi^{**}\colon B^{**} \to \mathbb{C}$ is in $B^* \subseteq (B^{**})^*$ and $(\varphi \circ V)^{**} = \varphi^{**} \circ T$. But this says that $T\colon A^{**} \to B^{**}$ is continuous with respect to $\sigma(A^{**}, A^*)$ and $\sigma(B^{**}, B^*)$.

In the special case of ???????? we have to require in addition also that $U(a^*) = U(a)^*$ and $V(a^*) = V(a)^*$ for all $a \in A$, such that $S := U^{**}$ and $T := V^{**}$ map selfadjoint elements to selfadjoint elements.

REMARK A.9.5. We consider on the class of C^*-algebras only "symmetric" Jordan morphisms $T\colon A \to B$ for C^*-algebras A and B, i.e., linear maps with the properties that $T(a^*) = T(a)^*$ for all $a \in A$ and $T(h^2) = T(h)^2$ for $h^* = h \in A$.

It follows then that $T(a^*b + b^*a) = T(a)^*T(b) + T(b)^*T(a)$ for all $a, b \in A$.

The second conjugate $T^{**}\colon A^{**} \to B^{**}$ of a C^*-Jordan morphism is again a C^*-Jordan morphism.

`PROOF of this ??:`

At first T and T^{**} are positive linear contractions on $A_{s.a.}$ and $(A^{**})_{s.a.}$, because $T(a) = T(a^{1'/2})T(a^{1/2}) \geq 0$ for $a \in A_+$ and $\chi(T(\cdot)) \in (A^*)_+$ for positive $\chi \in B^*$ gives that its natural extensions is the positive functional $\chi(T^{**}(\cdot)) \in A^*$ and is again positive on A^{**}_+ and T^{**} is $\sigma(A^{**}, A^*) - \sigma(B^{**}, B^*)$ continuous.

Let $0 \geq c \in A^{**}$ with $\|c\| = 1$. Then there exists a net of self-adjoint contractions $\{a_\lambda\} \subseteq A_+$ that converges to c *-ultra-strongly in A^{**}. Then $T(a_\lambda^2) \to T(c^2)$???????

If $\rho \in B^*$ is a state then $\rho(T(a_\lambda)T(a_\lambda)) \to$???

LEMMA A.9.6. *Let A and B C^*-algebras and $T\colon A \to B$ a C^*-Jordan morphism. Then the following properties of T are equivalent:*

(i) *T is 2-positive, i.e., $T \otimes \mathrm{id}_{M_2}\colon A \otimes M_2 \to B \otimes M_2$ is positive.*
(ii) *The linear map $T_2 := T \otimes \mathrm{id}_{M_2}\colon A \otimes M_2 \to B \otimes M_2$ is contractive, i.e., has norm ≤ 1.*
(iii) *T is C^*-algebra morphism.*

PROOF. It is somewhere calculated (further down?).

It holds $(T_2)^{**} = T_2^{**}$. Thus one can restrict the considerations to W*-algebras. □

REMARK A.9.7. Let A denote a C^*-algebra.

The open support projection $p_t \in A^{**}$ of $(a-t)_+ \in A_+$ is the unit element of $D_t^{**} \subseteq A^{**}$, where $D_t := \overline{(a-t)_+ A (a-t)_+}$ with $t \in [0, \|a\|]$. The projection p_t is contained in $C * (a)^{**} \subseteq A^{**}$ and is the unit element of $C * ((a-t)_+)^{**} \subseteq A^{**}$.

We can here replace A by $C^*(a)$ and A^{**} by the second conjugate $(C^*(a))^{**} \subseteq A^{**}$ of $C^*(a)$.

We denote the projection $p_0 \in A^{**}$ – respectively more generally the p_0 for a positive contraction in any W*-algebra M –, for $a \in A_+$ also by p_a, respectively by p_b for $b \in A_+$ (or $b \in M$), to distinguish the support projections of different elements $a, b \in A_+$.

If $a \neq 0$ and $t \in [0, \|a\|]$ then the projection p_t is a projection that is the smallest positive contraction $q \in A^{**}$ with the property

$$(\|(a-t)_+\|^{-1}(a-t)_+)^{1/n} \leq q \text{ for all } n \in \mathbb{N}\,.$$

This remains true for every W^*-algebra M (in place of the special case $M := A^{**}$):

The "support projection" $p_a \in M$ for some positive contraction $a \in M_+$ is the smallest positive contraction in M with $a^{1/n} \leq p_a$ for all $n \in \mathbb{N}$; i.e.:

Let $0 \leq x \leq 1$ in M with $a^{1/n} \leq x$ for all $n \in \mathbb{N}$ then $p_a \leq x$.

In particular, if N is another W*-algebra and $\psi \colon M \to N$ is an injective unital C^*-algebra mono-morphism from M into N and $E \colon N \to \psi(M) \subseteq N$ is a conditional expectation of N onto a C^*-subalgebra $\psi(M)$ of N, then $E(p_\psi(a)) = \psi(p_a)$, where $p_\psi(a)$ the support projection in N of $\psi(a) \in N$ and p_a the support projection in N of a positive contraction $a \in N$.

This applies to the natural unital conditional expectation from $N := M^{**}$ onto $M \subseteq M^{**}$, where ψ denotes here the canonical embedding of the W*-algebra M in its second conjugate N.

Let $0 \leq s < t \leq 1$, $a \in A_+$ with $\|a\| \leq 1$ and $p_s, p_t \in A^{**}$ the open support projection of $(a-s)_+$ and $(a-t)_+$ then $(a-t)_+ \leq (a-s)_+$, $p_s p_t = p_t$ and $(a-t)_+ = ((a-s)_+ - (t-s))_+ \leq (t-s) \cdot p_s + (a-t)_+$. It follows that $(a-s)_+ - (a-t)_+ \leq (t-s)p_s$. Moreover, $p_t((a-s)_+ - (a-t)_+) = (t-s)p_t \leq (a-s)_+ - (a-t)_+$.

Thus, $(t-s)p_t \leq (a-s)_+ - (a-t)_+ \leq (t-s)p_s$ for the support projections p_t for $(a-t)_+$ and p_s for $(a-s)_+$ and $0 \leq s \leq t \leq 0$.

LEMMA A.9.8. *Let A a C^*-algebra.*

(i) *Every $a \in A_+$ is contained in the norm closure of the convex set generated by the open support projections $p_t := p_{(a-t)_+} \in A^{**}$ of $(a-t)_+ \in A_+$ with $t \in (0, \|a\|)$.*
(ii) *All elements of A are contained in the (norm-)closed linear span of the open support projections $p_a \in A^{**}$ of elements $a \in A_+$.*
(iii) *Let $T \colon A \to M$ a bounded linear map into a W^*-algebra M, and let $\widetilde{T} \colon A^{**} \to M$ denote its unique natural extension to a $\sigma(A^{**}, A^*)$–$\sigma(M, M_*)$ continuous linear map (i.e., the normal extension of T).*

The point is:
It uses the natural conditional expectation from M^{**} onto M,
where M is considered as C- subalgebra of M^{**}.*

The linear map T is a Jordan morphism (cf. Remark A.9.5) if and only if, $\widetilde{T}(p_a)$ is an orthogonal projection in M for the open support projections $p_a \in A^{**}$ of each elements $a \in A_+$.

PROOF. (i): Let $a \in A_+$ with $\|a\| = 1$ and $p_{n,k} \in C*(a)^{**} \subseteq A^{**}$ the open support projection of $a_k := (a-(k-1)/n)_+ \in C*(a) \subseteq A$ for $k \in \{1, \ldots, n+1\}$. Then $a_\ell \leq a_k$ for $k \leq \ell$, and

$$a_{k+\ell}(a_k - a_{k+\ell}) = \|a_k - a_{k+\ell}\| a_{k+\ell} \,.$$

Then $a_1 = a$, $a_2 = (a-1/n)_+$, ..., $a_k = (a-(k-1)/n)_+$, ..., $a_n = (a-(n-1)/n)_+$, $a_{n+1} = 0$. Thus, $a = \sum_{k=1}^{n}(a_k - a_{k+1})$. Notice that $a_k \geq a_\ell$ for $\ell > k$.

We denote hereby $p_{n,k}$ the open support projection of a_k, $k = 1, \ldots, n+1$. Then $p_{n,n+1} = 0$. Thus

$$1/np_{n,k+1} \leq a_k - a_{k+1} \leq 1/np_{n,k} \,.$$

It follows that

$$(\sum_{k=1}^{n} p_{n,k+1}) \leq n \cdot a \leq (\sum_{k=1}^{n} p_{n,k}) \,.$$

Since $p_{n,n+1} = 0$, the difference of the left and right estimates is $= p_{n,1}$. It shows that $a \in A_+$ can be approximated arbitrarily well by scalar multiples of sums of open projections in A^{**}.

(ii): Each element $a \in A$ is the (complex) linear combination of at most 4 elements in A_+.

(iii): If $T\colon A \to M$ is a C^*-Jordan morphism and $a \in A_+$ is a positive contraction then $T(a) = T(a^{1/n})^n$ for all $n \in \mathbb{N}$. Thus, $T(a)^{1/n} = T(a^{1/n})$ for $n \in \mathbb{N}$.

Is $T^{**}\colon A^{**} \to M^{**}$ again C^*-Jordan???

Is the "natural" map $M^{**} \to M$ multiplicative?

The increasing sequence $T(a)^{1/n}$ converges in M to the support projection of $T(a)$ with respect to the $\tau(M, M_*)$-topology, that is stronger than the $\sigma(M, M_*)$-topology.

The increasing sequence of elements $a^{1/n}$ converge $\tau(A^{**}, A^*)$-strongly in A^{**} to the support projection $p_a \in A^{**}$ of a. Here $p_a A^{**} p_a$ is the $\tau(A^{**}, A^*)$ closure of the hereditary C^*-subalgebra $\overline{aAa}$

of ??????

In general the map $\widetilde{T}\colon A^{**} \to M$ is continuous with respect to the topologies $\sigma(A^{**}, A^*)$ and $\sigma(M, M_*)$, because $\widetilde{T} = P_M \circ T^{**}$ where $T^{**}\colon A^{**} \to M^{**}$ is the weakly continuous bi-dual map from T and $P_M\colon M^{**} \to M$

together τ-continuous ???

normal (!!! ???) W*-algebra epimorphism from M^{**} onto M.

Is this correct ????

$T\colon A \to M$ (positive because Jordan)

$T^{**}\colon A^{**} \to M^{**}$ σ-continuous.

(One could also use an orthogonality argument,

Or reduction to abelian case?

Take maximal commutative C^*-subalgebra C of A that contains the given element $a \in A_+$.

Then $C^{**} \subseteq A^{**}$ $T|C$ is a C^*-algebra morphism from C into M.

Second conjugate $T^{**}|C = (T|C)^{**} \colon C^{**} \to M^{**}$ The algebra $T(C) \subseteq M$ is commutative.

in particular $p_a \in A^{**} \to \widetilde{T}(p_a) \in M$

□

LEMMA A.9.9. *Let A and B denote C^*-algebras and suppose that $S, T \colon A^{**} \to B^{**}$ are $\sigma(A^{**}, A^*)$-$\sigma(B^{**}, B^*)$ continuous "symmetric" linear maps in the sense that $S(a^*) = S(a)^*$ and $T(a^*) = T(a)^*$ for all $a \in A$.*

(i) *Let $a, b \in A_+$ positive contractions with $a \in D := \overline{bAb}$ and $p, q \in A^{**}$ the open projection $p = pq \leq q$ corresponding to the hereditary C^*-subalgebras D and $E := \overline{aAa} \subseteq D$ of A.*

Then $S(p)T(1-q) \in A^{**}$ is contained in the $\sigma(A^{**}, A^*)$-closure of the set of elements $S(e)T(1-d)$ with positive contractions $e \in E_+$, $d \in D_+$ with $de = e$.

Moreover, each element $S(e)T(1-d)$ with positive contractions $d \in D_+$ and $e \in E_+$ and $de = e$ is in the $\sigma(A^{**}, A^*)$-closure of the set of elements $S(e)T(g)$ with positive contractions $g \in \overline{(1-d)A(1-d)}$.

(ii) *In particular, $S(p)T(1-q) = 0$ if $p, q \in A^{**}$ are open projections with the properties that $qp = q$ and for each contractions $a, b, c \in A_+$ with $cb = b$ and $ca = a$. holds $S(a)T(c-b) = 0$ for all positive contractions $a, b, c \in A_+$ with the properties that $a \leq p$ and $b \leq q$ and $cb = b$ $a(c-b) = 0$.*

?????????????????

$S((a - 1/n)_+)T(1 - f_m(b)) = 0$ for all $a \in A_+$ with $0 \leq a \leq p$ and $n, m \in \mathbb{N}$ with $(a - 1/n)_+(1 - f_m(a)) = 0$.

REMARKS A.9.10. Notice that Part (iii????) of Lemma A.9.9 contains the case $a = b$ and $D = E$.

The continuity properties of S and T are equivalent to the existence bounded linear maps $U, V \colon A \to B$ with $S := U^{**}$ and $T := V^{**}$. Because in general Banach space theory one has that a continuos map $V \colon A \to B$ from

In our case $U(a^*) = U(a)^*$ and $V(a^*) = V(a)^*$ for all $a \in A$, such that $S := U^{**}$ and $T := V^{**}$.

Proofs not ready, because of the partial continuity of ??? !!!

Ad(iii):

TEXT: Let $a, b \in A_+$ positive contractions with $a \in D := \overline{bAb}$ and $p, q \in A^{**}$ the open projection $p = pq \leq q$ corresponding to the hereditary C^*-subalgebras D and $E := \overline{aAa} \subseteq D$ of A.

Then $S(p)T(1-q) \in A^{**}$ is contained in the $\sigma(A^{**}, A^*)$-closure of the set of elements $S(e)T(1-d)$ with positive contractions $e \in E_+$, $d \in D_+$ with $de = e$.

We use the piece-wise linear continuous functions $\varphi_n(t)$ on $[0,1]$ given by $\varphi_n(t) := 2^{n+1}((t - 2^{-(n+1)})_+ - (t - 2^{-n})_+)$ for $n = 1, 2, \ldots$. This continuous functions are non-negative, $\varphi_n(0) = 0$, $\|\varphi_n\| = 1$, $\varphi_{n+1}\varphi_n = \varphi_n$ and φ_n converges on each interval $[s, 1] \subseteq (0, 1]$ uniformly to 1. For every $n \in \mathbb{N}$ and protection $P^* = P = P^2$ holds $\varphi_n(P) = P$.

Let $e_n := \varphi_n(a) \in E_+$, where a is a strictly positive contraction in E. Then the e_n are positive contractions that satisfy $e_m e_n = e_m$ for $m < n$ and the sequence $e_1, e_2, \ldots$ converges to the support projection $p \in A^{**}$ of a in the $\tau(A^{**}, A^*)$ topology, that is stronger than the $\sigma(A^{**}, A^*)$ topology on A^{**}. By our assumption that $S\colon A^{**} \to A^{**}$ is $\sigma(A^{**}, A^*)$-continuous shows that the sequence $S(e_1), S(e_2), \ldots$ converges to the element $S(p) \in A^{**}$ in the $\sigma(A^{**}, A^*)$ topology.

Let $p \leq q \in A^{**}$ denote the open support projection $E = \overline{aAa} \subseteq D = \overline{bAb}$. Since $e_n \leq p \leq q$, we get that $e_n + (1-e_n)^{1/2} q (1-e_n)^{1/2} = q$ for each $n \in \mathbb{N}$, i.e., $q - e_n = (1-e_n)^{1/2} q (1-e_n)^{1/2}$.

Let $f_n := \varphi_n(a) \in D_+$. This positive contractions satisfy $f_m f_n = f_m$ for $m < n$, and the increasing sequence $f_1, f_2, \ldots$ converges to the support p of a

Need to find contractions $d_n \in D_+$ (for $D = \overline{bAb}$) with $d_n e_n = e_n$ and $d_n \to q$ in A^{**}.

$d_{n-1} = e_n + (1-e_n)^{1/2} f_n (1-e_n)^{1/2}$

for suitable contractions $f_n \in D_+$ with $f_n \to q$ in A^{**} and $f_n^{1/2} e_n^{1/2} - e_n^{1/2} f_n$ and $e_n - f_n e_n$ small ...

Since $e_n \leq p \leq q$, we get $e_n + (1-e_n)^{1/2} q (1-e_n)^{1/2} = q$????

We call a projection $p \in A^{**}$ *open* in the following Lemma A.9.11 if $pA^{**}p$ is the $\sigma(A^{**}, A^*)$-closure of a hereditary C^*-subalgebra D of A. We say that p is a σ-unital open projection if the corresponding $D \subseteq A$ is σ-unital, i.e., if $D = \overline{dAd}$ for some $d \in A_+$. The following Lemma A.9.11 considers special cases of orthogonally preserving pairs of symmetric maps $V, W\colon A \to B$.

(It seem that the general theory of such pairs, without assuming the additional equivalent conditions (i)–(ix) in Lemma A.9.11 has to do with approximately inner gradings on C^*-algebras and questions of different nature.)

Lemma A.9.11. *Let A and B C^*-algebras and let $V\colon A \to B$ and $W\colon A \to B$ linear maps that are "symmetric" in the sense that $V(a^*) = V(a)^*$ and $W(b^*) = V(b^*)$. and are relatively orthogonality preversing that satisfy*

(1) $T := V^{**}(1_{A^{**}}) = W^{**}(1_{A^{**}}) \in B^{**}$ *and*

(2) $V(a_1)W(a_2) = 0$ *for all* $a_1, a_2 \in A_+$ *with* $a_1a_2 = 0$.

Then the following properties (i)–(ix) of V *and* W*, respectively of its bi-adjoints* V^{**} *and* W^{**}*, are equivalent:*

(i) $TV(a) = V(a)T$ *for all* $a \in A$.
(ii) $TW(a) = W(a)T$ *for all* $a \in A$.
(iii) $TV^{**}(p) = V^{**}(p)T$ *for all open projections* $p \in A^{**}$.
(iv) $TW^{**}(p) = W^{**}(p)T$ *for all open projections* $p \in A^{**}$.
(v) $TV^{**}(p) = V^{**}(p)T$ *for all* σ*-unital open projections* $p \in A^{**}$.
(vi) $TW^{**}(p) = W^{**}(p)T$ *for all* σ*-unital open projections* $p \in A^{**}$.
(vii) $V = W$.
(viii) $V(a)V(b) = 0$ *for all* $a, b \in A_+$ *with* $ab = 0$.
(ix) $W(a)W(b) = 0$ *for all* $a, b \in A_+$ *with* $ab = 0$.

Proof. □

Proposition A.9.12. *If* $V\colon A \to B$ *is symmetric and orthogonality preserving and has norm* $\|V\| \le 1$*, then there are two uniquely defined symmetric Jordan morphisms* $\psi_k\colon \mathrm{C}_0((0,1], A) \to B$ *(*$k \in \{1, 2\}$ *with the following properties:*

(i) $\psi_1(\mathrm{C}_0((0,1], A)) \cdot \psi_2(\mathrm{C}_0((0,1], A)) = \{0\}$ *and*
(ii) $V(a) = \psi_1(f_0 \otimes a) - \psi_2(f_0 \otimes a)$ *for all* $a \in A_+$.

Proof. Next proof wrong here?

By Lemma A.9.9 we get with $W = V$ that $W^{**}(1-p)V^{**}(p) = 0 = V^{**}(1-p)W^{**}(p)$

Thus $TW^{**}(p) = V^{**}(p)W^{**}(p)$, $V^{**}(p)T = V^{**}(p)W^{**}(p)$ and $TV^{**}(p) = W^{**}(p)V^{**}(p)$. Thus $TW^{**}(p) = V^{**}(p)T$. It follow that $TW(a) = V(a)T$ for all $a \in A$.

□

Lemma A.9.13. *Let* $\varphi\colon A \to B$ *an orthogonality preserving map in sense of Definition A.9.1. Then the restriction* $\psi\colon C^*(A, 1_{A^{**}}) \to B^{**}$ *of* $\varphi^{**}\colon A^{**} \to B^{**}$ *is again an* orthogonality preserving *map from* $C^*(A, 1_{A^{**}}) = A + \mathbb{C}\cdot 1_{A^{**}}$ *into* B^{**}.

Proof. We write here 1 for the unit element $1_{A^{**}}$ of A^{**}. If A is unital then nothing is to prove. If A is not unital then $A + \mathbb{C}\cdot 1 = C^*(A, 1) \subseteq A^{**}$ is natural isomorphic to the standard unification of A.

It implies: If $x^* = x$ and $y^* = y$ elements of A and $\alpha, \beta \in \mathbb{R}$ such that $(x + \alpha 1)(y + \beta 1) = 0$, then necessarily $\alpha\beta = 0$, and $xy = yx$.

If $\alpha = \beta = 0$ then $xy = 0$ and $\psi(x)\psi(y) = \varphi^{**}(x)\varphi^{**}(y) = \varphi(x)\varphi(y) = 0$, because φ preserves orthogonality.

By symmetry of the cases $\alpha \neq 0$ and $\beta \neq 0$ for $\alpha\beta = 0$, it suffices to consider the case $\alpha = 0$:
Let $z := -\beta^{-1}y$. It gives $zx = x = xz$, i.e., $(1-z)x = 0$.

If $\varphi^{**}(1-z)\varphi^{**}(x) = 0$, then $\varphi(x)(\varphi(y) + \beta\varphi^{**}(1)) = 0$, because $\beta\varphi^{**}(1 + \beta^{-1}y) = \varphi(y) + \beta\varphi^{**}(1)$ is equivalent to $\varphi^{**}(x)\varphi^{**}(1 + \beta^{-1}y) = 0$.

Suppose that $zx = x$, $x^* = x$ and $z^* = z$. Then also $xz = x$, $x_+ = zx_+$ and $x_- = zx_-$, because for example $x(x_+^{1/n}) \to x_+$, and, in the same way, $x_- = zx_-$.

A similar argument shows that $z_+x_+ - z_-x_+ = zx_+ = x_+$ implies $z_-x_+ = 0$ and $z_+x_+ = x_+$. If we take here x_- in place of x_+, then we get $z_-x_- = 0$ and $z_+x_- = x_-$.

It follows that $(z_+ - 1)_+x_+ = 0$ from $(z_+ - 1)x_+ = 0$, because $(z_+ - 1)$ and x_+ commute. (In general, if a selfadjoint element is orthogonal to a positive element, then its positive and negative parts are also orthogonal to the positive element.)

$z_+x = x$, $(z_+ - (z_+ - 1)_+)x = x$, $0 \geq v := 2(z_+ - (z_+ - 1/2)_+) \leq 1$ satisfies $v(z_+ - (z_+ - 1)_+) = (z_+ - (z_+ - 1)_+)$, $vx = x$ and $(1-v)^{1/2}x = 0$ in A^{**}

...

Positive contraction $e_\tau \in A_+$ suitably chosen ...

Need: $v + (1-v)^{1/2}e_\tau(1-v)^{1/2} \in A_+$ converges in A^{**} to 1 in $\sigma(A^{**}, A^*)$ topology.

Since ψ is linear, ...

□

Corollary A.9.14. *Let A and B C^*-algebras, $\varphi\colon A \to B$ an* orthogonality preserving

My blue Remarks:

Let $\varphi\colon A \to B$ a 2-positive and orthogonality preserving map.

Question:
Does there exists a C^*-algebra morphism $h\colon \mathrm{C}_0(0,1] \otimes A \to B$ and an element $g_\varphi \in \mathrm{C}_0(0,1]_+$ such that $\varphi(a) = h(g \otimes a)$ for all $a \in A$.

(Is there a certain uniqueness for such h and g?)

If A is not unital or B is not unital, then take $\varphi^{**}\colon A^{**} \to B^{**}$.

(It is here not clear if φ^{**} is still orthogonality preserving, – except for "open" projections in A^{**} corresponding to hereditary C^*-subalgebras of A. But φ^{**} maps selfadjoint elements to selfadjoint elements, respectively is positive, 2-positive or c.p. if φ has this properties.)

One has now to check if the (again 2-positive) restriction $\varphi^{**}|C^*(A \cup \{1_{A^{**}}\})$ of φ^{**} is again orthogonality preserving:

Let $a_1, a_2 \in A$ selfadjoint, $\alpha_1, \alpha_2 \in \mathbb{R}$ and suppose that A is not unital. If $x := a_1 + \alpha_1 \cdot 1_{A^{**}}$ and $y := a_2 + \alpha_2 \cdot 1_{A^{**}}$ are "orthogonal" in the sense $xy = 0$, then at least one of α_1 and α_2 is equal to 0. Therefore, up to change of indices and multiplication by a non-zero real number, this reduces all to the cases where $a_1a_2 = 0$ and $a_1a_2+a_1 = a_1(a_2+1) = 0$. Then $a_1a_2 = 0$ leads to $\varphi(a_1)\varphi(a_2) = 0$ by using the assumption that φ preserves orthogonality of selfadjoint elements. The case $a_1a_2 = -a_1$ induces that a_1 and a_2 are commuting selfadjoint elements and that $e^* = e := -a_2$ is a local unit for a_1.

Reduces to the case of $A_0 := C^*(a, e)$ with $ae = a$, $a^* = a$, $e^* = e$, $\varphi_0 \colon A_0 \to B$ linear, bounded, involution and orthogonality preserving. $C := A_0 + \mathbb{C} \cdot 1 \subseteq A^{**}$ (where 1 denotes here the unit element $1_{A^{**}}$ of A_0^{**}).

NEED that $\varphi_0(a)(\varphi^{**}(1_C)) = (\varphi^{**}(1_C))\varphi_0(a)$ and

??????????????????????

Let $T := \varphi^{**}(1_{A^{**}}) \in B^{**}$.

Does T commute element-wise with $\varphi(A)$?

Suppose (!!!) we have found a C^*-Jordan morphism $\psi \colon A \to \{T\}' \cap B^{**}$ with $\varphi(a) = T \cdot \psi(a)$, and suppose that φ is 2-positive on A.

Can we deduce that $\psi \colon A \to B^{**}$ is 2-positive if φ is 2-positive?

It would be enough to show that

$$\psi_\delta(a) := T_\delta\varphi(a)T_\delta$$

for $T_\delta := (T + \delta 1_{B^{**}})^{-1/2}$ is 2-positive for all $\delta \in (0, \|T\|/2)$.

(THIS ARE THE KEY QUESTION!)

Since $\psi(A)$ commutes ???????

It should reduce to the study of the images of the support projection $P \in A^{**}$ of positive elements in the Pedersen ideals of A:

They should have the important property $\varphi^{**}(1-P)\varphi^{**}(P) = 0$. It is then the key for all other study!

(Seems to be OK at least in case of positive maps φ, because if $a, b \in \mathrm{Ped}(A)_+$ are contractions with $ab = a$ and $b \le P$, then $(1-b)a = 0$ and $a \le P$, $(1-P) \le 1-b$. ...)

To obtains that $\varphi^{**}(1-b)\varphi^{**}(a) = 0$ we have to check if $(1-b)$ the $\sigma(A^{**}, A^*)$ - limit of elements $(1-b)^{1/2}c(1-b)^{1/2}$, where $c \in \mathrm{Ped}(A)_+$ with $\|c\| \le 1$.

All this should work also in case of self-adjoint non-positive case of $\varphi \colon A \to B$.

Precise estimates in self-adjoint case?

This could give that $\varphi^{**}(P) = T \cdot \Phi(P)$ with $\Phi(P)^2 = \Phi(P) = \Phi(P)^*$, where $T := \varphi^{**}(1_{A^{**}}) \in B^{**}$ is a *-preserving map (self-adjoint maps).

It seems that the Φ can be integrated to to a C^*-Jordan morphism from A to B^{**}.

It is not clear if $\Phi(A)B \subseteq B$ ...(likely not!)

Is this new bounded linear map again orthogonality preserving on self-adjoint elements?

(Seems to be by the above outlined decomposition for the images of the "compact" projections!)

General observations:
A C^*-Jordan morphism $\psi\colon A \to C$ is a linear map with $\psi(a^*) = \psi(a)^*$ for all $a \in A$, and satisfies $\psi(ib) = i\psi(b)$ and $\psi(b^2) = \psi(b)^2$ for all $b \in A_+$. In particular, ψ is a *positive* linear map: $\psi(A_+) \subseteq C_+$.

Then $\psi(b^2) = \psi(b)^2$ for all $b^* = b \in A_{s.a.}$. $A_{s.a.}$ is a real Jordan algebra with $[a, b] := (1/2) \cdot (ab + ba)$ for $a, b \in A_{s.a.}$, and $\psi|A_{s.a.}$ is a real Jordan algebra morphism from $A_{s.a.}$ into $C_{s.a.}$, because $ab + ba = (a + b)^2 - (a^2 + b^2)$. Clearly ψ is determined by its restriction to $A_{s.a.} = A_+ - A_+$.

Lemma A.9.15. *If $\psi\colon A \to C$ is a C^*-Jordan morphism then properties (i), (ii) and (iii) are equivalent:*

(i) *ψ is a C^*-morphism.*
(ii) *$\psi(a^*a) = \psi(a)^*\psi(a)$ for all $a \in A$.*
(iii) *$\psi(a^*a) \geq \psi(a)^*\psi(a)$ for all $a \in A$.*

Proof. The implications (i)⇒(ii)⇒(iii) are obvious.

(iii)⇒(ii): Always $\psi(a^*) = \psi(a)^*$ for C^*-Jordan morphisms ψ and

$$\psi(a)^*\psi(a) + \psi(a)\psi(a)^* = \psi(a^*a + aa^*) = \psi(a^*a) + \psi(aa^*)$$

Suppose $\psi(a^*a) \geq \psi(a)^*\psi(a)$ and $\psi(aa^*) \geq \psi(a)\psi(a)^*$, then this all together implies that the sum of the two positive elements $\psi(a^*a) - \psi(a)^*\psi(a)$ and $\psi(aa^*) - \psi(a)\psi(a)^*$ is zero. Thus, $\psi(a^*a) = \psi(a)^*\psi(a)$ and $\psi(aa^*) = \psi(a)\psi(a)^*$.

(ii) ⇒ (i): For C^*-Jordan morphisms ψ holds in general

$$\psi(h)\psi(k) + \psi(k)\psi(h) = \psi(hk + kh) = \psi(hk) + \psi(kh)$$

by using the equations $\psi((h + k)^2) = (\psi(h) + \psi(k))^2$, $\psi(h^2) = \psi(h)^2$ and $\psi(k^2) = \psi(k)^2$. It delivers the general formula:
$\psi(h)\psi(k) + \psi(k)\psi(h) = \psi(hk) + \psi(kh)$.

Now use $\psi(a^*a) = \psi(h)^2 + \psi(k)^2 + i(\psi(hk - kh))$ and
$\psi(a)^*\psi(a) = \psi(h)^2 + \psi(k)^2 + i(\psi(h)\psi(k) - \psi(k)\psi(h))$

The property $\psi(a^*a) = \psi(a)^*\psi(a)$ and that $\psi(hk - kh) = \psi(hk) - \psi(kh)$ imply the equation

$$\psi(h)\psi(k) - \psi(k)\psi(h) = \psi(hk) - \psi(kh)\,.$$

Add this to the general formula for C^*-Jordan morphisms (obtained by pentagon rule): $\psi(h)\psi(k) + \psi(k)\psi(h) = \psi(hk) + \psi(kh)$ and get $2\psi(h)\psi(k) = 2\psi(hk)$.

It says that ψ is multiplicative. Thus $\psi\colon A \to C$ is a C^*-algebras morphism. □

Remark: Let $T\colon A \to C$ a $\mathbb{C}$-linear map with $\|T\| \leq 1$, $T^{**}(1_{A^{**}}) = 1_{C^{**}}$ then T is positive.

LEMMA A.9.16. *Let A and C C^*-algebras, $T: A \to C$ a positive linear map (i.e., T is linear and $T(A_+) \subseteq C_+$).*

Then $\|T\| = \|T^{**}(1_{A^{**}})\|$ and $T(a)^2 \leq \|T\| \cdot T(a^2)$ for all $a^* = a \in A$.

If, moreover, T is 2-positive, in the sense that

$$T_2 := T \otimes \mathrm{Id}_{M_2} : A \otimes M_2 \to C \otimes M_2$$

is positive, then, for each $a \in A$,

$$T(a)^* T(a) \leq \|T\| \cdot T(a^* a) .$$

If T is positive and unital and $T \otimes \mathrm{id}_{M_2}$ has norm ≤ 1 on $A \otimes M_2$, then $T \otimes \mathrm{id}_{M_2}$ is positive.

Is generally a unital hermitian contraction T automatically positive:

Restrict T to $C^*(1, a)$ for $a \in A_+$, $\|a\| \leq 1$. Take pure state ρ on C with $\rho(T(a)) = \min\{t \in \mathrm{Spec}(T(a))\}$. Then $\rho \cdot T$ is a unital linear functional on A with norm ≤ 1.

(Is equivalent for positive T^{**}, to decompose T^{**} as $T^{**} = GS(\cdot)G$ with $G := T^{**}(1_{A^{**}})^{1/2}$ for some suitable positive $S: A^{**} \to C^{**}$ with $S(1) =$ support projection of G. Then require (!) that $S_2 := S \otimes \mathrm{id}_{M_2}$ has norm $= 1$. This requirement is equivalent to the 2-positivity of T.)

(The 2-positivity for positive T could be also equivalent to

$$Z \cdot \mathrm{diag}(T(a_{1,2}), T(a_{2,1})) \leq \mathrm{diag}(T(a_{11}), T(a_{22})) .$$

Here $Z \in M_2(\mathcal{M}(A))$ has entries $[1 - \delta_{j,k}]$, and $[a_{j,k}] \in M_2(A)_+$.

This should be equivalent to $\|(1/n + T(a_{1,1}))^{-1} T(a_{1,2})(1/n + T(a_{2,2}))^{-1}\| \leq 1$ and $\|(1/n + T(a_{2,2}))^{-1} T(a_{2,1})(1/n + T(a_{1,1}))^{-1}\| \leq 1$ ...

PROOF. Consider in place of A the commutative C^*-algebra $C^*(a)$. Then $T|C^*(a)$ is the restriction of the completely positive map $S := (T|C^*(a))^{**}$ from $C^*(a)^{**} \subseteq A^{**}$. Let $P \leq 1_{A^{**}}$ denote the unit-element of $C^*(a)^{**}$. Passage to linear sums of projections show that S is c.p. with norm $\|S\| = \|S(P)\| \leq \|T^{**}(1_{A^{**}})\| \leq \|T\|$ and therefore $T(a)^2 = S(a)^2 \leq \|S(P)\| S(a^2) \leq \|T\| T(a^2)$.

The map $T_2 := T \otimes \mathrm{id}_{M_2}$ has norm $\|T_2\| = \|T \otimes 1_2\| = \|T\|$ if T_2 is again positive (by the additional assumption). We can build for $a \in A$ the selfadjoint element $b^* = b := [a_{jk}]$ in $M_2(A) \cong A \otimes M_2$ with zero diagonal entries $b_{11} = b_{22} = 0$ and off-diagonal entries $b_{12} := a$ and $b_{21} := a^*$.

Can now apply the general formula for positive maps and self-adjoint elements: Get $T_2(b)^2 \leq \|T\| T_2(b^2)$ by positivity of T_2 and $\|T_2\| = \|T\|$. The upper left entries show that $T(a^*)T(a) \leq \|T\| \cdot T(a^* a)$. □

The following Proposition is a straight conclusion of above Lemmata A.9.15 and A.9.16.

Proposition A.9.17. *Let A and C C^*-algebras and $\psi\colon A \to C$ a C^*-Jordan morphism.*

Then ψ is a C^*-algebra morphism, if and only if, ψ is a 2-positive contraction, i.e., the algebraic tensor product

$$\psi_2 := \psi \otimes \mathrm{id}_{M_2}\colon A \otimes M_2 \to C \otimes M_2$$

is a positive contraction.

Suppose we have a C^*-Jordan morphism (or a C^*-morphism) $\pi\colon A \to B^{**}$ and $T \geq 0$ in B^{**} such that $\pi(A) \in \{T\}' \cap B^{**}$ and $T \cdot \pi(a) = \varphi(a) \in B$.

The C^*-Jordan morphism π is 2-positive (and therefore is a C^*-algebra morphism) if φ is 2-positive, because $\pi(\cdot)$ is point-wise limit of $(T + 1/n)^{-1/2}\varphi(\cdot)(T + 1/n)^{-1/2}$ for $n \to \infty$, and the maps $b \in B \mapsto (T + 1/n)^{-1/2}b(T + 1/n)^{-1/2}$ are completely positive.

(Is important to know, for my version of a complete proof!).

??? Let $D \subseteq B$ the hereditary C^*-subalgebra generated by $\varphi(A)$, i.e., $D := \overline{\varphi(A)B\varphi(A)}$, then it is likely that $T \in \mathcal{M}(D)$ (modulo annihilator of D?) and $\pi(a) \in \mathcal{M}(D)$ for all $a \in A$. ?????

10. The "socle" of a C^*-algebra.

Recall that the "socle" of Banach algebra A is the (algebraic) ideal of A generated by all elements $a \in A$ with aAa of finite linear dimension. In case of C^*-algebras A the socle(A) it is an algebraic *-ideal generated by all projections $p^* = p^2 = p \in A$ with pAp of finite dimension.

The closure of the socle of a C^*-algebra A is the largest closed ideal of A that is a c_0-sum of "elementary" C^*-algebras $\cong \mathbb{K}(\mathcal{H})$ with $\mathcal{H}$ of finite or infinite dimensions.

Lemma A.10.1. *Let ρ a pure state on A with $\rho(\mathrm{socle}(A)) = \{0\}$ and $a \in A_+$ with $\|a\| = 1 = \rho(a)$, then, for every $\delta > 0$ and $n \in \mathbb{N}$, in the commutant of $(a-(1-\delta))_+$ in $(a-(1-\delta))_+A(a-(1-\delta))_+$ contains pairwise orthogonal positive elements $a_1, \ldots, a_n$ with $\|a_i\| = 1$.*

Notice that the element $a \in A_+$ could be a projection $a = p$ and that then $(a - (1 - \delta))_+ = \delta p$ for all $\delta \in [0, 1]$.

Proof. If 1 is not isolated in the spectrum of a, then $a_1, \ldots, a_n$ exist by functional calculus applied to $(a - (1 - \delta/2))_+$. If 1 is isolated, then there are $\gamma > 0$, $b \in A_+$ and a projection $q \in A$ with $bq = 0$, $a = b + q$, $\|b\| = 1 - \gamma$. Then $(a - (1 - \delta))_+A(a - (1 - \delta))_+ = qAq$ for $0 < \delta < \gamma$. Since also $\rho(a^2) = 1$, we get $\rho(b(1 - b)) = \rho(a - a^2) = 0$, $\rho(b) = 0$ and $\rho(q) = 1$. It follows that $q \notin \mathrm{socle}(A)$. Thus any maximal commutative C^*-subalgebra C of qAq is infinite-dimensional, i.e., for every $n \in \mathbb{N}$ we find non-zero orthogonal $a_1, \ldots, a_n \in C_+$. □

REMARK A.10.2. Let $J := \overline{\mathrm{socle}(A)}$ and D a hereditary C^*-algebra of A, then $D \cap \mathrm{socle}(A) = \mathrm{socle}(D)$ and $D \cap J = \overline{\mathrm{socle}(D)}$.

Indeed: The elements $a \in \mathrm{socle}(A)$ are characterized by the property that a^*Aa is finite-dimensional. It implies $D\,\mathrm{socle}(A)D \subseteq D \cap \mathrm{socle}(A) = \mathrm{socle}(D)$. And it follows $\overline{\mathrm{socle}(D)} \subseteq DJD = J \cap D$ and $DJD \subseteq \overline{D\,\mathrm{socle}(A)D} \subset \overline{\mathrm{socle}(D)}$.

LEMMA A.10.3. *Let $J := \overline{\mathrm{socle}(A)}$, and $\psi\colon \mathrm{C}_0((0,1], M_n) \to A$ a C^*-morphism with $\|\pi_J \circ \psi(f_0 \otimes e_{1,1})\| = 1$, D the hereditary C^*-subalgebra of A generated by $\psi(f_0 \otimes 1)$, where $f_0(t) = t$ for $t \in [0,1]$.*

Then there exist an increasing continuous map $\lambda\colon [0,1] \to [0,1]$ with $\lambda(0) = 0$ and $\lambda(1) = 1$, and a C^*-morphism $\varphi\colon \mathrm{C}_0((0,1], M_n) \to D$ with $\varphi(f \circ \lambda) = \psi(f)$ for $f \in \mathrm{C}_0((0,1], M_n)$, such that 1 is not isolated *in the spectrum of $\varphi(f_0 \otimes 1)$ and $\psi(f_0(1 - f_0) \otimes 1_n)$ is contained in the hereditary C^*-subalgebra E generated by $\varphi((f_0(1 - f_0) \otimes 1_n)$. Moreover $\varphi(g - (g \circ \lambda)) \in E$ for $g \in \mathrm{C}_0((0,1], M_n)$.*

PROOF. We take $\varphi := \psi$ and $\lambda := f_0$ if 1 is not isolated in $\mathrm{Spec}(\varphi(f_0 \otimes 1))$. Thus we may suppose that 1 is isolated in $\mathrm{Spec}(\varphi(f_0 \otimes 1_n))$.

First we consider the case where $n = 1$ and where $e := \psi(f_0)$ is a strictly positive element of A:
Then there is $\mu \in (0,1)$ such that the hereditary C^*-algebra F of A generated by $(e - \mu)_+$ is unital with unit $p := 1_F = (1 - \mu)^{-1}(e - \mu)_+$. Then $pe = p = pe$ and $\|(1 - p)e\| \le \mu$. Since $\|\pi_J(e)\| = 1$, it follows that p is not in J. Let $C \subseteq F$ a maximal commutative C^*-subalgebra of F. The compact Hausdorff space $X := \mathrm{Prim}(C)$ can't be finite, because $p \in C \setminus J$ and the minimal idempotents of C are in the *socle* of $F = pAp$ and, therefore, are in J. Moreover, by the same argument, if the open subset U of $X := \mathrm{Prim}(C)$ corresponding to the ideal $C \cap J$ of C is closed in X, then the open and closed set $Y := X \setminus U$ (that is homeomorphic to $\mathrm{Prim}(C/J \cap C)$) is not finite. Thus, in any of this cases, there is a point $x_0 \in Y$ that is not isolated in X. It allows to find a function $h\colon \mathrm{Prim}(C) \to [0, (1 - \mu)/2]$ with $h(x_0) = 0$ such that there is a sequence $x_n \in X$ with $0 < h(x_n)$ and $h(x_n) \to 0$. This means that $0 \in \mathrm{Spec}(\pi_J(g))$, and that 0 is not isolated in $\mathrm{Spec}(g) \subseteq [0, (1 - \mu)/2]$ if $g \in C_+$ has Gelfand transform $\widehat{g} = h$.

Now let $a := e(1 - p) + (p - g) = e - g \in A_+$. Then $\mu e \le a \le e$, $\|\pi_J(a)\| = \|\pi_J(p) - \pi_J(g)\| = 1$, and 1 is not isolated in $\mathrm{Spec}(a)$. Moreover, $e(1 - e) = (1 - p)(e - e^2) \le e(1 - e) + (p - g) - (p - g)^2 = a(1 - a)$, i.e., $e(1 - e)$ is contained in the hereditary C^*-subalgebra that is generated by $a(1 - a)$. We denote by φ the (unique) C^*-morphism $\varphi\colon \mathrm{C}_0(0,1] \to A$ with $\varphi(f_0) = a$ and define a pice-wise linear *continuous* function $\lambda\colon [0,1] \to [0,1]$ by $\lambda(t) := t$ for $t \in [0, \mu]$, $\lambda(t) := 1$ for $t \in [(\mu + 1)/2, 1]$ and $\lambda|[\mu, (\mu + 1)/2]$ linear. Then $\varphi(\lambda) = \lambda(a) = e = \psi(f_0)$, and $\varphi(f \circ \lambda) = f \circ \lambda(a) = f(e) = \psi(f)$ for all $f \in \mathrm{C}_0(0,1]$.

In the *general case* we let D denote the hereditary C^*-subalgebra D of A that is generated by $\psi(f_0 \otimes 1_n) \in A_+$, and let $A_1 \subseteq D$ denote the hereditary C^*-subalgebra of A that is generated by $\psi(f_0 \otimes e_{1,1})$. Then $\psi_1(f) := \psi(f \otimes e_{1,1})$

defines a C^*-morphism from $\mathrm{C}_0(0,1]$ into A. There is a natural isomorphism θ from $A_1 \otimes M_n$ onto D such that $\theta(\psi_1(f) \otimes \alpha) = \psi(f \otimes \alpha)$ for all $f \in \mathrm{C}_0(0,1]$ and $\theta(a \otimes e_{1,1}) = a$ for all $a \in A_1$. Since $D \cap J = \overline{\mathrm{socle}(D)}$ and $A_1 \cap J = \overline{\mathrm{socle}(A_1)}$, the C^*-morphism $\psi_1\colon \mathrm{C}_0(0,1] \to A_1$ satisfies the assumptions for the case $n = 1$. With $\varphi_1\colon \mathrm{C}_0(0,1] \to A_1$ and $\lambda\colon [0,1] \to [0,1]$ constructed as above, we have that $\varphi := \theta \circ (\varphi_1 \otimes \mathrm{id}_n)$ and λ have the required properties. (Notice here that $\lambda(1) = 1$ and $\lambda(0) = 0$ implies $g - (g \circ \lambda) \in \mathrm{C}_0(0,1) \otimes M_n$ for all $g \in \mathrm{C}_0((0,1], M_n)$.) □

11. C^*-version of Cohen factorization

The following theorem is a C^*-version of the Cohen factorization theorem, given by G.K. Pedersen in [**621**, Thm. 4.1] with a proof on 13 lines. We give a more detailed proof. Only elementary knowledge on functional calculus and approximate units of C^*-algebras is used.

THEOREM A.11.1. *Suppose that a C^*-algebra A acts on the Banach space X from left with $\|a \cdot x\| \le \|a\|\|x\|$ and* $\mathrm{span}(A \cdot X)$ *dense in X.*

Then for given $\varepsilon > 0$ and $x \in X$, there exist $y \in \overline{\{a \cdot x\,;\ a \in A_+\}}$ and $e \in A_+$ with $e \cdot y = x$, $\|e\| \le 1$ and $\|y - x\| < \varepsilon$.

PROOF. Since the span of $A \cdot X$ is dense in X, any approximate unit (u_λ) for A will converge strongly to the operator 1 in $\mathcal{L}(X)$. We let $\widetilde{A} := A + \mathbb{C}1$ be the unitized C^*-algebra acting on X, and by induction (setting $a_0 := 1$ and $x_0 := x$) we define sequences (a_n) and (x_n) in $\widetilde{A}_+$ and X, respectively, by

$$a_n := a_{n-1} - 2^{-n}(1 - u_n)\,, \quad x_n := a_n^{-1} \cdot x\,.$$

It is easy to verify (by induction) that $a_n \ge 2^{-n}1$ (so that $\|2^{-n}a_n^{-1}\| \le 1$), and that $\mathrm{dist}(a_n, A) = 2^{-n}$.

Moreover

$$x_n - x_{n-1} = a_n^{-1}(a_{n-1} - a_n)a_{n-1}^{-1} \cdot x = 2^{-n}a_n^{-1}(1 - u_n) \cdot x_{n-1}\,.$$

Having chosen the u_n's properly we can therefore assume that $\|x_n - x_{n-1}\| < 2^{-n}\varepsilon$ for all n.

Let $e := \lim a_n$ and $y := \lim x_n$. Since $a_n \cdot x_n = x$ for all n, we have $ey = x$. Moreover, $\|x - y\| < \varepsilon$ by construction. Finally, $e \in A_+$ and $\|e\| \le 1$. In fact, $e = \sum_{n \ge 1} 2^{-n} u_n$. □

The additional condition $y \in \overline{\{a \cdot x\,;\ a \in A_+\}}$ is not mentioned in the original formulation of [**621**, Thm. 4.1], but can be seen from its proof, or can be shown from the decomposition of $x = e \cdot y$ and combination with the corresponding decomposition $e'f = e$ of e with $f \in \overline{C^*(e)_+ e}$ and a contraction $e' \in C^*(e)_+$.

Of course, Theorem A.11.1 works also for non-degenerate right-actions: Consider the left multiplication $a \cdot_\ell x := a \cdot x$ by $a \in A^{op}$, where A^{op} denotes the opposite C^*-algebra of A, i.e., A with opposite multiplication.

An easy separation argument and the fact that $\{e \in A_+ \,;\, \|e\| < 1\}$ is an approximate unit for A show that the non-degeneracy condition $\overline{\mathrm{span}(A \cdot X)} = X$ implies that each $x \in X$ is contained in the closure of $\{a \cdot x \,;\ a \in A_+, \|a\| < 1\}$.

In particular, $A \cdot x = \{0\}$ implies $x = 0$ if X is a non-degenerate Banach A-module.

It allows to extend the left-multiplication $(a, x) \in A \times X \mapsto a \cdot x \in X$ uniquely to a left multiplication $(b, x) \in \mathcal{M}(A) \times X \to X$ that is strictly continuous with respect to $b \in \mathcal{M}(A)$.

Since one can replace the Banach module X by $c_0(X)$ in Theorem A.11.1, this Theorem implies the following formally stronger result:

For every finite sequence $x_1, \ldots, x_n$ in X and $\varepsilon > 0$ there exists a positive contraction $e \in A_+$ and $y_1, \ldots, y_n \in X$ with $y_k \in \overline{\{a \cdot x_k \,;\ a \in A_+\}}$ such that $e \cdot y_k = x_k$, $\|y_k - x_k\| < \varepsilon$ and $\|e\| \leq 1$.

12. On centralizers in asymptotic coronas

The discrete version of asymptotic coronas for a C^*-algebra A is $A_\infty := \ell_\infty(A)/c_0(A)$.

LEMMA A.12.1. *The algebra $F_\infty(C, A) := (C' \cap A_\infty)/\operatorname{Ann}(C, A_\infty)$ is a unital C^*-algebra if C is any σ-unital C^*-subalgebra of A_∞.*

For a better understanding what is going on, notice here that an arguments in [**448**] can be modified to obtain the stronger result for *separable* C^*-subalgebras $C \subseteq A_\infty$ and $D := \overline{C(A_\infty)C}$ that there exists a natural isomorphism

$$(C' \cap A_\infty)/\operatorname{Ann}(C, A_\infty) \cong C' \cap \mathcal{M}(D)\,.$$

PROOF. If $e \in C_+$ is a strictly positive contraction then obviously $C \subseteq D := \overline{e \cdot A_\infty \cdot e} = \overline{C \cdot A_\infty \cdot C}$,

$$D' \cap A_\infty \subseteq C' \cap A_\infty \subseteq \{e\}' \cap A_\infty$$

and $\operatorname{Ann}(\{e\}, A_\infty) = \operatorname{Ann}(D, A_\infty) = \operatorname{Ann}(C, A_\infty)$. (We can suppose that $\|e\| = 1$, but this is not important.) Then $e = (e_1, e_2, \ldots) + c_0(A)$ with contractions $e_n \in A_+$.

Let $(g_1, g_2, \ldots)$ a sequence of positive contractions with $g_n \in C^*(e_n)$ and $\|e_n - g_n e_n\| < 1/n$.

Then $g := (g_1, g_2, \ldots) + c_0(A)$ satisfies $0 \leq g \leq 1$, $g - g^2 \in \operatorname{Ann}(\{e\}, A_\infty)_+$ and $g \in D' \cap A_\infty$. The latter because $gd = d = dg$ for all $d \in D$, i.e., the positive contraction d is a "local" unit for all elements in $D \supseteq C$. It follows that $g + \operatorname{Ann}(\{e\}, A_\infty)$ is a projection in $(D' \cap A_\infty)/\operatorname{Ann}(D, A_\infty)$.

Thus, g is a positive contraction in $D' \cap A_\infty \subseteq C' \cap A_\infty$ with $gc = c = cg$ for all $c \in C$ and $g + \operatorname{Ann}(C, A_\infty)$ is a projection in $(C' \cap A_\infty)_+/\operatorname{Ann}(C, A_\infty)$.

If $h \in C' \cap A_\infty$ and $c \in C$ then $(h - gh)c = 0 = c(h - gh)$ because $ghc = gch = ch = hc$ and $cgh = ch$, i.e., $(h - gh) \in \operatorname{Ann}(C, A_\infty)$.

We obtain $gh, hg \in C' \cap A_\infty$ for $h \in C' \cap A_\infty$ and $h - gh, h - hg, g - g^2 \in \operatorname{Ann}(C, A_\infty)$ in a similar way. It implies that $g + \operatorname{Ann}(C, A_\infty)$ is the unit element of $(C' \cap A_\infty)/\operatorname{Ann}(C, A_\infty)$. □

REMARK A.12.2. Let X a Polish l.c. space, that is not compact, and let $\omega \in \gamma X := \beta X \setminus X$ a point of the corona γX of X.

Recall that $\mathrm{C_b}(X, B)$ and $\mathrm{Q}(X, B) := \mathrm{C_b}(X, B)/\mathrm{C_0}(X, B)$ are C^*-bundles over βX respectively $\gamma X := \beta X \setminus X$, i.e., are algebras of continuous section of continuous fields of C^*-algebras B_ω (if $\omega \in \gamma X$) and $B_x = B$ (if $x \in X$).

In particular, $\mathrm{C_b}(X) \cong \mathrm{C}(\beta X)$ and $\mathrm{C_b}(X)/\mathrm{C_0}(X) \cong \mathrm{C}(\gamma X)$. $\mathrm{C_b}(X, B)$ is a $\mathrm{C}(\beta X)$-algebra, and $\mathrm{Q}(X, B)$ is a $\mathrm{C}(\gamma X)$-algebra. This explains the notations $\mathrm{C_b}(X, B)|Y$ and $\mathrm{Q}(X, B)|Z$ for closed subsets $Y \subseteq \beta X$ and $Z \subseteq \gamma Y$. We write also B_ω for $\mathrm{Q}(X, B)|\{\omega\}$ if $\omega \in \gamma X$.

Let $A \subseteq \mathrm{Q}(X, B)$ a *separable* C^*-subalgebra. We define

$$F(X;\, A, B) := \, (A' \cap \mathrm{Q}(X, B))/\operatorname{Ann}(A, \mathrm{Q}(X, B))\,.$$

It is always a *unital* algebra that contains a copy of $\mathrm{C}(\gamma X)$ in its center.

(*Indeed*: Clearly, $A' \cap \mathrm{Q}(X, B)$ is a $\mathrm{C}(\gamma X)$-algebra. Since $\operatorname{Ann}(A, \mathrm{Q}(X, B))$ is a hereditary C^*-subalgebraof $\mathrm{Q}(X, B)$, it is a $\mathrm{C}(\gamma X)$-algebra. An approximate unit of a separable C^*-subalgebra $C \subseteq \mathrm{C_b}(X, B)$ with $C|\gamma X = A$ allows to construct a positive contraction $f \in \mathrm{C_b}(X, B)_+$ with $\|f(x)\| = 1$ for each $x \in X$ and $ea = a = ae$ for all $a \in A$, where $e := f|\gamma X$.)

Let $D_A := \overline{A\, \mathrm{Q}(X, B) A}$ denote the σ-unital C^*-subalgebra of $\mathrm{Q}(X, B)$ that is generated by A, and let

$$\mathcal{N}(D_A) := \{\, f \in \mathrm{Q}(X, B)\, ;\; fD_A \cup D_A f \subseteq D_A \,\}\,.$$

Then (obviously) $A' \cap \mathrm{Q}(X, B) \subseteq \mathcal{N}(D_A)$, and the natural C^*-morphism from $A' \cap \mathrm{Q}(X, B)$ into $\mathcal{M}(D_A)$ has kernel $\operatorname{Ann}(A, \mathrm{Q}(X, B)) = \operatorname{Ann}(D_A, \mathrm{Q}(X, B))$. The image is contained in $A' \cap \mathcal{M}(D_A)$. It is not hard to show that it is really all of $A' \cap \mathcal{M}(D_A)$, because $\operatorname{Ann}(A, \mathrm{Q}(X, B)) \subseteq A' \cap \mathrm{Q}(X, B)$ and the natural *-monomorphism

$$\mathcal{N}(D_A)/\operatorname{Ann}(A, \mathrm{Q}(X, B)) \to \mathcal{M}(D_A)$$

is surjective. The latter can be deduced from the facts that A contains a strictly positive contraction e of D_A, and each element of $\mathcal{M}(D_A)_+$ is the sum $T + S + a$, where $a \in A$, and $T, S \in \mathcal{M}(D_A)_+$ are in $\mathcal{M}(D_A)$ strictly convergent sequences of mutually orthogonal positive elements of A.

Thus, the natural *-isomorphism from $\mathcal{N}(D_A)/\operatorname{Ann}(A, \mathrm{Q}(X, B)) \to \mathcal{M}(D_A)$ defines a natural $C(\gamma X)$-algebra isomorphism

$$F(X;\, A, B) \cong A' \cap \mathcal{M}(D_A)\,.$$

This isomorphism allows to see, that $F(X;\, \cdot, \cdot)$ is a "stable" invariant, i.e., $F(X;\, A \otimes \mathbb{K}, B \otimes \mathbb{K})$ is naturally $\mathrm{C}(\gamma X)$-module isomorphic to $F(X;\, A, B)$. In fact this follows easily from the natural $\mathrm{C}(\gamma X)$-module isomorphisms

$$(D_A) \otimes \mathbb{K} \cong D_{(A \otimes \mathbb{K})}$$

and

$$(A \otimes \mathbb{K})' \cap \mathcal{M}(D_A \otimes \mathbb{K}) \cong A' \cap \mathcal{M}(D_A)\,.$$

Let $\pi_Y \colon \mathrm{Q}(X,B) \to \mathrm{Q}(X,B)|Y = \mathrm{C_b}(X,B)|Y$ denote the natural $\mathrm{C}(\gamma X)$-modular epimorphism, and denote by $D|Y$ the image $\pi_Y(D) \subseteq \mathrm{Q}(X,B)|Y$ of a subset $D \subseteq \mathrm{Q}(X,B)$, in particular, $f|Y := \pi_Y(f)$.

We generalize $F(X;\, A,B)$ to closed subsets $Y \subseteq \gamma X$:

$$F(X,Y;\, A,B) := ((A|Y)' \cap (\mathrm{Q}(X,B)|Y)) / \operatorname{Ann}((A|Y),(\mathrm{Q}(X,B)|Y))\,.$$

Again one can show that $F(X,Y;\, A,B)$ is a stable invariant in the above considered sense. Our old definition of $F(X;\, A,B)$ becomes $F(X,\gamma X;\, A,B)$ in the new terminology.

The map π_Y maps $\operatorname{Ann}(A,\mathrm{Q}(X,B))$ into $\operatorname{Ann}((A|Y),(\mathrm{Q}(X,B)|Y))$, and maps $A' \cap \mathrm{Q}(X,B)$ into $(A|Y)' \cap (\mathrm{Q}(X,B)|Y)$.

It is not clear, for which Y one has $(A' \cap \mathrm{Q}(X,B))|Y = (A|Y)' \cap (\mathrm{Q}(X,B)|Y)$ see Questions A.12.3.

We have $\operatorname{Ann}(A,\mathrm{Q}(\mathbb{N},B))|Y \neq \operatorname{Ann}((A|Y),(\mathrm{Q}(\mathbb{N},B)|Y))$ for $B := \mathbb{C}$ and $Y = \{\omega\} \subseteq \gamma\mathbb{N}$ a suitable free ultrafilter and $A = C^*(f)$ for non-zero non-negative $f \in \mathrm{Q}(\mathbb{C}) = \ell_\infty(\mathbb{N})/c_0(\mathbb{N})$ with $0 \in \operatorname{Spec}(f)$ not isolated and $\omega \in \gamma\mathbb{N}$ a boundary point of the support of f.

It follows, that π_Y does *not* necessarily define an isomorphism from $F(X;\, A,B)|Y$ onto $F(X,Y;\, A,B)$. In particular, the fibers $F(X;\, A,B)|\omega$ of the $\mathrm{C}(\gamma X)$-algebra can not be identified with $F(X,\{\omega\};A,B)$.

The situation is less complicate if $A \subseteq B$. In particular, the algebras $F(X,Y;\, A,A)$ are stable invariants that have fibers $F(X,\{\omega\};A,B)$ with $\omega \in Y$.

Questions A.12.3. Let $A' \cap \mathrm{Q}(X,B)$ and $F(X;A,B)$ as in Remark A.12.2.

Here are some questions related to a possible or impossible reduction of properties of asymptotic invariants $F(X;A,B)$ to properties of its fibers $F(\omega;A,B)$:

When $(A' \cap \mathrm{Q}(X,B))|Y = (A|Y)' \cap (\mathrm{Q}(X,B)|Y)$?

What are the fibers $F(X;A,B)_\omega$ of $F(X;A,B)$?

Is it clear from the definition of $F(X;A,B)$ that $F(\{\omega\};A,B)$ is a quotient of $F(X;A,B)|\{\omega\}$?

Let $a,b \in \mathrm{Q}(X,B)$ self-adjoint contractions with commuting $\pi_\omega(a)$ and $\pi_\omega(b)$. Does there exist commuting self-adjoint $a',b' \in \mathrm{Q}(X,B)$ with $\pi_\omega(a') = \pi_\omega(a)$ and $\pi_\omega(b') = \pi_\omega(b)$?

Let $f,g \in \mathrm{C_b}(X,B)$ self-adjoint contractions with $a = f + \mathrm{C}_0(X,B)$ and $b = g + \mathrm{C}_0(X,B)$. When we can find self-adjoint $f_0 \in \mathrm{C}_0(X,B)$ and a continuous function $\mu\colon X \to [1, \infty)$ such that, for $f_1 = f + f_0$,

$$h(x) := (2\mu(x))^{-1} \int_{-\mu(x)}^{\mu(x)} e^{itf_1(x)} g(x) e^{-itf_1(x)}\, \mathrm{d}t$$

has the property $\pi_\omega(b') = \pi_\omega(b)$ and $ab' = b'a$ for $b' := h + \mathrm{C}_0(X, B)$?

Is there a "universal" continuous function $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\tau(0) = 0$, such that τ is increasing and has the following property?

If $a, b \in C_+$ are contractions with $\|ab\| < \varepsilon$, then, for every $\delta > 0$, there is a contraction $d \in C$ with $d(a - \delta)_+ = 0$ and $\|db - bd\| + \|d^*bd - b\| \leq \tau(\varepsilon)$.

Continuous maps into canonical commutation relations could give counterexamples?

Remark A.12.4. Let $\omega \in \beta(\mathbb{N})$, $f \in \ell_\infty(\mathbb{N})_+$ with $f(\omega) = 0$. It is not difficult to see that there is a sequence $S = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ (where $n_k < n_{k+1}$), such that $\lim_k f(n_k) = 0$, i.e., $f|S \in c_0(S)$.

(On finds a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ with $n_k \in X_k := f^{-1}[0, 1/k] \subseteq \mathbb{N}$, because each X_k is an *infinite* subset of $\mathbb{N}$.)

A subset S with $f|S \in c_0(S)$ can be found in the ultrafilter defined by ω, i.e., with $\chi(S)(\omega) = 1$, if and only if, $\omega \in \gamma\mathbb{N}$ is in the interior of $\beta\mathbb{N} \cap f^{-1}(0)$.

Remark A.12.5. Suppose that X and Y are non-compact σ-compact locally compact spaces, that X is finite-dimensional, and that $\psi\colon X \to Y$ is a proper continuous map.

Then ψ extends naturally to a continuous map $\beta\psi\colon \beta X \to \beta Y$ with $\beta\psi(\gamma X) = \beta\psi(\beta X) \cap \gamma Y$. Take $\omega \in \gamma X := \beta X \setminus X$ and let $\omega' := \beta\psi(\omega)$.

Now let B a C^*-algebra, and $\mathcal{S} \subseteq \mathrm{CP}_{\mathrm{in}}(B)$ a convex set of approximately inner completely positive contractions, such that $\mathcal{S}$ is closed in point-norm topology.

If, for each (point-norm) continuous map $x \in X \mapsto V_x \in \mathcal{S}$, the map $V_\omega\colon B_\omega \to B_\omega$ is ideal preserving, then, for each (point-norm) continuous map $y \in Y \mapsto W_y \in \mathcal{S}$, the map $W_{\omega'}\colon B_{\omega'} \to B_{\omega'}$ is ideal preserving.

Notice that every σ-compact non-compact l.c. space Y has a proper continuous map $\psi\colon Y \to \mathbb{R}_+$. This leads to the following question.

Question A.12.6. Let $\omega \in \beta(\mathbb{R}_+) \setminus \mathbb{R}_+$, and B a C^*-algebra.

Suppose that $V_\omega\colon B_\omega \to B_\omega$ is ideal preserving for every bounded (point-norm) continuous map $t \in \mathbb{R}_+ \mapsto V_t \in \mathrm{CP}_{\mathrm{in}}(B)$.

Does B satisfy the condition (i) of Definition 2.0.4 ?

The problem here is the following: Let $n \in \mathbb{N} \mapsto V_n \in \mathrm{CP}_{\mathrm{in}}(B)$ is a given sequence of approximately inner c.p. contractions. Can it be extended to $t \in \mathbb{R}_+ \mapsto V_t \in \mathrm{CP}_{\mathrm{in}}(B)$ point-norm continuous, with same properties of V_ω on B_ω for $\omega \in \gamma(\mathbb{R}_+)$ as for V_ω with $\omega \in \gamma(\mathbb{N})$?

13. Order unit spaces and pre-ordered semigroups

Order-unit spaces are the spaces of continuous affine functions on compact convex subsets of locally convex *real* vector spaces. We use them to derive some results on the extension of states of pre-ordered abelian semigroups with order unit. We denote by $\mathbb{Q}$ the rational numbers (and not the quaternions) during Section 13. The results are basic, but they are also known and proved with other or similar methods a long time ago.

DEFINITION A.13.1. An **order unit space** is a triple (B, B_+, e), where B is a *real* Banach space, B_+ is a closed convex cone in B and $e \in B_+$ is a distinguished element of the cone B_+, such that $-e \notin B_+$ and for every $x \in B$ holds:
The vectors $te + x$ and $te - x$ are both in B_+, if and only if, $t \geq \|x\|$.

B is partially ordered by

$$x \leq y \ \Leftrightarrow y - x \in B_+ \,.$$

A **state** of B is a *monotone* (i.e., order preserving) $\mathbb{R}$-linear map $\lambda\colon B \to \mathbb{R}$ from B into $\mathbb{R}$ with $\lambda(e) = 1$.

We denote by $E(B) \subseteq B^*$ the set of states of B equipped with the $\sigma(B^*, B)$-topology. Then $E(B)$ is a compact convex set with this topology (cf. Part (ii) of Proposition A.13.2).

PROPOSITION A.13.2. *Let $\mathbf{B} = (B, B_+, e)$ and $\mathbf{B_1} = (B_1, (B_1)_+, e_1)$ order unit spaces (in particular real Banach spaces), and let $Y \subseteq B$ denote an additive subgroup of B with $e \in Y$.*

(i) *Closed linear subspaces L of B with $e \in L$ are again order unit spaces, if we let $L_+ := B_+ \cap L$.*
(ii) *An additive map $\lambda\colon B \to \mathbb{R}$ is positive (i.e., $\lambda(B_+) \subseteq \mathbb{R}_+$), if and only if, λ is $\mathbb{R}$-linear and bounded with $\|\lambda\| = \lambda(e)$.*

*In particular, the positive states functionals $\lambda \in B^*$ with $\lambda(e) = 1$ and $\|\lambda\| \leq 1$, and $E(B)$ is a compact convex subset of B^* with *weak topology (i.e., $\sigma(B^*, B)$-topology).*
(iii) *For each $x \in B$ there is a state λ with $|\lambda(x)| = \|x\|$. In particular, B is in a natural way an order unit subspace of* $\mathrm{C}(E(B))$.
(iv) *For every continuous linear functional $f \in B^*$ there are (not necessarily unique) states $\lambda_1, \lambda_2 \in E(B)$ and $\alpha, \beta \in \mathbb{R}_+$ with $f = \alpha\lambda_1 - \beta\lambda_2$ and $\alpha + \beta = \|f\|$.*
(v) *An element $x \in B$ is in the interior $(B_+)^\circ$ of B_+, if and only if, there is $n \in \mathbb{N}$ with $e \leq nx$.*
(vi) *An additive subgroup Y of B with $e \in Y$ separates the* states *of B, if and only if, $\bigcup_n (1/n)Y = \{(1/n)x\,;\ x \in Y\}$ is dense in B.*
(vii) *Suppose that $\phi\colon B \to B_1$ is a unital additive map, (i.e., $\phi(e) = e_1$ and $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in B$), and let $Y \subseteq B$ an additive subgroup such that $\{(1/n)x\,;\ x \in Y\}$ is dense in B.*

The following properties (1)–(5) *are equivalent and imply that* ϕ *is* $\mathbb{R}$-*linear.*

(1) ϕ *is monotone (i.e., is order preserving:* $\phi(B_+) \subseteq (B_1)_+$ *).*
(2) $\rho \circ \phi \in E(B)$ *for every state* $\rho \in E(B_1)$.
(3) ϕ *is a contraction.*
(4) ϕ *is continuous, and, for* $x \in Y$, *the existence of* $n \in \mathbb{N}$ *with* $e \leq nx$ implies *the existence of* $m \in \mathbb{N}$ *with* $e_1 \leq m\phi(x)$.
(5) $\phi((B_+)^\circ) \subseteq (B_1)_+$.

(viii) *If* Y *is as in* (vii), *and the unital additive map* $\phi\colon B \to B_1$ *is monotone, then the following properties are equivalent:*
(a) $B_+ = \phi^{-1}(\phi(B) \cap (B_1)_+)$.
(b) *For every* $\lambda \in E(B)$ *exists* ρ *of* B_1 *with* $\lambda = \rho \circ \phi$.
(c) ϕ *is* isometric.
(d) *For* $x \in Y$, *the existence of* $m \in \mathbb{N}$ *with* $e_1 \leq m\phi(x)$ implies *the existence of* $n \in \mathbb{N}$ *with* $e \leq nx$.
(e) $b \in (B_+)^\circ$, *if and only if,* $\phi(b) \in ((B_1)_+)^\circ$.

(ix) *Suppose that a subset* $F \subseteq E(B)$ *is* norming *for* B *(or that* F *is* norming *on a rationally dense additive subgroup* Y *of* B *in the sense that* $\|y\| = \sup_{f\in F} |f(y)|$ *for each* $y \in Y$*).*

Then the convex hull of F *is* $\sigma(B^*, B)$*-dense in* $E(B)$.

In particular, this implies: If K *is a compact convex subset of a locally convex real vector space* V, *then the space* $\mathrm{Aff}_c(K) \subseteq \mathrm{C}(K)$ *of all continuous* affine *functions* $f\colon K \to \mathbb{R}$ *is an order unit space, and* $E(\mathrm{Aff}_c(K))$ *is naturally isomorphic to* K.

(x) (B, B_+, e) *is naturally isomorphic to the ordered space* $\mathrm{Aff}_c(E(B)) \subseteq \mathrm{C}(E(B))$ *of all real-valued continuous affine functions on the (*weakly) compact convex subset* $E(B) \subseteq B^*$.

Proof. (i): $(L, L_+ := B_+ \cap L, e)$ satisfies the axioms of an order unit space, because the definition is "local", i.e., requires only the (norm-)completeness of the space and the cone, and that any (at most) two-dimensional subspace $\mathbb{R}e + \mathbb{R}x$ is an order unit space.

(ii): Since $-e \notin B_+$ and $e \in B_+$, it follows that $te - e, te + e \in B_+$, if and only if, $t \geq 1$. Thus, $\|e\| = 1$.

Let $U := \{x \in B\,;\ \|x\| < 1\}$ denote the open unit ball of B. The sets $\delta e + B_+$ ($\delta > 0$) are contained in the interior of B_+, because $e + U \subseteq B_+$ and $\delta e + B_+ \subseteq \delta(e + U) + B_+$.

Let $\lambda\colon B \to \mathbb{R}$ an additive map with $\lambda((B_+)^\circ) \subseteq [0, \infty)$. Then λ is $\mathbb{Q}$-linear. Since $re \pm x = (r - \|x\|)e + (\|x\|e - x) \in (B_+)^\circ$ for rational $r > \|x\|$, we get $0 \leq \lambda(re + x) = r\lambda(e) \pm \lambda(x)$ for all rational numbers $r > \|x\|$. Thus, $\|\lambda\| < \infty$ and $\lambda(e) \geq \|\lambda\|$. On the other hand, $\|\lambda\| = \|\lambda\| \cdot \|e\| \geq \lambda(e)$.

Let $b \in B$, then $b \in B_+$, if and only if, $\|(\|b\|e) - b\| \le \|b\|$. Indeed: If $b \neq 0$, then $a := \|b\|^{-1}b \in B_+$, if and only if, $e - (e-a) = a \in B_+$ and $e + (e-a) \in B_+$, if and only if, $\|e - a\| \le 1$.

Let $\lambda \in B^*$ with $\|\lambda\| = \lambda(e)$. Then $|\,\lambda(e)\|b\| - \lambda(b)\,| \le \lambda(e)\|b\|$ for all $b \in B_+$, which implies that $\lambda(b) \ge 0$.

Since $\|e\| = 1$, there exists $\lambda \in B^*$ with $\lambda(e) = 1$, by Hahn-Banach extension, i.e., $E(B) \neq \emptyset$.

The subset $E(B)$ of $\lambda \in B^*$ with $\|\lambda\| \le 1$ and $\lambda(e) = 1$ is a *weakly closed subset of the unit ball $(B^*)_{\le 1}$ of B^*. The latter is compact in the *weak topology ($:= \sigma(B^*, B)$-topology).

(iii): It suffices to consider the case $\|b\| = 1$.
Let $\delta := 2 - \max(\|e-b\|, \|e+b\|) \ge 0$. Then $(1-\delta)e \pm b = (2-\delta)e - (e \mp b) \in B_+$, which implies $(1-\delta) \ge \|b\| = 1$ and $\delta = 0$. It follows that $\|e - b\| = 2$ or $\|e + b\| = 2$. We may suppose $\|e+b\| = 2$, because we can replace b by $(-b)$. By Hahn-Banach extension, there is $\lambda \in B^*$ with $\|\lambda\| = 1$ and $\lambda(e+b) = 2$. It follows that $\|\lambda\| = \lambda(e) = 1 = \lambda(b)$. Thus, $\lambda \in E(B)$ by Part (ii), and $\|b\| = |\lambda(b)|$.

(iv): It suffices to consider the case of $f \in B^*$ with $\|f\| = 1$. Let $T(\lambda_1, \lambda_2, \alpha) := \alpha\lambda_1 - (1-\alpha)\lambda_2$ for $\lambda_1, \lambda_2 \in E(B)$ and $\alpha \in [0,1]$. Then T is a (*weakly) continuous map from the compact set $K := E(B) \times E(B) \times [0,1]$ into the (*weakly) compact set $B^*_{\le 1} := \{g \in B^*\,;\ \|g\| \le 1\}$. In particular, the image $T(K)$ of T is closed. Straight calculation shows that $T(K)$ is also convex and $T(K) = -T(K)$. Since the subset $E(B) = T(E(B) \times E(B) \times \{1\})$ is norming for B, a standard Hahn-Banach separation argument shows that $T(K) = (B^*)_{\le 1}$. In particular, there are $\alpha, \beta \in [0, \infty)$ and $\lambda_1, \lambda_2 \in E(B)$ with $\alpha + \beta = \|f\| = 1$ and $\alpha\lambda_1 - \beta\lambda_2 = f$.

(v): If $x \in (B_+)^\circ$ then $x - (1/n)e \in (B_+)^\circ$ for sufficiently big $n \in \mathbb{N}$, in particular $nx - e \in B_+$. Conversely, if $nx - e \in B_+$, then $x = (1/n)e + y$ with $y := x - (1/n)e \in B_+$. Thus, $x \in y + (1/n)(e + U) \subseteq B_+$ for the open unit ball U of B, cf. proof of Part (ii). Hence, $x \in (B_+)^\circ$.

(vi): Let $\lambda_1, \lambda_2 \in E(B)$. $\lambda_1(x) = \lambda_2(x)$ for all $x \in Y$, if and and only if, $Z := \{(1/n)x\,;\ x \in Y\}$ is in the kernel of $\lambda_1 - \lambda_2$. If Z is dense in B, then $\lambda_1 = \lambda_2$, and the elements of Y separate the states.

Now let Y an additive subgroup of B with $e \in Y$. Then $Z := \{(1/n)x\,;\ x \in Y\} = \mathrm{span}_{\mathbb{Q}}(Y)$ is norm-dense in $\mathrm{span}_{\mathbb{R}}(Y)$. Thus, if Z is not dense in B, then there exists $f \in B^*$ with $\|f\| = 1$ and $f(x) = 0$ for all $x \in Y$. By Part (iv), there are states $\lambda_1, \lambda_2 \in E(B)$ and $\alpha \in [0,1]$ with $f = \alpha\lambda_1 - (1-\alpha)\lambda_2$. Since $e \in Y$, it follows that $0 = f(e) = 1 - 2\alpha$. Hence, $\lambda_1(x) = \lambda_2(x)$ for $x \in Y$ and $\lambda_1 \neq \lambda_2$.

It follows, that Z must be dense in B if Y separates the states.

(vii): The additivity of ϕ implies that $\phi(rb) = r\phi(b)$ for all rational numbers $r \in \mathbb{Q}$. Thus, if ϕ satisfies (3), then ϕ is $\mathbb{R}$-linear. The implications (1)$\Rightarrow$(5) is obvious.

(5)⇒(2): The map $\rho \circ \phi\colon B \to \mathbb{R}$ is additive and $\rho \circ \phi(e) = \rho(e_1) = 1$ for $\rho \in E(B_1)$. If $\phi((B_+)^\circ) \subseteq (B_1)_+$, then $\rho \circ \phi$ is an additive map from B to $\mathbb{R}$ that is not negative on $(B_+)^\circ$. It follows that $\rho \circ \phi$ is continuous by an argument in the proof of Part (ii). Then $\rho \circ \phi(B_+) \subseteq [0, \infty)$, and Part (ii) shows $\rho \circ \phi \in E(B)$.

(2)⇒(3): If $\rho \circ \phi \in E(B)$ for every state $\rho \in E(B_1)$, then $\|\phi(b)\| = \sup_{\rho \in E(B_1)} |\rho \circ \phi(b)| \leq \|b\|$ by Parts (ii) and (iii).

(3)⇒(1): Let $x \in B_+$ and $m, n \in \mathbb{N}$ with $m/n > \|x\|$. Then $me \pm (me - nx) \in B_+$ and, thus, $\|me - nx\| \leq m$. It follows $\|me_1 - n\phi(x)\| \leq m$. Thus $n\phi(x) = me_1 - (me_1 - n\phi(x)) \in (B_1)_+$.

(1)⇒(4): Let $x \in Y$ and $n \in \mathbb{N}$ with $nx - e \in B_+$, then $n\phi(x) - e_1 \in (B_1)_+$. The map ϕ is continuous, because (1) implies (3).

(4)⇒(5): Let $Z := \bigcap_n (1/n)Y = \operatorname{span}_{\mathbb{Q}}(Y)$. If $z = (1/k)y \in Z \cap (B_+)^\circ$ with $y \in Y$ and $k \in \mathbb{N}$, then $y \in Y \cap (B_+)^\circ$. By Part (v), there is $n \in \mathbb{N}$ with $ny - e \in B_+$. By assumptions, it follows that there is $m \in \mathbb{N}$ with $m\phi(y) - e_1 \in (B_1)_+$. In particular, $\phi(z) = (1/(mk))((m\phi(y) - e_1) + e_1) \in (B_1)_+$. Hence, $\phi(\,(B_+)^\circ \cap Z\,) \subseteq (B_1)_+$.

Since ϕ is continuous by assumptions, ϕ maps the closure M of $(B_+)^\circ \cap Z$ into $(B_1)_+$. By assumption, Z is dense in B. Thus Z is dense in each open subset of B, in particular $(B_+)^\circ \subseteq M$.

(viii): Since ϕ is unital and increasing (i.e., $\phi(B_+) \subseteq (B_1)_+$), ϕ is a unital contraction by Part (vii). By (v), $x \in (B_+)^\circ$, if and only if, there is $n \in \mathbb{N}$ with $nx - e \in B_+$. Thus $n\phi(x) - e_1 \in (B_1)_+$, and $\phi(x) \in ((B_1)_+)^\circ$, if $x \in (B_+)^\circ$. It shows that, under our assumptions, the properties (a), (c) and (e) are equivalent to:

(a') $\phi(B_+) \supset (B_1)_+ \cap \phi(B)$.
(c') $\|\psi(x)\| \geq \|x\|$ for all $x \in B$.
(e') $\phi((B_+)^\circ) \supset ((B_1)_+)^\circ$.

(c)⇒(e): Let $x \in B$ with $\phi(x) \in ((B_1)_+)^\circ$. There is $m \in \mathbb{N}$ with $\phi(b) := m\phi(x) - e_1 \in (B_1)_+$ for $b := mx - e \in B$, cf. Part (v). In the proof of Part (ii) we have seen that $\phi(b) \in (B_1)_+$ implies $\|te_1 - \phi(b)\| \leq t$ for $t := \|\phi(b)\| = \|b\|$. Thus $\|\|b\|e - b\| = \|te_1 - \phi(b)\| \leq \|b\|$, which implies $b = nx - e \in B_+$, as shown in proof of Part (ii). Thus, $x \in (B_+)^\circ$ by Part (v).

(e)⇒(a): Let $\phi(x) \in (B_1)_+$, then $\phi(x) + \delta e_1 \in ((B_1)_+)^\circ$ for each $\delta > 0$, by Part (v). It follows $x + \delta e \in B_+$ for each $\delta > 0$, i.e., $x \in B_+$.

(a)⇒(c): If $B_+ = \phi^{-1}(\phi(B) \cap (B_1)_+)$, then we get from $\|\phi(b)\|e_1 \pm \phi(x) \in (B_1)_+$, that $\|\phi(b)\|e \pm x \in B_+$. Thus $\|x\| \leq \|\phi(x)\|$ for all $x \in B$, and ϕ is isometric.

(c)⇒(b): The map ϕ defines an isometric unital isomorphism from B onto the closed subspace $L := \phi(B)$ with $e_1 \in L$. If $\lambda \in E(B)$, then $\rho' := \lambda \circ \phi^{-1}\colon L \to \mathbb{R}$ is in $E(L)$. There is an extension $\rho \in (B_1)^*$ of ρ' with $\|\rho\| = \|\rho'\| = 1$. Thus, $\rho \in E(B_1)$ by Part (ii), and $\lambda \circ \phi = \lambda' \circ \phi = \rho$.

(b)⇒(c): By Part (iii), for $b \in B$,

$$\|b\| = \sup_{\lambda \in E(B)} |\lambda(b)| = \sup_{\rho \in E(B_1)} |\rho(\phi(b))| = \|\phi(b)\| \,.$$

(e)⇒(d): Suppose that $x \in Y$ and $m \in \mathbb{N}$ satisfy $e_1 \leq m\phi(x)$. Then $\phi(x) \in ((B_1)_+)^\circ$ by Part (v), and (e) implies that $x \in (B_+)^\circ$. Again, Part (v) implies that there is $n \in \mathbb{N}$ with $e \leq nx$.

(d)⇒(c): Let $y \in Y$ and $p, q \in \mathbb{N}$ with $(p/q) > \|\phi(y)\|$. Then $pe_1 \pm q\phi(y) \in (B_1)_+$. Thus $\phi(x_k) - e_1 \in (B_1)_+$ for $x_k := (p+1)e + (-1)^k qy \in Y \,,\ k \in \{1, 2\}$. By assumption (d), there are $n_1, n_2 \in \mathbb{N}$ with $n_k x_k - e \in B_+$. Since $(n_k(p+1) - 1)/(n_k q) \leq (p+1)/q$, we get Thus $((p+1)/q)e \pm y \in B_+$, and $(p+1)/q \leq \|y\|$. Since $\inf\{(p+1)/q\,;\ p/q > t\} = t$ for each $t \in \mathbb{R}_+$, we obtain $\|\phi(y)\| \geq \|y\|$ for all $y \in Y$. This carries over to all $z \in Z = \bigcup_n (1/n)Y$, i.e., $\|z\| \leq \|\phi(z)\|$ for all $z \in Z$. It follows $\|b\| = \|\phi(b)\|$ for all $b \in B$, because Z is dense in B and $\|\phi\| \leq 1$.

(ix): Suppose that $Y \subseteq B$ is an additive subgroup such that $\mathrm{span}_{\mathbb{R}}(Y)$ is dense in B, and that $F \subseteq E(B)$ satisfies $\|y\| = \sup_{f \in F} |f(y)|$ for each $y \in Y$. Let $M(b) := \sup_{f \in F} |f(b)|$ for $b \in B$. Then $M(b)$ is a semi-norm on B with $M(b) \leq \|b\|$, the set $Z := \{(1/n)x\,;\ x \in Y\} = \mathrm{span}_{\mathbb{Q}}(Y)$ is norm-dense in B, and $M(z) = \|z\|$ for all $z \in Z$. Thus, $M(b) = \|b\|$ on all $b \in B$, i.e., F is norming for B.

Suppose now that F is norming on B. Let $G \subseteq E(B)$ denote the *weak closure of the the convex hull of F. Then G is convex, compact, and is norming on B.

Suppose that $G \neq E(B)$. Then, by Hahn-Banach separation, there exist $b \in B$ and $f \in E(B)$ such that $f(b) \notin \{g(b)\,;\ g \in G\}$. Since G is compact and convex, there are $\alpha, \beta \in \mathbb{R}$ with $\{g(b)\,;\ g \in G\} = [\alpha, \beta]$. Let $c := 2b - (\alpha + \beta)e$, then $f(c) \notin \{g(c)\,;\ g \in G\} = [\alpha - \beta, \beta - \alpha]$. Thus, $\|c\| = \sup_{g \in G} |g(c)| = \beta - \alpha$ and $|f(c)| > \|c\|$, which contradicts $f \in E(B)$.

Let K a compact convex subset of a locally convex vector space V. Then the continuous linear functionals $f \in V^*$ separate the point of K and $f|K \in \mathrm{Aff}_c(K)$. The set $\mathrm{Aff}_c(K)$ of continuous affine functions on K is a closed subspace of $\mathrm{C}(K)$ that contains 1.

The natural map $k \in K \mapsto \delta_k E(\mathrm{C}(K)) \subseteq \mathrm{C}(K)^*$ is continuous. The restrictions $\widehat{k}$ of δ_k to the elements of $\mathrm{Aff}_c(K)$ are in the state space $E(\mathrm{Aff}_c(K))$ of $\mathrm{Aff}_c(K)$. The map $k \mapsto \widehat{k}$ is continuous, affine (= convex) and injective. Thus, $\widehat{K}$ is a norming closed convex subset of the state space of $\mathrm{Aff}_c(K)$. Thus, $k \mapsto \widehat{k}$ is surjective. It follows that $k \mapsto \widehat{k}$ is an affine topological isomorphism from K onto $E(\mathrm{Aff}_c(K))$.

(x): The natural map $B \ni b \mapsto \widehat{b} \in \mathrm{Aff}_c(E(B)) \subseteq \mathrm{C}(E(B))$ with $\widehat{b}(\lambda) := \lambda(b)$. Is unital and isometric by Part (iii). By Part (ix), the natural map from $E(B)$ into $E(\mathrm{Aff}_c(E(B)))$ is bijective. It follows that $\widehat{B} \subseteq \mathrm{Aff}_c(E(B))$ separates the states of $\mathrm{Aff}_c(E(B))$. Now Part (vi) shows that $\widehat{B} = \mathrm{Aff}_c(E(B))$. Thus, $b \mapsto \widehat{b}$ defines a natural unital isometric isomphism from (B, B_+, e) onto the order unit space $\mathrm{Aff}_c(E(B)) \subseteq \mathrm{C}(E(B))$. □

Lemma A.13.3. *Let V be a real vector space, $K \subseteq V$ a convex cone in V and $u \in K$ such that $-u \notin K$ and that, for every $x \in V$ there exists $n = n(x) \in \mathbb{N}$ with $nu - x \in K$.*
Define $P(x) := \inf\{m/n\,;\ m, n \in \mathbb{N},\ mu \pm nx \in K\}$.

Then P is a seminorm on V.

Let V_0 the space of $v \in V$ with $P(v) = 0$, let $\pi(x) := x + V_0$ the quotient map $\pi\colon V \to V/V_0$ and define a Norm $\|\cdot\|$ on V/V_0 by $\|\pi(x)\| = P(x)$.

(i) *The completion B of the normed space V/V_0 becomes an order unit space with order unit $e := \pi(u) = u + V_0$ and $B_+ := \overline{\pi(K)}$.*
(ii) *(B, B_+, e) is naturally order-unit isomorphic to the space of affine functions on the $\sigma(V', V)$-compact convex set of linear maps $\lambda\colon V \to \mathbb{R}$ with $\lambda(K) \subseteq \mathbb{R}_+$ and $\lambda(u) = 1$.*
(iii) *$\pi(x)$ is in the interior of B_+, if and only if, there is $n \in \mathbb{N}$ with $nx-u \in K$.*
(iv) *Suppose that X is an additive subgroup of of V, such that $u \in X$ and V is the real linear span of X, and that $\psi\colon X \to B_1$ is an additive map from X into an order unit space $(B_1, (B_1)_+, e_1)$ with $\psi(u) = e_1$.*

 If the existence of $n \in \mathbb{N}$ with $nx - e \in K$ implies the existence of $m \in \mathbb{N}$ with $m\psi(x) - e_1 \in (B_1)_+$, then there is a unique unital linear contraction $\eta\colon B \to B_1$ with $\eta(\pi(x)) = \psi(x)$ for all $x \in X$.

 This is in particular the case, if the map $\psi\colon X \to B_1$ is order preserving, i.e., if $\psi(x) \le \psi(y)$ if $y - x \in K$.
(v) *The map $\eta\colon B \to B_1$ in (iv) is isometric, if and only if, for each $x \in X$, the existence of $n \in \mathbb{N}$ with $nx - e \in K$ is equivalent to the existence of $m \in \mathbb{N}$ with $m\psi(x) - e_1 \in (B_1)_+$.*

Proof. Clearly $P(x) = P(-x)$, $P(u) = 1$ (by $-u \notin K$) and $P(0) = 0$. If $t \in \mathbb{R}_+$ satisfies $tu \pm x \in K$, then $su \pm x = (tu \pm x) + (s - t)u \in K$ for all $s \ge t$. Thus, $P(x) = \inf\{t \ge 0\,;\ tu \pm x \in K\}$. Since, for $t, s > 0$, $tu \pm x \in K \quad \Leftrightarrow (st)u \pm sx \in K$, we get $P(sx) = sP(x)$ for $s > 0$ and $x \in V$.
If $t_1, t_2 \ge 0$ and $t_1u \pm x, t_2u \pm y \in K$ implies $(t_1 + t_2)u \pm (x + y) \in K$. Thus $P(x + y) \le P(x) + P(y)$. Thus, P is a semi-norm on V.

(i): Since $(P(x)+\varepsilon)u \pm x \in K$ for all $\varepsilon > 0$, we have $(\|\pi(x)\|+\varepsilon)e \pm \pi(x) \in B_+$ for all $x \in V$. The map $y \mapsto (\|y\| + \varepsilon)e - y$ is continuous on B and takes values in B_+ for all $y \in \pi(V) = V/V_0$. Since B_+ is closed, we get $\|y\|e - y \in B_+$ for all $y \in B$. It follows $te - y = (\|y\|e - y) + \in B_+$ for all $y \in B$ and $t \ge \|y\|$. In particular, $e + U \subseteq B_+$ for the open unit ball U of B, and e is contained in the interior of B_+.

If $y \in B$ and $s \ge 0$ satisfies $se \pm y \in B_+$ then for every $\varepsilon > 0$, there are $x_1, x_2 \in K$ and $z \in V$ with $\|se + y - \pi(x_1)\| < \varepsilon$, $\|se - y - \pi(x_2)\| < \varepsilon$ and $\|y-\pi(z)\| < \varepsilon$. It implies $P(su+z-x_1) < 2\varepsilon$, $P(se-z-x_2) < 2\varepsilon$ and $\|y\| \le P(z)+\varepsilon$.

It follows $(2\varepsilon)u + (su + z - x_1) \in K$ and $(2\varepsilon)u + (su - z - x_2) \in K$. Since $x_1, x_2 \in K$, we get $(2\varepsilon) + s \geq P(z)$ and, finally, $s \geq \|y\| - 3\varepsilon$. Thus, $se \pm y \in B_+$ and $s \geq 0$ imply $s \geq \|y\|$, and (B, B_+, e) is an order unit space.

(ii): We have $(P(x)+\varepsilon)u \pm x \in K$ for $x \in V$ and all $\varepsilon > 0$. Thus, $|\lambda(x)| \leq P(x)$ for all linear maps $\lambda\colon V \to \mathbb{R}$ with $\lambda(u) = 1$ and $\lambda(K) \subseteq \mathbb{R}_+$. Since V/V_0 is dense in B, there is a unique linear functional $\rho\colon B \to \mathbb{R}$ with $\rho \circ \pi = \lambda$ and $\|\rho\| \leq 1$. The functional ρ is positive and unital, because $\pi(K)$ is dense in B_+ and $\rho(e) = \lambda(u) = 1$.
Clearly, $\lambda := \rho \circ \pi$ is a linear map from V to $\mathbb{R}$ with $\lambda(K) \subseteq \mathbb{R}_+$ and $\lambda(u) = 1$, if $\rho\colon B \to \mathbb{R}$ is a positive linear functional on B with $\rho(e) = 1$.
The $\sigma(\mathrm{Lin}(V, \mathbb{R}), V)$-topology on the convex set of linear maps $\lambda\colon V \to \mathbb{R}$ with $\lambda(K) \subseteq \mathbb{R}_+$ and $\lambda(u) = 1$ gives it a structure of a compact convex set $E(V, K, u)$. The (algebraic and affine) isomorphism $E(B) \to E(V, K, u)$, given by $\rho \mapsto \rho \circ \pi$, is is an affine homeomorphism, because $\pi(V)$ is dense in B. By Proposition A.13.2(ix), the order unit space (B, B_+, e) is naturally order-unit isomorphic to the space of real-valued continuous affine functions on the compact convex set $E(B)$ of states on B.

(iii): If $nx - u \in K$, then $n\pi(x) - e \in B_+$, which implies that $\pi(x)$ is in the interior of B_+ by Proposition A.13.2(v).

Suppose that $\pi(x)$ is in the interior of B_+, then there is $n \in \mathbb{N}$ with $\pi(x) - (2/n)e \in B_+$, i.e., $\pi(nx - 2u) \in B_+$. Since B_+ is the closure of $\pi(K)$, there exists $y \in V$ and $k \in K$ with $nx - 2u = y + k$ and $P(y) = \|\pi(y)\| < 1$. It follows that $u + y \in K$ (by definition of $P(y)$). Hence $nx - u = (y + u) + k \in K$.

(iv): Suppose that $\psi\colon X \to B_1$ is an additive map with $\psi(u) = e_1$, and suppose that, for each $x \in X$, the existence of $n \in \mathbb{N}$ with $nx - u \in K$ implies the existence of $m \in \mathbb{N}$ with $m\psi(x) - e_1 \in (B_1)_+$.

Let $p, n \in \mathbb{N}$ with $p/n > P(x) = \|\pi(x)\|$, then $pu \pm nx \in K$, and $((p+1)u \pm nx) - e \in K$. Thus, there is $m \in \mathbb{N}$ with $m((p+1)e_1 \pm n\psi(x)) - e_1 \in (B_1)_+$. It follows, that $(m(p+1) - 1)e_1 \pm mn\psi(x) \in (B_1)_+$. Hence $\|\psi(x)\| \leq (p+1-1/m)/n \leq p/n + 1/n$. Since we can approximate $P(x) \geq 0$ from above by rational p/n with arbitrary large denominator n, we get $\|\psi(x)\| \leq P(x)$ for all $x \in X$.

The $\mathbb{Q}$-linear hull of X is given by $W := \bigcup_n (1/n)X$. The map $\psi\colon X \to B_1$ naturally extends to W by $\psi_0((1/n)x) := (1/n)\psi(x)$. The extension is well-defined and $\mathbb{Q}$-linear, because $(nm)^{-1}(mpx \pm nqy) = (1/n)(px) \pm (1/m)(qy)$ maps to $(nm)^{-1}(mp\psi(x) \pm nq\psi(y)) = (1/n)(p\psi(x)) \pm (1/m)(q\psi(y))$. We have $P(w) \geq \|\psi_0(w)\|$ for all $w \in W$, because this holds for $w \in X$. In particular, $\psi_0(w) = 0$ if $w \in V_0$.

Consider the set $Y := \pi(X) \subseteq B$. It satisfies the assumptions on Y of Parts (vi, vii, viii) of Proposition A.13.2, because V is the real span of X. In particular, the rational span $Z := \bigcup_n (1/n)\pi(X)$. of Y is dense in the real span of X, hence is dense in B. Let $z = (1/n)\pi(x) \in Z$.

We define $\eta_0(z) := \psi_0(w)$ for $z \in Z$ and $w \in W$ with $\pi(w) = z$. The definition is correct, because $\pi(w') = \pi(w)$ is equivalent to $P(w' - w) = 0$ by definition of V_0 and $\pi\colon V \to B$. Thus $\|\psi_0(w' - w)\| \leq P(w' - w) = 0$ and $\psi_0(w') = \psi_0(w)$. We have $\|z\| = P(w) \geq \|\psi_0(w)\| = \|\eta_0(z)\|$.

The map $\eta_0\colon Z \to B_1$ satisfies $\eta_0 \circ \pi = \psi_0$. It follows that η_0 is $\mathbb{Q}$-linear and contractive.

Since V is the real span of X, and since $\pi(V)$ is dense in B, the rational span Z of Y is dense in B. Thus, the unital contractive $\mathbb{Q}$-linear map $\eta_0\colon Z \to B_1$ uniquely extends to a unital contractive $\mathbb{R}$-linear map $\eta\colon B \to B_1$. This map satisfies $\eta \circ \pi(x) = \psi(x)$.

Since $\eta\colon B \to B_1$ is a unital linear contraction, the map η is increasing.

(v): Let $\eta\colon B \to B_1$ an isometric linear map with $\eta(\pi(x)) = \psi(x)$ for all $x \in X$. Then $\eta(e) = \eta(\pi(u)) = \psi(u) = u_1$. Since η unital and contractive, $\eta(B_+) \subseteq (B_1)_+$, by Proposition A.13.2(vii). If $nx - u \in K$, then $n\pi(x) - e \in B_+$ and $\eta(n\pi(x) - e) = n\psi(x) - e_1 \in (B_1)_+$. If there is $m \in \mathbb{N}$ with $\eta(m\pi(x) - e) = m\psi(x) - e_1 \in (B_1)_+$, then $m\pi(x) - e \in B_+$ by Proposition A.13.2(viii,a).

Conversely, suppose that $\psi\colon X \to B_1$ is additive and unital, and suppose that, for each $x \in X$, the existence of $n \in \mathbb{N}$ with $nx - u \in K$ is equivalent to the existence of $m \in \mathbb{N}$ with $m\psi(x) - e_1 \in (B_1)_+$. Then, by part (iv), there is a unique unital linear contraction $\eta\colon B \to B_1$ with $\eta(\pi(x)) = \psi(x)$ for all $x \in X$.

The set $Y := \pi(X)$ satisfies the assumptions of Proposition A.13.2(viii,d): $Z = \bigcup_n (1/n)Y$ is dense in B, and for $y \in Y$,the existence of $m \in \mathbb{N}$ with $e_1 \leq m\eta(y)$ implies the existence of $n \in \mathbb{N}$ with $e \leq ny$. Indeed: If $e_1 \leq m\eta(y)$ and $x \in X$ with $y = \pi(x)$, then $m\psi(x) - e_1 = m\eta(y) - e_1 \in (B_1)_+$, which implies, by assumption on X, the existence of $n \in \mathbb{N}$ with $nx - u \in K$, i.e., $ny - e = \pi(nx - u) \in B_+$. Thus, the unital linear contraction η is isometric by the equivalence of properties (viii,c) and (viii,d) in Proposition A.13.2(viii). □

Now we use the above elementary material on order-unit spaces for a study of scaled abelian semigroups.

Definition A.13.4. Let S denote an abelian semigroup. S is **preordered** by the relation $x <_S y$ if the relation is transitive and invariant under addition, i.e., $x <_S z$ if $x <_S y$ and $y <_S z$, $x + w <_S y + w$ for all $w \in S$. We do not require that $x <_S x$, therefore we write $x \leq_S y$ if $x <_S y$ *or* $x = y$. The preordered semigroup $(S, <_S)$ is **partially ordered** if $x \leq_S y$ and $y \leq_S x$ imply $x = y$. (If it is clear, which S and preorder $<_S$ is ment, we simply write $<$ and $\leq$ for $<_S$ respectively $\leq_S$.)

An element $u \in S$ is an **order unit** in the preordered abelian semigroup $(S, <_S)$ if $x \leq x + u$ for all $x \in S$, and, for every $y \in S$, there is $n = n(y) \in \mathbb{N}$ such that $y \leq nu$ and $u \leq y + nu$ ([7]). Usually there are many different order units in S.

[7]The properties $x \leq x + u$ and that $u \leq y + nu$ follow automatically, if S has a zero element 0 with $0 \leq x$ for all $x \in S$.

An abelian preordered semigroup $(S\,,\,<_S\,,\,u)$ with a *distinguished* order unit $u \in S$ will be called a **scaled** (abelian) semigroup.

A **state** on S is an order preserving additive map $\lambda\colon S \to \mathbb{R}$ with $\lambda(u) = 1$.

LEMMA A.13.5. *Suppose that V and W are vector spaces over the* rational numbers $\mathbb{Q}$.

(i) *The natural $\mathbb{Z}$-modul morphism $V \otimes_{\mathbb{Z}} W \to V \otimes_{\mathbb{Q}} W$ is an isomorphism.*
(ii) *$\eta\colon v \mapsto v \otimes_{\mathbb{Z}} 1$ is a natural $\mathbb{Q}$-linear monomorphism from V into the real vector space $V \otimes_{\mathbb{Z}} \mathbb{R} = V \otimes_{\mathbb{Q}} \mathbb{R}$.*
(We identify V with its image in $V \otimes_{\mathbb{Q}} \mathbb{R}$.)
(iii) *Let $U \subseteq V \otimes_{\mathbb{Z}} \mathbb{R}$ an open subset with respect to an locally convex topology $\mathbb{O}$ on $V \otimes_{\mathbb{Z}} \mathbb{R}$. Then $V \cap U$ is dense in U with respect to $\mathbb{O}$.*
(iv) *For every subset $X \subseteq V$, and for every (not necessarily separated) locally convex topology on $V \otimes_{\mathbb{Z}} \mathbb{R}$, the rational convex hull* $\mathrm{conv}_{\mathbb{Q}}(X)$ *of X in V is dense in the real convex hull* $\mathrm{conv}_{\mathbb{R}}(X)$ *of X in $V \otimes_{\mathbb{Z}} \mathbb{R}$.*
(v) $\mathrm{conv}_{\mathbb{R}}(X) \cap V = \mathrm{conv}_{\mathbb{Q}}(X)$ *holds in $V \otimes_{\mathbb{Z}} \mathbb{R}$.*

The observations (i)-(v) imply the following observation:

Suppose that $T\colon V \to W$ is an additive map, $\nu\colon W \to [0,\infty)$ is a semi-norm on $W \otimes_{\mathbb{Z}} \mathbb{R}$, $X \subseteq W$ is a subset of W, and that there exists $v_1 \in V \otimes_{\mathbb{Z}} \mathbb{R}$, such that $(T \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}})(v_1)$ is in the interior of $\mathrm{conv}_{\mathbb{R}}(X) \subseteq W \otimes_{\mathbb{Z}} \mathbb{R}$ *with respect to the topology on $W \otimes_{\mathbb{Z}} \mathbb{R}$ defined by ν.*

Then there exists $v_2 \in V$ such that $T(v_2) \in W$ is in the interior of $\mathrm{conv}_{\mathbb{Q}}(X)$ *with respect to ν.*

PROOF. Since the tensor products are the *algebraic* tensor products, the statements reduces to V, W of finite dimension (over $\mathbb{Q}$, i.e., $V \cong \mathbb{Q}^n$ and $W \cong \mathbb{Q}^m$ as vector spaces over $\mathbb{Q}$), and to finite subsets X of $V \cong \mathbb{Q}^n$.

Then the proofs automatically reduce to the following easy observations:

(i): There are natural isomorphisms $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, and, therefore, also natural isomorphisms

$$\mathbb{Q}^n \otimes_{\mathbb{Z}} \mathbb{Q}^m \cong \mathbb{Q}^{nm} \cong \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{Q}^m\,.$$

(ii): By Part (i), there is a natural isomorphism $V \otimes_{\mathbb{Z}} \mathbb{R} \cong V \otimes_{\mathbb{Q}} \mathbb{R}$, because $\mathbb{R}$ is a $\mathbb{Q}$-vector space. In case $V = \mathbb{Q}^m$, the natural isomorphism induces the natural embedding of $\mathbb{Q}^m$ in $\mathbb{R}^m$. The injectivity of $\eta\colon v \mapsto v \otimes_{\mathbb{Z}} 1$ then follows from the injectivity of natural embedding $\mathbb{Q}^n \to \mathbb{R}^m = \mathbb{Q}^m \otimes_{\mathbb{Z}} \mathbb{R}$.

(iii): The finest locally convex topology on $\mathbb{R}^n$ is identical with the topology induced by the norm $\|(x_1,\ldots,x_n)\|_\infty := \max_j |x_j|$. Thus, every open subset U of $\mathbb{R}^n$ with respect to any (not necessarily separated) locally convex topology $\mathbb{O}$ of $\mathbb{R}^n$ is the union of products $I_1 \times \cdots \times I_n$ of open intervals $I_k \subseteq \mathbb{R}$. It follows that the vectors with rational coordinates are dense U with respect to to the topology $\mathbb{O}$, because $U_1 \cap U = \emptyset$ if $U_1 \in \mathbb{O}$ and $U_1 \cap U$ does not contain a vector with rational coordinates.

(iv): Any $\mathbb{R}$-convex combination $\sum_{j=1}^{m} \alpha_j v_j = p \in \mathbb{R}^n$ of $v_1, \ldots, v_m \in \mathbb{Q}^n$ can be approximated by a convex combination of the $v_1, \ldots, v_m$ with rational $\alpha_j \geq 0$ with $\sum_j \alpha_j = 1$, because the rational points are dense in the $(m-1)$-dimensional standard simplex $S \in \mathbb{R}^m$ (by part (iii) applied to the inverse $T^{-1}(S)$ of a suitable affine injective map $T\colon \mathbb{R}^{m-1} \to \mathbb{R}^m$ with $T(0) = (1/m, \ldots, 1/m)$ and rational entries in its coefficient matrix).

(v): $\mathrm{conv}_{\mathbb{R}}(X) \cap V = \mathrm{conv}_{\mathbb{Q}}(X)$ comes from the case where $V \cong \mathbb{Q}^n$. Let $X = \{v_1, \ldots, v_m\} \subseteq \mathbb{Q}^n$ is a finite subset, and $\sum_j \alpha_j v_j = v_0 \in \mathbb{Q}^n$, with $\alpha_j \in (0, \infty)$, $\sum \alpha_j = 1$. Then we can form the matrix $A \in M_{m,n+1}(\mathbb{Q})$ with columns $a_j = (v_j, 1)^\top$, and let $b \in M_{n+1,1}(\mathbb{Q})$ the column $b = (v_0, 1)$. Then $A \cdot x_0 = b$ with $x_0 = [\alpha_1, \ldots, \alpha_n]^\top \in M_{n,1}(\mathbb{R})$. It follows that the matrices A and the extended matrix (A, b) have same rank (calculated in $\mathbb{R}$). Since both matrices have rational entries, the ranks are also equal if calculated in Q. Thus there is a solution of $A \cdot x_1 = b$ with $x_1 = [\beta_1, \ldots, \beta_n]^\top \in M_{n,1}(\mathbb{Q})$. If $B \in M_{n+1,m}(\mathbb{Q})$ is a generalized inverse of A then $T\colon y \in M_{m,1}(\mathbb{R}) \mapsto (1_m - BA)y + Bb$ is an affine map from $\mathbb{R}^m \cong M_{m,1}(\mathbb{R})$ onto the set of all solutions $x \in M^{(}_{m,1}\mathbb{R})$ of $Ax = b$. The set $T^{-1}(0, \infty)^m$ is an open convex subset of $\mathbb{R}^m$ that is not empty, because $(0, \infty)^m$ is open in $\mathbb{R}^m$ and $x_0 \in (0, \infty)^m$ by assumption. By part (ii), we get that $\mathbb{Q}^m \cap T^{-1}(0, \infty)^m$ is not empty, i.e., there is $x_2 \in (0, \infty)^m \cap \mathbb{Q}^m$ with $Ax_2 = b$. This means that v_0 is a rationally-convex combination of $v_1, \ldots, v_m$. □

The following key lemma is an observation of B. Blackadar and M. Rørdam [**86**]. It provided a useful "almost" cancellation property.

LEMMA A.13.6. *If $(S, <_S, u)$ is a scaled abelian semigroup, then, for $x, y \in S$, there exists $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$, if and only if, there are $n \in \mathbb{N}$ and $z \in S$ with $nx + u + z \leq ny + z$.*

PROOF. If $mx + (p+1)u \leq my + pu$, then we let $n := m$ and $z := pu$. To show the non-trivial direction, it suffices to show that $x + u + z \leq y + z$ implies the existence of $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$, because we can rename nx and ny by x and y in the statement.

If $x_j, y_j, z \in S$ satisfy $x_j + z \leq y_j + z$ $(j = 1, 2)$, then

$$x_1 + x_2 + z \leq x_1 + y_2 + z \leq y_1 + y_2 + z\,.$$

Induction gives that $nx + z \leq ny + z$ for all $n \in \mathbb{N}$ if $x + z \leq y + z$.

If $x_1, x_2 \in S$, then there is $n \in \mathbb{N}$ with $x_1 \leq x_2 + nu$: Indeed, there are $n_1, n_2 \in \mathbb{N}$ with $a \leq n_1 u$, $b \leq n_2 u$ and $u \leq b + n_2 u$. It follows $n_1 b \leq b + (n_1 - 1) n_2 u$ and $a \leq b + nu$ for $n := (2n_1 - 1)n_2$.

Let $x, y, z \in S$ with $x + u + z \leq y + z$. There are $k, \ell \in \mathbb{N}$ such that $u \leq z + ku$ and $z \leq ku$, $u \leq y + \ell u$, $y \leq \ell u$. It follows $z \leq k(y + \ell u)$. Let $x' = x + \ell u$, $y' = y + \ell u$. Then $z \leq ky'$, $x' + u + z \leq y' + z$, thus $nx' + nu + z \leq ny' + z$ for all $n \in \mathbb{N}$. Take above $a := kx' + u$ and $b := z$, then there is $n \in \mathbb{N}$ with $kx' + u \leq nu + z$. We get

$$(n + k)x' + u = nx' + kx' + u \leq nx' + nu + z \leq ny' + z \leq (n + k)y'\,.$$

It implies $mx + (p+1)u \leq my + pu$ for $m := n + k$, $p := (n+k)\ell$. □

PROPOSITION A.13.7. *Suppose that $(S, <, u)$ is a scaled abelian semigroup such that $m, n \in \mathbb{N}$ and $mu \leq nu$ imply $m \leq n$.*

Then there is an order unit space $B(S) = (B, B_+, e)$ and a unital additive map $\gamma\colon S \to B$ with $\gamma(u) = e$, such that, for every state λ on $(S, <, u)$, there is exactly one state ρ on B with $\lambda = \rho \circ \gamma$. Furthermore:

- (i) *Natural universality: If B_1 is an oder unit space, and $\psi\colon S \to B_1$ is monotone, unital and additive, then there is a unique monotone unital linear map $T\colon B(S) \to B_1$ with $T \circ \gamma = \psi$.*
- (ii) *There are equivalent:*
  - (ii.1) *$\gamma(y) - \gamma(x)$ is in the interior of B_+.*
  - (ii.2) *There exist $m, p \in \mathbb{N}$ such that $mx + (1+p)u \leq my + pu$.*
  - (ii.3) *$\lambda(x) < \lambda(y)$ for all states λ of S.*
- (iii) *An additive unital map ψ from S to an order unit space $(B_1, (B_1)_+, e_1)$ is monotone, if and only if, the inequality $x+u \leq y$ in S implies $\psi(x) \leq \psi(y)$ for $x, y \in S$ ([8]).*
- (iv) *For a* monotone *additive unital map $\psi\colon S \to B_1$ are equivalent:*
  - (iv.1) *There is an isometric unital map $I\colon B \to B_1$ with $I(\gamma(x)) = \psi(x)$ for $x \in S$.*
  - (iv.2) *If there is $n \in \mathbb{N}$ with $n\psi(x) + e_1 \leq n\psi(y)$, then there exist $m, p \in \mathbb{N}$ with $mx + (p+1)e \leq my + pe$ (i.e., $\gamma(y) - \gamma(x)$ is in the interior of B_+ by (ii)).*
  - (iv.3) *For every state λ on S there exists a state ρ on B_1 with $\lambda = \rho \circ \psi$.*

PROOF. Let $\mathrm{Gr}(S)$ denote the Grothendieck group of S, and let

$$\mathrm{Gr}(S)_+ := \{[y] - [x]\,;\ x, y \in S,\ x \leq_S y\}\,.$$

Then for each $g \in \mathrm{Gr}(S)$ there are $s, t \in S$ with $g = [s] - [t]$.

By Definition A.13.4, there are $m, n \in \mathbb{N}$ with $s \leq nu$, $u \leq t + nu$ and $t \leq nu$. We get $g = ([nu] - [t]) - ([nu] - [s]) \in \mathrm{Gr}(S)_+ - \mathrm{Gr}(S)_+$ and $g \leq g + ([nu] - [s]) = [nu] - [t] \leq 2n[u]$. Thus, $(\mathrm{Gr}(S), \mathrm{Gr}(S)_+, [u])$ is a scaled abelian group with order $g \leq h \Leftrightarrow g - h \in \mathrm{Gr}(S)_+$ and order unit $[u]$.

It turns out for $x, y \in S$ that $[x] \leq [y]$ in $\mathrm{Gr}(S)$, if and only if, there is $z \in S$ with $x + z \leq y + z$. Indeed, $x + z \leq y + z$ implies $[y] - [x] \in \mathrm{Gr}(S)_+$. If $[y] - [x] \in \mathrm{Gr}(S)_+$ then there are $x_1, x_2, s \in S$ with $x_1 \leq x_2$ and $y + x_1 + s = x + x_2 + s =: t$. Thus $x + z = t + x_1 \leq t + x_2 = y + z$ for $z := x_1 + x_2 + s$.

By Lemma A.13.6 it follows that there is $n \in \mathbb{N}$ with $n[x] + [u] \leq n[y]$ in $\mathrm{Gr}(S)$ (i.e., that there exists $z \in S$ with $nx + z + u \leq ny + z$), if and only if, there are $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$ in S.

We define a pre-ordered $\mathbb{R}$-vector space (V, K, w) by $V := \mathrm{Gr}(S) \otimes_Z \mathbb{R}$, $w := [u] \otimes_{\mathbb{Z}} 1$ and K is the convex cone of $V := \mathrm{Gr}(S) \otimes_Z \mathbb{R}$ generated by $\mathrm{Gr}(S)_+ \otimes_{\mathbb{Z}} 1$.

[8] It implies then that $\phi(y) - \phi(x)$ is in the interior of the positive cone $(B_1)_+$.

Since $\mathbb{Q}\otimes_Z\mathbb{R} = \mathbb{Q}\otimes_\mathbb{Q}\mathbb{R} = \mathbb{R}$, we get from Lemma A.13.5, that $\mathrm{Gr}(S)\otimes_Z\mathbb{Q} \to V$ is injective and that $(\mathrm{Gr}(S)\otimes_Z\mathbb{Q})\cap K = \bigcup_n(\mathrm{Gr}(S)_+\otimes_\mathbb{Z} 1/n)$. Hence, $(g\otimes 1) \in K$, if and only if, there exists $n \in \mathbb{N}$ with $ng \in \mathrm{Gr}(S)_+$. In particular, $([y]-[x])\otimes 1 \in K$, if and only if, there are $m \in \mathbb{N}$ and $z \in S$ with $mx + z \leq my + z$.

Suppose $-w = ([u] - [2u]) \otimes 1 \in K$, then there is $m \in \mathbb{N}$ (w.l.o.g.) with $2mu + z \leq mu + z$ in S. By LemmaA.13.6, there are $n, q \in \mathbb{N}$ with $n((2m-1)u) + (q+1)u \leq nmu + qu$, i.e., $ku \leq_S \ell u$ with $mn + q =: \ell < k := n(2m-1) + q + 1$. This contradicts our assumption that $ku \leq \ell u$ implies $k \leq \ell$.

Since $-w \notin K$, (V, K, w) is a scaled $\mathbb{R}$-vector space, as considered in Lemma A.13.3. We define, $B(S) := (B, B_+, e)$ as the order unit space that is obtained from the scaled $\mathbb{R}$-vector space (V, K, w) in Lemma A.13.3, and let $\gamma(x) := \pi([x] \otimes_\mathbb{Z} 1)$, where $\pi\colon V \to B$ is the quotient map $v \in V \mapsto v + V_0 \in V/V_0 \subseteq B$ as considered in Lemma A.13.3.

Since $g \in G \mapsto g \otimes_\mathbb{Z} 1 \in V$ and π are monotone and unital, we get that $\gamma\colon S \to B$ is a monotone, unital additive map. Thus $\lambda := \rho \circ \gamma$ is a monotone additive unital map, if ρ is a state of $B(S)$.

It is obvious, that every state $\lambda\colon S \to \mathbb{R}$ uniquely factorizes through $\mathrm{Gr}(S)$ and then through (V, K, w), i.e., for every state λ of S there is a unique state ρ' on (V, K, w) and then a unique state ρ on (B, B_+, e) (by Lemma A.13.3(iv) with $(B_1, (B_1)_+, u_1) := (\mathbb{R}, \mathbb{R}_+, 1))$ with $\rho(\gamma(x)) = \rho'([x] \otimes 1) = \lambda(x)$ for all $\lambda \in E(S)$. It follows from Lemma A.13.3(ii), that there is a unique isometric unital order isomorphism ι from B onto $\mathrm{Aff}_c(E(S))$ with $\iota(\gamma(x)) = \iota(\pi([x] \otimes 1))(\lambda) = \lambda(x)$ for all $x \in S$.

(i): Suppose that $\psi\colon S \to B_1$ is monotone, unital and additive map into an order unit space B_1. Since B_1 is an $\mathbb{R}$-vector space, the map factorizes through a linear map $[\varphi]\colon V = \mathrm{Gr}(S) \otimes_\mathbb{Z} \mathbb{R} \to B_1$ with $\psi(x) = [\psi]([x])$ for $x \in S$.

The map $[\psi]$ is unital and order preserving, and $X := \mathrm{Gr}(S) \otimes_Z \mathbb{Q}$ satisfies the assumptions of Lemma A.13.3(iv).

Have to show:

There is a unique monotone unital linear map $T\colon B(S) \to B_1$ with $T \circ \gamma = \psi$.

That means: Existence by construction ... and proof of uniqueness ... e.g. by use of separating states on B_1.

If $x \leq y$ then $mx + u \leq (mx + u)$ for any $m \in \mathbb{N}$. It follows that there are $n, q \in \mathbb{N}$ with $n(m\phi(x)) + (q+1)v \leq n(m\phi(x) + v) + qv$. Thus $nm\lambda(\phi(x)) \leq (n-1) + nm\lambda(\phi(x))$ and $\lambda(\phi(x)) \leq \lambda(\phi(x)) + (1/m)$ for all $m \in \mathbb{N}$ and all states λ on $(B_1, (B_1)_+, e_1)$.

`to be filled in` ?? □

Corollary A.13.8. *Let* $(S\,, <\,, u)$, $(T\,, <\,, v)$ *scaled abelian semigroups, and* $\phi\colon S \to T$ *an additive map with* $\phi(u) = v$.

If $mv \leq_T nv$ implies $m \leq n$ (for $n, m \in \mathbb{N}$), then ϕ has the following properties (i) and (ii).

(i) *$\lambda \circ \phi$ is a state of S for each state λ of T,*
if and only if,
for $x, y \in S$, the inequality $x + u \leq y$ implies the existence of $n, q \in \mathbb{N}$ with $n\phi(x) + (q+1)v \leq n\phi(y) + qv$.

(ii) *For every state ρ of S there is a state λ of T with $\rho = \lambda \circ \phi$,*
if and only if,
for $x, y \in S$, the existence of $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$ is equivalent to the existence of $n, q \in \mathbb{N}$ with $n\phi(x) + (q+1)v \leq n\phi(y) + qv$.

Proof. (i): Suppose that for each $x, y \in S$ with $x + u \leq y$ there are $n, q \in \mathbb{N}$ with $n\phi(x) + (q+1)v \leq n\phi(y) + qv$, then $1/n \leq \lambda(\phi(y)) - \lambda(\phi(x))$ for all states $\lambda \colon T \to \mathbb{R}$ on $(T, <_T, v)$. Thus $x + u \leq y$ implies $\lambda(\phi(x)) \leq \lambda(\phi(y))$. Since $(\mathbb{R}, \mathbb{R}_+, 1)$ is an order unit space, we get from Part(iii) of Proposition A.13.7, that the additive unital maps $\lambda \circ \phi \colon S \to \mathbb{R}$ are monotone, i.e., that the λ are states on $(S\, ,<_S\, ,u)$.

Suppose, conversely, that $\lambda \circ \phi$ is a state on $(S, <_S\, , u)$ for all states λ on $(T, <_T\, , v)$. If $x + u \leq y$ then $\lambda(\phi(y) - \phi(x)) \geq 1$ for all states λ on $(T, <_T\, , v)$. This implies the existence of $n, q \in \mathbb{N}$ such that $n\phi(x) + (l+q)v \leq n\phi(y) + qv$, by part (ii) of Proposition A.13.7,

(ii): By part (i) and by Proposition A.13.7, there is a unique unital monotone linear map $\psi \colon B(S) \to B_1 := \mathrm{Aff}_c(E(T)) \cong B(T)$ with $\psi(\gamma_S(x))(\lambda) = \lambda(\phi(x))$ for all $x \in S$ and $\lambda \in E(T)$.

Suppose that $n\phi(x) + (p+1)v \leq n\phi(xy) + pv$ in T implies the existence of $m, p \in \mathbb{N}$ with $mx + (p+1)e \leq my + pe$ in S. If there is $k \in \mathbb{N}$ with $k\psi(x) + e_1 \leq k\psi(y)$ in $(B(T), B(T)_+, e_T)$ then $1/k \leq \lambda(\phi(y)) - \lambda(\phi(x))$ for every state λ on T. If follows the existence of $n, q \in \mathbb{N}$ with $n\phi(x) + (p+1)v \leq n\phi(xy) + pv$ in T, by part (ii) of Proposition A.13.7. By assumption, there are $m, p \in \mathbb{N}$ with $mx + (p+1)e \leq my + pe$.

Thus, $\psi \colon B(S) \to \mathrm{Aff}_c(E(T))$ satisfies the condition in part (iv.2) of Proposition A.13.7(iv). It implies that for every state ρ on S there is a state ρ' on $\mathrm{Aff}_c(E(T))$ with $\rho(x) = \rho'(\psi(\gamma_S(x)))$ for $x \in S$. Since $E(T)$ is compact and convex, it follows that there is $\lambda \in E(T)$ with $\rho'(\gamma_T(y)) = \lambda(y)$ for all $y \in T$. Thus,

$$\rho(x) = \rho'(\psi(\gamma_S(x)) = \rho'(\gamma_T(\phi(x)) = \lambda(\phi(x))\,.$$

By Proposition A.13.7, there is a state λ on T such that $\rho'(\gamma_T(y)) = \lambda(y)$ for all $y \in T$.

$\gamma(y) - \gamma(x)$ is in the interior of $B(S)_+$, iff, there are $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$.

For every state λ on S there exists a state ρ on B_1 with $\lambda = \rho \circ \psi$.

Suppose that for every state ρ on S there is a state λ on T with $\lambda \circ \phi = \rho$, then ψ is isometric by

????????????? reference?

For every state ρ of S there is a state λ of T with $\rho = \lambda \circ \phi$, if and only if, for $x, y \in S$, the existence of $m, p \in \mathbb{N}$ with $mx + (p+1)u \leq my + pu$ is equivalent to the existence of $n, q \in \mathbb{N}$ with $n\phi(x) + (q+1)v \leq n\phi(y) + qv$.

Let $\eta \colon T \to B(T) \cong \mathrm{Aff}_c(E(T))$ denote the natural unital additive map from T into is universal order unit space $(B(T), B(T)_+, e_T)$ (cf. Proposition A.13.7), and let $\psi := \eta \circ \phi$.

By part (iii) of Proposition A.13.7, the additive unital map $\psi \colon S \to B(T)$ from S to the order unit space $(B(T), B(T)_+, e_T)$ is monotone, if and only if, the inequality $x + u \leq y$ in S implies $\phi(x) \leq \phi(y)$ for $x, y \in S$.

`to be filled in` ??

A.13.7 □

Corollary A.13.9. *If $(S_1, <, u)$ is a scaled subgroup of $(S, <, u)$ (with induced order), then $B(S_1) \to B(S)$ is isometric. In particular, every state of S_1 extends to a state of S.*

Proof. Consider the inclusion map $\phi \colon S_1 \ni x \mapsto x \in S$ and let $x, y \in S_1$. There are $n, p \in \mathbb{N}$ with $nx+(p+1)u \leq ny+pu$ in S_1, if and only if, $n\phi(x)+(p+1)u \leq n\phi(y) + pu$. Now apply part (ii) of Corollary A.13.8 and Proposition A.13.7. □

Corollary A.13.10. *Suppose that the scaled abelian semigroup $(S, <, u)$ has a zero element 0 with $0 \leq x$ for all $x \in S$.*

If $a \in S$ satisfies $\lambda(a) = 0$ for all states λ of S, then there exists $n(a, u) \in \mathbb{N}$ such that

$$2na \leq nu \quad \text{for all } n \geq n(a, u)\,.$$

Proof. The proof includes the case, that $\phi \equiv 0$ is the only monotone additive map $\phi \colon S \to \mathbb{R}$.

If $\phi(x) \equiv 0$ is the only monotone linear map $\phi \colon S \to \mathbb{R}$, then there is no state on the sub-semigroup $T = \mathbb{N} \cdot u$ of S with induced preorder and order-unit u (by Corollary A.13.9). It follows that there is $m \in \mathbb{N}$ with $(m+1)u \leq mu$, because otherwise there is a state on $(T, <_S |T, u)$ that extends to $(S, <_S, u)$ by Corollary A.13.9. We get $(n+\ell)u \leq nu$ for all $n \geq m$ and $\ell \in \mathbb{N}$ (by induction over ℓ). There is $k \in \mathbb{N}$ with $a \leq ku$. Thus, $2na \leq 2nku \leq nu$ for all $n \geq m$.

If $mu \leq nu$ always implies $m \leq n$, then, by Proposition A.13.7(ii), there are $m, p \in \mathbb{N}$ with $3ma + (p+1)u \leq (m-1)u + (p+1)u$, because $1 \leq \lambda(u) - \lambda(4a)$ for all states λ of S.

It implies $q3ma + (p+1)u \leq q(m-1)u + (p+1)u$ for all $q \in \mathbb{N}$ (by induction over q, cf. proof of Lemma A.13.6). There is $k \in \mathbb{N}$ with $a \leq ku$ and $k \geq p+1$. We get $(q3m+1)a \leq (q(m-1)+k)u$ for all $q \in \mathbb{N}$. If $n \in \mathbb{N} \cap [q(m-1)+k, (q3m+1)/2]$ then $2na \leq nu$. Let $q \geq (2m-3+2k)/(m+2)$ then $q3m+1 \geq 2((q+1)(m-1)+k)$.

It follows that

$$[n_0, \infty) \subseteq \bigcup_{q \in \mathbb{N}} [q(m-1)+k, (q3m+1)/2]$$

for $n_0 := q_0(m-1)+k$ with $q_0 := 1 + [(2m-3+2k)/(m+2)]$. □

14. Approximate intertwining of inductive limits

We have to work sometimes with C^*-systems that are inductive limits of completely positive contractions on non-unital C^*-algebras.

Some text missing? ??

We write $B_\infty := \ell_\infty(B)/c_0(B)$ for a Banach space B. Recall that B_∞ is a C^*-algebra in case that B is a C^*-algebra. Then B_∞ is the same as B_∞ in the terminology $\mathrm{Q}(X,B) := \mathrm{C_b}(X,B)/\,\mathrm{C}_0(X,B)$ used in Chapter 6 for locally compact spaces X (here $X := \mathbb{N}$). More generally, we define B_∞ for a sequence $B_1, B_2, \ldots$ of Banach algebras or C^*-algebras by

$$B_\infty := (\prod_n B_n)/(\bigoplus_n B_n) \;=\; \ell_\infty(B_1, B_2, \ldots)/c_0(B_1, B_2, \ldots).$$

A sequence of contractions $V_n \colon A_n \to B_n$ defines a contraction $V_\infty \colon A_\infty \to B_\infty$ by

$$V_\infty((a_1, a_2, \ldots) + c_0(A_1, A_2, \ldots) := (V_1(a_1), V_2(a_2), \ldots) + c_0(B_1, B_2, \ldots)$$

for $(a_1, a_2, \ldots) \in \ell_\infty(A_1, A_2, \ldots)$, a norm-bounded sequence with $a_n \in A_n$.

to be filled in: basics on $\mathrm{indlim}(T_n \colon B_n \to B_{n+1})$??

The following lemmata collect some basic facts on inductive limits of linear contractions on Banach spaces and c.p. contractions on C^*-algebras. We use notations $\oplus_n B_n = c_0(B_1, B_2, \ldots) \subseteq c_0(\mathcal{L}(\mathcal{H}))$ and $\Pi_n B_n = \ell_\infty(B_1, B_2, \ldots) \subseteq \ell_\infty(\mathcal{L}(\mathcal{H}))$ for closed subspaces of $\mathcal{L}(\mathcal{H})$ with sufficiently big Hilbert space $\mathcal{H}$ (which plays only an intermediate role).

Lemma A.14.1. *Suppose that $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are Banach spaces, and that $S_n \colon A_n \to A_{n+1}$ and $T_n \colon B_n \to B_{n+1}$ are contractions.*

There is a natural isometric embedding $\mathrm{indlim}(T_n \colon B_n \to B_{n+1}) \subseteq B_\infty$ *given by the maps $\eta_n \colon B_n \to B_\infty$ defined by*

$$\eta_n(b) := (0, \cdots, 0, b, T_n(b), T_{n+1}T_n(b), \ldots) + (\oplus_n B_n)\,,$$

where $b \in B_n$ is on the n-th position. We write also $T_{n,\infty}$ instead of η_n.

Let $V_n \colon A_n \to B_n$, $W_n \colon B_n \to A_{n+1}$ contractions, *and let $S_n := W_n V_n$, $T_n := V_{n+1} W_n$.*

only up to small ε_n ?

Then there is a natural isometry

$$\mathrm{indlim}(S_n \colon A_n \to A_{n+1}) \cong \mathrm{indlim}(T_n \colon B_n \to B_{n+1})$$

given by $V_\infty|\operatorname{indlim}(S_n\colon A_n \to A_{n+1})$ *and* $W_\infty|\operatorname{indlim}(T_n\colon B_n \to B_{n+1})$*, where* $V_\infty\colon A_\infty \to B_\infty$ *and* $W_\infty\colon B_\infty \to A_\infty$ *are given on representatives of elements in* A_∞ *and* B_∞ *by*

$$(a_1, a_2, \ldots) \mapsto (V_1(a_1), V_2(a_2), \ldots)\,,$$

respectively

$$(b_1, b_2, \ldots) \mapsto (0, W_1(b_1), W_2(b_2), \ldots)\,.$$

Suppose now that the A_n *and* B_n *are operator spaces (respectively are* C^**-spaces or* C^**-algebras). The above given isomorphism is completely isometric (respectively completely positive, respective is a* C^**-algebra isomorphism), if, for every* $k \in \mathbb{N}$ *there is* $n(k) \in \mathbb{N}$ *such that* $V_n \otimes \mathrm{id}_k$ *and* $W_n \otimes \mathrm{id}_k$ *are contractive (respectively are positive, i.e., have positive second conjugates; respective the* V_n*,* W_n *are* C^**-morphisms).*

Proof. We have $T_{k,\infty}(V_k(a)) = V_\infty(S_{k,\infty}(a))$ for $a \in A_k$ because, for $n > k$ and $a \in A_k$,

$$V_n(W_{n-1}V_{n-1}(\cdots(W_kV_k(a))\cdots)) = T_{n-1}(\cdots(T_k(V_k(a))\cdots))\,.$$

Thus V_∞ maps $\operatorname{indlim}(S_n\colon A_n \to A_{n+1})$ into $\operatorname{indlim}(T_n\colon B_n \to B_{n+1})$.

A similar argument shows that W_∞ maps $\operatorname{indlim}(T_n\colon B_n \to B_{n+1})$ into $\operatorname{indlim}(S_n\colon A_n \to A_{n+1})$.

Straight calculation shows that $W_\infty \circ V_\infty = S_\infty$ and $V_\infty \circ W_\infty = T_\infty$, for the maps $S_\infty\colon A_\infty \to A_\infty$ and $T_\infty\colon B_\infty \to B_\infty$ given on representatives by

$$(a_1, a_2, \ldots, a_n, a_{n+1}\ldots) \mapsto (0, S_1(a_1), S_2(a_2), \ldots, S_{n-1}(a_{n-1}), S_n(a_n), \ldots)\,,$$

and

$$(b_1, b_2, \ldots, b_n, b_{n+1}\ldots) \mapsto (0, T_1(b_1), T_2(b_2), \ldots, T_{n-1}(b_{n-1}), T_n(b_n), \ldots)\,.$$

The operators S_∞ and T_∞ are contractions and fix the images of $\operatorname{indlim}(S_n\colon A_n \to A_{n+1}) \subseteq A_\infty$, respectively of $\operatorname{indlim}(T_n\colon B_n \to B_{n+1}) \subseteq B_\infty$. Indeed: On representatives one has $S_\infty(S_{n,\infty}(a_n)) = S_{n,\infty}(a_n)$ for $a_n \in A_n$.

Thus, the restriction of V_∞ defines an isometric linear isomorphism from $\operatorname{indlim}(S_n\colon A_n \to A_{n+1})$ onto $\operatorname{indlim}(T_n\colon B_n \to B_{n+1})$ with the restriction of W_∞ to $\operatorname{indlim}(T_n\colon B_n \to B_{n+1})$ as inverses.

Suppose now that the A_n and B_n are operator spaces, matrix operator systems, or C^*-spaces (all not necessarily unital). It follows, that this isomorphism is completely isometric (respectively are matrix-order isomorphisms) if V_∞ and W_∞ are completely contractive (respectively completely positive) on the images of the respective images of the inductive limits.

It suffices to suppose that for each $k \in \mathbb{N}$ there is $n(k) \in \mathbb{N}$ such that $V_n \otimes \mathrm{id}_k$ and $W_n \otimes \mathrm{id}_k$ are contractions (respectively are positive) for each $n \geq n(k)$. □

LEMMA A.14.2. *Suppose that* $B_1, B_2, \ldots$ *are separable Banach spaces and that* $\mathcal{G}_n \subseteq \mathcal{L}(B_n, B_{n+1})$ *are* subsets *of linear contractions from* B_n *into* B_{n+1}. *If, for each* $n \in \mathbb{N}$, *the operator* $T_n \colon B_n \to B_{n+1}$ *is in the point-norm closure of* $\mathcal{G}_n$ (= *closure of* $\mathcal{G}_n$ *in strong operator topology), then there exist* $g_1, g_2, \ldots \mathcal{G}$, *such that in* $\ell_\infty(B_1, B_2, \ldots)/c_0(B_1, B_2, \ldots)$ *the related inductive limits coincide:*

$$\text{indlim}(g_n \colon B_n \to B_{n+1}) = \text{indlim}(T_n \colon B_n \to B_{n+1}) .$$

We identify here the inductive limits with their canonical corresponding closed subspaces of $B_\infty := \ell_\infty(B_1, B_2, \ldots)/c_0(B_1, B_2, \ldots)$.

PROOF. There exist sequences $b_{1,n}, b_{2,n}, \ldots \in B_n$ that are dense in B_n, for each $n \in \mathbb{N}$.

We define $T_{m,n} \colon B_m \to B_n$ for $n \geq m$, by $T_{m,m} = \text{id}$ (identity map of B_m), $T_{n,n+1} := T_n$ $T_{m,n} := T_{n-1} \circ \cdots \circ T_{m+1} \circ T_m$ for $n > m$. We denote by

$$T_{m,\infty} \colon B_m \to X := \text{indlim}(T_n \colon B_n \to B_{n+1}) \subseteq B_\infty$$

the defining morphisms from B_m into X. Recall that for our realization $X \subseteq B_\infty$ of $\text{indlim}(T_n \colon B_n \to B_{n+1})$ holds

$$T_{m,\infty}(b) = (0, \ldots, 0, b, T_{m,m+1}(b), T_{m,m+2}(b), \ldots) + c_0(B_1, B_2, \ldots)$$

for $b \in B_m$. Then $T_{m,\infty} = T_{n,\infty} \circ T_{m,n}$.

The set $\{T_{m,\infty}(b_{k,m})\,;\ k, m \in \mathbb{N}\}$ is dense in $X := \text{indlim}(T_n \colon B_n \to B_{n+1})$, because X is the closure of $\bigcup_m T_{m,\infty}(B_m)$.

We find linear subspaces $L_n \subseteq B_n$ and $g_n \in \mathcal{G}_n$ of finite dimension, such that

(i) L_n contains the sets $T_{n-1}(L_{n-1}) \cup g_{n-1}(L_{n-1})$, $\{ b_{1,n}, b_{2,n}, \ldots, b_{n,n} \}$, and $\{T_{k,n}(b_{j,k})\,;\ 1 \leq k \leq n-1,\ j \leq n\}$, and $\{g_{k,n}(b_{j,k})\,;\ 1 \leq k < n-1,\ j \leq n\}$, where $g_{k,n} := g_{n-1} \circ \cdots \circ g_{k+1} \circ g_k$ for $1 \leq k < n-1$.
(ii) $\|T_n(b) - g_n(b)\| \leq 2^{-n}\|b\|$ for $b \in L_n$.

Then $T_n(L_n) \subseteq L_{n+1}$, $g_n(L_n) \subseteq L_{n+1}$, and $\text{indlim}(T_n \colon L_n \to L_{n+1})$ and $\text{indlim}(g_n \colon L_n \to L_{n+1})$ are the same closed subspaces of $(\prod_n L_n)/(\bigoplus_n L_n) \subseteq B_\infty$, because, $T_{m,\infty}(L_m) \subseteq T_{n,\infty}(L_n)$ and $g_{m,\infty}(L_m) \subseteq g_{n,\infty}(L_n)$ for $n \geq m$, and because, for $b \in L_m$,

$$\|T_{m,\infty}(b) - g_{m,\infty}(b)\| \leq \|b\| \cdot \sum_{n \geq m} \|(T_n - g_n)|L_n\| \ \leq\ \|b\| \cdot 2^{1-m} .$$

The set $\{T_{m,\infty}(b_{k,m})\,;\ m, k \in \mathbb{N}\}$ is dense in $\text{indlim}(T_n \colon B_n \to B_{n+1})$. The element $T_{m,n}(b_{k,m})$ is in L_n for $n > \max(k,m)$. Thus, $T_{m,\infty}(b_{k,m}) = T_{n,\infty}(T_{m,n}(b_{k,m})) \in T_{n+1,\infty}(L_{n+1})$. It shows that $\text{indlim}(T_n \colon L_n \to L_{n+1})$ is the same closed subspace of B_∞ as $\text{indlim}(T_n \colon B_n \to B_{n+1})$. Since also $g_{m,n}(b_{k,m})$ is in L_n for $n > \max(k,m)$, a similar argument shows that

$$\text{indlim}(g_n \colon L_n \to L_{n+1}) = \text{indlim}(g_n \colon B_n \to B_{n+1}) .$$

It is obvious that the resulting isomorphisms are, e.g. , C^*-algebra isomorphisms if the intertwining maps are e.g. C^*-morphisms. □

15. Intersection with sums of left and right ideals

The following Remark allows to circumvent in some cases the rather engaged general (semi-constructive) reduction to relatively weakly injective separable subspaces of operator spaces given in Section 14.

REMARK A.15.1. The sum $L+R \subseteq B$ of a closed left ideal L and a closed right ideal R of a C^*-algebra B is always a closed subspace of B, see Proposition A.15.2.

Thus, if D is a C^*-subalgebra of B with the *property* that $(D \cap L) + (D \cap R)$ is *dense* in $D \cap (L + R)$ then automatically $(D \cap L) + (D \cap R) = D \cap (L + R)$.

Density of $(D \cap L) + (D \cap R)$ in $D \cap (L + R)$ is a very special situation. The following trivial example shows this:

Let $B = \mathbb{C} \oplus \mathbb{C}$, $L := \mathbb{C} \oplus 0$, $R := 0 \oplus \mathbb{C}$ and $D := \mathbb{C} \cdot (1, 1)$. Then $D \cap (L+R) = D$ but $D \cap L = \{0\}$ and $D \cap R = \{0\}$.

But in the very particular case of Lemma B.14.1 where B is a C^*-algebra, $G \subseteq B$ separable subset, X is a closed left ideal and Y is a closed right ideal of B, we need only Part (2) of Sublemma B.14.2 to get a separable C^*-algebra D with $G \subseteq D$ and $(D \cap X) + (D \cap Y)$ dense in $D \cap (X + Y)$.

Then the equality follows from next Proposition A.15.2.

PROPOSITION A.15.2. *Let L a closed left ideal and R a closed right ideal of a C^*-algebra A. Consider the intersections $D := L^* \cap L$ and $E := R^* \cap R$. Then*

(i) *D and E are hereditary C^*-subalgebras A that are identical with the sets of products $L^* \cdot L = \{a^*b\,;\ a, b \in L\} = D$, respectively $R \cdot R^* = E$.*
(ii) *L and R are the element-wise products $L = A \cdot D$ and $R = E \cdot A$, where, e.g. , $A \cdot D := \{\, a \cdot d\,;\ d \in D\,, a \in A \,\}$.*
(iii) *Let $\eta := \pi_{L \cap R} \colon a \mapsto a + (L \cap R)$ denotes the quotient map from A onto $A/(L \cap R)$. For any $x \in L$ and $y \in R$ holds*

$$\max\big(\|\eta(x)\|,\, \|\eta(y)\|\big) \;\leq\; \|\eta(x+y)\| \;\leq\; 2\max\big(\|\eta(x)\|,\, \|\eta(y)\|\big)\,.$$

(iv) *The algebraic sum $L + R$ is a closed linear subspace of A.*
(v) *The second conjugate operator space of $A/(L+R)$ is naturally isomorphic to the ternary algebra $(1-q)A^{**}(1-p)$, where p and q are the "open" support projections in A^{**} for the hereditary C^*-subalgebras $D = L^* \cap L$ and $E = R^* \cap R$ of A. In particular,*

$$\|\pi_{L+R}(a)\| = \|(1-q)a(1-p)\| \qquad \textit{for all } a \in A\,. \tag{15.1}$$

Here L^* (respectively R^*) mean the right ideal (respectively left ideal) given by $L^* := \{a^*\,;\ a \in L\}$ and $R^* := \{a^*\,;\ a \in R\}$, – not the dual spaces. Our approach here ignores the general theory of operator spaces, because we try to derive all from basic knowledge about Functional analysis, as e.g. the "Krein bi-polar theorem" for convex subsets in locally convex vector spaces ... All that we show here prove in the same way the corresponding results for the operator-matrix norms, because e.g. $M_n(A/L) \cong M_n(A)/M_n(L)$ and $M_n(A) \cong M_n \otimes A$, ... etc. The following proof is not minimal, because contains some additional observations.

Proof. Clearly, $D := L^* \cap L = D^2$ because D is a C^*-algebra and every element of a C^*-algebra is the product of two elements, e.g. by using polar decomposition and roots of positive elements.

The closed left ideal L is a non-degenerate right module for the C^*-algebra D. Thus $L = L \cdot D$ by the Cohen factorization theorem A.11.1.

Then $A \cdot D \subseteq L$, $L = L \cdot D$ and $L \subseteq A$ give that $A \cdot D = L$. Since D is a hereditary C^*-subalgebra, we get $D = D^3 \subseteq DAD \subseteq D$. It follows $D = D^2 \subseteq L^* \cdot L = DA^2D \subseteq D$.

Similar considerations, – or passage to the left ideal $R^* := \{a^*\,;\ a \in R\}$ –, show also the corresponding results for closed right ideals of A and $E := R^* \cap R$.

Since $(L \cap R) \cdot D \subseteq L \cap R$ and $E \cdot (L \cap R) \subseteq L \cap R$ we get that

$$\operatorname{dist}(f \cdot x, L \cap R) \le \|f\| \cdot \operatorname{dist}(x, L \cap R)$$

for $f \in E$ and $x \in A$, and

$$\operatorname{dist}(x \cdot g, L \cap R) \le \|g\| \cdot \operatorname{dist}(x, L \cap R)$$

for $g \in D$, and $x \in A$.

Let $\eta := \pi_{L\cap R}$. Then obviously $\|\eta(x+y)\| \le \|\eta(x)\| + \|\eta(y)\| \le 2\max(\,\|\eta(x)\|,\, \|\eta(y)\|\,)$.

We show that $\max(\|\eta(x)\|,\, \|\eta(y)\|) \le \|\eta(x+y)\|$:

Let $x \in L$ and $y \in R$. Then $\eta((x+y)(x^*x)^{1/n}) = \eta(x(x^*x)^{1/n})$, because $y(x^*x)^{1/n} \in L \cap R$. We get an estimate

$$\operatorname{dist}((x+y)(x^*x)^{1/n}, L \cap R) \le \|x\|^{2/n} \operatorname{dist}((x+y), L \cap R)\,,$$

because $(x^*x)^{1/n} \in D$.

It follows that for all $n \in \mathbb{N}$, $\|\eta(x(x^*x)^{1/n})\| = \|\eta((x+y)(x^*x)^{1/n})\| \le \|x\|^{2/n}\|\eta(x+y)\|$. On the other hand, $\|\eta(x)\| \le \|\eta(x(x^*x)^{1/n})\| + \|x - x(x^*x)^{1/n}\|$.

Since $\lim_{n\to\infty} \|x\|^{2/n} = 1$ and $\lim_{n\to\infty} \|x - x(x^*x)^{1/n}\| = 0$, we obtain that $\|\eta(x)\| \le \|\eta(x+y)\|$. In the same way we get that $\|\eta(y)\| \le \|\eta(x+y)\|$.

It follows that the natural linear map from $B := L/(L\cap R) \oplus_\infty R/(L \cap R)$ onto the image $C := \eta(L+R) \subseteq A/(L \cap R)$ of the sum $L + R \subseteq A$ is a bijective linear map that maps the Banach space B onto the normed vector space C. The above given estimates show now that the norm on B induced by this map is equivalent to the given complete norm on B. Thus, B is also a Banach space with this new norm, and C is a Banach space with respect to the norm on C induced from $A/(L \cap R)$. This implies that C is closed in $A/(L\cap R)$. It follows that $L+R = \eta^{-1}(C)$ is closed in A.

Therefore $A/(L+R)$ is a well-defined Banach space. It is in a natural way an operator space, because we can in above observations A, L and R replace by $M_n(A) \cong M_n \otimes A$, $M_n(L)$ and $M_n(R)$.

The general theory of Banach spaces A shows that for closed linear subspaces X of A the bi-dual space $(A/X)^{**}$ is naturally isomorphic to $A^{**}/\mathrm{wcl}(X)$, where $\mathrm{wcl}(X)$ is the $\sigma(A^{**}, A^*)$-closure of X. It is identical with the bi-polar X^{oo} of X in A^{**}. It holds $\mathrm{dist}(a, X) = \mathrm{dist}(a, X^{oo})$ measured with the norm in A^{**}. Thus $\|\pi_X(a)\| = \|\pi_{X^{oo}}(a)\|$ for all $a \in A$.

In case of (matrix-normed) operator spaces this carries over to the matrix spaces $M_n(A)$, $M_n(X)$ and $M_n(A/X) \cong M_n(A)/M_n(X)$.

The $\sigma(A^{**}, A^*)$ closure of $L+R$ in A^{**} is $A^{**}p + qA^{**}$, where $p, q \in A^{**}$ are the "open" projections corresponding to the hereditary C^*-subalgebras D and E respectively.

The second conjugate of $A/(L+R)$ is the quotient of A^{**} by the bi-polar of $L+R$ in A^{**}. The bi-polar of $L+R$ is just the $\sigma(A^{**}, A^*)$-closure of $L+R$ in A^{**}. It contains the sum of $A^{**}p \cong L^{**}$ and $qA^{**} \cong R^{**}$, and it is not difficult to see that the unit-ball of $A^{**}p + q\,A^{**}$ is $\sigma(A^{**}, A^*)$-compact. Here p and q are the open projections in A^{**} corresponding to the hereditary C^*-subalgebras $D = L^* \cap L$ and $E := R^* \cap R$ of A. Thus, $\big(A/(L+R)\big)^{**}$ is natural isomorphic to $A^{**}/(A^{**}p + q\,A^{**})$.

It is easy to check that the norm of $\pi_{(A^{**}p+q\,A^{**})}(a)$ for $a \in A^{**}$ is equal to $\|(1-q)a(1-p)\|$.

In particular, $\|\pi_{L+R}(a)\| = \|(1-q)a(1-p)\|$ for all $a \in A$.

In every W*-algebra M holds: If $p, q \in M$ are projections, then

$$\mathrm{dist}(a, Mp + qM) = \|(1-q)a(1-p)\| \quad \text{for all } a \in M\,.$$

This is the case because $(1-q)a(1-p) - a \in Mp + qM$ and the map $T\colon M \ni a \mapsto (1-q)a(1-p)$ is a linear contraction with $T^2 = T$ and kernel $= Mp + qM$.

It follows that $\|\pi_{L+R}(a)\| = \mathrm{dist}(a, A^{**}p + qA^{**}) = \|(1-q)a(1-p)\|$ for all $a \in A$. □

COROLLARY A.15.3. *If L and R are closed left and right ideals of a C^*-algebra A and $B \subseteq A$ a separable C^*-subalgebra of A then there exists a separable C^*-algebra $C \subseteq A$ such that $B \subseteq C$ and*

$$B \cap (L+R) \subseteq (C \cap L) + (C \cap R)\,.$$

PROOF. By Proposition A.15.2(iv,v), the linear subspace $L+R$ is closed in A. Thus $B \cap (L+R)$ is a closed separable subspace of $L+R$ and of B. We find a dense sequence $\{b_1, b_2, \ldots\}$ in the unit-ball of the separable Banach space $B \cap (L+R)$. In $L+R$ are elements $c_n \in L$ and $d_n \in R$ such that $b_n = c_n + d_n$. The separable C^*-algebra C of A generated by B and $\{c_1, c_2, \ldots; d_1, d_2, \ldots\}$ has the properties that $B \subseteq C$, and that $C \cap L$ and $C \cap R$ are closed left and right ideals of C, and, by

Proposition A.15.2, $(C \cap L) + (C + R)$ is a closed subspace of C that contains a dense subset of $B \cap (L+R) \subset C$. Thus, $B \subseteq C$ and $B \cap (L+R) \subseteq (C \cap L) + (C \cap R)$. □

16. Surjectivity and control of perturbations

Let E and F be Banach spaces and T a bounded linear map from E into F. If X and Y are subsets of E, then $\mathrm{cl}(X)$ means the (norm-) closure of X, $\mathrm{wcl}(X) \subseteq E^{**}$ is the $\sigma(E^{**}, E^*)$-closure of X in the second conjugate space E^{**} of E, and $\mathrm{dist}_H(X, Y)$ denotes the *Hausdorff distance* between subsets $X, Y \subseteq E$ (or $X, Y \subseteq E^{**}$ or – more generally – subsets of any metric space E) that is given by

$$\mathrm{dist}_H(X, Y) := \sup\{\, \mathrm{dist}(a, Y), \mathrm{dist}(b, X)\, ;\ a \in X,\ \ b \in Y\,\}.$$

We denote by S the *open* unit ball of E.

`Does def. of` $f(x,T)$ `fit the case of` $T := \pi(c(.)d)$`?`

Suppose that there exists a function $f(t)$ of $t \in (0, 2]$ with the properties that $\lim_{t \to 0} f(t) = 0$ and that if $a, b \in S$ and $\|T(a) - T(b)\| < x$ then $\mathrm{dist}(a, S \cap T^{-1}(T(b)) \leq f(x)$.

This allows to show that T maps the closed unit ball $\overline{S}$ of E *onto* the closure of $T(S)$:

Since T is bounded, $T(\overline{S})$ is contained in the closure of $T(S)$.

Let $g_1, g_2, \ldots \in T(S) \subseteq F$ a Cauchy sequence that converges to some $h_\infty \in F$.

Find a subsequence $h_1, h_2, \ldots$ of (g_n) such that for each $e \in T^{-1}(h_n) \cap S$ there exists $e' \in T^{-1}(h_{n+1}) \cap S$ with $\|e' - e\| \leq f(\|h_{n+1} - h_n\|)$ and $\sum_n f(\|h_{n+1} - h_n\|) < \infty$.

The latter requires only that $\lim_{x \to 0} f(x) = 0$ if we pass to a suitable subsequence of (h_n) of (g_n).

Then one finds step-wise $e_n \in T^{-1}(h_n)$ with $\sum_n \|e_n - e_{n+1}\| < \infty$. It follows that (e_n) converges to some $e_\infty \in \overline{S}$ and $T(e_\infty) = h_\infty$.

We define a function $f(x, T)$ for $x \in (0, \infty)$ by

Definition A.16.1.

$$f(x, T) := \sup\{\, \mathrm{dist}_H\big(S \cap T^{-1}(T(a)),\ S \cap T^{-1}(T(b))\big)\, ;\ a, b \in S,\ \|T(a-b)\| \leq x\,\}.$$

Clearly $f(x, T)$ is an increasing function of $x \in (0, \infty)$. The *perturbation constant* $\mathrm{per}(T)$ will be defined by $\mathrm{per}(T) := \lim_{x \to 0} f(x, T)$.

Proposition A.16.2. *Let $x \mapsto f(x, T)$ the above defined function.*

(i) *$f(x, T)$ is a continuous increasing function on* $]0, \infty[$.
(ii) *$per(T) = 0$ implies* $T(\mathrm{cl}(S)) = \mathrm{cl}(T(S))$.
(iii) *$f(x, T) = f(x, T^{**})$ for every $x > 0$.*

Here T^{**} means the second adjoint operator of T. $f(x,T)$ is the infimum of all numbers $y > 0$ such that given $b \in S$, $d \in T(S)$ with $\|T(b) - d\| \leq x$ there exists a perturbation $b + h$ of b *inside* S such that $\|h\| \leq y$ and $T(b+h) = d$. Thus, $per(T) = 0$ says that a small perturbation inside $T(S)$ can be realized by a small perturbation inside S. The proofs and further results will be given in

Section **??** (= in old A).

Now let A be a C^*-algebra and L, R closed left and right ideals with open support projections l and r in A^{**} respectively, i.e., l, r are the open projections in A^{**} satisfying $A^{**}l = \mathrm{wcl}(L)$ and $rA^{**} = \mathrm{wcl}(R)$, cf. [**767**, vol.I, chp.III.,def.6.19,cor.6.20], [**616**, thm.3.10.7, prop.3,11.9, rem.3.11.10].

Put $q = 1 - l$ and $p = 1 - r$. Let b, d be elements of A^{**} in the multiplier algebra $\mathcal{M}(A)$ of A such that pbb^*p is invertible in $pA^{**}p$ with inverse g and qd^*dq is invertible in $qA^{**}q$ with inverse h.

The algebraic sum $R + L$ is a closed linear subspace of A, $\mathrm{wcl}(L+R) = A^{**}l + rA^{**}$ and $\|\pi_{L+R}(a)\| = \|(1-r)a(1-l)\|$ for all $a \in\in A$ by Proposition A.15.2.

We denote by $\pi = \pi_{L,R}$ the quotient map $A \to A/(L+R)$ given by $c \to c + L + R$ and denote by $\pi(b(.)d)$ the map given by $c \to bcd + L + R$ for $c \in A$.

Proposition A.16.3. *$f(x,T) \leq x(\|g\| \ \|h\|)^{1/2} + (2x(\|g\| \ \|h\|)^{1/2})^{1/2}$, where $T := \pi(b(.)d)$.*

The Proof follows from Corollaries **??** (old cor.A.2.9 ???) and **??** (old cor.A.3.2 ???) (cf. Section **??** old A.4?).

We obtain from Propositions A.16.2(ii) and A.16.3 immediately the following corollary:

Corollary A.16.4. *Under the above assumptions concerning A, L, R, b and d the quotient map $A \to A/(L+R)$ maps $b(\mathrm{cl}(S))d$ onto a closed set.*

In particular the quotient map maps the closed unit ball of A onto the closed unit ball of $A/(L+R)$.

For every positive integer n, the map $[b_{j,k}]_n \to [pb_{j,k}q]_n$ from $M_n(A^{**}) \cong M_n(A)^{**}$ onto $M_n(pA^{**}q)$ is a contraction and has kernel $M_n(A^{**}l + rA^{**}) = M_n(\mathrm{wcl}(L+R)) \cong \mathrm{wcl}(M_n(L+R))$; passage to the quotient space defines an isometric M_n–bimodule isomorphism from $(M_n(A)/M_n(L+R))^{**} \cong M_n(A^{**})/M_n(\mathrm{wcl}(L+R))$ onto $M_n(pA^{**}q)$. On the other hand there is a natural M_n–bimodule isomorphism from $M_n(A/(L+R))$ onto $M_n(A)/M_n(L+R)$. The matrix norms induced by these isomorphisms give $A/(L+R)$ the structure of a *matrix normed space* in the sense of

Effros [EF2]

such that the second conjugate matrix normed space is completely isometrically isomorphic to $pA^{**}q$, an operator subspace of A^{**} and C^*-triple system,

cf. Section **??**.

If moreover A is unital and $L = \{b^*: b \in R\}$ then $p = q$ and, under the above identifications, $A/(L+R) = pAp \subseteq pA^{**}p$ becomes a matrix order unit space in the sense of [**235**] with matrix order unit $p \in A/(L+R)$ such that the second conjugate matrix order unit space is just the unital C^*-algebra $pA^{**}p$, thus then $A/(L+R)$ is an operator space in the sense of [**245**]. More generally we call a matrix order unit space X a *C^*-system* if its second conjugate matrix order unit space X^{**} is unitally matrix order isomorphic to a unital C^*-algebra. Then X is an operator system in the sense of

??? ??? [C/E2],

i.e., a closed unital and selfadjoint linear subspace of a C^*-algebra together with the matrix order inherited from the inclusion. The C^*-algebra structure on X^{**} is uniquely determined as the second conjugate matrix order unit structure of the given one on X and we can define the *left multiplier algebra* $\mathcal{M}_\ell(X)$, the *right multiplier algebra* $\mathcal{M}_r(X)$ and the *multiplier algebra* $\mathcal{M}(X)$ of X as follows: $\mathcal{M}_\ell(X) := \{b \in X^{**}: bX \subseteq X\}$, $\mathcal{M}_r(X) := \{b \in X^{**}: Xb \subseteq X\}$, $\mathcal{M}(X) := \mathcal{M}_\ell(X) \cap \mathcal{M}_r(X)$. Here we identify X with its canonical and isometric image in X^{**} by the evaluation map $\mathrm{ev}_X : X \to X^{**}$. The algebras $\mathcal{M}_\ell(X)$ and $\mathcal{M}_r(X)$ are closed subalgebras of X^{**} contained in X (more precisely: in the image of $\mathrm{ev}_X : X \to X^{**}$), $\mathcal{M}_\ell(X) = \{b^*: b \in \mathcal{M}_r(X)\}$ and the multiplier algebra $\mathcal{M}(X)$ of X is a unital C^*-algebra which is unitally completely positively and completely isometrically contained in the operator system X. In our special case $A/(L+R)$ is identified with $pAp (\subseteq pA^{**}p \subseteq A^{**})$ and $\mathcal{M}_r(A/(L+R)) = \{pbp;\ b \in A, pApbp \subseteq pAp\}$, $\mathcal{M}(A/(L+R)) = \{pbp;\ b \in A, pbpAp + pApbp \subseteq pAp\}$.

Let D be a hereditary C^*-subalgebra of A (i.e. D closed, selfadjoint and $DAD \subseteq D$). An element $b \in A$ that satisfies $Db \subseteq D$ (respectively $bD \subseteq D$, $bD + Db \subseteq D$) a *right normalizer* (respectively *left normalizer*, *normalizer*) of D in A. The right normalizers, left normalizers, normalizers obviously form closed operator algebras $\mathcal{N}_r(D)$, $\mathcal{N}_\ell(D)$ and $\mathcal{N}(D)$ respectively, $\mathcal{N}_r(D) = \{b^*: b \in \mathcal{N}_\ell(D)\}$ and $\mathcal{N}(D) = \mathcal{N}_\ell(D) \cap \mathcal{N}_r(D)$ is a C^*-subalgebra of A. Notice that $L = A \cdot D$ and $R = D \cdot A$ are closed left and right ideals of A, respectively, whose support projections in A^{**} are equal that of D (i.e., are equal to the unit element of $D^{**} = \mathrm{wcl}(D) \subseteq A^{**}$). From definitions we see that $L = \mathrm{cl}(AD) = A \cdot D \subseteq \mathcal{N}_r(D)$, $R = \mathrm{cl}(DA) \subseteq \mathcal{N}_\ell(D)$ and $D \subseteq \mathcal{N}(D)$ are closed ideals of $\mathcal{N}_r(D)$, $\mathcal{N}_\ell(D)$ and $\mathcal{N}(D)$ respectively.

We define $A//D := A/(\mathrm{cl}(AD) + \mathrm{cl}(DA))$, the (unital) *quotient–C^*-system* of the unital C^*-algebra A *with respect to the hereditary C^*-subalgebra* D of A. We denote again by $\pi_D : A \to A//D$ the quotient map $b \mapsto b + \mathrm{cl}(AD) + \mathrm{cl}(DA)$. Now we are in position to state the main result of this section.

Theorem A.16.5. *Let A be a unital C^*-algebra. D a hereditary C^*-subalgebra of A, $A//D := A/(AD + DA)$ the quotient–C^*-system of A with respect to D and $\pi_D : A \to A//D$ the quotient map.*

(i) *The restriction of* π_D *to* $\mathcal{N}_r(D)$ *(resp. to* $\mathcal{N}_\ell(D)$*) is a Banach algebra epimorphism onto* $\mathcal{M}_r(A//D)$ *(resp. onto* $\mathcal{M}_\ell(A//D)$*) with kernel* $\mathrm{cl}(AD)$ *(resp.* $\mathrm{cl}(DA)$*),*

(ii) *for every positive integer* n, $\pi_D \otimes \mathrm{id}_n$ *maps the closed unit ball of*

$$\mathcal{N}_r(M_n(D)) = M_n(\mathcal{N}_r(D)) \subseteq M_n(A)$$

onto the closed unit ball of

$$M_n(\mathcal{M}_r(A//D)) \subseteq M_n(A//D) \cong M_n(A)//M_n(D)\,,$$

(iii) $\pi_D|\mathcal{N}(D)$ *is a* C^**-algebra epimorphism from the (two-sided) normalizer algebra* $\mathcal{N}(D) \subseteq A$ *of* D *onto the multiplier algebra* $\mathcal{M}(A//D)$ *of* $A//D$ *with kernel ideal* $\ker(\pi_D|\mathcal{N}(D)) = D$.

As a corollary we get the following which can be seen as alternative formulation of

REFERENCE ?

Theorem **??** in view of Proposition **??**.

Corollary A.16.6. *Let* A *and* C *be unital* C^**-algebras,* B *a unital closed subalgebra of* C *and* $V\colon A//D \to C$ *a unital completely isometric map such that* $B \cup \{b^*b\colon b \in B\} \subseteq Im(V)$*. Then there exists a unital closed subalgebra* E *of* A *such that*

(i) $E \cap (\mathrm{cl}(AD) + \mathrm{cl}(DA)) = \mathrm{cl}(AD)$,

(ii) $V \circ \pi_D|E$ *is a Banach algebra epimorphism form* E *onto* B *with kernel* $\mathrm{cl}(AD)$ *and*

(iii) *the induced map* $[V \circ \pi_D]^0\colon E/\mathrm{cl}(AD) \to B$ *is completely isometric, where* B *is equipped with the matrix norms induced by* C *and* $E/\mathrm{cl}(AD)$ *is equipped with the matrix norms induced by the inclusion* $A/\mathrm{cl}(AD) \subseteq A^{**}p$.

If moreover B *is a* C^**-subalgebra of* C *then there exists a unital* C^**-subalgebra* F *of* A *such that*

(iv) $F \cap (\mathrm{cl}(DA) + \mathrm{cl}(AD)) = D$ *and*

(v) $V \circ \pi_D|F$ *is a* C^**-algebra epimorphism from* F *onto* B *with kernel* D.

With other words and under the assumptions of Corollary A.16.6: B is a C^*-quotient algebra of a C^*-subalgebra of A if B is a C^*-subalgebra of C. Theorem **??** and Corollary **??** are proven in Section **??**.

??? `Give ref. to suitable place` In a forthcoming paper we shall show:

The assumptions of Corollary **??** *are satisfied with the CAR-algebra* $A := M_{2^\infty}$ *and with* D,C *and* V *suitably chosen if and only if* B *is separable and exact in the sense of* [KI2], cf. [KI4].

Notice that all proofs work both in real and complex case. There are shorter proofs if one is interested only in the results of Theorem **??**(iii) and Corollary **??**(iv,v).

The results presented here are essentially reworking of material which has been circulating as preprint since 1989. Results similar to those of Corollary ???1.3??? and Theorem ???1.4(i)??? are also contained in preprints of L. Brown [BR], where he gives proofs using different methods and estimates.

17. On perturbation of unitaries

`Change here all signs to Halmos unitary !`

If b is a contraction on a Hilbert space we denote by $U(b) \in M_2(A)$ the "Halmos unitary" matrix defined in Remark 4.2.4, i.e., the matrix

$$\begin{bmatrix} b\,, & -(1-bb^*)^{1/2} \\ (1-b^*b)^{1/2}, & b^* \end{bmatrix}.$$

We are going to prove the following Proposition A.17.1 and its Corollary A.17.2.

Proposition A.17.1. *Let A denote a complex (or real) unital C^*-algebra, u a unitary in A and p,q a pair of projections in A such that $\|puq\| < 1$. Then for every contraction b in pAq there exists a unitary $\tilde{u}$ in A such that $p\tilde{u}q = b$ and $\|u - \tilde{u}\| = \|U(puq) - U(b)\|$.*

Corollary A.17.2. *Let A be a unital C^*-algebra (real or complex), p,q nonzero projections in A and T the natural map from A onto pAq given by $T(a) = paq$ (a in A). Then the perturbation function $f(x,T)$ of T, cf. Definition A.16.1, can be estimated by $f(x,T) \le x + (2x)^{1/2}$.*

We need some preliminary lemmata. To simplify notation, let $\operatorname{diag}(a,b,\ldots)$ denote the diagonal matrix with diagonal elements $a,b,\ldots$, let $M(c)$ be the matrix

$$\begin{bmatrix} c\,, & -(p-cc^*)^{1/2} \\ (q-c^*c)^{1/2}, & c^* \end{bmatrix}$$

if c is a contraction in pAq and let us denote by Z the matrix

$$\begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix}.$$

Lemma A.17.3. *Let A be a C^*-algebra, p,q projections in A, a in pAq, d in qAp such that a and*

$$N := \begin{bmatrix} a\,, & -(p-aa^*)^{1/2} \\ (q-a^*a)^{1/2}, & d \end{bmatrix}$$

are contractions in $M_2(A)$.
Then $M(a)$ satisfies $M(a)^*M(a) = \operatorname{diag}(q,p)$, $M(a)M(a)^* = \operatorname{diag}(p,q)$.
If moreover $\|a\| < 1$ then $d = a^*$.

Proof. $(1-aa^*)^{1/2}p = p(1-aa^*)^{1/2} = (p-aa^*)^{1/2}$ and $q(1-a^*a)^{1/2} = (1-a^*a)^{1/2}q = (q-a^*a)^{1/2}$ because $aa^* \le p,\ \ aa^* \le q$ and $paq = a$. It follows that $a^*(p-aa^*)^{1/2} = (q-a^*a)^{1/2}a^*$, $(p-aa^*)^{1/2}a = a(q-a^*a)^{1/2}$, and $(q-a^*a)^{1/2}d = (1-a^*a)^{1/2}d$. Put $M := M(a)$. Using these identities straightforward computations show $M^*M = \operatorname{diag}(q,p)$ and $MM^* = \operatorname{diag}(p,q)$. Moreover

$$M^*N = \operatorname{diag}(q, p-aa^*-ad) + \operatorname{diag}(c,0)Z$$

where $c := a^*(p-aa^*)^{1/2} + (q-a^*a)^{1/2}d$. But $\|M^*N\| \le \|M^*\|\cdot\|N\| \le 1$ by the assumptions on N and the above observations concerning M. Looking to the $(1,1)$-element of $(M^*N)(M^*N)^*$ we obtain $\|q+cc^*\| \le 1$. On the other hand $c = qc$ by the above identities. Thus $q+cc^* = q(q+cc^*)q \le q$, i.e., $c=0$. It follows:

$$(1-a^*a)^{1/2}(-a^*) = -a^*(p-aa^*)^{1/2} = (q-a^*a)^{1/2}d = (1-a^*a)^{1/2}d\,.$$

If $\|a\|<1$ then $(1-a^*a)^{1/2}$ is invertible and $d=-a^*$. □

Lemma A.17.4. *Let be A, a, p and q as in Lemma A.17.3 and $\|a\|<1$. If the matrix*

$$V := \begin{bmatrix} a, & (p-aa^*)^{1/2}, & g \\ (q-a^*a)^{1/2}, & d, & h \\ f, & k, & e \end{bmatrix}$$

is a partial isometry in $M_3(A)$ such that $V^*V = \operatorname{diag}(q,p,s)$ and $VV^* = \operatorname{diag}(p,q,r)$ then $f=g=h=k=0$, $d=-a^*$, $e^*e=s$ and $ee^*=r$.

Proof. The upper left 2×2-sub-matrix of V must be a contraction in $M_2(A)$. By Lemma A.17.3, $d=-a^*$ because $\|a\|<1$. Using now the equality $VV^* = \operatorname{diag}(p,q,r)$ from $d=-a^*$ we get $gg^*=0$, $hh^*=0$, and using $V^*V = \operatorname{diag}(q,p,s)$, we obtain $f^*f=0$, $k^*k=0$. From $f=g=h=k=0$ it follows that $e^*e=s$ and $ee^*=r$. □

Lemma A.17.5. *Let p,q,r,s be projections in a unital C^*-algebra A and $v,w \in A$ such that $v^*v=q,\ \ ww^*=p,\ \ p+r+vv^*=1$ and $q+s+w^*w=1$. Put*

$$D := \begin{bmatrix} p, & v, & r \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix},\ E := \begin{bmatrix} q, & w^*, & s \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix},\ F(y) := \begin{bmatrix} y, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix}.$$

And $G(y) := D^*F(y)D$, $H(y) := E^*F(y)E$, $T(y) := D^*F(y)E$ if y is in A. Then, for each $y\in A$,

(i)

$$T(y) = \begin{bmatrix} pyq, & pyw^*, & pys \\ v^*yq, & v^*yw^*, & v^*ys \\ ryq, & ryw^*, & rys \end{bmatrix}.$$

(ii) *G and H are injective C^*-morphisms from A into $M_3(A)$ such that $G(1) = \operatorname{diag}(p,q,r)$ and $H(1) = \operatorname{diag}(q,p,s)$.*

(iii) *T is an isometry from A into $M_3(A)$ such that $T(y)T(y)^* = G(yy^*)$ and $T(y)^*T(y) = H(y^*y)$ for $y\in A$.*

(iv)

$$T(a + vb + cw + vdw + e) = \begin{bmatrix} a, & c, & 0 \\ b, & d, & 0 \\ 0, & 0, & e \end{bmatrix}$$

if $e \in rAs$, $a \in pAq$, $b \in qAq$, $c \in pAp$ *and* $d \in qAp$.

(v) *If* $y \in A$ *then the equality*

$$T(y) = \begin{bmatrix} a, & c, & 0 \\ b, & d, & 0 \\ 0, & 0, & e \end{bmatrix}$$

implies $y = a+vb+cw+vdw+e$ *with* $e \in rAs$, $a \in pAq$, $b \in qAq$, $c \in pAp$ *and* $d \in qAp$.

(vi) *An element* $y = a+vb+cw+vdw+e$ *satisfying the conditions of Part(iv) is a unitary of A if and only if the upper left* 2×2*–submatrix is a partial isometry (say* W*) in* $M_2(A)$ *with* $W^*W = \mathrm{diag}(q,p)$ *and* $WW^* = \mathrm{diag}(p,q)$ *and* e *is a partial isometry with* $e^*e = s$ *and* $ee^* = r$.

Proof. (i) is obvious.

(ii,iii): We denote $M_3(A)$ by B and the projection $\mathrm{diag}(1,0,0) = F(1)$ by P. Then F defines a unital C^*-algebra isomorphism from A onto PBP. By our assumptions $DD^* = P = EE^*$. Thus $z \to D^*zE$, $z \to D^*zD$ and $z \to E^*zE$ define linear isometries from PBP onto D^*DBE^*E, D^*DBD^*D and E^*EBE^*E respectively. Moreover the latter two are unital C^*-algebra isomorphisms.

$$T(y)T(y)^* = D^*F(y)EE^*F(y^*)D = D^*F(yy^*)D = G(yy^*)$$

and $T(y)^*T(y) = H(y^*y)$. Hence T is an isometry from A onto D^*DBE^*E and G, H are unital $*$-isomorphisms from A onto D^*DBD^*D and E^*EBE^*E respectively. Computations show that $G(1) = D^*D = \mathrm{diag}(p,q,r)$ and $H(1) = E^*E = \mathrm{diag}(q,p,s)$.

(iv): By the assumptions on p, q, r, s, w, v we have $ws = qs = rv = rp = pv = wq = 0$. Put $y := a + vb + cw + vdw + e$. Then by the assumptions on a, b, c, d, e we have $ry = e = ys = rys$. With $g = y - e$ by (i) it follows that

$$T(y) = \begin{bmatrix} pgq, & pgw^*, & 0 \\ v^*gq, & v^*gw^*, & 0 \\ 0, & 0, & e \end{bmatrix}.$$

Using $pv = wq = 0$ we get $pgq = paq = a$, $pgw^* = cww^* = cp = c$, $v^*gq = v^*vb = qb = b$ and $v^*gw^* = v^*vdww^* = qdp = d$.

(v): As we have seen in the proofs of (ii) and (iii), T defines an isometry from A onto $\mathrm{diag}(p,q,r)B\,\mathrm{diag}(q,p,s)$.

From $T(y) = \mathrm{diag}(p,q,r)T(y)\,\mathrm{diag}(q,p,s)$ we see that $a = paq$, $b = qbq$, etc. Put $z := a + vb + cw + vdw + e$. Then by (iv), $T(z) = T(y)$. But $\ker(T) = 0$, i.e. $y = z$.

(vi): By (ii), (iii) and (iv),

$$\begin{aligned}
&W^*W = \mathrm{diag}(q,p),\ \ WW^* = \mathrm{diag}(p,q),\ \ e^*e = s \ \text{ and } \ ee^* = r && \text{if and only if}\\
&T(y)^*T(y) = H(1) \ \text{ and } \ T(y)T(y)^* = G(1) && \text{if and only if}\\
&H(1-y^*y) = 0 \ \text{ and } \ G(1-yy^*) = 0 && \text{if and only if}\\
&y^*y = 1 = yy^* .
\end{aligned}$$

□

Proof of Proposition A.17.1. :
We put $a = puq$. Then $\|a\| < 1$, $(1-a^*a)^{-1/2}$ and $(1-aa^*)^{-1/2}$ exist. Let $v = (1-p)uq(1-a^*a)^{-1/2}$ and $w = (1-aa^*)^{-1/2}pu(1-q)$. We have $v^*v = (1-a^*a)^{-1/2}qu^*(1-p)uq(1-a^*a)^{-1/2} = (1-a^*a)^{-1/2}(q-a^*a)(1-a^*a)^{-1/2} = q$ because $qa^*a = a^*a = a^*aq$. Similarly we obtain $ww^* = p$. In particular w and v are partial isometries. By definitions of v and w we have $(1-p)v = v$, $w(1-q) = w$,

$$(1-p)uq = v(1-a^*a)^{1/2} = vq(1-a^*a)^{1/2} = v(q-a^*a)^{1/2} ,$$

and $pu(1-q) = (p-aa^*)^{1/2}w$. Thus

$$vv^* \le 1-p,\ \ w^*w \le 1-q,\ \ v^*uq = (q-a^*a)^{1/2}$$

and $puw^* = (p-aa^*)^{1/2}$. Put $r = 1-p-vv^*$ and $s = 1-q-w^*w$. Then p, v, r, q, w, s satisfy the assumptions of Lemma A.17.5 and $T(u)$ defined there satisfies $T(u)^*T(u) = H(u^*u) = H(1) = \mathrm{diag}(q,p,s)$, $T(u)T(u)^* = G(1) = \mathrm{diag}(p,q,r)$ by Lemma A.17.5(ii,iii). Moreover, by Lemma A.17.5(i) and the above equations, the Lemma A.17.4 applies to $T(u)$:

$$T(u) = \begin{bmatrix} a, & (p-aa^*)^{1/2}, & 0 \\ (q-a^*a)^{1/2}, & -a^*, & 0 \\ 0, & 0, & e \end{bmatrix}$$

with $e^*e = s$, $ee^* = r$. We put $\tilde{u} = b + v(q-b^*b)^{1/2} + (p-bb^*)^{1/2}w - vb^*w + e$. By Lemma A.17.5(iv),

$$T(\tilde{u}) = \begin{bmatrix} b, & (p-bb^*)^{1/2}, & 0 \\ (q-b^*b)^{1/2}, & -b^*, & 0 \\ 0, & 0, & e \end{bmatrix} .$$

By Lemma A.17.3 and Lemma A.17.5(vi), the element $\tilde{u}$ is a unitary in A. We have $\|u-\tilde{u}\| = \|T(u-\tilde{u})\| = \|M(a)-M(b)\|$.
Now if $c \in pAq$ then $U(c) = M(c) + \mathrm{diag}(1-p, 1-q)Z$.
Thus $\|u-\tilde{u}\| = \|U(a)-U(b)\| = \|U(puq)-U(b)\|$. □

Lemma A.17.6. *Let be a, b contractions and h, k positive selfadjoint operators on a Hilbert space. Then*

(i) $\|h^{1/2}-k^{1/2}\| \le \|h-k\|^{1/2}$,
(ii) $\|a^*a-b^*b\| \le 2\|a-b\|$,
(iii) $\|U(a)-U(b)\| \le \|a-b\| + (2\|a-b\|)^{1/2}$.

PROOF. (i): Let $t = \|h-k\|$. Then $h+t \leq (h^{1/2}+t^{1/2})^2$ and $k \leq h+t$. The function $g(t) = t^{1/2}$ is operator monotone on $[0,\infty)$, cf. [**767**, Prop.I.6.3].

Thus $k^{1/2} \leq (h+t)^{1/2} \leq h^{1/2}+t^{1/2}$ and we can interchange k and h in this inequality.

(ii): $a^*a - b^*b = a^*(a-b) + (a-b)^*b$.

(iii): Let

$$c := \operatorname{diag}\big((1-aa^*)^{1/2} - (1-bb^*)^{1/2},\ (1-a^*a)^{1/2} - (1-b^*b)^{1/2}\big)\,.$$

By (i) and (ii) we get $\|c\| \leq (2\|a-b\|)^{1/2}$.
We have $U(a) - U(b) = \operatorname{diag}(a-b, (a-b)^*) + cZ$. □

PROOF OF COROLLARY A.17.2. :
Let be $a, b \in A$ and $x > 0$ such that $pbq = b$, $\|a\| < 1$, $\|b\| < 1$ and $\|paq - b\| < x$. Define $P := \operatorname{diag}(p, 0)$, $Q := \operatorname{diag}(q, 0)$, $B = \operatorname{diag}(b, 0)$ and $u := U(a)$ (almost Halmos unitary). Then u is a unitary in $M_2(A)$ such that $PuQ = \operatorname{diag}(paq, 0)$, $\|PuQ\| \leq \|a\| < 1$ and $\|PuQ - B\| = \|paq - b\| < x$.

Moreover $PBQ = B$ and $\|B\| = \|b\| < 1$. By Proposition A.17.1 there exists a unitary $\tilde{u} \in M_2(A)$ such that $\|u - \tilde{u}\| = \|U(PuQ) - U(B)\|$ and $\operatorname{diag}(pcq, 0) = P\tilde{u}Q = B = \operatorname{diag}(b, 0)$ where c means the (1,1)–element of the unitary 2×2–matrix $\tilde{u}$. Looking at the (1,1)–element of $u - \tilde{u}$ by Lemma A.17.6(iii) we obtain $\|a - c\| \leq \|u - \tilde{u}\| \leq \|U(PuQ) - U(B)\| < x + (2x)^{1/2}$. On the other hand, $\|c\| \leq \|\tilde{u}\| = 1$ and $pcq = b$. Now let $0 < t < 2$. Put $s = 1 - (t/2)$ and $e = (p + s(1-p))c(q + s(1-q))$. Then $peq = pcq = b$, $\|e - c\| \leq 2(1-s) = t$, $\|a - e\| < t + x + (2x)^{1/2}$,

$$\|(p + s(1-p))cq\| \leq (1-s)\|b\| + s\|cq\| \leq 1 - (1-s)(1 - \|b\|)$$

and $\|e\| \leq 1 - (1-s)^2(1 - \|b\|) < 1$.
Thus, $f(x, T) \leq x + (2x)^{1/2}$ by Lemma 2.3(i) (of some other paper!) ?. □

18. Open projections, C^*-spaces and Kadison transitivity

Others:
$C*$-space,
(unital) $C*$-system,
Non-unital $C*$-system,
adding a unit,
inductive limits of $C*$-systems,
nuclear $C*$-systems: nuclearity criteria,
inductive limits with residually nuclear maps,

Check if ‘‘quasi-state’’ can be mixed up with ‘‘quasi-trace state’’
???

Partly TRANSPORT from old Chp. 2:

List of equivalent properties
of open projections is given below.

DEFINITION A.18.1. Let A a C^*-algebra.

The projection $p \in A^{**}$ is **open** (with respect to A) if $A^{**}p$ is the $\sigma(A^{**}, A)$-closure of the left ideal $L_p := \{a \in A\,;\ a(1-p) = 0\}$.

A projection $q \in A^{**}$ is **closed** if $p := 1 - q$ is an open projection.

The following proposition given equivalent characterizations of open projections $p \in A^{**}$, respectively of closed projections $q = 1 - p \in A^{**}$. (We use for a linear functional $\varphi \in A$ and its normal extension to A^{**} the same notation φ, i.e., we identify A^* and the predual $(A^{**})_*$ of the bi-dual W*-algebra A^{**} in a natural way.)

PROPOSITION A.18.2. *Let A a C^*-algebra, $p \in A^{**}$ a projection and $q := 1-p$. The following are equivalent:*

(i) *p is open (i.e., q is closed).*
(ii) *p is the (unique) smallest projection in A^{**} with $ap = a$ for all $a \in A_+$ with $a(1-p) = 0$.*
(iii) *The convex cone $C(q)$ of all normal positive functionals on A^{**} with $\varphi(q) = \|\varphi\|$ (i.e., with $\varphi(p) = 0$) is $\sigma(A_*, A)$-closed in A_*.*
(iv) *The convex subset*

$$Q(q) := \{\varphi \in (A_*)_+\,;\ \varphi(q) = 0\,,\ \|\varphi\| \leq 1\}$$

of the quasi-state space $Q(A) \subseteq A^*$ of A is $\sigma(A^*, A)$-closed (i.e., is a weakly closed face of $Q(A)$).
(v) *The left A^{**}-module A^*q is $\sigma(A^*, A)$-closed in A^*.*
(vi) *$A^{**}p$ is the $\sigma(A^{**}, A^*)$-closure of $A \cap (A^{**}p)$.*
(vii) *The projection p is the least upper bound in A^{**} for the set $\{a \in A_+\,;\ ap = a,\ \|a\| \leq 1\}$ of positive contractions.*
(viii) *The projection $p \in A^{**}$ is the support projection of the hereditary C^*-subalgebra $D \subseteq A$ of A defined by $D := \{a \in A\,;\ (aa^* + a^*a)q = 0\,\}$.*
(ix) This property is not? sufficient ?:

For every pure state φ of A with $\varphi(p) = 1$, (i.e., $\varphi(q) = 0$) there exists $a \in A_+$ with $\|a\| = 1$, $ap = p$ and $\varphi(a) = 1$.

(It delivers that $p \geq q$ for some open projection. Let $L_p = \{b \in A\ \ b(1-p) = 0\}$ it is a left ideal with open support projection

Hope? Perhaps next has to be added:

If ψ is a quasi-state on A with $\psi(p) = 1$ then there exists a pure state φ in the weakly closed face of $\mathrm{Q}(A)$ generated by ψ with $\varphi(p) \neq 0$.

Or: p is determined by its discrete part.

The equivalences of Parts (i)-(viii) are – at least implicitly – contained in the book of G.K. Pedersen, cf. [**616**, thm. 3.6.11,thm. 3.10.7] and its proofs, or are obvious reformulations of each other. We prove here only the equivalence of the Parts (viii) and (ix).

PROOF. (viii)⇒(ix): If $p \in A^{**}$ is an open projection and $\varphi \in A^*_+$ is a pure state on A with $\varphi(1-q) = 0$, then $\varphi|D$ is a state on D.

This restriction to D is pure on D: If ψ_1 and ψ_2 are states on D with $\psi_1+\psi_2 = 2\varphi$ then this happens also for the all (the unique) normal extensions $\overline{\psi_k}$ of ψ_k to states on A^{**} and implies that $\psi_1 = \psi_2 = \varphi$.

By ????? excision lemma??? reference ???? there exists for the pure state $\varphi|D$ an positive contraction $a \in D_+$ with $\varphi(a) = 1$. Clearly, $1 = \varphi(a) \leq \|a\| \leq 1$, and $ap = a$. because $a \in D_+$.

(ix)⇒(viii): Suppose that for every pure state φ of A with $\varphi(p) = 1$, (i.e., $\varphi(q) = 0$) there exists $a \in A_+$ with $\|a\| = 1$, $ap = p$ and $\varphi(a) = 1$.

Let $L := \{a \in A\,;\, a(1-p) = 0\} = \{a \in A\,;\, ap = a\}$. Then L is a closed left ideal of A and the positive contractions in L generate L. Clearly the open support projection $q \in A$ of the hereditary C^*-subalgebra $L^* \cap L$ satisfies again $(1-p)q(1-p) = 0$. Thus $qp = q$ and $r \leq q$ for all open projections $r \in A^{**}$ with $r(1-p) = 0$. □

Notice that Part (ix) of Proposition A.18.2 is a non-commutative version of the original definition of F. Hausdorff of open subsets $U \subseteq X$ of a metric space X: U is open, if and only if, for each point $x \in U$ there exists a (bounded) continuous function $f \in \mathrm{C_b}(X)_+$ with $f(x) = 1$, $\|f\| \leq 1$ such that the support of f is contained in U.

Notice that ???????

```
Before next:
Here we have first to show that
```
$a \in A \mapsto a + (R+L) \in A/(R+L)$ `maps`
```
the closed unit-ball onto the closed unit ball.!!!
```

One of the obvious consequences is the following Corollary is equivalent to the "advanced Kadison transitivity" in Lemma 2.1.15(i,ii).

COROLLARY A.18.3. *If $p \in A^{**}$ is a projection such that $pA^{**}p$ has finite dimension (as vector space) then p is a closed projection and there is a closed left-ideal $L \subseteq A$ such that $A^{**}(1-p)$ is the $\sigma(A^{**}, A^*)$-closure of L in A^{**}.*

In particular, the natural map $a \in A \to pap$ defines an isometric isomorphism from $A/(L^*+L)$ onto $pA^{**}p$, that maps the closed unit-ball of A onto *the closed unit-ball of $pA^{**}p$.*

It is well known for Banach spaces that π_X maps the open unit-ball of a Banach space A maps onto the open unit ball of A/X for each closed linear subspace of X.

Notice that $R + L$ is a closed subspace of a C^*-algebra A for any closed left-ideals L and right ideals R, cf. Proposition A.15.2.

The bipolar of $R+L$ ($=\sigma(A^{**},A^*)$-closure of $R+L$) is identical with $A^{**}P+QA^{**}$, where P and Q are the open support projections of the hereditary C^*-subalgebras $R^*\cap R$ and $L^*\cap L$. The second adjoint of $(A/(R+L))^{**}$ of $A/(R+L)$ is naturally isomorphic to $A^{**}/(A^{**}P+QA^{**})$. All this spaces are matrix-normed operator spaces and the canonical isomorphisms are complete isometries. It is easy to see that $A^{**}/(A^{**}P+QA^{**})$ is naturally completely isometric isomorphic to the ternary C^*-algebra $(1-Q)A^{**}(1-P)$.

REMARK A.18.4. The Proposition A.18.6 and Theorem **??**

main content:
$(A//D)\cap\mathcal{M}(A//D)\cong\mathcal{N}(D)/D$

yields an alternative, more general, conceptual and algebraic proof of the existence of a C^*-subalgebra $B\subseteq A$ with the property that $a\in B\mapsto pap\in pA^{**}p$ is a *-epimorphism onto $pA^{**}p$ if $pA^{**}p$ is finite-dimensional, as derived in Remark A.20.1 from the Kadison transitivity theorem.

In fact the proof is completely independent from the use of the non-commutative Lusin theorem [**616**, thm. 2.7.3] and is independent from all textbook versions of the Kadison transitivity theorem. It gives an alternative and “almost algebraic” proof of the Kadison transitivity theorem without any approximation arguments. (But uses the uniform Hölder continuity of that what we call “restricted perturbation”).

Let $p^*=p\in A^{**}$ a non-zero projection in the *socle* of A^{**}, i.e., $pA^{**}p$ is finite-dimensional. The set $S_p(A)$ of $f\in A^*$ with $\|f\|\le 1=f(p)$ is identical with the set of all normal states on $pA^{**}p$. The set of states on $pA^{**}p$ is compact in norm of $pA^*p\subseteq A^*$ if $pA^{**}p$ is finite dimensional.

Since the $\sigma(A^*,A)$-topology is Hausdorff and is continuous with respect to the norm on A^*, it follows that $S_p(A)$ is compact in the $\sigma(A^*,A)$-topology.

By ??? Proposition **??**, the $\sigma(A^*,A)$-compactness of $S_p(A)=C_p(A^{**})$

It seems better to consider the

quasi-states
in $Q(A)$?... No consequent terminology!!!

ρ on A with $\rho(1-p)=0$. And then require that this set is $\sigma(A^*,A)$ closed. Then apply then bi-polar theorem ...

Give -- or refer to -- Definition of the convex set
$C_p(M):=\{f\in M_*\,;\ \|f\|=1, f(p)=1\}$.

Open-ness of $1-p$ is not proven now!!!

implies that $p\in A^{**}$ is a closed projection, i.e., there is a closed left ideal $L\subseteq A$ such that $A^{**}(1-p)$ is the $\sigma(A^{**},A^*)$-closure of L.

Then p is a closed projection in A^{**} with respect to A, i.e., the hereditary C^*-subalgebra $D:=\{a\in A\,;\ ap=0=pa\}$ has $(1-p)$ as its *open* support projection

q_D of D in A^{**}, respectively $(1-p)A^{**}(1-p) \cong D^{**}$ is equal to the $\sigma(A^{**}, A^*)$-closure of $D \subseteq A^{**}$. We let $L := A \cdot D = \overline{\text{span}(A \cdot D)}$. It is a closed left-ideal of A with $\sigma(A^{**}, A^*)$-closure equal to $A^{**}(1-p)$.

If L is a closed left-ideal of A, and $q \in A^{**}$ is the *open* support projection of L, i.e.,

$$L^{**} \cong A^{**}q = \overline{L}^{\sigma(A^{**},A^*)},$$

then there is a natural isomorphism $pAp \cong (A/(R+L))^{**}$ for $R := \{a^* ;\ a \in L\}$ and $p := 1 - q_L$.

Let $D := R \cap L$. The C^*-space $A//D := A/(R+L) \cong pAp \subseteq A^{**}$ is c.p. and completely isometrically isomorphic to $pAp \subseteq pA^{**}p$ and $(A//D)^{**} \cong pA^{**}p$ as matrix-normed and matrix-ordered spaces. Since $pA^{**}p$ has finite dimension, it follows that $(A//D) \cong pAp = pA^{**}p$.

By Theorem ?? the quotient $\mathcal{N}(D)/D$ by D of the (two-sided) normalizer C^*-algebra $\mathcal{N}(D) := \{a \in A;\ aD \cup Da \subseteq A\}$ of D is naturally isomorphic to the intersection $pAp \cap \mathcal{M}(A//D)$ of $A//D \cong pAp$ with the C^*-subalgebra

$$\mathcal{M}(A//D) := \{T \in pA^{**}p;\ TpAp \cup pApT \subseteq pAp\}$$

of two-sided multipliers of the subset $A//D \subseteq (A//D)^{**} \cong pA^{**}p$.

Since $pAp \cong A//D$ is $\sigma(A^{**}, A^*)$ dense in $pA^{**}p \cong (A//D)^{**}$ and, in our special case, $pA^{**}p$ is of finite dimension, we get that $\mathcal{M}(A//D) = pAp = pA^{**}p$. Thus, the restriction $V|B$ to $B := \mathcal{N}(D)$ of the c.p. contraction $V: a \in A \mapsto pap \in pA^{**}p$ is a *-epimorphism from B onto $pA^{**}p \cong \mathcal{M}(A//D) \cong \mathcal{N}(D)/D$.

`New attempt Dec.2016:`

Define a projection $q = q^*q \in A^{**}$ as *open* projection if q is the support projection $q := q_D$ of a hereditary C^*-subalgebra D of A.

A projection $p \in A^{**}$ is - by definition - a *closed projection* if $1 - p \in A^{**}$ is open.

LEMMA A.18.5. *Let $p \in A^{**}$ a projection.*

The projection $q := 1 - p \in A^{**}$ is open, if an only if, the set $S_p(A)$ of $f \in A^*$ with $\|f\| = f(p) \leq 1$ is $\sigma(A^*, A)$-closed and for each pure state $g \in A^*$ with $g(p) = 0$ there exists an element $a \in A_+$ with $\|a\| = g(a) = 1$ and $f(a) = 0$ for all $f \in S_p(A)$.

PROOF. If the projection $1 - p$ is open then the set of $b \in A$ with $bp = 0$ is a closed left ideal L of A and $q := 1-p \in A^{**}$ is the support projection of $D := L^* \cap L$ (this by definition of open projections).

In particular $f(a) = 0$ for all $f \in S_p(A)$ and $a \in D$, and conversely $f(p) = 1$ for all normal states f on A^{**} with $f(D) = 0$. Clearly, the set of normal states with $f(D) = \{0\}$ is closed in the $\sigma(A^*, A)$-topology.

Thus, if $g \in A^*$ is a pure state with $g(p) = 0$ then $g|D$ is a pure state of D. It follows that there exists a contraction $a \in D_+$ with $g(a) = 1$, cf. [**616**, thm. 2.7.5]. Thus $f(a) = 0$ for all $f \in S_p(A)$.

Notice that generally $a \in A_+$ and $f(a) = 0$ for all $f \in S_p(A)$ is equivalent to $a(1-p) = a$.

The set $L := \{b \in A\,;\ f(b^*b) = 0\ \forall\ f \in S_p(A)\}$ is the maximal closed left-ideal of A with the property $L \cdot p = \{0\}$.

The support projection $Q \in A^{**}$ of the hereditary C^*-subalgebra $D := L^* \cap L$ of A is an open projection with $Qp = 0$.

Suppose that $Q \neq q := 1-p$. Then there exists a pure state $\rho \in A^*$ with $\rho(Q) = 0$ and $\rho(p) < 1$:

This is the case because $S_{1-Q}(A)$ is a $\sigma(A^*, A)$-compact convex set of states with the property that each extreme point f of $S_{1-Q}(A)$ is also an extreme point of of the quasi-state space of A, hence is a pure state of A with $f(Q) = 0$.

Each $f \in S_{1-Q}(A)$ is the $\sigma(A^*, A)$ limit of convex combinations of extreme points in $S_{1-Q}(A)$ (as it holds for every compact convex set). Thus, if there is no pure state $h \in A^*$ with $h(Q) = 0$ and $h(p) < 1$, then $Q = 1-p$, which contradicts our assumption $Q \neq q := 1-p$.

But if $h \in A^*$ is a pure state on A with $h(q) = 0$ and $h(p) < 1$, then we can consider the irreducible representation $R\colon A \to \mathcal{L}(\mathcal{H})$ corresponding to h. There is a cyclic vector $x \in \mathcal{H}$ with $\|x\| = 1$ such that $h(a) := \langle R(a)x, x\rangle$ for all $a \in A$.

Let $R\colon A^{**} \to \mathcal{L}(\mathcal{H})$ also denote the normalization of R. Then $R(Q)x = 0$ and $\|R(p)x\| < 1$. Thus $y := \gamma(x - R(p)x) \in \mathcal{H}$ with $\gamma := \|x - R(p)x\|^{-1}$ satisfies $R(Q)y = 0$, $R(p)y = 0$ and $\|y\| = 1$. The normal state $g(a) := \langle R(a)y, y\rangle$ on A^{**} is pure and satisfies $g(1 - (Q + p)) = 1$.

In particular, the assumption $1 - Q - p \neq 0$ gives a pure state on A with $g(Q) = 0$ and $g(p) = 0$.

But by assumption, there exists $a \in A_+$ with $\|a\| = 1 = g(a)$ and $ap = 0$. This implies $a \leq Q$ and $g(Q) \neq 0$.

Thus, $Q = 1-p$ (i.e., $1-p$ is open) if for each pure state $g \in A^*$ with $g(p) = 0$ there exists an element $a \in A_+$ with $\|a\| = g(a) = 1$ and $f(a) = 0$ for all $f \in S_p(A)$. □

Is it also equivalent to each of the following ???

The norm-closed left-ideal

$$L := (A^{**}(1-p)) \cap A = \{\, a \in A\,;\ ap = 0\,\}$$

of A is $\sigma(A^{**}, A^*)$-dense in $A^{**}(1-p)$.

This is equivalent to:

$A^*p := \{f \in A^*\,;\ f(ap) = f(a)\}$ is $\sigma(A^*, A)$-closed.

Clearly, L is a closed left-ideal of A and $L \subseteq A^{**}(1-p)$.

Let $q_L \in A^{**}$ denote the open support projection corresponding to L, i.e., $q_L := \sup\{d \in A_+\,;\ d \leq 1,\, d \in L\}$.

Then the $\sigma(A^{**}, A^*)$-closure of L is $A^{**}q_L$ and $q_L \leq 1 - p$.

Let L be $\sigma(A^{**}, A^*)$-dense in $A^{**}(1-p)$. Then $A^{**}q_L = A^{**}(1-p)$, hence $q_L = 1 - p$.

$A^*p = \{f \in A^*\,;\ f(ap) = f(a), \text{ for all}\, a \in A\,\} = \{f \in A^*\,;\ f(A(1-p)) = 0\}\,.$

It says that $A^*p = \{f \in A^*\,;\ f(L) = \{0\}\,\}$. The right side is $L^\perp$ in A^* and is there $\sigma(A^*, A)$-closed.

THE HARD stuff is the opposite direction:

Suppose $p \in A^{**}$ is a projection with the property that $A^* \cdot p$ is $\sigma(A^*, A)$-closed.

(Then $pA^*p = (p \cdot A^*) \cap (A^* \cdot p)$ is $\sigma(A^*, A)$-closed by $\sigma(A^*, A)$-continuity of $f \to \overline{f}$ on A^* given by the map $\overline{f}(a) := \overline{f(a^*)}$.)

Thus, there is closed subspace X of A such that ?????

$(A/X)^* \cong A^* \cdot p$???

?????

Let $L_p := \{a \in A\,;\ ap = 0\}$ i.e., $L_p = A \cap (A^{**}(1-p)) = \{a \in A\,;\ a = a(1-p)\}$.

And L_p is a closed left-ideal of A. Let Q denote the open support projection of A with $A^{**}Q = \overline{L_p}$ with respect to $\sigma(A^{**}, A^*)$-topology. We get $Q \leq (1-p)$.

Thus $p \leq 1 - Q$, and pA^*p is a subspace of $(A/(L_p^* + L_p))^* \cong (1-Q)A^{**}(1-Q)$

`HERE starts the difficulty of this rather special approach:`

Is $Q = 1 - p$????

Are p and Q determined by the there restrictions to the "discrete" type-I part of A^{**} ?

There are natural bijections between the following fairly different types of elements in A^{**}, quotient spaces of A, subspaces of A and certain subsets of A^*:

(a) closed projections in A^{**}

(b) open projections in A^{**}

(c) quotient C^*-spaces $A//D := A/(L^* + L)$ (with matrix order and matrix norms induced from A (respectively A^{**}).

(d) closed left-ideals $L \subseteq A$

(e) closed right-ideals $R \subseteq A$

(f) hereditary C^*-subalgebras D of A given by $D := L^* \cap L$ respectively $D := R^* \cap R$.

(g ??) hereditary $\sigma(A^*, A)$-closed convex sub-cones K of A^*_+. (The K should be of form $\mathbb{R}_+ \cdot C_p(A^{**})$ for some projection $p \in A^{**}$.)

(h ??) the set $S_p(A)$ of $f \in A^*$ with $\|f\| \leq 1 = f(p)$ is $\sigma(A^*, A)$-closed.

(if $1 - p$ is open ???).

Uses Lemma:

Each hereditary norm-closed sub-cone K of the positive part $(M_*)_+$ of the predual M_* of a W*-algebra M is of the form $K_q = (qM_*q)_+ \cong ((qMq)_*)_+$ for some projection $q \in M$.

$0 \neq p \in M := A^{**}$ is a closed projection, if and only if, the unit ball of K_p is $\sigma(A^*, A)$-closed, if and only if, $S_p(A) = C_p(M) := \{f \in M_* \, ; \; \|f\| \leq 1 = f(p)\}$ is $\sigma(A^*, A)$-compact.

In particular, every projection $p \in A^{**}$ with $pA^{**}p$ of finite dimension (as a vector-space over $\mathbb{R}$) is a closed projection.

This gives, e.g. by [**616**, thm. 1.5.2, thm. 3.10.7, cor. 3.10.8, prop. 3.11.9], the equivalences:

$D \subseteq A$ hereditary C^*-subalgebra,

$D \mapsto D_+$, D_+ is hereditary closed convex cone in A_+,

$M \subseteq A_+$ hereditary closed convex cone, build $M \mapsto L(M) := \{a \in A \, ; \; a^*a \in M\}$ then $L(M)$ is closed left-ideal of A.

$L \subseteq A$ closed left-ideal, $L \mapsto D_L := D(L) := L^* \cap L$, $D_L := L^* \cap L$ is a hereditary C^*-subalgebra of A.

Directly to L from D without passing through D_+:
Let $L(D) := D \cdot A$. To see that this a closed left-ideal simply apply Cohen factorization to the left Banach D-module $L := \overline{\mathrm{span}(D \cdot A)} \subseteq A$. It gives $L = D \cdot L \subseteq D \cdot A \subseteq L$.

The projection $p \in A^{**}$ is a *closed* projection for A, with $p := 1 - q$, if and only if,

$q \in A^{**}$ *open* projection for A.

A projection $q \in A^{**}$ is an *open* projection,

if and only if,

$q = \text{l.u.b.}\{a \in A_+ \, ; \; a \leq q\}$ in A^{**},

if and only if,

the closed left ideal $L_q := \{a \in A \, ; \; aq = a\}$ has $\sigma(A^{**}, A^*)$-closure $= A^{**}q$.

A non-zero projection $0 \neq p \in M := A^{**}$ is closed,

if and only if,

$S_p(A) = C_p(M) := \{f \in M_* \, ; \; \|f\| \leq 1 = f(p)\}$ with $M := A^{**}$ is $\sigma(A^*, A)$-closed,

if and only if,

$C_p(A^{**})$ is $\sigma(A^*, A)$-compact,

if and only if,

$q := 1 - p$ is open in A^{**},

if and only if,

$A^*p := \{f \in A^* ;\ f((\cdot)p) = f\}$ is closed in A^* with respect to $\sigma(A^*, A)$-topology.

Missing:

$D \mapsto A//D := A/(L^* + L)$,

$X := A//D \mapsto q := 1_{X^{**}} \in A^{**}$ *closed* projection, unit of the second conjugate $X^{**} \cong qA^{**}q$ of X. Given by $q := 1 - p_D$.

Passage to the *open* support projection $q_D \in A^{**}$ of D is given by $q_D := 1 - q$.

Proposition A.18.6. *Let A a C^*-algebra and $p \in A^{**}$ a (self-adjoint) projection.*

The following are equivalent:

(i) *p is a closed projection,*
(ii) *$1 - p$ is an open projection,*
(Definition of "open" ?
$1 - p$ is the l.u.b. in A^{**} of an upward directed family positive contractions in A?)
(iii) *$1 - p$ is the least upper bound of $\{a \in A_+ ;\ a \leq 1 - p\}$.*
(iv) *The convex subset $C_p(A^{**}) = S_p(A) := \{f \in A^* ;\ \|f\| \leq 1 = f(p)\}$ of A^* is $\sigma(A^*, A)$-closed.*
(v) *$C_p(A^{**})$ $(= S_p(A))$ is $\sigma(A^*, A)$-compact.*
(vi) *The closed left-ideal $L := \{a \in A ;\ ap = 0\}$ of A has $\sigma(A^{**}, A^*)$-closure equal to $A^{**}(1 - p)$.*

If $p \in A^{**}$ is a non-zero projection, then the $\sigma(A^*, A)$-compactness of $S_p := \{f \in A^* ;\ \|f\| \leq 1 = f(p)\}$ implies that $p \in A^{**}$ is a closed projection, i.e., there is a closed left ideal $L \subseteq A$ such that $A^{**}(1 - p)$ is the $\sigma(A^{**}, A^*)$-closure of L.

`Next is cited where !!!`
`Compare with following lemmata.`

The proof of the below stated Proposition A.21.4 concerning excision of pure states use a reduction method to the separable case that is established for arbitrary operator spaces and Banach spaces in full generality by the following Lemma B.14.1 and then applied for our basic observation on operator space quotients of C^*-algebras in Remark A.18.9.

Since we use them here the first time, we place here some "easy" facts – i.e., those that does not discuss the residually equivariant behavior of quotients by sums $L + R$ of left and right ideals L and R. We need reduction to separable subspaces with same perturbations conditions for quotients, because Ext-theory, KK-theory and our version of a stable but un-suspended E-theory (continuous versions of the approximate kind of E-theory used by Elliott and Rørdam) have to do with non-separable corona spaces.

We start with a very general observation:

LEMMA A.18.7. *Let* (R_1, ρ_1) *and* (R_2, ρ_2) *metric spaces of bounded diameters* ≤ 2, $T\colon R_1 \to R_2$ *a contractive map with* $T(R_1)$ *dense in* R_2.

Suppose that (R_1, ρ_1) *is complete (i.e., each Cauchy sequence in* R_1 *with respect to the metric* ρ_1 *has a limit in* R_1*).*

Need an increasing continuous function $\gamma \in \mathrm{C}([0,1], \mathbb{R})$ *with* $\gamma(0) = 0$, $t \leq \gamma(t)$ *and with the property:*

If $a, b \in R_1$ *then there exists for each* $\varepsilon > 0$ *an element* $c \in R_1$ *such that* $\rho_2(T(c), T(b)) < \varepsilon$ *and* $\rho_1(a, c) < \gamma(\rho_2(T(a), T(b))) + 2\varepsilon \leq 3\gamma(\rho_2(T(a), T(b)))$.

Suppose that there exists a function a continuous function λ *on* $[0, 2]$ *with* $\lambda(0) = 0$ *and the property that for* $p_1, p_2 \in R_1$ *and* $\varepsilon > 0$ *there exists* $p_3 \in R_1$ *with* $\rho_2(T(p_3), T(p_2)) < \varepsilon$ *and* $\rho_1(p_1, p_3) \leq \lambda(\rho_2(p_1, p_2)) + \varepsilon$ *????*

???? Then T *is surjective ????.*

PROOF. Find an decreasing sequence $r_1 > r_2 > \cdots$ in $(0, 1)$ such that $\sum_n \lambda(3^n r_n) < \infty$. In particular $\lim r_n = 0$.

Let $q \in R_2$. There exist $p_1, p_2, \ldots \in R_1$ with $\rho_2(p_n, q) < r_n/2$. Thus $\rho_2(p_m, p_n) \leq (r_m + r_n)/2$.

Find $P_{m,n} \in R_1$ with ???

??

Idea: ????

Take a point $q \in R_2$. Given any sequence $q_n \in T(R_1)$ with $\lim_n \rho_2(q_n, q) = 0$ "quickly enough", we can select a sub-sequence $r_k := q_{n_k}$ such that in the inverses $T^{-1}(B(r_k, \delta_k))$ of small balls around the r_k one finds inductively elements p_k that build a Cauchy sequence in R_1 ... □

Let E a Banach space and S the open unit ball of E. Recall that the **Hausdorff distance** between two bounded convex subsets K_1, K_2 of the Banach space E is given by

$$\mathrm{dist}_H(K_1, K_2) := \inf\{\, t \in [0, \infty)\,;\; K_1 \subseteq K_2 + tS \text{ and } K_2 \subseteq K_1 + tS \,\}\,.$$

DEFINITION A.18.8. Let E a real or complex Banach spaces $X \subseteq E$ a (not necessarily closed) linear subspace, S the open unit-ball of E and $t \in (0, \infty)$.

Define for $a, b \in S$ the Hausdorff distance between the intersection $K(a, X) := S \cap (a + X)$ and $K(b, X) = S \cap (b + X)$ of the affine spaces $a + X$ and $b + X$ with the open unit-ball S of E by

$$\rho(a, b) := \mathrm{dist}_H(K(a, X), K(b, X))\,.$$

We define the *function of restricted perturbation* $t \mapsto f(t;\, X)$ of the quotient map π_X by

$$f(t;\, X) := \sup\{\rho(a, b)\,;\; a, b \in S,\; \|\pi_X(a - b)\| \leq t\,\}$$

It turns out that $t \to f(t;\, L_1^* + L_2)$ is Hölder continuous near 0 if L_1 and L_2 are closed left ideals of a C^*-algebra E. The point is that

$$f(t;\, L_1^* + L_2) \le f(t;\, p_1 E^{**} + E^{**} p_2)$$

for the (open) support projections p_ℓ of the $\sigma(E^{**}, E^*)$ closures $E^{**}p_\ell$ of L_ℓ, an the latter is easily to estimate by an at zero Hölder continuous function of t.

The function $f(t;\, X)$ estimates in a symmetric way for given elements $a, b \in E$ with $\max(\|a\|\,, \|b\|) < 1$ the best value for the solution of the following "restricted perturbation" problem:

Let $a + X = \pi_X(a)$ and $b + X = \pi_X(b)$. What is the infimum $=: \lambda(a,b) \le \|a - b\|$ of $\|a - c\|$ over all $c \in b + X$ with $\|c\| < 1$?

Because then, for each $\varepsilon > 0$, we can find $c = c(\varepsilon) \in E$ with $\|c\| < 1$ and $\pi_X(c) = \pi_X(b)$ and

$$\|a - b\| + \varepsilon \ge \varepsilon + \lambda(a,b) \ge \|a - c\| \ge \|\pi_X(a) - \pi_X(b)\|\,.$$

We have that $\lambda(a,b) \le f(\|\pi_X(a) - \pi_X(b)\|; X)\,.$?????

`Need place for A CLEAR STATEMENT on the estimate of`

$$f(t;\, L_1^* + L_2) \le f(t;\, p_1 A^{**} + A^{**} p_2) \le t + (2t)^{1/2}$$

`!! ?? !!`

REMARK A.18.9. Let A denote a C^*-algebra, $L \subseteq A$ a closed left ideal, $R \subseteq A$ a closed right ideal, D a hereditary C^*-subalgebra of A. We use the notations $D_L := L^* \cap L$, $D_R := R^* \cap R$, $L_D := \overline{\mathrm{span}(A \cdot D)}$.

Let $\psi(\alpha) := \alpha + (2\alpha)^{1/2}$ for $\alpha \in [0, \infty)\,.$

??

Then there are the following elementary observations, cf. [**431**]:

(i) The vector space sum $R + L$ is a closed linear subspace of A.

The bi-duals and bi-polars in A^{**} are given by $L^{**} \cong (L^o)^o = A^{**}p_L$ where $p_L := p_D \in A^{**}$ is the open support projection of L respectively of $D := L^* \cap L$. Similarly $R^{**} = (R^o)^o$ for closed right ideals.

The bi-polar $\cong (R + L)^{**}$ in the W*-algebra A^{**} is identical with the (always $\sigma(A^{**}, A^*)$-closed) subspace

$$A^{**}q \,+\, pA^{**} = (1-p)A^{**}q + pA^{**}q + pA^{**}(1-q)$$

where $p, q \in A^{**}$ are the "open" support projections of the hereditary C^*-algebras $D_L := L^* \cap L$ and $D_R := R^* \cap R$ of A.

The bi-dual operator space $(A/(R+L))^{**}$ is naturally isomorphic to the *ternary algebra* (triple product algebra) $(1-q)A^{**}(1-p) \subseteq A^{**}$.

In particular, $(A/(L^* + L))^{**} \cong (1-q)A^{**}(1-q)$ is a W*-algebra.

The isomorphisms and natural embeddings are all completely isometric, and the quotient maps are completely contractive, because we can here alway replace A, R, L, D and p, q by $M_n(A), M_n(R), M_n(L), M_n(D)$ and $p \otimes 1_n$, $q \otimes 1_n$ for $n \in \mathbb{N}$.

The below considered constructions remain compatible via the obvious canonical completely isometric isomorphisms if we tensor by M_n, as e.g. $\mathcal{N}(D, A) \otimes M_n \cong \mathcal{N}(D \otimes M_n, A \otimes M_n)$ for $\mathcal{N}(D, A)$ in Part (iv).

(ii) Let $x, y \in A/(R+L)$ with $\|x\|, \|y\| \leq 1$ and $a \in A$ with $\|a\| \leq 1$ and $\pi_{R+L}(a) = x$, then there exist $b \in A$ with $\|b\| \leq 1$ that satisfies $\pi_{R+L}(b) = y$ and

$$\|a - b\| \leq \psi(\| y - x \|) \,.$$

In particular, we get the important fact that the quotient map π_{R+L} maps the closed unit-ball of A *onto* the closed unit-ball of $A/(R+L)$.

(iii) Let $D := L^* \cap L \subseteq A$ and $A//D := A/(L^* + L)$ the operator system defined by D.

Then $L := A \cdot D = \overline{\operatorname{span}(A \cdot D)}$ (because $D \subset L$ and L is a right D-module), and $(A//D)^{**}$ is an operator system that is naturally isomorphic to the W*-algebra $(1-q_D)A^{**}(1-q_D)$, where q_D denotes the open support projection of D in A^{**}.

(iv) Let $\mathcal{M}(A//D) \subset (1-p_D)A^{**}(1-p_D)$ denote the C^*-algebra of two-sided multipliers

$$T \cdot A//D \,,\; A//D \cdot T \quad \subseteq A//D$$

of the space $A//D \subseteq (1-p_D)A^{**}(1-p_D)$.

The natural epimorphism π_D from the two-sided normalizer algebra

$$\mathcal{N}(D, A) := \{a \in A \,;\; aD \cup Da \subseteq D\}$$

of D in A into

$$A//D := A/(L^* + L) \subseteq (1-p_D)A^{**}(1-p_D)$$

maps $\mathcal{N}(D, A)$ onto $(A//D) \cap \mathcal{M}(A//D)$, and has kernel

$$\operatorname{Ann}(D, A) := \{a \in A \,;\; aD = \{0\} = Da\} \,.$$

(v) An element $a \in A$ is in $\mathcal{N}(D, A)$ if and only if $\pi_{L^*+L}(a^*a) = \pi_{L^*+L}(a^*)\pi_{L^*+L}(a)$ and $\pi_{L^*+L}(aa^*) = \pi_{L^*+L}(a)\pi_{L^*+L}(a^*)$, i.e., if and only if a is in the multiplicative domain of π_{L^*+L}.

(vi) `The first part of this conclusion is also a corollary of a more elementary proposition: Corollary A.15.3!!`

For every separable C^*-subalgebra $G \subseteq A$ there exists a separable C^*-subalgebra $B \subseteq A$ such that $G \subseteq B$,

$$B \cap (L^* + L) = (B \cap L)^* + (B \cap L) \,,$$

and $B/(B \cap L) \to A/L$ is completely isometric, and (in addition) that $b \in B$ and $b(D \cap B) \subseteq D \cap B$ for $D := D_L$ implies $bD \subseteq D$ and the natural map $B//(B \cap D) \to A//D$ is completely isometric and defines a *-monomorphism

$$(B//(B \cap D)) \cap \mathcal{M}(B//(B \cap D)) \to (A//D) \cap \mathcal{M}(A//D) \,.$$

(vii) If A is unital, then naturally $\mathcal{M}(A//D) \cong \mathcal{N}(A,D)/\operatorname{Ann}(A,D)$ and B can be chosen such that in addition to the properties in (vi), B unital with $1_B = 1_A$, $\mathcal{N}(B, B\cap D) \subseteq \mathcal{N}(A,D)$ and $\operatorname{Ann}(B, B\cap D) = B\cap \operatorname{Ann}(A,D)$.

19. On non-unital C^*-systems

needed:
operator systems, $C*$-systems (definitions), nuclear operator system, c.p. and c.i. maps

needed statements:
indlim nuclear if maps are nuclear,
perhaps also:
indlim is $C*$-space, $B//D$ is $C*$-space,
nuclear OS is $C*$-space
definitions and listing of some result appear also further below.

?? ????

Matrix-normed operator spaces $(X, \|\cdot\|_n)$ can be defined as closed linear subspaces of $\mathcal{L}(\mathcal{H})$. An axiomatic characterization as Banach spaces X with a family of matrix norms $\|\cdot\|_n$ on $M_n(X) = X\otimes M_n$ has been given by E. Effros and Z.-J. Ruan, [**245**], that satisfy $\|a\oplus b\|_{m+n} = \max(\|a\|_m, \|b\|_n)$ and $\|\alpha c\beta\|_n \leq \|\alpha\|\|c\|_m\|\beta\|$ for $a, c\in M_m(X)$, $b\in M_n(X)$, $\alpha\in M_{n,m}$, $\beta\in M_{m,n}$, cf. [**245**]. If X is completely isometric embedded in $\mathcal{L}(\mathcal{H})$ then the Banach space second conjugate X^{**} of X is naturally embedded in the C^*-algebra $\mathcal{L}(\mathcal{H})^{**}$ as a $\sigma(\mathcal{L}(\mathcal{H})^{**}, \mathcal{L}(\mathcal{H})^*)$-closed linear subspace. One can see immediately that the norm-closed unit balls of the bi-dual Banach spaces

$$(M_n(X))^{**} \cong M_n(X^{**}) \subseteq M_n(\mathcal{L}(\mathcal{H})^{**}) = (M_n(\mathcal{L}(\mathcal{H}))^{**}$$

are just the $\sigma((M_n(X))^{**}, (M_n(X))^*)$ closures of the unit-balls of $M_n(X)$. Notice that this definition is completely independent from any chosen norm on $M_n(X^*) \cong M_n\otimes X^*$ that is equivalent on the subspaces $X^*\cong e_{ij}\otimes X^*$ to the usual dual Banach space norm on X^*, because the $\sigma((M_n(X))^{**}, (M_n(X))^*)$ topology depends only form the equivalence classes of norms on $M_n(X)^* = M_n(X^*) = M_n\otimes X^*$ and nothing else. For example, one can here use the usual Banach dual norms $\|\cdot\|_n^*$ on $M_n\otimes X^*$ that satisfy $\|f\oplus g\|^*_{m+n} = \|f\|^*_m + \|g\|^*_n$ instead of $\|f\oplus g\|^*_{m+n} = \max(\|f\|^*_m, \|g\|^*_n)$. This can be also abstractly described as the bi-dual matrix-normed space that is given by applying construction of matrix normed operator space X^* of X with matrix norms given the isomorphisms $M_n(X^*)\cong \operatorname{CB}(X, M_n)$ by E. Effros and Z.J. Ruan in [**245**, sec. 3.2]. The natural inclusion $M_n(X) \hookrightarrow M_n(X^{**})$ is isometric by [**245**, prop. 3.2.1]. But the above mentioned canonical isometry of $M_n(X^{**})\cong M_n(X)^{**}$ is a – formally – stronger statement.

Lemma A.19.1. *Let A a C^*-algebra, $\rho_1, \rho_2\in A^*$ Hermitian with $\|\rho_1\| = \|\rho_2\| = 1$. Suppose that there exists $b, c\in (A^{**})_+$ with*

$$\|b+c\| = \|b-c\| = \max(\|b\|, \|c\|) = 1 = \rho_1(b-c)$$

and $\rho_1(\beta b + \alpha c) = \rho_2(\beta b + \alpha c)$ *for all* $\alpha, \beta \in \mathbb{R}$. *Then for the polar decompositions* $\rho_j = \rho_{j+} - \rho_{j-}$ *holds* $\rho_1(b) = \rho_2(b) = \rho_{j+}(b) = \|\rho_{j+}\|$, *and* $\rho_1(c) = \rho_2(c) = -\rho_{j-}(c) = -\|\rho_{j-}\|$.

Proof. Recall that the polar decompositions $\rho_j = \rho_{j+} - \rho_{j-}$ of ρ_2 and ρ_2 satisfy $1 = \|\rho_{j+} + \rho_{j-}\| = \|\rho_{j+}\| + \|\rho_{j-}\|$.

Let $t_j := \rho_{j+}(b) + \rho_{j-}(c)$, $s_j := \rho_{j+}(c) + \rho_{j-}(b))$. Then $1 = \rho_j(b - c) = t_j - s_j$ and $1 \geq (\rho_{j+} + \rho_{j-})(b + c) = t_j + s_j$. They imply $s_j = 0$ and $\rho_j(b) = \rho_{j+}(b)$, $\rho_j(c) = -\rho_{j-}(c)$, for $j = 1, 2$.

On the other hand, $1 = \|\rho_{j+}\| + \|\rho_{j-}\| \geq t_j = 1$, $\|\rho_{j+}\| \geq \rho_{j+}(b)$ and $\|\rho_{j-}\| \geq \rho_{j-}(c)$, that gives $\rho_{j+}(b) = \|\rho_{j+}\|$ and $\rho_{j-}(c) = \|\rho_{j+}\|$. Now use that $\rho_1 = \rho_2$ on $\mathbb{R}b + \mathbb{R}c$. □

Remark A.19.2. A general (= not necessarily order-unital) **operator system** X is a closed linear subspace X of a C^*-algebra A such that X is invariant under passage to adjoints (i.e., $a \in X \Leftrightarrow a^* \in X$), and such that for each $n \in \mathbb{N}$, $a = a^* \in M_n(X) \subseteq M_n(A)$ and $\delta > 0$ there exist $b, c \in M_n(X)_+ := M_n(A)_+ \cap M_n(X)$ with

$$\|b + c\| = \max(\|b\|\,, \|c\|) \leq \|a\| \qquad \text{and} \qquad \|a - (b - c)\| < \delta\,.$$

If A is unital and $1 \in X$, then this property is trivially satisfied with $b := (\|a\|1 + a)/2$, $c := (\|a\|1 - a)/2$ and $\delta = 0$. The condition implies that functionals $f\colon M_n(X) \to \mathbb{C}$ with $f(T) \geq 0$ for $T \in M_n(X)_+$ satisfy $\|f\| = \sup\{f(T)\,;\, T \in M_n(X)_+ \ \|T\| = 1\}$, and (therefore) norm-preserving extensions of f to a hermitian functional $g\colon M_n(A) \to \mathbb{C}$ are positive functionals on $M_n(A)$. The condition also implies that hermitian linear functionals f on $M_n(X)$ have decompositions $f = f_+ - f_-$ into positive functionals f_+ and f_- on $M_n(X)$ that satisfy $\|f\| = \|f_+ + f_-\| = \|f_+\| + \|f_-\|$. The decompositions with this properties are not necessarily unique, but the norms $\|f_+\|$ and $\|f_-\|$ are independent from the chosen decomposition with this property, cf. Lemma A.19.1 and use the *weak compactness in $A^{**} \oplus A^{**}$ of the set of pairs (b, c) with $b, c \in X_+^{**} \subseteq A^{**}$ and $\|b + c\| \leq 1$.

It gives immediately, that the *weak closure ($\cong M_n(X^{**})$) of $M_n(X)$ in $M_n(A)^{**} \cong M_n(A^{**})$) contains a unique element E_n with the property $f(E_n) = \|f\|$ for each positive functional of f on $M_n(X)$. In this way, each $M_n(X^{**})_{sa}$ becomes an order-unit space, and X^{**} a unital operator system (with norms defined by the order-unit norms on $M_n(X^{**})$.

It allows to show that every completely positive contraction $V\colon X \to \mathcal{L}(H)$ extends to a completely positive contraction $V^e\colon A \to \mathcal{L}(H)$. In this way we get a "universal" c.p.c. map $V_u\colon X \to B$ for some suitable C^*-algebra B. We choose B such that $V_u(X)$ generates B and call $C^*_{\max}(X) := B$ (together with $V_u\colon X \to C^*_{\max}(X)$) the *universal C^*-algebra* generated by the (not necessarily unital) operator system X. There is a natural C^*-morphism $h\colon C^*_{max}(X) \to A$ such that $h \circ V_u(x) = x$ for all $x \in X$.

The induced norms on $M_n(X)$ and the closed cones of positive elements $M_n(X)_+$ provide X with a structure of matrix-norms and matrix orders ([9]).

Let, more generally, $X \subseteq A$ and $Y \subseteq B$ self-adjoint closed subspaces of C^*-algebras A and B, and let $T\colon X \to Y \subseteq B$ a linear map. We call T **completely positive** (respectively **completely contractive**, **completely isometric**) if the natural extensions $T \otimes \mathrm{id}_n$ of T to maps from $M_n(X) \cong X \otimes M_n$ to $M_n(Y)$ are positive (respectively contractive, isometric) for every $n = 1, 2, \ldots$. We write later **c.p.** , **c.c.** and **c.i.** maps (respectively).

We say that two self-adjoint closed subspaces $X \subseteq A \subseteq \mathcal{L}(H)$ and $Y \subseteq B \subseteq \mathcal{L}(K)$ are **c.i. and c.p. isomorphic** , if and only if, there are c.p. and c.c. maps $V\colon \mathcal{L}(H) \to \mathcal{L}(K)$ and $W\colon \mathcal{L}(K) \to \mathcal{L}(H)$ with $WV|X = \mathrm{id}_X$ and $VW|Y = \mathrm{id}_Y$.

The second conjugate space X^{**} of a selfadjoint closed subspace $X \subseteq A$ is naturally isomorphic to the $\sigma(A^{**}, A^*)$-closure of $X \subseteq A^{**}$. It is easy to check (because $M_n(X)^{**} \hookrightarrow M_n(A)^{**}$ is isometric) that the matrix-norms induced by $M_n(A^{**}) \cong M_n(A)^{**}$ on $M_n(X^{**})$ are the same as the second conjugate norms (i.e., bi-polar norms) on $M_n(X^{**})$ under the natural isomorphisms $M_n(X)^{**} \cong M_n(X^{**})$ (of vector spaces).

We say that inclusion $X \hookrightarrow A$ is **admissible** if the $\sigma(M_n(X)^{**}, M_n(X)^*)$-closure of $M_n(A)_+ \cap M_n(X)$ is the same as $M_n(A)_+^{**} \cap M_n(X)^{**}$ for each $n \in \mathbb{N}$. This is e.g. the case if every bounded positive functional on $M_n(A)_+ \cap M_n(X)$ extends to a positive functional on $M_n(A)_+$ with same norm. In general, the inclusion $X \hookrightarrow A$ is *not* admissible. But an arguments with bi-polars shows that $X \hookrightarrow A$ is admissible if X is an *operator system* (not necessarily unital) and is equipped with the matrix norms and orders induced on $M_n(X)$ by $M_n(A)$. On the other hand, there *always* exists a C^*-algebra B and c.p. and c.c. maps $V\colon X \to B$ and $W\colon B \to \mathcal{L}(H)$ such that $WV(x) = x$ for $x \in X$ and $V(X) \hookrightarrow B$ is admissible ([10])

If $X \subseteq A$ is *admissible* in A then an argument using the bi-polar shows:
$X^{**} \subseteq A^{**}$ is an operator system, if and only if, $X \subseteq A$ is an operator system.

Suppose that X is a self-adjoint closed subspace of A and that X^{**} (with bi-dual matrix norms and orders of the matrix norms coming from matrix orders of X) is c.i. and c.p. isomorphic to a C^*-algebra M . Then M is necessarily a W*-algebra.

Any (not necessarily unital) completely isometric and completely positive map V from a unital C^*-algebra C into an other C^*-algebra B (in our case with $B = A^{**}$) satisfies $V \otimes \mathrm{id}_n(c) \in M_n(B)_+$ if and only if $c \in M_n(C)_+$. (In fact, it is enough to require that C is a matrix ordered space with an order unit such that the matrix norms are given by the induced order norm: $\|c\| \leq 1$ for $c^* = c \in M_n(C)$ if and only if $-1 \otimes 1_n \leq c \leq 1 \otimes 1_n$.)

[9] One can give axiomatic descriptions of, not necessarily order-unital, operator systems independent from inclusions $X \hookrightarrow A$ into C^*-algebras – using ideas of Effros and Ruan [**245**].

[10] Here we consider A as C^*-subalgebra of $\mathcal{L}(H)$, and that V is a c.i. and c.p. isomorphism from X onto $V(X)$.

It follows: *If X^{**} is c.i. and c.p. isomorphic to a C^*-algebra, then X is a – not necessary order-unital – operator system*, ([11]).

Then any c.i. and c.p. embedding $V: X \hookrightarrow B$ into a C^*-algebra B satisfies that $M_n(B)_+ \cap V \otimes \mathrm{id}_n(M_n(X)) = V \otimes \mathrm{id}_n(M_n(A)_+ \cap M_n(X))$, and – therefore – that $V(X) \hookrightarrow B$ is admissible. In particular, the natural embedding $X \hookrightarrow X^{**}$ is admissible.

Since completely positive and completely isometric maps T from a C^*-algebra onto an other C^*-algebra is always a C^*-algebra isomorphism, we get that the C^*-algebra structure on X^{**} is uniquely determined if $X \subseteq A$ is admissible and X^{**} is c.i. and c.p. isomorphic to a C^*-algebra. We can define the **multiplier algebra** $\mathcal{M}(X)$ of a C^*-system X by

$$\mathcal{M}(X) := \{a \in X^{**}\,;\ aX \cup Xa \subseteq X\}\,.$$

It is a C^*-subalgebra of X^{**}. If $I: X \to Y$ is a c.i. and c.p. map from X onto a self-adjoint closed subspace $Y \subseteq B$ of a C^*-algebra B then Y is an operator system with Y^{**} c.i. and c.p. isomorphic to a C^*-algebra, $I^{**}: X^{**} \to Y^{**}$ is an isomorphism of C^*-algebras and $\mathcal{M}(I) := I^{**}|\mathcal{M}(X)$ is a C^*-isomorphism from $\mathcal{M}(X)$ onto $\mathcal{M}(Y)$.

Examples where X^{**} (together with its matrix norms and orders) is isomorphic to a C^*-algebra are:

(1) Let $X := B//D := B/(BD+DB) \subseteq A := (1-p_D)B^{**}(1-p_D)$ where D is a hereditary C^*-subalgebra of a C^*-algebra B and $p_D \in B^{**}$ is the "open" support projection of D with $D^{**} = p_D B^{**} p_D$ ([12]). The matrix norms and matrix orders on $M_n(X)$ that are induced by the inclusion $X \subseteq A$ are easily seen to be the same as those induced by the quotient maps $M_n(B) \to M_n(B//D) \cong M_n(B)//M_n(D)$. Therefore, X^{**} is naturally c.i. and c.p. isomorphic to the von Neumann algebra $(1-p_D)B^{**}(1-p_D)$.

(2) More generally, $Y \subseteq X$ and the quotients $X//Y := X/Z$ are in a natural manner operator systems (with second conjugates isomorphic to the C^*-algebras $pX^{**}p$ respectively $(1-p)X^{**}(1-p)$) if

(i) X^{**} is c.i. and c.p. isomorphic to a C^*-algebra,
(ii) the $\sigma(X^{**}, X^*)$-closure of Y in X^{**} is a hereditary C^*-subalgebra $pX^{**}p$ of X^{**} for a projection $p \in X^{**}$, and
(iii) Z is the set of $z \in X$ with $\rho(z) = 0$ for all positive functionals ρ on $A \supset X$ with $\rho(Y) = \{0\}$, i.e., $Z = X \cap (pX^{**} + X^{**}p)$.

(Then there is a natural c.i. and c.p. embedding $X//Y \subseteq A//D$ where D is the hereditary C^*-subalgebra of A generated by Y.)

[11] We consider X^{**} together with the second conjugate matrix norms on $M_n(X^{**}) \cong M_n(X)^{**}$ and the $\sigma(M_n(X)^{**}, M_n(X)^*)$-closure of $M_n(A)_+ \cap M_n(X)$ as matrix-order structure.

[12] The closed linear span $R \subseteq B$ of $DB := \{db\,;\ d \in D, b \in B\}$ is a non-degenerate left D-module. Thus $DR = R \subseteq DB \subseteq R$ by the Cohen factorization theorem, because D has a bounded approximate unit.

(3) A special case of (2) with $X//Y = X/Y$, $Y = Z$ is obtained if $Y \subseteq X$ is an M-ideal of X (i.e., the subspace $Y^\perp$ of X^* is ℓ_1-complemented in X^*: $X^* = Y^\perp + L$ with $\|f + l\| = \|f\| + \|l\|$ for $f, l \in X^*$ with $f(Y) = \{0\}$ and $l \in L$), because then there is a central projection p in the center of the W*-algebra X^{**} such that pX^{**} is the weak closure of Y in X^{**}.

(4) Inductive limits $\mathrm{indlim}(T_n : B_n \to B_{n+1})$ are C^*-systems, if $B_1, B_2, \ldots$ are C^*-algebras and $T_n : B_n \to B_{n+1}$ are c.p. contractions, cf. Lemma A.19.6(v).

(5) All nuclear operator systems (that are not necessarily unital), are C^*-systems (cf. Proposition A.19.4).

Definition A.19.3. Let A a C^*-algebra and $X \subseteq A$ a self-adjoint closed subspace of A. Then X is a (not necessarily with order-unit equipped) ***C*-system*** if X is a (not necessarily order-unital) operator system and there exists a c.i. and c.p. map I from a C^*-algebra M onto X^{**}, i.e., the second conjugate operator system X^{**} is c.i. and c.p. isomorphic to a C^*-algebra.

A similar definition for general *C^*-spaces* would be completely wrong, because there exist separable operator spaces that have injective C^*-algebras as second conjugate, but are not C^*-spaces and are not nuclear (are even not exact).

The unique C^*-algebra structure on X^{**} allows to define the **multiplier algebra** $\mathcal{M}(X) \subseteq X^{**}$ by

$$\mathcal{M}(X) := \{a \in X^{**}\, ;\, aX \cup Xa \subseteq X\}\,.$$

A closed subspace $X \subseteq A$ is **nuclear** if, for every $x_1, \ldots, x_n \in X$ and $\delta > 0$, there are $k \in \mathbb{N}$ and completely positive contractions $V : A \to M_k$, $W : M_k \to A$ with $W(M_k) \subseteq X$ and $\|WV(x_j) - x_j\| < \delta$ for $j = 1, \ldots, n$.

It is easy to see that nuclear X must be an operator system (that is not necessarily order-unital).

Below we see that X is moreover a C^*-system, cf. Lemma **??**. (??? lem:6.?nuc.cst.syst ???)

A C^*-system X is nuclear if X is a *nuclear subspace* of the C^*-algebra X^{**}.

Proposition A.19.4. *Let $X \neq 0$ a separable (not necessarily unital) matrix-ordered and matrix-normed space. TFAE:*

(i) *X is nuclear.*
(ii) *X is completely isometric and completely order isomorphic to the inductive limit of completely positive contractions $V_n : M_{k_n} \to M_{k_{n+1}}$.*
(iii) *X^{**} is completely isometric and completely order isomorphic to an injective W^*-algebra.*
(iv) *There exists hereditary C^*-subalgebras $D \subseteq E \subseteq M_{2^\infty}$ of the CAR-algebra $M_{2^\infty} = M_2 \otimes M_2 \otimes \cdots$ and a completely isometric matrix-order isomorphism from X onto $E//D := E/(E \cdot D + D \cdot E)$.*

REMARKS A.19.5. Before we start with the proof of Proposition A.19.4 we remind the reader that a (not necessarily unital) *operator system* X is a very special kind of much more general operator spaces defined and considered by E. Effros and Zh.-J. Ruan in [**245**].

The (as Banach space) separable operator systems are always completely isometric isomorphic to the quotient space $A/(R+L)$ of some separable C^*-algebra A by a sum $R+L$ of a closed right ideal R of A and and closed left ideal L of A. The algebraic sum is always closed, cf. Proposition A.15.2(iv)

An operator system has in addition to its matrix norms an involution map $x \in X \to x^* \in X$ and a matrix order structure with the property that the second order operator space X^{**} contains an order unit $1 \in X^{**}$ with the property that $-1_n \otimes 1_2 \leq x \otimes p_{12} + x^* \otimes p_{21} \leq 1_n \otimes 1_2$ is equivalent to $\|x\| \leq 1$ for $x \in M_n(X^{**})$. (Here the order unit $1 \in X^{**}$ is often not in X itself, and $p_{jk} \in M_2$ denote matrix units and $1_2 := p_{11} + p_{22}$.)

In this case holds: If X is an operator system and is in addition a *C^*-space* (as reminded below), then X^{**} is a W*-algebra.

A ***C^**-ternary space** (also called "***C^**-space**") X is by definition is an operator space that has the following two properties:

(c*-sp 1) The second conjugate operator-space X^{**} of X is completely isometric isomorphic to a ternary algebra (triple product algebra) pMq for some W*-algebra M and projections $p, q \in M$.

(c*-sp 2) The natural completely isometric embedding $\eta \colon X \hookrightarrow X^{**}$ of X into X^{**} is *weakly injective*, i.e., there is a completely contractive projection P from X^{4*} onto the $\sigma(X^{4*}, X^{***})$ closure of the image of $\eta^{**}(X)$ of the natural C^*-ternary morphism η from the W*-ternary algebra X^{**} into X^{4*}.

The general definition of **relative weakly injective** linear maps $T \colon X \hookrightarrow A$ into a C^*-ternary algebra A is that T is completely isometric and that there is a completely contractive $\sigma(A^{**}, A^*)$-continuous linear projection

$$P \colon A^{**} \to T^{**}(X^{**}) \subseteq A^{**}$$

with the property that $P \circ T^{**} = T^{**}$. The existence of this embedding shows that the operator space X is a C^*-space (C^*-ternary system).

An operator space X is **weakly injective** , if and only if, a suitable completely isometric linear embedding of X into $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is *relative weakly injective*.

All separable C^*-ternary spaces are completely isometric isomorphic to quotient operator spaces $C^*(T)/(L+R)$ of the universal C^*-algebra $C^*(T)$ generated by a single contraction T, with sum $L+R$ of a suitable closed left-ideal L and a closed right-ideal R of $C^*(T)$ (this follows easily from the results of [**472**]).

An operator system X is a (not necessarily unital) ***C^**-system** if X^{**} is completely isometric and order isomorphic to a W*-algebra and the natural inclusion map $\eta\colon X \hookrightarrow X^{**}$ is relative weakly injective in X^{**}.

It is known that all (non-unital) separable C^*-systems X admit a canonical unitization $\widehat{X}$ that is completely order isomorphic to $C^*(F_\infty)/(L^* + L)$ for some closed left ideal L of $C^*(F_\infty)$, cf. [**472**].

It is not difficult to see that all inductive limits $A_1 \to A_2 \to \cdots$ defined by completely positive unital maps $V_n\colon A_n \to A_{n+1}$ are C^*-systems, cf. [**472**].

Proof of Proposition A.19.4. (i)⇒(ii): The definition of nuclearity for separable operator systems X is equivalent to the existence of

(α) linear subspaces $X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ with $\dim(X_n) < \infty$ and $\bigcup_n X_n$ dense in X, and

(β) completely positive contractions $T_n\colon M_{k_n} \to X_{n+1}$ and $S_n\colon X_n \to M_{k_n}$ such that $\|(T_n \circ S_n)|X_n\| < 2^{-n}$.

It follows that $X = \text{indlim}_n(T_n \circ S_n)\colon X_n \to X_{n+1}$. One can check this in $X_\infty := \ell_\infty(X)/c_0(X)$ by comparing it in X_∞ on the subsets X_n with the inductive limit $X = \text{indlim}_n(\iota_n\colon X_n \to X_{n+1})$ for the natural inclusion maps ι_n from X_n into X_{n+1}. The asymptotic of the approximately commuting diagrams allow to see that

$$X \cong \text{indlim}_{n\to\infty} W_n := T_n \circ S_n\colon X_n \to X_{n+1}$$

by comparing the restrictions of the iterates

$$W_{k,\ell} := W_\ell \circ \ldots \circ W_k\colon X_k \to X_\ell$$

on $X_n \subseteq X_k$ for fixed $n \in \mathbb{N}$ and large $n < k < \ell$. It shows that the natural embedding in X modulo $c_0(X)$ in X_∞ coincides with this inductive limit, because it has a natural embedding into X_∞ given for $x \in X_n$ by

$$W_{n,\infty}(x) = (0, \ldots, 0, x, W_n(x), W_{n+1}W_n(x), \ldots) + c_0(X)\,.$$

The complete isometry from X onto the map

$$x \in X \mapsto (S_1(x), S_2(x), \ldots) + c_0(M_{k_1}, M_{k_2}, \ldots)$$

defines an a completely isometric and completely positive map from X onto the inductive limit defined by the sequence of c.p. contractions $V_n := S_{n+1} \circ T_n\colon M_{k_n} \to M_{k_{n+1}}$. The defining morphisms for the inductive limit of the V_n are given by

$$V_{n,n+\ell} := V_{n+\ell} \circ \ldots \circ V_{n+1} \circ V_n\colon M_{k_n} \to M_{k_{n+\ell}}$$

and for $a_n \in M_{k_n}$ by

$$V_{n,\infty}(a_n) = (0, \ldots, 0, a_n, V_n(a_n), V_{n+1}V_n(a_n), \ldots) + c_0(M_{k_1}, M_{k_2}, \ldots)\,.$$

Clearly this happens also for the map ???????

`To be filled in` ??

(ii)⇒(iv):

(iv)⇒(iii):

(iii)⇒(i): □

LEMMA A.19.6. *Suppose that $A, B_1, B_2, \ldots$ are C^*-algebra, that $T_n \colon B_n \to B_{n+1}$ $(n = 1, 2, \ldots)$ are completely positive contractions and that $X \subseteq A$ is a closed subspace of A, that is invariant under passage to adjoints, i.e., $X = X^* := \{x^* \ x \in X\}$.*

Then

(i) *Suppose that there is a c.p. contraction $V \colon A \to A^{**}$ such that $V(x) = x \in A \subseteq A^{**}$ for $x \in X$, $V(A)$ is contained in the $\sigma(A^{**}, A^*)$-closure of X in A^{**} and $(V \otimes \mathrm{id}_n)(M_n(A)_+)$ is contained in the $\sigma(M_n(A)^{**}, M_n(A)^*)$-closure of $M_n(A)_+ \cap M_n(X))$ in $M_n(A^{**}) \cong M_n(A)^{**}$.*

   Then X is a C^*-system.
(ii) *If X is a C^*-system and $\psi \colon X \to C$ is a c.i. and c.p. map into a C^*-algebra C such that $\psi(X)$ generates C, then there exists a unique *-algebra morphism φ from C into X^{**} with $\varphi(\psi(x)) = x$ for $x \in X$.*
(iii) *If, for every $x_1, \ldots, x_n \in X$ and $\delta > 0$, there is a completely positive contractions $V \colon A \to A$, such that $V(A) \subseteq X$ and $\|x_j - V(x_j)\| < \delta$ for $j = 1, \ldots, n$, then X is a (not necessarily unital) C^*-system.*
(iv) *$X \subseteq A$ is a C^*-system if X is a nuclear subspace of A.*
(v) $\mathrm{indlim}(T_n \colon B \to B) \subseteq \ell_\infty(B)/c_0(B)$ *is a C^*-system.*
(vi) *If the maps $T_n \colon B \to B$ are nuclear, then the C^*-system* $\mathrm{indlim}(T_n \colon B \to B)$ *is nuclear.*

PROOF. (vi): We show: If the B_n are separable (not necessarily unital) C^*-algebras and the maps $T_n \colon B_n \to B_{n+1}$ are nuclear c.p. contractions, then $\mathrm{indlim}_n(T_n \colon B_n \to B_{n+1})$ is a separable nuclear C^*-space.

This reduces to the case, where B_n is of finite dimension by Lemma A.14.2 and Lemma A.14.1.

In the non-separable case is $\mathrm{indlim}_n(T_n \colon B_n \to B_{n+1})$ the union of nuclear subspaces given by $\mathrm{indlim}_n(S_n \colon A_n \to A_{n+1})$ for separable C^*-subalgebras $A_n \subseteq B_n$ with $T_n(A_n) \subseteq A_{n+1}$, such that $S_n := T_n | A_n$ is still an nuclear map from A_n into A_{n+1}.

`to be filled in` ??

Let $X := \mathrm{indlim}_n(S_n \colon A_n \to A_{n+1})$, where the A_n are separable and the S_n are nuclear. Consider the natural embedding $X \subseteq \ell_\infty(A_1, A_2, \ldots)/c_0(A_1, A_2, \ldots)$. In the separable case we can find a increasing sequence $Z_1 \subseteq Z_2 \subseteq \cdots$ of finite-dimensional subspaces of X, and subspaces $Y_k \subseteq A_k$ of finite dimension, such that $\bigcup_k Z_k$ is dense in X, $S_k(Y_k) \subseteq Y_{k+1}$ and $Z_k = S_{k,\infty}(Y_k)$ for all $k \in \mathbb{N}$. Then we find ?????????? and

c.p. contractions $V_k \colon M_{\mu(k)} \to A_{k+1}$, $W_k \colon A_k \to M_{\mu(k)}$, such that $V_k \circ W_k \colon A_k \to A_{k+1}$ have the property that for $L_k := Y_k \cup V_{k-1}(M_{\mu(k-1)}) \subseteq A_k$ holds

$\|(V_k \circ W_k - S_k)|L_k\| < 8^{-k}$. It follows that $X = \mathrm{indlim}_{k\to\infty}(S_k \colon Y_k \to Y_{k+1})$ and $X = \mathrm{indlim}_{k\to\infty}(V_k \circ W_k \colon L_k \to L_{k+1})$ with matrix order and matrix norm inherited from $\ell_\infty(A_1, A_2, \ldots)/c_0(A_1, A_2, \ldots)$.

This implies, by Lemma A.14.1, that the matrix-normed and matrix-ordered space X is completely isometric and completely order isomorphic to $C := \mathrm{indlim}_{k\to\infty}(W_{k+1} \circ V_k \colon M_{\mu(k)} \to M_{\mu(k+1)}$. The $T_k := W_{k+1} \circ V_k$ are c.p. contractions.

By Proposition A.19.4 it follows that C and therefore X is a nuclear (not necessarily unital) C^*-system.

????? □

Remark A.19.7. "Non-unital" modifications of the proofs in [**438**] show that:

Compare next Definitions with those in work of J.Pisier or Effros–Lance ???

(1) There is a C^*-algebra C and a hereditary C^*-subalgebra $D \subseteq C$ such that $C//D \cong \mathrm{indlim}(T_n \colon B \to B)$ by a completely isometric and completely positive isomorphism.
(Use Stinespring dilation and [**438**, lem.2.5], compare the arguments in [**472**].)

(2) Every *separable* C^*-system X is isomorphic to $C//D$ for some separable C^*-algebra C and hereditary C^*-algebra $D \subseteq C$.
(Compare [**472**, prop.5] for the unital case.)

(3) Let X a C^*-system, i.e., X (considered e.g. as $X \subseteq \mathcal{L}(\mathcal{H})$ for some $\mathcal{H}$) is a matrix-ordered operator space with second conjugate X^{**} isometrically order isomorphic to a W*-algebra M *and* there is a normal conditional expectation P_X from M^{**} onto the $\sigma(M^{**}, M^*)$-closure of the image of the canonical map from X into M^{**} given by the natural map $X \to X^{****} \cong M^{**}$. This latter assumption is equivalent to the requirement that $X \subseteq X^{**}$ is relatively weakly injective in X^{**}, which can be expressed equivalently that the natural map from $X \otimes^{\max} C^*(F_2)$ to $X^{**} \otimes^{\max} C^*(F_2)$ is injective (in an isometric sense). Here $\otimes^{\max}$ is taken on the category of tannery algebras extended to operator spaces X with X^{**} isomorphic to a W*-ternary algebra pMq.

(To understand the latter extra requirement check by examples that P_X defines a normal c.p. contraction from M into M^{**} that is usually very different from both of the central normal embedding $M \cong p_M \cdot M^{**}$ and of the non-normal natural inclusion of M into M^{**}.)

Then: X is nuclear,

if and only if,

the second conjugate operator system X^{**} is an injective C^*-algebra,

if and only if,

the inclusion map from X into the W*-algebra X^{**} is weakly nuclear.

(Modify here [**438**, lem.2.8] for the implication

$$X^{**} \text{ injective } \Rightarrow X \text{ nuclear } .$$

The other directions follows from $(X^{**})^{op} \otimes^{\max} X = (X^{**})^{op} \otimes^{\min} X$ (where the tensor products are defined by the natural inclusion of X in the W*-algebra X^{**})

???????? Find reference ????????? by an argument of E. Effros and Ch. Lance [**241**, thm. ???] that uses here the standard representation of X^{**} and $(X^{**})^{op}$ in the sense of U. Haagerup).

If this is the case and X is separable then one can take $C := M_{2^\infty}$ in (2).

(4) Similar arguments show the analogs for C^*-operator spaces $X \cong C/(L+R)$ of the results (2) and (3) for separable C^*-spaces X, (by definition operator spaces X with with X^{**} completely isometric to pMq for some von-Neumann-algebra M and projections $p, q \in M$.)

Lemma A.19.8. *Suppose that D is a hereditary C^*-subalgebra of a C^*-algebra B with corresponding "open" support projection $p := p_D \in B^{**}$. Let*

$$\mathcal{M}(B//D) := \{a \in (B//D)^{**} \;:\; a(B//D) \cup (B//D)a \subseteq B//D\}.$$

(o) *The natural isomorphisms*

$$(B//D)^{**} \cong B^{**}/(pB^{**} + B^{**}p) \cong (1-p)B^{**}(1-p)$$

are unital and completely isometric. They equip $(B//D)^{**}$ with the unique von-Neumann algebra structure that is compatible with the bi-adjoint matrix order unit structure on $B//D$. It holds $B//D = (1-p)B(1-p)$ if we naturally identify $(1-p)B^{**}(1-p) \cong (B//D)^{**}$ and consider all operators as elements of $(1-p)B^{**}(1-p)$.

(i) *$\mathcal{M}(B//D)$ is a C^*-subalgebra of $(B//D)^{**}$, and $\mathcal{M}(B//D) \cap (B//D)$ is a closed ideal of $\mathcal{M}(B//D)$.*

The ideal $\mathcal{M}(B//D) \cap (B//D)$ is natural isomorphic to $\mathcal{N}(D)/D$ by the canonical epimorphism $B \ni b \mapsto b + R + L \in B//D = B/(R+L)$, where L and R denote the closures of BD respectively DB.

(ii) *An element $x \in (1-p)B^{**}(1-p)$ is in $\mathcal{M}(B//D) \cap (B//D)$, if and only if, $x, x^*x, xx^* \in B//D \subseteq (1-p)B^{**}(1-p)$.*

(iii) *If we identify $\mathcal{N}(D)/D$ and $\mathcal{M}(B//D) \cap (B//D)$ naturally via π_{R+L}, then, for every closed ideal J of B,*

$$\pi_D(\mathcal{N}(D) \cap J) = \pi_{R+L}(J) \cap \pi_D(\mathcal{N}(D)).$$

(iv) *Suppose that that $T \colon B//D \to C$ is a completely positive and completely isometric map from $B//D$ into a C^*-algebra C.*

*If a C^*-subalgebra A of C is contained in $T(B//D)$ then $\gamma := T^{-1}|A$ is a *-monomorphism from A into $B//D \cap \mathcal{M}(B//D) \cong \mathcal{N}(D)/D$.*

(v) *Let $\mathrm{Ann}(D, J) := \{b \in B\,;\; bD + Db \subseteq J\}$.*

If J is closed ideal of B, then $J \subseteq \mathrm{Ann}(D, J)$ and

$$\pi_J(\mathrm{Ann}(D, J)) = \mathrm{Ann}(\pi_J(D)) := \mathrm{Ann}(\pi_J(D), \{0\}).$$

In particular, $\pi_J(D)$ *is essential in* B/J*, if and only if,* $\mathrm{Ann}(D,J)=J$.

In particular, $\mathcal{N}(D)\cap\mathrm{Ann}(D,J)=\mathcal{N}(D)\cap J$ *for all* $J\lhd B$.

If $\pi_J(D)$ *is essential in* B/J *then*

$$\mathcal{N}(D)\cap\mathrm{Ann}(D,J)=\mathcal{N}(D)\cap J\,.$$

(vi) *Suppose that* $\mathrm{Ann}(D,J)=J$ *for all* $J\lhd B$*. Then there is a unique *-monomorphism*

$$\lambda\colon\,\mathcal{N}(D)\to\mathcal{M}(D)$$

such that $\lambda|D=\mathrm{id}_D$ *and*

$$\lambda(\mathcal{N}(D)\cap J)=\lambda(\mathcal{N}(D))\cap\mathcal{M}(D,D\cap J)$$

for all $J\lhd B$.

In particular,

$$\pi_D\circ\lambda(\mathcal{N}(D))\cap\pi_D(\mathcal{M}(D,D\cap J))=\pi_D\circ\lambda(\mathcal{N}(D)\cap J))\,.$$

(vii) *If* B *is separable, then there exist a completely positive contraction* $V\colon B//D:=B/(L+R)\to\mathcal{M}(D)/D$ *such that* $V|\mathcal{N}(D)=\pi_D\circ\lambda$ *and*

$$V\big(\pi_{L+R}(J)\big)\subseteq\pi_D\big(\mathcal{M}(D,D\cap J)\big)\qquad\textit{for all } J\lhd B\,.$$

Proof. Ad (o) and (i): If B is unital, then $\mathcal{M}(B//D)\subseteq B//D$ and the statements (o) and (i) are contained in [**437**, thm.1.4] (in the unital case).

If B is *non-unital*, we can pass to the unitization $\widetilde{B}=B+\mathbb{C}\cdot 1$ of B and have $\widetilde{B}^{**}\cong B^{**}\oplus\mathbb{C}$. Let e denote the unit of B^{**} in $\widetilde{B}^{**}$. Then $p\le e$, $B//D=(e-p)B(e-p)=(1-p)B(1-p)$ and

$$B//D\subseteq\widetilde{B}//D=(1-p)\widetilde{B}(1-p)\subseteq(e-p)B(e-p)\oplus\mathbb{C}\cdot(1-e)=B//D\oplus\mathbb{C}\cdot(1-e)\,.$$

It follows $\widetilde{B}//D=(1-p)B(1-p)+\mathbb{C}\cdot(1-p)=B//D+\mathbb{C}(1-p)$ and, therefore

$$B//D=(\widetilde{B}//D)\cap(e-p)B^{**}(e-p)\,,$$

and $\mathcal{M}(\widetilde{B}/D)\cap(e-p)B^{**}(e-p)=\mathcal{M}(\widetilde{B}//D)\cap B//D$.

If $x\in\mathcal{M}(B//D)\cap B//D$, then $x(B//D)+(B//D)x\subseteq B//D$ and $x(1-e)=(1-e)x=x\in\widetilde{B}//D$. Thus $x\in\mathcal{M}(\widetilde{B}//D)\cap B//D$. If $y\in\mathcal{M}(\widetilde{B}//D)$, then $y=x+z(1-p)$ with $x\in B//D$ and $z\in\mathbb{C}$. Since $(1-p)c=c=c(1-p)$ for $c\in\widetilde{B}//D$, we get $x(B//D)+(B//D)x\subseteq(\widetilde{B}//D)\cap(e-p)B^{**}(e-p)$. The right side is equal to $B//D$, which shows that $\mathcal{M}(\widetilde{B}//D)=(\mathcal{M}(B//D)\cap B//D)+\mathbb{C}\cdot(1-p)$.

The normalizer algebra $\mathcal{N}(D)\subseteq B$ of D in B, is the intersection of B with the normalizer algebra $\mathcal{N}(D)+\mathbb{C}\cdot 1$ of D in $\widetilde{B}$.

By [**437**, thm.1.4(iii), lem.4.9], the natural epimorphism from $\widetilde{B}$ onto the unital C^*-system $\widetilde{B}//D$ defines a unital *-epimorphism from $\mathcal{N}(D)+\mathbb{C}\cdot 1\subseteq\widetilde{B}=B+\mathbb{C}\cdot$ onto $\mathcal{M}(\widetilde{B}/D)=\mathcal{M}(B//D)+\mathbb{C}\cdot(1-p)\subseteq\widetilde{B}//D=B//D+\mathbb{C}\cdot(1-p)$ with kernel D. It follows that the natural epimorphism from B onto $B//D$ defines a *-epimorphism from $\mathcal{N}(D)$ onto $\mathcal{M}(B//D)\cap B//D$ with kernel $=D$.

Ad(ii): Let e with $e\widetilde{B}^{**}=B^{**}$ as in the proof of parts (o,i).

If $x, x^*x, xx^* \in B//D \subseteq \widetilde{B}//D$, then $x \in \mathcal{M}(\widetilde{B}//D) \cap B//D = \mathcal{M}(B//D) \cap B//D$ by [**437**, lem.4.8(ii)] and proof of part(i).

Conversely, $x, x^*x, xx^* \in B//D$ if x is in $\mathcal{M}(B//D) \cap B//D \subseteq (1-p)B^{**}(1-p)$, because $\mathcal{M}(B//D) \cap B//D$ is a closed ideal of the C^*-subalgebra $\mathcal{M}(B//D)$ of $(1-p)B^{**}(1-p)$ by part(i).

Ad(iii): Because π_D and $\pi_{L+R}|\mathcal{N}(D)$ are the same, $\pi_D(\mathcal{N}(D) \cap J)$ is contained in $\pi_D(\mathcal{N}(D)) \cap \pi_{L+R}(J)$.

If $a \in J$ and $\pi_{L+R}(a) \in \pi_D(\mathcal{N}(D))$, then there exists $b \in \mathcal{N}(D)$ with $b - a \in R + L$. If we apply the quotient map π_J from B onto B/J to b, then $\pi_J(b) \in \pi_J(R) + \pi_J(L)$, and $\pi_J(b)$ is in the normalizer algebra $\mathcal{N}(\pi_J(D))$.

Since $\pi_J(R)$ and $\pi_J(L)$ are the closed right and left ideals generated by $\pi_J(D)$, we have that $\mathcal{N}\big(\pi_J(D) \cap (\pi_J(R) + \pi_J(L))\big) = \pi_J(D)$. It follows $\pi_J(b) \in \pi_J(D)$, i.e., $b \in D + J$. If $b = d + c$ with $d \in D$ and $c \in J$, then $c = b - d \in \mathcal{N}(D) \cap J$ and $\pi_D(c) = \pi_D(a)$.

Ad(iv): Since T is completely positive and completely isometric, it follows that $\gamma\colon A \to B//D$ is completely positive and completely isometric. Thus, in $(B//D)^{**} \cong (1-p)B^{**}(1-p)$ holds $\gamma(a)^*\gamma(a) \le \gamma(a^*a)$ and $a^*a \le T^{**}(\gamma(a)^*\gamma(a)) \le a^*a$. It gives $\gamma(a^*a) = \gamma(a)^*\gamma(a)$ for $a \in A$, i.e., $\gamma\colon A \to \gamma(A) \subseteq (B//D)^{**}$ is a C^*-morphism. Hence $\gamma(A) \subseteq \mathcal{M}(B//D)$ by part (ii).

Ad(v): Let $J \lhd B$. We get $J \subseteq \mathrm{Ann}(D, J)$ from the definition $\mathrm{Ann}(D, J) := \{b \in B\,;\; bD + Db \subseteq J\}$.

An element $b \in B$ satisfies $\pi_J(bD) = \pi_J(b)\pi_J(D) = \{0\}$ if and only if $bD \subseteq J$. Thus,

$$\pi_J(\mathrm{Ann}(D, J)) = \mathrm{Ann}(\pi_J(D)) := \mathrm{Ann}(\pi_J(D), 0)\,.$$

It follows that $\mathcal{N}(D) \cap \mathrm{Ann}(D, J) = \mathcal{N}(D) \cap J$ if $\pi_J(D)$ is essential in B/J.

If $\pi_D(\mathcal{N}(D) \cap \mathrm{Ann}(D, J)) = \pi_D(\mathcal{N}(D) \cap J)$, and $\pi_J(bD) = \pi_J(b)\pi_J(D) = \{0\}$, then $b^*b \in \mathrm{Ann}(D, J)$

More????? `proof: Lemma A.19.8` ??

Ad(vi):

Ad(vii): □

20. Open projections and Kadison transitivity

Remark A.20.1. We use the following lemmata and the Lemma 2.2.3 in the proof of Proposition 2.2.5 and in proofs in Chapter 3.

The proofs do not allow something "up to ε" instead they require "precise" algebraic variants of the Kadison transitivity theorem (cf. [**616**, thm. 2.7.5]) for projections $p^* = p \in A^{**}$ with $pA^{**}p$ of finite linear dimension (i.e., p is in the C^*-algebra-"socle" of A^{**} considered as C^*-algebra).

The variant [**616**, thm. 2.7.5] of the Kadison transitivity theorem *implies* that, for each projection $q \in pA^{**}p$, there exists a self-adjoint contraction $a^* = a \in A$, $\|a\| \leq 1$ with $p - 2q = (1-2q)p = ap$.

It shows that the natural C^*-morphism $A_0 := p' \cap A \ni a \mapsto ap \in pMp$ is surjective.

The theorem [**616**, thm. 2.7.5] covers also that case of any projections p in the "socle" socle(A^{**}) of A^{**}, because there exists a normal non-degenerate *-representation $D\colon A^{**} \to \mathcal{L}(\mathcal{H})$ such that $D|pA^{**}p$ is faithful and $D(p)$ has finite rank, i.e., $\mathrm{Dim}(D(p)\mathcal{H}) < \infty$, cf. Lemma 2.1.15(i).

Here the (french) notation "socle" ("pedestal", "base"?) of a Banach *-algebra B is defined (since ca. 1950) by

$$\mathrm{socle}(B) := \{ b \in B\,;\ \mathrm{Dim}(b^*Bb) < \infty \}\,.$$

See ?????

It gives that for each selfadjoint contraction $b^* = b \in pA^{**}p = pAp$ there is a self-adjoint *contraction* $a^* = a$ in A with $pa = ap = b$.

Notice that all this happens also in the cases of real C^*-algebras, C^*-ternary algebras etc., e.g. in the case of real C^*-algebras, the same result with skew-adjoint elements $b^* = -b \in A$:

Then $c := e_{12} \otimes b - e_{21} \otimes b$ is self-adjoint in $(p \oplus p)M_2(A)(p \oplus p) = M_2(pAp)$.

A self-adjoint contraction $a^* = a \in M_2(A)$, $a(p \otimes p) = (p \otimes p)a$ and $a(p \otimes p) = c$ satisfies for its entries

$$[a_{jk}] = e_{11} \otimes a_{11} + e_{12} \otimes a_{12} + e_{21} \otimes a_{21} + e_{22} \otimes a_{22}$$

that $a_{12}^* = a_{21}$, $a_{11}^* = a_{11}$, $a_{22}^* = a_{22}$, $pa_{jk} = a_{jk}p$, $pa_{11} = 0$, $pa_{22} = 0$, $pa_{12} = b$, $pa_{12}^* = -b$. Thus, $a := (1/2)(a_{12} - a_{12}^*)$ is skew-adjoint and $pa = ap = b$.

It follows that p commutes with a and $pap^2 = pap$. Hence, a is in the **multiplicative domain** $\mathrm{Mult}(V) \subseteq A$ of the completely positive map

$$V\colon a \in \widetilde{A} \to pap \in pA^{**}p\,.$$

Since the finite-dimensional real C^*-algebra $pA^{**}p$ is the linear span of its projections, the restriction of V to the C^*-subalgebra $\mathrm{Mult}(V)$ of A is a *-epimorphism from $\mathrm{Mult}(V)$ onto $pA^{**}p$.

In particular, for each irreducible representation $d\colon A \to \mathcal{L}(\mathcal{H})$, we get that for $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ there exists $e \in A$ with $d(e)x = y$ *and* $\|e\| = 1$. Then $\psi(e^*e) = 1$ if ψ denotes the pure state $\psi(a) := \langle d(a)x, x\rangle$.

Compare Remark A.18.4 for an alternative proof of the existence of a C^*-subalgebra $B \subseteq A$ such that $B \ni b \mapsto pbp \in pA^{**}p$ is a *-epimorphism.

21. On excision of pure states

We need non-separable and "precise" versions of well-known excision results e.g. in [**6**]. Here we consider only excision for pure states on C^*-algebras A. The idea is, to show that every separable C^*-subalgebra is contained in an eventually bigger separable C^*-subalgebra with the property that the restriction of the pure state to this bigger one is again a pure state. For use in proofs, recall here that all hereditary C^*-subalgebras D of separable C^*-algebras are σ-unital.

HERE some comments (important ??, check it, erase unnecessary discussions):

Before the proofs of some Lemmata and the Proposition A.21.4 let us mention some of the conclusions from them. One of it is the following important property:

For every C^*-algebra A and any pure state φ on A, the separable C^*-subalgebras B of A with the property that $\varphi|B$ is again an irreducible state on B build an upward directed net of separable C^*-subalgebras of A such that each separable C^*-subalgebra of A is contained in one member of this net.

In particular, A is the (algebraic, set-theoretic) union of this special separable C^*-subalgebras, and one can combine it with a countable number of other properties of separable C^*-subalgebras of A, as considered in other sections in the Appendices A and B, e.g. B can be chosen that it is relative weakly injectivity in A, that all ideals of B are intersections of ideals of A with B .... And we can see that A is the set-theoretic union of the upward directed family of all separable C^*-algebras that have all this countably many properties together...

The sequence $(e^n)_{n\geq 1}$ of elements in $B \subseteq A$ with $G \subseteq B$ in Proposition **??** "excises" in a very sharp sense the restriction $\varphi|B$ of the pure state φ. If one allows to do the same with some more general nets of positive contractions for some state φ on A then this φ can be "excised", if and only if, φ is in the $\sigma(A^*, A)$-closure of the pure states on A – compare [**6**, def. 2.1, prop. 2.2] for this more general excision property, which is not equivalent to our demand of accuracy for our definition of "strict excision".

Here we equip the reader with necessary information on this topic to understand our methods for a detailed study of sufficient conditions on simple C^*-algebras A that cause pure infiniteness of A. (In fact we provide a big list of equivalent properties for simple C^*-algebras in Sections 2 to 4 of Chapter 2 that each imply or are equivalent to pure infiniteness.)

The here constructed special fixed element e is a contraction that "supports" φ sharply if A is separable and $G = A$, i.e., then the extreme point φ of the quasi-state space A, i.e., the set of *all* linear functional on A of norm one, is the unique *peak point* for suitable $e \in A_+$ and $e \leq 1$ with $A = \overline{eAe}$.

For given pure states φ on separable or non-separable A there are many positive contractions e of norm $\|e\| = 1$ with the property that φ is a peak point of $e \in A_+$ if A has no sub-quotient of type I, e.g. e^t for $t \in (0, \infty)$.

We use another terminology that describes all with help of quotient C^*-spaces $A/(L^* + L)$ build from C^*-algebras by using left-ideals L because those anyway play a role in some of our proofs e.g. for the embedding theorem.

For non-separable A one gets usually only a convex subset of positive contraction $k \in A_+$ with $\varphi(k) = 1$. Then $\varphi(k^2) = 1$. If $\varphi \neq \psi$ for some other pure state ψ then there exists a positive contraction $k \in A_+$ with

$$\min(\varphi(k), \psi(k)) < \max(\varphi(k), \psi(k)) = 1$$

by application of Kadison transitivity to projections in $p \in A^{**}$ with $pA^{**}p$ isomorphic to M_2 or $\mathbb{C} \oplus \mathbb{C}$.

This follows e.g. from Remark A.18.9 by considering the C^*-algebra

$$A/\big((L_\varphi \cap L_\psi)^* + (L_\varphi \cap L_\psi)\big)\,.$$

It is an, – at most 4-dimensional and at least 2-dimensional –, quotient C^*-space of A (which is automatically a C^*-algebra because of its finite dimension as a vector space).

Lemma A.21.1. *Let E a C^*-algebra and $J \triangleleft E$ a non-zero σ-unital closed ideal of E such that E/J is one-dimensional.*

Then E contains a strictly positive contraction $e \in E_+$ such that $\pi_J(e) = 1_{E/J}$ and that $e - e^2$ is a strictly positive element of J.

Proof. Let $\psi(t) := \min(t, 1)$ for $t \in [0, \infty)$. Since E/J is one-dimensional, there is unique element $d \in (E/J)_+$ with $\|d\| = 1$, i.e., $d = 1_{E/J}$. Let $g \in E$ a lift of d, i.e., $\pi_J(g) = d^{1/2} = d$. Then $c := \psi(g^*g)$ is a positive contraction $c \in E_+$ with $\pi_J(c) = d = 1_{E/J}$. By assumption, J_+ contains a strictly positive element h. We may suppose that $\|h\| = 1/2$, – otherwise replace h by $\|2h\|^{-1}h$.

We have now $c \in E_+$ with $0 \leq c \leq 1$ in $E + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(E) \subseteq E^{**}$ and $\pi_J(c) = 1_{E/J}$ and a strictly positive element $h \in J_+$ for J with norm $\|h\| = 1/2$. Then $1/2 \leq 1 - h \leq 1$ and $2 \cdot 1 \leq (1 - h)^{-1}$ in $\mathcal{M}(E)$. (Here 1 denotes the unit of $\mathcal{M}(E)$.)

This implies that $c + h$ is strictly positive in E and $c + h \leq (1 - h)^{-1} \in \mathcal{M}(E)$. We define a positive contraction by

$$e := (1 - h)^{1/2}(c + h)(1 - h)^{1/2} \quad \in E_+\,.$$

The element e is strictly positive in E because $(1 - h)^{1/2}$ is invertible in $\mathcal{M}(E)$ and $c + h$ is strictly positive in E.

Obviously $(1 - h)^{1/2}h(1 - h)^{1/2} = h - h^2 \in J_+$ and $c^{1/2}hc^{1/2} \in J_+$. Thus $\pi_J(e) = \pi_J(c) = 1_{E/J}$ by the MvN-equivalence

$$(1 - h)^{1/2}c(1 - h)^{1/2} \sim_{MvN} (c - c^{1/2}hc^{1/2})\,.$$

It implies $e - e^2 \in J_+$ because $0 \leq e \leq 1$.

If we use that $0 \leq c \leq 1$ to get $e \leq (1 - h)^{1/2}(1 + h)(1 - h)^{1/2} = 1 - h^2$, i.e., that $h^2 \leq 1 - e$. It implies that $e^{1/2}h^2e^{1/2} \leq e - e^2 \in J_+$.

The contraction h^2 is strictly positive in J because h is strictly positive in J. Now observe that if J is a closed ideal, h^2 is strictly positive in J and e is strictly positive in E, then $e^{1/2}h^2e^{1/2}$ is strictly positive in J. □

DEFINITION A.21.2. Let A C^*-algebra and ρ a state on A. We say that ρ has the **excision property** if there exists a net of positive contractions $\{e_\tau\}_{\tau\in\Sigma} \subseteq A_+$ that satisfy that $\rho(e_\tau) = 1$ for all $\tau \in \Sigma$ and

$$\lim_{\tau\in\Sigma} \| \, e_\tau a e_\tau \, - \, \rho(a)e_\tau^2 \, \| \, = 0 \, .$$

We say that a state ρ on A has the **strict excision property** if A contains a positive contraction $e \in A_+$ with $\rho(e) = 1$ such that

$$\lim_{n\to\infty} \|e^n a e^n - \rho(a)e^{2n}\| = 0 \quad \text{for all } \; a \in A \, .$$

It is not difficult to see that a state ρ on A with the property that there exists $e \in A_+$ with $\|e\| \leq 1$, $\rho(e) = 1$ and $\lim_n \|e^n a e^n - \rho(a)e^{2n}\| = 0$ must be necessarily a pure state on A:
The assumptions show that $\|e\| = 1$, i.e., there is a pure state λ on A with $\lambda(e) = 1$. Then $\lambda(xe^k) = \lambda(e^k x) = \lambda(x)$ for all $x \in A$, and this implies $\lim_n \, (\lambda(a) - \rho(a)) \, = 0$, for all $a \in A$, i.e., $\rho = \lambda$ is pure.

LEMMA A.21.3. *Let B separable C^*-algebra, ρ a state on B.*

The state ρ is a pure state on B, if and only if, there exists $e \in B_+$ with $\|e\| = 1$, $\rho(e) = 1$ and

$$\{b \in B; \, \rho(b) = 0\} \, \subseteq \, \overline{(e - e^2)B} + \overline{B(e - e^2)} \, . \tag{21.1}$$

In particular, then ρ has the strict excision property *in Definition A.21.2, i.e., there there exists $e \in B_+$ with $\rho(e^n) = 1 = \|e\|$ for all $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} \|e^n b e^n - \rho(b)e^{2n}\| = 0 \quad \textit{for all } \; b \in B \, .$$

PROOF. Suppose that ρ is a state on B and $e \in B_+$ with $\rho(e) = 1 = \|e\|$. Then $1 \leq \rho(e)^2 \leq \rho(e^2) \leq \rho(e) = 1$, It follows that $0 \leq e - e^2 \leq e$ and $0 \leq \rho((e - e^2)^2) \leq \rho(e - e^2) = 0$ imply that $\rho((e - e^2)b^*b(e - e^2)) = 0$ for all $b \in B$. Thus, $\rho(b^*b) = 0$ for all $b \in L := \overline{B(e - e^2)}$. It says that the closed left ideal L is contained in $L_\rho := \{b\, ; \; b \in B, \rho(b^*b) = 0\}$, i.e., $L \subseteq L_\rho$. It is easy to see that always $L_\rho^* + L_\rho \subseteq \rho^{-1}(0)$. If we combine this containment with the containment $\rho^{-1}(0) \, \subseteq \, L^* + L$ in (21.1), then we get that

$$\rho^{-1}(0) = L_\rho^* + L_\rho \, .$$

But this shows that the state ρ is a pure state, because [**616**, prop. 3.13.6.(i,ii)] says that ρ is pure state on B, if and only if, $L_\rho^* + L_\rho = \rho^{-1}(0)$, i.e., ρ is a pure state, if and only if, $L_\rho^* + L_\rho$ is equal to the null-space of ρ.

Now let ρ be a pure state on the separable C^*-algebra B. We are going to show that there exists a strictly positive contraction $e \in B_+$ with $\rho(e) = 1$, that satisfies with $L := \overline{B(e-e^2)}$ the equation

$$\rho^{-1}(0) \,=\, L^* + L\,.$$

In particular, then e is a strictly positive contraction in B, $L^* + L + \mathbb{C}\cdot e = B$, and $e - e^2$ is a strictly positive element of $L^* \cap L$.

A special case of the Kadison transitivity theorem shows then the existence of $c \in B_+$ with $\|c\| = \rho(c) = 1$, cf. [**616**, thm. 2.7.5, thm. 3.13.2(vi)]. – or use our more engaged Lemma 2.1.15(ii,iii) for that.

The positive contraction $c \in B_+$ satisfies then $1 = \rho(c)^2 \le \rho(c^2) \le \rho(c) = 1$. Thus, $\rho(c^2) = 1$ and $\rho(c - c^2) = 0$.

Let $h\colon B \to \mathcal{L}(\mathcal{H})$ the corresponding irreducible representation with cyclic vector $v \in \mathcal{H}$ corresponding to ρ, i.e., $\|v\| = 1$ and $\langle h(b)v, v\rangle = \rho(b)$ for $b \in B$. Then $\|c\| = 1 = \rho(c) = \rho(c^2)$ and $c \ge 0$ imply that $h(c)v = v$, because $\|(1 - h(c))v\|^2 = \langle h(c)v, h(c)v\rangle + \langle v, v\rangle - \langle v, h(c)v\rangle - \langle h(c)v, v\rangle = \rho(c^2) + 1 - 2\rho(c) = 0$, – here with $\mathrm{id}_{\mathcal{H}}$ is denoted by 1.

It follows that $\rho(b) = \rho(bc) = \rho(cbc) = \rho(cb)$ for all $b \in B$. In particular, $\rho(cb^*bc) = \rho((bc)^*bc) = \rho(b^*b)$ for all $b \in B$.

Thus, the closed left ideal $L := \{b \in B\,;\ \rho(b^*b) = 0\}$ of B has the property $L \cdot c \subseteq L$. Notice that $B = L^* + L + \mathbb{C}\cdot c$, because $L^* + L$ is the kernel of the pure state ρ and $\rho(c) = 1$.

Since B is separable, the hereditary C^*-subalgebra $D := L^* \cap L$ of B contains a strictly positive element $g \in D_+$ for D. The positive part D_+ of D consists of all $b \in B_+$ with $\rho(b) = 0$. In particular, $c - c^2 \in D_+$. It follows that $g + c$ is a strictly positive element of B.

The relation $L \cdot c \subseteq L$ impies that $D \cdot c \subseteq D$ and $c \cdot D \subseteq D$. It says that the $E := D + \mathbb{C}\cdot c$ is a C^*-subalgebra of B, D is a closed ideal of E, $\rho|E$ is a character on E with kernel equal to D and $\rho(\xi \cdot c) = \xi$ for all $\xi \in \mathbb{C}$, i.e., E/D is one-dimensional. If we identify $\mathbb{C}$ naturally with $\pi_D(E) = E/D$, then the quotient map π_D is just the character $\rho|E$.

By Lemma A.21.1, E contains a strictly positive contraction $e \in E_+$ such that $\pi_D(e) = 1_{E/D}$ and that $e - e^2$ is a strictly positive element of $D = L^* \cap L$ that is the kernel of $\rho|E$ – if we adjust the identification of $\mathbb{C}$ with E/D such that $1 \in \mathbb{C}$ corresponds to $1_{E/D}$, i.e., $\rho|E$ maps e and c in $D + \mathbb{C}\cdot c$ onto the unit-element of $E/D \cong \mathbb{C}\cdot\rho(c) = \mathbb{C}$ by the character $\rho|E$ with $\rho(e) = \rho(c) = 1$.

Now we return to $B = L^* + L + \mathbb{C}\cdot c$. We got also $B = L^* + L + \mathbb{C}\cdot e$, because $\rho(e) = \rho(c)$ and $L^* + L$ is the kernel of ρ. Since $e - e^2$ is a strictly positive element of $D = L^* \cap L$ (by above construction of e), it follows that $L = B \cdot D = \overline{B \cdot (e - e^2)}$ and $L^* = \overline{(e - e^2)\cdot B}$.

Finally we estimate the norms $\|e^n b e^n - \rho(b) e^{2n}\|$ for large $n \in \mathbb{N}$:
Each element $b \in B = L^* + L + \mathbb{C}\cdot e$ can be written as $b = b_1 + b_2 + \alpha\cdot e$ with $b_1^* \in L$, $b_2 \in L$, $\alpha \in \mathbb{C}$. Then $\rho(b) = \alpha$ and the equation $L = \overline{Be(1-e)}$ implies

that, for given $\delta \in (0,1)$, there exist $c_1, c_2 \in B$ with $\|b_1 - e(1-e)c_1\| < \delta$ and $\|b_2 - c_2 e(1-e)\| < \delta$. Notice that $\|e^n c_1 (1-e)e^n\| \leq \|c_1\| \|(1-e)e^n\|$ and use that $\lim_{n\to\infty} \|e^n(1-e)\| = 0$ to get that $\lim_n \|e^n b e^n - \rho(b)e^{2n}\| \leq 3\delta$ for arbitrary $\delta \in (0,1)$. □

The following Proposition is essentially [**93**, lem. 2.14]. But we give here together with Lemma A.21.3 a more detailed proof, because it plays a basic role for proofs of some other results.

Proposition A.21.4. *Let φ be a pure state on a C^*-algebra A and $G \subseteq A$ a separable C^*-subalgebra.*

Then there exist a separable C^*-subalgebra $B \subseteq A$ and an element $e \in B_+$ with the properties $G \subseteq B$, $\|e\| = 1$, $\varphi(e) = 1$ and

$$\{b \in B;\, \varphi(b) = 0\} \subseteq \overline{(e-e^2)B} + \overline{B(e-e^2)}\,.$$

In particular, $\varphi|B$ is a pure state on B and has the strict excision property in Definition A.21.2 by Lemma A.21.3.

It shows that every pure state on a C^*-algebra A has the excision property *of Definition A.21.2.*

Proof of Proposition A.21.4. Let $\varphi \in A^*_+$ a pure state, i.e., an extreme point of the convex set of positive linear functionals on A of norm $= 1$. There are many equivalent properties of a state φ that imply that φ is a pure state, e.g. by [**616**, prop. 3.13.6(i,ii)]: The closed left ideal $L := L_\varphi$ of A defined by $L := L_\varphi := \{a \in A;\, \varphi(a^*a) = 0\}$ satisfies $\varphi^{-1}(0) = L^* + L$, if and only if, φ is a pure state on A.

`compare with text:`

Then a special case of the Kadison transitivity theorem shows the existence of $c \in A_+$ with $\|c\| = \varphi(c) = 1$, e.g. use [**616**, thm. 2.7.5, thm. 3.13.2(vi)] or use our farer going Lemma 2.1.15(ii,iii) to get such $c \in A_+$.

The purity of φ implies that

$$\{a - \varphi(a)c;\; a \in A\} = L^* + L = \ker(\varphi)\,,$$

cf. [**616**, thm. 3.13.2(iv), prop. 3.13.6] for the second equation. It yields that $P\colon A \to A$ defined by $P(a) := a - \varphi(a)c$ maps A into $\ker(\varphi)$ and satisfies $P(a) = a$ for all $a \in \ker(\varphi)$, $\|P\| \leq 2$ and $P^2 = P$, i.e., is a linear projection of norm $\|P\| \leq 2$ from A onto $\ker(\varphi)$.

The maps $a \in A \mapsto \varphi(a) \cdot c \in A$ and id_A are completely positive contractions. Thus, the map $P(a) := a - \varphi(a)c$ on A has a cb-norm $\|P\|_{cb} \leq 2$, in particular $\|P\| \leq 2$. Obviously, $\varphi(P(a)) = 0$ for all $a \in A$, – i.e., $P(A) \subseteq \varphi^{-1}\{0\}$ –, and $\varphi(a) = 0$ implies $P(a) = a$. Thus, $\varphi^{-1}\{0\} \subseteq P(A)$. It shows that

$$P(A) = \varphi^{-1}\{0\} = L^* + L \quad \text{and} \quad P^2 = P\,.$$

In particular, $P(a) = a$ if and only if $a \in L^* + L$. The same arguments (– but with $b \in B$ in places of a –) show that $P(B) = B \cap (L^* + L)$ for every linear subspace B of A with $c \in B$ ([13]).

The projection map P has following property: If $B_1 \subseteq B_2 \subseteq \cdots$ and $B := \overline{\bigcup_n B_n}$ then $P(B) = \overline{\bigcup_n P(B_n)}$.

Now let $G \subseteq A$ a separable C^*-subalgebra of A, and let B_1 the separable C^*-subalgebra of A generated by $G \cup \{c\}$. Then $(L^* + L) \cap B_1 = P(B_1)$, in particular $P(B_1) \subseteq B_1$.

By the – here very crucial – Corollary A.15.3, there exists a separable C^*-subalgebra $B_2 \subseteq A$ with the properties that $B_1 \subseteq B_2$ and

$$B_1 \cap (L^* + L) \subseteq (L^* \cap B_2) + (L \cap B_2)\,.$$

We can repeat this construction and argument by induction, and get an increasing sequence of separable C^*-subalgebras $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ of A with the property

$$B_n \cap (L^* + L) \subseteq (L^* \cap B_{n+1}) + (L \cap B_{n+1}) \subseteq B_{n+1} \cap (L^* + L)\,.$$

Let B denote the closure of $\bigcup_{n \in \mathbb{N}} B_n$. Then, for all $n \in \mathbb{N}$,

$$B_n \cap (L^* + L) \subseteq (L^* \cap B) + (L \cap B) \subseteq B \cap (L^* + L)\,.$$

(Use here that $(L \cap B) \subseteq B \cap (L^* + L)$ and $L \cap B_{n+1} \leq L \cap B$.)

The space $\bigcup_n B_n \cap (L^* + L)$ is dense in $B \cap (L^* + L)$:
$P(B_n) = B_n \cap (L^* + L)$ and $P(B) = B \cap (L^* + L)$, because $c \in B_n \subseteq B$. Since P is a bounded projection and $\bigcup_n B_n$ is dense in B we get that $\bigcup_n P(B_n)$ is dense in $P(B)$.

The sum $(L^* \cap B) + (L \cap B)$ is the sum of a closed left and closed right ideal of B. Sums $K^* + K := \{x^* + y\,;\ x, y \in K \subseteq B\}$ build by closed left ideals $K \subseteq B$ of B are always closed subspaces of B by Proposition A.15.2.

We get that $P(B) \subseteq (L^* \cap B) + (L \cap B)$, because $B_n \cap (L^* + L) = P(B_n) \subseteq B_{n+1} \cap L^* + B_{n+1} \cap L$ and $\|P\| \leq 2$.

It follows that $P(B) = B \cap (L^* + L) = B \cap \varphi^{-1}\{0\}$ is equal to $(L^* \cap B) + (L \cap B)$ and is the kernel of the restriction $\psi := \varphi|B$ of φ to B. Notice that $b \in L \cap B$, if and only if, $\psi(b^*b) = 0$, i.e., $L \cap B = L_\psi$, where $L_\psi := \{b \in B\,;\ \psi(b^*b) = 0\}$. The characterization of pure states on B, given by the above cited [**616**, prop. 3.13.6(i,ii)], shows that $\psi := \varphi|B$ is a pure state on B, because ψ satisfies the purity criterium $\psi^{-1}(0) = L_\psi^* + L_\psi$.

Notice that $G \subseteq B_1 \subseteq B$ and that B is separable.

---

[13] Notice here that it could be that $\varphi|B$ is not a pure state on a C^*-subalgebra B of A with $c \in B$ ... except in some special cases where $\varphi|B$ is the only state ρ on B with $\rho(c) = 1$. For example, take $A := M_2$, $c := 1_2$ and let $B \subseteq A$ the diagonal elements in A. Then $\varphi([\alpha_{jk}]) = 2^{-1} \sum_{jk} \alpha_{jk}$ is a pure state on A but not on B.

By Lemma A.21.3 there exists an element $e \in B_+$ with the properties $\|e\| = 1$, $\varphi(e) = 1$ and

$$\{b \in B;\, \varphi(b) = 0\} \subseteq \overline{(e - e^2)B} + \overline{B(e - e^2)}\,,$$

and B has the strict excision property of Definition A.21.2.

Since we can do this selection procedure for every given separable subset G of A with with respect to a given fixed pure state of A, we can see that each pure state of A has the (in general not “strict”) excision property of Definition A.21.2. □

22. Some technical functions on $[0, 1]$

We use often for some very elementary approximation technics the increasing piecewise linear functions $h_n \in \mathrm{C}_0(0, 1]_+$ with break points at 2^{-n}, $n = 0, 1, 2, \ldots$, given by

$$h_n(t) := \min(1, \max(2^n t - 1, 0)) \quad \text{for } t \in [0, 1],\ n = 0, 1, \ldots\ .$$

In the C^*-algebra $\mathrm{C}_0(0, 1]$ and $n \in \mathbb{N}$ they are given by

$$h_0 := 0\,,\ h_1 := (2f_0 - 1)_+\,,\ \ldots\,,\ h_n := (2^n f_0 - 1)_+ - (2^{n-1} f_0 - 1)_+\,,$$

where $f_0 \in \mathrm{C}_0(0, 1]$ denotes the identity map $f_0(t) := t$ on $(0, 1]$.

The below listed properties of the h_n can be easily seen on the intervals $[2^{-n}, 2^{-(n-1)}]$, because the functions h_n are linear on two of this intervals and take outside them only the values 0 and 1:

(i) $h_n|[0, 2^{-n}] = 0$, $h_n|[2^{-n}, 2^{-(n-1)}]$ is the linear map

$$h_n(t) := (2^{-(n-1)} - 2^{-n})^{-1}(t - 2^{-n}) \quad \text{for } t \in [2^{-n}, 2^{-(n-1)}]$$

and $h_n|[2^{-(n-1)}, 1] = 1$. In particular, $h_n \in \mathrm{C_c}(0, 1]_+ \subset \mathrm{C}_0(0, 1]_+$ for all $n \in \mathbb{N}$, and $h_0 := 0$ on $[0, 1]$.

(ii) The set $(2^{-n}, 1]$ is the open support of h_n in $[0, 1]$, and is contained in $h_{n+1}^{-1}(1) = [2^{-n}, 1]$. In particular,

$$0 \le h_{n+1} h_n = h_n \le h_{n+1} \le 1 \quad \text{for all } n \in \mathbb{N}\,.$$

(iii) The non-negative function $h_{n+1} - h_n$ has support in the open interval $(2^{-(n+1)}, 2^{-(n-1)})$, i.e., is zero on $[0, 2^{-(n+1)}]$ and $[2^{-(n-1)}, 1]$, is linear increasing on $[2^{-(n+1)}, 2^{-n}]$ with value $(h_{n+1} - h_n)(2^{-n}) = 1$ and is linear decreasing on $[2^{-n}, 2^{-(n-1)}]$.

In particular, $h_{n+2}(t) - h_{n+1}(t) = 1 - h_{n+1}(t)$ for $t \in [2^{-(n+1)}, 2^{-n}]$.

(iv) The restriction to $[2^{-(n+1)}, 2^n]$ of the, – in $\mathrm{C}_0(0, 1]$ absolute convergent and on all intervals $[1/n, 1]$ uniformly convergent –, series

$$\sum_{n=0}^{\infty} 2^{-n}(h_{n+1}(t) - h_n(t))$$

is on $[2^{-(n+1)}, 2^n]$ just equal to the increasing linear function

$$2^{-(n+1)}(1 + h_{n+1})|[2^{-(n+1)}, 2^n]$$

with values $2^{-(n+1)}$ and 2^{-n} at the end-points.

Thus, the series $\sum_{n=0}^{\infty} 2^{-n}(h_{n+1} - h_n)$ converges uniformly on $[0,1]$, and converges *absolute* in the Banach space $C_0(0,1]$ to $f_0(t) := t$, because

$$\sum_{n=0}^{\infty} 2^{-n}\|h_{n+1} - h_n\| \leq 2\,.$$

(v) The h_n have the othogonality property $h_n(h_{n+2} - h_{n+1}) = 0$.

(vi) For each $\gamma \in (0,1]$ only finitely many functions $h_n - h_{n-1}$ have support in $[\gamma, 1]$. In particular, $\sum_{n\geq 1}(h_n - h_{n-1})$ converges on each interval $[\gamma, 1]$ uniformly to 1.

(vii) The functions $\psi_n := (h_n - h_{n-1})^{1/2} \in C_0(0,1]$ (applied to $t = f_0(t)$) are well-defined, have support in in $[2^{-n}, 2^{-n}4]$.

In particular, $\psi_n\psi_m = 0$ for $|n-m| > 1$.

(viii) The sum $\sum_{n=1} \psi_n^2$ converges in the multiplier algebra $\mathcal{M}\big(C_0(0,1]\big) \cong C_b\big((0,1]\big)$ strictly to 1, because $\sum_{n=1}(h_n - h_{n-1})$ converges on each interval $[2^{-n}, 1]$ uniformly to $1 = \lim_n h_n$. (See Remark 5.1.1(2) concerning strict convergence.)

An Application of the functions $h_1, h_2, \ldots$ is the below given simple Example A.22.1. It shows that the $\approx$-classes in $W(A)$ and the corresponding $\approx$-classes in $\mathrm{Cu}(A)$ are not the same for $A := C_0(0,1]$. Notice that the Pedersen ideal of this C^*-algebra A is $C_c(0,1]$ and our "small Cuntz semigroup $\mathrm{CS}(A)$" defined and used in

where ????????

is also different from $W(A)$ and $\mathrm{Cu}(A)$. (But here $\mathrm{Cu}(A)$ has "good comparison" because of $\mathrm{Dim}((0,1]) = 1$!)

Check calculations and notations in example:
Seems to be no real progress with comparison.
But it is likely to be true ... Perhaps pass to $M_{2n}(C_0(0,1])$ ... and combine with the h_n

EXAMPLE A.22.1. The $\approx$-classes in $W(A)$ and the corresponding $\approx$-classes in $\mathrm{Cu}(A)$ are not the same for $A := C_0(0,1]$.

Consider $A := C_0(0,1]$ and define $g_n(t) := \big(2^{-n}(h_{n+1}(t) - h_n(t))\big)^{1/2}$. Let $T := \sum_{n=1}^{\infty} g_n \otimes p_{1,n}$.

Then $T \in C_0(0,1] \otimes \mathbb{K}$ and $TT^* = (\sum g_n^2) \otimes p_{11} = f_0 \otimes p_{11}$ for $f_0(t) := t$, but the "almost diagonal" matrix

`check formula for` T^*T `!!!`

$$T^*T = \sum_n g_n g_{n+1} \otimes p_{n,n+1} + g_n g_{n-1} \otimes p_{n+1,n} + \sum_n 2^{-n}(h_{n+1} - h_n) \otimes p_{nn}$$

is not contained in $\bigcup_n M_n(A)$.

23. K_1-injectivity and generalized Kuiper theorem

Compare with related topics in Section 2 of Chapter 4
in particular with the squeezing property in Definition 4.2.14.
It is the basic observation used in the proof of Part(c) of
Proposition 4.2.15.

We reformulate some technical lemmata of J. Cuntz and N. Higson in [**172**] and [**180**], that we use for example to derive generalizations of Kuiper's Theorem [**505**] in the spirit of J. Cuntz, N. Higson [**180**] and J. Mingo [**557**] by generalizing their ideas to C^*-algebras that have our "squeezing" Property (sq) of Definition 4.2.14.

We modify ideas in [**180**] to get a proof of the following easier applicable Lemma.

Lemma A.23.1. *If B is a σ-unital and stable, then, for every separable C^*-sub-algebra C of $\mathcal{M}(B)$, $e \in C$ and $\varepsilon > 0$, there exist isometries $S, T \in \mathcal{M}(B)$ with $T^*CS \subseteq B$, $T^*S = 0$ and $\|T^*eS\| < \varepsilon$.*

Proof. The stability of B causes that for each given $b \in B$ and $\varepsilon > 0$ there exists an isometry $r \in \mathcal{M}(\mathbb{K}) \subseteq \mathcal{M}(B)$ such that $\|r^*br\| < \varepsilon$ (with r depending on b and ε).

Thus, if we later are only interested in showing that $\mathcal{M}(B)$ has the "squeezing" Property (sq) in Definition 4.2.14 for $A := \mathcal{M}(B)$, then it is enough to find, for a given $a \in \mathcal{M}(B)$, isometries $S, T \in \mathcal{M}(B)$ with $T^*aS \in B$ and $S^*T = 0$.

The claimed property $S^*T = 0$ is not important for this proof, because we could replace a considered sequence $b_1, b_2, \ldots$ of contractions in $\mathcal{M}(B)$ anyway by $Q^*b_1R, Q^*b_2R, \ldots$ with isometries $Q, R \in \mathcal{M}(B)$ with $Q^*R = 0$ for this proof, because the isometries QS and RT have the additional orthogonality property $(RT)^*(QS) = 0$.

We describe now a selection method that shows for a given separable C^*-subalgebra C of $\mathcal{M}(B)$ the existence of isometries $S, T \in \mathcal{M}(B)$ with orthogonal ranges and the property that $S^*CT \subseteq B$.

Let $c_1, c_2, \ldots \in \mathcal{M}(B)$ a sequence of contractions in $\mathcal{M}(B)$. Likewise this could be a sequence that is dense in the unit ball of a given separable C^*-subalgebra C of $\mathcal{M}(B)$ and with $c_1 := c$ the particular element that should be squeezed below $\varepsilon > 0$ with help of "squeezing isometries" S and T in $\mathcal{M}(B)$, i.e., with the additional property $\|S^*cT\| < \varepsilon$.)

We describe an inductive selection method for the construction of isometries $S, T \in \mathcal{M}(B)$ with $S^*T = 0$ and $S^*c_nT \in B$ for all $n \in \mathbb{N}$, such that we can apply Remark 5.1.1(2) concerning strict convergence on $\mathcal{M}(B)$ for σ-unital stable B:
The stability of B is equivalent to the existence of a sequence $s_1, s_2, \ldots \in \mathcal{M}(B)$ of isometries with the property that $\sum_k s_k s_k^*$ converges strictly to $1 \in \mathcal{M}(B)$, i.e.,

with respect to the strict topology on $\mathcal{M}(B)$, cf. Remark 5.1.1(8) and Lemma 5.1.2. In particular, the sequence $k \in \mathbb{N} \to \|s_k^* b\| + \|b s_k\|$ converges to 0 for each $b \in B$.

Describe first the selection process.
Because now comes an other observation
about producing new isometries:

The sums $\sum_n s_{\lambda(n)} s_{\lambda(n)}^*$ converge in $\mathcal{M}(B)$ strictly to a projection if $\lambda \colon n \in \mathbb{N} \to \lambda(n) \in \mathbb{N}$ is an injective map. Moreover, the sum $\sum_n s_{\lambda_n} f_n$ converges strictly to an isometry $S_\lambda \in \mathcal{M}(B)$ if $f_1, f_2, \ldots \in \mathcal{M}(B)$ is any sequence with the property that $\sum_n f_n^* f_n$ converges strictly to 1 in $\mathcal{M}(B)$ by Remark 5.1.1(2).

??

We use the following method to construct isometries in $\mathcal{M}(B)$:
Let $e \in B_+$ a strictly positive contraction with $\|e\| = 1$. Apply the functions $h_n, \psi_n \in \mathrm{C}_0(0,1]_+$ as defined in Section 22 with $\psi_n = (h_n - h_{n-1})^{1/2}$ to $e \in A$, i.e., $f_n := \psi_n(e) = (h_n(e) - h_{n-1}(e))^{1/2} \in A_+$. The $\sum_n f_n^2$ converges strictly in $\mathcal{M}(A)$ to $1_{\mathcal{M}(A)}$ and $\sum_n 2^{-n} h_n(e)$ is absolute convergent to e in A.

< Check above?

Thus $S := \sum_{n=1}^{\infty} s_{\lambda(n)} f_n$ converges strictly to an isometry in $\mathcal{M}(A)$ if $\sum_{n=1} s_n s_n^*$ converges strictly to $1_{\mathcal{M}(A)}$ by (two-fold application of) Remark 5.1.1(2).

If $\kappa(n) \in \mathbb{N}$ is another strictly increasing sequence, then for each $b \in \mathcal{M}(A)$ the sequence

$$\sum_{m,n} f_n s_{\kappa(m)}^* b s_{\lambda(n)} f_n$$

unconditional ??? Has to be checked !!!

is strictly convergent in $\mathcal{M}(A)$ with limit $= T^* b S \in \mathcal{M}(A)$ for the isometries $T := \sum_{n=1}^{\infty} s_{\kappa(n)} f_n$ and above defined S.

Let $b_1, b_2, \ldots \in \mathcal{M}(A)_+$ a sequence of positive contractions.

We find inductively a sequences of even numbers $k_n \in 2 \cdot \mathbb{N}$ with $k_1 := 2$, $k_n < k_{n+1}$ and odd numbers $\ell_n \in 1 + 2 \cdot \mathbb{N}$ with $\ell_n < \ell_{n+1}$ that satisfy

$\|(s_{\ell_n})^* b_j s_{k_m} f_m\| < 4^{-n}$ for all $j \le n$ and $m < n$ and $\|f_m (s_{\ell_m})^* b_j s_{k_n}\| < 4^{-n}$ for all $j \le n$ and $m < n$.

Beginning step:

Since $b_j s_2 f_1 \in A$ for $j \in \{1, 2\}$ there exists an odd number $\ell_1 > 1$ with $\|s_{\ell_1}^* b_j s_2 f_1\| < 1/4$ for $j \in \{1, 2\}$.

Then use that $f_1 s_{\ell_1}^* b_j$ are in A for $j \le 3$. It follows that there exists even $k_2 \in \mathbb{N}$ with $k_2 > 2$ and $\|f_1 (s_{\ell_1})^* b_j s_{k_2}\| < 4^{-n}$ for all $j \le 3$.

Still to define/arrange the selection procedure!

Let $m \in \mathbb{N}$ an odd natural number, and given isometries s_{k_j} , $j = 1, \ldots, m$ then find an even index $\ell_m \in \mathbb{N}$ such that $\ell_m > \ell_{m-1}$ and $\|s^*_{\ell_m} b_n s_{k_j} f_j\| < 4^{-n}$ for all $n \leq m + 2$ and $j < m$.

Then go one step higher and select the next odd index k_{m+1} in a similar way.

We select below suitable sequences $f_n \in B_+$ and injective maps $\kappa \colon \mathbb{N} \to \mathbb{N}$ with odd values and $\lambda \colon \mathbb{N} \to \mathbb{N}$ with even values such that $S^*_\lambda b_n S_\kappa \in B$ for all $n \in \mathbb{N}$.

Let $e \in B_+$ a strictly positive element with $\|e\| = 1$. The stability of B implies that 0 is not isolated in the spectrum $\mathrm{Spec}(e) \subseteq [0, 1]$. We take the piecewise linear functions $h_n(\xi) := \min(1, \max(2^n \xi - 1, 0))$ (with break points at 0, 2^{-n}, 2^{1-n} and 1), that is defined in Section 22 (and studied there in every detail), and consider the non-negative functions $\varphi_n(\xi) := \big(h_n(\xi) - h_{n-1}(\xi)\big)^{1/2}$ for $n = 1, 2, \ldots$, where we let $h_0 := 0$. Notice that $\varphi_m \cdot h_n = \varphi_m$ for $m \leq n - 1$.

Then the series $\sum_n \varphi_n(\xi)^2 \xi$ converges uniformly on $[0, 1]$ to f_0, where $f_0(\xi) := \xi$, cf. Property (iv) in Section 22.

It follows that the contractions $f_n := \varphi_n(e) \in A_+$ and $e_n := h_{n+1}(e) \in A_+$ have the property that $f_m e_n = f_m$ for all $m \leq n$ and $\sum_n f_n^2 e$ converges in norm inside B to $e \in B$.

Refere to Property (iv???) in Section 22???

Hence, $\sum_n f_n^2$ converges strictly to $1_{\mathcal{M}(B)}$ in $\mathcal{M}(B)$.

The above observations imply that $S_\kappa := \sum_n s_{\kappa(n)} f_n$ is an isometry in $\mathcal{M}(A)$ for each strictly increasing map $\kappa \colon \mathbb{N} \ni n \mapsto \kappa(n) \in \mathbb{N}$ for the above defined $f_n \in A_+$.

If the ranges of two injective maps $\kappa \colon \mathbb{N} \to \mathbb{N}$ and $\lambda \colon \mathbb{N} \to \mathbb{N}$ are disjoint then the isometrics S_λ and S_κ have orthogonal ranges. (And if the intersection $\kappa(\mathbb{N}) \cap \lambda(\mathbb{N})$ is a finite subset of $\mathbb{N}$ then $S^*_\kappa S_\lambda \in B$.)

Start with $k_1 := 1$, and define inductively strictly increasing maps $\kappa \colon n \mapsto k_n \in \mathbb{N}$ with odd k_n and $\lambda \colon n \mapsto \ell_n \in \mathbb{N}$ with even ℓ_n such that

$$\|e_{n+2} s^*_{\ell_m} b_j s_{k_{n+1}}\| < \varepsilon \cdot 2^{-(n+3)}/(n+1)$$

for all $m \leq n$ and $j \leq n + 2$, and $k_{n+1} > \max(k_n, \ell_n)$ and that in the next step

$$\|s^*_{\ell_{n+1}} b_j s_{k_m} e_{n+2}\| < \varepsilon \cdot 2^{-(n+4)}/(n+2)$$

for all $m \leq n + 1$ and $j \leq n + 2$, and $\ell_{n+1} > \max(\ell_n, k_n)$.

It is possible to find the desired odd number k_{n+1} and then the desired even number ℓ_{n+1}, because the sum $\sum_n s_n s^*_n$ of the range projections converges strictly to 1 and this implies, e.g. for the finitely many elements $e_{n+2} s^*_{\ell_m} b_j \in B$ with $m \leq n+1$ and $j \leq n+2$ that $\lim_{\nu \to \infty} \|e_{n+2} s^*_{\ell_m} b_j s_\nu\| = 0$.

Estimates to be checked again.
In particular the case $j = 1$. ??

For each $j \in M$ the element $S^*_\lambda b_j S_\kappa \in \mathcal{M}(B)$ is contained in B:

For fixed b_j in the sequence $(b_1, b_2, \ldots)$ the summands $X_{1,j} := f_1 s_{\ell_1}^* b_j s_{k_1} f_1$ and partial sums

$$X_{\ell,j} := \sum_{m,n=1}^{\ell} f_n s_{\ell_n}^* b_j s_{k_m} f_m$$

are in B. For each b_j $\sum_\ell \|X_{\ell+1,j} - X_{\ell,j}\| < \infty$, because $\|X_{\ell+1,j} - X_{\ell,j}\| < \varepsilon \cdot 2^{-(n+1)}$ for $n \geq j$. It implies convergence in B itself.

In the particular case $b_1 := c$, we get $\|S_\lambda^* c S_\kappa\| < \varepsilon$, because we have managed that the norm of $X_{1,1}$ and of all $(X_{\ell+1,1} - X_{\ell,1})$ is small enough to get the desired estimate.

To see that $X := \lim_n S_n \in B$ and $S^* b_j T \in \mathcal{M}(B)$ are equal for it suffices to see that $cS^* b_j T d = cXd$ for all $c, d \in B$. But this is evident even with $c = d := e$ and easy to see

It follows $S^* C T \subseteq B$ and S, T are isometries with $S^* T = 0$ if the sequence $b_1, b_2, \ldots$ is dense in the positive part of the unit ball of $C \subseteq \mathcal{M}(B)$.

□

Remark A.23.2. Lemma A.23.1 implies that $A := \mathcal{M}(B)$ has the "squeezing" Property (sq) of Definition 4.2.14. Therefore, all non-zero quotients of $A \otimes^{\max} D$ for unital C^*-algebras D are K_1-bijective, by Proposition 4.2.15.

This covers the results of [**180**], e.g. that $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}_0(\mathcal{M}(B))$ and that all pointed homotopy groups $\pi_n(\mathcal{U}(\mathcal{M}(B)), 1)$ are trivial ([14]).

M. Mingo [**557**] obtained the following generalization of Kuiper's theorem in the special case where $B = C \otimes \mathbb{K}$ for unital C^*-algebras C.

Theorem A.23.3 (J.Cuntz, N. Higson,[**180**]). *Let B σ-unital stable C^*-algebra, and A a unital C^*-algebra.*

The K_*-groups of $\mathcal{M}(B) \otimes A$ and $\mathcal{M}(B) \otimes^{\max} A$ are trivial, and any quotient algebra of $\mathcal{M}(B) \otimes^{\max} A$ is K_1-injective.

The unitary groups $\mathcal{U}$ of the algebras $\mathcal{M}(B) \otimes^{\max} A$ are connected (with respect to the operator-norm topology), and have only trivial homotopy groups $\pi_n(\mathcal{U}, 1) = 0$.

Every continuous map from a locally finite CW-complex X into $\mathcal{U}$ is homotopic in $\mathcal{U}$ to the constant map $X \ni x \mapsto 1$.

Proof. The C^*-algebra $\mathcal{M}(B)$ has the "squeezing" property (sq) of Definition 4.2.14 by Lemma A.23.1. It causes the property (sq) for all C^*-algebras $(\mathcal{M}(B) \otimes^{\max} A) \otimes \mathrm{C}(X)$, all unital C^*-algebras A and compact Polish spaces X (e.g. let $X := S^n$). The property (sq) of $\mathcal{M}(B)$ implies K_1-injectivity of this algebras.

[14] The there deduced contractibility of the non-separable (!) Banach-space manifold $\mathcal{U}(\mathcal{M}(B))$ requires some set-theoretic discussion that is not given in detail. We doubt that it is possible without using a version of the Axiom of Choice (which is not provable), that should be mentioned.

The infinite repeat $D := \delta_\infty \otimes^{\max} \mathrm{id}_A$ satisfies $D \oplus \mathrm{id} \cong D$. It implies that $\mathrm{K}_*(\mathcal{M}(B) \otimes^{\max} A) = 0$ for all C^*-algebras.

The K_1-triviality and K_1-injectivity for all $X = S^n$ imply together that every continuous map from a locally finite CW-complex into $\mathcal{U}$ is homotopic in $\mathcal{U}$ to the constant map 1. □

The following Corollary A.23.4 (see J.Cuntz, N. Higson,[**180**]) of Theorem A.23.3 uses that a complete metrizable space that is locally uniformly homeomorphic to a (not necessarily separable !) Banach space is homotopic to a (locally finite) CW-complex, cf. [**556**] for open subsets of *separable* Banach spaces and [**542**, chp. IV, lem.5.2, cor.5.5] for the *non-separable* case ([15]). Then the vanishing of the – by 1 – pointed homotopy groups yields that $\mathcal{U}(C)$ is contractible if $\mathcal{U}(\mathrm{C}(S^n) \otimes C) = \mathcal{U}_0(\mathrm{C}(S^n) \otimes C)$ for each $n \in \mathbb{N}$. It implies:

Corollary A.23.4. *If B is stable and σ-unital and A is unital, then the unitary groups $\mathcal{U}(\mathcal{M}(B) \otimes A)$ and $\mathcal{U}(\mathcal{M}(B) \otimes^{\max} A)$ are contractible.*

Remark A.23.5. The following independent conclusion does not use arguments from [**542**] (dependent from additional axioms on set theory?):

If A is unital and B is stable and σ-unital, then each C^*-quotient $C \neq \{0\}$ of $\mathcal{M}(B) \otimes^{\max} A$ satisfies $\mathcal{U}(C)/\mathcal{U}_0(C) \cong \mathrm{K}_1(C)$.

Indeed: The quotients C have Property (sq) of Definition 4.2.14 and are K_1-bijective by Proposition 4.2.15.

24. Tensor Intersection lemma

Next lemma to App.B? To Reduction to separable case?

In the following lemma $X \otimes^\nu Z$ means the closure of $X \odot Z$ in the completion $A \otimes^\nu B$ of the algebraic tensor product $A \odot B$ with respect to a C^*-norm $\|\cdot\|_\nu$ on $A \odot B$. Notice that, e.g. in the case of the maximal C^*-tensor product $A \otimes^{\max} B$ of C^*-algebras A and B, one has for C^*-subalgebras $C \subset A$ not necessarily that the natural C^*-morphism $C \otimes^{\max} B \to A \otimes^{\max} B$ is injective. This C^*-morphism is injective for all C^*-algebras B if and only if C is relatively weakly injective in A (i.e., if there is a c.p. map $V : A \to C^{**}$ with $V(c) = c$ for $c \in C \subseteq A$), because – otherwise – the natural *-epimorphism $C \otimes^{\max} B \to \overline{C \odot B} \subset A \otimes^{\max} B$ is not injective for $B = C^*(F_2)$.

The following Lemma is the *Intersection Lemma* [**438**, lem. 3.9].

Lemma A.24.1. *Let A, B Banach spaces and $N : A \odot B \to \mathbb{R}_+$ a norm on $A \otimes B$ with $N(a \otimes b) \leq \gamma_1 \|a\| \cdot \|b\|$ for some $\gamma_1 < \infty$. Let $A \otimes_N B$ denote the completion of $A \odot B$ w.r.t. N. Furthermore let $X \subset A$ and $Y \subset B$ closed linear*

[15] [**542**] does not say what kind of set theory axioms are precisely needed in case of open subsets of non-separable Banach spaces.

subspaces such that there is net of linear maps $T_\mu\colon B \to Y$ *and* $\gamma_2 < \infty$ *with* $N((\mathrm{id} \otimes T_\mu)(z)) \le \gamma_2 N(z)$ *for* $z \in A \odot B$*, and* $\|T_\mu(y) - y\| \to 0$ *for all* $y \in Y$ *. Then*

$$\overline{X \odot B} \cap \overline{A \odot Y} = \overline{X \odot Y}$$

in the completion of $A \odot B$ *w.r.t.* N*.*

In particular,

$$(C \otimes^\nu B) \cap (A \otimes^\nu Y) = C \otimes^\nu Y$$

for C^**-algebras* A*,* B*,* C^**-norms* $\|\cdot\|_\nu$ *on the algebraic tensor product* $A \odot B$*, any* C^**-subalgebra* $C \subset A$ *and sums* $Y = L + R \subseteq B$ *(respectively intersection* $Y = L \cap R$*) of closed right ideals* R *and closed left ideals* R *of* B*.*

Proof. Let $z \in \overline{X \odot B} \cap \overline{A \odot Y}$ and $\varepsilon > 0$. Define $\delta := \varepsilon/(3 + 2\gamma_2)$. We find $z_1 \in X \odot B$ and $z_2 \in A \odot Y$ with $N(z_j - z) < \delta$ for $j = 1, 2$. Let $a_k \in A$ and $y_k \in Y$ with $z_2 = \sum_k a_k \otimes y_k$. By assumption, there is T_μ with $N(z_2 - (\mathrm{id} \otimes T_\mu)(z_2)) \le \gamma_1 \sum_k \|a_k\| \|T_\mu(y_k) - y_k\| < \delta$. Since $(\mathrm{id} \otimes T_\mu)(z_1) \in X \otimes Y$ and

$$z - (\mathrm{id} \otimes T_\mu)(z_1) = (z - z_2) + (z_2 - (\mathrm{id} \otimes T_\mu)(z_2)) + (\mathrm{id} \otimes T_\mu)(z_2 - z_1)\,,$$

we get $\mathrm{dist}(z, X \odot Y) \le \delta + \delta + \gamma_2(2\delta) < \varepsilon$, – with respect to the semi-norm N on $A \odot B$. □

25. Characterization of closed ideals by inner automorphisms

Lemma A.25.1. *Let* A *a (complex)* C^**-algebra and let* $X \subseteq A_+$ *a hereditary, closed convex cone in* A_+*,* $\exp(-ih) X \exp(ih) \subseteq X$ *for all* $h = h^* \in A$ *with* $\|h\| < \pi$*, then* X *is the positive part* J_+ *of a closed ideal* J *of* A*.*

Proof. `There should be also some reference to text books`
e.g. `Pedersen, Kadison, Dixmier, ???  ...`

It is "easy to see" that the hereditary C^*-subalgebra J of A generated by the linear span of X has the property that X is dense in $J_+ := A_+ \cap J$. Thus, $X = J_+$.

The hereditary C^*-subalgebra J is an ideal of A, because it is invariant under conjugation with unitaries $\exp(ih)$ (and $\exp(-ih)$) from any set $S \subseteq A$ of elements $h = h^*$ that has dense linear span in A, because then the open support projection of the hereditary C^*-subalgebra J is necessarily in the center of A^{**}. (In case of a "real" C^*-algebra A one has here to take as S a suitable set of $k \in A$ with $k^* = -k$ and to apply that $\exp(-k) X \exp(k) \subseteq X$ and to rediscover X.)

Therefore it suffices for a proof to show that the linear subspace $J := (X - X) + i(X - X)$ is a hereditary C^*-subalgebra and that $J_+ = X$ if X is a hereditary sub-cone of A_+ that satisfies $\exp(-ih) X \exp(ih) \subseteq X$ for all $h^* = h \in A$.

We show that a hereditary, closed convex cone $X \subseteq A_+$, has the property $A_+ \cap (X - X) = X$ and that this implies that $X - X$ is closed in A and, therefore, $J := (X - X) + i(X - X)$ is a closed linear subspace of A with the property $J_+ := A_+ \cap J = X$.

Moreover we show that $X = A_+ \cap L_X$ for the subset $L_X \subseteq A$ defined by $L_X := \{a \in A\,;\ a^*a \in X\,\}$ and that L_X is a closed left-ideal of A. It implies that $J = L_X^* \cap L_X$, i.e., that J is a hereditary C^*-subalgebra of A.

If $x_1 - x_2 \geq 0$, $x_2 \in A_+$ and $x_1 \in X$ then $x_1 \geq x_1 - x_2 \geq 0$. It follows $x_1 - x_2 \in X$ because X is hereditary.

Let $L_X := \{a \in A\,;\ a^*a \in X\}$. Then L_X is a closed left ideal of A and $X = A_+ \cap L_X$.

Indeed: Use that $x^2 \leq \|x\|x$ and $(a-b)^*(a-b)+(a+b)^*(a+b) = 2(a^*a+b^*b)$ and that $X + X \subset X$, $2X = X$, X is hereditary and closed. In particular $x \in A_+ \cap L_X$ for all $x \in X$, because $x^2 \in X$ for all $x \in X$ by $x^2 \leq \|x\|x$.

Conversely, if $y \in A_+ \cap L_X$ then $y^2 \in X$. But $\delta(y-\delta)_+ \leq y^2$ for all $\delta \in (0,1)$. It yields that $(y-\delta)_+ \in X$ for each $\delta \in (0,1)$. Since X is closed, it results that $X = A_+ \cap L_X$. In particular $(X - X) + i(X - X) = L_X^* \cap L_X$ is a hereditary C^*-subalgebra of A. □

26. On the Corona Factorization Property (CFP)

Proposition A.26.1. *Let A denote a non-zero simple σ-unital C^*-algebra of real rank zero.*

If $A \otimes \mathbb{K}$ contains an infinite projection and A has the corona factorization property (CFP), then A is purely infinite.

By *definition*, a C^*-algebra A has the **Corona Factorization Property** (CFP), if and only if, every *full* projection P in the multiplier algebra $\mathcal{M}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$ is properly infinite, i.e., $P \oplus_{S,T} P \precsim P$ in $\mathcal{M}(A \otimes \mathbb{K})$ for isometries $S, T \in \mathcal{M}(A \otimes \mathbb{K})$ with $S^*T = 0$.

Here a projection $P \in \mathcal{M}(A \otimes \mathbb{K})$ is *full* in $\mathcal{M}(A \otimes \mathbb{K})$ if the ideal J of $\mathcal{M}(A \otimes \mathbb{K})$ generated by P is equal to $\mathcal{M}(A \otimes \mathbb{K})$.

This is obviously equivalent to the property that a finite Cuntz sum $Q := P \oplus P \oplus \cdots \oplus P \in \mathcal{M}(A \otimes \mathbb{K})$ and an isometry $R \in \mathcal{M}(A \otimes \mathbb{K})$ exists with the property $RR^* \leq Q$.

`How to produce full projections?` By sequences of properly infinite projections in $A \otimes \mathbb{K}$?

Others from old Part in Part B:

We discuss the corona factorization property (CFP) for some classes of σ-unital C^*-algebras of real rank zero.

Does the Elliott conjecture hold for simple, separable, unital, nuclear C^*-algebras A of real rank zero that have no non-trivial lower semi-continuous 2-quasi-trace?

The latter property (no non-trivial l.s.c. 2-quasi-traces) is equivalent to the existence of a properly infinite projection in $A \otimes \mathbb{K}$, and we can pass to some algebra $p(A \otimes M_n)p$ that contains a unital copy of $\mathcal{O}_2$.

A more general open question is:

Has every separable nuclear simple C^*-algebra A of real rank zero the following "corona factorization property"?

(CFP): Every full projection in $P \in \mathcal{M}(A \otimes \mathbb{K})$ is properly infinite.

(Then the projection P is the range $P = TT^*$ of an isometry $T \in \mathcal{M}(A \otimes \mathbb{K})$, because $\mathcal{M}(A \otimes \mathbb{K})$ has trivial K_*-groups.)

(It is equivalent to:
If $p \in \mathcal{M}(A \otimes \mathbb{K})$ is a projection with $1 \precsim (p \oplus p)$ then $1 \precsim p$.)

Every simple purely infinite σ-unital C^*-algebra A has property (CFP) because, – for σ-unital (!) A –, the stable corona $\mathrm{Q}^{\mathrm{s}}(A) := \mathcal{M}(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$ is simple (and then automatically purely infinite), if and only if, A is simple and purely infinite, cf. Corollary 2.2.11(i). It implies that $\mathcal{M}(A \otimes \mathbb{K})$ is (strongly) purely infinite. Since $\mathrm{K}_*(\mathcal{M}(A \otimes \mathbb{K})) = \{0\}$ it follows that each full projection in $\mathcal{M}(A \otimes \mathbb{K})$ is the range of an isometry.

For simple σ-unital $A \otimes \mathbb{K}$ and every non-zero projection $p \in A \otimes \mathbb{K}$, $p(A \otimes \mathbb{K})p \otimes \mathbb{K} \cong A \otimes \mathbb{K}$. Thus we may suppose that A is unital if A is simple and p.i.

An ideal system preserving isomorphism $p(A \otimes \mathbb{K})p \otimes \mathbb{K} \cong A \otimes \mathbb{K}$ exists if A is σ-unital and $A \otimes \mathbb{K}$ contains a projection $p \in A \otimes \mathbb{K}$ that generates $A \otimes \mathbb{K}$ as a closed ideal.

It comes from $P := \delta_\infty(p) \in \mathcal{M}(A \otimes \mathbb{K})$ as $P(A \otimes \mathbb{K})P \cong p(A \otimes \mathbb{K})p \otimes \mathbb{K}$ and the fact that there is an isometry $s \in \mathcal{M}(A \otimes \mathbb{K})$ with $ss^* = P$.

Let $T \in \mathcal{M}(A \otimes \mathbb{K})_+$ a positive contraction with $\|\pi_{A\otimes\mathbb{K}}(T)\| = 1$. Define with matrix units $e_{jk} \in \mathbb{K}$ and identify $c_0(\mathbb{C})$ with $c_0 := C^*(e_{11}, e_{22}, \cdots) \subset \mathbb{K}$ and let $g_n := \sum_{k=1}^{n} 1 \otimes e_{kk} \in 1 \otimes c_0$.

We find an increasing quasi-central approximate unit for $C^*(T) \subseteq \mathcal{M}(A \otimes \mathbb{K})$ in the *convex hull of the approximate unit* consisting of the projections $g_n \in A \otimes \mathbb{K}$. This can be seen from our cf. Lemma B.23.1 or arguments in the proof for [**616**, thm. 3.12.14] and combine them with the arguments for [**616**, cor. 3.12.15,cor. 3.12.16].

To be shown:
We find projections $p_n \in c_0 \subset \mathbb{K}$ with $p_n := \sum_{k=r_n}^{t_n} e_{kk}$ where $t_{n-1} + 1 < r_n \leq t_n$ are in $\mathbb{N}$, $\|(1 \otimes p_n)T(1 \otimes p_n)\| \geq (n-1)/n$, $p_n p_m = 0$ for $m \neq n$, $(1 \otimes p_n)g_n = 0$, and $\|(1 \otimes p_k)T(1 \otimes p_n)\| < 1/n^2$ for $k < n$.

Notice here that for $T \in \mathcal{M}(A \otimes \mathbb{K})_+$ and $q := \sum_{k=1}^{n-1} p_k \in \mathbb{K}$ holds $(1 \otimes q)T = \lim_\ell (1 \otimes q)T1 \in \mathcal{M}(A \otimes \mathbb{K})$ It has to be checked if there exists the p_n!

Notice that $\sum_n 1 \otimes p_n$ converges strictly to a projection $Q \in (1_{\mathcal{M}(A)} \otimes \mathcal{M}(\mathbb{K})) \subseteq \mathcal{M}(A \otimes \mathbb{K})$.

If such $p_1, p_2, \ldots$ exist, then there are contractions $d_n \in A \otimes \mathbb{K}$

(likely such that $d_n^* d_n \leq 1 \otimes e_{nn}$ and $d_n \in (1 \otimes p_n)(A \otimes \mathbb{K})(1 \otimes e_{nn})$)

with $1 \geq d_n^*(1 \otimes p_n)T(1 \otimes p_n)d_n \geq (1 - 2/n)(1 \otimes e_{nn})$. Check Conjecture: $\sum_n (1 \otimes p_n)d_n(1 \otimes e_{nn})$ is strictly convergent to a contraction $S \in \mathcal{M}(A \otimes \mathbb{K})$ with $1 - S^*TS \in A \otimes \mathbb{K}$.

One can find/use a partial isometries $U \in \mathcal{M}(\mathbb{K})$ with $e_{nn}U^* = e_{n,r_n}$, i.e., with $e_{nn}U^* = e_{n,n}e_{n,r_n}$ where $e_{r_n,r_n} \leq p_n$ for all $n \in \mathbb{N}$. The sum $\Gamma := \sum_n (1 \otimes p_n)d_n(1 \otimes e_{n,\ell_n})$ converges strictly in $\mathcal{M}(A \otimes \mathbb{K})$ and defines a contraction. Let $S := \Gamma \circ U$.

Then S is a contraction in $\mathcal{M}(A \otimes \mathbb{K})$ such that $1 - S^*TS \in A \otimes \mathbb{K}$ and this shows that $\mathrm{Q^s}(A)$ is simple if A is σ-unital, simple and purely infinite. See above attempt.

The strong pure infiniteness of $\mathcal{M}(A)$ is known if A is s.p.i. and σ-unital!

More general: (With $\mathbb{K} := \mathbb{K}(\ell_2(\mathbb{N}))$.) If σ-unital A is s.p.i. then $\mathcal{M}(A \otimes \mathbb{K})$ is s.p.i. (It requires to show that $\mathcal{M}(A)$ is s.p.i. if A is σ-unital and A is s.p.i.)

It implies in particular that A has (CFP), because each (non-zero) projection in $\mathcal{M}(A \otimes \mathbb{K}) \setminus (A \otimes \mathbb{K})$ is properly infinite if the stable corona $\mathrm{Q^s}(A)$ is purely infinite, because $\mathrm{K}_0(\mathrm{Q^s}(A)) = 0$.

The stable corona $\mathrm{Q^s}(A)$ is simple if and only if A σ-unital, simple and p.i. if and only if $\mathrm{Q^s}(A)$ is simple and purely infinite, cf. Corollary 2.2.11.

Check:

Are some relations between [**670**, prop. 9.3] (on CFP) and our non-existence of "infinitesimal" sequences??!!

(Is not very likely.)

Corona Factorization Property (CFP):

"Every *full* projection in the multiplier algebra $\mathcal{M}(A \otimes \mathbb{K})$ of $A \otimes \mathbb{K}$ is properly infinite."

(Who has really introduced this definition?)

If $T_1, \ldots, T_n \in \mathcal{M}(A \otimes \mathbb{K})$ exists with $\sum_{k=1}^n T_k^* P T_k = 1$, then P is properly infinite, i.e., there exists $S \in \mathcal{M}(A \otimes \mathbb{K})$ with $1 - S^*PS \in A \otimes \mathbb{K}$.

???? Is Def. of what???

Is it equivalent to the following if A is σ-unital?:

(CFPI) := "Corona-full are properly infinite" defined by:

"Every full projection of $\mathrm{Q^s}(A)$ is properly infinite."

Implies (CFP), because if P is a full projection in $\mathcal{M}(A \otimes \mathbb{K})$ then $\pi_{A \otimes \mathbb{K}}(P)$ is a full projection in $\mathrm{Q^s}(A)$. By (CFPI) there exists a contraction $R \in \mathcal{M}(A \otimes \mathbb{K})$ with $1 - R^*PR \in A \otimes \mathbb{K}$. We can find an isometry $X \in \mathcal{M}(\mathbb{K})$ such that $\|1 - (1 \otimes X)^* R^* PR(1 \otimes X)\| < 1/4$. Thus, there exists an element $Y \in \mathcal{M}(A \otimes \mathbb{K})$ with

$Y^*PY = 1$. Then $T := PY$ is an isometry with $TT^* \leq P$. It follows that P is properly infinite. Now use that $\mathrm{K}_0(\mathcal{M}(A \otimes \mathbb{K})) = \{0\}$ and get that $[P] = [1]$, i.e., there is an isometry $Z \in \mathcal{M}(A \otimes \mathbb{K})$ with $ZZ^* = P$.

(Similar arguments show that each full element of $\mathcal{M}(A \otimes \mathbb{K})_+$ dominates the range of an isometry if this is the case for every full *element* of $\mathrm{Q}^{\mathrm{s}}(A)$.)

Thus, we can show that (CFPI) implies (CFP).

Consider now the property (of ?????):

If $1 \precsim P \oplus P$ *then* $1 \precsim P$ *for projections* $P \in \mathcal{M}(A \otimes \mathbb{K})$.

This property implies (CFP), because a projection P is full in $\mathcal{M}(A \otimes \mathbb{K})$ if there exist $n \in \mathbb{N}$ such that $[1] \leq 2^n[P]$ in $\mathrm{Cu}(\mathcal{M}(A \otimes \mathbb{K}))$.

There is a projection $Q \in \mathrm{Cu}(\mathcal{M}(A\otimes\mathbb{K}))$ with $[Q] = 2^{n-1}[P]$. Thus $1 \precsim Q\oplus Q$. reduces to $[1] \leq [Q] = 2^{n-1}[P]$.

It follows also from (CFP), because $1 \precsim P\oplus P$ implies that P is a full projection in $\mathcal{M}(A \otimes \mathbb{K})$.

More on the question if (CFP) implies (CFPI) (in case that A has real rank zero):

Let $T \in \mathcal{M}(A \otimes \mathbb{K})_+$ a positive contraction, with $\pi(T) = T + (A \otimes \mathbb{K})$ a projection in $\mathrm{Q}^{\mathrm{s}}(A)$ for $\pi := \pi_{A\otimes\mathbb{K}}$.

When does there exist $Y_k := 1\otimes X_k$ that such that $\pi(Y_k)$ is unitary and $Y_1^*TY_2$ is a partial isometry?

If $\pi(T)$ is a full projection in $\mathrm{Q}^{\mathrm{s}}(A)$ the there are elements $S_1, \ldots S_n \in \mathcal{M}(A\otimes\mathbb{K})$ with $1 - \sum_k S_k^*TS_k \in A \otimes \mathbb{K}$

There exists an isometry $I \in 1 \otimes \mathcal{M}(\mathbb{K})$ with $\sum_k I^*S_k^*TS_kI = 1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(K)}$.

Seems that we need a stronger property, e.g. that every full positive contraction T in $\mathcal{M}(A \otimes \mathbb{K})$ with $\pi(T)$ a projection is properly infinite ...

Then $(1 - T)T \in A \otimes \mathbb{K}$.

Let $G := f(T)$ for $f(t) := 1$ for $t \geq 1/4$ and $f(t) := 0$ for $t \leq 1/8$ and $f(t) := 8t - 1$ on $[1/8, 1/4]$. Then $\pi(G) = \pi(T)$, $G(T - 1/4)_+ = (T - 1/4)_+$ and $G \leq 4T$.

If $\pi(T)$ is full in $\mathrm{Q}^{\mathrm{s}}(A)$, then $\pi(sTs^*)$ is full in $\mathrm{Q}^{\mathrm{s}}(A)$.

But it is not clear if sTs^* can be "extended" to a projection in $\mathcal{M}(A\otimes\mathbb{K})$ such that ????

Let $S := (T - 1/2)_+^{1/2} \in \mathcal{M}(A \otimes \mathbb{K})$, and $a_0 \in A_+$, $b_0 \in \mathbb{K}$ strictly positive contractions. Notice $GS = S = SG$. And let $D := \overline{S(A \otimes \mathbb{K})S}$ the hereditary C^*-subalgebra of $A \otimes \mathbb{K}$ generated by the strictly positive element $S(a_0 \otimes b_0)S$ of D.

We find $d_1, \ldots, d_n \in \mathcal{M}(A \otimes \mathbb{K})$ with $\sum_j d_j^*Sd_j = 1$. Thus, $L := (A \otimes \mathbb{K})S$ is not contained in a non-trivial closed ideal of A.

Thus, D a "full" hereditary C^*-subalgebra of $A \otimes \mathbb{K}$. $Gd = d = dG$ for all $d \in D$.

`Next ''green'' to be shown:`

We find a projection $P \in \mathcal{M}(A \otimes \mathbb{K})$ and an element $z \in A \otimes \mathbb{K}$ such that $zz^* = S(a_0 \otimes b_0)S$ and z^*z is a strictly positive element of $P(A \otimes \mathbb{K})P$.

Does there exists an element $Y \in \mathcal{M}(A \otimes \mathbb{K})$ with $YY^* \geq P$ and $Y^*YG = Y^*Y$? Is P full in $\mathcal{M}(A \otimes \mathbb{K})$?

The true question is:
Can we find for each full projection Q in $\mathrm{Q}^{\mathrm{s}}(A)$ a full projection $P \in \mathcal{M}(A \otimes \mathbb{K})$ such that $\pi(P) \precsim Q$.

`OK. Above seems not to work.  !!!`

Other approach:

Let $d_1, \ldots, d_n \in \mathcal{M}(A \otimes \mathbb{K})$ (w.l.o.g.) with $\sum_k \pi(d_k)^* \pi(T) \pi(d_k) = 1$ and $\| \sum_k d_k^* d_k \| \leq 1$.

Here "w.l.o.g.", because we can replace the d_k by $e_k := T^{1/2} d_k S$, with some suitable isometry $S \in \mathcal{M}(A \otimes \mathbb{K})$.

Then $\sum_k \pi(e_k)^* \pi_(e_k) = 1$, and we can lift the column $[\pi(e_1), \ldots, \pi(e_n)]^\top$, which defines a partial isometry in $M_n(\mathrm{Q}^{\mathrm{s}}(A))$, of norm $= 1$ to a contraction in $\mathcal{M}(A \otimes \mathbb{K})$.

Then there exists isometry $S \in 1_{\mathcal{M}(A)} \otimes \mathcal{M}(\mathbb{K})$ with $\|1 - \sum_k S^* d_k^* T d_k S\| < 1/4$ and $1 - \sum_k S^* d_k^* T d_k S \in A \otimes \mathbb{K}$.

Let $D := (\sum_k S^* d_k^* T d_k S)^{-1/2}$. Then $1 = \sum_k R_k^* T R_k$ in $\mathcal{M}(A \otimes \mathbb{K})$ for $R_k := d_k S D$. ...

Suppose (!) now that $T \in \mathcal{M}(A \otimes \mathbb{K})$ itself is a full projection and that each full projection in $\mathrm{Q}^{\mathrm{s}}(A)$ is properly infinite.

Then we can take $n = 1$ and get isometry $R \in \mathcal{M}(A \otimes \mathbb{K})$ with $R^*TR = 1$, i.e., $T \geq RR^*$, $R^*R = 1$.

Thus, T is full and infinite, $R\mathcal{O}_2 R^* \in T\mathcal{M}(A \otimes \mathbb{K})T$, $RsTs^*R^* \leq T$ and $RtTt^*R^* \leq T$. Thus $[T] + [T] \leq [T] \leq [1]$, $[1] \leq [T]$.

Can we lift each projection in $\mathrm{Q}^{\mathrm{s}}(A)$ to a projection in $\mathcal{M}(A \otimes \mathbb{K})$?

Is not possible in general, because the elements of the K_0-groups are represented by projections and $\mathrm{K}_0(\mathcal{M}(A \otimes \mathbb{K})) = 0$.

Then we can consider the case ?????

If every closed (two-sided) ideal of A has the Corona Factorization Property, then we say that A has the *strong Corona Factorization Property*.

Or is this the "strong" (CFP)? : See the green text further below!

It is known that A has property (CFP) if $A' \cap A_\omega$ does not have a character, cf. [**467**, thm. 4.2]. But the converse implication is wrong:

The example [**467**, ???] is a simple purely infinite exact unital separable C^*-algebra A that has property (CFP) but where $A' \cap A_\infty$ has a character.

Where it is shown that $A' \cap A_\infty$ has a character for A in example ????

Thus the opposite direction is wrong.

Give citation for ''no character'' on $A' \cap A_\infty$ of example. Likely in [467]

It follows from [**467**, thm. 4.2] that for every σ-unital C^*-algebra A the algebra $A \otimes \mathcal{Z}$ has property (CFP), because of $\mathcal{Z} \otimes \mathcal{Z} \otimes \cdots \cong \mathcal{Z}$.

This is because one can show that each separable C^*-subalgebra of $\mathcal{M}(\mathbb{K} \otimes A \otimes \mathcal{Z})$ commutes modulo $\mathbb{K} \otimes A \otimes \mathcal{Z}$ with a unital copy of $\mathcal{Z}$.

Check now outlined proof
of property (CFP) for $A \otimes \mathcal{Z}$!

The preparations for [**467**, thm. 4.2] and the many needed technical facts from other papers make it useful to give a more elementary proof of property (CFP) for the algebras $A \otimes \mathcal{Z}$.

A less engaged proof can be based on the property $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$ and that there is a unital C^*-morphism $\eta \colon \mathcal{Z} \to \mathcal{E}(\mathcal{Z}, \mathcal{Z}) \subseteq \mathrm{C}([0,1], \mathcal{Z} \otimes \mathcal{Z})$ with $\pi_0 \circ \eta(a) = 1 \otimes a$ and $\pi_1 \circ \eta(a) = a \otimes 1$.

Thus, in the stable corona each "full" projection P is also properly infinite, because – by the above mentioned property – the elements $t_1, \ldots, t_n \in \mathcal{M}(\mathbb{K} \otimes A \otimes \mathcal{Z})$ with $t_1^* P t_1 + \ldots + t_n^* P t_n = 1$ (calculated modulo $\mathbb{K} \otimes A \otimes \mathcal{Z}$) and a unital copy of $\mathcal{O}_2$ generate a separable C^*-subalgebra of $\mathcal{M}(\mathbb{K} \otimes A \otimes \mathcal{Z})$ that commutes modulo $\mathbb{K} \otimes A \otimes \mathcal{Z}$ with a unital copy of $\mathcal{Z}$.

Let $B := C^*(P, t_1, \ldots, t_n, s_1, s_2; R)$ the *free* C^*-algebra – with relations R given by $s_1^* s_1 = s_2^* s_2 = s_1 s_1^* + s_2 s_2^* = 1$, $P^* = P = P^2$ and $\sum_k t_k^* P t_k = 1$, $\|t_k\| \leq 1$ –. Then one can see that $1 \otimes P$ is in the C^*-algebra tensor product $\mathcal{Z} \otimes C^*(P, t_1, \ldots, t_n, s_1, s_2; R)$ a properly infinite full projection:

$(n+1)[1] = [1] \leq n[P] \leq n[1] = [1]$

After $\mathcal{Z}$-tensoring this becomes $[1] = [P]$ in the Cuntz semigroup $W(\mathcal{Z} \otimes C^*(P, t_1, \ldots, t_n, s_1, s_2; R))$ and carries then over to the stable corona.

Look to the explicit formula in the green below given calculation.

What about replacing $\mathcal{Z}$ by infinite tensor products of $A(n,2)$ with fixed $n \geq 3$, where $A(n,2) := C^*(a_1, \ldots a_n, b_1, \ldots b_n, R)$ with relations (R) given by:

$$(R) := \{ \sum_{k=1}^{n} a_k^* a_k = 1,\ b_j^* a_j = 0,\ b_j^* b_j = a_j^* a_j,\ j = 1, \ldots, n \} .$$

The $A(n,p)$ are similar defined and are contained ???? in suitable tensor products of $A(n,2)$:

Instead $a_1, \ldots a_n, b_1, \ldots b_n$ we consider $a_{k\ell}$ ($k = 1, \ldots, n$, $\ell = 1, \ldots, p$) with $\sum_{k=1}^n a_{k,1}^* a_{k,1} = 1$, $a_{k,\ell}^* a_{k,m} = \delta_{\ell,m} a_{k,1}^* a_{k,1}$.

Which elements of $\mathcal{O}_\infty \otimes A(n,p)$ are full in $\mathcal{O}_\infty \otimes A(n,p)$???

In particular, those of form $1 \otimes X$ with X looking how ???

$\sum_{\ell=1} t_\ell^* P t_\ell = 1$ Gets only elements Y_k with

$Y_k^*(P \otimes 1)Y_k = 1 \otimes a_{k,1}^* a_{k,1}$. (Does not work so simple as it was hoped.)

Question:

Can we prove $\mathcal{Z}$-absorption for separable simple unital *nuclear* A if A satisfies (CFP)?

There exist a separable simple *exact* unital C^*-algebra A that is p.i. (thus $\mathrm{Q}^s(A) := \mathcal{M}(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$ is simple and s.p.i., ... give citation for simplicity and p.i. of $\mathrm{Q}^s(A)$!!!), but $A' \cap A_\omega$ has a character:

The algebra $A := C^*(F_2) \otimes \mathcal{R}$ is an example with is property.

Give citation for the character on $A' \cap A_\omega$ in case $A := C^*(F_2) \otimes \mathcal{R}$!!!

Where is the algebra $\mathcal{R}$ defined ?

It follows (from which conditions ?) that each full projection in $\mathcal{M}(A \otimes \mathbb{K})$ is the range of an isometry in $\mathcal{M}(A \otimes \mathbb{K})$. Thus, this algebra A has the (CFP) but $A' \cap A_\infty$ has a character.

Has (at least) $A' \cap A_\infty$ a character if A is a simple unital separable and *nuclear* C^*-algebra that does not satisfy (CFP)?

It is known that separable unital A has (CFP) if $A' \cap A_\infty$ has no character, see: [**467**, thm. 4.3].

But it is not known if stably infinite simple A with (CFP) is infinite ?????

Does (CFP) of simple separable A imply that $A' \cap A_\infty$ has no character?

Answer: No! Counterexample with exact A is in [**467**]. (But this counterexample is not nuclear!)

What happens for exact simple separable unital p.i. A? Has A (CFP) ?? (Answer: Yes). Notice that simple p.i. A are s.p.i. !!!

Proposition **??** says that $\mathcal{M}(A \otimes \mathbb{K})$ s.p.i. if A is a σ-unital s.p.i. C^*-algebra. Thus, s.p.i. σ-unital A have property (CFP).

Part (2) of Theorem [**499**, thm.3.1] says that property (CFP) for separable B passes to non-zero quotients.

This shows that P is properly infinite modulo $\mathbb{K} \otimes A \otimes \mathcal{Z}$.

By Remark 5.10.3, this implies that $P = TT^*$ for some isometry $T \in \mathcal{M}(\mathbb{K} \otimes A \otimes \mathcal{Z})$.

(Reference? What about converse in simple nuclear case?)

(Here "full" means equivalently that a finite Cuntz sum $P \oplus P \oplus \ldots \oplus P$ of P in $\mathcal{M}(A \otimes \mathbb{K})$ majorize 1, which implies that this sum is properly infinite.)

`Open question:`

Are all simple (separable, nuclear) stably infinite C^*-algebras of real rank zero automatically purely infinite?

Long standing question? Compare Chp. 2.!

By [**581**, cor. 5.16], a separable simple C^*-algebra of real rank zero with the Corona Factorization Property (CFP) is either stably finite or purely infinite.

This was also proved in an unpublished paper by S. Zhang, who didn't explicitly introduce/mention the (CFP). (He simply considered something like it – or something equivalent to it – for granted).

S. Zhang: (Reference ??? unpublished)

If A is a simple C^*-algebra of real rank zero with Property (CFP), then A is either stably finite or purely infinite.

(One can pass to the case of σ-unital simple C^*-algebras, by passage to suitable separable subalgebras to drop the separability assumption by passing to a suitable separable C^*-subalgebra.)

The example $A_0 := C^*_{red}(F_2) \otimes \mathcal{R}$ is simple, exact and p.i. (thus is s.p.i.). Simple σ-unital purely infinite C^*-algebras have the (CFP) (cite for proof?). But $F(A_0)$ has a character. Thus (CFP) does not imply that $F(A)$ has no character if A is exact, simple and p.i.

A result for non-simple σ-unital C^*-algebra A of real rank zero says:

[**581**, cor. 5.15]: Let A be a σ-unital C^*-algebra of real rank zero, and let p be a projection in A such that the m-fold direct sum $p \oplus p \oplus \ldots \oplus p$ is properly infinite (in $M_m(A)$) for some natural number m. Then p itself is properly infinite if one of the following two conditions (i) or (ii) below hold:

(i) Every ideal in A has the (CFP)
(ii) A has the (CFP) and p is a full projection in A.

(The Part (i) should immediately follow from Part (ii).)

It comes from:

[**581**, thm. 5.13]: Let A be a σ-unital C^*-algebra of real rank zero. Then $V(A)$ has the strong Corona Factorization Property (for monoids), if and only if, every ideal I in A has the Corona Factorization Property (for C^*-algebras).

Is here A suppose to be stable? [**581**, lem. 5.6]: Let A be a σ-unital C^*-algebra, let P be a properly infinite, full projection in $\mathcal{M}(A)$ and let $p \leq P$ be a projection in A. Then $P - p$ is properly infinite and full in $\mathcal{M}(A)$.

[**581**, lem. 5.7]: Let A be a C^*-algebra, let $\{p_n\}$ and $\{q_n\}$ be sequences of pairwise orthogonal projections in A such that the sums $P = \sum_{n=1}^{\infty} p_n$ and $Q = \sum_{n=1}^{\infty} q_n$ are strictly convergent in the multiplier algebra $\mathcal{M}(A)$, and hence define projections P and Q in $\mathcal{M}(A)$.

(i) Suppose that there are sequences $\{k_n\}$ and $\{\ell_n\}$ of natural numbers such that $1 \le k_1 < \ell_1 < k_2 < \ell_2 < k_3 < \cdots$, and such that

$$[p_n] \le [q_{k_n}] + [q_{k_n+1}] + \ldots + [q_{l_n}]$$

for all $n \in \mathbb{N}$. Then $P \precsim Q$ in $\mathcal{M}(A)$.

(ii) If $P \precsim Q$ in $\mathcal{M}(A)$, then for every natural number k there exists a natural number ℓ such that

$$[p_1] + [p_2] + \ldots + [p_k] \le [q_1] + [q_2] + \ldots + [q_\ell]$$

in $V(A)$.

Proof of (i): For each n, let $s_n \in A$ be a partial isometry with

$$s_n^* s_n = p_n\,, \quad s_n s_n^* \le q_{k_n} + q_{k_n+1} + \ldots + q_{l_n}\,.$$

As the sums $\sum p_n$ and $\sum q_n$ are strictly convergent, it follows that the sum $S := \sum_{n=1}^{\infty} s_n$ is strictly convergent in $\mathcal{M}(A)$. Hence $P = S^*S \sim SS^* \le Q$.

From [**467**, thm. 4.3] (When central sequence algebras have a character.) we get:

Theorem 4.3:

Let A be a unital separable C^*-algebra such that the central sequence algebra $F(A)$ has no characters. Then A has the strong Corona Factorization Property.

D. Kucerovsky and P.W. Ng show in [**499**, thm. 3.1] that the quotient of any separable C^*-algebra with the Corona Factorization Property again has the Corona Factorization Property.

"It follows from this result that the quotient of any separable C^*-algebra with the strong Corona Factorization Property again has the strong Corona Factorization Property."

It was shown in [**670**, prop. 6.3] that $A \otimes_{\max} (\bigotimes_{\max} D)$ has the strong Corona Factorization Property.

(For D with what kind of properties??? Likely D has to be unital and without characters???)

The corollary [**581**, cor. 5.16]: says that a separable simple C^*-algebra of real rank zero with the Corona Factorization Property is either stably finite or purely infinite.

(It shows that we could add the (CFP) to the assumptions of **(Q1)** then the answer becomes positive. Moreover (Q1.implies.CFP for A), (Q1.implies.pi of A), (Q1.implies.$F(A)$ has no character) and (Q1.implies. ... are equivalent positive answers.)

(It was also proved in an unpublished paper by S. Zhang, but without explicit realizing / recognition that something like the (CFP) is needed.)

(One can drop the separability assumption by passing to suitable separable C^*-subalgebras.)

There exist exact simple separable unital C^*-algebras, e.g. $C^*_\rho(F_2)\otimes\mathcal{R}$, that are purely infinite but have not the (CFP). (Because $F(C^*_\rho(F_2)\otimes\mathcal{R})$ has a character.)

If A is separable simple and σ-unital and $F(A)$ has not a character then A has (CFP).

(Recall: (CFP) = "Corona factorization property")

`END of (CFP) Discussion !`

27. On dense algebraic Ideals

The following definition is perhaps not identical with the definition given by Pedersen in his book [**616**] on C*-algebras A ... He requires (!) that his dense ideal is "hereditary".

We consider later the set $A_{0,+}$ of all ε-cut downs of positive elements in A_+ and show that this set is contained in any dense algebraic ideal J of A. We do not know if the Pedersen ideal is identical with the algebraic ideal A_0 of A generated by all cut-down elements $(a-\varepsilon)_+$ for $a\in A_+$.

The ideal A_0 is dense in A and is contained in any dense algebraic ideal of A.

Since the Pedersen ideal of A contains all the cut-down elements $(a-\varepsilon)_+$ (for $a\in A_+$ and $\varepsilon>0$), it is likely that they are the same: Find a proof ...

Consider first the case $A:=C_0(0,\infty)$ ... ???

But ?????

Are semi-finite (2-)quasi-traces on the Pedersen ideal l.s.c.?

Thm. 1.3.3.: (in Pedersen Book) The set A_+ is a closed real cone in A_{sa}, and $x\in A_+$, if and only if, $x=y^*y$ for some $y\in A$.

Thm. 1.3.5.: (in Pedersen Book) If $0\le x\le y$ then $a^*xa\le a^*ya$ for all $a\in A$ and $\|x\|\le\|y\|$.

Prop. 1.4.5.: (in Pedersen Book) Let $x\in A$ and $x^*x\le a\in A_+$. If $0<\alpha<1/2$, then there exists $u\in A$ with $\|u\|\le\|a^{(1/2)-\alpha}\|$ such that $x=ua^\alpha$.

(The element $u\in A$ is not necessarily unitary.)

Sec. 5.6.: The minimal dense ideal. (But Pedersen considered only "hereditary" algebraic ideals. The true minimal dense ideal is the ideal ideal

Thm. 5.6.1.: (in Pedersen Book, p.175) !!!!!!! For each C*-algebra A there is a dense hereditary ideal $K(A)$, which is minimal among all dense ideals. (of A).

My question: Is it identical with the algebraic ideal of A that is generated by the set of all ε-cut-downs $(a-\varepsilon)_+$? (Here $a\in A_+$ and $\varepsilon\in(0,\|a\|)$.)

Proof of Pedersen Thm. 5.6.1: Let $K((0,\infty)):=$ set of all continuous function on $(0,\infty)$ with compact support (inside $(0,\infty)$). Define

$$K(A)_0:=\{f(x):x\in A_+,f\in K((0,\infty))_+\}.$$

Let

$$K(A)_+ := \{x \in A_+ : x \leq \sum_{k=1}^{n} x_k,\ x_k \in K(A)_0\}$$

so that $K(A)_+$ is the smallest hereditary cone in A_+ containing $K_0(A)$.

If $K(A)$ denotes the linear span of $K(A)_+$ we conclude as in (Pedersen Book) Proposition(5.1.3) that $K(A)$ is a hereditary (!!!) *-algebra with $(K(A))_+ = K(A)_+$.

Since $u^*f(x)u = f(u^*xu)$ for any unitary $u \in \widetilde{A}$ we have $u^*K(A)_0u = K(A)_0$, and (as in subsection 5.2.1 of Pedersen book) this implies that $K(A)$ is an ideal:

If u is a unitary in $\widetilde{A} = A + \mathbb{C}1$ then

$$4u^*x = 4(x^{1/2}u)^*x^{1/2} = \sum_{k=0}^{3} i^k(1 + i^k u)^* x(1 + i^k u)$$

(the polarization identity), ... ?

Let (f_n) be a sequence in $K(A)$ ...

Now use that each element of $\widetilde{A}$ is a combination of unitaries ...

28. Dimension-Functions in Sense of J.Cuntz

The following definition of Dimension functions (is quoted to J.Cuntz).

Definition A.28.1. Let $D\colon X_A \to ???$ be a (non-negative) semi-finite *Dimension function* on $X_A := \bigcup_n M_n(\mathrm{Ped}(A))$ "in sense of J.Cuntz".

Claim: $D(a) = D(a^*a)$, $D(a) \leq D(b)$ if $a \precsim b$.

$$D^*(a) := \sup_{\varepsilon>0} D((a-\varepsilon)_+)$$

Then

- α $0 \leq D(a) \leq D^*(a)$ for all $a \in X$,
- β D^* is again a Dimension function,
- γ $D - D^*$ is a Dimension function,
- δ If D is a Dimension function, then then the kernel $\mathrm{Ker}(D)$ satisfies

$$\mathrm{Ker}(D) = (\bigcup_n M_n(\mathrm{Ped}(A))) \cap (\bigcup_n M_n(J))$$

  for some (closed ?) ideal J of A.
- ε ? $(D - D^*)^* = D^* - D^* = 0$?

Has it following property?: $D(a) \neq 0$ implies $D(b) \neq 0$ for some b.

Are semi-finite (2-) quasi traces on the "Pedersen ideal" l.s.c.?

Are semi-finite Dimension functions (in the sense of J.Cuntz) "lower semi-continuous"?

APPENDIX B

Exact C*-algebras and examples

We discuss in this Appendix B some facts related e.g. to the following questions:

(0) Passage to suitable separable subalgebras that preserve important properties of the algebra.

(1) Monotone maps between partially ordered sets.

(2) Some properties of exact C^*-algebras.

(3) Example of a nuclear and approximately inner c.p. map that is not residually nuclear, and its relation to the possible existence of group vN-algebras that are not weakly exact. (Is there an example of Ozawa of vN-alg. that is not "weakly exact"?)

(4) Example of a (simple?) unital nuclear separable C^*-algebra A and a unital *-monomorphisms $\iota\colon A \hookrightarrow \mathcal{O}_2$ of A into the Cuntz algebra $\mathcal{O}_2$ such that there is no conditional expectation from $\mathcal{O}_2$ onto $\iota(A)$.

It shows that there exist unital *-monomorphisms ι_1 and ι_2 of a (simple?) unital nuclear separable C^*-algebra A with the property that if we tensor the ι_k with $\mathrm{id}_{\mathcal{O}_2}$, then the unital *-monomorphisms $\mathrm{id}_{\mathcal{O}_2} \otimes \iota_k \colon \mathcal{O}_2 \otimes A \to \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ of the nuclear C^*-algebras $\mathcal{O}_2 \otimes A$ are not unitarily equivalent, – even not equivalent if we change it by an automorphism of $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ and an automorphism of $\mathcal{O}_2 \otimes A$, because the existence/non-existence of conditional expectations onto the image remains unchanged if we tensor with $\mathcal{O}_2$ and apply any of this operations.)

(5) Some open questions concerning p.i. algebras.

(6) Examples of D where $\mathbb{O}(\mathrm{Prim}(D))$ is not the projective limit of lattices $\mathbb{O}(X_n)$ of Hausdorff l.c. spaces X_n with *lower s.c. and monotone upper s.c.* maps from $\mathbb{O}(X_{n+1})$ into $\mathbb{O}(X_n)$.

(7) A family of pi-sun C^*-algebras that exhausts the UCT-classes up to KK-equivalence.

(8) This is more a *conjecture* than a given proof:

If the "exponential length" of all unitaries in $\mathcal{U}_0(A)$ in a unital A is bounded by some constant $\gamma \in (0, \infty)$ then $\mathcal{U}_0(A)$ is uniformly contractible, i.e., there exists an (uniformly) continuous map

$$\psi \colon \mathcal{U}_0(A) \times [0,1] \to \mathcal{U}_0(A)$$

such that $\psi((u,0)) = 1$ and $\psi((u,1)) = u$

(The proof uses [**553**, thm. 3.2"], because

$$\phi(u) := \{(h_1,\ldots,h_n)\in A^n\,;\; h_k^* = -h_k\,,\; u=\exp(h_1)\cdot\ldots\cdot\exp(h_n)\,\}$$

is an l.s.c. carrier $\phi\colon \mathcal{U}_0(A)\mapsto \mathcal{F}(A^n)$. if $\phi(u)$ is non-empty on each $u\in\mathcal{U}_0(A)$.

`The l.s.c.-property of` ϕ `has to be checked again !!!`

Open and at the same time closed subsets of complete metric spaces are complete metric spaces, hence are para-compact.

We use this to get an alternative proof of the uniform contractibility of $\mathcal{U}(\mathcal{M}(A))$ for all σ-unital stable C^*-algebras A. (NOW STILL OPEN QUESTION !!!).

1. Operations on Multiplier algebras (1)

List of properties of Multiplier algebras:

A σ-unital (and $e\in A_+$ strictly positive contraction):

If $e\in B\subseteq A$ then the natural C^*-morphism $h\colon B\to A$ given by the inclusion map defines defines a *-monomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(A)$. The image consists of all $T\in\mathcal{M}(A)$ with $Te, eT\in B$.

For each separable C^*-subalgebra C of $\mathcal{M}(A)$ exists separable $B\subseteq A$ with $e\in B$, and $eC\cup Ce\in B$.
In particular $C\subseteq\mathcal{M}(B)\subseteq A$.

If $J\subseteq A$ is closed ideal then A/J has the strictly positive element $f:=\pi_J(e)$.

If $D\subseteq\mathcal{M}(A/J)$ is a separable C^*-subalgebra, then there exists a separable C^*-subalgebra B of A such that $e\in B$ and $Df\cup fD\subseteq\pi_J(B)$. Thus, $D\subseteq\mathcal{M}(\pi_J(B))\subseteq\mathcal{M}(A/J)$. Then the natural epimorphism $\pi_J|B$ from B onto $\pi_J(B)$ defines an epimorphism from $\mathcal{M}(B)\subseteq\mathcal{M}(A)$ onto $\mathcal{M}(\pi_J(B))$, and $D\subseteq\mathcal{M}(\pi_J(B))\subseteq\mathcal{M}(A/J)$. In particular, $D\subseteq\mathcal{M}(\pi_J)(\mathcal{M}(A))$. Thus, $\mathcal{M}(\pi_J)\colon\mathcal{M}(A)\to\mathcal{M}(A/J)$ is surjective.

`Compare with Remark B.1.1 !!!`

REMARK B.1.1 (Non-commutative Tietze extension). Let A a σ-unital C^*-algebra, i.e., there exists a an element $e\in A_+$ with $\|e\|=1$ that is a strictly positive element of A.

Then for every closed ideal J of A the natural (on bounded parts strictly continuous) C^*-morphism $\mathcal{M}(\pi_J)\colon\mathcal{M}(A)\to\mathcal{M}(A/J)$ is surjective.

The kernel of $\mathcal{M}(\pi_J)$ is $\mathcal{N}(A,J) := \{T\in\mathcal{M}(A)\,;\; Te, eT\in J\}$.

In case where A is separable, this is [**616**, prop. 3.12.10].

If A is not separable and $e\in A_+$ as strictly positive element with $\|e\|=1$ then, for every unital separable C^*-subalgebra $B\subseteq\mathcal{M}(A/J)$ there exists a separable C^*-subalgebra $E\subseteq A$ with $e\in E$ and $BD\cup DB\subseteq D$ for $D:=\pi_J(E)\cong E/(J\cap E)$.

The strictly continuous extension $\mathcal{M}(\pi_{J\cap E})\colon\mathcal{M}(E)\to\mathcal{M}(E/(J\cap E))$ of the epimorphism $\pi_{J\cap E}\colon E\to E/(J\cap E)\cong D$ is surjective by [**616**, prop. 3.12.10].

The algebra B is contained in $\mathcal{M}(D) \subseteq \mathcal{M}(A/J)$ – if we naturally identify $\mathcal{M}(D)$ with the two-sided multipliers of D in $\mathcal{M}(A/J)$. In the same way $\mathcal{M}(E)$ is naturally isomorphic to algebra the two-sided multipliers of E in $\mathcal{M}(A)$. The strictly continuous extension $\mathcal{M}(\pi_J)\colon \mathcal{M}(A) \to \mathcal{M}(A/J)$ of π_J maps $\mathcal{M}(E)$ into $\mathcal{M}(D)$. It becomes there under natural identification of $E/(J \cap E)$ with D the same as the natural strictly continuous unital C^*-morphism $\mathcal{M}(\pi_{E\cap J})$ from $\mathcal{M}(E)$ into $\mathcal{M}(E/(J \cap E))$.

Thus, $B \subseteq \mathcal{M}(A/J)$ is contained in the image of $\mathcal{M}(\pi_J)\colon \mathcal{M}(A) \to \mathcal{M}(A/J)$.

Above has to be discussed / reformulated

2. Actions and monotone maps

The Chapter 0 of [**321**] could be a reference for this section, but all results mentioned here are elementary exercises for the reader. Recall that a set Y with transitive, reflexive and antisymmetric relation $x \leq y$ is a *partially ordered set* (poset). We need only very special posets, namely those that are order isomorphic or order anti-isomorphic to the lattice of the ideals (or: ideal lattice) of a ring.

Obviously, the family $\mathbb{O}(X)$ of open subsets of a T_0 space X and the family $\mathbb{F}(X)$ of closed subsets of X, and the set X itself are all partially ordered sets with the inclusions $U \subseteq V$, $F \subseteq G$, and $x \leq y$ if and only if $x \in \overline{\{y\}}$.

The order reversing isomorphisms $F \mapsto X \setminus F$ and $U \mapsto X \setminus U$ show that $\mathbb{F}(X)$ is just the opposite ordered space of $\mathbb{O}(X)$.

Subsets Z of $\mathbb{O}(X)$ or $\mathbb{F}(X)$ have unique $\inf Z$ a and $\sup Z$, i.e., they are complete lattices. Moreover $\mathbb{O}(X)$ is a *Heyting algebra* (also called *frame*), because

$$W \cap \bigcup_\gamma U_\gamma = \bigcup_\gamma W \cap U_\gamma \,.$$

The infimum $\bigwedge\{U_\gamma\}$ is the interior $(\bigcap_\gamma U_\gamma)^\circ$ of the intersection of a family $\{U_\gamma\} \subseteq \mathbb{O}(X)$. The supremum in $\mathbb{F}(X)$ of a family of closed subsets is the closure of its union.

A closed subset F of a T_0 space X is *prime* if it is not the union of two closed subsets $F_1, F_2 \subseteq F$, that are both different from F. Equivalently this means: If $F \subseteq Z_1 \cup Z_2$, then $F \subseteq \overline{Z_1}$ or $F \subseteq \overline{Z_2}$. A T_0 spaces X is **sober** (also called point-wise complete, or *point-complete*) if each prime closed subset $F \subseteq X$ is the closure of a point of X.

For any T_0 space X there is a unique (up to natural isomorphisms) *sober* space X^c and an embedding $\eta\colon X \to X^c$ such that $U \in \mathbb{O}(X^c) \mapsto \eta^{-1}(U) \in \mathbb{O}(X)$ is a lattice isomorphism. X^c is called the **sobrification** of X.

In particular, each *sober* T_0 space X is completely determined likewise by each of the lattices $\mathbb{O}(X)$ or $\mathbb{F}(X)$.

The sobrification $\mathrm{Prim}(A)^c$ of the space of primitive ideals $\mathrm{Prim}(A)$ is naturally isomorphic to $\mathrm{prime}(A)$ (the space of prime ideals of A) if A is a C^*-algebra.

If A is a C^*-algebra, then there are natural order isomorphisms $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(X))$ and order reversing isomorphism from the lattice of closed ideals $\mathcal{I}(A)$ onto onto $\mathbb{F}(A)$.

$\mathcal{I}(A)$, $\mathbb{O}(X)$, $\mathbb{F}(X)$ `as posets`

lower s.c. and upper s.c. maps

case $\mathcal{I}(A)$ (lattice of closed sets of

notation $\uparrow X$, $\uparrow t$, $\downarrow X$, $\downarrow t$

`Galois connections`

Definition B.2.1. Let X and Y posets and $\phi\colon Y \to X$ and $\psi\colon X \to Y$ maps that satisfy:

(i) ϕ and ψ are monotone (order preserving), and
(ii) the relations $\phi(y) \geq x$ and $y \geq \psi(x)$ are equivalent for all $(x, y) \in X \times Y$.

Then (ϕ, ψ) is called **Galois connection** or *adjunction* between X and Y.

The map ϕ is the *upper adjoint* and ψ is the *lower adjoint*.

Let $\mathcal{C} \subseteq \mathrm{CP}(A, B)$ a matricially operator-convex cone. Consider the lower semi-continuous action $\Psi_{\mathcal{C}}\colon \mathcal{I}(B) \to \mathcal{I}(A)$ and the upper semi-continuous action $\Phi_{\mathcal{C}}\colon \mathcal{I}(A) \to \mathcal{I}(B)$ as defined in Definition **??**. Then $(\Psi_{\mathcal{C}}, \Phi_{\mathcal{C}})$ is a Galois connection, and $\Psi_{\mathcal{C}}$ is the upper adjoint and $\Phi_{\mathcal{C}}$ is the lower adjoint of the connection.

The following proposition equips us with the in Chapters 3 and 6 needed observation on Galois connections.

Proposition B.2.2. *Let X and Y partially ordered sets and $\phi\colon Y \to X$ and $\psi\colon X \to Y$ monotone (increasing) maps.*

(i) *The formula $\psi(x) := \inf \phi^{-1}(\uparrow x)$ for $x \in X$ defines a lower adjoint of ϕ (such that (ϕ, ψ) is a Galois connection), if $\phi(\inf Z) = \inf \phi(Z)$ so far as $\inf Z$ exists in X, and Y is a complete lattice, or Y is a complete semi-lattice and $\phi\colon Y \to X$ is* co-final *(i.e., for all $x \in X$ there is $y \in Y$ with $x \leq \phi(y)$ — or equivalently: $\phi^{-1}(\uparrow x) \neq \emptyset$).*
(ii) *If (ϕ, ψ) is a Galois connection, then $\psi(x) := \min \phi^{-1}(\uparrow x)$ for all $x \in X$ and $\phi(y) := \max \psi^{-1}(\downarrow y)$ for all $y \in Y$.*
   In particular, they determine each other.
(iii) *If (ϕ, ψ) is a Galois connection, then the upper adjoint ϕ is* inf*-preserving and the lower adjoint ψ is* sup*-preserving.*
(iv) *`compare products with`* id
(v) *`when` $\psi \circ \phi$ `or` $\psi \circ \phi$ `is identity?`*

`definitions and uniqueness of adjoints`

uniqueness of the upper and lower Galois adjoints

in particular, the upper and the lower map of a Galois connection determine each other uniquely.

A reference could be [**321**, chp. 0].

3. Topological actions

Let $\mathcal{L}$ denote a set (or a lattice), X and Y topological spaces, $\Psi_1\colon \mathcal{L} \to \mathbb{O}(X)$ and $\Psi_2\colon \mathcal{L} \to \mathbb{O}(Y)$ map (increasing if $\mathcal{L}$ is a lattice). We say that an increasing map $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ is **fitting** (or **Ψ_1-Φ_2 compatible**) if $\Psi(\Psi_2(\ell)) \supset \Psi_1(\ell)$ for $\ell \in \mathcal{L}$.

Recall that a map $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ is is *lower semi-continuous* if the image $\Psi(V)$ of the interior $V := (\bigcap_\alpha V_\alpha)^\circ$ of the set $\bigcap_\alpha V_\alpha$ is the interior of $\bigcap_\alpha \Psi(V_\alpha)$, for each family $\{V_\alpha\}$ of open subsets $V_\alpha \subseteq Y$. Lower s.c. actions Ψ are increasing.

Lemma B.3.1. *Let $\mathcal{L}$ denote a set (or a lattice), X and Y topological spaces, $\Psi_1\colon \mathcal{L} \to \mathbb{O}(X)$ and $\Psi_2\colon \mathcal{L} \to \mathbb{O}(Y)$ maps (increasing if $\mathcal{L}$ is a lattice).*

(i) *Let S and A sets and $(\psi,\alpha) \in S \times A \mapsto W_{\psi,\alpha} \in \mathbb{O}(X)$ a map. Then*

$$\left(\bigcap_\alpha \left(\bigcap_\psi W_{\psi,\alpha}\right)^\circ\right)^\circ = \left(\bigcap_\psi \left(\bigcap_\alpha W_{\psi,\alpha}\right)^\circ\right)^\circ .$$

(ii) *If $S \neq \emptyset$ is a set of lower semi-continuous maps $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$, then the map*

$$\Psi'\colon \mathbb{O}(Y) \ni V \mapsto (\bigcap_{\Psi\in S} \Psi(V))^\circ \in \mathbb{O}(X)$$

is lower semi-continuous.

(iii) *There is a minimal element $\Psi'\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ in the set S of the fitting lower s.c. maps, i.e., $\Psi'(V) \subseteq \Psi(V)$ for every open subset V of Y and every fitting l.s.c. map $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$.*
Moreover, $\Psi'(V) \supset \bigcup_{\ell\in\mathcal{L}(V)} \Psi_1(\ell)$ for every open subset V of Y, where $\mathcal{L}(V) := \{\ell \in \mathcal{L}\,;\ \Psi_2(\ell) \subseteq V\}$.

(iv) *If $\Psi_2(\ell_1) \subseteq \Psi_2(\ell_2)$ implies $\Psi_1(\ell_1) \subseteq \Psi_1(\ell_2)$ for $\ell_1, \ell_2 \in \mathcal{L}$, and if $x \in \bigcap_\alpha \Psi_1(\ell_\alpha) \neq \emptyset$ implies the existence of $\ell \in \mathcal{L}$ with $\Psi_2(\ell) \subseteq \Psi_2(\ell_\alpha)$ for all $\alpha \in A$ and $x \in \Psi_1(\ell)$, then the minimal fitting map Ψ' in part (iii) satisfies $\Psi'(V) = \bigcup_{\ell\in\mathcal{L}(V)} \Psi_1(\ell)$ for each $V \in \mathbb{O}(Y)$, and $\Psi'(\Psi_2(\ell)) = \Psi_1(\ell)$ for all $\ell \in \mathcal{L}$.*

(v) *An increasing map $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ is lower semi-continuous,*
if and only if,
the map

$$\lambda\colon Y \ni y \to \lambda(y) := X \setminus \Psi(Y \setminus \overline{\{y\}}) \in \mathbb{F}(X)$$

from Y to $\mathbb{F}(X)$ satisfies $\lambda(y) \subseteq \overline{\bigcup_{z\in Z} \lambda(z)}$ for each subset $Z \subseteq Y$ and $y \in \overline{Z}$,
if and only if,
the function $g(y) := \sup f(\lambda(y))$ is lower semi-continuous for every bounded lower semi-continuous function $f\colon X \to [0,\infty)$.

PROOF. (i): Since $(\bigcap_\psi W_{\psi,\alpha})^\circ$ is contained in $W_{\psi,\alpha}$ for each (ψ,α), we get that $\bigcap_\alpha(\bigcap_\psi W_{\psi,\alpha})^\circ$ is contained in $\bigcap_\alpha W_{\psi,\alpha}$ for each $\psi\in S$. Thus, the interior of $\bigcap_\alpha(\bigcap_\psi W_{\psi,\alpha})^\circ$ is contained in $(\bigcap_\alpha W_{\psi,\alpha})^\circ$ for each $\psi\in S$. This implies that

$$(\bigcap_\alpha \left(\bigcap_\psi W_{\Psi,\alpha}\right)^\circ)^\circ \;\subseteq\; (\bigcap_\psi \left(\bigcap_\alpha W_{\psi,\alpha}\right)^\circ)^\circ .$$

The interior of $\bigcap_\psi(\bigcap_\alpha W_{\psi,\alpha})^\circ$ is contained in $\bigcap_\alpha(\bigcap_\psi W_{\psi,\alpha})^\circ$ by a similar argument.

(ii): Let $\{V_\alpha\}_{\alpha\in A}$ a family of open subsets of Y, and let $W_{\Psi,\alpha}:=\Psi(V_\alpha)$ for $\Psi\in S$ and $\alpha\in A$. Then $\Psi'((\bigcap_\alpha V_\alpha)^\circ)$ is the interior of $\bigcap_\Psi\Psi\big((\bigcap_\alpha V_\alpha)^\circ\big)=\bigcap_\Psi(\bigcap_\alpha W_{\Psi,\alpha})^\circ$ (by lower semi-continuity of $\Psi\in S$). On the other hand, $\bigcap_\alpha\Psi'(V_\alpha)=\bigcap_\alpha(\bigcap_\Psi W_{\Psi,\alpha})^\circ$ by definition of Ψ' and of $W_{\Psi,\alpha}$. Now apply part (i) and the definition of lower semi-continuity.

(iii): Let $\Psi_0(V):=X$ for all open subsets $V\subseteq Y$ (including the case $V=\emptyset$). It is lower s.c. and is fitting. Thus, the set S of fitting l.s.c. maps $\Psi\colon\mathbb{O}(Y)\to\mathbb{O}(X)$ is not empty.

Let $\Psi'(V)\in\mathbb{O}(X)$ denote the interior of the set $\bigcap_{\Psi\in S}\Psi(V)$ for each $V\in\mathbb{O}(Y)$. Then, $\Psi_1(\ell)\subseteq\Psi'(\Psi_2(\ell))$ for all $\ell\in\mathcal{L}$ and $\Psi'\colon\mathbb{O}(Y)\to\mathbb{O}(X)$ is lower semi-continuous by part (ii). Since $\Psi'\in S$, it is the minimal l.s.c. map $\Psi\in S$. Since Ψ' is fitting, $\Psi_1(\ell)\subseteq\Psi'(V)$ for each $\ell\in\mathcal{L}$ with $\Psi_2(\ell)\subseteq V$.

(iv): We define $\mathcal{L}(V):=\{\ell\in\mathcal{L}\,;\,\Psi_2(\ell)\subseteq V\}$. Then, $\bigcap_\alpha\mathcal{L}(V_\alpha)=\mathcal{L}((\bigcap_\alpha V_\alpha)^\circ)$, $\ell\in\mathcal{L}(\Psi_2(\ell))$ for $\ell\in\mathcal{L}$, and $\mathcal{L}(V_1)\subseteq\mathcal{L}(V_2)$ for $V_1\subseteq V_2$.

Now let $\Phi(V):=\bigcup_{\ell\in\mathcal{L}(V)}\Psi_1(\ell)$ for $V\in\mathbb{O}(Y)$. Then $\Psi_1(\ell)\subseteq\Phi(\Psi_2(\ell))$ for $\ell\in\mathcal{L}$, $\Phi(V)\subseteq\Psi'(V)$ for all $V\in\mathbb{O}(Y)$ (by Part (iii)), and $\Phi(V_1)\subseteq\Phi(V_2)$ for $V_1\subseteq V_2$. In particular, $\Phi((\bigcap_\alpha V_\alpha)^\circ)$ is contained in the interior of $\bigcap_\alpha\Phi(V_\alpha)$.

The reversed inclusion can be derived from the rather strong assumption, that $x\in\bigcap_\alpha\Psi_1(\ell_\alpha)\neq\emptyset$ implies the existence of $\ell\in\mathcal{L}$ with $\Psi_2(\ell)\subseteq\Psi_2(\ell_\alpha)$ for all $\alpha\in A$ and $x\in\Psi_1(\ell)$.

In particular Φ is lower s.c. and fitting. Thus, $\Phi(V)\supset\Psi'(V)$.

Since $\ell_2\in\mathcal{L}(\Psi_2(\ell_2))=\{\ell_1\in\mathcal{L}\,;\,\Psi_2(\ell_1)\subseteq\Psi_2(\ell_2)\}$, we have $\Psi_1(\ell_1)\subseteq\Psi_1(\ell_2)$ for $\ell_1\in\mathcal{L}(\Psi_2(\ell_2))$, if $\Psi_2(\ell_1)\subseteq\Psi_2(\ell_2)$ implies $\Psi_1(\ell_1)\subseteq\Psi_1(\ell_2)$. Thus $\Psi'(\Psi_2(\ell))=\Psi_1(\ell)$ in this case.

(v): We write $\lambda(Z)$ for the union $\bigcup_{z\in Z}\lambda(z)\subseteq X$.

Suppose that $\Psi\colon\mathbb{O}(Y)\to\mathbb{O}(X)$ is an increasing lower semi-continuous map, i.e.,

$$(\bigcap_\tau\Psi(U_\tau))^\circ=\Psi((\bigcap_\tau U_\tau)^\circ)\,.$$

Let $Z\subseteq Y$, then $y\in\overline{Z}$, if and only if,

$$Y\setminus\overline{\{y\}}\supset Y\setminus\overline{Z}=(\bigcap_{z\in Z}Y\setminus\overline{\{z\}})^\circ\,.$$

It follows that

$$\Psi(Y \setminus \overline{\{y\}}) \supset (\Psi(\bigcap_{z\in Z} Y \setminus \overline{\{z\}}))^\circ .$$

The right side is the same as $X \setminus \overline{\lambda(Z)}$, by the rule $X \setminus (\bigcap_\tau V_\tau)^\circ = \overline{\bigcup_\tau (X \setminus U_\tau)}$. Thus, $\lambda(y) \subseteq \overline{\lambda(Z)}$ for each subset $Z \subseteq Y$ and $y \in \overline{Z}$.

Suppose that the map

$$\lambda \colon Y \ni y \to \lambda(y) := X \setminus \Psi(Y \setminus \overline{\{y\}}) \in \mathbb{F}(X)$$

from Y to $\mathbb{F}(X)$ satisfies:
$\lambda(y) \subseteq \overline{\lambda(Z)}$ for each subset $Z \subseteq Y$ and $y \in \overline{Z}$.
Let $f \colon X \to [0,\infty)$ l.s.c., and let $g(y) := \sup f(\lambda(y))$. Then $W := f^{-1}[0,t]$ is closed, and $y \in Z := g^{-1}[0,t]$, if and only if, $\lambda(y) \subseteq W$. Since W is closed and $\lambda(Z) \subseteq W$ we have $\lambda(y) \subseteq W$ for all $y \in \overline{Z}$. Thus, $g^{-1}[0,t]$ is closed for all $t \in [0,\infty)$, i.e., g is l.s.c.

Suppose that the function $g(y) := \sup f(\lambda(y))$ is lower semi-continuous for every bounded lower semi-continuous function $f \colon X \to [0,\infty)$.
Consider the map $F \colon \mathbb{F}(Y) \to \mathbb{F}(X)$ given by $F(Z) := X \setminus \Psi(Y \setminus Z)$. F is increasing. $F(\overline{\{y\}}) = \lambda(y)$.

We must show that $\bigcup F(Z_\tau)$ has closure $F(\overline{\bigcup Z_\tau})$ for each family $\{Z_\tau\}$ of closed subsets of Y. Since F is increasing,

$$\overline{\bigcup F(Z_\tau)} \subseteq F(\overline{\bigcup Z_\tau}) .$$

Let $Z := \bigcup Z_\tau$. Then $Z = \bigcup_{z\in Z} \overline{\{z\}}$, and

$$\bigcup F(Z_\tau) \supset \bigcup_{z\in Z} F(\overline{\{z\}}) = \lambda(Z)$$

holds because F is increasing. Let $U := X \setminus \overline{\lambda(Z)}$ and let $f \colon X \to [0,1]$ the characteristic function of U. Then $g(y) := \sup f(\lambda(y))$ is lower s.c. and the *closed* subset $G := g^{-1}(0)$ is given by $y \in Y$ with $\lambda(y) \in \overline{\lambda(Z)}$. In particular, G contains the closure of Z. We get $F(\overline{Z}) \subseteq X \setminus U$. Thus,

$$F(\overline{\bigcup Z_\tau}) = F(\overline{Z}) \subseteq \overline{\lambda(Z)} \subseteq \overline{\bigcup F(Z_\tau)} .$$

□

4. Some properties of exact C^*-algebras

DEFINITION B.4.1. We say that a C^*-subalgebra $E \subseteq A$ is a **regular subalgebra** of A, if, for all closed ideals J_1, J_2 of A,

(i) $E \cap (J_1 + J_2) = (E \cap J_1) + (E \cap J_2)$, and
(ii) $J_1 = J_2$ if $E \cap J_1 = E \cap J_2$.

In other words this says, that $\Psi^{\mathrm{up}}_{A,E}(J) := E \cap J$ defines a continuous action of $X := \mathrm{Prim}(A)$ on E in the sense of Definition 1.2.6, because of (i), and this action is injective if considered as a map from $\mathbb{O}_X \cong \mathcal{I}(A)$ into $\mathcal{I}(E)$, by (ii).

It is shown in [**464**] that $A \otimes \mathcal{O}_2$ contains a regular Abelian C^*–subalgebra E for every separable nuclear C^*-algebra A. We obtain in Chapter 12, that this implies the same for separable exact C^*-algebras A.

The Lemma 2.2.3 yields another useful result:

Proposition B.4.2. *Suppose that A and B are separable C^*-algebras, and that A is exact.*

(i) *Every primitive ideal J of $A \otimes B$ is a sum $J = (J_1 \otimes B) + (A \otimes J_2)$, where J_1 and J_2 are primitive ideals of A and B, respectively.*
(ii) *Every closed ideal J of $A \otimes B$ is the closure of the sum of the family of all elementary ideals $J_1 \otimes J_2 \subseteq J$ containing in J, where $J_1 \subseteq A$ and $J_2 \subseteq B$ are closed ideals.*

Here ''elementary'' ideals are defined as the tensor products $J_1 \otimes J_2$??

(iii) *The 1-1-map from $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ onto $\mathrm{Prim}(A \otimes B)$ which is given by (i) is a homeomorphism from the Tychonoff product of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ onto $\mathrm{Prim}(A \otimes B)$.*
(iv) *If $E \subseteq A$ and $F \subseteq B$ are C^*-subalgebras which separate the closed ideals of A and B, in the sense that e.g. $E \cap J_1 = E \cap J_2$ implies $J_1 = J_2$, then again $E \otimes F$ separates the closed ideals of $A \otimes B$.*

If, moreover, E and F are regular subalgebras of A and B in the sense of Definition B.4.1, then $E \otimes F$ is a regular subalgebra of $A \otimes B$.

Proof. (i): The tensor product $d = d_1 \otimes d_2$ of irreducible representations d_1 of A and d_2 of B is irreducible. d defines a faithful representation of $(A/J_1) \otimes (B/J_2)$, where J_1 and J_2 are the kernels of d_1 and d_2. Since A and, therefore, A/J_1 and J_1 are exact, the kernels of $A \otimes B \to A \otimes (B/J_2)$, of $(A/J_1) \otimes B \to (A/J_1) \otimes (B/J_2)$ and of $J_1 \otimes B \to J_1 \otimes (B/J_2)$ are $A \otimes J_2$, $(A/J_1) \otimes J_2$ and $J_1 \otimes J_2$, respectively. Since exact C^*-algebras are locally reflexive, the kernels of $A \otimes B \to (A/J_1) \otimes B$ and of $A \otimes (B/J_2) \to (A/J_1) \otimes (B/J_2)$ are $J_1 \otimes B$ and $J_1 \otimes (B/J_2)$. Thus the kernel of d is $(J_1 \otimes B) + (A \otimes J_2)$, as the 3×3-lemma shows.

Thus $\lambda\colon (J_1, J_2) \mapsto J_1 \otimes B + A \otimes J_2$ maps $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ into $\mathrm{Prim}(A \otimes B)$.

A general irreducible representation d of $A \otimes B$ defines commuting factorial representations d_1 of A and d_2 of B, such that $d(a \otimes b) = d_1(a)d_2(b)$. By a result of J. Dixmier, the kernels J_1 of d_1 and J_2 of d_2 are primitive. $(J_1 \otimes B) + (A \otimes J_2)$ is contained in the kernel of d, and d defines a C^*-norm on the algebraic tensor product $A/J_1 \odot B/J_2$ which majorize the spatial norm, cf. [**704**, prop.1.20.5,prop.1.22.7]. Thus the kernel of d is $(J_1 \otimes B) + (A \otimes J_2)$.

(ii):

Check again

Let J_0 be the closure of the sum of all elementary ideals which are contained in J. Suppose that there exists $d \in J_+$ such that d is not in J_0. Then, by (i), there exists primitive ideals $J_1 \subseteq A$ and $J_2 \subseteq B$ such that $J_0 \subseteq J_1 \otimes B + A \otimes J_2$ and $A \otimes B \to (A/J_1) \otimes (B/J_2)$ maps d into a non-zero positive element of $(A/J_1) \otimes (B/J_2)$. Thus there exist pure states φ on A and ψ on B such that $\varphi \otimes \psi(J_0) = 0$ and $\varphi \otimes \psi(d) > 0$. By Lemma 2.2.3, there exists $z \in A \otimes B$, $e \in A_+$ and $f \in B_+$ with $z^* z \in dAd$, $e \otimes f = zz^*$, $\varphi(e) > 0$ and $\psi(f) > 0$. Hence $e \otimes f \in J$, $\varphi \otimes \psi(e \otimes f) > 0$ and the elementary ideal generated by $e \otimes f$ is contained in J_0. A contradiction to the choice of φ and ψ.

(iii): Every open subset of the Tychonoff product $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ is the union of cartesian products $Y \times Z$ of open subsets Y of $\mathrm{Prim}(A)$ and Z of $\mathrm{Prim}(B)$. They correspond to closed ideals $K_1 \subseteq A$ and $K_2 \subseteq B$, and λ maps $Y \times Z$ onto the open subset of $\mathrm{Prim}(A \otimes B)$ which corresponds to $K_1 \otimes K_2$. Thus, by (ii), λ maps the open subsets of the Tychonoff product $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ onto the open subsets of $\mathrm{Prim}(A \otimes B)$.

(iv): The natural map $J \mapsto E \cap J$ defines a monotone map $\Psi_1 := \Psi^{\mathrm{up}}_{A,E}$ from the open subsets of $\mathrm{Prim}(A)$ into the open subsets of $\mathrm{Prim}(E)$ ($\cong$ closed ideals of E), which satisfies the conditions (ii),(iii) and (iv) of Definition 1.2.6. E and F separate the closed ideals of A and B, if and only if, Ψ_1 and $\Psi_2 := \Psi^{\mathrm{up}}_{B,F}$ are injective.

We consider $\Psi_3 := \Psi^{\mathrm{up}}{}_{A \otimes B, E \otimes F}$ as a map from the open subsets of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ into the open subsets of $\mathrm{Prim}(E) \times \mathrm{Prim}(F)$. By the monotony and by condition (ii) of Definition 1.2.6 for Ψ_3, it is enough to check the injectivity of the map Ψ_3 on the base of the topology of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. This base is given by the cartesian products of open subsets of $\mathrm{Prim}(A)$ and of $\mathrm{Prim}(B)$. $\Psi_3(Y \times Z) = \Psi_1(Y) \times \Psi_2(Z)$, for open subsets Y of $\mathrm{Prim}(A)$ and Z of $\mathrm{Prim}(B)$, because, for closed ideals $J \subseteq A$ and $K \subseteq B$, $(E \otimes F) \cap (J \otimes K) = (E \cap J) \otimes (F \cap K)$. The latter identity follows from the exactness of $E \subseteq A$. The cartesian product map $\Psi_1 \times \Psi_2$ is injective, because Ψ_1 and Ψ_2 are injective.

Now suppose that, moreover, that E is regular in A and that F is regular in B. E and F are regular, if and only if, moreover, Ψ_1 and Ψ_2 satisfy condition (i) of Definition 1.2.6, e.g., $\Psi_1(Z_1 \cup Z_2) = \Psi_1(Z_1) \cup \Psi_1(Z_2)$ for open subsets Z_1 and Z_2 of $\mathrm{Prim}(A)$. We want to deduce that $\Psi_3(X_1 \cup X_2) = \Psi_3(X_1) \cup \Psi_3(X_2)$ for open subsets X_1 and X_2 of $\mathrm{Prim}(A \otimes B)$.

By the monotony and by condition (ii) of Definition 1.2.6 for Ψ_3, it suffices to show that, for open subsets Y_k of $\mathrm{Prim}(A)$ and Z_k of $\mathrm{Prim}(B)$,

$$\Psi_3\Big(\bigcup_{1 \le k \le n} Y_k \times Z_k\Big) \subseteq \bigcup_{1 \le k \le n} \Psi_3(Y_k \times Z_k)\,.$$

Note that $\Psi_1(\mathrm{Prim}(A)) = \mathrm{Prim}(E)$ and $\Psi_2(\mathrm{Prim}(B)) = \mathrm{Prim}(F)$.

If $J \subseteq A$ and $K \subseteq B$ are closed ideals, then the natural epimorphism from $A \otimes B$ onto $(A/J) \otimes (B/K)$ has kernel $A \otimes K + J \otimes B$, because A is exact and, therefore, locally reflexive. The same applies to $J \cap E$, $K \cap F$, E and F, and $E/(E \cap J) \otimes F/(F \cap K)$ is a subalgebra of $A/J \otimes B/K$. Thus

$$((A \otimes K) + (J \otimes B)) \cap (E \otimes F) = (E \otimes F) \cap (A \otimes K) + (E \otimes F) \cap (J \otimes B),$$

which means that

$$\Psi_3(Y \times \mathrm{Prim}(B)) \cup \Psi_3(\mathrm{Prim}(A) \times Z) = \Psi_3((Y \times \mathrm{Prim}(B)) \cup (\mathrm{Prim}(A) \times Z))\,.$$

Let (I_1, I_2) be a point of $\mathrm{Prim}(E) \times \mathrm{Prim}(F)$ which is not in in the union of the sets $\Psi_1(Y_k) \times \Psi_2(Z_k)$ for $k = 1, \ldots, n$. Then there are subsets S and T of $\{1, \ldots, n\}$, with $S \cup T = \{1, \ldots, n\}$, such that I_1 is not in $\bigcup_{k \in S} \Psi_1(Y_k)$ and I_2 is not in $\bigcup_{k \in T} \Psi_2(Z_k)$.

Since E and F are regular subalgebras, we have $\Psi_1(Y) = \bigcup_{k \in S} \Psi_1(Y_k)$ and $\Psi_2(Z) = \bigcup_{k \in T} \Psi_2(Z_k)$, where $Y := \bigcup_{k \in S} Y_k$ and $Z := \bigcup_{k \in T} Z_k$.

Thus (I_1, I_2) is not in $(\Psi_1(Y) \times \mathrm{Prim}(F)) \cap (\mathrm{Prim}(E) \times \Psi_2(Z))$. The latter is the same as $\Psi_3((Y \times \mathrm{Prim}(B)) \cap (\mathrm{Prim}(A) \times Z))$.

But Ψ_3 is monotonous, and $\bigcup Y_k \times Z_k \subseteq (Y \times \mathrm{Prim}(B)) \cap (\mathrm{Prim}(A) \times Z)$. □

COROLLARY B.4.3. *Suppose that $A_1, A_2, \ldots$ is a sequence of separable exact C*-algebras (with A_n unital for $n \geq n_0$).*

(i) *There is a natural homeomorphism from the Tychonoff product* $\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2) \times \ldots$ *onto* $\mathrm{Prim}(A_1 \otimes A_2 \otimes \ldots)$.
(ii) *If B_n are regular C*-subalgebras of A_n, $n = 1, 2, \ldots$, then $B_1 \otimes B_2 \otimes \ldots$ is a regular C*-subalgebra of $A_2 \otimes A_2 \otimes \ldots$.*

PROOF. Let $A := A_1 \otimes A_2 \otimes \ldots$, $C_n := A_1 \otimes A_2 \otimes \ldots \otimes A_n$, $D_n := A_{n+1} \otimes A_{n+2} \otimes \ldots$, and let 1_n denote the unit element of D_n. Then, for every closed ideal J of A, A/J is the inductive limit of $\pi_J(C_n \otimes 1_n)$. Thus J is the inductive limit both of $J_n \otimes 1_n$ and, therefore, of $J_n \otimes D_n$, where $J_n \subseteq C_n$ is defined by $J_n \otimes 1_n = (C_n \otimes 1_n) \cap J$.

(i): By Proposition B.4.2, we get that every primitive ideal of A is the closure of the sum of ideals $C_{n-1} \otimes J_n \otimes D_n$, where J_n is a primitive ideal of A_n for $n = 1, 2, \ldots$.

Conversely, let $J_n \subseteq A_n$ a sequence of primitive ideals. We find pure states ψ_n on A_n which define irreducible representations with kernel J_n. By [**704**, prop. 4.3.4], the infinite tensor product of pure states is pure. Thus, the kernel of the natural epimorphism from A onto the infinite tensor product of the quotients A_n/J_n is primitive. This shows the set-theoretic isomorphism of $\mathrm{Prim}(A)$ with the Tychonoff product of the sequence $\mathrm{Prim}(A_n)$. The topological isomorphism means that one can find for both a bases of the topology which is mapped by the set-theoretic isomorphism onto each other. By definition of the Tychonoff product this means that every ideal of A is the closure of a sum of ideals $J_1 \otimes \ldots \otimes J_n \otimes D_n$. This follows from Proposition B.4.2 and the remark at the beginning of the proof.

(ii): By, Proposition B.4.2(iv), for $n = 1, 2, \ldots$, $G_n := B_1 \otimes \ldots \otimes B_n \otimes 1_n$ is a regular subalgebra of $C_n \otimes 1_n$. This and $G_n \subseteq G_{n+1}$ imply that the closure $B_1 \otimes B_2 \otimes \ldots$ of $\bigcup_n G_n$ is regular in the closure A of $\bigcup_n C_n$. □

If we combine Proposition B.4.2 with the criteria in [**432**], we get:

Corollary B.4.4. *A separable C^*-algebra A is exact, if and only if, for every separable C^*-algebra B, the natural map from $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ to $\mathrm{Prim}(A \otimes B)$, which is induced by the tensor product of irreducible representations, is a topological isomorphism. The isomorphism is the same as in Proposition B.4.2(i).*

Remark B.4.5.
(1) A lower s.c. quasi-trace $\tau \colon A_+ \to [0, \infty]$ is 2-additive if and only if it is an integral $\tau(a) = \int_{0_+}^{\infty} D((a-t)_+)\, \mathrm{d}t$ of a lower s.c. *sub-additive* dimension function $D \colon A \to [0, \infty]$. One gets D back from τ by $D(a) = \sup_{\delta > 0} \tau(f_\delta(a^* a))$, where $f_\delta(t) := \min(\delta^{-1}(t-\delta)_+, 1)$ for $\delta > 0$.

(2) Every lower semi-continuous 2-quasi-trace $\tau \colon A \to [0, \infty]$ on an exact C^*-algebra is additive. (In case of a unital A this is a result of Haagerup and Thorbjørnsen, [**342**] and [**348**, cor. 9.14]. It extends to the non-unital case by the below given Lemma B.4.6.)

(3) Every lower semi-continuous dimension function $D \colon A \to [0, \infty]$ on A integrates to a l.s.c. 2-quasi-trace τ_D on A_+ by $\tau_D(a) = \int_{0_+}^{\infty} D((a-t)_+) \mathrm{d}t$.

(4) A stable simple C^*-algebra A contains a (non-zero) properly infinite projection, if and only if, there is no non-zero dimension function D on the Pedersen ideal of A (cf. Blackadar and Cuntz [**78**]).

(??? This should be also my remark in the 1994 book from Canada workshop ???)

(5) Summing up (1)–(4), we get:
If A is simple and exact, then $T^+(A) = \{0\}$ if and only if A is stably infinite.
In particular, *simple stably projection-less exact C^*-algebras have always non-zero* additive *traces, cf.* [**441**]. *The $T^+(A) = \{0\}$ implies for simple nuclear A that $A \otimes \mathbb{K}$ contains a (non-zero) properly infinite projection.*

Lemma B.4.6. *Let $\tau \colon A_+ \to [0, \infty]$ a lower semi-continuous 2-quasi-trace.*

(i) *The map τ and its dimension function $D \colon A \to [0, \infty]$ are order monotone on A_+, and D with respect to Cuntz majorization $a \precsim b$ on A_+.*
(ii) *If, for every contraction $c \in A_+$ with $\tau(c) < \infty$, the restriction $\tau | B_+$ is additive for $B := \{a \in A\, ;\ ac = a = ca\}$, then τ is additive on A_+.*
(iii) *If $\tau(A_+) \subseteq [0, \infty)$ then $\gamma := \sup\{\tau(a)\, ;\ \|a\| \leq 1,\ a \in A_+\} < \infty$ and $\widetilde{\tau}(b + z1) := z\gamma + \tau(b_+) - \tau(b_-)$ (for $b + z1 \in \widetilde{A}_+$) defines a bounded 2-quasi-trace on the unitization $\widetilde{A}$ of A.*

PROOF. (i): Let $D\colon A \to [0,\infty]$ the l.s.c. dimension function corresponding to τ, i.e.,

$$D(a) := \limsup_{\delta>0,\gamma>0} \tau((((a^*a)-\delta)_+)^\gamma)$$

and let $0 \leq a \leq b$ or $a \precsim b$.

Then $(a-t)_+^3 \leq (a-t)_+(b-t)_+(a-t)_+ \leq \|b\|(a-t)_+^2$, because $(b-t)_+ \leq b \leq \|b\|$ by $(b-t)_+ = (b-t)+(b-t)_- \leq b$ and $(a-t)_+ - (a-t)_- = a-t \leq b-t \leq (b-t)_+$ and the order preserving map $X \mapsto (a-t)_+X(a-t)_+$ on $X^* = X$ implies $(a-t)_+^3 \leq (a-t)_+(b-t)_+(a-t)_+$.

It induces: For $\varepsilon > \delta > 0$ there is $x \in A$ with $x^*x = f_\varepsilon(a)$ and $xx^*f_\delta(b) = xx^*$. It follows $\tau(f_\delta(a)) \leq \tau(f_\epsilon(b)) \leq D(b)$. Thus $D(a) \leq D(b)$. From $D(a^\alpha) = D(a)$ and the monotony and centrality of D it follows $D((a-t)_+) \leq D((b-t)_+)$ for all $t > 0$ if $0 \leq a \leq b$. Since $\tau(a) = \int_{0+}^\infty D((a-t)_+)\,dt$, we get from it that $\tau(a) \leq \tau(b)$,

(ii): Let $a, b \in A_+$ with $\tau(a) + \tau(b) < \infty$.

Then ?????

show that $\tau((a-\varepsilon)_+ + (b-\varepsilon)_+) < \infty$???? `Fill Proof in !!! ??`

(iii): Suppose that $\gamma = \infty$ then there exists positive contractions $a_1, a_2, \ldots \in A_+$ with $\tau(a_n) > 4^n$. Then $a := \sum 2^{-n}a_n$ is a positive contraction in A with $a_n \leq 2^n a$, and $4^n < 2^n\tau(a)$ for all $n \in \mathbb{N}$ by part (i). Thus $\tau(a) = \infty$, contradicting ????? that ???

If $b \in A$ and $z \in \mathbb{C}$ then $b + z1 \geq 0$ implies $z \geq 0$, $b^* = b$ and $\|b_-\| \leq z$. Thus $\widetilde{\tau}(b + z1) \geq 0$ for $b + z1 \geq 0$. If $0 \leq b_j + z_j1$ $(j = 1, 2)$ commute, then b_1 and b_2 commute. The additivity of τ on $C^*(b_1, b_2)$ induces the additivity of $\widetilde{\tau}$ on $C^*(b_1, b_2, 1)_+$.

It follows that $\widetilde{\tau}$ is a bounded local quasi-trace on $\widetilde{A}$ with bounded lower semi-continuous rank-function $\widetilde{D}(x + z1) = \gamma$ if $z \neq 0$ and $\widetilde{D}(x) = D(x)$ for $x \in A$ and $z \in \mathbb{C}$.

??????????????????? check

`the point is the existence of` $\widetilde{D_2}$ `!!` □

5. Approximate divisible algebras

REMARK B.5.1. Recall that a separable C^*-algebra B is called **approximately divisible** if there exist a sequence of unital *-morphisms $h_n\colon M_2 \oplus M_3 \to \mathcal{M}(B)$ with $\lim_{n\to\infty} \|h_n(a)b - bh_n(a)\| = 0$ for all $a \in M_2 \oplus M_3$ and $b \in B$.

$B := A \otimes \mathcal{O}_\infty$ is approximately divisible for every separable C^*-algebra A.

(Proof: $C^*(s_1, s_2, \cdots) \cong \mathcal{O}_\infty \cong \mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$ by Corollary F(ii), – or the argument following Corollary H –, and $M_2 \oplus M_3$ is unitally contained in $\mathcal{O}_\infty = C^*(s_1, s_2, \ldots)$.

This is because we can take $M_3 := C^*(s_i s_j^*\,;\, 1 \leq i, j \leq 3)$ and $M_2 := C^*(t_k t_\ell^*\,;\, 1 \leq k, \ell \leq 2)$, where t_1, t_2 are suitable partial isometries with $t_1 t_1^* + t_2 t_2^* = p := 1 - (s_1 s_1^* + s_2 s_2^* + s_3 s_3^*)$ and $t_1^* t_1 = t_2^* t_2 = q := 1 - (s_1 s_1^* + s_2 s_2^*)$.

The existence of t_1 and t_2 follows from $\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, with $[p] = -2$ and $[q] = -1$ and the pure infiniteness of $p\mathcal{O}_\infty p$, e.g. by cf. Lemma 4.2.6.

In fact, the projections $r := s_4 q s_4^* + s_5 q s_5^*$ and p satisfy $r \leq p$ and $-2 = [p] = [r] \in \mathrm{K}_1(\mathcal{O}_\infty)$. Thus p and r are MvN-equivalent in $\mathcal{O}_\infty$. Let $z \in \mathcal{O}_\infty$ a partial isometry with $zz^* = p$ and $z^* z = r$. The partial isometries $t_1 := zs_4 q$, $t_2 := zs_5 q$ have the desired property.

It is known that the infinite tensor product $(M_2 \oplus M_3)^{\otimes\infty} := (M_2 \oplus M_3) \otimes (M_2 \oplus M_3) \otimes \cdots$ contains the *GICAR*-algebra unitally, which is a simple unital AF-algebra defined by the "Pascal triangle" with multiplicities given by the integer coefficients of the polynomials $P_n(t) := (t+1)^n$, cf. [**270**].

Every infinite-dimensional simple unital AF-algebra contains the Jiang-Su algebra $\mathcal{Z}$, [**391**, cor. 6.3]. It implies that $A \otimes \mathcal{Z} \cong A$ for all "approximately divisible" A, i.e., those A that have a unital C^*-morphism of $M_2 \otimes M_3$ into $F(A) := (A' \cap A_\omega)/\operatorname{Ann}(A, A_\omega)$. (The $F(A)$ is a"stable" invariant of separable C^*-algebras A in the sense that $F(A) \cong F(A \otimes \mathbb{K})$.)

It seems that one can replace the assumption of approximate divisibility in most applications by the weaker requirement that A tensorial absorbs $\mathcal{Z}$, i.e., that $A \otimes \mathcal{Z} \cong A$. The main reason could be something like the pull-back proposition [**690**, prop. 6.5] of M. Rørdam.

E.g., *separable exact A is* strongly *p.i., if $A \cong A \otimes \mathcal{Z}$ and every l.s.c. trace $\tau\colon A_+ \to [0, \infty]$ takes only values in $\{0, \infty\}$*. (For *separable nuclear A holds also the converse, because then $A \cong A \otimes \mathcal{O}_\infty$ and one can use that $\mathcal{O}_\infty \otimes \mathcal{Z} = \mathcal{O}_\infty$*.)

Is above partly shown in Chapter 2?
That $A \otimes \mathcal{Z}$ s.p.i. for nuclear separable
trace-less A is [690, thm. 5.2].
The proof in [690, thm. 5.2] shows
that this holds also for exact trace-less A,
and more generally for all C^*-algebras
that have only trivial l.s.c. traces. ???????

Further topics. To be filled in. ??

6. Positions of C^*-subalgebras in $\mathcal{O}_2$

There exist a separable unital nuclear C^*-algebra A and a unital monomorphisms $i\colon A \to \mathcal{O}_2$ of A into the Cuntz algebra $\mathcal{O}_2$, such that, there does not exist a conditional expectation from $\mathcal{O}_2$ onto $i(A)$.
But all separable unital nuclear C^*-algebras A have a unital $*$-monomorphism

$\iota\colon A \to \mathcal{O}_2$ such that there exists a conditional expectation from $\mathcal{O}_2$ onto $\iota(A)$, i.e., a completely positive contraction $E\colon \mathcal{O}_2 \to \iota(A)$ with $E \circ \iota = \iota$.

There is an interesting connection to some conjecture on equivalence of embeddings that are in "sufficiently general position" (see below).

Weaker results on embeddings $\iota\colon A \hookrightarrow \mathcal{O}_2$, as there are (for example):
One finds always for separable unital nuclear C^*-algebras A at least one unital embedding $\iota\colon A \hookrightarrow \mathcal{O}_2$ such that there is a conditional expectation E from $\mathcal{O}_2$ onto $\iota(A)$.

On the other hand, for any other embedding $\kappa\colon A \hookrightarrow \mathcal{O}_2$, there is a norm-continuous path $t \in [0,\infty) \mapsto U(t)$ into the unitary group of $\mathcal{O}_2$ such that $U(0) = 1$ and $\lim_{t\to\infty} \|U(t)^*\iota(a)U(t) - \kappa(a)\|$ for all $a \in A$. One can use this to show that for the (more general type of) embedding $\kappa\colon A \hookrightarrow \mathcal{O}_2$ there is a norm-continuous path $t \in [0,\infty) \mapsto V(t)$ into the *isometries* in $\mathcal{O}_2$ such that $\lim_{t\to\infty} \|V(t)^*\kappa(a)V(t) - \kappa(a)\|$ for all $a \in A$ and $\lim_{t\to\infty} \operatorname{dist}(V(t)^*bV(t), \kappa(A)) = 0$ for all $b \in \mathcal{O}_2$. (This could replace the non-existing conditional expectations onto $\kappa(A)$ by expectations in an approximate sense.)

The examples below produce also counter-examples

`to the above question ???`,

but are also counter-examples to the following more interesting question (which was suggested by the `above recalled results` and by the example of finite-dimensional A):

We call a separable unital nuclear C^*-algebra A **transportable in $\mathcal{O}_2$**, if, for any two unital *-monomorphisms $i\colon A \hookrightarrow \mathcal{O}_2$, and $j\colon A \hookrightarrow \mathcal{O}_2$ in *general position* there is an automorphism ψ of $\mathcal{O}_2$ with $\psi \circ i = j$.

We say that a unital *-monomorphisms $i\colon A \hookrightarrow \mathcal{O}_2$, is **in general position** if the commutant $i(A)' \cap \mathcal{O}_2$ of the image $i(A)$ in $\mathcal{O}_2$ contains a copy of $\mathcal{O}_2$ unitally.

Let $i\colon \mathcal{O}_2 \to \mathcal{O}_2$ and $j\colon \mathcal{O}_2 \to \mathcal{O}_2$ in general position. Is there an automorphism of $\mathcal{O}_2 \otimes \mathcal{O}_2$ that conjugates $i \otimes \mathrm{id}$ and $i \otimes \mathrm{id}$?

Case: $\mathcal{O}_2 \otimes \mathrm{C}[0,1] \to \mathcal{O}_2$ unital and injective and $\mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ an isomorphism?

Is there an automorphism of $\mathcal{O}_2 \otimes \mathcal{O}_2$ that maps $1 \otimes i(A)$ onto $1 \otimes j(A)$?

Obviously all unital *-monomorphisms of finite-dimensional C^*-algebras A are unitary equivalent in $\mathcal{O}_2$. It implies that each *-monomorphism of A is in general position, and that A is "transportable" inside $\mathcal{O}_2$.

We do not know if separable unital AF-algebras A are transportable in $\mathcal{O}_2$. Perhaps one could find a unital endomorphism $\iota\colon M_{2^\infty} \hookrightarrow M_{2^\infty}$ such that there does not exist a conditional expectation from M_{2^∞} onto $\iota(M_{2^\infty})$. (Compare the below given reasoning.)

Then the ??????

The below given examples show that C[0,1] is not transportable in $\mathcal{O}_2$:

There exists nuclear A and unital embeddings $i\colon A \hookrightarrow \mathcal{O}_2$ and $j\colon A \hookrightarrow \mathcal{O}_2$ such that there is a conditional expectation from $\mathcal{O}_2$ onto $i(A)$, but that there does not exist a conditional expectation from $\mathcal{O}_2$ onto $j(A)$ (see the examples discussed below). It yields that for $i'(a) = i(a) \otimes 1$ and $j'(a) = j(a) \otimes 1$ there does not exist a *-isomorphism ψ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ ($\cong \mathcal{O}_2$) with $\psi \circ i' = j'$ (see the explanation below).

Let A a separable unital nuclear C^*-algebra.

First:
If one uses the proof of the existence of embeddings $j\colon A \hookrightarrow \mathcal{O}_2$ with help of the generalized Voiculescu-Weyl-vonNeumann theorem 5.4.1 (with $\mathcal{O}_2 \otimes \mathbb{K}$ in place of the compact operators $\mathbb{K}$) and with help of $\mathrm{Ext}(A, \mathcal{O}_2) = 0$, then one gets (as an additional result) that the unital embedding $j\colon A \hookrightarrow \mathcal{O}_2$ can be found such that there is a conditional expectation P from $\mathcal{O}_2$ onto $j(A)$. (In fact the conditional expectation can be chosen "extreme", i.e., that P is an extreme point in the convex set of contractive linear maps from $\mathcal{O}_2$ into $\mathcal{O}_2$.)

Second:
If $i\colon A \hookrightarrow \mathcal{O}_2$ is an other unital embedding such that there does not exist a conditional expectation from $\mathcal{O}_2$ onto $i(A)$, then there does not exist an isomorphism ψ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with $\psi(j(a) \otimes 1) = i(a) \otimes 1$ for $a \in A$, because otherwise $E(b) := id \otimes f(\psi(P(b) \otimes 1))$ for $b \in \mathcal{O}_2$ and a (fixed pure) state f on $\mathcal{O}_2$ defines a conditional E expectation from $\mathcal{O}_2$ onto $i(A)$.

Third:
If A is a nuclear C^*-subalgebra of a separable unital nuclear C^*-algebra B such that $1_B \in A$ and such that there does not exist a conditional expectation from B onto A (We list below some examples), then:
If $k\colon B \hookrightarrow \mathcal{O}_2$ is a unital embedding and if we define $i := k|A$ as the restriction of k to A, then there can not exist any conditional expectation E from $\mathcal{O}_2$ onto $i(A) = k(A)$, because otherwise $k^{-1} \circ E \circ k$ would be a conditional expectation from B onto A.

Some Examples:
(o) Suppose that D is a hereditary C^*-subalgebra of a unital C^*-algebra B such that the normalizer algebra $\mathcal{N}(D) := \{b \in B\,;\, bD + Db \subseteq D\}$ is different from $A := D + \mathbb{C}1$ and that $bD \neq \{0\}$ for all $b \in B_+ \setminus \{0\}$. Then there does not exist a conditional expectation E from B onto A.
(Here $\mathbb{C}$ denotes the complex numbers.)
Indeed, $(E(b) - b)^*(E(b) - b)d = 0$ for all $b \in \mathcal{N}(D)$, $d \in D$, i.e., $E|\mathcal{N}(D) = \mathrm{id}$.
It yields e.g. the following two examples (i) and (ii):

(i) Consider $\mathcal{O}_2$ as a C^*-subalgebra of $\mathcal{L}(\ell_2)$ (by some unital *-representation). Let $\mathbb{K}$ denote the compact operators, and let $A := \mathbb{K} + \mathbb{C}1$, $B := \mathbb{K} + \mathcal{O}_2$.

(ii) Let $B := \mathcal{T}$, $A := \mathbb{K} + \mathbb{C}1$, where $\mathcal{T}$ denotes the Toeplitz algebra, generated as C^*-algebra by the Toeplitz operator T (i.e. the unilateral shift of ℓ_2).

(iii) Consider the natural continuous epimorphism σ from the Cantor space $\Omega := \{0,1\}^\infty$ onto $[0,1]$ given by

$$\sigma\colon (a_1, a_2, ...) \mapsto \sum_{n=1}^{\infty} a_n 2^{-n} \,.$$

Then $\gamma\colon f \in \mathrm{C}[0,1] \mapsto \gamma(f) := f \circ \sigma \in \mathrm{C}(\Omega)$ gives a unital embedding of $\mathrm{C}[0,1]$ into $\mathrm{C}(\Omega) = \bigotimes_{n=1}^{\infty}(\mathbb{C} \oplus \mathbb{C}) \subseteq M_{2^\infty}$.

The pairs $A \subseteq B$ with $A := \gamma(\mathrm{C}[0,1])$ and $B := \mathrm{C}(\Omega)$ or with $B := M_{2^\infty}$ have the property that there does not exist a conditional expectation from B onto A, because the continuous map σ is not open.

(It is an exercise to check that σ is *not* open, and that this implies the non-existence of the conditional expectation onto A .)

Some Remarks:

A related open question is:
Suppose that $i, j\colon A \hookrightarrow \mathcal{O}_2$ are unital embeddings such that there are *extremal* conditional expectations from $\mathcal{O}_2$ onto $i(A)$, respectively onto $j(A)$. Is there an automorphism ψ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with $\psi(i(a) \otimes 1) = j(a) \otimes 1$ for all $a \in A$?

It is likely that there exists *simple* separable nuclear unital C^*-algebras $A \subseteq B$ such that $1_B \in A$ and such that there does not exist conditional expectation from B onto A. (Perhaps, even with $A \cong B \cong M_{2^\infty}$?)
Then it would follow that there are unital endomorphisms $i\colon \mathcal{O}_2 \hookrightarrow \mathcal{O}_2$ such that there does not exist a conditional expectation from $\mathcal{O}_2$ onto $i(\mathcal{O}_2)$, because $\mathcal{O}_2 \cong A \otimes \mathcal{O}_2 \subseteq B \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. (This would show that the position of $i(\mathcal{O}_2)$ in $\mathcal{O}_2$ could be very random.)

(It seems likely that $\mathcal{O}_2$ is not transportable in $\mathcal{O}_2$).

7. Quotients of nuclear maps

Example of a nuclear and approximately inner c.p. map that is not residually nuclear.

Its relation to the existence of group vN-algebras that are not weakly exact.

Here we explain the reason for principal difficulties with nuclear maps (coming from the existence of non-exact discrete groups), and we point out, that nuclear and approximately inner (respectively nuclear and Ψ-equivariant) c.p. maps are in general not residually nuclear if they are ideal-system preserving.

We introduce also some tools that allow to overcome this problems at least in special cases.

LEMMA B.7.1. *Suppose that M is a W^*-algebra with faithful normal state ρ and let $b_1, b_2, \ldots \in M_+$.*

(i) *If* $\sum_n \rho(b_n) \leq 1$, *then for each* $\varepsilon > 0$ *there exists a projection* $p \in M$ *such that*

$$\rho(1-p) \leq \varepsilon \quad \text{and} \quad \| \sum_{n=1}^{k} p b_n p \| \leq \varepsilon^{-1} \quad \text{for all } k \in \mathbb{N}. \tag{7.1}$$

(ii) *Let* $d, c_1, c_2, \ldots \in M$ *contractions such that* $\sum_n \rho(b_n) \leq 1$ *for the elements* $b_n := (d - c_n)^*(d - c_n) \in M_+$.

If $p \in M$ *is a projection that satisfies the inequalities* (7.1) *of part* (i) *for this* $b_1, b_2, \ldots \in M_+$ *then*

$$\lim_n f(p c_n p) = f(pdp) \quad \text{for all } f \in M^* .$$

Proof. (i): The functions $g_t(x) := x/(1 + t^{-1}x) = t(1 - 1/(1 + t^{-1}x))$ on $[0, \infty)$ are operator monotone continuous functions, cf. [**616**, sec. 1.3.7]. They satisfy $g_s(x) \leq g_t(x) \leq x$ for $s \leq t$, $g_t(x) \leq t$ for all $x \geq 0$, and $|x - g_t(x)| \leq t^{-1}\alpha^2$ for $x \in [0, \alpha]$. Let $a_k := \sum_{n=1}^{k} b_n$, then $0 \leq g_t(a_k) \leq g_t(a_{k+1})$, $\|g_t(a_k)\| \leq t$,

more ?? check again?

It follows that the increasing sequence $(g_t(a_k))_k$ of positive elements of M has an ultra-weak limit (at first only an ultra-weak cluster point) $A(t) \in M_+$ with $\|A(t)\| \leq t$ and $g_t(a_k) \leq A(t)$. Then $d_k := (A(t) - g_t(a_k))^{1/2} \in M_+$ is a bounded sequence with $\lim_k \psi(d_k^* d_k) = 0$ for all positive normal functionals ψ on M. It follows that (d_k) converges ultra-strongly to zero, and implies that d_k^2 also converges ultra-strongly to zero, i.e., $A(t)$ is uniquely determined and is the ultra-strong limit of $(g_t(a_k))_k$ in M.

Since $g_t(a_k) g_s(a_k) = g_s(a_k) g_t(a_k)$, $g_s(a_k) \leq g_t(a_k)$ for $s \leq t$ and $\rho(g_t(a_k)) \leq \rho(a_k) \leq 1$, we get for the strong limits $\lim_k g_t(a_k) = A(t) \in M_+$ that $A(t)A(s) = A(s)A(t)$, $A(s) \leq A(t)$ for $s \leq t$ and $\rho(A(t)) \leq 1$ for all $t \in [0, \infty)$. Let C denote the commutative W*-subalgebra of M generated by 1 and $A(m)$ ($m \in \mathbb{N}$). Let $q_m \in C$ the support projection of $(A(m) - \varepsilon^{-1})_+$, i.e., $q_m = 1 - p_m$, where p_m is the support of the annihilator of $(A(m) - \varepsilon^{-1})_+$ in C. Then $q_m \leq q_{m+1}$ and $\varepsilon^{-1} q_m \leq A(m)$. Thus, $\rho(q_m) \leq \varepsilon$ for all $m \in \mathbb{N}$. It follows that $p := 1 - \bigvee_m q_m \in C$ satisfies $pA(m)p = A(m)p \leq \varepsilon^{-1} p$ for all $m \in \mathbb{N}$ and $\rho(1 - p) = \sup_m \rho(q_m) \leq \varepsilon$. We get $p g_m(a_k) p \leq pA(m)p \leq \varepsilon^{-1} p$ for all $k, m \in \mathbb{N}$. If we fix (arbitrary) $k \in \mathbb{N}$ and let m tend to ∞, then this shows that $p a_k p \leq \varepsilon^{-1} p$ for all $k \in \mathbb{N}$.

(ii): If f is a (not-necessarily normal) state on M, then $|f(p(d - c_n)p)|^2 \leq f(p b_n p)$. Thus, $\sum_n |f(pdp) - f(p c_n p)|^2 \leq \varepsilon^{-1} f(p)$. In particular $\lim_n f(p c_n p) = f(pdp)$. Finally use that M^* is the linear span of the states on M. □

Lemma B.7.2. *Suppose that* M *is a W*-algebra,* A *is a separable C*-algebra and* $\mathcal{C} \subseteq \mathrm{CP}(A, M)$ *a matrix operator-convex cone.*

If $V : A \to M$ *is in the point-ultraweak closure of* $\mathcal{C}$, *then there is a net of projections* p_μ *such that* $p_\mu V(\cdot) p_\mu$ *is in the point-norm closure of* $\mathcal{C}$ *and the net* $\{p_\mu\}$ *converges strongly to* 1_M.

*In particular, if $e \in A_+$ satisfies $V(e) = 0$, then there is a net of contractions $T_\mu \in \mathcal{C}$ in the point-*norm *closure of $\mathcal{C}$ with $T_\mu(e) = 0$ such that T_μ converges in point-ultra-strong topology to V.*

Proof. We can replace in the statement $\mathcal{C} \subseteq \mathrm{CP}(A, M)$ by its point-norm closure $\overline{\mathcal{C}}$, which is again an m.o.c. cone. Then $\mathcal{C}$ is a hereditary sub-cone of $\mathrm{CP}(A, M)$ in the sense that $W_1+W_2 \in \mathcal{C}$ implies $W_1, W_2 \in \mathcal{C}$, cf. Corollary 3.6.28. Let R denote the set of $b \in M$ such that $b^*V(\cdot)b \in \mathcal{C}$. Since $(b+c)^*V(\cdot)(b+c)+(b-c)^*V(\cdot)(b-c) = 2b^*V(\cdot)b+2c^*V(\cdot)c$ and $\mathcal{C}$ is hereditary, point-norm closed and invariant under composition with inner c.p. maps, it follows that R is a (norm-) closed right-ideal of M. The hereditary C^*-subalgebra $D := R^* \cap R$ of M contains an approximate unit p_σ of D consisting of projections. Let R_1 denote the weak closure and let $E \in M$ the projection with $EM = R_1$. Then p_σ converges ultra-strongly to E, $p_\sigma V(\cdot)p_\sigma \in \mathcal{C}$, and $\{p_\sigma V(\cdot)p_\sigma\}_\sigma$ converges to $EV(\cdot)E$ in point-ultra-strong topology. In particular, $E = \bigvee\{p\,;\ p \in R, p^2 = p = p^*\}$. It remains to show that $E = 1_M$.

Since for every non-zero projection $P \in M$ there is a non-zero countably decomposable projection $Q \in M$ with $Q \leq P$, it suffices to show that for every countably decomposable projection Q there is a projection $p \leq Q$ with $p \in R$. If we replace M, V and $\mathcal{C}$ by QMQ, $QV(\cdot)Q$ and $Q\mathcal{C}Q := \{QW(\cdot)Q\,;\ W \in \mathcal{C}\}$, then the proof reduces to the case, where M has a faithful normal state ρ.

We introduce the norm $\|b\|_\rho := \rho(b^*b)^{1/2}$ on M. The topology defined by the norm $\|\cdot\|_\rho$ coincides on *bounded parts* of M with the (ultra-)strong topology on M.

A separation argument shows that that $V\colon A \to M$ is moreover in the point-**ultra-strong* closure of of $\mathcal{C}$ if V is in the point-ultra-weak closure of $\mathcal{C}$. We may suppose that $\|V\| \leq 1$, then Lemma 3.1.8 implies that V can be approximated in point-strong topology by contractions $W \in \mathcal{C}$.

Let $\mathcal{C}_1$ the convex set of $W \in \mathcal{C}$ with $\|W\| \leq 1$, and let (a_n) denote a sequence that is dense in the unit ball of A. We find $W_n \in \mathcal{C}_1$ with $\|V(a_k)-W_n(a_k)\|_\rho < 8^{-n}$ for $k \leq n$. Let $b_n := \sum_{k\leq n}(V(a_k)-W_n(a_k))^*(V(a_k)-W_n(a_k))$. Then $\sum \rho(b_n) \leq 1$. Thus, for each $\varepsilon > 0$, there is a projection $p := p(\varepsilon) \in M$ with $\rho(1-p) \leq \varepsilon$ and $\sum_{n=1}^\infty pb_np \leq \varepsilon^{-1}p$, cf. Lemma B.7.1(i). It follows from Lemma B.7.1(ii) that $\lim_n f(pW_n(a_k)p) = f(pV(a_k)p)$ for every bounded linear functional $f \in M^*$ on M and every k. Since $\mathcal{C}_1 \subseteq \mathcal{L}(A, M)$ consists of contractions and since $\{a_1, a_2, \ldots\}$ is dense in the unit ball of A, the map $pV(\cdot)p$ is in the point-$\sigma(M, M^*)$ closure of the convex subset $p(\mathcal{C}_1)p := \{pW(\cdot)p\,;\ W \in \mathcal{C}_1\}$. Thus $pV(\cdot)p$ is the point-norm closure of $p(\mathcal{C}_1)p \subseteq \mathcal{C}$, i.e., $pV(\cdot)p \in \mathcal{C}$, $p \in R$. □

Remark B.7.3. A special case of Lemma B.7.2 is:
Suppose that M is a W^*-algebra, A is separable, $V\colon A \to M$ is weakly nuclear and that $e \in A_+$ satisfies $V(e) = 0$. Then there is a net of (norm-)nuclear contractions $T_\mu\colon A \to M$ with $T_\mu(e) = 0$ and $T_\mu \to V$ in point-ultrastrong topology.
It is in general not possible to find factorable maps T_μ with this property. See the following Remark B.7.4.

REMARK B.7.4. Let $C \subseteq \mathcal{M}(D)$ and $V: C \to B$ with $V(C \cap D) = \{0\}$ a nuclear map. Then $V_D(c+d) := V(c)$ for $c \in C$ and $d \in D$ is a well-defined completely positive map from the C^*-algebra $C + D$ into B, because $(C+D)/D \cong C/(C \cap D)$ naturally.

When $V_D: C + D \to B$ is again a nuclear map ? (It is an equivalent formulation of the general quotient problem for nuclear maps in Remark 3.1.2(iv), as our considerations below indicate.)

From [**238**] and [**438**] it follows that V_D is nuclear if C is exact (see Remark B.7.8). Clearly, V_D is nuclear if $[V]_{C \cap D}: C/(C \cap D) \to B$ is nuclear.

For a long time it was expected that V_D is always nuclear. But N. Ozawa [**597**] has shown that the von Neumann algebra generated by the regular representation of a non-exact discrete group is not weakly exact. M. Gromov [**335**] has shown the existence of discrete non-exact groups. (And Guentner and Kaminker [**338**] and Ozawa [**592**] discovered this connections between Gromov's work to non-exactness.)

By Definition 3.1.1, a map $T: C \to N$ from a C^*-algebra C to a von Neumann algebra N is **weakly nuclear** if T is the point-wise weak limit of factorable maps ([1]), compare Definition 3.1.1.

By [**597**] there exists a von-Neumann algebra that is not weakly exact. That means (equivalently) that there exists a von Neumann factor N of type II_1 with separable predual N_* and a weakly nuclear maps T from a separable C^*-algebra C into N and a closed ideal J of C such that $T(J) = \{0\}$ and that $[T]_J: C/J \to N$ is not weakly nuclear.

It implies that the same happens with a (norm-)nuclear map $V: C \to N$ with $V(J) = \{0\}$ (in place of T): An Egoroff type argument (see B.7.2) shows that every weakly nuclear c.p. contraction $T: C \to N$ from a separable C^*-algebra C into a von Neumann algebra with separable predual N_* is the point-strong limit of a sequence of (norm-)nuclear c.p. contractions $V_n: C \to N$ with $V_n(J) = \{0\}$. It follows that $[V_n]_J: C/J \to N$ is not weakly nuclear for almost all $n \in \mathbb{N}$. In particular, $[V]_J: C/J \to N$ is not nuclear for suitable $n \in \mathbb{N}$ and $V := V_n$.

Let $I := \mathrm{Ann}(J)$ denote the annihilator of J in C, let $p \in C^{**}$ be the support projection of J in the second conjugate C^{**} of C and consider $E := C^*(C, p) \subseteq C^{**}$, $D := I + pE$. Then $E = C + pE = C + D$ is naturally a C^*–subalgebra of $\mathcal{M}(D)$, $a \in C \to (1_p)a \in E$ defines a C^*-morphism $W: C/J \to C + D = E$ such that $[V]_J = V_D \circ W$. Thus $V_D: C + D \to N$ can not be (weakly) nuclear.

In a similar way one finds a C^*-subalgebra $C_1 \subseteq B := C \oplus N \overline{\otimes} \mathcal{L}(\ell_2)$ and an inner c.p. contraction $W: B \to B$ such that $(W|C_1): C_1 \to B$ is nuclear but not residually nuclear (with respect to the action of $\mathrm{Prim}(B)$ on C_1): Stinespring dilation and application of an infinite repeat lead to a *-representation ρ of C into the von Neumann algebra tensor product $N \overline{\otimes} \mathcal{L}(\ell_2)$ and a projection $e \in \mathcal{L}(\ell_2)$ of

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[1] It is also a point-$\tau(N, N_*)$-limit of factorable maps, because the set of factorable maps is an operator convex cone.

rank one, such that $V(a) \otimes e = (1 \otimes e)\rho(a)(1 \otimes e)$ for all $a \in C$. Let $\rho_1(a) := (a, \rho(a)) \in B$, $C_1 := \rho_1(C)$ and $W(b) := (0, 1 \otimes e)b(0, 1 \otimes e)$ for $b \in B$. Then $W|C_1$ is nuclear and is inner in B, but is not residually nuclear on $C_1 \subseteq B$.

We use the following Lemmata B.7.5 and B.7.7 and the Definitions B.7.6 to bypass the above mentioned problems at least under certain good conditions.

LEMMA B.7.5. *Let $V : A \to \mathcal{M}(B) \subseteq B^{**}$ a completely positive map from A into the multiplier algebra $\mathcal{M}(B)$ of B. Then the following are equivalent:*

(i) *For every $d \in B$ the completely positive map $a \in A \mapsto d^*V(a)d \in B$ is nuclear.*
(ii) *V is a weakly nuclear map from A to the W^*-algebra B^{**} in sense of Definition 3.1.1 if V is considered as a completely positive map from A to the second conjugate B^{**} of B*

PROOF. One can see from the definition (of weakly nuclear maps), that the weakly nuclear maps build a point-weakly closed m.o.c. cone $\subseteq \mathcal{L}(A, B^{**})$. Since the unit-ball B is *-strongly dense in the unit-ball of B^{**} (by Kaplansky density theorem), a map $V : A \to B^{**}$ is weakly nuclear, if and only if, the maps $a \mapsto d^*V(a)d$ are weakly nuclear for every $d \in B$.

`Old Prop. 3.1.9(i) has been changed. Compare again!!`
`No! Next is not Part of Proposition 3.1.9(i-iv) !!!`

By Proposition 3.1.9(i), the map $a \in A \mapsto d^*V(a)d \in B$ is nuclear, if and only if, it is weakly nuclear as a map from A to B^{**}, cf. Proposition 3.1.9(i). □

DEFINITION B.7.6. We call a completely positive map $V : A \to \mathcal{M}(B)$ **weakly nuclear** if V satisfies the equivalent conditions (i) and (ii) of Lemma B.7.5.

An example of a weakly nuclear completely positive map is given by $B := \mathbb{K}(\mathcal{H})$, $V : A \to \mathcal{L}(\mathcal{H})$ a faithful *-representation of A. It is nuclear if and only if A is exact, cf. Remark 3.1.2(ii) and Corollary 5.6.3.

Let $D \subseteq B$ be a hereditary C^*-subalgebra of a C^*-algebra B. D is a **corner** of B, if there is a projection $p \in \mathcal{M}(B)$ such that $D = pBp$. Certainly the (two-sided) annihilator $\{b \in B : bD = 0 = Db\}$ of D is just the orthogonal corner $(1-p)B(1-p)$ of pBp. A corner is σ-unital if B is σ-unital, because the compression pbp of a strictly positive element b of B is a strictly positive element of pBp. The unital, in particular σ-unital, Calkin algebra $\mathrm{Q}(\mathbb{K})$ contains hereditary C^*-subalgebras which are not σ-unital.

We say that a *-subalgebra $E \subseteq B$ **generates a corner** of B if the closure D of EBE is a corner of B. The annihilator of E in B is then a corner of B, and is therefore σ-unital if B is σ-unital.

A hereditary C^*-subalgebra $D \subseteq B$ is **full** , if it generates B as a closed ideal. A theorem of L.G. Brown [**107**] says that $B \otimes \mathbb{K}$ and $D \otimes \mathbb{K}$ are isomorphic if B is σ-unital and D is a σ-unital full hereditary C^*-subalgebra of B. The isomorphism is induced in the sense of Remark 2.3.1 by an element $z \in B \otimes \mathbb{K}$ such that z^*z is an strictly positive element of $B \otimes \mathbb{K}$ and zz^* is a strictly positive element of $D \otimes \mathbb{K}$, cf. Corollary 5.5.6.

A subalgebra D of a C^*-algebra A will be called an **essential** subalgebra if zero is the only left annihilators of D in A, i.e., if $aD = \{0\}$ implies $a = 0$.

It is easy to see, that an ideal I of a C^*-algebra A is essential if and only if it has non-zero intersection with every non-zero ideal of A.

An extension E, i.e. an exact sequence $0 \to J \to E \to A \to 0$ (defined by the epimorphism $\eta \colon E \to A$), is **semisplit** if there is a completely positive contraction $V \colon A \to E$ with $\eta \circ V = \mathrm{id}_A$.

LEMMA B.7.7. *Suppose C is C^*–subalgebras of $\mathcal{M}(D)$, and that $V \colon C \to \mathcal{M}(B)$ is a weakly nuclear completely positive contraction. Then:*

(i) $\widetilde{V}(c + z1) := V(c) + z1$ *is a unital weakly nuclear map from the (outer) unitization $\widetilde{C}$ of C into $\mathcal{M}(B)$.*
(ii) *If $C \cap D$ generates a corner of D, then every weakly nuclear contraction $V \colon C \to \mathcal{M}(B)$ with $V(C \cap D) = \{0\}$ and with $V(C \cap (1 + D)) \subseteq \{1\}$ extends naturally to a unital weakly nuclear map $W \colon D + C + \mathbb{C}1 \to \mathcal{M}(B)$ with $W(D) = 0$ by $W(d + c + \xi 1) := V(c) + \xi 1$ for $d \in D$, $c \in C$ $1 \in \mathcal{M}(D)$ and $\xi \in \mathbb{C}$.*

PROOF. (i): For a contraction $d \in C$ holds $V(d^*d) \le 1$, $b^*V(d^*cd)b + zb^*b = b^*V(d^*(c + z)d)b + zb^*(1 - V(d^*d))b$, and $c + z1 \mapsto z \mapsto zb^*(1 - V(d^*d))b$ is a nuclear c.p. map, because 1 is not in C if the unit element 1 is outer adjoined. Therefore, for contractions $d \in C$, $c + z1 \mapsto b^*V(d^*cd)b + zb^*b$ is nuclear as a sum of two nuclear maps. The map $c + z1 \mapsto b^*\widetilde{V}(c + z1)b$ is the point-norm limit of those maps.

(ii): Let $C_1 := D + C + \mathbb{C}1 \subseteq \mathcal{M}(D)$.

If $1 \in C + D$ then $C_1 = D + C$ and $C_1/D \cong C/(D \cap C)$. If $1 \notin D + C$ then $C_1 = \widetilde{D + C}$ (respectively $C_1/D \cong \widetilde{C/(D \cap C)}$) is the outer unitization of $D + C$ (respectively of $C/(D \cap C)$). The condition $V(C \cap (1 + D)) \subseteq \{1\}$ says that $[V]_{D \cap C} \colon C/(D \cap C) \cong (D + C)/D \to \mathcal{M}(B)$ is unital if $1 \in C + D$. We can define in both cases a unital c.p. map $W \colon C_1 \to \mathcal{M}(B)$ by $W := [V] \circ \pi_D$, respectively by its extension to the outer unitization of $C + D$. Here $[V] := [V]_{D \cap C} \colon (C + D)/D \cong C/(D \cap C) \to \mathcal{M}(B)$ is well-defined because $V(D \cap C) = \{0\}$. Thus, W is a well-defined, unital c.p. map with $W(D) = \{0\}$. In both cases holds

$$W(d + c + \xi 1_{\mathcal{M}(D)}) = W(c + \xi 1) = V(c) + \xi 1_{\mathcal{M}(B)} \ .$$

It suffices to prove that for every $b \in B$ the c.p. map $W_b \colon c + d \mapsto b^*V(c)b$ is nuclear, because the outer unitization of W is then also weakly nuclear.

Let $p \in \mathcal{M}(D)$ the orthogonal projection such that pDp is the closure of $(C \cap D)D(C \cap D)$. Then $E := (1 - p)D(1 - p)$ is the (two-sided) annihilator $\{d \in D\, ;\ d(C \cap D) = 0 = (C \cap D)d\}$ of $C \cap D$. Therefore $CE \subseteq E$, $EC \subseteq E$ and

$pc(1-p) = 0$, i.e., $pc = cp$ for $c \in C$. Thus $F := E + C$ is a C^*-algebra, E is an ideal of $F \subseteq \mathcal{M}(D)$, F commutes with p and $pF = pC$. $C \cap E = C \cap (D \cap E) = (C \cap D) \cap E = 0$. Therefore $T|F$ is the composition of $F \to F/E \cong C$ with $c \in C \to b^*V(c)b$. Thus $T|F$ is nuclear and $T(F \cap D) = 0$.

Let X be any C^*-algebra. By the criteria in Remark 3.1.2(i), the algebraic tensor product $\mathrm{id}_X \odot (T|F)\colon X \odot F \to X \odot B$ extends to completely positive map S_1 from the minimal (=spatial) tensor product $X \otimes F$ into the maximal C^*-algebra tensor product $X \otimes^{\max} B$. Since $S_1(X \odot (F \cap D)) = 0$, we have that $S_1(X \otimes (F \cap D)) = 0$. Since F commutes with p, $F \cap D = p(F \cap D)p + (1-p)(F \cap D)(1-p)$. But $E = (1-p)(F \cap D)(1-p)$ and $C \cap D \subseteq p(F \cap D)p \subseteq p(C \cap D)p = C \cap D$. Thus $F \cap D = (C \cap D) + E$. It follows that D is the closure of $D(F \cap D)$.

It is now the essential point of the proof that, therefore, the distance lemma [**438**, lem. 3.9] applies and gives the identity $(X \otimes F) \cap (X \otimes D) = X \otimes (F \cap D)$ in the C^*-algebra $X \otimes (C + D)$ for every C^*-algebra X.

It follows $(X \otimes (C + D))/(X \otimes D) \cong (X \otimes F)/(X \otimes (F \cap D))$, because $X \otimes D$ is a closed ideal of $F \otimes (C + D)$.

Let $S_2\colon (X \otimes F)/(X \otimes (F \cap D)) \to X \otimes^{\max} B$ be the completely positive map that is induced by S_1. The composition of $X \otimes (C + D) \to (X \otimes F)/(X \otimes (F \cap D))$ with the completely positive map S_2 defines a completely positive map S_3 from $X \otimes (C + D)$ into $X \otimes^{\max} B$. The restriction of S_3 to the algebraic tensor product is just $\mathrm{id}_X \odot T$.

Since S_3 can be found for every C^*-algebra X, T is nuclear by the criteria in Remark 3.1.2(i). □

Remark B.7.8. To understand the essential point of the proof of Lemma B.7.7(ii), the reader should note, that the intersection formula $(F \otimes C) \cap (F \otimes D) = F \otimes (C \cap D)$ holds for every C^*-algebra D and $C \subseteq \mathcal{M}(D)$, *if and only if, F is exact*. This happens even if we consider here only separable C and require, in addition, that D is a simple, purely infinite and separable C^*-algebra.

The intersection formula it true for every pair of C^*-algebras $C, D \subseteq \mathcal{L}(\mathcal{H})$, if and only if, F satisfies the slice map property (S) of S. Wassermann. The Property (S) implies exactness, but it is unknown if (S) is equivalent to exactness.

On the other hand, for every C^*-algebra F, every locally reflexive C^*-algebra $C \subseteq \mathcal{M}(D)$, one can show that $(F \otimes C) \cap (F \otimes D) = F \otimes (C \cap D)$. (The ideal $C \cap D$ of C is a locally reflexive C^*-subalgebra of D.)

Thus our proof of Lemma B.7.7(ii) also shows that the natural extension V_D of V to $C + D$ with $V_D(D) = 0$ is weakly nuclear if C is locally reflexive in the sense of [**238**] (without any assumptions about the behavior of $C \cap D$ in D). All exact C^*-algebras are locally reflexive, cf. [**432**, rem. on p. 71] (compare [**802**] for the case of separable C^*-algebras).

8. Questions about pure infiniteness

compare and sort following questions !! Below, we list some open questions concerning the verification of pure infiniteness, and K_1-injectivity.

QUESTION B.8.1. Denote by $A * B$ the unital universal free C^*-algebra of unital A and B, and define the *free joint* $\mathcal{E}_{\text{free}}$ of A and B by

$$\mathcal{E}_{\text{free}}(A,B) := \{f \in \mathrm{C}([0,1], A * B)\, ;\; f(0) \in A * 1\,,\; f(1) \in 1 * B\,\}\,.$$

(1) Suppose that A is non-unital, separable and *locally purely infinite.*
Does there exist $n \in \mathbb{N}$ such that the unit of $M_n(\mathcal{M}(A))$ is properly infinite?
(It is the case for weakly purely infinite A, or if $\mathcal{M}(A)$ itself is locally purely infinite. in the sense of Definition 2.0.3)

(2) Suppose that A is unital and weakly p.i. (respectively locally p.i.) and that $1_{A/J}$ is properly infinite for every non-zero closed ideal $\{0\} \neq J \lhd A$.
Is 1_A properly infinite if A is not prime *?*
It leads to the –equivalent– question, whether or not n-purely infinite quotients of the unital C^*-subalgebra $\mathcal{E}_{\text{free}}(\mathcal{O}_\infty, \mathcal{O}_\infty)$ of $\mathrm{C}([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$ have properly infinite unit elements. See the farer going question (8) below.

(3) Let $D := \{z \in \mathbb{C}\, ;\; |z| \leq 1\}$ and $S^1 = \partial D$. Then $\mathrm{C}(D)$ is naturally isomorphic to the (unital) algebra $\mathrm{cone}(\mathrm{C}(S^1))$ and the corresponding epimorphism from $\mathrm{C}(D)$ onto $\mathrm{C}(S^1)$ is given by $\varphi(f) := f|S^1$. Let $f_0(z) := z$ and $u := \varphi(f_0)$. The unitary u defines an inner automorphism $\sigma(x) := u^*xu$ of $\mathcal{O}_\infty * \mathrm{C}(S^1)$. One can show ([2]) that a positive answer to the following question implies a positive answer to question (2):
Does the pull-back

$$(\mathcal{O}_\infty * \mathrm{C}\,(D)) \oplus_{\pi,\sigma\circ\pi} (\mathcal{O}_\infty * \mathrm{C}\,(D))$$

of the unital epimorphisms

$$\pi := \mathrm{id} * \varphi\colon\; (\mathcal{O}_\infty * \mathrm{C}\,(D)) \to \left(\mathcal{O}_\infty * \mathrm{C}\left(S^1\right)\right)$$

and of $\sigma \circ \pi$ have a properly infinite unit?
(Unfortunately, it is likely that (3) has a negative answer. And it seems that the same happens for $\mathcal{O}_2$ – in place of $\mathcal{O}_\infty$.)

(4) Suppose that A is a purely infinite (non-simple) algebra, and $a \in A_+$.
Are the (unital!) fibers of the upper (!) semi-continuous C^*–bundle

$$F(C^*(a), A) := (\{a\}' \cap A_\omega)/\, \mathrm{Ann}(a, A_\omega)$$

over $\mathrm{Spec}(a)$ *purely infinite?*
(The fibers are 2-p.i. by [**443**].)

More precise and actual citation?

[2] by [**448**, prop.1.6,rem.1.15(2), prop.A.4], because two unital *-endomorphisms of $\mathcal{O}_\infty$ are approximately unitarily equivalent by Theorem B

(5) *Are the fibers* B_t *(of* $F(C^*(a), A)$ *,* $t \in \mathrm{Spec}(a)$ *)* K_1*-injective ?*
(I.e., is $\mathcal{U}_0(B_t)$ the kernel of the natural map $\mathcal{U}(B_t) \to \mathrm{K}_1(B_t)$.)

(6) *Is* $\mathcal{O}_\infty * \mathrm{C}(S^1)$ K_1*-injective ?*
With other words:

Does there exist $v \in \mathcal{U}_0(\mathcal{O}_\infty * \mathrm{C}(S^1))$ *with* $ut_1 = vt_1$*, where* $t_1, t_2, \ldots$ *(respectively u) are the canonical generators of* $\mathcal{O}_\infty$ *(respectively of* $\mathrm{C}(S^1)$*)?*
If the answer is positive, then $\mathcal{U}(B)/\mathcal{U}_0(B) \cong \mathrm{K}_1(B)$. for every unital algebra B that contains a a copy of $\mathcal{O}_\infty$ unitally. (The answer could be negative. The question is possibly related to question (3). See also Proposition 4.3.6(iv).)

(7) *Is* $\mathcal{O}_\infty * \mathcal{O}_\infty$ K_1*-injective?*
Note that (6) and (7) are related, because unital copies of $\mathcal{O}_\infty$ in a unital C^*–algebra A are unitarily homotopic by a continuous path $u(t) \in \mathcal{U}(A)$, and one can manage that $[u(t)]_1 = 0$ in $\mathrm{K}_1(A)$.
(Since $h_i \colon \mathcal{O}_\infty \to A$, $i = 1, 2$, are unital, it follows from Theorem B – and from the UCT for $\mathcal{O}_\infty$ – that there is a continuous path $u(t)$ in the unitaries of A such that $\lim_{t\to\infty} \|u(t)^* h_1(b)u(t) - h_2(b)\| = 0$.)

Perhaps, $\mathcal{O}_\infty * \mathcal{O}_\infty$ and $\mathcal{O}_\infty * \mathrm{C}(S^1)$ are KK-equivalent.

At least, there are unital *-morphisms $\alpha \colon \mathcal{O}_\infty * \mathcal{O}_\infty \to \mathcal{O}_\infty * \mathrm{C}(S^1)$ $(\alpha(a * 1) := a * 1$ and $\alpha(1 * a) := (1 * U)^*(a * 1)(1 * U))$ and $\beta \colon \mathcal{O}_\infty * \mathrm{C}(S^1) \to \mathrm{C_b}(\mathbb{R}_+, \mathcal{O}_\infty * \mathcal{O}_\infty)$ $(\beta(a * 1) = a^*1$, $\beta(U) := u)$ such that $\pi \circ \beta \circ$ is the natural – constant – embedding from $\mathcal{O}_\infty * \mathcal{O}_\infty$ into $\mathrm{Q}(\mathbb{R}_+, \mathcal{O}_\infty * \mathcal{O}_\infty)$.)

(8) *Does* $\mathcal{E}_{\mathrm{free}}(\mathcal{O}_\infty, \mathcal{O}_\infty)$ *contain a non-trivial projection?*
Is the unit of $\mathcal{E}_{\mathrm{free}}(\mathcal{O}_\infty, \mathcal{O}_\infty)$ *properly infinite?*

(9) Let J a separable weakly purely infinite C^*–algebra such that there is a sequence of positive contractions $e_1, e_2, \ldots \in J_+$ and contractions $v_{n,k}$, such that $e_n v_{m,k} = v_{m,k}$, $v_{n,k} e_m = v_{m,k}$, $e_n e_m = e_m$, for $m < n$, $k, l \in \mathbb{N}$, $v_{m,k}^* v_{n,l} = \delta_{k,l}\delta_{m,n} e_n^2$ for $k, l, m, n \in \mathbb{N}$, moreover, we suppose that (e_n) is an approximate unit of J.
Is the unit $1_{\mathcal{M}(J)}$ *of* $\mathcal{M}(J)$ *infinite?*

(10) Suppose, in addition to the assumptions of (9), that J of (9) is an ideal of a unital C^*-algebra A and that there is a *-monomorphism $\varphi \colon \mathrm{C}_0(0, 1] \otimes \mathcal{O}_\infty \hookrightarrow A$ with $\varphi(f_0 \otimes 1) = \sum_n 2^{-n} e_n$.
Is the unit of A *properly infinite?*

A positive answer would imply that weakly purely infinite algebras are purely infinite, and that purely infinite algebras A satisfy the assumption of the question (12).

(11) Suppose that $a_1, a_2 \in A_+$ are *stable* elements of A (i.e., $D_k := \overline{a_k A a_k}$ is stable for $k = 1, 2$).

Is $a_1 + a_2$ a properly infinite element of A?

A positive answer would imply that locally purely infinite algebras are purely infinite, and that every purely infinite algebra A satisfies the assumption of the question (12).

(But it is even unknown whether or not sums of σ-unital stable ideals have a properly infinite strictly positive element.

What happens in the case where D_1 and D_2 are ideals of A and where A is moreover locally purely infinite?)

(12) Suppose that $F(C, A) := (C' \cap A_\omega)/\operatorname{Ann}(C, A_\omega)$ has a properly infinite unit for every separable commutative C^*-subalgebra C of A.
Is A strongly purely infinite?

(13) Suppose that A is p.i. *Is* $\mathrm{C}([0,1], A)$ *p.i.?*. (It is 2-p.i.)

Special case:

Let A a purely infinite separable nuclear C^*-algebra. *Is* $\mathrm{C}([0,1], A)$ *purely infinite?*.
For simple A the answer is positive, because simple p.i. algebras are strongly p.i.

(14) *Are local p.i. C^*-algebras A with the global Glimm halving property purely infinite?*
(A is p.i. if and only if A is l.p.i. and the sum $D = \overline{(D_1 + D_2)A(D_1 + D_2)}$ of any two stable hereditary C^*-subalgebras D_1 and D_2 of A contains a family $(D_\tau)_\tau$ of stable hereditary C^*-subalgebras of A such that each element of the Pedersen ideal of D can be approximated by elements from $\bigcup_\tau A(D_\tau)A$ – only elements of the union, not sums of them! –.)

The next question asks what happens with pure infiniteness if we have a conditional expectation of finite index onto a subalgebra.

Question B.8.2. Suppose that $A \subseteq B$ (both not necessarily simple) and that $P\colon B \to A$ is a *conditional expectation from B onto A of finite index*, i.e., $P|A = \mathrm{id}_A$, $\|P\| = 1$, and there exists $0 < \lambda < \infty$ with $\lambda^{-1} b \leq P \otimes \mathrm{id}_n(b)$ for all $b \in (B \otimes M_n)_+$ and $n \in \mathbb{N}$. (Note that the existence of P implies that A is unital if and only if B is unital, and that then P must be necessarily unital. More generally, A is a non-degenerate C^*-subalgebra of B in the non-unital case. This can be seen by passage to the second conjugate P^{**}.)

What about of the following statements (1)–(6) concerning versions of pure infiniteness?

(1) Every l.s.c. 2-quasi-trace of A is trivial, if and only if, every l.s.c. 2-quasi-trace of B is trivial.

(2) A is locally p.i., if and only if, B is locally p.i.

(3) A is p.i. (respectively is weakly p.i., strongly p.i.), if and only if, B is p.i. (respectively is weakly p.i., strongly p.i.).

(4) If B is unital, then 1_A is stably properly infinite (respectively is properly infinite), if and only if, 1_B is stably properly infinite (respectively is properly infinite).

(5) There are a *-monomorphism $h\colon B \to A \otimes \mathbb{K}$ and a conditional expectation $P'\colon D \to h(B)$ of finite index from the hereditary C^*-algebra D of $A \otimes \mathbb{K}$ generated by $h(B)$ onto $h(B)$.

(6) If $\varphi\colon A \to F$ is a *-epimorphism onto a finite AW*-factor F, then there exists a finite AW*-algebra G with finite-dimensional centre and a *-epimorphism $\psi\colon B \to G$, a unital *-morphism $\epsilon\colon F \to G$ and a conditional expectation $E\colon G \to \epsilon(F)$ of finite index, such that $\epsilon \circ \varphi = \psi|A$ and $\psi \circ P = E \circ \psi$.

(Alternatively, and almost equivalent: If the multiplier algebra $\mathcal{M}(X) \cap X$ of a not-necessarily unital C^*-system X contains a finite AW*-factor $F \subseteq \mathcal{M}(X) \cap X$ and if there is a conditional expectation from X onto F of finite index, then $X = \mathcal{M}(X)$ and X is a finite AW*-algebra with finite centre.)

(7) For every $b \in B_+$ there are $n \in \mathbb{N}$, a projection $q \in M_n$ and $a \in (A \otimes M_n)_+$ such that $b \otimes q$ is Cuntz-equivalent to a in $B \otimes M_n$. (At least, if $J = (A \cap J)B(A \cap J)$ for each closed ideal J of B, and $I = A \cap \overline{\mathrm{span}(BIB)}$ for each closed ideal of I of A.)

QUESTIONS B.8.3. *Are inductive limits A of pi(n) algebras $A_1, A_2, \ldots$ again pi(n)?*
Can the ultra-product $\prod_\omega A_k$ of pi(n) algebras A_k $(k = 1, 2, \ldots)$ have a non-zero finite-dimensional quotient? (Equivalently: *Is the c_0-sum of pi(n) C^*-algebras $\bigoplus_k A_k$ a pi(n) algebra?*)
Is there a general function $f\colon \mathbb{N} \to \mathbb{N}$ such that $A \otimes \mathbb{K}$ is pi$(f(n))$ if A is pi(n)?

The implication
$\{\ A$ is pi$(n)\ \} \Rightarrow \{\ A$ has property pi-m for *some* $m \geq n\ \}$
(m depending on A) was shown by an indirect argument (showing that A_ω is l.p.i. and, therefore, A_ω is traceless). Our there given argument does not indicate a way to find a uniform bound for m (if $n \in \mathbb{N}$ is given).

Fix $n \in \mathbb{N}$. Suppose that A is σ-unital, has no non-zero finite-dimensional quotient A/J of dimension $\leq n$, and that for each $a \in A_+$, positive $b \in \mathrm{span}(AaA)$ and $\varepsilon > 0$ there are $d_1, \ldots, d_n \in A$ such that $\|b - \sum_j d_j^* a d_j\| < \varepsilon$.

Further let $e \in A$ a strictly positive element of A. Then *each hereditary C^*-subalgebra $A_k := \overline{(e - 1/k)_+ A (e - 1/k)_+}$ is pi(n)*, by Lemma 2.12.4(iii).

Hence, positive answers to the above (equivalent) questions, would imply, that one can replace the condition (ii) in Definition 2.0.4 by the (possibly weaker) condition that A has no quotient A/J of dimension $\leq n^2$.

QUESTION B.8.4. *Is the unit 1_F of a unital weakly purely infinite C^*-algebra F properly infinite?*

By definition, F is pi(n) for some $n \in \mathbb{N}$. We know at least that $1_F \otimes 1_m$ is properly infinite for some $m \in \mathbb{N}$, because pi(n) implies that F is l.p.i.

Proof of Corollary 2.7.18

Proposition **??**, that there exists $m \geq n$ such that $a \otimes 1_m$ is properly infinite for every element $0 \neq a \in A$. In particular $1_A \otimes 1_m$ is properly infinite.

A positive answer would imply that every weakly p.i. algebra is purely infinite. Moreover, this would imply that each weakly p.i. algebra is commutant-p.i. in the sense of Definition 2.16.6.

QUESTION B.8.5. Is every commutant-p.i. algebra A (cf. Def. 2.16.6), strongly purely infinite?

QUESTION B.8.6. Suppose that Prim(A) is isomorphic to the Hilbert cube $[0, 1]^\infty$ and that each fiber (= each simple quotient) is isomorphic to $\mathcal{O}_2$.
Is A purely infinite?
(The question considers a special case of [**462**, ques. 4.8].)
Is A purely infinite if A is weakly purely infinite?

QUESTION B.8.7. l.p.i. implies w.p.i.?

w.p.i. implies p.i.?

p.i implies l.c.p.i.?

l.c.p.i. implies c.p.i.?

c.p.i. implies s.p.i.?

(Is special case of [**462**, ques. 4.8].)

`Merge below blue questions into above questions`

??????

QUESTION B.8.8. Is the unit of $\mathcal{M}(\overline{aAa})$ p.i. if $a \in A_+$ is properly infinite in A? ([**462**, ques.3.10])

A special open question is: *Is the unit of $\mathcal{M}(A)$ properly infinite if A is a separable purely infinite algebra?*

QUESTION B.8.9. $a, b \in A_+$ *commuting* and $(a - t)_+$ and $(b - t)_+$ properly infinite or zero for all $t > 0$. Is $a + b$ properly infinite?

(It is a special case of [**462**, ques.3.10].)

QUESTION B.8.10. Let A a separable unital C^*-algebra and that J and K are closed ideals of A with $J \cap K = \{0\}$. Suppose that the units $1_{A/J}$ and $1_{A/K}$ are properly infinite in A/J and A/K. It is easy to see that $1_A \otimes 1_2$ is properly infinite in $M_2(A)$.

Is 1_A properly infinite in A ?

If the unitary group of $A/(J + K)$ is connected, then the answer is positive. (Notice that $A = A/(J \cap K)$ is just the pull-back $A/J \oplus_{\mu,\nu} A/K$ of the epimorphisms $\mu \colon A/J \to A/(J + K)$ and $\nu \colon A/K \to A/(J + K)$.)

The question considers a test case of the question whether the proper infiniteness of a projection $p \in A$ can be verified by looking only to $p + J$ in all *prime* quotients A/J.

QUESTION B.8.11. Suppose that A is separable and is w.p.i. and $a, b \in A_+$ are *stable* contractions such that $a + b$ is strictly positive in A.

Does there exist a stable element $c = c(\gamma) \in A_+$ such that $(a + b - \gamma)_+$ is in the closed ideal of A generated by c?

The question is equivalent to the question, if every w.p.i. algebra is p.i. (And then, automatically, every p.i. algebra is c.p.i.)

QUESTION B.8.12. Is $\mathcal{M}(A)$ weakly purely infinite if A is weakly purely infinite and σ-unital?

QUESTION B.8.13. Is the unit of $\mathcal{M}(A)$ properly infinite if A is purely infinite and σ-unital?

9. On the properties "IR" and "stable rank one"

It's all related to the almost ridiculous Part (xvii) of Proposition 2.2.1.

`ALL HAS TO BE SORTED!!`

`DESIRE OF Remark 2.2.2:`

See Section 9 in Appendix B for the definitions of "Property *IR*" and "stable rank one", and for an explanation, why Part (xvii) of Proposition 2.2.1 is one of the needed observations for the proof of the Brown-Friis-Rørdam alternative for simple C^*-algebras with Property *IR* between pure infiniteness and "stable rank one".

`BEGIN OF LIST OF MATERIAL`

DEFINITION B.9.1. A C^*-algebra B has **stable rank one** if each element of $B + \mathbb{C} \cdot 1_{\mathcal{M}(B)}$ is in the operator-norm closure of the invertible elements in $C^*(B, 1_{\mathcal{M}(B)}) = B + \mathbb{C} \cdot 1_{\mathcal{M}(B)} \subseteq \mathcal{M}(B)$.

Obviously, C^*-algebras B that contain an infinite projection do not have this is never the case for any simple purely infinite C^*-algebras.)

A is simple, and has the property that each element T of $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ that is not in the norm-closure of invertible elements of $A + \mathbb{C} \cdot 1$ must be right-invertible or left-invertible in $A + \mathbb{C} \cdot 1$, – which would be the correct interpretation of Property *IR* in the special case of simple C^*-algebras A – then this property holds also for $(A \oplus +\mathbb{C} \cdot 1_{\mathcal{M}(A)}) \oplus \mathbb{C}$ and for any non-zero hereditary C^*-subalgebra $D \subseteq A$, i.e., also for $D + \mathbb{C} \cdot 1 \subseteq A + \mathbb{C} \cdot 1$ holds that any element that can not be approximated by invertible elements in $D + \mathbb{C} \cdot 1$ is left- or right-invertible (but is not invertible).

This passage to hereditary C^*-subalgebras is a key point and is equivalent to the Morita equivalence of property *IR* on the class of simple C^*-algebras.

It seems that it suffices to consider: passage to $M_2(A)$, to inductive limits, and to (non-zero) corners of (simple) C^*-algebras, to get Morita equivalence of property *IR*.

The same happens with property “real rank zero”.

as the property that there exists a hereditary C^*-subalgebra

there exist elements in $\widetilde{A} := A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ that are not in the closure of of the invertible elements ... ,if and only if, A contains an infinite projection. Thus ????

Each element in $A + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$ that is not in the closure of the set of invertible elements in $A + \mathbb{C} \cdot 1$ has a left or a right inverse (but not both of them) and $A + \mathbb{C}$ has not stable rank one, i.e., the invertible elements are not dense in $A + \mathbb{C}$. (Or is $M_n(A + \mathbb{C})$ to consider also? And its stabilizations ...? Is it a stable problem?)

This gives immediately that $A + \mathbb{C}$ contains a non-unitary isometry ... Delivers a projection in A. Then to every non-zero hereditary C^*-subalgebra of A...?

It carries over to $D + \mathbb{C}$ for all non-zero hereditary C^*-subalgebras D of A (by stabilization ?)

`ABOVE TEXT VERY RANDOM!`
`COMPARE WITH BELOW`
`GIVE ONLY SHORT SUMMARIES ...`

The C^*-algebras with stable rank one are – in a sense – opposites to (weak) purely infinite C^*-algebras, ... at least in case of simple C^*-algebras.

Definition B.9.2. A C^*-algebra B has “stable rank one” (denoted by “*SR1*”) if each element of $B + \mathbb{C} \cdot 1_{\mathcal{M}(B)} \subseteq \mathcal{M}(B)$ can be approximated in norm by invertible elements in $B + \mathbb{C} \cdot 1$.

It is easy to see from the definition that the class of C^*-algebras with stable rank one are invariant under following operations:

(i) forming direct sums $B_1 \oplus B_2$,
(ii) passage to quotients B/J,
(iii) forming of inductive limits, e.g. B has SR1 if there are C^*-subalgebras $B_1 \subset B_2 \subset \cdots \subset B$ of C^*-subalgebras B_n with SR1 such that $\bigcup_n B_n$ is dense in B,
(iv) pBp has SR1 if B has SR1 and $p^* = p \in \mathcal{M}(B)$ is a projection.
(v) $M_2(B)$ has SR1 if B has SR1. (use a modification of the classical Gauss algorithm here in an approximate manner). It follows that $M_n(B)$ has SR1 (in conjunction with Part (iv)).
(vi) $B \otimes \mathbb{K}$ has SR1 if B has SR1.
(vii) B has SR1 and separable and A is a separable C^*-algebra that is Morita equivalent to B then A has SR1.

WHAT about hereditary subalgebras/ideals?

The following Lemma B.9.3 is due to L.G. Brown [**109**].

LEMMA B.9.3. *Let D a hereditary C^*-subalgebra of B and $d \in D+\mathbb{C}\cdot 1 \subseteq \mathcal{M}(B)$.*

Suppose that there exists a sequence (b_n) of elements in $b_n \in B+\mathbb{C}\cdot 1$ that are invertible in $B+\mathbb{C}\cdot 1$ and satisfy $\lim_n \|b_n - d\| = 0$. Then there exists a sequence of $c_n \in D+\mathbb{C}\cdot 1$ that are invertible in $D+\mathbb{C}\cdot 1$ and satisfy $\lim_n \|c_n - d\| = 0$.

It allows to show that (non-zero) simple C^*-algebras A are purely infinite if A has not SR1 but satisfies property *IR*.

`Next is my Definition!!  Better name needed?`

DEFINITION B.9.4. We say (for a moment and only here) that an element $x \in B+\mathbb{C}\cdot 1_{\mathcal{M}(B)} \subseteq \mathcal{M}(B)$ is "*well-behaved*" if, for every closed ideal $J \subseteq B$ of B, the element $\pi_J(x)$ is invertible in $(B+\mathbb{C})/J$ if $\pi_J(x)$ is left or right invertible in $(B+\mathbb{C})/J$.

If B is simple, then this says that $x \in B+\mathbb{C}\cdot 1_{\mathcal{M}(B)}$ is "well-behaved" if x is invertible if it is left or right invertible.

If, for example, $\pi_J(x)$ is neither left- nor right-invertible for all closed ideals J of B, then x is well-behaved

Notice that P. Friis and M. Rørdam have 1996 in [**305**] the following property *IR* that sounds a bit like a weakening of the property *SR1* ???, but is rather different.

DEFINITION B.9.5. A C^*-algebra B has the property *IR* if each element $b \in B+\mathbb{C}\cdot 1 \subseteq \mathcal{M}(B)$, that is "well-behaved" in sense of Definition B.9.4, is in the norm-closure of the invertible elements in $B+\mathbb{C}\cdot 1$.

Thus property IR causes:
If $b \in B+\mathbb{C}\cdot 1$ is not in the norm-closure of the invertible elements in $B+\mathbb{C}\cdot 1$ and B has property IR, then there exists an closed ideal J of B such that $\pi_J(b)$ is left- or right-invertible but is not invertible in $(B+\mathbb{C}\cdot 1)/J$.

The point is now the following: `Is it the Proposition 2.2 ???`

If $b = c + \lambda 1_{\mathcal{M}(B)}$ (with $c \in B$) can be approximated by invertible elements in $B+\mathbb{C}\cdot 1_{\mathcal{M}(B)}$ and D is a hereditary C^*-subalgebra of B with $c \in D$, then $c+\lambda 1_{\mathcal{M}(D)}$ can be approximated by invertible elements of $c+\lambda 1_{\mathcal{M}(D)}$

(Lemma of Brown, which one?).

THEOREM B.9.6. *Simple C^*-algebras with property* IR *have stable rank one or are purely infinite.*

For the proof see [**305**] and [**109**].

Can proceed as follows:

If A is simple and has property IR and A has not stable rank one.

Then every non-zero hereditary C^*-subalgebra D of A has not stable rank one, because “stable rank one” is invariant under Morita equivalence.

So, if A has not stable rank one, then D can not have stable rank one.

If D has not stable rank one then there exists $d \in D$ such that d or $d + 1_{\mathcal{M}(A)}$ can not be approximated by invertible elements in $D + \mathbb{C} \cdot 1$.

The point is now, that one must show that d (or $d+1$) can not be approximated by invertible elements in $A+\mathbb{C}\cdot 1$ if d (or $d+1$) can not be approximated by invertible elements in $D + \mathbb{C} \cdot 1 \subseteq \mathcal{M}(A)$.

– This is because *IR* is not proven to be Morita equivalent so far –

Possible way:

1.) Passage to $M_2(A)$, 2.) Passage to corner, 3.) to inductive limits, ...

If there is no passage to σ-unital hereditary C^*-subalgebras, this does not help ...

How to check if *IR* passes to hereditary C^*-subalgebras D: All closed ideals of D are intersections $J \cap D$ with closed ideals J of A.

Let $d \in D \subseteq A$ well-behaved in $D + \mathbb{C} \cdot 1_{\mathcal{M}(D)}$, is it then also well-behaved in $A + \mathbb{C} \cdot 1_{\mathcal{M}(D)}$?

And conversely?

Suppose that $\pi_J(d) + z1$ is approx-invertible in $A + \mathbb{C} \cdot 1$, the also approx-invertible in $D + \mathbb{C} \cdot 1$? And conversely?

(Is it not so, that there exists $\rho > 0$ such that $d - z1_{\mathcal{M}(A)}$ is not invertible for every $z \in \mathbb{C}$ with $|z| < \rho$?

Otherwise, we find a zero-sequence $z_n \in \mathbb{C}$ with $d - z_n$ invertible in for every $n \in \mathbb{N}$ and $|z_n| \to 0$. Then d is in the closure if the invertible elements in $D + \mathbb{C} \cdot 1$.

Multiplying d or $d + 1$ with a constant we may suppose that $d - z1$ is not invertible for all $z \in \mathbb{C}$ with $|z| \leq 1$.

That implies that, e.g. , d can not be approximated by invertible elements in $A + \mathbb{C}1$ because ...

If T is invertible, then $(T - z)$ is invertible for all $z \in \mathbb{C}$ with $|z| < \|T^{-1}\|$??

and $\|T - d\| < \gamma$, implies $\mathrm{Spec}(T) \subseteq (\gamma \circ D_2) + \mathrm{Spec}(d)$,

$\|1 - T^{-1}d\| \leq \|T^{-1}\| \cdot \|T - d\|$

?????

Now in the class of C^*-algebras ?????

(Other idea:
Try to show directly that $D+\mathbb{C}1$ has also property *IR*, because then we get directly that d (or $d+1$) are right or left invertible. It is a sort of invariance of property *IR* under Morita equivalence.)

(But a Lemma/Preposition of L.G. Brown gives this!).

L.G. Brown:

`Is it the proposed Proposition 2.2 ???`

The intersection of $D + \mathbb{C} \cdot 1_{\mathcal{M}(A)}$ with the operator-norm closure in $\mathcal{M}(A)$ of the invertible elements in $A + \mathbb{C} \cdot 1_{\mathcal{M}(A)}$ is equal to the operator-norm closure of the invertible elements in $D + \mathbb{C} \cdot 1$ in $\mathcal{M}(A)$.

All ???? inside ????? in $\mathcal{M}(A)$?

If ????

If A has property IR*, this implies* that d (or $d+1$) is right-invertible or left-invertible in A or $A + \mathbb{C} \cdot 1_{\mathcal{M}(A)}$.

If $d+1$ is not invertible but is right-invertible in $A + \mathbb{C}1$, it implies that there exists right-inverse $R = e + z1 \in A + \mathbb{C} \cdot 1_{cM}$ for $d + 1_{\mathcal{M}}$ shows that $(e+z1)(d+1) = 1$ (respectively $(e+z1)d = 1$).

Thus, $\|e + z1\|^2 (1+d)^*(d+1) \geq 1$ and $d+1$ (respective d) is right invertible in $D + \mathbb{C}1$ by $[(1+d)^*(d+1)]^{-1}(1+d)^*$. Thus, $D + \mathbb{C} \cdot 1$ contains a non-unitary isometry ...

Perhaps there exists a proof that works without Lemma of Brown?

The "opposite" direction is "Corollary 1.11":
Any purely infinite simple C^*-algebra satisfies property IR.

`Here is a collection of excerpts:`

Material for Part (xvii) of Proposition 2.2.1!!!

ON BROWN CRITERIUM:
M. Rieffel: "Stable rank one":
$\mathrm{GL}(\widetilde{A})$ is dense in $\widetilde{A}$.

For a unital C^*-algebra A, P. Friis and M. Rørdam [**305**] defined $R(A)$ as follows:
The element a is in $R(A)$ if and only if there does not exist a (closed two-sided) ideal I such that $\pi_I(a)$ is one-sided invertible but not invertible, where $\pi_I \colon A \to A/I$ is the quotient map.

They then said that A satisfies IR if $R(A)$ is in the (norm) closure of $\mathrm{GL}(A)$.

Of course it is obvious that $GL(A) \subseteq R(A)$, since no element that is one-sided invertible but not invertible (in some quotient algebra of A) can be approximated by invertible elements of A.
For non-unital A, they said A satisfies IR if $\widetilde{A}$ does. Here $\widetilde{A} = A$ if A is unital and $\widetilde{A} = C^*(A, 1) \subseteq \mathcal{M}(A)$ if A is not unital.

For expository purposes we introduce a formally weaker property, but in Proposition 2.2 we will show it is equivalent.
An arbitrary C^*-algebra A satisfies IR_0 if every element of $R(A^+) \cap (1 + A)$ is in

the closure of $\mathrm{GL}(A^+)$.

Obviously IR_0 is equivalent to IR in the unital case, but for non-unital A we are leaving open (for now) the possibility that $R(\widetilde{A}) \cap A$ is not contained in the closure of $\mathrm{GL}(A^+)$.

Lemma 1.2. (cf. [4, Theorem 3.5]).

Let B be a proper hereditary C^*-subalgebra of a unital C^*-algebra A, and identify B^+ with $B + \mathbb{C}1$. For t in $1 + B$, if $t \in GL(A)$, then $t \in GL(B^+)$.

Proposition 1.9.

If A is the direct limit of an upward directed family $\{B_i\}$ of hereditary C^*-subalgebras, and if each B_i satisfies IR, then A satisfies IR.

Corollary 1.10.

If A has an approximate identity of projections, in particular if A is of real rank zero, then A satisfies IR if and only if pAp satisfies IR for each projection p in A.

Corollary 1.11.

Any purely infinite simple C^*-algebra satisfies property IR.

Theorem 2.5.

Let I be a closed two-sided ideal of a C^*-algebra A. Then A has IR if and only if both I and A/I have IR and invertibles lift from $\widetilde{A}/I$ to $\widetilde{A}$.

Theorem 3.2.

Let A be a C^*-algebra of real rank zero. Then A has IR, if and only if, whenever p and q are projections in A generating the same ideal, then $(1 - p) \sim (1 - q)$ in A^+ implies $p \sim q$.

Proposition 4.1.

If A is a simple C^*-algebra, then A has IR if and only if either A has stable rank one or A is purely infinite.

My comment: “stable rank one” is in case of simple C^*-algebras what ? ...

It was pointed out in [**305**] that stable rank one implies IR

(“Stable rank in this sense was introduced by Rieffel [9] and stable rank one means that $GL(\widetilde{A})$ is dense in $\widetilde{A}$.”)

(Thus, in case of simple A, excluding “stable rank one” means that there exist elements in $\widetilde{A}$ that can not be approximate by invertible elements.)

Rieffel showed in [9] that C^*-algebras of stable rank one satisfy a stronger cancellation property: If p and q have the same image in $\mathrm{K}_0(A)$, then $p \sim q$. Of course $p \sim q$ implies that p and q generate the same ideal I and have the same image in $K0(I)$. We show next that stable IR implies a

?????

Proof.

For the direction not already proved, assume that A has IR and is not of stable rank one. We need to show that every non-zero hereditary C^*-subalgebra B of A contains a non-zero projection and that every non-zero projection is infinite.

For the first, note that B cannot have stable rank one, since B is strongly Morita equivalent to A and the stable rank one property is preserved by strong Morita equivalence. We may assume B is not unital, and thus $\widetilde{B}$ can be identified with $B + \mathbb{C} \cdot 1 \subseteq \widetilde{A}$. Then there must be $t \in 1 + B$ such that $t \notin R(\widetilde{B})$ (since B has IR by Proposition 1.4). Since the only relevant quotient of $\widetilde{B}$ is $\widetilde{B}$ itself, t is one-sided invertible but not invertible, and B contains a proper isometry u. Then $1 - uu^*$ is the desired non-zero projection in B. For the second, let p be a non-zero projection in A, and let $B = pAp$. The same sort of reasoning as above shows that there is a proper isometry in B, whence p is infinite. □

`Decide what is really needed from below!!`

Some More Text from L.G. Brown, to support the relation of this this Dichotomy Property with the rather simple property in Part (xvii) of Proposition 2.2.1:

Text comes mainly from [**109**]:

For a non-unital C^*-algebra A, $\widetilde{A}$ denotes the result of adjoining an identity, and $\widetilde{A}$ if A is unital. Let A^+ denote $\widetilde{A}$ if A is non-unital and $A^+ := A \oplus \mathbb{C}$ if A is unital.

DEFINITION OF $R(A)$ and IR:
For a unital C^*-algebra A, Friis and Rørdam [**305**] defined $R(A)$ as follows:

The element a is in $R(A)$, if and only if, there does not exist a (closed two-sided) ideal J such that $\pi_J(a)$ is one-sided invertible but not invertible, where $\pi_J \colon A \to A/J$ is the quotient map. They then said that A satisfies IR if $R(A)$ is in the (norm) closure of $GL(A)$.

Of course it is obvious that $\overline{GL(A)} \subseteq R(A)$, since no element that is one-sided invertible but is not invertible (in some quotient algebra of A) can be approximated by invertible elements of A.

For non-unital A, they said A satisfies IR if $\widetilde{A}$ does.

We introduce a formally weaker property, but in Proposition 2.2 we will show it is equivalent.

An arbitrary C^*-algebra A satisfies IR_0 if every element of $R(A^+) \cap (1 + A)$ is in the closure of $GL(A^+)$.

Obviously IR_0 is equivalent to IR in the unital case, but for non-unital A we are leaving open (for now) the possibility that $R(\widetilde{A}) \cap A$ is not contained in the closure of $GL(\widetilde{A})$.

Lemma [**305**, lem. 4.3] shows that direct products (also known as ℓ_∞-direct sums) of C^*-algebras with IR have IR, and the proof contains the assertion that IR

passes to ideals (see Remark 1.5 below for more on this). We proceed to generalize the latter result. Our proof relies on Proposition 2.2, but the result is included in this section for expository purposes.

Lemma 1.2.
([**114**, thm. 3.5]).
Let B be a proper hereditary C^*-subalgebra of a unital C^*-algebra A, and identify B^+ with $B + \mathbb{C} \cdot 1$, where 1 means 1_A. For all t in $1 + B$ holds: If $t \in \overline{GL(A)}$, then $t \in \overline{GL(B^+)}$.

Proposition 1.4.
Any hereditary C^*-subalgebra of a C^*-algebra with IR also has IR.

(HERE A CRITICAL REMARK TO [**305**, lem. 4.3])

Remark 1.5.
The argument given in the proof of [**305**, lem. 4.3] for the fact that any ideal I in a C^*-algebra with IR also has IR actually shows only that I has IR_0. Of course this is remedied by our Proposition 2.2, but it is not hard to see, without using any results from the present paper, that [**305**, lem. 4.3] is correct as stated.

Proposition 1.6.
Let B be a hereditary C^*-subalgebra of a unital C^*-algebra A. Then $(1 + B) \cap R(A) \subseteq R(B + \mathbb{C} \cdot 1)$.

DEFINITION OF $R(A)$ is where ???

Example 1.7.
The last result is not true for elements of B; i.e., $B \cap R(A)$ need not be contained in $R(B + \mathbb{C} \cdot 1)$.

To see this let A_0 be a non-unital purely infinite simple C^*-algebra, let $A := \widetilde{A_0}$, and let $B := pAp$ for a non-zero projection p in A_0. If u is a proper isometry in B, then $u \notin R(B + \mathbb{C} \cdot 1)$, since $B + \mathbb{C} \cdot 1 \cong B \oplus C$, but $u \in R(A)$, since the only relevant quotient of A is A itself.

Lemma 1.8.
If A is the direct limit of a directed family $\{B_i\}$ of hereditary C^*-subalgebras, and if each B_i satisfies IR_0, then A satisfies IR_0.

Proposition 1.9.
If A is the direct limit of
an upward directed

family $\{B_i\}$ of hereditary C^*-subalgebras, and if each B_i satisfies IR, then A satisfies IR.

`What is correct??`

If A is the direct limit of
a directed family $\{B_i\}$ of hereditary C^*-subalgebras, and if each B_i satisfies IR_0, then A satisfies IR_0.

10. Uniform global Glimm halving

REMARK B.10.1. Let $A_\omega := \ell_\infty(A)/J_\omega$, and $B \subseteq A_\omega$ a C^*-subalgebra. We define

$$F(B, A) := (B' \cap A_\omega)/\operatorname{Ann}(B, A_\omega)$$

and the special case $F(A) := F(A, A)$.

One can formulate "uniform variations" of the global Glimm halving property for A, respectively of the non-existence of characters on A:

(i) There is (universal) $n \in \mathbb{N}$ such that, for every $a \in A_+$, $c \in A_+$ with $\|c\| \leq 1$, $ca = a$, and $\varepsilon > 0$, there exists $b, d_1, \ldots, d_n \in A$, $b \in aAa$, $b^2 = 0$, $\|b\| = 1$, $\|[b, a]\| < \varepsilon$, $\|[d_j, a]\| < \varepsilon$, $\sum d_j^* bb^* d_j = c$.
(ii) For all $a^* = a \in A_\omega$, $F(C^*(a), A)$ has the global Glimm halving property (Notice here that $F(C^*(a_+), A) \oplus F(C^*(a_-), A) \cong F(C^*(a), A)$.)
(iii) A_ω has the global Glimm halving property, i.e., for all $a \in (A_\omega)_+$, $\varepsilon > 0$, there exist $b_{\varepsilon,a} \in a(A_\omega)_+ a$ with $b_{\varepsilon,a}^2 = 0$, $(a - \varepsilon)_+ \in I(b_{\varepsilon,a})$ (i.e., for every $\delta \in (0, \varepsilon)$, there exist $n = n(\delta)$ and $d_1, \ldots, d_n \in A_\omega$ such that $\sum_j d_j^*(b_{\varepsilon,a}^* b_{\varepsilon,a}) d_j = (a - (\varepsilon + \delta))_+$).
(iv) There is (universal) $n \in \mathbb{N}$ such that for $a, c \in A_+$ with $\|c\| = 1$ and $ca = a$ there exists $b \in A$ with $b^2 = 0$, $cb = bc = b$ and $d_1, \ldots, d_n \in A$ with $\sum d_j^*(bb^* + b^*b) d_j = a$.
(v) No hereditary C^*-subalgebra of A_ω has a character $\neq 0$. (It suffices to consider σ-unital hereditary subalgebras.)
(vi) There exists (universal) $n \in \mathbb{N}$, such that for all $a, c \in A_+$, $\|c\| = 1$, $ca = a$ there are $b_1, \ldots, b_n, d_1, \ldots, d_n \in A$ with $b_j^2 = 0$, $\|b_j\| = 1$, $cb_j = b_j c = b_j$, $cd_j = d_j c = d_j$, and $\sum d_j^*(b_j^* b_j + b_j b_j^*) d_j = a$.

Then (i)⇔(ii), (iii)⇔(iv), and (v)⇔(vi). Moreover (iv)⇒(vi).

11. Examples that exhaust the UCT-class

The following Lemma B.11.1 shows that every $\mathbb{Z}_2$-graded countable Abelian group $G = G_0 \oplus G_1$ is isomorphic to $\mathrm{K}_*(C_0(X))$ for some Polish l.c. space X that is locally homeomorphic to an at most 3-dimensional CW-complex – optical not visibly because some of them can be considered as topological subspaces of $\mathbb{R}^7$ but not of $\mathbb{R}^6$ or the they can be in the "visible" class of subsets of $\mathbb{R}^3$. In particular, *each separable C^*-algebra in the UCT class is* KK*-equivalent to* $\mathrm{C}_0(X)$ *for at least one of such* X.

LEMMA B.11.1. *Let W denote the compact Alexandroff (one-point)-completion of $\mathbb{N} \times \mathbb{R}$. If G is a countable Abelian group, then there exists a continuous map*

$\varphi\colon W \to W$ such that $\varphi(\infty) = \infty$ and that $\mathrm{K}^1(X) = 0$ and $\mathrm{K}^0(X) \cong G$ for $X := Y \setminus ((0,1] \times \{\infty\})$ where Y denotes the mapping cone of

$$Y := ((0,1] \times W) \cup_\varphi W$$

of φ.

Proof. Let $\mathbb{Z}^\infty$ denote the free Abelian group with free generators $y_1, y_2, \ldots$. If $x_1, x_2, \ldots \in G$ is a sequence that generates G, then there is an epimorphism $\lambda\colon \mathbb{Z}^\infty \to G$ with $\lambda(y_p) := x_p$. We can suppose that 0 appears in the sequence (x_n) infinitely often. Then the kernel $F \subseteq \mathbb{Z}^\infty$ of λ is not finitely generated. By [**308**, thm. 14.2], there is an isomorphism $\mu\colon \mathbb{Z}^\infty \to F$ from $\mathbb{Z}^\infty$ onto F.

It is not difficult to see that there is an automorphism σ of $\mathbb{Z}^\infty$ (given by an inductively selected new basis $y'_1, y'_2, \ldots$ of $\mathbb{Z}^\infty$), such that $\nu := \mu \circ \sigma\colon \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ has the property that

$$\nu(y_p) = \sum_{j=k_p}^{l_p} g_{p,j} y_j$$

with $k_p \leq l_p$, $g_{p,j} \in \mathbb{Z}$ and $\lim_{p\to\infty} k_p = \infty$.

It follows that it suffices to define suitable continuous maps φ_p from $C_p := \{p\} \times S^1 \cong (\mathbb{R} \cup \{\infty\})$ (where we identify $\mathbb{R}$ with $\{e^{is}\,;\ s \in (0, 2\pi)\}$) onto the rosette

$$W_p := (\{k_p, k_p+1, \ldots, l_p - 1, l_p\} \times \mathbb{R}) \cup \{\infty\}\,.$$

The latter is the identification of the points $(j,1)$ (representing our infinite point of W_p) in the $m_p := 1 + l_p - k_p$ disjoint copies $C_j = \{j\} \times S^1$ of S^1. There exists maps $\varphi_p\colon C_p \to W_p$ that satisfy $\varphi_p(\infty) := \infty$ and that the maps

$$\mathrm{K}^1(\varphi_p)\colon\ \mathrm{K}^1(W_p) \to \mathrm{K}^1(C_p)$$

satisfy $\mathrm{K}^1(\varphi_p)[U_j] = g_j[V_p]$ in $\mathrm{K}^1(W_p)$. Here $U_j\colon W_p \to S^1$ and $V_p\colon C_p \to S^1$ are defined by $U_j(k,z) := 1$ if $k \neq j$ and $U_j(j,z) := z$, and by $V_j(j,z) := z$. The map $\varphi_p\colon C_p \to W_p$ can be defined as follows:
Select $t_0 = 0 \leq t_1 \leq \cdots \leq t_{m_p} = 2\pi$ with $t_{j-1} = t_j$ if $g_j = 0$ and $t_{j-1} < t_j$ if $g_j \neq 0$. If $g_j = 0$, let $\varphi_p((p, e^{t_j 2\pi i})) := (j,1) = \infty$. If $g_j \neq 0$, then define homeomorphisms $\gamma_j\colon [t_{j-1}, t_j] \to [0, 2|g_j|\pi]$ by the affine maps $\gamma_j(s) := (s - t_{j-1})(t_j - t_{j-1})^{-1} 2 g_j \pi$ if $g_j > 0$ and $\gamma_j(s) := (t_j - s)(t_j - t_{j-1})^{-1} 2|g_j|\pi$ if $g_j < 0$. Now define $\varphi_p((p, e^{2\pi s})) := (j, e^{i\gamma_j(s)})$??? for $s \in [t_{j-1}, t_j]$ (where we identify all $(j,1)$ with ∞).

Then we define $\varphi\colon W \to W$ by $\varphi((p,z)) := \varphi_p((p,z))$ -

Let $Y := ((0,1] \times W) \cup_\varphi W$ denote the mapping cone of $\varphi\colon W \to W$. It follows, that $X := Y \setminus ((0,1] \times \{\infty\})$ is locally a 2-dimensional CW-complex, that $\mathrm{K}^1(W) = \mathrm{K}^1(\mathbb{N} \times \mathbb{R}) \cong \mathbb{Z}^\infty$ and that $\mathrm{K}^1(\varphi)\colon \mathrm{K}^1(\mathbb{N} \times \mathbb{R}) \to \mathrm{K}^1(\mathbb{N} \times \mathbb{R})$ defines the monomorphism $\nu\colon \mathbb{Z}^\infty \to \mathbb{Z}^\infty$. In particular,

$$\mathrm{K}_1(\mathrm{C}_0(\varphi))\colon\ \mathrm{K}_1(\mathrm{C}_0(\mathbb{N} \times \mathbb{R})) \to \mathrm{K}_1(\mathrm{C}_0(\mathbb{N} \times \mathbb{R}))$$

is an injective map with co-kernel isomorphic to G.

The Mayer-Vietoris sequence shows that the 6-term K_*-sequence of the short-exact sequence

$$\mathrm{C}_0((0,1)\times(\mathbb{N}\times\mathbb{R})\to \mathrm{C}_0(X)\to \mathrm{C}_0(\mathbb{N}\times\mathbb{R})$$

(defined by a pull-back) has injective boundary map

$$\partial\colon\ \mathrm{K}_1(\mathrm{C}_0(\mathbb{N}\times\mathbb{R}))\to \mathrm{K}_0(\mathrm{C}_0((0,1)\times(\mathbb{N}\times\mathbb{R}))\cong \mathrm{K}_1(\mathrm{C}_0(\mathbb{N}\times\mathbb{R}))$$

given by $\mathrm{K}_1(\mathrm{C}_0(\varphi))$. Since $\mathrm{K}_1(\mathrm{C}_0((0,1)\times(\mathbb{N}\times\mathbb{R}))=\mathrm{K}_0(\mathrm{C}_0(\mathbb{N}\times\mathbb{R}))=0$, we get $\mathrm{K}^1(X)=\mathrm{K}_1(\mathrm{C}_0(X))=0$ and $\mathrm{K}^1(X)=\mathrm{K}_0(\mathrm{C}_0(X))\cong G$. □

REMARK B.11.2. If the group G is finitely generated then one can take for X with $\mathrm{K}^0(X)\cong G$ and $\mathrm{K}^1(X)=0$ likewise:

(i) $X :=$ a finite disjoint union of points and of pointed Moore spaces

$$(\mathbb{D}/\sim_n)\setminus\{0\}$$

(For construction of Moore spaces see [**692**, ex.12.2]), or – alternatively –

(ii) $X :=$ the mapping cone of a continuous map from a not necessarily finite cloverleaf $(A\times\mathbb{R})\cup\{\infty\}$ into a not necessarily finite cloverleaf (or a point) $(B\times\mathbb{R})\cup\{\infty\}$, where A is a finite set of cardinality $|A|=$ minimal number of generators of G and $|B|\le|A|$ (and B can be empty).

12. S.p.i. subalgebras B with Hausdorff $\mathrm{Prim}(B)$

We say that a C^*-algebra D *can be locally approximated by C^*-subalgebras B with Hausdorff primitive ideal space* $\mathrm{Prim}(B)$, if, for every finite subset $X\subseteq D$ and for each $\varepsilon>0$, there is a C^*-subalgebra $B:=B(X,\varepsilon)\subseteq D$ of D such that $\mathrm{Prim}(B)$ is Hausdorff and $\mathrm{dist}(x,B):=\inf_{b\in B}\|x-b\|<\varepsilon$ for every $x\in X$.

It is likely that Theorem ??K?? can be used to show, that this property can be reformulated as a property of the primitive ideal space $\mathrm{Prim}(D)$ alone, provided that D is separable and nuclear and $D\cong D\otimes\mathcal{O}_2\otimes\mathbb{K}$:

If $B\subseteq D$ has Hausdorff $Y:=\mathrm{Prim}(B)$, then the action $\mathcal{I}(D)\cong\mathbb{O}(\mathrm{Prim}(D))\to\mathbb{O}(Y)$ of $\mathrm{Prim}(D)$ on Y is l.s.c. and monotone upper s.c. Conversely those actions define C^*-morphisms from $\mathrm{C}_0(Y,\mathcal{O}_2\otimes\mathbb{K})$ into D.

Consider all local approximations of this type.

Is the above local "Hausdorff" approximation property equivalent to the following property:

Property (???): *The family of adjoint maps from* $\mathrm{C}_0(Y)_+$ *to the Dini functions on* $\mathrm{Prim}(D)$ *contain in its images every finite set of Dini functions on* $\mathrm{Prim}(D)$ *up to* $\varepsilon>0$.

We give an almost minimal example D, that can *not* be locally approximated by C^*-subalgebras B with Hausdorff primitive ideal space $\mathrm{Prim}(B)$, and contains regular abelian C^*-subalgebras with rather different properties.

Proofs are not complete ??

Remark B.12.1. Let $A := \{f \in \mathrm{C}([0,1], M_2)\,;\ f(1) \in \Delta\}$ where Δ denotes the diagonal matrices of M_2.

We fix a unital monomorphism $h_1 \colon M_2(\mathcal{O}_\infty) \to \mathcal{O}_2$ and a non-degenerate monomorphism $h_2 \colon (\mathcal{O}_2 \oplus \mathcal{O}_2) \otimes \mathbb{K} \to \mathcal{O}_\infty \otimes \mathbb{K}$.

Further let $D := A \otimes \mathcal{O}_2 \otimes \mathbb{K}$, $D_1 := \{f \in \mathrm{C}([0,1], \mathcal{O}_2)\,;\ f(1) \in h_1(\Delta \otimes \mathcal{O}_\infty)\}$, and let

$$D_2 := \{f \in \mathrm{C}([0,1], \mathcal{O}_\infty \otimes \mathbb{K})\,;\ f(1) \in h_2((\mathcal{O}_2 \oplus \mathcal{O}_2) \otimes \mathbb{K})\}\,.$$

Then:

(1) The abelian C^*-subalgebra $B := \mathrm{C}([0,1], \Delta)$ of A is *regular* in A and in D, but $B \hookrightarrow A$ does not define a $\mathrm{KK}(\mathrm{Prim}(A), \cdot, \cdot)$ equivalence.
(2) D can not be locally approximated by C^*-subalgebras of D that have Hausdorff primitive ideal spaces.
(3) The algebras $D_0 := D$, D_1 and D_2 have same primitive ideal spaces, and contain regular abelian C^*-algebras C_0, C_1 respectively C_2, such that the inclusion maps define $\mathrm{KK}(\mathrm{Prim}(D_k); \cdot, \cdot)$-equivalences $(k = 0, 1, 2)$.

Proof. We embed A into D by $A \ni a \mapsto a \otimes r \in D := A \otimes \mathcal{O}_2 \otimes \mathbb{K}$, where r is a non-zero projection in $\mathcal{O}_2 \otimes \mathbb{K}$. Notice that $\mathrm{Prim}(A) \cong \mathrm{Prim}(D)$ by restriction of primitive ideals $I \lhd D \mapsto I \cap A \lhd A$.

The two characters χ_1 and χ_2 on Δ define characters on A and, therefore, define closed points x_1 and x_2 of $\mathrm{Prim}(A)$ (and of $\mathrm{Prim}(D)$). Since $[0,1)$ is naturally homeomorphic to the primitive ideal space of the closed ideal $\mathrm{C}_0([0,1), M_2)$ of A, there is an open subset U of $\mathrm{Prim}(A)$ that is homeomorphic to $[0,1)$. The closed complement is $\mathrm{Prim}(A) \setminus U = \{x_1, x_2\}$, where x_1 and x_2 denote the kernels of χ_1 respectively χ_2.

Since $A \subseteq \mathrm{C}([0,1],\, M_2)$, the set-valued map $\mu(t) := \{t\}$ for $t \in [0,1)$ and $\mu(1) := \{x_1, x_2\}$ from $[0,1] = \mathrm{Prim}(\mathrm{C}([0,1],\, M_2))$ into $\mathrm{Prim}(A)$ is lower semi-continuous and is montone upper semi-continuous. The corresponding map

$$\Phi_0 \colon \mathbb{O}([0,1]) \;\to\; \mathbb{O}(\mathrm{Prim}(A)) \cong \mathbb{O}([0,1) \cup \{x_1, x_2\})$$

is given by $\Phi_0(V) := V$ if $1 \notin V$ and $\Phi_0(V) := (V \cap (0,1]) \cup \{x_1, x_2\}$.

Clearly, $B \cong \mathrm{C}([0,1]) \oplus \mathrm{C}([0,1])$ is a commutative C^*-subalgebra of A (and, hence of D) that separates the ideals of A (and, hence, that of D).

The set-valued map Φ from $\mathrm{Prim}(A)$ into the closed subsets of $\mathrm{Prim}(B) \cong [0,1] \times \{1,2\}$ satisfies $\Phi(x_j) = \{(1,j)\}$ for $j = 1,2$ and $\Phi(t) = \{(t,1), (t,2)\}$ for $t \in [0,1)$.

The montone upper semi-continuity and the lower semi-continuity of the corresponding map Ψ from the lattice of open subsets of $\mathrm{Prim}(A)$ to the lattice of open subsets of $\mathrm{Prim}(B)$ imply that in good cases (e.g. if B separable and exact)

????????????????

we may suppose that $B \subseteq A \otimes \mathcal{O}_2$

We have for the injective map $\Psi \colon \mathbb{O}(\mathrm{Prim}(A)) \to \mathbb{O}(\mathrm{Prim}(B))$, that an open subset of W of $\mathrm{Prim}(B) \cong [0,1] \times \{1,2\}$ is in the image of Ψ, if and only if, $(t,1), (s,2) \in W$ and $s,t \in [0,1)$ imply that $(t,2),(s,1) \in W$.

It follows ????

that the image of Ψ consists of the family of open subsets of $V \times \{1,2\} \setminus G$ with V an open subset of $[0,1]$ and G a subset of $\{(1,1),(1,2)\}$.

If I is a closed ideal of A, then $(I \cap B)$ contains a strictly positive element of I: This is clear, if $I \subseteq \mathrm{C}_0([0,1), M_2)$, because then $I = \mathrm{C}_0(Y, M_2)$ for some open subset of Y of $[0,1)$.

In the other cases, there is an open subset Y of $[0,1]$ with

????????????????????????

Suppose now that for every finite subset $Z \subseteq D$. and every epsilon there is C^*-subalgebra F of D such that $\mathrm{dist}(z,F) < \varepsilon$ for all $z \in Z$.

The two projections $p = 1 \otimes e_{11} \otimes r$ and $q = 1 \otimes e_{22} \otimes r$ of B and its sum $1 \otimes 1_2 \otimes r$ are (up to MvN-equivalence) the only non-zero projections in D.

Indeed: If $Q \in D$ is a projection, it defines a continuous map $Q(t)$ from $[0,1]$ to $M_2 \otimes \mathcal{O}_2 \otimes \mathbb{K}$ with $Q(1) \in \Delta \otimes \mathcal{O}_2 \otimes \mathbb{K}$. If we let $P(t) := Q(1)$ for $t \in [0,1]$, then Q and P are in D and

$Q_s := Q(s + t^s)$

?????????

is a homotopy that connects Q and P in the projections of D.

Now use that all non-zero projections in $\Delta \otimes \mathcal{O}_2 \otimes \mathbb{K}$ are MvN-equivalent to the above mentioned three,

????

We have: *If. P, Q, R_1, R_2 are projections in D such that R_1 is MvN-equivalent to R_2 in B, $R_1 + P = p$ and $R_2 + Q = q$, then $R_1 = R_2 = 0$.*

Indeed: The R_i must be in the intersection of the primitive ideals defined by the characters χ_k on A, Thus, the R_i are in the ideal $\mathrm{C}_0([0,1), M_2) \otimes \mathcal{O}_2 \otimes \mathbb{K}$ of D, which contains only the zero projection.

???????????

`check last until here`

Since ?????

The reason should be the contradictory observation, that such an inductive limit decomposition eventually would lead to a continuous map f from $[0,1]$ into the Hausdorff space of subsets of $\{1,2,3\}$ which is not constant:

Let $C := \mathrm{C}([0,1], M_2) \otimes \mathcal{O}_2$. To see such a map, the reader should remark that then also $A \otimes \mathcal{O}_2$ has such an inductive limit decomposition by unital subalgebras with Hausdorff primitive ideal spaces. Thus, eventually, one finds, at least after application of an inner automorphism of $A \otimes \mathcal{O}_2$, a C^*-subalgebra B of $A \otimes \mathcal{O}_2$ which has Hausdorff primitive ideal space $Y := \mathrm{Prim}(B)$ and contains the constant diagonal projections $p = \mathrm{diag}(1,0)$ and $q = \mathrm{diag}(0,1)$ of $M_2 \cap A \subseteq C$, and contains also a contraction b of A with $\|bb^*(0) - p\| < 1/8$ and $\|b^*b(0) - q\| < 1/8$. In particular, B is unital. The two primitive ideals I_1 and I_2 of $A \otimes \mathcal{O}_2$ corresponding to the point 1 of $\mathrm{Prim}(C) \cong [0,1]$ separate p and q. Thus there are primitive ideals $J_1 \supset I_1 \cap B$ and $J_2 \supset I_2 \cap B$ of B with $p \in J_1$, $q \notin J_1$, $p \notin J_2$ and $q \in J_2$. Every primitive ideal J_3 of B which contains the intersection of B with the kernel of the evaluation map $g \mapsto g(0)$ contains neither p nor q. Let Z_1 and Z_2 be the open and closed subsets of Y which correspond to the norm functions $\widehat{p}$ and $\widehat{q}$ respectively. Since $p + q = 1$, $Z_1 \cup Z_2 = Y$. Therefore, the open and closed subsets $W_1 := Z_1 \setminus Z_1 \cap Z_2$, $W_2 := Z_2 \setminus Z_1 \cap Z_2$ and $W_3 := Z_1 \cap Z_2$ define a partition of Y with $J_n \in W_n$, $n \in \{1,2,3\}$.

Since $\mathrm{Prim}(B)$ is Hausdorff and B is unital, we get that there are four orthogonal projections P, Q, R_1, R_2 in B such that R_1 is MvN-equivalent to R_2 in B, $R_1 + P = p$ and $R_2 + Q = q$.

Above, we have seen, that this can not happen in $A \otimes \mathcal{O}_2$.

??????

`To be explained` ??

We define a continuous and open map λ from Y onto $\{1,2,3\}$ by $\lambda(x) := n$ for $x \in W_n$ and $n \in \{1,2,3\}$.

Then

$$f \colon t \in [0,1] \mapsto \lambda(Y \setminus \Psi^{\mathrm{up}}_{B,C}([0,1] \setminus \{t\}))$$

is a map from $[0,1]$ into the Hausdorff space of subsets of $\{1,2,3\}$, and $f(0) = \{3\}$, $f(1) = \{1,2\}$, which leads to a contradiction. Recall here that $\Psi^{\mathrm{up}}_{B,C}$ is a set-valued upper semi-continuous and monotone lower semi-continuous map. Thus f satisfies $\overline{f(S)} = f(\overline{S})$ for every subset S of $[0,1]$. □

`Next proofs not complete` ??

Remark B.12.2. Let A a unital C^*-algebra with Hausdorff $\mathrm{Prim}(A)$ and $p, q \in A$ projections with $\|p + q - 1\| < 1$ such that $\{(p-q)^2\}' \cap A$ has again Hausdorff primitive ideal space.

Then A decomposes into a direct sum $A \cong B \oplus C$ such that $C^*(\pi_2(p), \pi_2(q), 1_C)$ is a commutative C^*-subalgebra of C, $\pi_2(q) = 1_C - \pi_2(p)$, and that $1 \otimes C^*(\pi_1(p), \pi_1(q), 1_B)$ is contained in a 2-homogenous C^*-subalgebra $\cong \mathrm{C}(Y, M_2)$ of $\mathcal{O}_2 \otimes B$ with Y an open and closed subset of $X \times [0, \pi/2]$.

`Next has to be checked !`

Moreover, it can be managed that the induced unital C^*-morphism

$$h\colon C^*(P,Q,1) \to 1 \otimes C^*(\pi_1(p), \pi_1(q), 1_B)$$

extends to a unital C^*-morphism $\mathrm{C}([0,\pi/2], M_2) \to \mathcal{O}_2 \otimes B$.

`Above seems not to be obvious !!`
`Check if extension exists for irreducible rep's of` B `!!`

Let $D := \{f \in \mathrm{C}([0,1], M_2)\,;\ f(1)\,\text{diagonal}\}$. The quotient

$$\mathcal{O}_2 \otimes (C^*(P,Q,1)|[\pi/4,\pi/2]) \cong \mathcal{O}_2 \otimes D$$

of $\mathcal{O}_2 \otimes C^*(P,Q,1)$ does not contain a unital C^*-subalgebra A that satisfies $1 \otimes (P|[\pi/4,\pi_2]) \in A$ and $\operatorname{dist}(1 \otimes (Q|[\pi/4,\pi_2]); A) < 1/32$ and has *Hausdorff* primitive ideal space.

This non-existence implies that the C^*-algebras

$$\mathbb{K} \otimes \mathcal{O}_2 \otimes [\mathrm{C}_0([0,\pi/2), M_2) + \mathbb{C}P + \mathbb{C}(1-P)] \cong \mathbb{K} \otimes \mathcal{O}_2 \otimes D$$

and

$$\mathbb{K} \otimes \mathcal{O}_2 \otimes C^*(P,Q,1)$$

can not be an inductive limit of C^*-algebras A_n with Hausdorff primitive ideal spaces $\operatorname{Prim}(A_n)$.

It follows that the lattice $\mathcal{I}(C^*(P,Q,1)) \cong \mathbb{O}(\operatorname{Prim}(C^*(P,Q,1)))$ of open subsets of the (non-coherent) primitive ideal space $\operatorname{Prim}(C^*(P,Q,1))$ of $C^*(P,Q,1)$ can not be an algebraic limit of lattices $\mathbb{O}(X_n)$ for locally compact Hausdorff spaces X_n with respect to lower s.c. and monotone upper s.c. actions of X_{n+1} on X_n.

Proof. The C^*-algebra $A_0 := \mathrm{C}([0,\pi/2], M_2)$ and its C^*-subalgebra $C^*(P,Q,1)$ can be described by a universal C^*-algebra $C^*(R,S)$ with following universal relations:

R is a partial isometry, i.e., $RR^*R = R$, with relations $R^2 = 0$ and $(R-R^*)^4 = -(R-R^*)^2$. And S satisfies $0 \le S^* = S \le -(R-R^*)^2$ and $SR = RS$.

In particular, $-(R-R^*)^2$ is the unit element of $C^*(R,S)$, $C^*(R) \cong M_2$, S is in the centre of $C^*(R,S)$ and S has spectrum $\operatorname{Spec}(S) \subseteq [0,1]$. It follows that $\arcsin(S)$ is a well-defined positive element in the centre of $C^*(R,S)$, has spectrum in $[0,\pi/2]$ and $C^*(\arcsin(S)) = C^*(S)$. It follows that the natural C^*-morphism from $M_2 \otimes \mathrm{C}(\operatorname{Spec}(\arcsin(S)))$ onto $C^*(R, \arcsin(S)) = C^*(R,S)$ is an isomorphism.

To define a unital C^*-morphism λ from $C^*(R,S)$ onto $\mathrm{C}([0,\pi/2], M_2) \cong M_2 \otimes \mathrm{C}[0,\pi/2]$ we can choose the matrix-unit $R(\varphi) := e_{21}$ and $S(\varphi) := \sin(\varphi)1_2$. This epimorphism shows in particular that $\operatorname{Spec}(S) = [0,1]$ and $\operatorname{Spec}(\arcsin(S)) = [0,\pi/2]$. Notice that $\lambda(S^2) = (P-Q)^2$ is a positive contraction in the centre of A_0 and $\lambda(S) = |P-Q|$.

An inverse $\mu\colon \mathrm{C}([0,\pi/2], M_2) \to C^*(R,S)$ of λ can be defined by the natural epimorphism from $\mathrm{C}([0,\pi/2], M_2) = M_2 \otimes \mathrm{C}(\operatorname{Spec}(\arcsin(S)))$ onto $C^*(R,S)$, i.e., is determined by a $\mu(e_{2,1}) := R$ and $\mu(f_0 \cdot 1_2) := \arcsin(S)$ for $f_0(t) := t$ on $[0,\pi/2]$.

Then $\mu(M_2) = C^*(R) \subseteq C^*(R,S)$, $\mu(P) = R^*R \in C^*(R)$, $\mu(Z) = R - R^*$, $\mu((P-Q)^2) = S^2$, $\mu(H) = \arcsin(S)\cdot Z$, and $\mu(Q) = \exp(-H)R^*R\exp(\mu(H)$.

Let A a unital C^*-algebra and $p,q \in A$ projections such that p and $1-p$ are full projections in $\{(p-q)^2\}' \cap A$. Let $B := \mathcal{O}_2 \otimes (\{(p-q)^2\}' \cap A)$. Then there exists a partial isometry $V \in B$ with $V^*V = 1\otimes p$ and $VV^* = 1 \otimes (1-p)$, because 1_B is properly infinite and $1\otimes p$ and $1\otimes(1-p)$ are full and properly infinite projections in $\mathcal{O}_2 \otimes B$, cf. Lemma 4.2.6(ii).

If we let $T := 1 \otimes |p-q|$, then $U(1\otimes q) = (1\otimes p)U$ for $U := \exp(\arcsin(T)\cdot (V - V^*))$ and there is a unique unital C^*-morphism $\rho\colon C^*(R,S) \to B \subseteq \mathcal{O}_2 \otimes A$ with $\rho(R) = V$ and $\rho(S) := T$. Then $\rho(Z) = \rho(R+R^*) = V + V^*$, $\rho((P-Q)^2) = \rho(S^2) = T^2 = 1\otimes (p-q)^2$,

$$\rho(H) = \rho(\arcsin(S)\cdot Z) = (1\otimes \arcsin(|p-q|))(V - V^*)\,.$$

`Why` $\rho(Q) = 1\otimes q$ `???`

It follows that $\rho(P) = \rho(R^*R) = 1\otimes p$, $\rho(H) = \rho(\arcsin(S)\cdot(R-R^*)) = (1\otimes \arcsin(|p-q|))\cdot(V-V^*)$, and

$$\rho(Q) = \exp(-\rho(H))(1\otimes p)\exp(\rho(H)) = 1\otimes q\,.$$

Let A a unital C^*-algebra with Hausdorff primitive ideal space $X := \mathrm{Prim}(A)$ and $p \in A$ a projection. Then X is compact and the generalized Gelfand transforms $\widehat{p} : x \in X \mapsto \|\pi_x(p)\| = \|p + J_x\| \in [0,1]$ and

$$\widehat{1-p} : x \in X \mapsto \|\pi_x(1-p)\| = \|(1-p) + J_x\| \in [0,1]$$

are continuous functions on X with values in $\{0,1\}$ such that $\max(\widehat{p}\,, \widehat{1-p}\,) = 1$, because Dini functions on compact Hausdorff spaces X are continuous on X.

The support U_1 of $\widehat{p}$ (respectively U_2 of $\widehat{1-p}$) is the open subset of X corresponding to the closed ideal J_1 of A generated p (respectively J_2 generated by $1-p$). Since $1-\widehat{p}$ (respectively $1-\widehat{1-p}$) is continuous and non-negative, its support is open. Thus, U_1 and U_2 are open and compact subsets of X. In particular U_1 and U_2 are also closed subsets of the Hausdorff space X. Then $V := U_1 \cap U_2$, $W_1 := U_1 \setminus V$ and $W_2 := U_2 \setminus V$ are disjoint open and closed subsets of the compact Hausdorff space X with $X = V \cup W_1 \cup W_2$.

Since the J_k correspond to $U_k = W_k \cup V$, the ideal $K := J_1 \cap J_2$ has open and closed support V in $X = \mathrm{Prim}(A)$. Recall that $1 \in J_1 + J_2 = A$, $\pi_{J_1}(p) = 0$ and $\pi_{J_2}(p) = 1$.

Let I denote the closed ideal of A corresponding to the open and compact subset V of X. Then $I + K = A$, $J_1 + J_2 = A$ and $K \cap I = \{0\}$. If we define $B_1 := A/I$, $D_1 := A/J_1$ and $D_2 := A/J_2$. Then $\pi_I|K$ is an isomorphism from K onto B_1, and $\pi_K|I$ defines an isomorphism from I onto $C_1 := A/K \cong D_1 \oplus D_2$, where the isomorphism from C_1 onto $D_1 \oplus D_2$ is induced by $a \in A \mapsto \pi_{J_1}(a) \oplus \pi_{J_2}(a)$. In particular, $\pi_I(p)$ and $\pi_I(1-p)$ are full projections of B_1. Since $\pi_{J_1}(p) \oplus \pi_{J_2}(p) = 0 \oplus 1$, it follows that $\pi_K(p)$ is in the centre of the unital C^*-algebra C_1.

`Is next the critical case?`

Let $q \in A$ another projection and and let $Y := \arcsin(\mathrm{Spec}(|p-q|))$. Recall that there is a natural unital C^*-morphism from $\mathrm{C}(X \times Y)$ into $\{(p-q)^2\}' \cap A$ which makes $\{(p-q)^2\}' \cap A$ into a $\mathrm{C}(F)$-algebra for some closed subset F of $X \times Y$. Our assumptions imply that the $\mathrm{C}(F)$-algebra $\{(p-q)^2\}' \cap A$ is in a natural way a C^*-bundle over $X \times \arcsin \mathrm{Spec}(|p-q|)$.

Then $\pi_K(q)$ and $\pi_K(p)$ commute in C_1, and we can repeat the arguments with $(\pi_I(\{(p-q)^2\}' \cap A), \pi_I(q))$ in place of (A, p).

We get finally a direct sum decomposition $A \cong B \oplus C$ such that $\pi_1 \colon A \to B$ and $\pi_2 \colon A \to C$ have the property that $\pi_1(p)$, $1-\pi_1(p)$, $\pi_1(q)$ and $1-\pi_1(q)$ are projections in B which are full in $\{\pi_1(p-q)^2\}' \cap B$, and that $\pi_2(qp) = \pi_2(pq)$. Notice that $B = \pi_1(A)$ and $\{\pi_1(p-q)^2\}' \cap B$ have Hausdorff primitive ideal space Y that is homeomorphic to an open and closed subset of $X \cong \mathrm{Prim}(A)$, respectively of $X \times \arcsin(\mathrm{Spec}(|p-q|))$.

To simplify notation we suppose from now on that A itself has the property that $p, 1-p, q, 1-q$ are full projections (and that $X := \mathrm{Prim}(A)$ is Hausdorff) and that $(p-q)^2$ is an essential element of the centre of A, i.e., that $\mathrm{Ann}((p-q)^2, A) = \{0\}$.

In the considered case, we still have that our additional assumption $\|(p+q)-1\| = \|(1-p)-q\| < 1$ remains in charge in the new A, – our renamed B and $\pi_1(p), \pi_1(q)$.

If we apply part (v) of Lemma 4.1.3 to $1-p, q$ in place of p, q then we get $-h^* = h \in C^*(p,q,1) \subseteq A$ with $\|h\| = \arcsin\|p+q-1\| < \pi/2$ and $php = 0 = (1-p)h(1-p)$ such that $\exp(-h)(1-p)\exp(h) = q$. Thus $h = a - a^*$ for some $a \in A$ with $a = pa(1-p)$ and $\|a\| = \arcsin\|p+q-1\|$.

Need ???

the existence of $c \in \mathcal{O}_2 \otimes pA(1-p)$ with $cc^* = 1 \otimes p$, $c^*c = 1 \otimes (1-p)$ and $b \in A_+$ with $bp = pb$, $bq = qb$ and $a = bc$

???????

We can use that $1 \otimes (p-q)^2$ is in the centre of $\mathcal{O}_2 \otimes A$ and that $1 \otimes p$ and $1 \otimes (1-p)$ are both full and properly infinite projections of $\mathcal{O}_2 \otimes A$.

`Explicit formula for` $H|[0,\psi]$ `for study of the inclusion` $C^*(P,Q,1)|[0,\pi/4] \subseteq \mathrm{C}([0,\pi/4], M_2)$.

An explicit formula for $H|[0,\psi] \in C^*(P,Q)|[0,\psi]$ for $\psi \in [0,\pi/2)$ is given by $H(\varphi) := \varphi(e_{2,1} - e_{1,2})$.

Notice that

$$PQ(\varphi)(1-P) = P\exp(-H)(\varphi)P \cdot P\exp(H)(\varphi)(1-P) = -\cos(\varphi)\sin(\varphi)e_{1,2}\,.$$

and

$$PQ(\varphi)(1-P) - (1-P)Q(\varphi)P = \cos(\varphi)\sin(\varphi)Z$$

and $(P - Q(\varphi))^2 = \sin(\varphi)^2 1_2$, because

$$Q(\varphi) - P = [\cos(\varphi), -\sin(\varphi)]^\top [\cos(\varphi), -\sin(\varphi)] - P = \sin(\varphi) I$$

with $I = [\alpha_{jk}]$ with $\alpha_{11} = -\sin(\varphi) = -\alpha_{22}$, $\alpha_{12} = \alpha_{21} = -\cos(\varphi)$, i.e., orthogonal with determinant -1, $I^2 = 1_2$. Similarly: $(P + Q(\varphi) - 1)^2 = (Q(\varphi) - (1-P))^2 = \cos(\varphi)^2 1_2$.

I.e., always $(p-q)^2 + (p+q-1)^2 = 1$, $|p-q|$ and $|p+q-1|$ are in the centre of $C^*(p,q,1)$.

Alternatively one can use symmetry of sin and cos at $\pi/4$ $(= 45°)$.

$H := a - a^*$, $a = -\varphi e_{1,2}$ for $a = CPQ(1-P)$ with $C := D^{-1} \arcsin |P-Q|$ where $D := (1 - (P-Q)^2)^{1/2} \cdot |P-Q|$

?????????.

Let $A \subseteq \mathcal{O}_2 \otimes (C^*(P,Q,1)|[\pi/4, \pi/2])$ a unital C^*-subalgebra such that $p_0 := 1_{\mathcal{O}_2} \otimes P|[\pi/4,\pi/2] \in A$. Suppose that $X := \mathrm{Prim}(A)$ is Hausdorff. Since A is separable and unital, X is second countable and compact, i.e., is a compact Polish space.

Each quotient C^*-algebra of $\mathrm{C}[\pi/4,\pi/2] \otimes A$ has again Hausdorff primitive ideal space. Thus, the C^*-subalgebra $C^*(\mathrm{C}[\pi/4,\pi/2] \cdot 1, A)$ generated by A and $\mathrm{C}[\pi/4, \pi_2] \cdot 1$ is a again a unital C^*-subalgebra of $\mathcal{O}_2 \otimes (C^*(P,Q,1)|[\pi/4,\pi/2])$ that contains p_0 and has again a Hausdorff primitive ideal space. Thus, we can suppose in addition that $1 \otimes \mathrm{C}[\pi/4,\pi/2]$ is contained in A.

If we use that $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$, then we can tensor here all algebras in question again by $\mathcal{O}_2$ and suppose moreover that $\mathcal{O}_2 \otimes A \cong A$. Then A is purely infinite and the MvN-equivalence classes of projections in A are in 1-1-correspondence with the open and compact subsets of X.

Since $A \subseteq \mathcal{O}_2 \otimes C^*(P,Q,1)|[\pi/4, 1\pi/2]$ is contained in $\mathrm{C}([\pi/4,\pi/2], \mathcal{O}_2 \otimes M_2) \cong \mathcal{O}_2 \otimes \mathrm{C}[\pi/4,\pi/2]$ and $1 \otimes \mathrm{C}[\pi/4,\pi/2] \subseteq A$ is contained in the centre of $\mathcal{O}_2 \otimes (C^*(P,Q,1)|[\pi/4,\pi/2])$, the C^*-algebra A is C^*-bundle over $[\pi/4,\pi/2]$, i.e., is the C^*-algebra of continuous sections of a continuous field

$$\varphi \in [\pi/4, \pi/2] \mapsto A_\varphi \subseteq \mathcal{O}_2 \otimes M_2 \subseteq \mathcal{O}_2\,.$$

The same happens with the C^*-subalgebras p_0Ap_0 and $(1-p_0)A(1-p_0)$ of A, because, they are invariant under multiplications with $1 \otimes f$ where $f \in \mathrm{C}[\pi/4,\pi/2]$.

The generalized Gelfand transforms $\widehat{a}(J) := \|a + J\|$ for $J \in \mathrm{Prim}(A) \cong X$ are Dini functions on X. The class of Dini functions on Hausdorff spaces X coincides with the bounded non-negative continuous functions $f \in \mathrm{C}_0(X)$ with σ-compact locally compact open support $f^{-1}(0,\infty)$ in X. The Gelfand transforms $\widehat{q} \in \mathrm{C}_0(X)$ of projections $q \in A$ are the characteristic functions of open and compact subsets of X corresponding to the closed ideal $\overline{\mathrm{span}(AqA)}$ generated by q.

Let $J_1 \subseteq A$ the closed ideal of A generated by $p := 1 \otimes (P|[\pi/4, \pi/2])$, $J_2 \subseteq A$ the closed ideal generated by $1 - p = 1 \otimes (1 - P)|[\pi/4, \pi/2]$ and $U_1, U_2 \subseteq X$ are the corresponding open subsets.

Their characteristic functions are given by $\widehat{p}$ and by $\widehat{1-p} = 1 - \widehat{p}$. It implies that U_1 and U_2 are compact (and open) and satisfy $U_1 \cup U_2 = X$.

The set $F := X \setminus (U_1 \cap U_2)$ is an open and compact subset of X and is a metric space with $F_1 := F \cap U_1$ and $F_2 := F \cap U_2$, that are disjoint open and closed subsets of F with $F = F_1 \cup F_2$.

`To be filled in ??`

Suppose that the lattice $\mathcal{I}(C^*(P, Q, 1))$ is an algebraic limit of the lattices $\mathbb{O}(X_n)$, where the corresponding actions are l.s.c. and monotone upper s.c.

Then the corresponding actions

$$\Psi_n : \mathbb{O}(X_{n+1}) \to \mathbb{O}(X_n)$$

are induced from suitable C^*-morphisms

$$\mathrm{C}_0(X_n, \mathbb{K} \otimes \mathcal{O}_2) \to \mathrm{C}_0(X_{n+1}, \mathbb{K} \otimes \mathcal{O}_2)$$

and $\mathbb{K} \otimes \mathcal{O}_2 \otimes C^*(P, Q, 1)$ is isomorphic to an inductive limit of $\mathrm{C}_0(X_n, \mathbb{K} \otimes \mathcal{O}_2)$ coming from suitable compatible C^*-morphisms

$$\phi_n : \mathrm{C}_0(X_n, \mathbb{K} \otimes \mathcal{O}_2) \to \mathbb{K} \otimes \mathcal{O}_2 \otimes C^*(P, Q, 1) .$$

For some n_0 there exists a unitary $u \in \mathcal{M}(\mathbb{K} \otimes \mathcal{O}_2 \otimes C^*(P, Q, 1))$ and projections $r, s \in \mathrm{C}_0(X_n, \mathcal{O}_2)$ such that $s \leq r$ $e_{11} \otimes 1 \otimes 1 = u^* \phi_{n_0}(e_{11} \otimes r)u$, $e_{11} \otimes 1 \otimes p = u^* \phi_{n_0}((e_{11} \otimes s)u$ and $\mathrm{dist}(\, q;\ \phi_{n_0}(e_{11} \otimes (r\, \mathrm{C}_0(X_{n_0}, \mathcal{O}_2)r)) < 1/32$.

`The proof depends from above un-verified local uniqueness !!!`

□

13. Approximate locally liftable maps

Let $J \triangleleft A$ a closed ideal. We say that a completely contractive linear map $T : C \to A/J$ (e.g. $T = \mathrm{id}_{A/J}$) is *ε-approximately locally liftable* if for every finite-dimensional subspace $X \subseteq C$ there is a completely contractive linear map $S = S_{(T,X,\varepsilon)} : X \to A$ with $\|\pi_J \circ S - \eta_X\| \leq \varepsilon$. The map T is *approximately locally liftable* if T is ε-approximately locally liftable for all $\varepsilon > 0$. The map T is *locally liftable* if T is ε-approximately locally liftable for $\varepsilon = 0$.

Let $A \subseteq \mathcal{L}(\mathcal{H})$ and denote by F the hereditary C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ that is generated by J. Then $A \cap F = J$, F is approximately injective (in the sense of [**238**]), and $A + F$ is C^*-algebra with the intersection properties $(F \otimes D) \cap (A \otimes D) = J \otimes D$ and $((A + F) \otimes D)/(F \otimes D) = (A \otimes D)/(A \otimes J)$ (because J contains an approximate unit of F). The approximation arguments in [**238**] show that on each separable C^*-subalgebra $C_1 \subseteq C$ there is a complete contraction $S : C_1 \to A + F$ with $\pi_F \circ S = T|C_1$ if $T : C \to A/J$ (and then $T : C \to (A + F)/F$) is approximately locally liftable.

In fact, it follows (by using arguments in [**238**]) that $T: C \to A/J$ is approximately locally liftable (into A), if and only if, $T: C \to (A+F)/F$ is locally liftable, if and only if, on every separable C^*-subalgebra $C_1 \subseteq C$ there is a completely contractive map $S_1: C_1 \to A+F$ with $\pi_F \circ S_1 = T|C_1$.

If we apply this to $T = \mathrm{id}_{A/J}$, then we get: $\mathrm{id}_{A/J}$ *is approximately locally liftable, if and only if,* $J \otimes D \to A \otimes D \to (A/J) \otimes D$ *is exact for every* C^**-algebra* D.

For exact A (and more generally for "locally reflexive" A in the sense of [**238**]) it is true that $W: A/J \to B$ is nuclear if $V = W \circ \pi_J$ is nuclear.
Indeed, more generally, one can check with help of the criteria (i) in Remark 3.1.2.

????? that the identity map $\mathrm{id}_{A/J}$ of A/J is *locally* liftable, in the sense that for every subspace $X \subset A/J$ of finite dimension and every $\varepsilon > 0$ there exists a completely $(1+\varepsilon)$-contractive map $T: X \to A$ with $\pi_J \circ T = \mathrm{id}_X$, then this is sufficient to prove the nuclearity of W if $W \circ \pi_J$ is nuclear on A:
This can be seen by an (elementary but unpleasant) approximation and extension argument, or by the following more conceptual argument.
If $\mathrm{id}_{A/J}$ is locally liftable, then (the reader can check that) `Give argument for proof!`

$$J \otimes D \to A \otimes D \to (A/J) \otimes D$$

is an exact sequence for each C^*-algebra D.

If A is locally reflexive, then this sequence is exact, – almost directly by the definition of (matricial) local reflexivity.

One can show that the local reflexivity of A is equivalent to the (metric) exactness of all sequences

$$L \otimes D \to A \otimes D \to (A/L) \otimes D$$

for all closed left-ideals $L \subseteq A$ and *all* C^*-algebras D.

`Give refs or arguments for proof of this!`

14. Reductions to separable subspaces (sep 1)

The following lemma on "reductions to the separable case" is used in some cases, where we need to apply a kind of "excision" methods. Such type of reductions can be managed by iterated application of the variant [**553**, prop. 7.2] by E. Michael of the Bartle-Graves theorem [**52**, thm. 4]. Our elementary "separable" selection of sequences does not involve the axiom of choice for sets of uncountable cardinalities (as all selections principles do implicitly!), but we use the Open Mapping Theorem that is an application of the (almost – but not really – "constructive") Baire category theorem.

Lemma B.14.1. *Let* B *denote a Banach space, and let* $X, Y \subset B$ *closed linear subspaces of* B*, such that the (algebraic) sum* $X + Y$ *is closed in* B*.*

Then for every separable linear subspace $G \subseteq B$ there exist a separable closed linear subspace D of B such that D has following properties:

(i) $G \subseteq D$,
(ii) $(D \cap X) + (D \cap Y) = D \cap (X + Y)$,
(iii) $\operatorname{dist}(d, V) = \operatorname{dist}(d, V \cap D)$ *for all* $d \in D$, *where* V *means each of the closed subspaces* $X + Y$, X, Y *and* $X \cap Y$ *of* B.

If B is a C-algebra, then a separable C*-subalgebra D of B with properties* (i)–(iii) *can be found.*

If B is an operator space, and $M_n(X)$ and $M_n(Y)$ are equipped with the induced matrix norms then a separable closed subspace D of B can be found such that D satisfies in addition the matrix versions (iii*) *of Part* (iii)*:*

(iii*) $\operatorname{dist}(d, M_n(V)) = \operatorname{dist}(d, M_n(V \cap D))$ *for all* $d \in M_n(D)$, $n \in \mathbb{N}$, *where* V *stands for each of the spaces* $X \cap Y$, X, Y *and* $X + Y$.

That the linear space $X+Y$ of B is closed in B is a necessary extra assumption because there exist closed linear subspaces X and Y e.g. of $\mathrm{C}[0,1]$ and of $\mathcal{L}_1[0,1]$ such that the algebraic sum $X + Y$ is not closed.

Give Refs. to counter examples !

The proof of Part (ii) uses the open mapping theorem.

We prove first a sub-lemma that describes an inductive selection procedure.

Sublemma B.14.2. *Let B a closed linear subspace of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and X, Y closed linear subspaces of B such that $X + Y$ is closed in B.*

Let $G \subseteq B$ a separable subset of B.

Then there are separable closed linear subspaces $G_1, G_2, \ldots$ of B that have the following properties:

(1) $G \subseteq G_n \subseteq G_{n+1}$ *for all* $n \in \mathbb{N}$
(2) $G_n \cap (X + Y)$ *is contained in the closure of the linear subspace* $(G_{n+1} \cap X) + (G_{n+1} \cap Y)$ *of* $G_{n+1} \cap (X + Y)$.
(3) $\|(\mathrm{id}_n \otimes \pi_{V \cap G_{n+1}})(b)\| \leq \|(\mathrm{id}_n \otimes \pi_V)(b)\|$ *for all* $b \in M_n(G_n)$, *where* V *stands for each of the four closed subspaces* X, Y, $Z := X + Y$ *and* $X \cap Y$ *of* B, *and* $\mathrm{id}_n \otimes \pi_V$ *denotes the quotient map* $M_n(B) \mapsto M_n(B)/M_n(V)$.

If B is a C-algebra, then separable C*-subalgebras G_n of B with properties* (1)–(3) *can be found for* $n = 1, 2, \ldots$.

Proof of Sublemma B.14.2. The Property (3) splits into the 4 cases where $V := X$, $V := Y$, $V := X + Y$ or $V := X \cap Y$. Each of (1), (2) and of the 4 cases of (3) can be considered separately for any given separable subspace $G_n \subseteq B$ (in place of any given separable $G \subseteq B$), i.e., we get six separable subspaces $G_{n+1,1}, G_{n+1,2}, G_{n+1,3}, \ldots, G_{n+1,6}$ of B that are as desired in steps (1),(2) and (3) independently. Then any separable subspace $G_{n+1} \subseteq B$ that contains the six subspaces $G_{n+1,k}$ $(k = 1, \ldots, 6)$ satisfies the conditions (1)–(3) with respect to the given G_n.

If, moreover, B is a C^*-algebra, then this shows that we can take as G_{n+1} any separable C^*-subalgebra of B that by the $G_{n+1,k}$ $(k = 1, \ldots, 6)$.

Therefore it suffices to find independently for given separable $G \subseteq B$ (e.g. for $G := G_n$ with before constructed $G_n \subseteq B$) and for each property (k), $k = 1, 2, 3$, and each part of (3) a "solution" $G_{n+1} \supseteq G_n$ with the in (1)–(3) listed properties.

Ad(1): It is an "input" condition: Given any separable subset G, e.g. $G := G_n \supseteq B$, then any separable subspace $G_{n+1,1}$ of B with $G \subseteq G_{n+1,1}$ satisfies (1).

Ad(2): Let G any closed separable linear subspace of B, (e.g. $G := G_n$ if G_n is constructed before). Then $G \cap (X + Y)$ is separable and closed. Take a dense sequence $z_1, z_2, \ldots \in G \cap (X + Y)$, and find $x_n \in X$ and $y_n \in Y$ with $z_n = x_n + y_n$. Let F any closed separable subspace of B that contains the closed linear span of the elements of G and of $\{x_1, y_1, x_2, y_2, \ldots\}$. Then $G \cap (X + Y)$ is contained in the *closure* of the linear subspace $(F \cap X) + (F \cap Y)$ of $F \cap (X + Y)$. And we can take $G_{n+1,2} := F$.

Ad(3): Recall that $\|(\mathrm{id}_n \otimes \pi_V)(b)\| = \mathrm{dist}(b, M_n(V))$ for all $b \in M_n(B)$.

Given any G (e.g. $G := G_n$, a before selected G_n) and let V one of the 4 listed spaces (resp. operator spaces) $X, Y, X + Y, X \cap Y$ in B. Take a sequence of matrices $g_1, g_2, \ldots$ that is dense in the unit ball of $M_n(G)$. For each $g_k \in M_n(G)$ we find $h_k \in M_n(V)$ such that

$$\|g_k - h_k\| \le (1 + 2^{-k}) \cdot \mathrm{dist}(g_k, M_n(V))\,.$$

We can take as $G_{n+1,k}$ $(k \in \{1, 2, 3, 4\}$ here depending on the considered $V)$ any separable closed linear subspace of B that contains all entries of the matrices $h_1, h_2, \ldots$, because it is not difficult to see that it has then the property that $\mathrm{dist}(g, M_n(G_{n+1,k} \cap V)) = \mathrm{dist}(g, M_n(V))$ for all $g \in M_n(G)$ ([3]).

If this is done for all $V \in \{X, Y, X + Y, X \cap Y\}$ then any separable closed linear subspace G_{n+1} of B that contains $G_{n+1,1} \cup \ldots \cup G_{n+1,4}$ has the property that, for all $g \in M(G)$ holds $\mathrm{dist}(g, M_n(G_{n+1}) \cap M_n(V)) = \mathrm{dist}(g, M_n(V))$ for the considered spaces $V \subseteq B$, □

Proof of Lemma B.14.1. If B is a Banach space B, and $X \subseteq B$ and $Y \subseteq B$ are closed linear subspaces, then we could consider B as a closed subspace of the C^*-algebra $\ell_\infty(M)$ for a set M of suitable cardinality. Thus it suffices to consider in the following always operator spaces, i.e., closed linear subspaces of a given closed linear subspace $B \subseteq \mathcal{L}(\mathcal{H})$.

Let G, $G_1, G_2, \ldots$ as in Sublemma B.14.2, define $G_\infty := \bigcup_n G_n$ and $D := \overline{G_\infty}$. Notice that Part (3) of Sublemma B.14.2 is equivalent to $\mathrm{dist}(b, M_n(V \cap G_{n+1})) = \mathrm{dist}(b, M_n(V))$ for all $b \in M_n(G_n)$. It implies $\mathrm{dist}(b, M_n(V \cap D)) = \mathrm{dist}(b, M_n(V))$

[3] Use here that for each contraction $g \in M_n(G)$ and $\varepsilon > 0$ there is an infinite subset R of $\mathbb{N}$ such that $\|g - g_k\| < \varepsilon$ for each $k \in R$.

for $b \in M_n(G_\infty)$ and each $n \in \mathbb{N}$. This carries over to all $b \in M_n(D)$ and $V \cap D$ in place of $V \cap G_\infty$, because $M_n(D)$ is the norm-closure of $M_n(G_\infty)$. It implies that for every $n \in \mathbb{N}$ and $V \subseteq B$ ($V \in \{X + Y, X, Y, X \cap Y\}$) holds that the natural injective linear map $\mathrm{id}_n \otimes \pi_V$ from $M_n(D)/M_n(D \cap V)$ into $B/M_n(V)$ of norm ≤ 1 is a complete isometry, as one can see from the formula

$$\mathrm{dist}(b, M_n(D \cap V)) \leq \mathrm{dist}(b, M_n(V)) \quad \text{for all } b \in M_n(D) .$$

It turns out that D is a closed linear subspace of B that satisfies all conditions (i), (ii) and (iii*) in Lemma B.14.1:

Clearly condition (i) follow from $G \subseteq G_1$.

If, in addition, B is a C^*-algebra, then the G_n in Conditions (1)–(3) of Sublemma B.14.2 can be chosen as C^*-subalgebras of B. I.e. we can then suppose that all G_n ($n = 1, 2, \ldots$) are C^*-algebras, that $G_\infty := \bigcup_n G_n$ is a *-subalgebra of B, and that $D := \overline{G_\infty}$ is a separable C^*-subalgebra of B.

The condition (iii) follows from the Property (3) of Sublemma B.14.2 for the operator system D. The isomorphisms $M_k \cong pM_np$ for all projections $p \in M_n$ with rank $= k < n$, shows that the operator space $D := \overline{\bigcup_n G_n}$ satisfies

$$\mathrm{dist}(d, M_n(V \cap D)) = \mathrm{dist}(d, M_n(V)) \quad \text{for all } d \in M_n(D),$$

for each $V \in \{X + Y,\, X,\, Y,\, X \cap Y\}$ and every $n \in \mathbb{N}$, because this is true for all $d \in G_k$ with $k > n$, by Part (3) of Sublemma B.14.2.

Clearly the sum $(D\cap X)+(D\cap Y)$ is contained in $D\cap(X+Y)$, because $D\cap X$ and $D \cap Y$ are closed subspaces of the closed subspace $D \cap (X + Y)$ of B.

We show that the linear space $(D \cap X) + (D \cap Y)$ is *dense* in $D \cap (X + Y)$:

Property (2) of Sublemma B.14.2 with $V := X + Y$ and the there constructed $G_n \subseteq G_{n+1} \subseteq \cdots$ satisfy

$$G_n \cap (X + Y) \subseteq \overline{(G_{n+1} \cap X) + (G_{n+1} \cap Y)} \subseteq \overline{(D \cap X) + (D \cap Y)} .$$

It implies that

$$(\bigcup_n G_n) \cap (X + Y) \subseteq \overline{(D \cap X) + (D \cap Y)} \subseteq D \cap (X + Y) .$$

Thus, $(D \cap X) + (D \cap Y)$ is dense in $D \cap (X + Y)$ if $(\bigcup_n G_n) \cap (X + Y)$ is dense in $D \cap (X + Y)$.

To show the latter, let $V := X + Y$ and $y \in D \cap V$, Then $y \in D$, $\pi_V(y) = 0$ and, by density of $\bigcup_n G_n$ in D, there exist a sequence $n_k \in \mathbb{N}$ with $n_k < n_{k+1}$ and elements $d_k \in G_{n_k}$ such that $y = \lim_k d_k$ and $\lim_k \pi_V(d_k) = 0$.

Recall that any closed subspace V of B holds $(\mathrm{id}_n \otimes \pi_V)(p_{11} \otimes b) = p_{11} \otimes \pi_V(b)$ for $b \in B$ for the quotient map $\pi_V \colon B \to B/V$ from B, i.e.,

$$\mathrm{dist}(b, V) = \|\pi_V(b)\| = \|(\mathrm{id}_n \otimes \pi_V)(p_{11} \otimes b)\| .$$

Thus, if we take $n := n_k$ in Part (3) of Sublemma B.14.2, then we obtain that $\|\pi_{G_{1+n_k} \cap V}(d_k)\| \leq \|\pi_V(d_k)\|$ for $d_k \in G_{n_k}$. It follows that there exists $e_k \in$

$G_{1+n_k} \cap V$ with $\|e_k - d_k\| \leq \|\pi_V(d_k)\|$. Hence $\lim_k \|e_k - d_k\| = 0$ and $\lim_k \|e_k - y\| = 0$, i.e., $y \in \overline{\bigcup_n (G_n \cap V)}$.

It says that $\bigcup_n (G_n \cap V)$ is dense in $D \cap V$ and implies that the subspace $(D \cap X) + (D \cap Y)$ is *dense* in $D \cap (X + Y) = D \cap V$.

Next we show that the dense linear subspace $(D \cap X) + (D \cap Y)$ of $D \cap (X+Y)$ is closed in $D \cap (X+Y)$. This shows then the predicted equality $(D \cap X) + (D \cap Y) = D \cap (X + Y)$ in Part(ii), and finishes the proof.

From now on let $V := X \cap Y \subseteq B$.

By Part (3) of Sublemma B.14.2 we have that $\mathrm{dist}(g, G_{n+1} \cap V) \leq \mathrm{dist}(g, V)$ for $g \in G_n$. Thus, $\mathrm{dist}(g, D \cap V) = \mathrm{dist}(g, V)$ for all $g \in G_\infty$. Since G_∞ is dense in D it follows that the restriction to D of the quotient map $\pi_V : b \in B \to b + V \in B/V$ defines an linear *isometry* η from the quotient space $D/(D \cap V)$ into B/V with kernel $D \cap V = D \cap X \cap Y$. In particular, $\eta | (D \cap X)/(D \cap V)$ and $\eta | (D \cap Y)/(D \cap V)$ are isometries into the subspaces X/V respectively Y/V of $(X + Y)/V$. We denote this restrictions of η by η_1 and η_2.

Recall that the space $D \cap V$ is a closed subspace of each of the closed subspaces $D \cap X$, $D \cap Y$ and $\overline{(D \cap X) + (D \cap Y)} = D \cap (X + Y)$. Thus, if the linear space $\eta(D \cap X) + \eta(D \cap Y)$ is closed in $(X + Y)/V$ then $(D \cap X) + (D \cap Y)$ is closed in $D \cap (X + Y)$, because

$$(D \cap X) + (D \cap Y) = \eta^{-1}(\eta(D \cap X) + \eta(D \cap Y)) \,.$$

This implies finally the predicted equation $(D \cap X) + (D \cap Y) = D \cap (X + Y)$.

The natural linear map T from the Banach space $X/V \oplus_\infty Y/V$ with norms $\|(x + V, y + V)\| := \max\big(\|\pi_V(x)\|, \|\pi_V(y)\| \big)$ onto $(X + Y)/V$ that is given by

$$T(x + V, y + V) := (x + y) + V$$

is surjective and has norm ≤ 2. The T is also injective because $x + y = v \in V$ implies that $x = v - y \in X \cap Y = V$, thus $x, y \in V$. The Open Mapping Theorem shows that the inverse $T^{-1} : (X+Y)/V \to X/V \oplus_\infty Y/V$ of the linear isomorphism T is bounded. In particular, T maps each closed subspace of $X/V \oplus_1 Y/V$ onto a closed subset of $(X + Y)/V$.

The map $\eta_1 \oplus \eta_2$ from $(D \cap X)/(D \cap V) \oplus_\infty (D \cap Y)/(D \cap V)$ into $X/V \oplus_\infty Y/V$ is an isometry. Thus, its image is a closed subspace of $X/V \oplus_1 Y/V$. It follows that T maps this image onto a closed subspace of $(X + Y)/V$. Calculation shows that $T \circ (\eta_1 \oplus \eta_2)$ maps $(D \cap X)/(D \cap V) \oplus_\infty (D \cap Y)/(D \cap V)$ onto $\eta(D \cap X) + \eta(D \cap Y)$. Thus, $\eta(D \cap X) + \eta(D \cap Y)$ is closed in $(X + Y)/V$. □

15. Passage to separable subalgebras (sep 2)

We define for $a, b \in A$ and $\varepsilon > 0$ minimal numbers $n := n(b, a, \varepsilon)$ with the property that there exist $c_1, \ldots, c_n, d_1, \ldots, d_n \in A$ with $\delta := \|b - \sum_k d_k^* a c_k\| < \varepsilon$.

The latter inequality implies by Lemma 2.1.9 that there exists a rational number ρ and $\ell \in \mathbb{N}$ with $\delta < \rho < \varepsilon$ such that there exist $c_1, \ldots, c_n \in A$ with $(b-\rho)_+ = \sum_k c_k^* a c_k$ if $a, b \in A_+$.

`New more precise text:`

We define numbers $n(b, a, \varepsilon; B)$ and $\gamma(b, a, \varepsilon; B) \geq n(b, a, \varepsilon; B)$ (the latter only in case $a, b \in B_+$ and $\|a\| = \|b\| = 1$) for C^*-subalgebras B of A and non-zero $a, b \in B$ with b in the closed ideal $J(a)$ of B generated by a, by letting $n := n(b, a, \varepsilon; B)$ the minimal number $n \in \mathbb{N}$ such that there exist $c_1, \ldots, c_n, d_1, \ldots d_n \in B$ with $\|b - \sum_{j=1}^n d_j^* a c_j\| < \varepsilon$. We can here take $d_j = c_j$ if $a, b \in B_+$, cf. Remark B.15.1. Moreover $n(b, a, \varepsilon) = n((b^*b)^{1/2}, (a^*a)^{1/2}, \varepsilon)$.

The number $\nu := \nu(b, a; \varepsilon)$ for $a, b \in B_+$ is the smallest number $\nu \in \mathbb{N}$ with the property that there exists *contractions* $c_1, \ldots, c_\nu \in B$ such that $\|b - \sum_{j=1}^n d_j^* a c_j\| < \varepsilon$.

`What about definition of above defined` γ`?`

Equivalently: There exists $X, Y \in M_n(B)$ with $\|b \otimes p_{11} - Y^*(a \otimes 1_n)X\| < \varepsilon$. This is because we can replace here $X, Y \in M_n(B)$ by the columns $[c_1, \ldots, c_n]^\top := X \cdot (1 \otimes p_{11})$ and $[d_1, \ldots, d_n]^\top := Y \cdot (1 \otimes p_{11})$ and vice versa. Here 1 denotes the unit element of the multiplier algebra of $\mathcal{M}(A)$.

Remark B.15.1. If $a, b \in B_+$ and there exists by $c_1, \ldots, c_n, d_1, \ldots, d_n \in A$ with $\|b - \sum_j d_j^* a c_j\| < \varepsilon$ then there exists $f \in A$ such that $e_j := (c_j + d_j)f$ satisfies $\|b - \sum_j e_j^* a e_j\| < \varepsilon$.

Indeed, if $a, b \in B_+$ then we can use that for the positive elements $g := b \otimes p_{11}$ and $h := a \otimes 1_n$ holds also $\|2g - (Y^*hX + X^*hY)\| < 2\varepsilon$. It implies that there exists $\delta \in (0, \varepsilon)$ such that $0 \leq 2g \leq (Y^*hX + X^*hY) + 2\delta(1 \otimes 1_n)$.

Clearly, by $X^*hY = 0$, $(Y^*hX + X^*hY) + (X^*hX + Y^*hY) = (X+Y)^*h(X+Y)$, which gives that $0 \leq g \leq T^*hT + \delta \cdot (1 \otimes n)$ for $T := \sqrt{2}(X+Y) \in M_n(A)$. By Lemma 2.1.9, there exists a contraction $d \in M_n(A)_+$ with $d^*(T^*hT)d = (g - (\varepsilon+\delta)/2)_+$. In particular, $\|g - Z^*hZ\| < \varepsilon$ for $Z := Td$.

Thus, with $[e_1, \ldots, e_n]^\top := Z(1 \otimes p_{11})$ we get $\|b - \sum_{j=1}^n e_j^* a e_j\| < \varepsilon$.

`Important are Parts (iii) and (iv)`

Proposition B.15.2. *For every C^*-algebra A and every separable subset $X \subset A$ of A there exists a separable C^*-subalgebra B of A with $X \subset B$ such that B has the following additional properties:*

(i) *The natural C^*-morphism $C \otimes^{\max} B \to C \otimes^{\max} A$ is injective for all separable C^*-algebras C (i.e., B is "relative weakly injective" in A),*
(Are properties (i) and (iii) equivalent ?)

(ii) *The natural maps $(C \otimes^{\min} B)/(Y \otimes^{\min} B) \mapsto (C \otimes^{\min} A)/(Y \otimes^{\min} A)$ are isometric for all separable C^*-algebras C and closed subspaces Y of C.*
??? Or better ?:
Require metric version of the "intersection property":

$(Y \otimes^{\min} A) \cap (C \otimes^{\min} B) = Y \otimes^{\min} B$.

Let $Z_1 \subseteq Z_2$ and $Z_3 \subseteq Z_2$ operator spaces (closed linear subspaces of $\mathcal{L}(\ell_2)$).

What has to be required on this spaces?

that the natural map from $Z_1/(Z_1 \cap Z_3)$ to Z_2/Z_3 is completely isometric ???

(iii) *For each $k \in \mathbb{N}$, $a, b \in M_k(B)_+$ and $\delta > 0$ holds that for every $c_1, \ldots, c_n \in M_k(A)$ there exists $d_1, \ldots, d_n \in M_k(B)$ with $\|d_j\| \leq \|c_j\|$ for $j = 1, \ldots, n$ and*

$$\|b - \sum_k d_k^* a d_k\| \leq \delta + \|b - \sum_k c_k^* a c_k\| .$$

(iv) *For every $a, b \in B$ and $\varepsilon > 0$, $n(b, a, \varepsilon; B) = n(b, a, \varepsilon; A)$ and $\gamma(b, a, \varepsilon; B) = \gamma(b, a, \varepsilon; A)$ if $a, b \in B_+$ and $\|a\| = \|b\| = 1$.*

(Where $n(\ldots)$ and $\gamma(\ldots)$ are defined ?)

(v) *$(B \cap I_1) + (B \cap I_2) = B \cap (I_1 + I_2)$ for any closed ideals I_1 and I_2 of A.*

??? ($\supseteq$ is in question!!! and by out-devision of $I_1 \cap I_2$ we can suppose that $l_1 \cdot I_2 = \{0\}$. But the question is if one finds separable B_m that satisfies at least one of the properties and this B_m are contained in the B_n with $n > m$ ... then the inductive limits gives this "all properties" B ...)

In particular, every closed ideal J of B is the intersection $J = B \cap I$ of a closed ideal I of A with B.

PROOF. `To be filled in ??`

??? Reduce (v) to σ-unital A with approximate unit in B and $A = I_1 + I_2$ generated as ideals by $B \cap (I_1 + I_2)$. ???

$B \subseteq A \subseteq A^{**}$ leads to $B^{**} \subseteq A^{**}$ in the sense that B^{**} is naturally isomorphic to the $\sigma(A^{**}, A^*)$-closure of $B \subseteq A^{**}$ in A^{**}.

It seems that B is "relative weakly injective" in A, if and only if there exists a c.p. contraction P from A into B^{**}

(A positive contraction would be enough, because its restriction can be normalised to a completely positive contraction S from A^{**} into $B^{**} \subseteq A^{**}$ with the property that $S(b) = b$ for all $b \in B \subseteq A^{**}$.

Since one can do this with $B \otimes M_n \subseteq A \otimes M_n$, one can obtain by cluster-point arguments that there is a completely positive contraction T from A into $B^{**} \subseteq A^{**}$ with the property that $T(b) = b$ for all $b \in B$.

□

16. Reductions to separable cases (sep 3)

An important technical property in the study of residually nuclear maps between non-simple C^*-algebras is the notion of local reflexivity for closed subspaces $B \subseteq \mathcal{L}(\mathcal{H})$ introduced by Effros and Haagerup in [**238**]:

DEFINITION B.16.1. A closed subspace $B \subseteq \mathcal{L}(\mathcal{H})$ (i.e., an operator space with its system of matrix norms) is **locally reflexive** if, for every finite-dimensional subspaces $X \subseteq B^{**}$, $F \subseteq B^*$, and $\varepsilon > 0$ there is a linear map $T\colon X \to B$ that satisfies the following properties (i) and (ii):

(i) $|f(T(x)) - x(f)\| \leq \varepsilon \|x\| \cdot \|f\|$ for all $x \in X$, $f \in F$, and
(ii) T is completely contractive, i.e., $\|T \otimes \mathrm{id}_n\| \leq 1$ for all $n \in \mathbb{N}$ for the maps $T \otimes \mathrm{id}_n\colon X \otimes M_n \to B \otimes M_n$ with norms on $X \otimes M_n$ induced from $B^{**} \otimes M_n \subseteq \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n)^{**}$.

REMARK B.16.2. One can find for each $n \in \mathbb{N}$ a linear map $T\colon X \to B$ that satisfies property (i) and $\|T \otimes \mathrm{id}_n\| \leq 1$ by the local reflexivity of the *Banach* spaces $B \otimes M_n$, but in general such T does not exist that it satisfies also $\|T \otimes \mathrm{id}_{n+k}\| \leq 1$ for $k \in \mathbb{N}$.

Does it say that there could exist $X_k \subseteq B^{**}$ and $Y_k \subseteq B^*$ of finite dimension such that,
for all $T_k \in L(X_k, B)$ with (i) holds that
$\lim_{k \to \infty} \|T_k \otimes \mathrm{id}_k\| = \infty$?

All exact C^*-algebras (or, more generally, 1-exact operator spaces) are locally reflexive. It is not known if locally reflexive C^*-algebras are exact in general.

Local reflexivity passes to C^*-quotients and C^*-subalgebras (in particular, $\prod_n M_n$ is *not* locally reflexive, because it contains $C^*(F_\infty)$ as a C^*-subalgebra).

The class of locally reflexive C^*-algebras A is invariant under tensor products with $\mathbb{K}$ and M_n.

But it is so far unknown if tensor products of locally reflexive A with M_{2^∞} (and – therefore – also min-tensor products with arbitrary exact C^*-algebras) again become locally reflexive. It is also not known if the class of locally reflexive C^*-algebras are invariant under inductive limits (at least for suitably “controlled” nets of C^*-morphisms?).

Local reflexivity is *not* preserved under passage to stable coronas, asymptotic coronas, multiplier algebras, and ultra-powers of A if A is not sub-homogenous (i.e., if A has only irreducible representations of dimension $\leq n$ for some $n \in \mathbb{N}$).

Indeed, it is not difficult to see that stable coronas, asymptotic coronas, multiplier algebras, and ultra-powers of A are sub-Stonean C^*-algebras. Recall that sub-Stonean C^*-algebras C have the property that for orthogonal positive contractions $a, b \in C_+$ there are orthogonal positive contractions $c, d \in C_+$ with $ca = a$ and $db = b$.

Sub-Stonean C^*-algebras C have the property that $B_1(\ell_\infty) = V(B_1(C))$ for every c.p. contraction $V\colon C \to \ell_\infty$ with $V(B_1(C)) \cap c_0$ dense in $B_1(c_0)$ (where $B_1(X)$ means the closed unit ball of a Banach space X).

If a C^*-algebra C with the latter property contains a C^*-subalgebra D, that has $\bigoplus_n M_n$ as a quotient, then C contains a C^*-subalgebra E, such that E has a quotient that is isomorphic to $\prod_n M_n = \ell_\infty(\mathbb{C}, M_2, M_3, \ldots)$.

The following more general property of a C^*-algebra C, stated as follows, is equivalent to the property that C has no non-zero closed ideal of type-I:

For every non-zero positive $e \in C_+$ in the C^*-algebra C, the σ-unital hereditary C^*-subalgebra $\overline{eCe}$ contains a C^*-subalgebra $D \subseteq \overline{eCe}$ such that $\prod_n M_n = c_0(\mathbb{C}, M_2, M_3, \ldots)$ is a quotient of the C^*-algebra D.

A non-zero C^*-algebra C is sub-homogenous, i.e., there exists $n_C \in \mathbb{N}$ such that all irreducible representations of C have dimension $\leq n_C$, if and only if, there does not exist a C^*-subalgebra $D \subseteq C$ with the property that $\prod_n M_n$ is a quotient C^*-algebra of D. (For a proof of this criteria it suffices to consider only the type-I algebras by Glimm's sub-quotient theorem, saying that each separable non-type-I C^*-algebra C contains a hereditary C^*-subalgebra D such that $C = D \cdot C + C \cdot D + \mathcal{N}(D, C)$ and $\mathcal{N}(D, C)/D \cong M_{2^\infty}$).

The *full* group C^*-algebras $C^*(G)$ of a lattices G in non-compact semi-simple Lie groups are not exact (and even not locally reflexive) because they all contain a copy of $C^*(F_2)$ and every separable unital C^*-algebra is a quotient of its subalgebra $C^*(F_\infty)$.

The reduced group C^*-algebras $C^*_{red}(G)$ are all exact for G in this class of discrete groups.

REMARK B.16.3. Suppose that $A \subseteq M$ is a C^*-subalgebra of a W*-algebra M. Then centre of the *-ultra-strong closure of A in M contains a family of mutually orthogonal projections $\{p_\kappa\}$ such that $Q := 1 - \sum_\kappa p_\kappa$ satisfies $QA = AQ = \{0\}$, and that each p_κ can be decomposed in M into a sum of a family mutually orthogonal countably decomposable projections in M of cardinality $\leq$ density of A.

(It suffices to consider the case where $M = A^{**}$.)

What was the original remark ??? Good for? ??

QUESTIONS B.16.4. Given a semi-finite properly infinite W*-algebra M and a separable C^*-subalgebra $A \subseteq M$.

Does there exist a family of mutually orthogonal countably decomposable projections $p_\gamma \in A' \cap M$ with $\sum p_\gamma = 1$ such that following properties (i)–(iii) are satisfied?

(i) p_γ is properly infinite,
(ii) for every weakly l.s.c. trace $\tau \colon N_+ \to [0, \infty]$ (on the positive part of $N := \{p_\gamma\, ;\ \gamma \in \Gamma\}' \cap M$) there is a weakly l.s.c. trace $\tau' \colon M_+ \to [0, \infty]$ with $\tau | A_+ = \tau' | A_+$,
(iii) for every projection p in the centre of N there is a projection q in the centre of M with $\|ap\| = \|aq\|$ for all $a \in A$.

Does there exist a separable unital C^*-subalgebra $A \subseteq C \subseteq M$ (with M a W*-algebra) such that following properties (i)–(iv,v????) are satisfied

(i) $\mathcal{O}_2$ is unitally contained in C, $\delta_\infty(C) \subseteq C$ and $\delta_\infty \oplus \mathrm{id}$ unitarily equivalent to δ_∞ by unitaries in C (for the weakly continuous $\delta_\infty \colon M \to M$).
(ii) Every closed ideal I of C is the intersection of a closed ideal of M with C.
(iii) For every projection p in the centre of M there is q in the centre of C with $\|ap\| = \|aq\|$ for all $a \in A$.
(iv) $C \cap K$ is an essential ideal of C.
(v) $C \cap K$ is stable (????????).

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    local lifting, local lifting property,
locally liftable exact sequences
    weakly injective algebras,
relatively weakly injective subalg.,
approximately injective algebras
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Recall that $A \otimes B$ denotes the *minimal* C^*-algebra tensor product of A and B.

LEMMA B.16.5. *Let A and B C^*-algebras, where A is separable and B is σ-unital.*

(i) *If X is a separable Banach space, $\mathcal{C} \subseteq \mathcal{L}(X, B)$ a point-norm closed set of contractions, J a σ-unital closed ideal of B. If for all $S, T \in \mathcal{C}$ and for every contraction $a \in J_+$, the element*

$$aS(\cdot)a + (1 - a^2)^{1/2}T(\cdot)(1 - a^2)^{1/2}$$

is again in $\mathcal{C}$, then $\pi_J \circ \mathcal{C}$ is point norm closed in $\mathcal{L}(X, B/J)$.
(ii) *A c.p. map $V \colon A \to B/J$ is locally c.p. liftable (or c.b. liftable ???) if*

$$J \otimes \mathcal{L}(\ell_2) \to B \otimes \mathcal{L}(\ell_2) \to (B/J) \otimes \mathcal{L}(\ell_2)$$

is exact. If, moreover ?????. See Effros –Haagerup.
(iii) *B is locally reflexive, if and only if,*

$$J_\tau \otimes \mathcal{L}(\ell_2) \to B_\tau \otimes \mathcal{L}(\ell_2) \to B^{**} \otimes \mathcal{L}(\ell_2)$$

is exact, if and only if, the natural C^*-morphism form $B^{**} \otimes^{\max} \mathcal{L}(\ell_2)$ into $(B \otimes \mathcal{L}(\ell_2))^{**}$ factorizes over $B^{**} \otimes^{\min} \mathcal{L}(\ell_2)$.
(iv) *More ??????????????*

PROOF. (i): We use metrics on $\mathcal{L}(X, B)$ and $\mathcal{L}(X, B/J)$ that define the point-norm topologies: Let $x_1, x_2, \ldots \in X$ a sequence that dense in the unit-ball of X. Then let $\rho_0(T, S) := \sum_n 2^{-n}\|T(x_n) - S(x_n)\|$ and similarly defined ρ_1 on $\mathcal{L}(X, B/J)$. Obviously $\rho_1(\pi_J \circ T, \pi_J \circ S) \le \rho_0(T, S)$ and $\rho_0(T, S) \le \|T - S\|$.

Moreover, if $E := \{e_\tau\} \subset J_+$ is a with respect to the separable C^*-subalgebra $C^*(T(X) \cup S(X)) \subseteq B$ quasi-central approximate unit, then $\rho_1(\pi_J \circ T, \pi_J \circ S)$ is equal to

$$\inf\{\, \rho_0\big((1-e_\tau)^{1/2}T(\cdot)(1-e_\tau)^{1/2}, (1-e_\tau)^{1/2}S(\cdot)(1-e_\tau)^{1/2}\big)\ ;\ e_\tau \in F\,\}\,.$$

Let $\mathcal{C} \ni T_n \colon X \to B$ such that $\pi_J \circ T_n$ converges in point-wise to some map $S \colon X \to B/J$.

Can select $n_k \in \mathbb{N}$ that $\rho_1(\pi_J \circ T_{n_k}, \pi_J \circ T_{n_k}) < 4^{-k}$. Then we define a new sequence $R_k \in \mathcal{L}(X, B)$ with the property that $\rho_0(R_k, R_{k+1}) < 2^{-k}$ with help step by step modification by replacing the $T_{n_{k+1}}$ by

$$R_{k+1} := (1-f_k)^{1/2} T_{n_{k+1}}(\cdot)(1-f_k)^{1/2} + f_k^{1/2} R_k(\cdot) f_k^{1/2}$$

with suitably chosen $f_k \in F \subseteq J_+$. It follows that the $R_k \in \mathcal{C}$ converge in point-norm on X to a map $R \in \mathcal{C}$ with $\pi_J(R) = S$.

To be filled in ??

□

17. Singly generated separable C^*-algebras (sep. 4)

Lemma B.17.1. *Let E denote a unital C^*-algebra with properly infinite unit and $s_1, s_2 \in E$ isometries with orthogonal ranges: $s_j^* s_k = \delta_{j,k} 1$, and let $e \in E_+$ denote a positive contraction.*

Then the singly generated C^*-subalgebra $C^*(X) = C^*(X^*, X)$ of E generated by the element

$$X := X(s_1, s_2, e) := s_1 \cdot \big(1 + s_1(2+e)^{-1}s_2^* + s_2(2+e)^{-1}s_1^*\big)$$

is identical with the C^*-subalgebra $C^*(s_1, s_2, e) \subseteq E$ generated by the elements $\{s_1, s_2, e\} \subseteq E$.

Proof. Obviously $X(s_1, s_2, e) \in C^*(s_1, s_2, e) \subseteq E$ by definition of X and $C^*(s_1, s_2, e) = C^*(s_1, s_2, a)$ for $a := (2+e)^{-1}$ by $e = a^{-1} - 2$.

Recall that $s_1, s_2 \in A$, $s_j^* s_k = 1_A$ and that $a := (2+e)^{-1} \in A_+$ is invertible with estimates $1/3 \leq a \leq 1/2$.

We define $Y := s_1(1 + s_1 a s_2^* + s_2 a s_1^*)$. Consider $f := s_1 a s_2^* + s_2 a s_1^*$. Then $f^* = f$, $f s_1 = s_2 a$ and $f^2 = s_1 a^2 s_1^* + s_2 a^2 s_2^*$. Thus $\|f\| \leq 1/2$, and it follows that $1/2 \leq h := 1 + f \leq 3/2$.

This element h allows to rewrite the above defined Y as $X = s_1 h$.

We show that the C^*-subalgebra $C^*(X) = C^*(X^*, X)$ of A generated by X contains s_1, s_2 and a, i.e., that $C^*(X) = C^*(s_1, s_2, a)$. Then $1 \in C^*(X)$ and $e = a^{-1} - 2 \in C^*(X)$, i.e., $C^*(X) = C^*(s_1, s_2, e)$.

The above discussion allows to see the following formulas: $X^*X = h^2 \geq 1/4$, $s_1 = X(X^*X)^{-1/2} = Xh^{-1} \in C^*(X)$, $1 = s_1^* s_1$, $f = h - 1 = (X^*X)^{1/2} - 1$ and $f s_1 = s_2 a$. Thus, $s_1, f, 1 \in C^*(X)$ and $a^2 = a s_2^* s_2 a = s_1^* f^2 s_1$. Since, by assumptions, $a \geq 1/3$ we get that $a, a^{-1}, e = a^{-1} - 2 \in C^*(X)$, and finally that $s_2 = f s_1 a^{-1} \in C^*(X)$. □

Compare next Proposition with [**770**, thm.2.3] and the Remark B.17.3 concerning other attempts to prove Proposition B.17.2.

PROPOSITION B.17.2. *If E is a separable C^*-algebra with a properly infinite unit element, then there exists a unital C^*-morphism $\psi\colon \mathcal{O}_\infty \to E$ and E is singly generated as C^*-algebra.*

PROOF. Let $s_1, s_2 \in E$ isometries with $s_1^* s_2 = 0$. Then let $t_k := s_1^k s_2$ for $k = 1, 2, \ldots$. The elements $t_k \in E$ satisfy that $t_j^* t_k = \delta_{j,k} 1_E$ for $j, k \in \mathbb{N}$, i.e., the $t_1, t_2, t_3, \ldots$ define a unital C^*-morphism from $\mathcal{O}_\infty$ into E.

By separability of E, there exists a norm-dense sequence $a_1, a_2, \ldots$ in the positive contractions in E. The positive contraction

$$e := \sum_{k=1}^{\infty} s_k (2^k + a_k)^{-1} s_k^* \in E_+$$

has the property that the C^*-subalgebra $C^*(t_1, t_2, e)$ of E generated by $\{t_1, t_2, e\}$ is identical with E, because $s_k, 1_E \in C^*(t_1, t_2)$ for all $k \in \mathbb{N}$ and $(2^k + a_k)^{-1} = s_k^* e s_k$ gives that $a_k = (s_k^* e s_k)^{-1} - 2^k 1_E$ is in $C^*(t_1, t_2, e)$.

Thus, E is singly generated by the element $X(t_1, t_2, e) \in E$ defined in Lemma B.17.1. □

REMARK B.17.3. The authors H. Thiel and W. Winter of [**770**, thm.2.3] quote Nagisa [**567**] who had referenced the result [**770**, thm.2.3] (and thus Proposition B.17.2) to the author (E.K.) of this book. Probably M. Nagisa was informed that the author wrote in some evening discussion (or in a break between talks) sometimes in the 1990th in MF Oberwolfach a formula on the blackboard in the discussion room that did show that every properly infinite unital ([4]) separable C^*-algebra is singly generated. It could be that this proof was different from the here given proof of Proposition B.17.2, but it was definitely different from all of the (incomplete) proof of [**770**, thm.2.3]. Thus, unfortunatly we couldn't use it as citation here, had to give our own proof. H. Thiel and W. Winter quote for the proof of [**770**, thm.2.3] an argument in the proof of [**580**, thm. 9]. But this quoted argument does not carry over to an element $Y := \sum_{k\geq 1}(s_k a_k s_k^* + 1/2^k s_k)$ with spectrum of $a_k = a_k^*$ in $[4^{-n}/2, 4^{-n}]$ as considered in their proof, because they claim – and need/use it (!) in their proof –

`Check next red text carefully again!!!`

`Perhaps, the constructions give two self-adjoint elements that generates all?`

... that the spectrum of Y is contained in $\{0\} \cup \bigcup_{k=1}^{\infty} [4^{-n}/2, 4^{-n}]$. But this is not the case in general, e.g. consider the there given formula in case where $A := \mathcal{O}_\infty$

[4]It is the case if and only if the algebra contains a copy of $\mathcal{O}_\infty$ unitally!

with generating isometries $s_1, s_2, \ldots$ satisfying $s_j^* s_k = \delta_{j,k}$ the spectrum of the element $\sum_k 4^{-k} s_k s_k^* + 2^{-k} s_k$ is exactly equal to $\{0\} \cup \bigcup_{k\geq 1}(4^{-k} + 2^{-k} D)$, where D is the unit-disc, in particular the spectrum is not contained in $\{0\} \cup \bigcup_{k\geq 1}[4^{-k}/2, 4^{-k}]$. Unfortunately this gap in the arguments for the proof of [**770**, thm.2.3] given by H. Thiel and W. Winter harms then any further argument of their proof.

Further details concerning sums of invertible positive elements and non-unitary isometries $s_1, s_2, \ldots \in E$ with $s_j^* s_k = \delta_{j,k}$:
Let us consider elements $X \in E$ defined by

$$X := \sum_{k\geq 1} (s_k a_k s_k^* + \gamma_k s_k)\,,$$

where the a_k are positive with $\mathrm{Spec}(a_k) \subseteq [\alpha_k, \beta_k]$ for "suitable" $0 < \beta_{k+1} < \alpha_k < \beta_k$. One can show that the spectrum of such element X is contained $\{0\} \cup \bigcup_{k\geq 1}([\alpha_k, \beta_k] + \gamma_k D)$ where $D := \{z \in \mathbb{C};\ |z| \leq 1\}$ is the unit disc.

Below we give a more precise formula for $\mathrm{Spec}(X)$ that allows to deduce that our, in Remark B.17.3 mentioned, special example $Y := \sum_{n=1}^{\infty} 4^{-n} s_n s_n + 2^{-n} s_n$ in $\mathcal{O}_\infty$ it is precisely equal to $\{0\} \cup \bigcup_n (4^{-n} + 2^{-n} D)$.

Let $P_n := \sum_{k=1}^{n} s_k s_k^*$ and consider stepwise the "triangular" decompositions of X as

$$X = P_n X P_n + (1 - P_n) X (1 - P_n) + P_n X (1 - P_n)\,.$$

It allows to identify X with an upper triangular matrix in $M_2(E)$ with entries $X_{11} = P_n X P_n$, $X_{12} = P_n X(1 - P_n)$ and $X_{22} = (1 - P_n) X (1 - P_n)$. This is possible, because all X of the above defined type satisfy $(1 - P_n) X P_n = 0$ for this projections P_n. It implies that

$$\{0\} \cup \mathrm{Spec}(X) = \{0\} \cup \mathrm{Spec}(P_n X P_n) \cup \mathrm{Spec}((1 - P_n) X (1 - P_n))\,.$$

If we proceed with $n \in \mathbb{N}$ by induction then we can see that this leads to

$$\mathrm{Spec}(X) \cup \{0\} = \{0\} \cup \bigcup_{k=1}^{\infty} \mathrm{Spec}(a_k + \gamma_k s_k)\,.$$

Here we have used that $(s_k a_k s_k^* + \gamma_k s_k) s_k s_k^* = s_k (a_k + \gamma_k s_k) s_k^*$. Using that $\mathrm{Spec}(S) = D$ for every non-unitary isometry $S \in E$, this leads to the estimate $\mathrm{Spec}(a_k + \gamma_k s_k) \subseteq \{0\} \cup (\mathrm{Spec}(a_k) + \gamma_k D)$.

In the special case where X is build with scalars $a_k = \beta_k \cdot 1_E$, and $\lim_k \beta_k = 0$ and $\lim_k \gamma_k = 0$ it gives the precise equation

$$\mathrm{Spec}(X) \;=\; \{0\} \cup \bigcup_k (\beta_k + \gamma_k D)\,.$$

This applies in particular to the above defined special element $Y \in \mathcal{O}_\infty$.

An other special case is the case where $\lambda := \sum_k \gamma_k^2 < \infty$ and $\lim_k \beta_k = 0$ we get that $S := \lambda^{-1/2} \sum \gamma_k s_k$ is an isometry in E and with $a := \sum_k s_k a_k s_k$ the C^*-algebra $C^*(a, S)$ contains automatically all s_k and a_k, i.e., is identical with $C^*(a_1, a_2, \ldots; s_1, s_2, \ldots)$.

Our conclusion is that the basic question for the study of above discussed phenomena should be:
Under which circumstances holds that $a, S \in C^*(a+S) \subset \mathcal{L}(\ell_2)$ for a given positive invertible operator $a \in \mathcal{L}(\ell_2)$ and a non-unitary isometry $S \in \mathcal{L}(\ell_2)$?

Here $\ell_2 := \ell_2(\mathbb{N})$ and the isometry S should have an infinite dimensional co-kernel $(1 - SS^*)\ell_2$, e.g. up to unitary equivalence $S \in \mathcal{L}(\ell_2)$ can be defined with an ONB by $Se_n := e_{2n}$.

More calculations. To be checked again.
No final decision obtained if the weakly separated case works:

Even if one selects the positive numbers more carefully, namely such that

$$([\alpha_k, \beta_k] + \gamma_k D) \cap ([\alpha_\ell, \beta_\ell] + \gamma_\ell D) = \emptyset ,$$

for $k \neq \ell$, then we still have to struggle to obtain at least two of the isometries to get the rest of all. This "weak" disjointness holds, e.g. in the case where we take $\alpha_k := 2^{-2k}$, $\beta_k := 2\alpha_k$ and $\gamma_k := (\alpha_k - \beta_{k+1})/4 = \alpha_k/8$, because then

$$\alpha_{k+1} = \alpha_k/4 < \beta_{k+1} = \alpha_k/2 < \alpha_k < 2\alpha_k = \beta_k$$

and

$$\beta_{k+1} + \gamma_{k+1} < \alpha_k - \gamma_k \ .$$

It follows from $\alpha_{k+1} = \alpha_k/4$ $\beta_{k+1} = \alpha_k/2$ and $\gamma_k = \alpha_{k+1}/2 = \alpha_k/8$.

This particular choice causes at least that there exist unitaries $u_1, u_2, \ldots \in C^*(X)$ that have the very weak property that at least $s_k u_k \in C^*(X)$, because the above defined very particular α_k, β_k and γ_k satisfy the additional conditions $\gamma_k < \alpha_k^2/(2\beta_k)$. (obtained form $\beta_k = 2\alpha_k$ and $\gamma_k = \alpha_k/8$).

It is not unlikely that there is some possible repair of the arguments in the proof of [**770**, thm.2.3] if one takes the γ_k very small in relation to to the lower bounds $\alpha_k > 0$ of invertible elements ... but we doubt this.

But then one can only find (not necessarily selfadjoint) idempotents $P_k \in C^*(X)$ with $P_k X = s_k a_k s_k^* + \gamma_k s_k$.

An other calculation gives $s_k a_k s_k^* + \gamma_k s_k = Y + Z$ with $Y := s_k(a_k + \gamma_k s_k)s_k^*$ and $Z := \gamma_k s_k(1 - s_k s_k^*)$ satisfy $ZY = 0$, $Z^2 = 0$, $(1 - s_k s_k^*)Z = 0$ and $(1 - s_k s_k^*)Y = 0$. In particular,

$$\{0\} \cup \mathrm{Spec}(Y + Z) = \{0\} \cup \mathrm{Spec}(Y) = \mathrm{Spec}(a_k + \gamma_k s_k)$$

for the invertible positive contraction a_k .

18. On asymptotically estimating functions

Let $A := \mathcal{L}(\ell_2)$, and let $F\colon (a, b) \in A \times A \to A$ and $G\colon (a, b) \in A \times A \to A$ with the properties that $G(a, b) = 0$ implies that $F(a, b) = 0$ and that there exists a bund $\gamma \in (0,\infty)$ such that $\max\{\,\|F(a,b)\|,\ \|G(a,b)\|\,\} \le \gamma$ for all contractions $a,b \in A$. We require moreover that for $(a_1,a_2,\ldots),(b_1,b_2,\ldots) \in \ell_\infty(A)$ holds

$$F(\,\pi_{c_0(A)}(a_1,a_2,\ldots),\pi_{c_0(A)}(b_1,b_2,\ldots)\,) \;=\; \pi_{c_0(A)}(F(a_1,b_1),F(a_2,b_2),\ldots)$$

– by suitable "local" re-imbedding of parts of $\ell_\infty(A)/c_0(A)$ into $\mathcal{L}(\ell_2)$ –, and

$$G(\,\pi_{c_0(A)}(a_1,a_2,\ldots),\pi_{c_0(A)}(b_1,b_2,\ldots)\,) \;=\; \pi_{c_0(A)}G(a_n,b_n)\|\,.$$

Likewise, for F is would be enough to have

$$\|F(\,\pi_{c_0(A)}(a_1,a_2,\ldots),\pi_{c_0(A)}(b_1,b_2,\ldots)\,)\| \ge \liminf_n \|F(a_n,b_n)\|\,,$$

and for G that

$$\|G(\,\pi_{c_0(A)}(a_1,a_2,\ldots),\pi_{c_0(A)}(b_1,b_2,\ldots)\,)\| \;\le\; \limsup_n \|G(a_n,b_n)\|\,.$$

Then there is a general method for such F and G that decides if there exists an increasing continuous function $\varphi \in \mathrm{C}_0(0,1]_+$ with $\|F(a,b)\| \le \varphi(\|G(a,b)\|)$ for all contractions $a,b \in A$:

Fix $t \in (0,1]$ and take the supremum $S(t)$ over the norms $\|F(a,b)\|$ for all contractions a,b with $\|G(a,b)\| \le t$ and call it ??? ...

It is certainly $\le \gamma$ and $t \mapsto S(t)$ is increasing.

There exist increasing continuous functions $\varphi(t)$ on $(0,1]$ with $S(t) \le \varphi(t)$ and $\lim_{t\to 0}\varphi(t) = \lim_{t\to 0} S(t)$.

If $0 < \mu := \lim_{t\to 0} S(t)$, then there exist sequences of contractions $a_1,a_2,\ldots$ and $b_1,b_2,\ldots$ with $\|G(a_n,b_n)\| < 2^{-n}$ and $\|F(a_n,b_n)\| \ge \mu - 2^{-n}$.

19. An elementary matrix norm bound

A simple explanation for the estimate in Remark 2.1.10 goes as follows:

We can write each matrix $a = [a_{jk}] \in M_n(A)$ as a sum of n sub-matrices $\mathrm{RL}(\ell)$ or as sum of n sub-matrices $\mathrm{LR}(\ell)$ that have the property that $\mathrm{RL}(\ell)^*\mathrm{RL}(\ell)$ and $\mathrm{LR}(\ell)^*\mathrm{LR}(\ell)$ are diagonal matrices. Thus the sub-matrices have norms $\|\mathrm{RL}(\ell)\|$ and $\|\mathrm{LR}(\ell)\|$ that is $\le$ the maximum of the norms of there entries a_{jk}. Then use that

$$a = \sum_{\ell=1}^{n} \mathrm{RL}(\ell) = \sum_{\ell=1}^{n} \mathrm{LR}(\ell)\,.$$

Such sub-matrices can be selected as follows:
$\mathrm{RL}(\ell)$ for $\ell \in \{1,\ldots,n\}$ the sub-matrix given with entries a_{jk} at the places (j,k) with $j+k \;=\; \ell \bmod (n)$ i.e., with the elements

$$\{\,a_{1,\ell-1},a_{2,\ell-2},\ldots,a_{\ell-2,2},a_{\ell-1,1}\,;\;\; a_{\ell,n},a_{\ell+1,n-1},\ldots,a_{n-1,\ell+1},a_{n,\ell}$$

on its given places and zero's otherwise. It is on 45^o degree from right to left downward directed lines.

Similarly let LR(ℓ) for $\ell \in \{1, \ldots, n\}$ denote the sub-matrix with entries on the left to right downward directed lines, parallel to the main diagonal LR(n) = diag($a_{11}, a_{22}, \ldots, a_{nn}$):
The sub-matrix LR(ℓ) is non-zero only on places (j, k) with $j - k = \ell \bmod (n)$:

$$\{\, a_{1,\ell}, a_{2,\ell+1}, \ldots , a_{\ell-1,n-1}, a_{\ell,n} \,;\; a_{\ell+1,1}, a_{\ell+2,2}, a_{\ell+3,3}, \ldots , a_{n-1,\ell-1}, a_{n,\ell}$$

All sub-matrices RL(ℓ) and LR(ℓ) have the property that the products

??????

Only the case of the right-to-left downward diagonal $a_{1,n}, a_{2,n-1}, \ldots, a_{n,1}$ (with $j+k = 1 \bmod (n)$) splits not into two – in orthogonal boxes contained – sub-matrices.

The same happens with the lines that are parallel to the main diagonal $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$.

20. On flip-invariant subalgebras of $A \otimes A$

Let A a C^*-algebra and let $\gamma \colon A \otimes A \to A \otimes A$ the unique *-automorphism given by $\gamma(a \otimes b) = b \otimes a$ for $a, b \in A$. We call it the **flip** automorphism on $A \otimes A$.

It would be helpful to know more on the fix-point algebras of the flips on $\mathcal{O}_2 \otimes \mathcal{O}_2$, $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ and $\mathcal{P}_\infty \otimes \mathcal{P}_\infty \cong (1 - s_1 s_1^*)\mathcal{O}_\infty(1 - s_1 s_1^*) = {\mathcal{O}_\infty}^{st}$.

Also on $\mathcal{Z} \otimes \mathcal{Z}$?

More generally what about the fix-point algebra of the flip on $\mathcal{O}_n \otimes \mathcal{O}_n$, $M_{n^\infty} \otimes M_{n^\infty}$? Are they simple?

The flip algebra $(A \otimes A)^\gamma$ is nuclear if (and only if ??) A is nuclear.

Is $(A \otimes A)^\gamma$ also simple ??? (seems to be if we can say something on the case of tensorial $\mathcal{Z}$ absorption, but the flip algebras in $M_n \otimes M_n$ or $\mathbb{K} \otimes \mathbb{K}$ are not simple.)

if A is simple and unital without quotient of finite dimension? The fix-point algebras of the flips on $M_{n^\infty} \otimes M_{n^\infty}$ are simple (except possible the case $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$), and they have all a central sequence of copies of $\mathcal{E}_2$ (the latter with exception of $M_{n^\infty} \otimes M_{n^\infty}$ or of the Jiang-Su algebra).

It makes them pi-sun C^*-algebras if the fix-point algebras of the flip on $M_{n^\infty} \otimes M_{n^\infty}$ is simple ????

and `Why this?`

The inductive limit for the latter is give by unital $M_{f(k)} \oplus M_{g(k)} \to M_{f(k+1)} \oplus M_{g(k+1)}$. This happens with some "over-crossing" and $M_{f(k)} \oplus M_{g(k)}$ is unitally contained in $M_{n^k} \otimes M_{n^k}$, in particular $n^{2k} = f(k) + g(k)$). $f(k) := n^k(n^k + 1)/2$, $g(k) := n^k(n^k - 1)/2$.

Iteration rules are given by $f(k+1) := \beta_n(f(k), g(k)) := f(k)n(n+1)/2 + g(k)n(n-1)/2$,

$g(k+1) := \gamma_n(f(k), g(k)) := f(k)n(n-1)/2 + g(k)n(n+1)/2$.

It comes from the rule for eigenvalues of the flip-unitary for $M_n \otimes M_n$ on $\ell_2(n) \otimes \ell_2(n)$:
Plus: $n + (n^2 - n)/2 = n(n+1)/2$ and Minus: $(n^2 - n)/2 = n(n-1)/2$.

Check of $f(k) + g(k) = n^{2k}$ is OK:

$$g(k+1) = n^k(n^k+1)n(n-1)/4 + n_k(n^k-1)n(n+1)/4 = n^{k+1}(n^{k+1}-1)/2,$$

and

$$f(k+1) := f(k)n(n+1)/2 + g(k)n(n-1)/2 = n^{k+1}(n^{k+1}+1)/2.$$

Is there a variant of next prop.
for $A = B \otimes \mathcal{Z}$?

Proposition B.20.1. *Let $A := B \otimes \mathcal{O}_\infty$ and let $\gamma \colon A \otimes A \to A \otimes A$ the flip isomorphism $\gamma(a_1 \otimes a_2) := a_2 \otimes a_1$.*

Then the fix-point algebra $(A \otimes A)^\gamma$ has only ideals I that are intersections of ideals J of $A \otimes A$ with $(A \otimes A)^\gamma$:

$$I = (A \otimes A)^\gamma \cap J.$$

The flip-algebra $(A \otimes A)^\gamma$ contains in its multiplier algebra a central sequence of unital copies of $\mathcal{O}_\infty$.

In particular, $(A \otimes A)^\gamma$ is simple and purely infinite if B is simple and exact.

Proof. It is easy to see (why $\subseteq$??) that for each closed ideal I of $A \otimes A$ the set $I + \gamma(I)$ is a γ-invariant closed ideal of $A \otimes A$ with

$$(A \otimes A)^\gamma \cap (I + \gamma(I)) = (A \otimes A)^\gamma \cap I.$$

We show that if $0 \leq b \in (A \otimes A)^\gamma$ and $c \in (A \otimes A)_+$ is contained in the ideal I of $A \otimes A$ generated by b then the element $c + \gamma(c)$ is in the closed ideal J of $(A \otimes A)^\gamma$ generated by b:

Let $0 \leq b \in (A \otimes A)^\gamma$ with $\|b\| = 1$, $c \in (A \otimes A)_+$ in the ideal of $A \otimes A$ generated by b and $\varepsilon \in (0, 1/4)$. Then there exists $g_1, \ldots, g_n \in A \otimes A$ with $\|c - \sum_{k=1}^n g_k^* b g_k\| < \varepsilon$. Define $\eta := \varepsilon/(n^2+1)(1+C^2)$, where $\delta := \max\{\|g_1\|, \ldots, \|g_n\|\}$.

Since b can be approximated by elements in the algebraic tensor product $A \odot A$ there exist isometries $s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n} \in \mathcal{M}(A)$ with mutually orthogonal ranges and the property that $\|(s_k \otimes s_{n+k})b - b(s_k \otimes s_{n+k})\| < \eta$ for above defined η and $k \in \{1, \ldots, n\}$.

The norms of $(s_k \otimes s_{n+k})^* b (s_k \otimes s_{n+k}) - b$ and of $(s_j \otimes s_{n+j})^* b (s_k \otimes s_{n+k})$ for $j \neq k$ are less than η.

Now let $h := \sum_k (s_k \otimes s_{n+k}) g_k$. The norm of $\delta^{-2}\big(h^*bh - \sum_{k=1}^n g_k^* b g_k\big)$ can be estimated by

$$\sum_k \|(s_k \otimes s_{n+k})^* b (s_k \otimes s_{n+k}) - b\| + \sum_{j \neq k} \|(s_j \otimes s_{n+j})^* b (s_k \otimes s_{n+k})\| \leq n^2 \eta .$$

Recall that $\gamma(h) = \sum_k (s_{n+k} \otimes s_k) \gamma(g_k)$.

It follows that $\gamma(h)^* h = 0$, $\|\gamma(h)^* b \gamma(h) - \gamma(c)\| < \varepsilon$ and

$$\|\gamma(h)^* b h\| \leq \delta^2 \big(\sum_{k,\ell=1}^{n} \|(s_{n+j} \otimes s_j)^* b (s_k \otimes s_{n+k})\| \big) \leq \delta^2 n^2 \eta .$$

It gives that

$$\| (c + \gamma(c)) - (h + \gamma(h))^* b (h + \gamma(h)) \| \leq 2(\varepsilon + \delta^2 n^2 \eta) < 4\varepsilon .$$

Notice that $c \leq c + \gamma(c)$ and that $c + \gamma(c)$ and $h + \gamma(h)$ are in $(A \otimes A)^\gamma$.

Thus, each ideal of the fix-point algebra $(A \otimes A)^\gamma$ of the flip automorphism on $A \otimes A$ is the intersection $J \cap (A \otimes A)^\gamma$ of an ideal J of $A \otimes A$ with $(A \otimes A)^\gamma$.

Let $(s_n), (t_n) \subset \mathcal{M}(A)$ sequences of isometries with $\lim_{n\to\infty}\big(\|s_n a - a s_n\| + \|t_n a - a t_n\|\big) = 0$ for each $a \in A$, and $s_n^* t_n = 0$, then $s_n \otimes s_n, t_n \otimes t_n \in \big(\mathcal{M}(A) \otimes \mathcal{M}(A)\big)^\gamma \subseteq \mathcal{M}\big((A \otimes A)^\gamma\big)$, the isometries $(s_n \otimes s_n)_n$ and $(t_n \otimes t_n)_n$ are central sequences for all elements of $A \otimes A$. □

Question B.20.2. What happens if $A \cong B \otimes \mathcal{E}(M_{2^\infty}, M_{3^\infty})$?

What happens in case that A is approximately divisible?

Corollary B.20.3. *If A is a pi-sun C*-algebra then its flip-algebra $(A \otimes A)^\gamma$ is again a pi-sun algebra.*

Proof. It is clear that $(A \otimes A)^\gamma$ is separable, unital and nuclear because $V(a) := (1/2)(a + \gamma(a))$ is a unital conditional expectation from the separable, unital and nuclear algebra $A \otimes A$ onto $(A \otimes A)^\gamma$. By Corollary F(ii), $A \cong A \otimes \mathcal{O}_\infty$ for all pi-sun algebras A. Thus Proposition B.20.1 applies to A and gives that $(A \otimes A)^\gamma$ is simple and contains a central sequence of copies of $\mathcal{O}_\infty$. (Recall here that $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes \cdots$ and that $C^*(s_n^k t_n, k = 1, 2, \ldots)$ is isomorphic to $\mathcal{O}_\infty$ for each pair of isometries s_n, t_n with $s_n^* t_n = 0$.) □

21. On ideal structure of crossed products

Remark B.21.1. Let X a *non-compact* locally compact Hausdorff space. Then X is a *completely regular* space, i.e., for each closed subset $F \subseteq X$ and $x \in X \setminus F$ there exists $g \in \mathrm{C}_0(X)_+$ with $g(x) = 1$, $\|g\| = 1$ and $g(F) = \{0\}$. We denote by $\beta(X)$ the Stone-Cech compactification of X (maximal compact Hausdorff space with X as a dense subspace). Recall that it can be defined equivalently by the property that each continuous map $\phi\colon X \to Z$ from X into a compact (Hausdorff) space Z uniquely extends to a continuous map $\beta(\phi)\colon \beta(X) \to Z$. An other description of $\beta(X)$ is the the set of characters of $\mathrm{C_b}(X)$ with $\sigma(\mathrm{C_b}(X)^*, \mathrm{C_b}(X))$-topology, i.e., as Gelfand character space of $\mathrm{C_b}(X)$. Thus, there is a natural isomorphism $\mathrm{C_b}(X) \cong \mathrm{C}(\beta(X))$ by Gelfand-Naimark. and each continuous map from X into a compact metric space M – e.g. a closed disk – extends to a continuous map from $\beta(X)$ into M.

Let A a C^*-algebra and let $Y \subset \gamma(X)$ a non-empty compact subset of the corona space $\gamma(X) := \beta(X) \setminus X$ of X.

If we use the natural isomorphism $\mathrm{C}(\beta(X)) \cong \mathrm{C_b}(X)$ then $\mathrm{C_b}(X, A)$ becomes the $\mathrm{C}(\beta(X))$-algebra of continuous sections in a suitable continuous field of C^*-algebras $\{A_y\, ;\, y \in \beta(X)\}$ over $\beta(X)$. This is because the norm function map $f \in \mathrm{C_b}(X, A) \mapsto N(f) \in \mathrm{C_b}(X)$ defined by $N(f)(x) := \|f(x)\|$. It satisfies with $N(f)h = N(hf)$ for $h \in \mathrm{C_0}(X)$. This defines $\mathrm{C_b}(X, A)$ as continuous field over $\beta(X)$:

(1.) $\mathrm{C_b}(X, A)$ is a $\mathrm{C_b}(X)$-algebra:

If $g\colon X \to A$ in $\mathrm{C_b}(X, A)$ and $f \in \mathrm{C_b}(X)$, then $(f \cdot g) \in \mathrm{C_b}(X, A)$ for $(f \cdot g)(x) = f(x)g(x)$.

(2.) There is a natural norm function $N\colon g \in \mathrm{C_b}(X, A) \to N(g) \in \mathrm{C_b}(X)_+$ that satisfies $N(f \cdot g) = |f| \cdot N(g)$, $N(g_1 + g_2) \leq N(g_1) + N(g_2)$ and $N(g^* g) = N(g)^2$.

The property (1.) and (2.) together impliy that $\mathrm{C_b}(X, A)$ the algebra of continuous sections for the continuous field $(\, A_\omega\, ;\, \omega \in \beta(X)\,)$. Here A_ω is the quotient defined the C^*-seminorm $N(\cdot)(\omega)$ and is isomorphic to $\mathrm{C_b}(X, A)/(J_\omega \cdot \mathrm{C_b}(X, A))$ with J_ω equal to the kernel of the evaluation character $f \mapsto f(\omega)$ on $\mathrm{C_b}(X)$ for $\omega \in \beta(X)$. For our applications it is an important point that $\mathrm{C_b}(X, A)$ is also a lower s.c. field of C^*-algebras. and not only an upper s.c. field of C^*-algebras over the compact space $\beta(X)$.

If Y is a closed subspace of $\beta(X)$, then we can form the "restriction" C^*-algebra $\mathrm{C_b}(X, A)|Y := \mathrm{C_b}(X, A)/J_Y$, where J_Y is the ideal of bounded A-valued continuous functions $f \in \mathrm{C_b}(X, A)$ with the property that the function $g(x) := \|f(x)\|$ in $\mathrm{C_b}(X) = \mathrm{C}(\beta(X))$ satisfies $g|Y = 0$, i.e., J_Y is the closed ideal of $\mathrm{C_b}(X, A)$ generated $\mathrm{C_0}(\beta(X) \setminus Y) \cdot \mathrm{C_b}(X, A)$.

One can here also think of $\mathrm{C_b}(X, A)$ as an ideal of $\mathrm{C_b}(X, \widetilde{A}\,)$ where $\widetilde{A}$ denotes A if A is unital and $A + \mathbb{C}1 \subseteq \mathcal{M}(A)$ otherwise

We write always A_ω for $\mathrm{C_b}(X, A)|Y$ if $Y = \{\omega\}$ for a point $\omega \in \beta(X) \setminus X$ and, sometimes, also A^Y for $\mathrm{C_b}(X, A)|Y$.

(There are good reasons for the requirement $Y \cap X = \emptyset$ on Y.)

Well-known *examples* are $A_\infty := \mathrm{C_b}(X, A)|Y$ for $X = \mathbb{N}$, $Y := \beta(\mathbb{N}) \setminus \mathbb{N}$, $A_\omega := \mathrm{C_b}(X, A)|Y$ for $X = \mathbb{N}$, $Y := \{\omega\}$ where $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ is a point (corresponding to a free ultra-filter on $\mathbb{N}$) or $\mathrm{Q}(X, A) := \mathrm{C_b}(X, A)|Y$ for $Y := \gamma(X) = \beta(X) \setminus X$.

Let $f_a(x) := a$ for $x \in X$ and $a \in A$. The map $A \ni a \mapsto f_a(x) \in \mathrm{C_b}(X, A)$ defines a natural *-monomorphism from A into $\mathrm{C_b}(X, A)$ with $J_Y \cap A = \{0\}$ for all non-empty $Y \subset \beta(X) \setminus X$.

Therefore, there are natural *-monomorphisms $A \hookrightarrow \mathrm{C_b}(X, A)|Y$. We identify A with the image of A under this embedding. The hereditary C^*-subalgebras $A\,\mathrm{C_b}(X, A)A$ (respectively $A(\mathrm{C_b}(X, A)|Y)A$) of $\mathrm{C_b}(X, A)$ (respectively of $\mathrm{C_b}(X, A)|Y$) will be denoted by E_A or $E(X, A)$ (respectively $E_A|Y$ or $E(X, A)|Y$).

It turns out that σ-unital A (or $E(X, A)|Y$) is *essential* in $\mathrm{C_b}(X, A)|Y$ for at least one $Y \neq \emptyset$, if and only if, A is unital.

In particular, A is a *degenerate* C^*-subalgebra of $\mathrm{C_b}(X, A)|Y$ if A is non-unital but σ-unital (and X is non-compact).

It follows that there are natural unital *-monomorphisms $\varphi_1 \colon \mathcal{M}(A) \subset \mathrm{C_b}(X, \mathcal{M}(A))|Y \to ????$ and $\varphi_2 \colon \mathcal{M}(A) \to \mathcal{M}(E_A|Y)$. The morphism φ_2 is strictly continuous, because $\varphi_2|A$ is essential in E_A (by definition of $E_A|Y$).

The natural morphism $\varphi \colon \mathcal{M}(A) \subset \mathrm{C_b}(X, \mathcal{M}(A))|Y$ is unital and injective and satisfies $\varphi(a) \in A \subset \mathrm{C_b}(X, A)|Y$.

The algebra $\mathrm{C_b}(X, A)|Y$ is an ideal of the unital algebra $\mathrm{C_b}(X, \mathcal{M}(A))|Y$, therefore, there is a natural unital *-morphism $\psi \colon \mathrm{C_b}(X, \mathcal{M}(A))|Y \to \mathcal{M}(\mathrm{C_b}(X, A)|Y)$, with kernel $\pi(I)$, where π denotes the natural epimorphism from $\mathrm{C_b}(X, \mathcal{M}(A))$ onto $\mathrm{C_b}(X, \mathcal{M}(A))|Y$ and I is the ideal of $g \in \mathrm{C_b}(X, \mathcal{M}(A))$ with $gf \in J_Y$ for all $f \in \mathrm{C_b}(X, A)$.

It holds $\varphi(\mathcal{M}(A)) \cap \ker(\psi) = \{0\}$, because $(\varphi(b)f)(x) = af(x)$ for $b \in \mathcal{M}(A)$, $x \in X$, and $bf|Y = 0$ for all $b \in \mathrm{C_b}(X, A)$ implies $bA = \{0\}$. Thus, $\psi \circ \varphi \colon \mathcal{M}(A) \to \mathcal{M}(\mathrm{C_b}(X, A)|Y)$ is a unital *-monomorphism with $\psi \circ \varphi(a) = a \in A \subset \mathrm{C_b}(X, A)|Y$. Thus, $\psi \circ \varphi(\mathcal{M}(A)) \subset \mathcal{N}(\mathrm{C_b}(X, A)|Y, E_A)$ and the induced unital *-morphism from $\mathcal{M}(A)$ into $\mathcal{M}(E_A)$ is the unique strictly continuous extension of the non-degenerate *-monomorphism $A \to E_A$.

If $T \in \mathcal{L}(A, B)$ is a bounded linear map, then $T^X(f)(x) := T(f(x))$ defines a bounded linear map $T^X \colon \mathrm{C_b}(X, A) \to \mathrm{C_b}(X, B)$ with $T^X(J_Y) \subset T^X(J_Y)$ for $Y \subset \beta(X)$, i.e., $T^X(f)|Y = 0$ if $f|Y = 0$. It allows to define the linear quotient map $T^Y \colon \mathrm{C_b}(X, A)|Y \to \mathrm{C_b}(X, B)|Y$, i.e., T^Y satisfies $T^Y(f|Y) = T^X(f)|Y$ and $T^Y(a) = T(a)$ for all $a \in A$.
The map $\mathcal{L}(A, B) \ni T \to T^Y \in \mathcal{L}(\mathrm{C_b}(X, A)|Y, \mathrm{C_b}(X, B)|Y)$ is linear and isometric. If $\alpha \colon A \to B$ is a *-morphism (respectively *-monomorphism, *-isomorphism, completely positive) then $\alpha^Y \colon \mathrm{C_b}(X, A)|Y \to \mathrm{C_b}(X, B)|Y$ is a *-morphism (respectively *-monomorphism, *-isomorphism, completely positive) with $\alpha^Y(a) = \alpha(a) \in B$ for all $a \in A$. If $T(a) = cad$ for $c, d \in \mathcal{M}(A)$, then $T^Y \colon \mathrm{C_b}(X, A)|Y \to \mathrm{C_b}(X, A)|Y$ satisfies $T^Y(f|Y) = \psi \circ \varphi(c) \cdot (f|Y) \cdot \psi \circ \varphi(d)$ for $f \in \mathrm{C_b}(X, A)$, and $T^Y(f|Y) = c \cdot (f|Y)d$ if $f|Y \in E_A = A(\mathrm{C_b}(X, A)|Y)A$.

In particular, if $\alpha \in \mathrm{Aut}(A)$ and $M := (\mathrm{C_b}(X, A)|Y)^{**}$ p the support projection of E_A in M, then $\alpha^Y \in \mathrm{Aut}(\mathrm{C_b}(X, A)|Y)$, $\alpha^Y(A) = A$, $\alpha^Y(E_A) = E_A$, $(\alpha^Y)^{**}(A' \cap M) = A' \cap M$ and $(\alpha^Y)^{**}(p) = p$.

Moreover, if $\alpha = \mathrm{Ad}(u)$ (i.e., $\alpha(a) = uau^*$ for $a \in A$, where $u \in \mathcal{M}(A)$ is unitary), then $\alpha^Y = \mathrm{Ad}(U)$ for $U = V|Y$, where $V(x) := u$ for $x \in X$.

??????????????????????

LEMMA B.21.2. *Let $u \in \mathcal{M}(A)$ unitary, and $B := \mathrm{C_b}(X, A)|Y$ where $Y \neq \emptyset$ is a closed subset of $\beta(X) \setminus X$ (e.g. if $X = \mathbb{N}$ then $B := A_\infty$, $B := A_\omega$ are of this kind).*

*The *-isomorphism $\mu := \mathrm{Ad}(u)^Y \in \mathrm{Aut}(B)$ (cf. Remark B.21.1) is induced on B by $\gamma_u(f)(t) := \mathrm{Ad}(u)(f(t)) := uf(t)u^*$ for $t \in \mathbb{R}_+$ and $f \in \mathrm{C_b}(\mathbb{R}_+, A)$, and is given by the inner automorphism $\mathrm{Ad}(U)\colon B \ni b \mapsto UbU^* \in B$, for the unitary $U := \psi \circ \varphi(u)$ in $\psi(\varphi(\mathcal{M}(A))) \subset \psi(\mathrm{C_b}(X, \mathcal{M}(A)) \subset \mathcal{M}(B)$.*

In particular, $\mathrm{Ad}(U)(a) = \mathrm{Ad}(u)(a)$ for all $a \in A$.

*The second conjugate $\mathrm{Ad}(U)^{**}\colon B^{**} \to B^{**}$ is given by $\mathrm{Ad}(U)^{**}(b) = \eta(U)b\eta(U)^*$, where $\eta\colon \mathcal{M}(B) \to B^{**}$ means the natural strictly–*ultra-strongly continuous unital monomorphism.*

The $\sigma(B^{**}, B^*)$-closure of A in B^{**} is naturally W-isomorphic to A^{**} via a natural isomorphism $\eta_1\colon A^{**} \to B^{**}$. Let $q = \eta_1(1) \in B^{**}$ denote the unit of $\eta_1(A^{**}) \subset B^{**}$, and identify $\mathcal{M}(A)$ naturally with its image in A^{**}.*

Then

(i) *$\eta_1(a) = q \cdot \eta \circ \psi \circ \varphi(a)$ for all $a \in \mathcal{M}(A)$.*
(ii) *$A' \cap B^{**} = \eta_1(A^{**})' \cap B^{**}$ and $\mathrm{Ad}(U)^{**}(A' \cap B^{**}) = A' \cap B^{**}$.*
(iii) *$\mathrm{Ad}(U)^{**}(q) = q \in A' \cap B^{**}$ and $q\mathrm{Ad}(U)^{**}(b) = qb$ for all $b \in A' \cap B^{**}$, In particular $ab = a\mathrm{Ad}(U)^{**}(b)$ for all $a \in A$ and $b \in A' \cap B^{**}$.*

PROOF. **??**

$pB^{**}p$ is the second conjugate of ABA,

????? □

LEMMA B.21.3. *Suppose that $A \subset B := \mathrm{C_b}(X, A)|Y$ is as in Remark B.21.1, $\alpha \in \mathrm{Aut}(A)$, and $q \in B^{**}$ denotes the support projection of the hereditary C*-subalgebra $ABA = E_A|Y$ of B. Let $\beta := \alpha^Y : B \to B$ denote the natural extension of α to B (cf. Remark B.21.1).*

Then $\beta(q) = q$, q is in the centre of $A' \cap B^{**}$, and $\beta(A' \cap B^{**}) = A' \cap B^{**}$.

If there are $\kappa \in (0, 1)$ and a unitary $u \in \mathcal{M}(A)$ with $\|\alpha - \mathrm{Ad}(u)\| \leq 2\kappa$, then $\|\beta(p) - p\| \leq \kappa$ for all projections $p \in (A' \cap B^{**})q$.

In particular, if p is a projection in $A' \cap B^{**}$, then $a\beta^{**}(p)p = 0$ for all $a \in A$ implies that $qp = 0 = q\beta^{**}(p)$.

PROOF. The linear map $\beta - \mathrm{Ad}(U)$ is equal to the natural extension $(\alpha - \mathrm{Ad}(u))_Y$ of $\alpha - \mathrm{Ad}(u) \in \mathcal{L}(A, A)$ to a bounded linear operator of $\mathrm{C_b}(\mathbb{R}_+, B)|Y$, where $U \in \mathcal{M}(B)$ is defined as in Lemma B.21.2. It follows $\|\beta - \mathrm{Ad}(U)\| \leq 2\kappa$.

Since $\beta(a) = \alpha(a)$ for all $a \in A$, we get $\beta^{**}(A' \cap B^{**}) = A' \cap B^{**}$ and $\beta^{**}(q) = q$. The projection q is in the centre of B^{**}, because q is in the $\sigma(B^{**}, B^*)$-closure of A and commutes with A.

By Lemma B.21.2, $\mathrm{Ad}(U)^{**}(b) = b$ for all $b \in (A' \cap B^{**})q$. Thus, for projections $p \in (A' \cap B^{**})q$,

$$2\|p - \beta^{**}(p)\| = \|(q - 2p) - \beta^{**}(q - 2p)\| \le 2\kappa\,.$$

The equation $A\beta^{**}(p)p = \{0\}$ implies $\beta^{**}(qp)qp = 0$. But then $\|\beta^{**}(qp) - qp\| \le \kappa < 1$ implies that $0 = qp = q\beta^{**}(p) = \beta^{**}(qp)$. □

22. On semi-projective relations

Next Def. makes no sense !? In this formulation... only?

Definition B.22.1. An operator-polynomial relation $R\colon F^n \to B_\infty$ (on elements of a subset F of a C^*-algebra A) is *semi-projective* on F^n if there exists for each element $X \in F^n$ with $R(X) = 0$ a sequence $X_k \in F^n$ with $R(X_k) = 0$ and $\lim_k \|X_k - X\| = 0$.

23. Non-2-sub-additive quasi-traces exist.

We list some more details than given at the reference places:

Where is the "reference place"?

The unital type-I C^*-algebra A is a (very special) unital extension E:

$$0 \to c_0(\mathbb{K}) \to E \to_\psi \mathrm{C}([0,1]^2) \to 0\,.$$

It means that the closed ideal J generated by all commutators in the unital C^*-algebra E is isomorphic to $c_0(\mathbb{K}) \cong c_0 \otimes \mathbb{K}$. The commutative quotient E/J of E is isomorphic to $\mathrm{C}([0,1]^2)$ and, if $\psi\colon E \to \mathrm{C}([0,1]^2)$ is the defining map for the extension, then the (central) unital quasi-trace τ_E on E is given by $\tau_A \circ \pi_{c_0(\mathbb{K})}$, where τ_A is the Aarnes quasi-state on $\mathrm{C}([0,1]^2)$.

The following says: $C := \mathrm{C}([0,1]^n)$ is a "projective" C^*-algebra ? In the sense: If $\psi : C \to B/J$ is a unital *-monomorphism, then there exists a unital *-morphism $\phi : C \to B$ with $\phi = \pi_J \circ \psi$.

But what happens with the question concerning additivity, if μ is a quasi trace then for all strictly commuting elements a, b in the image holds additivity ... ???

THE NEW OBSERVATION IS:

Because $\mathrm{C}([0,1]^2)$ is projective inside the class of unital Abelian C^*-algebras one gets that an extension with $\ell_\infty(\mathbb{K})$ can be modified to an extension by $c_0(\mathbb{K})$ such that $\pi_{c_0(\mathbb{K})}(C^*(S,T)) \cap \ell_\infty(\mathbb{K}) = \{0\}$.

It gives an extension $0 \to c_0(\mathbb{K}) \to E \to \mathrm{C}([0,1]^2)$ that has the property that

(1) E is a type I C^*-algebra with Abelian quotient of Dimension $= 2$.

(2) This extension has the property that every local unital quasi-trace τ on $C([0,1]^2)$ is additive on $\tau|C^*(x,1)_+$ for all normal elements $x \in E$.

(3) The composition $\tau_E := \tau_A \circ \pi_{c_0(\mathbb{K})}$ is a is additive on each commutative C^*-subalgebra of E. But is not 2-sub-additive.

Why is it not 2-sub-additive ??

(4) There exist elements $x, y \in E_+$ with $\tau_E(x) = 0$, $\tau_E(y) = 0$ and $\tau_E(x+y) = 1$.

The restriction map

$$C_0(X) \ni f \mapsto f|Z \in C(Z) \cong C([0,1]^2)$$

defines an epimorphism from $A/J \cong C_0(X)$ onto $C([0,1]^2)$ and has the property that ?????????

It is not unlikely that one can manage that that $X = Z \cong [0,1]^2$ using the index-preserving compressions $(T^*)^n(\cdot)T^n$ on the Toeplitz algebra

$$0 \to \mathbb{K}(\ell_2) \to \mathcal{T} = C^*(T) \to \ Cf(S_1)\,.$$

Then one would get a honest extension $0 \to c_0(\mathbb{K}) \to A \to C([0,1]^2)$.

`But we need that` $\|[(T^*)^n a T^n, (T^*)^n b T^n]\|$ `becomes small if` $ab - ba \in \mathbb{K}(\ell_2)$
`for positive contraction` $a, b \in \mathcal{L}(\ell_2)$ `with` $ab - ba \in \mathbb{K}$.

We use the following Lemma:

Lemma B.23.1. *Let $a, b \in \mathcal{L}(\ell_2)$ positive contractions that satisfy $ab - ba \in \mathbb{K}$, $p \in \mathbb{K}$ a projection and $\varepsilon(0, 1/2]$. Then there exists a positive contraction $c \in \mathbb{K}$ such that $p \leq c$, $\|ca - ac\| < \varepsilon$, $\|cb - bc\| < \varepsilon$ and*

$$\|(1-c)a(1-c)^2b(1-c) - (1-c)a(1-c)^2b(1-c)\| < \varepsilon\,.$$

I.e. $(1-c)a(1-c)$ and $(1-c)b(1-c)$ commute up to ε and are compact perturbations of a and b.

Proof. It is essentially the existence of an approximate unit $\{e_n \in A_+\,;\ n = 1, 2, \ldots\}$ in σ-unital C^*-algebras A that satisfies $e_n e_{n+1} = e_n$, $\|e_n\| = 1$ and is quasi-central with respect to a separable C^*-subalgebra $B \subseteq \mathcal{M}(A)$, – in the sense that $\lim_n \|be_n - e_n b\| = 0$ for all $b \in B$.

The proof uses obvious modifications of the proof of [**616**, thm. 3.12.14], and has to observe that the arguments for [**616**, cor. 3.12.15,cor. 3.12.16] work also for σ-unital A and separable $B \subseteq \mathcal{M}(A)$. It shows that we can find a sequence (e_n) with this properties inside the (algebraic) convex hull of any given countable approximate unit $\{p_1, p_2, \ldots\}$ of σ-unital C^*-algebra A that have the additional property $p_n p_{n+1} = p_n$. (Such an approximate unit $\{p_n\,;\ n \in \mathbb{N}\}$ of A exists in every σ-unital C^*-algebra A.)

In our special case we apply this to $A := \mathbb{K}(\ell_2)$ and take the projections $p_n \in A$ of rank $n \in \mathbb{N}$ as approximate unit of A, that maps $\ell_2(\mathbb{N})$ to the linear span of the first n elements of the canonical basis of $\ell_2(\mathbb{N})$. Now let $B := C^*(\mathbb{K}(\ell_2), a, b) \subseteq \mathcal{L}(\ell_2) = \mathcal{M}(A)$. $\square$

REMARK B.23.2. Let $f_0 \in \mathrm{C}([0,1]) = C^*(f_0, 1)$ denote the identity map $f_0(t) := t$ for all $t \in [0,1]$ and let $1 \in \mathrm{C}([0,1])$ denote the function that is constant equal to 1.

The functions $g, h \in \mathrm{C}([0,1]^2) = \mathrm{C}([0,1]) \otimes \mathrm{C}([0,1])$ are the coordinate maps $g(x,y) = x$ and $h(x,y) = y$, corresponding to $f_0 \otimes 1$ and $1 \otimes f_0$ in the tensor product notation.

Notice that $\mathrm{C}([0,1]^2) = C^*(g, h, 1)$ and that this means in the tensor notation that $\mathrm{C}([0,1]) \otimes \mathrm{C}([0,1]) = C^*(f_0 \otimes 1, 1 \otimes f_0, 1 \otimes 1)$.

LEMMA B.23.3. *The algebra* $\mathrm{C}([0,1]^2)$ *is the universal C^*-algebra with respect to the above given relations and has following properties:*

If A is a unital C^*-algebra that is generated by commuting positive contactions $S, T \in A$ and 1_A, then there exists a unique C^*-algebra morphism $\varphi\colon \mathrm{C}([0,1]^2) \to A$ *with $\varphi(g) = S$, $\varphi(h) = T$ and $\varphi(1) = 1_A$.*

In particular, if $S, T \in A$ are commuting positive contractions in a unital C^*-algebra A and if $\psi\colon C^*(S, T, 1_A) \to \mathrm{C}([0,1]^2)$ is a unital C^*-morphism with $\psi(S) = f$ and $\psi(T) = g$ then ψ is an isomorphism form $C^*(S, T, 1_A)$ onto $\mathrm{C}([0,1]^2)$.

PROOF. The universality of $\mathrm{C}([0,1]^2)$ in the class of unital commutative C^*-algebras is easy to see, because $\mathrm{C}([0,1])$ is the universal unital C^*-algebra generated by a positive contraction, and $\mathrm{C}([0,1])$ is nuclear. This implies that any pair of element-wise commuting unital C^*-morphisms $\phi_1, \phi_2\colon \mathrm{C}([0,1]) \to A$ define a unital C^*-morphism $\varphi\colon \mathrm{C}([0,1]^2) \to A$ with $\varphi(g) = \phi_1(f_0)$ and $\varphi(h) = \phi_2(f_0)$.

Let $\psi\colon C^*(S, T, 1_A) \to \mathrm{C}([0,1]^2)$ a unital C^*-morphism with $\psi(S) = g$ and $\psi(T) = h$. Then the universal epimorphism φ from $\mathrm{C}([0,1]^2)$ onto $C^*(S, T, 1_A)$ satisfies $\psi \circ \varphi = \mathrm{id}$ of $\mathrm{C}([0,1]^2)$, because $\psi \circ \varphi$ is unital, $\psi \circ \varphi(g) = g$ and $\psi \circ \varphi(h) = h$. Therefore ψ must be surjective and φ injective. Thus ψ and φ are isomorphisms. $\square$

Let $\gamma_n\colon S^1 \to [0,1]^2$ a sequence of piecewise smooth continuous map that contains in its uniform closure the set of all continuous maps $\gamma\colon S^1 \to [0,1]^2$ that are piecewise smooth Jordan curves ([5]).

Denote by $\mathcal{T} \subset \mathcal{L}(\ell_2)$ the Toeplitz algebra. Notice that $\mathbb{K} \subset \mathcal{T}$ and that there is a natural isomorphism from $\mathrm{C}(S^1)$ onto $\mathcal{T}/\mathbb{K} \subset \mathcal{C}$, where $\mathcal{C} := \mathcal{L}(\ell_2)/\mathbb{K}$ denotes the *Calkin* algebra.

The maps γ_n define a sequence of unital C^*-morphisms $\varphi_n\colon \mathrm{C}([0,1]^2) \to \mathrm{C}(S^1) \subset \mathcal{C}$ and, therefore, a unital C^*-morphism $\Phi\colon \mathrm{C}([0,1]^2) \to \ell_\infty(\mathrm{C}(S_1)) \subset \ell_\infty(\mathcal{C})$.

[5] I.e. the closure of the set of maps γ_n – but considered as positive contractions in $\mathrm{C}(S^1) \oplus \mathrm{C}(S^1)$ – contains all those maps $\gamma\colon S^1 \to [0,1]^2$ that are piecewise smooth Jordan curves.

The C^*-morphism Φ is faithful on $\mathrm{C}([0,1]^2)$, and is a unital C^*-monomorphism, because for every point $x \in [0,1]^2$ there exists a piece-wise smooth Jordan curve $\gamma\colon S^1 \to [0,1]^2$ with $x \in \gamma(S^1)$.

The generators $\{1, \Phi(g), \Phi(h)\} \subset \ell_\infty(\mathrm{C}(S_1))$ of can be lifted to $1 \in \ell_\infty(\mathcal{T}) \subset \ell_\infty(\mathcal{L}(\ell_2))$ and to, not necessarily commuting, positive contractions $X, Y \in \ell_\infty(\mathcal{T})$. The positive contractions $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ satisfy $XY - YX \in \ell_\infty(\mathbb{K})$. It follows that the n-th entries with $\pi_{\mathbb{K}}(x_n) = \varphi_n(g)$ and $\pi_{\mathbb{K}}(y_n) = \varphi_n(h)$ can be modified step by step by a compact perturbation with elements in $1 + \mathbb{K}$, as considered and shown in Lemma B.23.1. We get new positive contractions s_n and t_n with $x_n - s_n, y_n - t_n \in \mathbb{K}$ and $\|s_n t_n - t_n s_n\| < 2^{-n}$. The new positive contractions $S := (s_1, s_2, \ldots)$ and $T := (t_1, t_2, \ldots)$ in $\ell_\infty(\mathcal{T}) \subset \ell_\infty(\mathcal{L}(\ell_2))$ have the properties that $S - X, T - Y \in \ell_\infty(\mathbb{K})$,

$$ST - TS \in c_0(\mathbb{K}) \subset \ell_\infty(\mathbb{K}) \subset \ell_\infty(\mathcal{T}),$$

$S + \ell_\infty(\mathbb{K}) = \Phi(g)$ and $T + \ell_\infty(\mathbb{K}) = \Phi(h)$. The new positive contractions $S + c_0(\mathbb{K})$ and $T + c_0(\mathbb{K})$ are commuting positive contractions in $\ell_\infty(\mathcal{T})/c_0(\mathbb{K})$ and the commutative unital C^*-algebra $D \subset \ell_\infty(\mathcal{T})/c_0(\mathbb{K})$ generated by them (and 1) maps onto the C^*-algebra $\Phi(\mathrm{C}([0,1]^2)) \subseteq \ell_\infty(\mathrm{C}(S^1))$ in a way that $S + c_0(\mathbb{K})$ maps to $\Phi(g)$ and $T + c_0(\mathbb{K})$ maps to $\Phi(h)$. By the injectivity of Φ, it follows from Lemma B.23.3 that $\pi_{\ell_\infty(\mathbb{K})}$ maps the commutative C^*-algebra $\pi_{c_0(\mathbb{K})}(C^*(S,T,1))$ onto $\Phi(\mathrm{C}([0,1]^2))$ in a way that the canonical that the described generators are respected.

??????? Alternative description: ?????

Since $ST - TS \in c_0(\mathbb{K})$ it follows that $\pi_{c_0(\mathbb{K})}(S)$ and $\pi_{c_0(\mathbb{K})}(T)$ generate commutative C^*-subalgebra A of $\ell_\infty(\mathcal{T})/c_0(\mathbb{K})$ with the property that it is generated by two commuting positive contraction, that gives

?????

It follows, that $B := C^*(S,T,1) + c_0(\mathbb{K})$ is a C^*-algebra with the property $B/c_0(\mathbb{K}) \cong \mathrm{C}([0,1]^2)$, because $C^*(S,T,1) \cap \ell_\infty(\mathbb{K}) = C^*(S,T,1) \cap c_0(\mathbb{K})$.

`To be checked if we have moreover ???`

For all normal elements $b \in B$ (respectively $b \in C^*(S,T,1) + \ell_\infty(\mathbb{K})$) holds that the function $f := \pi_{c_0(\mathbb{K})}(b)$ (respectively the function $f := \pi_{\ell_\infty(\mathbb{K})}(b)$) in $\mathrm{C}([0,1]^2)$ has zero winding numbers, – with respect to alls closed curves $\gamma\colon S^1 \to [0,1]^2$ in $[0,1]^2$ and all points in $\mathbb{C} \setminus f(\gamma(S^1))$. (It is here not of interest if clockwise or anti-clockwise parametrized.)

The extensions have the property that

$$f \in \mathrm{C}([0,1]^2) \cong (C^*(S,T,1) + \ell_\infty(\mathbb{K}))/\ell_\infty(\mathbb{K}) \subseteq \ell_\infty(\mathrm{C}(S^1))$$

has only zero winding numbers,

???? if and only if, ???? "if" is O.K.
???? But what about the converse ?:
???? Suppose that f has only zero winding

???? numbers. Then f can be approximately
???? factorized over 1-dimensional spaces:
???? Means if X is compact and metric space:
???? If $f \in \mathrm{C}(X)$ and $\varepsilon > 0$,
???? there exists $g\colon X \to Y$
???? with Y 1-dimensional, and
???? $h\colon Y \to \mathbb{C}$ with ???? $\|h \circ g - f\| < \varepsilon$. ????

if there is a normal element in $x \in \ell_\infty(\mathbb{K}) + C^*(S, T, 1)$, i.e., x satisfies $x^*x = xx^*$, with $f = \pi_{\ell_\infty(\mathbb{K})}(x)$ ([6]).

The latter is the case, if and only if, there exists, – for a given $y \in C^*(S, T, 1) + c_0(\mathbb{K}) \subset \ell_\infty(\mathcal{T})$ with $\pi_{c_0(\mathbb{K})}(y) = f$ –, an element $z \in \ell_\infty(\mathbb{K})$ with the property that $z + y$ is normal: $(z + y)^*(z + y) = (z + y)(z + y)^*$. The element y satisfies $y^*y - yy^* \in c_0(\mathbb{K})$ by commutativity of $(C^*(S, T, 1) + c_0(\mathbb{K}))/c_0(\mathbb{K})$.

(If one could here manage that $y^*z, zy^* \in c_0(\mathbb{K})$, then one gets at least that $z^*z - zz^* \in c_0(\mathbb{K})$, but we don't know that.)

It leads to the following question about contractions with index zero in the Toeplitz algebra $\mathcal{T}$:

Question B.23.4. Do we need for the determination of the winding numbers only to look for invertible elements in $\mathrm{C}(S^1)$?

Let $f \in \mathrm{C}([0, 1])$. We need for the conclusion that the restriction $\tau_A | \mathrm{C}(f, 1)_+$ of the Aarnes quasi-state τ_A on $\mathrm{C}([0, 1])_+$ to the positive part of $\mathrm{C}(f, 1)_+$ is there additive, the following property of the given $f \in \mathrm{C}([0, 1])$:

For *each* continuous map $\gamma\colon S^1 \to [0, 1]^2$ and $z \notin f(\gamma(S_1))$ the winding number $w(f \circ \gamma, z)$ is equal to zero.

It can happen that $z \in f \circ \gamma_1(S_1)$ for some other Jordan curve $\gamma_1(S_1) \subseteq [0, 1]^2$ ... So z can be in the spectrum of some other factor $f \circ \gamma_1(S_1) \subseteq [0, 1]$.

But we have at least the positive direction: If there exists a normal element $g \in C^*(S, T, 1) + \ell_\infty(\mathbb{K})$ with $\pi_{\ell_\infty(\mathbb{K})}(g) = f$ then all winding numbers are zero.

Question: Suppose that one of it is not zero. Does there exist a non-additive quasi-state on $\mathrm{C}(f, 1)_+$?????

Does there exist for an increasing function $g \in \mathrm{C}_0((0, 1])$ with the following property?:

Let $X \in \mathcal{T}$ a contraction with index $\mathrm{index}(X) = 0$ and $\|X^*X - XX^*\| < \varepsilon$ then there exists $Y \in \mathcal{T}$ with $Y^*Y = YY^*$, $X - Y \in \mathbb{K}$ and $\|X - Y\| \leq g(\varepsilon)$.

Can take here contraction $X \in \mathcal{T}$ with polar decomposition $ZA = X$ for positive self-adjoint $A := (X^*X)^{1/2} \in \mathcal{T}_+$ and $Z \in \mathcal{L}(\mathcal{H})$ an isometry with index zero. It shows that one can replace Z by a unitary

[6] We can here suppose that $\|x\| = \|f\| = 1$, because we can f multiply with a positive scalar without changing the winding numbers of f and the normality of x, because all happens in the commutative C^*-algebra $C^*(x, x^*)$ and its quotients.

????. Get ???

Unfortunately, we do not know if this implies also the existence of a normal element $y \in C^*(S,T,1) + c_0(\mathbb{K})$.

Is this is equivalent (!) to the property that all the indices of the $f := \pi_{c_0(\mathbb{K})}(y)$ are zero?

It can be represented by a normal element in B [or only that there is an element x in $\ell_\infty(\mathbb{K})$ such that $f + x$ is normal].

[Need only that the indices vanish !!!! This should not change by compact perturbations ...]

(This is true if we consider the bigger $C^*(S,T) + \ell_\infty(\mathbb{K})$ instead.

Have then to show: If we have a normal representative in $C^*(S,T) + \ell_\infty(\mathbb{K})$ then we have a normal representative in B.)

??? Similar observations ?? show that the map $\mu := \tau_A \circ \varphi \colon B_+ \to [0,\infty)$ from the free product $B := \mathrm{C}[0,1] * \mathrm{C}[0,1]$ into $[0,\infty)$ is a unital quasi-trace that has additive restrictions $\mu|C_+$ for each commutative C^*-subalgebra C of B. It requires only the additivity on C_+ for all $C := C^*(a+ib)$ with commuting $a, b \in B_+$.

`Check next red and blue again! Look to the original notes and compare.`

The non-subadditive quasi-measure on μ_A on $[0,1]^2$ and the corresponding non-2-sub-additive quasi-state τ_A on $\mathrm{C}([0,1]^2)_+$, given by J. Aarnes in [**3**], has the property that there exists $g, h \in \mathrm{C}([0,1]^2)_+$ with $\tau_A(g) = 0$, $\tau_A(h) = 0$ and $\tau(g+h) > 2$. And it has the – for us important – property that the restriction $\tau_A|C^*(1,f)_+$ is additive, if and only if, the function $f\colon [0,1]^2 \to \mathbb{C}$ has zero "winding numbers" with respect to closed (Jordan) paths in $P \subset [0,1]^2$ and points in $\mathbb{C}\backslash f(P)$.

Now take the universal unital free product C^*-algebra $B := \mathrm{C}([0,1]) * \mathrm{C}([0,1])$ and the natural unital C^*-algebra epimorphism $\varphi\colon B \to \mathrm{C}([0,1]^2)$. One can show that all images $f = \varphi(b) \in \mathrm{C}([0,1]^2)$ of normal elements, i.e., $b \in B$ with $b^*b = bb^*$, have only zero winding numbers. (The proof uses the natural relation between winding numbers of parametrized closed curves and the index of operators in the Toeplitz algebra $0 \to \mathbb{K}(\ell_2) \to \mathcal{T} \to \mathrm{C}(S^1) \to 0$.) It implies that the restriction $\mu|B_+$ of $\mu := \tau_A \circ \varphi$ is a unital quasi-trace of $B = \mathrm{C}([0,1]) * \mathrm{C}([0,1])$. This μ can not be 2-sub-additive because otherwise τ_A must be 2-sub-additive on $\mathrm{C}([0,1]^2)_+$, by using here that the set of elements $x \in B$ with $\mu(x^*x) = 0$ must be a closed ideal of B if μ is 2-sub-additive.

One can explore the relation between winding numbers of elements of $\mathrm{C}([0,1]^2)$ and indices of the Toeplitz algebra $\mathcal{T}$ even more and take a sequence $\psi_n\colon S^1 \to [0,1]^2$ of pice-wise smooth closed curves $\psi_n\colon S^1 \to [0,1]^2 \cong [0,1] + i[0,1] \subseteq \mathbb{C}$ that are uniformly dense in the set of all closed curves $\psi\colon S^1 \to [0,1]^2$. It defines a sequence of unital C^*-epimorphisms $\varphi_n\colon \mathrm{C}([0,1]^2) \to \mathrm{C}(S^1) \cong \mathcal{T}/\mathbb{K}(\ell_2)$.

The latter sequence φ_n can be explored to show that there exist l.s.c. quasi-traces τ on certain type I C^*-algebras A:

It is an extension $0 \to J \to A \to_\phi \mathrm{C}([0,1]^2) \to 0$ with an ideal J that is an extension of $\mathrm{C}_0((0,1))$ (or of $\mathrm{C}_0(0,1]$?) by $c_0(\mathbb{K})$, that has the property that $\mu_A \circ \phi$ is not a 2-sub-additve, but is bounded and unital, is defined on all of A_+, and has additive restrictions to each commutative C^*-subalgebras of A. Moreover, there exists no "bound" $\gamma \in (0,\infty)$ with the property $\tau(a+b) \leq \gamma \cdot \max(\tau(a), \tau(b))$, because it can happen that $\tau(a+b) > 0$ – but $\mu(a) = 0 = \mu(b)$ for some non-commuting $a, b \in A_+$.

`HERE is an older summary:` J.F. Aarnes [**3**] described a ???????

There exists a unital extension $0 \to c_0(\mathbb{K}) \to A \to \mathrm{C}([0,1]^2) \to 0$ such that τ is given by $\tau = \tau_A \circ \phi$, where τ_A denotes the example of a globally non-additive unital quasi-state on $\mathrm{C}([0,1]^2)$ that is additive on each C^*-subalgebra $C^*(f,1)$ where $f\colon [0,1]^2 \to \mathbb{C}$ factorizes approximately over a tree (that is in this case of J.F. Aarnes described in [**3**]).

`It should be better to give a citation, not an explanation?!!`

Much more easy to handle, despite not of of type I, is the unital free product $B := \mathrm{C}[0,1] * \mathrm{C}[0,1]$ and the non-2-subadditive quasi-state $\tau_0 := \tau_A \circ \pi$, where here π denotes the natural C^*-epimorphism π from B onto $\mathrm{C}([0,1]) \otimes \mathrm{C}[0,1]) = \mathrm{C}([0,1]^2)$.

The key observation uses a suitable chosen unital *-monomorphism η from $\mathrm{C}([0,1]^2)$ into

$$\ell_\infty(\mathrm{C}(S^1)) \cong \ell_\infty(\mathcal{T})/\ell_\infty(\mathbb{K}),$$

where here $\mathcal{T} := C^*(T) \subset \mathcal{L}(\ell_2)$ means the Toeplitz algebra generated by the forward shift $Te_n = e_{n+1}$ on $\ell_2(\mathbb{N})$, and the defining countable family of maps from $\mathrm{C}([0,1]^2)$ to $\mathrm{C}(S^1)$ are given by a uniformly dense sequence in all pice-wise linear continuous maps (could also take smooth maps) from S^1 into $[0,1]^2 = [0,1] + i[0,1] \subseteq \mathbb{C}$. For each map $\gamma\colon S^1 \to [0,1]^2$ define a unital C^*-morphism $\widehat{\gamma}\colon f \mapsto f \circ \gamma$ All this C^*-morphisms lift to unital C^*-morphisms from B into $\mathcal{T}$, and define a unital C^*-morphism from B into $\ell_\infty(\mathcal{T})$.

Then the free product $B = \mathrm{C}[0,1] * \mathrm{C}[0,1]$ has an "universal lifting property" that implies that all unital C^*-morphisms $h\colon B \to \mathcal{C}$ into the Calkin algebra $\mathcal{C} := \mathcal{L}(\ell_2)/\mathbb{K}(\ell_2)$ lift to unital C^*-morphisms $H\colon B \to \mathcal{L}(\ell_2)$.

By using the BDF theory, this implies that each image of each normal element maps to an element with the property that all entries have zero indices ...

????

BDF = Brown Douglas Fillmore

supports an index argument for the characterization of normal operators (modulo compact perturbation) among the essentially normal operators:

An operator $S \in \mathcal{L}(\ell_2)$ is a compact perturbation of a normal operator, if and only if, all indices of $S - \lambda$ are equal to zero for all $\lambda \notin \mathrm{Spec}(\pi_{\mathbb{K}}(S))$, cf. [**207**, cor. IX.7.4].

This implies that all winding-numbers for $\pi(S) \in \mathrm{C}([0,1]^2)$ are zero.

The construction shows that a function $f \in \mathrm{C}([0,1]^2)$ is the image $\pi(g)$ of a normal element $g \in B$, if and only if, f can be uniformly approximated by functions $f_n \in \mathrm{C}([0,1]^2)$ that factorize over a finite tree T_n (which is then automatic homotopic to the root of T_n), i.e., $f_n = \varphi_n \circ \psi_n$ with $\psi_n \colon [0,1]^2 \to T_n$ and $\varphi_n \colon T_n \to [0,1]^2$, $\lim_n \|f_n - f\| = 0$.

An important property of such tree-factorable functions f_n is that they are homotopic inside there images $f_n([0,1]^2) \subset \mathbb{C}$ to a constant map (or at least "inside any neighbourhood of its image", by a simpler proof).

It is not difficult to check that the Aarnes quasi-state μ_0 on $\mathrm{C}([0,1] \times [0,1])$ becomes additive on the subalgebras $C^*(f) \subset \mathrm{C}([0,1]^2)$ generated by f with this special property (being a uniform limit of tree-factorable functions on $[0,1]^2$). Thus the restriction of $\tau_0 := \mu_0 \circ \pi$ every abelian C^*-subalgebra of $B := \mathrm{C}[0,1] * \mathrm{C}[0,1]$ is additive. But $\mu_0 \circ \pi$ can not be additive, because π is surjective and the Aarnes quasi-state μ_0 is not additive on $\mathrm{C}([0,1]^2)$.

An application of quasi-diagonal elements shows that this implies the existence of a unital type I C^*-algebra A that has a non-additive quasi-state is an extension $c_0(\mathbb{K}) \to A \to \mathrm{C}_0((0,1], \psi(B))$ by modification of a C^*-morphism ψ from the unitization of $\mathrm{C}_0((0,1], B)$ into $\ell_\infty(\mathcal{T}) \subseteq \ell_\infty(\mathcal{L}(\ell_2))$ constructed from the C^*-morphism that we have described above.

This equivalent properties of τ are long known for a sort of "normal" quasi-states on finite AW*-algebras M, and the proof of the equivalence of the three properties of τ goes over variants of suitable ultra-power constructions that give on suitable hereditary C^*-algebras $D \subseteq A$ with $\tau | D_+$ bounded a C^*-morphism $\varphi \colon D \to M$ where M is a hereditary C^*-subalgebra of an AW*-algebra such that all maximal abelian C^*-subalgebras of M are hereditary C^*-subalgebras of Abelian W*-algebras.

24. Central sequence algebras, the ε-test and its application.

The here given formulation and proof of the ε-test Lemma given in the Appendix of [**448**] corrects also a really obvious(!) typo in the there given formulation and proof of this Lemma.

To each free filter ω on the natural numbers $\mathbb{N}$ and to each C^*-algebra A one can associate the ultrapower A_ω and the central sequence C^*-algebra $A_\omega \cap A'$ as follows:

Let $c_\omega(A)$ denote the closed two-sided ideal of the C^*-algebra $\ell^\infty(A)$ of bounded sequences from A given by

$$c_\omega(A) = \left\{(a_n)_{n\geq 1} \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\right\}.$$

We use the notation $\lim_{n\to\omega} \alpha_n$ (and sometimes just $\lim_\omega \alpha_n$) to denote the limit of a sequence $(\alpha_n)_{n\geq 1}$ along the filter ω. This limit exists for all free ultra-filters ω on $\mathbb{N}$.

The ultrapower A_ω is defined to be the quotient C^*-algebra $\ell^\infty(A)/c_\omega(A)$; and we denote by π_ω the quotient mapping $\ell^\infty(A) \to A_\omega$. Let $\iota\colon A \to \ell^\infty(A)$ denote the "diagonal" inclusion mapping $\iota(a) = (a, a, a, \dots) \in \ell^\infty(A)$, $a \in A$; and define $\iota_\omega := \pi_\omega \circ \iota\colon A \to A_\omega$. Both mappings ι and ι_ω are injective. We shall often suppress the mapping ι_ω and view A as a sub-C^*-algebra of A_ω. The relative commutant, $A_\omega \cap A'$, then consists of elements of the form $\pi_\omega(a_1, a_2, a_3, \dots)$, where $(a_n)_{n\geq 1}$ is a bounded asymptotically central sequence in A. The C^*-algebra $A_\omega \cap A'$ is called a *central sequence algebra*.

We shall most often insist that the free filter ω is an ultrafilter, and we avoid using $A^\infty := \ell^\infty(A)/c_0(A)$ and $A_\infty := A' \cap A^\infty$, which have similar properties and produce similar results (up to different selection procedures). One of the reasons is that we need an *epimorphism* from A_ω onto the W*-algebra $N^\omega := \ell_\infty(N)/c_{\tau,\omega}(N)$ (to be defined below), cf. Theorem B.24.3 below, where N is the weak closure of A in the GNS representation determined by a tracial state τ on A. The sequence algebra $N^\infty := \ell_\infty(N)/c_{\tau,0}(N)$ (with $c_{\tau,0}(N)$ the bounded sequences in N with $\lim_n \|a\|_{2,\tau} = 0$) is not a W*-algebra. Another reason for preferring free ultra-filters to general free filters is that, for $A \neq \mathbb{C}$, simplicity of A_ω is equivalent to pure infiniteness and simplicity of A. The algebras A_ω are the fibers of the continuous field A_∞ with base space $\beta(\mathbb{N}) \setminus \mathbb{N}$. The structure of this bundle appears to be complicated.

Recall from [**448**] that if B is a C^*-algebra and if A is a separable sub-C^*-algebra of B_ω, then we define the central sequence algebra

$$F(A, B) := (A' \cap B_\omega)/\mathrm{Ann}(A, B_\omega).$$

This C^*-algebra has interesting properties. For example, $F(A, B) := F(A \otimes \mathbb{K}, B \otimes \mathbb{K})$. We let $F(A) := F(A, A)$. The C^*-algebra $F(A)$ is unital if A is σ-unital. If A is unital, then $F(A) = A' \cap A_\omega$. We refer to [**448**] for a detailed account on the C^*-algebras $F(A, B)$ and $F(A)$. We shall often, in the unital case, denote the central sequence algebra $A_\omega \cap A'$ by $F(A)$.

We have the following useful selection principle for sequences in the filter ω with a countable number conditions from [**448**, Lemma A.1]. For completeness we add a proof ([7]).

[7] We correct here a misleading typo in the original proof given in [**448**]!.

LEMMA B.24.1 (The ε-test). *Let ω be a free ultrafilter. Let $X_1, X_2, \dots$ be any sequence of sets and suppose that, for each $k \in \mathbb{N}$, we are given a sequence $(f_n^{(k)})_{n\geq 1}$ of functions $f_n^{(k)}\colon X_n \to [0,\infty]$.*

For each $k \in \mathbb{N}$ define a new function $f_\omega^{(k)}\colon \prod_{n=1}^\infty X_n \to [0,\infty)$ by

$$f_\omega^{(k)}(s_1, s_2, \dots) := \lim_{n\to\omega} f_n^{(k)}(s_n), \qquad (s_n)_{n\geq 1} \in \prod_{n=1}^\infty X_n.$$

Suppose that, for each $m \in \mathbb{N}$ and each $\varepsilon > 0$, there exists $s = (s_1, s_2, \dots) \in \prod_{n=1}^\infty X_n$ such that $f_\omega^{(k)}(s) < \varepsilon$ for $k = 1, 2, \dots, m$. It follows that there is $t = (t_1, t_2, \dots) \in \prod_{n=1}^\infty X_n$ with $f_\omega^{(k)}(t) = 0$ for all $k \in \mathbb{N}$.

PROOF. For each $n \in \mathbb{N}$ define a decreasing sequence $(X_{n,m})_{m\geq 0}$ of subsets of X_n by $X_{n,0} = X_n$ and

$$X_{n,m} = \big\{ s \in X_n\,;\ \max\{f_n^{(1)}(s), \dots, f_n^{(m)}(s)\} \,<\, 1/m \big\},$$

for $m \geq 1$. We let $m(n) := \sup\{m \leq n\,;\ X_{n,m} \neq \emptyset\}$; and for each integer $k \geq 1$, let $Y_k := \{n \in \mathbb{N}\,;\ k \leq m(n)\}$. Fix some $k \geq 1$. By assumption there exists $s = (s_n) \in \prod_n X_n$ such that $f_\omega^{(j)}(s) < 1/k$ for $1 \leq j \leq k$. This entails that there exists a set $Z_k \in \omega$ such that $f_n^{(j)}(s_n) < 1/k$ for $1 \leq j \leq k$ and for all $n \in Z_k$. This again implies that $X_{n,k} \neq \emptyset$ for all $n \in Z_k$; which finally shows that $m(n) \geq \min\{k, n\}$ for all $n \in Z_k$. It follows that $Z_k \setminus \{1, 2, \dots, k-1\} \subseteq Y_k$, from which we conclude that $Y_k \in \omega$ (because ω is assumed to be free). Now,

$$\lim_{n\to\omega} \frac{1}{m(n)} \;=\; \liminf_{n\to\omega} \frac{1}{m(n)} \;\leq\; \inf_k \sup_{n\in Y_k} \frac{1}{m(n)} \;\leq\; \inf_k \frac{1}{k} = 0.$$

By definition of $m(n)$ we can find $t_n \in X_{n,m(n)} \subseteq X_n$ for each $n \in \mathbb{N}$. Put $t = (t_n)_{n\geq 1}$. Then $f_n^{(k)}(t_n) \leq 1/m(n)$ for all $k < m(n)$ by definition of $X_{n,m(n)}$, so

$$f_\omega^{(k)}(t) \;=\; \lim_{n\to\omega} f_n^{(k)}(t_n) \;\leq\; \lim_{n\to\omega} \frac{1}{m(n)} \;=\; 0,$$

for all $k \geq 1$ as desired. □

Let N be a W*-algebra with separable predual and let τ be a faithful normal tracial state on N. Consider the associated norm, $\|a\|_{2,\tau} = \tau(a^*a)^{1/2}$, $a \in N$, on N. Let N^ω denote the W*-algebra $\ell_\infty(N)/c_{\omega,\tau}(N)$, where $c_{\omega,\tau}(N)$ consists of the bounded sequences $(a_1, a_2, \cdots)$ with $\lim_\omega \|a_n\|_{2,\tau} = 0$. (As mentioned above, if ω is a free filter, which is not an ultrafilter, then N^ω is not a W*-algebra.)

REMARK B.24.2. Since the pre-dual N_* of the W*-algebra N is separable, the algebra $N' \cap N^\omega$ contains a copy of the hyper-finite II_1 factor $\mathcal{R}$ with separable predual, if and only if, for each $k \in \mathbb{N}$ there exists a sequence of unital *-homomorphisms $\psi_n\colon M_k \to N$ such that $\lim_{n\to\infty} \|[\psi_n(t), a]\|_{2,\tau} = 0$ for all $t \in M_k$ and $a \in N$, if and only if, there is a *-homomorphism $\varphi\colon M_2 \to N' \cap N^\omega$ such that $t \in N \mapsto t \cdot \varphi(1) \in N^\omega$ is faithful. (It suffices to consider elements t in the centre of N.)

This equivalence was shown by Dusa McDuff, [**550**], in the case where N is a *factor*. She gets moreover that $N \cong N\overline{\otimes}\mathcal{R}$, if and only if, N has a central sequence that is not hyper-central. Such a II_1 factor is called a *McDuff factor*.

The result below was proved by Y. Sato in [**707**, Lemma 2.1] in the case where A is nuclear. We give here an elementary proof of this useful result, that does not assume nuclearity of A. The result implies that the central sequence C^*-algebra $A_\omega \cap A'$ has a subquotient isomorphic to the hyperfinite II_1 factor $\mathcal{R}$ whenever A has a factorial trace so that the corresponding II_1 factor, arising from GNS representation with respect to that trace, is a McDuff factor. In Remark **??** in the next section we show how one can give an easier proof of the theorem below using that the kernel J_τ of the natural *-morphism $A_\omega \to N^\omega$ is a so-called σ-ideal. The proof given below does not (explicitly) use σ-ideals.

THEOREM B.24.3. *Let A be a separable unital C^*-algebra, τ be a faithful tracial state on A, let N be the weak closure of A under the GNS representation of A with respect to the state τ, and let ω be a free ultrafilter on $\mathbb{N}$. It follows that the natural morphisms*

$$A_\omega \to N^\omega\,, \qquad A' \cap A_\omega \to N' \cap N^\omega$$

are surjective.

PROOF. Let π_A and π_N denote the quotient mappings $\ell^\infty(A) \to A_\omega$ and $\ell^\infty(N) \to N^\omega$, respectively. Denote the canonical map $A_\omega \to N^\omega$ by Φ, and let $\widetilde{\Phi}\colon \ell^\infty(A) \to N_\omega$ denote the map $\Phi \circ \pi_A$.

We show that $\Phi\colon A_\omega \to N^\omega$ is surjective: If $x = \pi_N(x_1, x_2, \dots)$ is an element in N^ω, then, by Kaplansky density theorem, there exists $a_k \in A$ with $\|a_k\| \leq \|x_k\|$ and with $\|a_k - x_k\|_{2,\tau} \leq 1/k$. It follows that $(a_1, a_2, \dots) \in \ell^\infty(A)$ and that $\widetilde{\Phi}(a_1, a_2, \dots) = x$.

To prove that the natural map $A' \cap A_\omega \to N' \cap N^\omega$ is surjective, it suffices to show that if $b = (b_1, b_2, \dots) \in \ell^\infty(A)$ is such that $\widetilde{\Phi}(b) \in N^\omega \cap N'$, then there is an element $c \in \ell^\infty(A)$ such that $\pi_A(c) \in A_\omega \cap A'$ and $\widetilde{\Phi}(c) = \widetilde{\Phi}(b)$. Let such a $b \in \ell^\infty(A)$ be given, put $B = C^*(A, b) \subseteq \ell^\infty(A)$, and put $J = (\widetilde{\Phi})^{-1}(0) \cap B$. Notice that an element $x = (x_1, x_2, \dots) \in \ell^\infty(A)$ belongs to the kernel $(\widetilde{\Phi})^{-1}(0)$ of $\widetilde{\Phi}$ if and only if $\lim_{n\to\omega} \|x_n\|_{2,\tau} = 0$. Notice also that $ba - ab \in J$ for all $a \in A$.

Let $(d^{(k)})_{k\geq 1}$ be an increasing approximate unit for J consisting of positive contractions which is asymptotically central with respect to the separable C^*-algebra B. Then

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|(1 - d^{(k)})\big(ba - ab\big)(1 - d^{(k)})\| \\ &= \lim_{k\to\infty} \|(1 - d^{(k)})b(1 - d^{(k)})a - a(1 - d^{(k)})b(1 - d^{(k)})\| \end{aligned}$$

for all $a \in A$.

We use the ε-test (Lemma B.24.1) to complete the proof:
Let $(a_k)_{k\geq 1}$ be a dense sequence in A. Let each X_n be the set of positive contractions

in A. Define $f_n^{(k)}: X_n \to [0,\infty)$ by

$$f_n^{(1)}(x) = \|x\|_{2,\tau}, \qquad f_n^{(k+1)}(x) = \|(1-x)b_n(1-x)a_k - a_k(1-x)b_n(1-x)\|, \ k \geq 1 .$$

Notice that $f_\omega^{(1)}(d^{(\ell)}) = \lim_{n\to\omega} \|d_n^{(\ell)}\|_{2,\tau} = 0$ for all ℓ because each $d^{(\ell)}$ belongs to J. Note also that

$$f_\omega^{(k)}(d^{(\ell)}) = \|(1-d^{(\ell)})b(1-d^{(\ell)})a_k - a_k(1-d^{(\ell)})b(1-d^{(\ell)})\|.$$

It is now easy to see that the ε-test in Lemma B.24.1 is satisfied, so there exists a sequence $d = (d_n)_{n\geq 1}$ of positive contractions in A such that $f_\omega^{(k)}(d) = 0$ for all k. As $f_\omega^{(1)}(d) = 0$ we conclude that $\widetilde{\Phi}(d) = 0$.

Put $c = (1-d)b(1-d)$. Then $\widetilde{\Phi}(c-b) = 0$, so $\widetilde{\Phi}(c) = \widetilde{\Phi}(b)$. Since $f_\omega^{(k)}(d) = 0$, we see that $ca_k - a_kc = 0$ for all $k \geq 2$. This shows that $\pi_A(c) \in A_\omega \cap A'$. □

Correction to:

'*The UCT, the Milnor Sequence, and a Canonical Decomposition of the Kasparov Groups*' Author: Schochet C.L.

Source: K-theory, Volume 14, Number 2, June 1998 , pp. 197-199(3)

Publisher: Springer

Abstract: In this note we correct a mistake in K-Theory 10 (1996), 49–72. In that paper we asserted that under bootstrap hypotheses the short exact sequence (injective at the beginning and surjective at the end):

$$\underset{\longleftarrow}{\lim}^1 \operatorname{Hom}_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \underset{\longleftarrow}{\lim} \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which arises in the computation of $KK_*(A,B)$ is a split sequence. This is not always the case. Thus $KK_*(A,B)$ decomposes into the three components

$$\operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)), \qquad \underset{\longleftarrow}{\lim} \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

and

$$\underset{\longleftarrow}{\lim}^1 \operatorname{Hom}_{\mathbb{Z}}(K_*(A_i), K_*(B)).$$

However, this is a decomposition in the sense of composition series, not as three direct summands. The same correction applies to the Milnor sequence. If there is no prime p for which both $K_*(A)$ and $K_*(B)$ have p-torsion then the decomposition is indeed as direct summands. The other results of the paper are unaffected.
Keywords: KK-theory; Kasparov groups; Universal Coefficient Theorem; Milnor sequence; KK-filtration; fine structure

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APPENDIX C

Temporary (!!) monitor and To-Do List

1. Circulation of Changes

We still have no example of a nuclear separable C^*-algebra that is purely infinite but is not strongly purely infinite !!! (2014, first discussed ?).

On simple C*-algebras A this two properties are the same! (For obvious reason!)

Def. equivalent to "purely infinite" (p.i.):

The element $a \in A_+$ is "purely infinite" if $a \oplus a \precsim a$ (inside $M_2(A)$ and means there that $\mathrm{diag}(a, a) \precsim \mathrm{diag}(a, 0)$). Here one can take also non-positive $a \in A$ with $a \oplus a \precsim a$, because $a \approx a^* a \approx (a^* a)^* = a(a^*)$.

This is equivalent to the property of $a \in A$ that If $b \in A_+$ and $b \neq 0$ is in the closed ideal $I(a)$ of A generated by a, then $b \oplus a \precsim a$.

(This is equivalent to $a \oplus a \precsim a$ and is the original definition of a being "p.i." given by J. Cuntz ! Give here precise Citation of Definition in paper of Cuntz!!!)

The Definition C.1.1 is equivalent to $a \oplus a \precsim a$ for all $a \in A_+$.

In general $a \in A_+$ is called "infinite" if there exists non-zero $x \in A$ with $x \oplus a \precsim a$.

Lemma:
, Let A a C*-algebra and $X \subseteq A$ a countable subset of A. The there exists a *(*separable) C*-subalgebra B of A with the properties that $X \subseteq B$ and for all elements $c, d \in \mathbb{K}(\ell_2) \otimes B$ holds:

$c \precsim d$ in $\mathbb{K}(\ell_2) \otimes B$, if and only if, $c \precsim d$ in $\mathbb{K}(\ell_2) \otimes A$.

This separable C*-subalgebra B of A is mostly much bigger then the C*-subalgebra generated by the separable X, but B is still a separable C*-algebra.

Moreover one can manage (by suitable enlarging of B by its construction) that B is nuclear if A is nuclear.

(But notice here that exactness is anyway a hereditary property! And remind that exactness is the same as *Subnuclearity*.) Give here citation!

This implies not only that elements of B are infinite (resp. purely infinite) in B, if and only if, this elements are infinite (resp. purely infinite) in A (which is usually a weaker property). This property of B implies also that each closed ideal J of B is the intersection $I \cap B = J$

Definition C.1.1. A (non-zero) C*-algebra A is called *purely infinite*, if and only if,

(i) The C*-algebra A has no characters, and
(ii) for every element $a, b \in A_+ \setminus 0$ with b in the closed ideal $I(a)$ of A, generated by a, there is a sequence of elements $c_1, c_2, \ldots \in A$ with $\lim_n c_n^* a c_n = b$. (In the sense $\lim_n \|b - c_n^* a c_n\| = 0$.)

Observations concerning Definition C.1.1:

Part (ii) of the Definition C.1.1 of purely infinite (p.i.) C*-algebras A (also called "properly infinite" C*-algebras ?) is equivalent to:
For each $d_1, d_2 \in A$, $a \in A_+$ and $\varepsilon > 0$ there exists $d_3 \in A$ with

$$\|d_1^* a d_1 + d_2^* a d_2 - d_3^* a d_3\| < \varepsilon\,.$$

This equivalence can be shown by complete induction, because each element $b \in I(a)_+$ for the closed ideal $I(a)$ of A generated by a can be approximated arbitrary near by elements of the kind

$$e_1^* a e_1 + e_2^* a e_2 + \cdots + e_n^* a e_n \in I(a)_+\,.$$

We discuss here the "permanence properties" of this definition of purely infinite (= properly infinite ?) C*-algebras:

Proposition C.1.2. *The class of purely infinite C*-algebras A as defined in Definition C.1.1 has following permanence properties:*

(i) *Quotients, ideals, and hereditary C*-subalgebras,*
(ii) *Inductive limits of purely infinite C*-algebras,*
(This is still not proven !!! But every positive element $a \in A_+$ of an inductive limit is the limit of a sequence of properly infinite positive elements. It seems that the limit of p.i. elements is p.i., which gives the proof.)
(iii) *Every separable subset X of A is contained in a separable purely infinite C*-subalgebra B of A. Moreover one can find such B with the property that each ideal J of B is the intersection $IcapB = J$ of a closed ideal of B.*
(Then X contained in B, that can be much bigger than $C^*(X) \subseteq A$.
Can we manage to find such a C-subalgebra B that is in addition relatively weakly injective in A? Unfortunately: It is Not known!?)*

(But what about extensions?)

Observations:

Question: Can we prove that (all sorts of) purely infinite C*-algebras have the Cuntz property:

For all $a \in A_+$ holds $a \oplus a \precsim a$.

This property should work also for inductive limits.

If this is shown than one must show that Cuntz-pi of A implies that A is a p.i. C*-algebra.

(But this depends from the definition of "p.i." etc.)

(It is then to show that hereditary C*-subalgebras of D of A have no characters. But that is equivalent to the property that $\rho(A) \cap \mathbb{K}(H) = \{0\}$ for each irreducible representation $\rho : A \mapsto \mathcal{L}(H)$ of A holds that $\rho(A) \cap \mathbb{K}(H) = 0$.)

Remark C.1.3. Notice that, moreover, one can manage to find separable C*-subalgebras A_τ of A in part (iv) of Proposition C.1.2 that are in addition relatively weakly injective in A.

This follows from the general observation that each separable C*-subalgebra B of A is contained in a separable C*-subalgebra D of A that is relatively weakly injective in A, i.e. there exist a completely positive contraction $E\colon A \to D^{**}$ with $E(d) = d$ for all $d \in D$.

Where is this (= the latter) shown ?

Proof. Ad(i): Suppose that A is a p.i. C*-algebra, and $J \subset A$ a (non-zero) closed ideal. Then J and A/J can not have no characters.

(See considerations below.)

The part (ii) of Definition C.1.1 is moreover shown for all hereditary C*-subalgebras.

Ad(ii): The case of inductive limits:

Since, by part (i), the properties of purely infinite C*-algebras pass to (non-zero) quotients we can suppose that A is the inductive limit of C*-subalgebras $A_\tau \subseteq A$ of A and the A_τ are purely infinite.

If A is a C*-algebra with a character ξ then every (e.g. separable) C*-subalgebra B of A is in the kernel of ξ or the restriction $\xi|B$ to B is a character. Thus, if A is an inductive limit of C*-algebras A_τ without characters, then A has no character.

Suppose that A is an *inductive limit* of an upward directed net of C*-subalgebras $A_\tau \subseteq A$ and each A_τ is purely infinite. Then for each $a, b \in A_+$ with the property that b is contained in the closed ideal J of A generated by a, and each chosen $\delta \in (0,1)$ (small enough) there exists $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in A$ with $\|b - \sum_k d_k^* a d_k\| < \delta$.

The problem is about the estimates ...

can be simplified if we use that we must show only that there exists for each $d_1, d_2 \in A$ and $\varepsilon > 0$ an element $d_3 \in A$ with the property that

$$\|d_3^* a d_3 - d_1^* a d_1 - d_2^* a d_2\| < \varepsilon\,.$$

Let $\gamma > 0$ (sufficiently small) and find $b, e_1, e_2, e_3 \in A_\tau$, with norms of $a - b$, $d_j - e_j$ $(j = 1, 2)$ with norms below γ, and

$$\|e_3^* b e_3 - e_1^* b e_1 - e_2^* b e_2\| < \gamma\,.$$

then ?????????

??? This gives ???:

For each $b \in A_+$ and each positive element c of the closed ideal J generated by b there exists a sequence of elements $x_n \in A$ such that $\|c - x_n^* b x_n\|$ converges to zero.

Have the same problem: norm- estimate $\|e_3^* b e_3 - e_3^* a e_3\|$ and a way to come from a to b ...

$d_j - e_j$ for $j = 1, 2$, $a - b$ have to be estimated:

Now we take $\gamma > 0$ small enough such that

$e - b$, $a - f$, $g_k - d_k$

$\|e - b\|$, $\|a - f\|$, $\|g_k - d_k\| < \gamma$ can be chosen arbitrary small?

Find $A_\tau \subseteq A$, such that it contains ????

Find $h \in A_\tau$ with $\|h^* f h - \sum_k g_k^* f g_k\|$ (arbitrary small !) say $< \mu$ for some before given $\mu > 0$.

The problem is: $\|h\|$ and $\|h^* f h - h^* a h\| <$???.

???? we can find $A_\tau \subseteq A$ and $e, f, ? \in A_\tau$????

Has to be studied more careful !!! It seems that the cut-down possibility $(a-\varepsilon)_+$ plays a role ...

(1) If $J \subseteq A$ is a nonzero closed ideal of A with $J \neq A$ then J and A/J have no characters:

Obviously all (closed) ideals J of A and quotients A/J have no characters if A has no characters, because each character ξ of A/J defines the (non-zero) character $a \mapsto \chi(a) := \xi(\pi_J(a))$ on A. But this can not exist by property (i) of A.

And if η is a character of J with kernel K. Then K is a closed ideal of A and $J/K \subseteq A/K$ is a closed ideal of A/K that is one-dimensional. Thus, $J/K = \mathbb{C} \cdot p$ for some projection $p \in A/K$. Since J/K is a closed ideal of A/K this implies that $x \mapsto xp \in \mathbb{C} \cdot p$ for $x \in A/K$ defines a character on A/K. Above we have seen that the quotients A/K can not have a character. Thus, J can not have a character.

So far we have seen that closed ideals and quotients of A can not have a character.

(The question about characters of hereditary C*-subalgebras of A will be considered further below in ????? ....)

Here is the general observation concerning this direction:

Lemma:
Let $D \subseteq A$ a hereditary C*-subalgebra of A (i.e. $D = DAD = \overline{DAD}$) and $I \subseteq D$ a closed ideal of D. Then the closed ideal $J \subseteq A$ of A generated by the hereditary C*-subalgebra I of A has the property that $J \cap D = I$.

Remark:
The same happens with von-Neumann algebras M: Let $P^*P = P \in M$ a projection and Q a projection in the centrum $c(PMP)$ of PMP. Then let R in the centre $c(M)$ of M the smallest projection with the property $RQ = Q$, which means equivalently that $RM = RMR$ is the weak closure (i.e. the $\sigma(M, M_*)$-closure) of the linear span of MQM.

Then R is the smallest projection in the centrum $c(M)$ of M with the property that $RQ = Q$, and it turns out that R, P and Q satisfy that $RP = Q$.

(3) Now let $D \subseteq A$ a (non-zero) hereditary C*-sub-algebra of A. Suppose that D has a character, say η. Let $I \subseteq D$ denote the kernel-ideal of η. Then the closed ideal J of A generated by I has the property that $J \cap D = I$ (It is an exercise Lemma! Cited above.) and that – therefore – $\mathbb{C} \cong D/I \subseteq A/J$ is a one-dimensional hereditary C*-subalgebra of A/J that is naturally isomorphic to $\mathbb{C}$

and is needed for what ???

Thus, A/F contains an isomorphic image of some algebra of compact compact operators ... on a Hilbert space (of dimension ≥ 1). But this was excluded by the requirements that no hereditary C*-sub-algebra of A has a character.

(0) The elements c_n in Part (ii) can be taken in $\overline{aAb}$ because the Part (ii) of this definition can be expressed equivalently:

If nonzero $b \in A_+$ is contained in the closed ideal of A generated by $a \in A_+$ then, for every $0 < \varepsilon < \|b\|$, there exists $\delta > 0$ and $d \in A$ with $d^*(a-\delta)_+d = (b-\varepsilon)_+$.

In particular, the Part(ii) passes automatically to all (non-zero) hereditary C*-subalgebras $D \subseteq A$ of A, i.e., if A is p.i., then D satisfies Part (ii) of Definition C.1.1.

Let $J \subseteq A$ a closed ideal and $x, y \in (A/J)_+$ such that y is in the closed ideal of $x \in (A/J)_+$, and $\varepsilon \in (0, \|y\|)$.

Then there exist elements $a \in A_+$ and $d_1, \ldots d_n \in A$ with $\pi_J(a) = x$ and $\|\pi_J(b) - y\| < \delta$ for $b := d_1^*ad_1 + \cdots + d_1^*ad_1$ and $\delta := \varepsilon/3$. By Part (ii) of Definition C.1.1 there exists $c \in A$ with $\|c * ac - b\| < \delta$. Thus, $\|\pi_J(c)^*x\pi_J(c) - y\| < \varepsilon$.

The Parts (i) is also satisfied for D: Suppose that D has a character ϕ, then there exists a contraction $e \in D_+$ with $\phi(e) = 1$. It extends to a pure state ρ on A. (Because in general states of D extend to states on A. Then the extreme points of this extensions turn out to be pure state on A.)

And then ???

$x, y \in (A/J)_+$ □

(XXXX????)

It is not clear if strong pure infiniteness and the pure infiniteness are really different !!!

Here is a definition that is equivalent to "strongly purely infinite" (s.p.i.):

If $S = [a_{j,k}] \in M_2(A)_+$ then there exists a sequence of diagonal matrices $T_n = s_n \oplus t_n \in M_2(A)$ such that $T_n^* S T_n$ converges to the diagonal matrix $a_{1,1} \oplus a_{2,2}$. (Gives $s_n^* a_{1,1} s_n \approx a_{1,1}$, $t_n^* a_{2,2} t_n \approx a_{2,2}$, $s_n^* a_{1,2} t_n \approx 0$, and $t_n^* a_{2,1} s_n \approx 0$.)

Gives with $a_{j,k} := a$ for all cases $j, k \in \{1, 2\}$ that a is properly infinite in A, because $s_n^* a s_n \approx a$, $t_n^* a t_n \approx a$ $s_n^* a t_n \approx 0$. It is equivalent $a \oplus a \precsim a$, by considering the case $a_{1,1} := a$, $a_{i,j} = 0$ for $(i, j) \neq 0$.

Need to show my and others notations and definitions!

Definition of: "approximately divisible" ?? where ??? Def. of: "???" etc.

`Some old citation changes:`

Lemma **??** (Old Ref. lem:2.pure.state.excision) was A.1.24 or: A.old.2.4 ??

Lemma **??** (Old Ref. lem:2.pure.state.excision) or: A.old.2.4 also in old: A.8.1 Only 1-cited in proof of Lemma 2.2.3.

Lemma A.6.1 old A.1.9.

Lemmas = Lemmata **??** – ??? now named ??? (Old Ref. lem:2.pure.state.excision)

New 2.1.22 = old.. lem:A.old.3.4c, also ??? old A.1.24 (=lem:A.old.2.4) - old A.1.28,

2. Proposed changes (1)

Change it now, because it can produce misunderstandings !

1.) General notations for matrices (to be decided and then changed where different notation was used):

`Adjust in all chapters the use of`

$$M_{m,n}(A) \cong A \otimes M_{m,n}(\mathbb{C}) \cong M_{m,n}(\mathbb{C}) \otimes A$$

`by this rules:`

The columns with n-entries should be in $(1, n)$ -matrices ... $\in M_{n,1}(A)$ ... not in $M_{1,n}(A)$!!!

Lin. Alg. book of Boseck S.76: Matrix of type (m, n) has m rows and n columns. Set of this matrices denoted by $\mathcal{A}_{m,n}$. $\mathcal{A}_{m,r} \cdot \mathcal{A}_{r,n} \subset \mathcal{A}_{m,n}$.

M. Koecher: $K^{(m,n)}$ set of matrices with m rows and n columns. $K^{(m,r)} \cdot K^{(r,n)} \subset K^{(m,n)}$.

H. Anton: $m \times n$-matrix has m rows and n columns. $m \times r$-matrix multiplied with $r \times n$-matrix is $m \times n$-matrix. Transposed matrix of A denoted by A^T.

H.-J. Kowalsky: (m,n) -matrix has m rows and n columns.

G. Strang: $m \times n$-matrix has m rows and n columns:

All have first rows: $a_{1,1}, a_{1,2}, a_{1,3}, \ldots$ next row: $a_{2,1}, a_{2,2}, a_{2,3}, \ldots$

TO DO LIST: from Nov. 2014!!! Not done July/Aug 2021!!!

Adjust in all chapters of the book the use of $\sim$ and $\approx$!!!
And the same with $[a]_\sim$, $\langle a\rangle_\sim$, $[a]_\approx$, $\langle a\rangle_\approx$.
Also adjust all the matrix calculations to the above
cited terminology. ??

Chp 1. Careful check of Introduction.

Ad Chp.2:

Ad 2.1. Shorten: bundle version of $\mathrm{C}(X, A)$ with simple A.

Ad 2.2. Other parts only as overview article.

Ad Chp. 3. Check what is really needed later.

It would be good if we could show that:

Question:

Let A denote a separable exact unital C^*-algebra and consider a unital nuclear c.p. contraction $V\colon A \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ with $V(a^*a) - V(a)^*V(a) \in \mathcal{O}_2 \otimes \mathbb{K}$.

Find a c.p. extension $W\colon A \otimes \mathcal{O}_2 \to \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})$ with $W(a \otimes 1) - V(a) \in \mathcal{O}_2 \otimes \mathbb{K}$ for $a \in A$ and $W(x^*x) - W(x)^*W(x) \in \mathcal{O}_2 \otimes \mathbb{K}$ for $x \in A \otimes \mathcal{O}_2$.

Perhaps, we can replace $\mathcal{O}_2$ by the infinite tensor product $\mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ and try to adjust V with help of suitable elements of $1 + \mathcal{O}_2 \otimes \mathbb{K}$ ... ???

In particular it should be interesting for **nuclear** A. (It is true if A is in the UCT-class. Perhaps here is a critical point if canceling the important UCT-assumption.)

($<--$ Why it is true in UCT-class ?)

For exact A the question is less interesting because we can replace A by $A \otimes \mathcal{O}_2$ at beginning of the embedding-proof for exact A in $\mathcal{O}_2$.

Since $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$, it is likely that $\mathcal{O}_2$ has the property that any separable C^*-subalgebra of $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ commutes with a unital copy of $\mathcal{O}_2$ in $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$.

($\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ means here the "stable" corona $\mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})/\mathcal{O}_2 \otimes \mathbb{K}$ of $\mathcal{O}_2$?)

It could replace homotopy invariance considerations, because otherwise one gets only unital embedding of $A \otimes \mathcal{O}_2$ that are "extremal" in $\mathcal{O}_2$ for nuclear A but are not extremal for A itself. (What means here "extremal"?)

It seems to be necessary to study following questions:

Question: If A is a separable unital nuclear C^*-algebra and $\varphi\colon A \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ a unital *-monomorphism where

$$\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2) := \mathrm{Q}(\mathcal{O}_2 \otimes \mathbb{K}) := \mathcal{M}(\mathcal{O}_2 \otimes \mathbb{K})/(\mathcal{O}_2 \otimes \mathbb{K})$$

It is likely that φ extends to $\psi\colon A \otimes \mathcal{O}_2 \to \mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$ with $\psi(a \otimes 1) = \varphi(a)$.

"Likely" only ???

Look what happens with the unital c.p. maps V from M_n to $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$. Try to find an "arbitrary" small perturbation W of V that commutes with a unital copy of $\mathcal{O}_2$ in $\mathrm{Q}^{\mathrm{s}}(\mathcal{O}_2)$.

Alternatively we could proof that:

Nuclear *-monomorphisms ϕ of exact unital separable A into s.p.i. "corona" spaces C extends to $A \otimes \mathcal{O}_\infty$, i.e., that $\phi(A)' \cap C$ has a properly infinite unit.

(Where this is proved? Could prove?)

(What about c.p. maps from M_n to C? By doubling the rows that define this maps ?)

Should be a consequence of a conclusion of a generalized Weyl–von-Neumann theorem that could prove that ϕ dominates $\phi \oplus \phi$. If C is s.p.i., e.g. in the case where C is the corona $C = \mathrm{Q}^{\mathrm{s}}(B)$ of a σ-unital strongly p.i. C^*-algebra B.

It gives that ϕ extends at least to $\psi\colon A \otimes \mathcal{O}_\infty \to C$ with $\psi(a \otimes 1) = \phi(a)$. Remind here the definition of the Cuntz algebra:

$$\mathcal{O}_\infty := C^*(s_1, s_2, \ldots ; s_j^* s_k = \delta_{jk} 1)\,.$$

Then $\psi(1 \otimes p)$ for $p = 1 - s_1 s_1^*$ is the unit of a copy of $\mathcal{O}_2$ in $p\mathcal{O}_\infty p$, and $q := \psi(1 \otimes p)$ is a full properly infinite element of C with $0 = [q] \in \mathrm{K}_0(C)$. (The embedding γ of $\mathcal{O}_2$ in $p\mathcal{O}_\infty p$ will be denoted by $\gamma\colon b \in \mathcal{O}_2 \mapsto \gamma(b) \in p\mathcal{O}_\infty p$ and satisfies $\gamma(1) = p$.)

If, in addition, the algebra C has a properly infinite unit 1 with $0 = [1] \in \mathrm{K}_0(C)$ (– as it is the case e.g. for all stable coronas $C := \mathrm{Q}^{\mathrm{s}}(B)$ or for quotients C of $\mathrm{Q}^{\mathrm{s}}(B)_\infty$ –) then there is an isometry $T \in C$ with $TT^* = q$. (In fact, the requirement that $\psi(1 \otimes p)$ is full and properly infinite implies that 1 is properly infinite in C.)

We can define a new unital $\psi_0\colon A \otimes \mathcal{O}_2 \to C$ by $\psi_0(a \otimes b) := T^*\psi(a \otimes \gamma(b))T$.

Then $a \mapsto \phi_0(a) := \psi_0(a \otimes 1)$ is a nuclear unital *-monomorphism that commutes with a copy of $\mathcal{O}_2$ unitally contained in $\phi_0(A)' \cap C$.

We can define ϕ back from this map by

$$\phi(a) = \psi(a\otimes 1) = \psi(1\otimes s_2)^*\psi(a\otimes p)\psi(1\otimes s_2) = (\psi(1\otimes s_2)^*T)\psi_0(a\otimes 1)(T^*\psi(1\otimes s_2))\,.$$

Notice that $V := T^*\psi(1 \otimes s_2)$ is an isometry with $1 - VV^*$ full and properly infinite in $\mathcal{O}_2$.

The remaining problem is the following:

Suppose that the "liftable" copy (in sense of given split extension) $\lambda\colon A \otimes \mathcal{O}_2 \to$???? ???

Now suppose that there exists a ("liftable") nuclear unital *-monomorphism $\lambda\colon A\otimes\mathcal{O}_2 \to C$ with the property that there exists sequences of isometries $t_1, t_2, \ldots$ and $r_1, r_2, \ldots$ such that, for all $a \in A$, $\lambda(a\otimes 1) = \lim_n t_n^* \phi_0(a) t_n$ and $\phi_0(a) = \lim_n r_n^* \lambda(a\otimes 1) r_n$.

Apply a "fitting together" procedure to get with properties of the "corona algebra" that there are isometries t and r with $r^*\lambda(a\otimes 1)r = \phi_0(a)$ and $s^*\phi_0(a)s = \lambda(a\otimes 1)$ for $a \in A$. Since both commute with (different) unital copies of $\mathcal{O}_2$ in its commutants it follows from Corollary 4.3.7 that ϕ_0 and $\lambda((\cdot)\otimes 1)$ are unitary equivalent in C. (If C is a stable corona, then this equivalence is given by a product of exponentials.)

In particular: check the material of concurring sections.

Use the beamer presentations (Muenster !!! details???), compare with its contents.

Add needed case:

$A \subseteq \mathcal{M}(D)$, $D \subseteq \mathcal{M}(B)$ non-degenerate (necessary ??? only for $\mathcal{M}(D) \subseteq \mathcal{M}(B)$, could be e.g. enough that DBD is a corner of B), D s.p.i.)

Then residually nuclear $V\colon A \to B$
– with respect to the action

$$J \in \mathcal{I}(B) \mapsto A \cap \mathcal{M}(D, D \cap \mathcal{M}(B, J))$$

–, is 1-step approximately inner in $\mathcal{M}(B)$.

(Or is section 2.2 ?? of Chp.???? 2? the right place for the proof of this?).

We could use the minimality of the m.o.c.c. of residually nuclear c.p. maps among the cones with same invariant ideals:

It is enough to consider separable A, because non-separable A is the inductive limit of a net of separable C^*-sub-algebras $A_\tau \subseteq A$ with the following additional properties:

For each countable subset $X \subseteq A$ there is an separable A_τ with $X \subseteq A_\tau$.

Each closed ideal of A_τ is the intersection of a closed ideal of A with A_τ, and A_τ is residually relatively weakly injective in A in the sense that $(A_\tau/(A_\tau \cap J)) \otimes^{\max} F$ is naturally a C^*-subalgebra of $(A/J) \otimes^{\max} F$ for each C^*-algebra F.

(It seems to turn out that it suffices to consider here only separable C^*-algebras F ?)

If A s.p.i. then A_τ can be chosen such that A_τ is in addition s.p.i.

It implies that the restriction $V|A_\tau$ of V to A_τ of each residually nuclear map $V\colon A \to B$ with respect to some l.s.c. action Ψ of $\operatorname{Prim} B$ on A is also residually nuclear with respect to the induced action of $\operatorname{Prim} B$ on A_τ, i.e., for $I := \Psi(J)$ and $I_\tau := I \cap A_\tau := \Psi_\tau(J)$ holds: $V(I_\tau) \subseteq J$ and $[V|A_\tau]\colon A_\tau/I_\tau \to B/J$ is nuclear, because $[V|A_\tau]$ is equivalent to the restriction of $[V]\colon A/I \to B/J$ to $\pi_I(A_\tau) \subseteq A/I$ by the isomorphism $[\pi_I]\colon A_\tau/I_\tau \to \pi_I(A_\tau)$.

If B_τ is a separable C^*-subalgebra of B such that $V(A_\tau) \subseteq B_\tau$, "symbolic" invariance $H_\Psi(A_\tau)B_\tau \otimes \mathbb{K} \subseteq B_\tau \otimes \mathbb{K}$, each closed ideal of B_τ is the intersection of a closed ideal of B with B_τ and B_τ is residually weakly injective in B, then $V|A_\tau\colon A_\tau \to B_\tau$ is still residually nuclear with respect to the "induced" action $\Phi\colon \mathcal{I}(B_\tau) \to \mathcal{I}(A)$ given by $\Phi(J) := \Psi(BJB) \cap A_\tau$.

Here we simplify notation:

BJB denotes the closed linear span of the set of products b_1cb_2 with $c \in J$ and $b_1, b_2 \in B$. Indeed, $J = B_\tau \cap BJB$,

$$V(\Phi(J)) \subseteq V(A_\tau) \cap BJB \subseteq B_\tau \cap BJB = J$$

and $[V]\colon A_\tau/\Phi(J) \to B_\tau/J$ is nuclear, because $V(A_\tau) \subseteq B_\tau$, $[V]\colon A_\tau/\Phi(J) = A_\tau/(A_\tau \cap BJB) \to B/BJB$ is nuclear, and there is a natural isomorphism from B_τ/J onto $\pi(B_\tau) \subseteq B/BJB$ which is relatively weakly injective in B/BJB by assumption.

Another question is to find suitable separable C^*-sub-algebras B_τ of B such that for a given action $H(A_\tau)B_\tau \subseteq B_\tau$ and, for $J \in \mathcal{I}(B)$,

$$(H|A_\tau)^{-1}(\mathcal{M}(B_\tau, J \cap B_\tau)) = A_\tau \cap H^{-1}(\mathcal{M}(B, J))\,.$$

More generally: Let $E \subseteq \mathcal{M}(B)$ and $F \subseteq B$ separable C^*-subalgebras, $\mathcal{S}$ a countable family of closed ideals of B.

Find a separable C^*-subalgebra $C \subseteq B$ that contains F and satisfies $EC \subseteq C$, each closed ideal I of C is the intersection $I = C \cap J$ of a closed ideal J of B with C, and $E \cap \mathcal{M}(B, J) = H^{-1}(\mathcal{M}(C, C \cap J))$ for all closed ideals J of B (for the *-homomorphism $H\colon E \to \mathcal{M}(C)$ with $H(e)c = ec$ for $e \in E$ and $c \in C$).

What about the desire that C is relatively injective in B? Or is approximately injective in B?

Can we find such C with the additional property that there exists a normal conditional expection from B^{**} (or from D^{**} with $D := \overline{CBC}$) onto C^{**}? Perhaps, this seems to be equivalent to the injectivity of the natural morphism

$$C \otimes F \mapsto B \otimes F$$

for all (separable) C^*-algebras F.

Reduce to separable case? Here necessary?

The idea would be to compare the composition $\mathcal{C}_{\mathrm{nuc}}$ of the l.s.c. action Ψ_1 of $\operatorname{Prim} B$ on D composed with $\mathcal{C}_{\mathrm{nuc}}$ of the l.s.c. action Ψ_2 of $\operatorname{Prim} D$ on A with the $\mathcal{C}_{\mathrm{nuc}}$ of the l.s.c. action of $\operatorname{Prim} B$ on A:

If the composition $\Psi_2 \circ \Psi_1$ defines the same action as Ψ, and if the m.o.c. cone $\mathcal{C}_{\mathrm{nuc}}(\Psi_2; A, D)$ and $\mathcal{C}_{\mathrm{nuc}}(\Psi_1; D, B)$ separate Ψ_2 respectively Ψ_1, then the minimality of $\mathcal{C}_{\rm nuc}(\Psi; A, B)$ among the m.o.c.c. with same induced action delivers that

$$\mathcal{C}_{\rm nuc}(\Psi; A, B) \subseteq \mathcal{C}_{\rm nuc}(\Psi_2; A, D) \circ \mathcal{C}_{\rm nuc}(\Psi_1; D, B)\,.$$

Since – at the present state of text – we do not know that (for *separable* A, B and D) every l.s.c. action Ψ between there ideal lattices can be separated by the corresponding m.o.c.c. of Ψ-residually nuclear maps, we get some more work:

We have to show that we can the question reduce to the cases of (other) separable A, B and D with the additional property that B and D contain *regular* abelian C^*-subalgebras C.

WHERE is "regular C^*-subalgebra" defined?

Perhaps by the property that each pure state can be extended to a pure state in a unique manner?

We fix a residually nuclear c.p. map $V\colon A \to B$ with respect to an action Ψ of $\mathrm{Prim}(B)$ that has the following property:

The – lower semi-continuous – action $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is given by $\Psi(J) := H^{-1}(\mathcal{M}(B, J))$ where $H\colon A \to \mathcal{M}(B)$ is a *-homomorphism with the property that H – in the topology of point-wise strict convergence – 1-step innerly approximately factorizes through $H_1\colon A \to \mathcal{M}(D)$ and $H_2\colon D \to \mathcal{M}(B)$ in a sense that the maps $a \mapsto b^*H(a)b$ can be approximated in point-norm topology of maps of the form $a \mapsto c^*H_2(d^*H_1(a)d)c$ for suitable $c \in B$ and $d \in D$ depending on $b \in B$. Moreover we require that D is strongly p.i.

Since it suffices to show that for given $a_1, \ldots, a_n \in A_+$, and $\varepsilon > 0$ there is $b \in B$ with $\|V(a_j) - b^*H(a_j)b\| < \varepsilon$ for $j = 1, \ldots, n$, it suffices to consider suitable separable C^*-sub-algebras A_τ, B_τ and D_τ of A, B and D, such that $a_1, \ldots, a_n \in A_\tau$ and that the A_τ, B_τ, D_τ same property the restrictions of H, H_1 and H_2 have the same

This can be done by replacing first A by its "good" separable C^*-algebras A_τ that contains $a_1, \ldots, a_n$ – as described above. Since $V(a_j)$ is contained in the closed ideal generated by $b^*H(a_j)b$ $(b \in B)$ and is – at the same time – also in the closed ideal generated by $c^*H_2(d^*H_1(a_j)d)c$ $(c \in B,\ d \in D)$ we find a separable C^*-subalgebra $A_\tau \ni a_1, \ldots a_n$ of A, a separable C^*-subalgebra B_τ of B and a separable strongly p.i. subalgebra D_τ of D such that $H(A_\tau)B_\tau \subseteq B_\tau$, $H_2(D_\tau)B_\tau \subseteq B_\tau$, $H_1(A_\tau)D_\tau \subseteq D_\tau$ and that $a \mapsto b^*H(a)b$ can be approximated by the restrictions to A_τ.

Then we replace D and B by suitable σ-unital hereditary C^*-subalgebras D_τ and B_τ such that $H_1(A_\tau)D_\tau = D_\tau$ and $H_2(D_\tau)B_\tau = B_\tau$.

This is possible, because ?????

The following covers only the case where the action of prime(B) on A is also monotone upper s.c.

Can't see any reason for this!

The good approach to separable D_τ and B_τ is the study of the Dini-function:

X Dini-space, Y locally q-compact sober T_0 space, e.g. $X := \mathrm{Prim}(A)$, $Y := \mathrm{prime}(B)$. $\Psi\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ l.s.c. and monotonous upper s.c. Then (equivalently) each Dini-function f on X defines a a Dini function on Y.

Since the set of Dini-functions on X (e.g. $X = \mathrm{Prim}(A)$) is uniformly separable, one has to find in a C^*-algebra corresponding to Y a sequence of elements that correspond to a dense sequence in the image of the Dini functions on X.

Then one enlarge it to a suitable B_τ with all the needed properties.

`Older approach:`

(1) Reduction to A separable, D and B σ-unital. Then all can be reduced to the case where D, B are also separable, by careful selection of the separable full C^*-sub-algebras of B and D that have the same "semi-metrics" $\rho(a,b;n) \in [0,\infty]$ than those coming from B respectively from D.

(Use here: If A separable, then $V(A) \subseteq B$ generates a separable C^*-subalgebra.)

(2) Then consider the maps that factorize over abelian C^*-subalgebras of $D_\omega \subseteq \mathcal{M}(B)_\omega \to \mathcal{M}(B_\omega)$.

(2') Or (perhaps better), consider equivariant c.p. maps $A \to D_\omega$ that factorize over abelian C^*-subalgebras of D_ω. Approximate them by 1-step inner c.p. maps.

(3) Then go from separable abelian C^*-subalgebras $C \subseteq D_\omega \to \mathcal{M}(B_\omega)$ equivariant to B_ω and approximate the residually nuclear c.p. maps by 1-step inner maps.

Use here the embedding $C \subseteq C \otimes \mathcal{O}_\infty \subseteq D_\omega$

(4) Show that the maps from (1)-(3) define the l.s.c. action of $\mathcal{I}(B_\omega) \to \mathcal{I}(A)$ of $\mathrm{prime}(B_\omega)$ on A.

(5) Show that the point norm closure of the set of maps $A \to B_\omega$ defined by (1)-(3) is an m.o.c. cone. (The convexity via Cuntz criterion has to be shown).

(6) All? ???

Passage to the non-degenerate case by compressions:

B and D first by the hereditary C^*-subalgebras $C := ADAB \cdot (B_\omega) \cdot BADA$ and $E := AD \cdot (ADA)_\omega \cdot DA$ (that is naturally contained in the multiplier algebra $\mathcal{M}(C)$ of C) of the ultra-powers of B and E, to obtain separable "regular" abelian C^*-subalgebras, that are suitable for the problem.

One of the points is to show that residual nuclear $V\colon A \to B$ is also residual nuclear as map from A into C – one with respect to the l.s.c. action Ψ_1 of $\mathcal{I}(B)$ on $\mathcal{I}(A)$, the other with respect to the l.s.c. action Ψ_2 of $\mathcal{I}(C)$ on $\mathcal{I}(A)$ –, which needs a check that $I := \Psi_2(J) = \Psi_1(J \cap B)$. Then $[V]_I\colon A/I \to B/(J \cap B) \subseteq C/J$ is again nuclear, and it follows that $V\colon A \to C$ is again residually nuclear.

Now fix a residually nuclear map from $V: A \to B$. We find separable C^*-algebras F of C and G of E such that $AG = G$, $GF = F$, G is strongly p.i., G and F contain regular Abelian C^*-subalgebras, each closed ideal of G is the intersection of a closed ideal of E with G, each closed ideal C

????????????????

Concerning work on Chp. 4. :

4a) Perhaps: Study of K_1-injectivity on extra place?

Give *short !!!* proof of Cuntz lemma on " full properly infinite projections " ?

4b) Clean presentation of the “basic proposition”? Now a Theorem ?

4c) Again case: $A \subseteq E$ (via H_0) J ideal of E, $V: A \to J$ 1-step inner (approximately ?).

Suppose $E \supset A$ separable, for all separable $X \subseteq E$ and $Y \subseteq J$, there exists a positive contraction $e \in E$ (?) with $ex = xe$ for all $x \in X$ and $ey = y = ye$ for all $y \in Y$.

How the Grothendieck group looks like?

When there is a “majorizing” element? At least of larger semigroup?

Consider an abelian semigroup S with following property:

There exists $x_1 \in S$ with:

For each $x \in S$ there exists $y \in S$ with $x + y = x_1$. Then there exists x_0 with $2x_1 + x_0 = x_1$. Thus $2x_0 + 2x_1 = x_0 + x_1$. Then $x + y + x_0 + x_1 = x_0 + 2x_1 = x_1$ shows that $S + x_1$ is a subgroup of the semigroup S, and is isomorphic to the Grothendieck group of S and its "zero" element is equal $x_0 + x_1$.

$(h_1: A \to J) \in S(A, E, H_0, J)$ (last good notation?) with property:

If $[h] \in S(A, H_0, E)$ with $h(A) \subseteq J$ we get/require that $[h] \in S(A, h_1, E)$.

Special case: $x = s^*t^*H_0ts$, $y =$???, $(1 - ss^*)t^*H_0(\cdot)t(1 - ss^*) \in S(A, H_0, E)$.

implies $h_1 \oplus h_1 \precsim h_1$?

Does it imply the existence and uniqueness of h_0 ??

Ad Chp. 5

Ad 5a) Partial results for $A \subseteq \mathcal{O}_2$ perhaps only to list?

Anyway now shifted to Appendix B ?

At least of consequences of the more general results (on WvN, Ext)?

5b) More systematic “stable” section. Select only the later needed, e.g. for asymptotic morphisms, Rørdam’s criteria ...

5c) $\mathrm{Ext}(\mathcal{C}\,;\ A, B)$ in case B s.p.i. “Kasparov for $\mathcal{C}$”.

5d) Homotopy invariance (even of $\mathrm{Ext}(A, \mathcal{O}_2) = \mathrm{Ext}_{nuc}(A, \mathcal{O}_2)$) follows first from Kasparov's homotopy invariance of KK -theory,

and from $\delta_2 \sim \mathrm{id}$ on $\mathcal{O}_2$:

Perhaps one shows (before) that the unital endomorphism δ_2 is *unitarily homotopic* to identity map of $\mathcal{O}_2$.

5e)

$\mathrm{Ext}(\mathcal{C}\,;\, A, B)$ in front of other Ext-notion?

5f) Characterization of "zero" element of $\mathrm{Ext}(\mathcal{C}\,;\, A, B)$:

B σ-unital and stable,

$H\colon\ A \otimes \mathcal{O}_2 \hookrightarrow \mathcal{M}(B)/B$,

A stable, A exact, inclusion H nuclear, in case of $\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(\Psi;\, A, B)$.

But H must be fitting with the related "action" Ψ.

It must be underlined that "stability" of the corresponding extension algebra is only granted if the map H dominates zero.

(This is always reached if one tensors – again ? – with $\mathbb{K}$.)

One problem is: When $\mathrm{Q}^s(B)$ is strongly p.i? Same with $\mathcal{M}(B)$?

Positive if $B \cong B \otimes \mathcal{O}_\infty$ ($\cong B \otimes \mathcal{D}_\infty$). Then moreover every separable C^*-subalgebra of B "commutes approximately" with a unital copy of $\mathcal{O}_\infty$ in B_∞.

This should carry over to $\mathrm{Q}^{\mathrm{s}}(B)$. (Still a conjecture!)

Equivariant H:

$\pi_B(\mathcal{M}(B, J)) \cap H(A \otimes 1) = H((A \cap \mathcal{M}(B, J)) \otimes 1)$ for all closed ideals J of B.

It should follow that $a \mapsto H(a \otimes 1)$ is unitary equivalent to $\pi_B \circ H_0\colon A \to \mathrm{Q}^s(B)$.

Important:

Is it better to start with the classification of $A \otimes \mathcal{O}_2 \to \mathrm{Q}^s(B)$ nuclear?

It is not so useful in case of A nuclear and $B = \mathbb{K} \otimes \mathcal{O}_2$, because it would not allow to show that the lift is the range of a c.p. projections that is an ?? what ?? in the set of unital positive maps.

5g) Is it possible to prove the triviality of
$\mathrm{Ext}(\mathcal{C};\, A \otimes \mathcal{D}_2, B)$ and $\mathrm{Ext}(\mathcal{C};\, A, B \otimes \mathcal{D}_2)$ directly? Not using KK ? (Here $\mathcal{D}_2 = \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$ and the point is that we do not borrow the homotopy - invariance from KK.)

Partly from suitable material from Chapter 3?

A general Theorem about unitary equivalence in $\mathrm{Q}(B)$ (for B stable and σ-unital) of morphisms $h_1, h_2\colon A \otimes \mathcal{O}_2 \hookrightarrow \mathrm{Q}(B)$ would be useful.

Minimally necessary: $h_1(a \otimes 1)$ and $h_2(a \otimes 1)$ generate the same closed ideal of $\mathrm{Q}(B)$ for each $a \in A_+$.

There is (by some Corollary in 4) a unitary in $\mathrm{Q}(B)$ that makes them unitary equivalent. The $\mathrm{Q}(B)$ is K_1-bijective.

In case $B = \mathcal{O}_2 \otimes \mathbb{K}$ we can use that $B \cong B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \cdots$. It gives that any separable C^*-subalgebra of $\mathrm{Q}(\mathcal{O}_2 \otimes \mathbb{K})$ commutes with a unital copy of $\mathcal{O}_2$. Since it commutes also with some countable approximations by c.p. maps of a C^*-morphism from a separable C^*-algebra A into $\mathrm{Q}(\mathcal{O}_2 \otimes \mathbb{K})$, and since $\mathrm{Q}(\mathcal{O}_2 \otimes \mathbb{K})$ has trivial K_*-groups

Similar problems appear in the study of Prim-space extensions. (By getting them manageable by tensor them $\otimes \mathcal{O}_2$.)

E.g. if $H_k(a) := h_k(a \otimes 1)$ have weakly nuclear c.p. lifts, A separable and exact,

$H_1(a)$ and $H_2(a)$ generate the same ideal of $\mathrm{Q}(B)$ for each $a \in A_+$.

If A is stable (respectively the corresponding general extension stable), then it should be possible to give the equivalence by a unitary in $\mathcal{U}_0(\mathrm{Q}(B))$.

The needed basic informations have to be developed (in parts) in Chapters 3 and 4.

Chp. 6:

Embedding Theorem needs the characterization of the "big zero-element" of $\mathrm{Ext}(\mathcal{C}\,;\ A, B)$.

A exact, stable and separable, B stable, σ-unital and strongly p.i.

$\mathcal{C} = \mathrm{CP}_{\mathrm{nuc}}(A,B) \cap \mathrm{CP}(\Psi; A, B)$ with $\Psi \colon \mathbb{O}(\mathrm{Prim}(B)) \to \mathbb{O}(\mathrm{Prim}(A))$ l.s.c. and *monotone* u.s.c. (that is weaker than u.s.c. ???)

Requires to use and study $\mathcal{C} \otimes \mathrm{id}_{\mathcal{O}_2}$ What is needed really and minimally?

Chp 7.

New chapter 7:

equal to old Chapter 10 (or 11?)

plus first parts of old Chapter 9.

More ????

3. List of Changes: labels and names

(Wanted: Ref. to Definition of "properly infinite element" $c \in A_+$?) for Reference 'sec2:basics.factorize.infinite' ??? Possible: 1

LaTeX Warning: Reference 'NOref:WvN.Aomega.implies.WvN.A.?' on page 48 undefined on input line 5174. (?? seems not to exist in book ??? chapter 12 ??)

LaTeX Warning: Reference 'thm:3.?black.out??PrfUses:WvN.implies.Spi' on page 48 undefined on input line 5187. [48]

eksec3..Part1.tex [236] Chapter 3

4. Ref's and Cite to be fixed — not up to date

5. Temporary references / compare with my way

What is the precise definition of "Cuntz semigroup"?

From `Black.Robert.Tik.Toms.Wint.2010` [**85**]:

sec:2.2. The Cuntz semigroup. Let A be a C*-algebra. Let us consider on $(A \otimes \mathbb{K})_+$ the relation $a \precsim b$ if $v_n b v_n^* \to a$ (i.e., the sequence $(v_n b v_n^*)_n$ converges to a for the considered $a, b \in (A \otimes \mathbb{K})_+$) for some sequence (v_n) in $A \otimes \mathbb{K}$. Let us write $a \sim b$ if $a \precsim b$ and $b \precsim a$. In this case we say that a is Cuntz equivalent to b. Let $\mathrm{Cu}(A)$ denote the set $(A \otimes \mathbb{K})_+ / \sim$ of Cuntz equivalence classes. We use $\langle a \rangle$ to denote the class of a in $\mathrm{Cu}(A)$. It is clear that

$$\langle a \rangle \leqslant \langle b \rangle \;\Leftrightarrow\; a \precsim b$$

defines an order on $\mathrm{Cu}(A)$. We also endow $\mathrm{Cu}(A)$ with an addition operation by setting

$$\langle a \rangle + \langle b \rangle := \langle a' + b' \rangle ,$$

where a' and b' are orthogonal and Cuntz equivalent to a and b respectively (the choice of a' and b' does not affect the Cuntz class of their sum). The semigroup $\mathrm{W}(A)$ is then the sub-semigroup of $\mathrm{Cu}(A)$ of Cuntz classes with a representative in $\bigcup_n M_n(A)_+$.

Alternatively, $\mathrm{Cu}(A)$ can be defined to consist of equivalence classes of countably generated Hilbert modules over A. The equivalence relation boils down to isomorphism in the case that A has stable rank one, but is rather more complicated in general. We do not require the precise definition of this relation in the sequel, and so omit it; the interested reader may consult [5] or [1] for details.

[1] P. Ara, F. Perera, and A. S. Toms. *K-theory for operator algebras. Classification of C*-algebras*, Commun. Contemp. Math. To appear.(2008?)

[5] Coward, K.T., Elliott, G.A., Ivanescu, C., [2008], *The Cuntz semigroup as an invariant for C*-algebras*, J. Reine Angew. Math. **623**, 161–193.

" We note, however, that the identification of these two approaches to $\mathrm{Cu}(A)$ is achieved by associating the element $\langle a \rangle$ to the class of the Hilbert module $a\ell_2(A)$. If X is a countably generated Hilbert module over A, then we use $[X]$ to denote its Cuntz equivalence class; with this notation the sub-semigroup $\mathrm{W}(A)$ is identified with those classes $[X]$ for which X is finitely generated. "

(E.K.: I can not see how this fits bijectively together with the $\precsim$ classes in $A \otimes \mathbb{K}$...)

2.4. Functionals and Cu.

Let S be a semigroup in the category Cu. A functional on S is a map $\lambda\colon S \to [0,\infty]$ that is additive, order preserving, preserves suprema of increasing sequences and satisfies $\lambda(0) = 0$. We use $F(S)$ to denote the functionals on S. We will make use of a lemma, established in [**268**].

3. THE RADIUS OF COMPARISON

3.1.Original definition.

The *radius of comparison* was originally introduced in [18 ??] as an invariant for unital C*-algebras. (18 is cited in [**268**])

Let A be a unital C*-algebra, and let $\mathrm{QT}_2^1(A)$ denote the set of normalized 2-quasi-traces on A. The radius of comparison of A, denoted by $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ with the property that $a, b \in \sqcup_{n=1}^{\infty} M_n(A)$ (or noted $a, b \in \bigsqcup_{n=1}^{\infty} M_n(A)$) satisfy $a \precsim b$ whenever

$$d_\tau(\langle a\rangle) + r \,<\, d_\tau(\langle b\rangle), \quad \tau \in \mathrm{QT}_2^1(A)\,. \tag{5.1}$$

By the results of Subsection 2.4 (??) (of which paper/book ??), this is equivalent to the demand that $x, y \in \mathrm{Cu}(A)$ satisfy $x \leqslant y$ whenever

$$\lambda(x) + r < \lambda(y)\,,$$

for all $\lambda \in F(\mathrm{Cu}(A))$ which are normalized in the sense that $\lambda(\langle 1_A\rangle) = 1$. The motivation for this definition comes from the stability properties of topological vector bundles.

4.4.?? When is $\mathrm{W}(A)$ hereditary?

Recall that $W(A)$ is the sub-semigroup of $\mathrm{Cu}(A)$ of elements $\langle a\rangle$ with $a \in M_n(A)_+$ for some n. In fact, $\mathrm{W}(A)$ is the original definition of the Cuntz semigroup. Here we consider the question of when this sub-semigroup is hereditary, i.e., has the property that if $x \leqslant y$ in $\mathrm{Cu}(A)$ and $y \in \mathrm{W}(A)$, then $x \in \mathrm{W}(A)$.

We prove that finite "radius of comparison" suffices. This result was previously unknown, even in the case of "strict comparison".

Theorem 4.4.1.???

Let A be a C*-algebra for which the projections in every quotient of $A \otimes \mathbb{K}$ are finite. Let $a \in A_+$ be strictly positive and suppose that $r_{A,a} < k \in N$. If $\langle b\rangle \in \mathrm{Cu}(A)$ is such that

$$\lambda(\langle b\rangle) \,\leqslant\, n\lambda(\langle a\rangle) \;\text{ for all }\; \lambda \in F(\mathrm{Cu}(A))\,,$$

for some $n \in \mathbb{N}$, then b is Murray–von-Neumann equivalent to an element of $M_{2(n+k)}(A)_+$.

Lemma 4.4.2.???

If b_1 and b_2 are Murray–von-Neumann equivalent to elements in $M_n(A)$ then $b_1 + b_2$ is Murray–von-Neumann equivalent to an element in $M_{2n}(A)$.

(Case $n = 1$ checked ???)

Question 4.4.3.???

Is there a C*-algebra for which $\mathrm{W}(A)$ is not a hereditary subset of $\mathrm{Cu}(A)$?

6. Collection of Definitions of pure infinite C*-algebras

In the separate paper is a shorter and more transparent (??) calculation of the inequalities below?

`Beginning here: Collections of Defs for p.i.`

This is NOW (2022 !!!) only for my private use!!! (improving of estimates started July 2021)

It could be useful for the study of extensions of p.i. algebras ???

The below considered estimates are used to prove and improve the following estimate:

Let $a, b \in A_+$ positive contractions, then, using polar decomposition of $a-b$ in positive and negative parts $(a-b)_+$ and $(a-b)_- = (b-a)_+$ of

$$a - b = (a-b)_+ - (a-b)_- \,.$$

It gives that $a - (a-b)_+ = b - (b-a)_+$, by using that $(a-b)_- = (b-a)_+$. This symmetry allows to estimates the norms of $a - (a-b)_+$ and $b - (b-a)_+$ by using the norms of $(a+b) - |a-b|$ and $b - (a-b)_- = b - (b-a)_+$. Since $|a-b| = (a-b)_+ + (a-b)_-$ it follows that

$$a + b - ((a-b)_+ + (a-b)_-) = (a - (a-b)_+) + (b - (a-b)_-)$$

Since the two terms at the right side are equal and

$$|a-b| = \sqrt{(a-b)^2} = (a-b)_+ + (a-b)_- \,,$$

we get here immediately that

$$2(a - (a-b)_+) = (a+b) - |a-b| = 2(b - (b-a)_+)\,.$$

If we use that $|a-b| = \sqrt{(a-b)^2}$ and $a+b$ are (positive) roots of the positive operators $X := (a-b)^2$ and $Y := (a+b)^2$. For all positive operators $X, Y \in A_+$ holds

$$2\|a - (a-b)_+\| = \|Y^{1/2} - X^{1/2}\| \leq (\|Y - X\|)^{1/2}\,.$$

We get in our special case that $Y - X = 2(ab + ba)$,

$$\|Y^{1/2} - X^{1/2}\| \leq \|Y - X\|^{1/2} \leq 2\|ab\|^{1/2},$$

and finally that

$$\|b - (b-a)_+\| = \|a - (a-b)_+\| = (1/2)\|(a+b) - |a-b|\| \leq \|ab\|^{1/2}$$

PROOF of $\|Y^{1/2} - X^{1/2}\| \leq \|Y - X\|^{1/2}$? (Refer to Pedersen book, concerning citation on operator monotone functions. But there exits other papers on this – cite them also!).

$Y \leq X + \gamma \cdot 1$ with $\gamma := \|Y - X\|$ implies

$$Y^{1/2} \leq (X + \gamma)^{1/2} \leq X^{1/2} + \gamma^{1/2}$$

The estimate $(\beta + \gamma)^{1/2} \leq \beta^{1/2} + \gamma^{1/2}$ for $\beta, \gamma \in (0, \infty)$ is easy to see by taking on both sides quadrants. Thus $\|Y^{1/2} - X^{1/2}\| \leq \|Y - X\|^{1/2}$, and in our special case where X and $\gamma \cdot 1$ we get ??????)

In this special case we get $Y - X = 2(ab + ba)$, and $\|Y - X\| \leq 4\|ab\|$, hence

$$\|(a - (a - b)_+)\| = \|(b - (b - a)_+)\| \leq \|ab\|^{1/2} .$$

See below, but with other notations, because of desired generality.

We use here the "operator monotony": It is e.g. given in the book *C*-algebras and their automorphism groups* of G.K. Pedersen:

Prop. (1.3.8): Let $X, Y \in \mathcal{L}(H)$ positive operators. If $0 < \|Y - X\| \leq 1$, then

The operator function $t \mapsto t^{\beta}$ is operator monotone on $\mathbb{R}_+$ for $\beta \in (0, 1)$.

We use this in case $\beta = 1/2$ for the proof of following Result:

Lemma: For all positive operators $X, Y \in \mathcal{L}(H)_+$ holds the inequality of norms:

$$\|Y^{1/2} - X^{1/2}\| \;\leq\; \|Y - X\|^{1/2} .$$

And, therefore,

$$\|Y - X\| \;\leq\; \|Y^2 - X^2\|^{1/2}$$

Proof: Notice that this follows by the symmetries in X and Y, – as e.g. $\|Y - X\| = \|X - Y\|$ that we consider here as the $\|Y - X\| \cdot ??? \cdot \mathrm{id}_H$ –, from the general observation that the operator monotony (which says that $0 \leq a \leq b$ implies $0 \leq a^{1/2} \leq b^{1/2}$) implies that

$$Y^{1/2} \leq X^{1/2} + \|Y - X\|^{1/2} ,$$

because $Y \leq (X + \|Y - X\| \cdot \mathrm{id}_H)$ and, always, $(X + \|Y - X\|)^{1/2} \leq X^{1/2} + \|Y - X\|^{1/2}$, because $C * (X, \|Y - X\| \cdot \mathrm{id}_H)$ is commutative.

By symmetry between X and Y it follows that

$$X^{1/2} \leq Y^{1/2} + \|Y - X\|^{1/2}$$

using $\|Y - X\| = \|X - Y\|$. Thus,

$$\|X^{1/2} - Y^{1/2}\| \leq \|Y - X\|^{1/2} .$$

So, it is enough to apply the operator monotony with $\beta = 1/2$ to the obvious inequality $Y \leq X + \|Y - X\|$. We get that $Y^{1/2} \leq (X + \|Y - X\|)^{1/2}$ by operator monotony. And,- similarly -, $X^{1/2} \leq (Y + \|Y - X\|)^{1/2}$.

The operator X and the scalar $\|Y - X\|$ (– here considered as the operator $\|Y - X\| \cdot \mathrm{id}_H \in \mathcal{L}(H)_+$ –) are in a commutative C*-subalgebra of $\mathcal{L}(H)$. There, in the commutative C*-algebra $C^*(X, \|Y - X\| \cdot \mathrm{id}_H) \subseteq \mathcal{L}(H)$, all monotone increasing continuous functions apply to the positive elements in commutative C*-algebras as order-monotone function ... like: $0 \leq Z^* = Z \mapsto Z := X + \|Y - X\|$. Thus, $(X + \|Y - X\|)^{1/2} \leq X^{1/2} + \|Y - X\|^{1/2}$,

By taking squares one can see that this is in the commutative C*-subalgebra $C^*(X, \|Y - X\| \cdot 1)$ equivalent to the obvious inequality

$$X + \|Y - X\| \leq X + \|Y - X\| + 2(\|Y - X\| \cdot X)^{1/2}.$$

?????????

This works because for all nonnegative numbers(!) u and v holds that

$$(u + v)^{1/2} \leq u^{1/2} + v^{1/2},$$

because we have then that $u + v \leq u + v + 2(uv)^{1/2}$.

This carries over to all positive and elements $u, v \in A_+$ ... ???

Here is something missing:

It follows for positive contractions A, B that

$$\|(A + B) - \sqrt{(A - B)^2}\| \leq 2\|AB\|^{1/2}$$

for positive contractions A, B. ??

(Above it is better explained!)

Want to know:

$$\|A - (A - B)_+\| \leq 2\|AB\|^{1/2}$$

It comes from $C := A + B$ and $D := |A - B|$ and

$$C^2 - D^2 = (A^2 + AB + BA + B^2) - (A^2 - AB - BA + B^2) = 2(AB + BA)\,.$$

Thus, $\|C^2 - D^2\| \leq 4\|AB\|$ and

$$\|C - D\| \leq (\|C^2 - D^2\|)^{1/2} \leq 2\|AB\|^{1/2}\,.$$

Above calculations have the following desired application:

Let $0 \leq A, B \leq 1$ in a C*-algebra then

$$2(A - (A - B)_+) = (A + B) - \sqrt{(A - B)^2} = (A + B) - |A - B|$$

It follows that

$$\|A + B - |A - B|\| \leq 2\|AB\|^{1/2}$$

and $\|A - (A - B)_+\| \leq \|AB\|^{1/2}$.

Notice here also that: $AB = 0$ if and only if $A - (A - B)_+ = 0$.

The above shown later used application is:

$$\|B - (A - B)_-\| = \|A - (A - B)_+\| \leq \|AB\|^{1/2}\,.$$

!!! Other topic now below !!!:

7. On passage to suitable separable (!) C*-subalgebras

Let $X \subseteq A$ a countable (or norm-separable) subset of a C*-algebra A, then there exists a separable C*-subalgebra B of A that contains X, and such that $X \subseteq B \subseteq A$ have following properties:

1.) $b_1 \precsim_{M_n(B)} b_2$, if and only if, $b_1 \precsim_{M_n(A)} b_2$ for all $b_1, b_2 \in M_n(B)$ and $n \in \mathbb{N}$.

2.) $B \subseteq A$ is exact, if A is exact.

((2.) Means "Exactness is hereditary". Where is it proved ? Give reference and/or outline of proof.)

Let (more general) B a C*-subalgebra of an "exact" C*-algebra A. $A \otimes^{min} C$ is defined by the minimal C*-norm on the algebraic tensor product $A \otimes C$.

The algebras $B \otimes^{min} J --> B \otimes^{min} C --> B \otimes^{min} C/J$ are natural subalgebras of the algebras

$$A \otimes^{min} J --> A \otimes^{min} C --> A \otimes^{min} C/J$$

Notice here that $A \otimes^{min} C --> A \otimes^{min} C/J$ and $B \otimes^{min} C --> B \otimes^{min} C/J$ are always surjective, and are well-defined from the algebraic tensor products $A \otimes C --> A \otimes C/J$ by completion with the minimal C*-norms.

We see elow that $B \otimes^{min} J$ is the kernel of $B \otimes^{min} C --> B \otimes^{min} C/J$ if $A \otimes^{min} J$ is the kernel of $A \otimes^{min} C --> A \otimes^{min} C/J$.

If the kernel of $B \otimes^{min} C --> B \otimes^{min} C/J$ would be different from $B \otimes^{min} J$ then it must be bigger than $B \otimes^{min} J$. But if A is exact, then this kernel is always contained in $(A \otimes^{min} J) \cap (B \otimes^{min} C)$.

If we take approximate units $e_\sigma \in B_+$ and $f_\tau \in J_+$ of B and of J, then

$$Y_{\sigma,\tau} := (e_\sigma \otimes f_\tau) X (e_\sigma \otimes f_\tau)$$

converges to X for each element of $X \in (A \otimes^{min} J) \cap (B \otimes^{min} C)$. But, each $Y_{\sigma,\tau}$ is contained in $B \otimes^{min} J$. Thus,

$$(A \otimes^{min} J) \cap (B \otimes^{min} C) = B \otimes^{min} J$$

and it shows that $B \subseteq A$ is exact, if A is exact.

Notice here that there is a natural isomorphism $A \otimes^{min} C/J \cong (A \otimes^{min} C)/(A \otimes^{min} J)$ for exact C*-algebras A.

The question is equivalent to:

Is the kernel $(A \otimes^{min} J) \cap B \otimes^{min} C \subset A \otimes^{min} C$ of $B \otimes^{min} C --> (A \otimes^{min} C)/(A \otimes^{min} J)$ bigger than $B \otimes^{min} J$?

In general this kernels are contained in

$$F(B, C; J) := \{x \in B \otimes^{min} C; \varphi \otimes \mathrm{id}(x) \in J\}?$$

And we can ask the stronger question: Is $F(B, C; J) = (A \otimes^{min} J) \cap B \otimes^{min} C$ under which circumstances?

If one can it reduce (!) to the separable case for A the one can use that $B \subseteq A \subseteq \mathcal{O}_2$ proves exactness of B.

This would then require to show that exactness is preserved under inductive limits ... in addition.

(give Ref's !!!)

3.) If A is nuclear, then B can be chosen nuclear, by an iteration argument for B becoming a suitable inductive limit $X \subseteq B_1 \subseteq B_2 \subseteq \cdots$ and let B the closure of $\bigcup_n B_n$.

(One can moreover manage that B is approximately injective, weakly injective etc. if A has such properties, also (??) slice-map properties (??) could be earned from A ...??? Check this carefully !!!). (Below is an other reduction attempt ...)

Now let A be a *separable* C*-algebra and let J a closed ideal of A, and $a_0 \in J$ a strictly positive contraction for J, i.e. with

$$J = \overline{A a_0 A} = \overline{a_0 A a_0} = \overline{a_0 J a_0}\,.$$

Assume (from now on) that a_0 is properly infinite, i.e., $a_0 \oplus a_0 \precsim_J a_0$ in J (and then also $a_0 \oplus a_0 \precsim_A a_0$ and vice versa). Notice that $a_0 \oplus a_0 \precsim_B a_0$ also for every C*-subalgebra B of A with $J \subseteq B$.

Suppose that A/J contains an infinite element $x \in (A/J)_+$, i.e., there exists nonzero $y \in (A/J)_+$ with $x \oplus y \precsim_{A/J} x$. Then the closed ideal I of A/J generated by all the elements $y \in (A/J)_+$ with $x \oplus y \precsim x$ contains (by separability of A/J) a strictly positive contraction $z \in I_+$. (And z is non-zero because because I_+ is non-zero.)

This implies that $z(A/J)z$ and $(A/J)z(A/J)$ are both dense in I.

It turns out that $x \oplus z \precsim_{A/J} x$ and – since I is an ideal where z is a strictly positive contraction – that also $z \oplus z \precsim_{A/J} z$, i.e., that z is properly infinite in A/J.

We can lift z to a positive contraction $a_1 \in A_+$ with $\pi_J(a_1) = z$.

Then can consider the positive element $e := a_0 + (1 - a_0)^{1/2} a_1 (1 - a_0)^{1/2}$ with $\pi_J(e) = a_1$ and $e \geq a_0$.

Thus, $J \subseteq \overline{eAe}$ and $\pi_J(\overline{eAe}) = I$.

Is e infinite? Even this is not clear!

Is $a_0 \oplus e \precsim e$ and is $a_1 \oplus e \precsim e$?

Can we also take $a_0 + a_1^{1/2}(1 - a_0)a_1^{1/2}$???

Here some minimal conditions that let hope that A is p.i. (=: all nonzero elements are "infinite"):

(0) This part (0) is true for all C*-algebras:

The J. Cuntz relation $a \precsim_A b$ is compatible with with the corresponding relation in hereditary C*-subalgebras:

If $D \subseteq A$ is a hereditary C*-subalgebra of A and $a, b \in D$ satisfy $a \precsim_A b$ in A, then also $a \precsim_D b$ in D.

If $c_n b d_n$ converges to a in A where $c_n, d_n \in A$, then one finds $e_n c_n f_n, g_n d_n h_n \in \overline{rAr} \subseteq D$, for $r := a^*a + aa^* + b^*b + bb^*$, such that $(e_n c_n f_n) b (g_n d_n h_n)$ converges to $a \in \overline{rAr}$.

It is possible to find for every separable subspace $X \subseteq A$ (or every countable subset $Y \subseteq A$ of A) a separable C*-subalgebra $C \subseteq A$ that contains X (respectively contains Y) such that for $a, b \in \mathbb{K} \otimes C$ holds $a \precsim b$ in $\mathbb{K} \otimes C$, if and only if, $a \precsim b$ in $\mathbb{K} \otimes C$. Remind here $a \precsim_D b$ for positive elements $a, b \in D$ in a C*-algebra D means that there exists a sequence (d_n) in D with $a = \lim_{1 \to \infty} \precsim d_n^* b d_n$.

(In addition one can C select as a nuclear C*-subalgebra of A if A is nuclear, this is automatic for exactness, but it is an *open question* if $C \supseteq X$ can be always chosen that there exists in addition a conditional expectation from A^{**} onto C^{**}.)

(0) Original definition of pure infiniteness of a C*-algebra A is that $a \oplus a \precsim_A a$ for all $a \in A_+$. (It says that there exists sequences (c_n) and (d_n) in A such that $c_n^* a c_n$ and $d_n^* a d_n$ converge both to a, but $c_n^* a d_n$ converges to 0.)

(1) Minimal requirements (!) to the idea of "p.i." C*-algebras are:

(1a) For every hereditary C*-subalgebra D of A and every pure state ρ on D there exists elements $d, e \in D_+$ with $\rho(e) > 0$ and $e \oplus d \precsim d$.

(Then A contains a closed ideal J with $e \in J$, and $f \oplus d \precsim d$ for all $f \in J$.)

Notice here that the question if every non-zero positive contraction $b \in A_+$ is properly infinite (p.i.) is equivalent to the following question:

If $D := \overline{bAb}$ is the hereditary C*-subalgebra of A that contains b as strictly positive element of D, then for each element c of the closed ideal J of A generated by D there exists sequences (e_n) and (f_n) in A with $c = \lim e_n b f_n$.

Here one can pass to suitable separable C*-subalgebras $B \subseteq A$ of A, or of D, with the property that for $b, c \in B$ holds $b \precsim_B c$, if and only if, $b \precsim_A c$. This implies also that each (closed) ideal of B is the intersection of B with a closed ideal of A.

The same can be done with $A \otimes \mathbb{K}(\ell_2)$ and its C*-subalgebra $B \otimes \mathbb{K}(\ell_2)$.

To exclude also the cases $\mathbb{C} \cong D$ or $\mathbb{K} \cong D$ for hereditary C*-subalgebras of A/J one has to suppose that ??????

But we must to say much more! For example:
If E is a hereditary C*-subalgebra ... ????

Then B satisfies again (1a) and the question if every positive contraction of b of B ?????

(1c) Possibly (formally) "stronger" as (1a) is the following:

For every closed ideal $J \neq A$ (that can be $J = \{0\}$) and (non-zero) hereditary C*-subalgebra D of A/J, the algebra D contains an "infinite" element $0 \neq d \in D_+$ (i.e. that d is non-zero).

It is defined by the property that there exists non-zero $e \in D_+$ with $e \oplus d \precsim_D d$, i.e. it means that there exists a sequences of elements $f_n, g_n \in D$ with $\lim_n f_n * df_n = d$, $\lim_n g_n * df_n = 0$ and $\lim_n g_n * dg_n = e$.

??? Thus, ???? $e \oplus$??? is in the ideal of A/J one-step map by d ?????

Since ????

(Or is here also $e \in A/J$ allowed?

Yes is allowed, because the elements $e \in A/J$ with $e \oplus d \precsim_{A/J} d$ build a closed ideal K of A/J. The intersection $K \cap D$ of K with D is identical with .... ?????)

In particular, then each (!) nonzero hereditary C*-sub-algebra D of A has no character.

Equivalently expressed: A has no irreducible representation that contains in its image non-zero compact operators on a Hilbert space. Or: A does not contain two ideals $I \subseteq J \subseteq A$ such that J/I is isomorphic to the algebra of compact operators $\mathcal{K}(\mathbb{H})$ on an infinite dimensional or (non-zero) finite dimensional Hilbert space $\mathbb{H}$.

We call $a \in A_+$ *infinite* if there exists some non-zero element $b \in A_+$ such that $a \oplus b \precsim a$. Here we use the Cuntz-majorization $x \precsim y$ in $A \otimes \mathbb{K}$, and $a \oplus b$ means the the 2×2 matrix in $M_2(A)$ with diagonal entries a and b.

Let the following be a "suitable"(?) definition of a "locally purely infinite" C*-algebra A:

For every closed ideal J of A and (nonzero) hereditary C*-sub-algebra D of A/J there exists an infinite element in D (which is then infinite A/J), but for hereditary C*-sub-algebras D of A/J this is the same as being infinite in D).

Questions:

(Q1) Is this class of C*-algebras invariant under passage to hereditary sub-algebras and quotients?

(Q2) Is every separable subset X of A contained in a (suitable) separable C*-sub-algebra B of A that is again a "locally purely infinite C*-algebra"?

The reduction to the separable case is very important. One should take those separable C*-sub-algebras $B \subseteq A$ with the property that elements in $B \otimes \mathbb{K}$ have the same $\precsim --$ relations in $B \otimes \mathbb{K}$ as in $A \otimes \mathbb{K}$.

But the important point is: If $b \in B$ is infinite in A then b should be infinite in B. ... And this for all hereditary sub-algebras ...

It seems "easy" to show that inductive limits of locally purely infinite C*-algebras are again locally purely infinite.

The point is: How looks like a hereditary subalgebra of a quotient of this inductive limit?

Is it itself an inductive limit? (Of useful things).

On reduction to the separable case:

Let $X \subseteq A$ a separable subset of A.

Then there exists a separable C*-sub-algebra B of A that contains X and has the property that for all $a, b \in B$ holds: $a \precsim_B b$, if and only if, $a \precsim_A b$. It implies also that each closed ideal of B is is an intersection of a closed ideal of A with B.

(But in general B does not separate the ideals of A.)

8. MORE on infinite elements

Let A a C*-algebra ...

Let A a separable C*-algebra and D a hereditary C*-sub-algebra. Let $h \in D$, denote by $I(h)$ the closed ideal of elements $a \in A$ with $a \oplus h \precsim_A h$, then the intersection $I(h) \cap D$ has the property that $b \in (I(h) \cap D)$, if and only if, $b \oplus h \precsim_D h$.

(Check this again! Seems to work.)

Moreover $I(h)$ is generated by $I(h) \cap D$.

If $I(h)$ or $I(h) \cap D$ contain a strictly positive contraction $c \in A_+$ (i.e. cAc dense in $I(h)$, or cAc is dense in $I(h) \cap D$) then c is properly infinite and $I(h) = I(c)$ respectively $I(h) \cap D = I(c) \cap D$.

Such element $c \in A_+$ exists if A or D is separable.

It should be equivalent to require that for each ????

A question of Bruce Blackadar:

Is it possibly true that the classifiable simple C*-algebras are precisely the (simple) inductive limits of sequences of *semiprojective* (separable) nuclear C*-algebras?

E.K.: What is the Definition of *semiprojective* C*-algebras?

Kind of approximate lifting property? ????

9. On ideals generated by n-homogenous elements

Let A a C*-algebra and H_n the (smallest) hereditary C*-subalgebra of A that is generated (or only "defined") by the n-homogenous elements in A_+.

Definition: An element $a \in A_+$ is n-homogenous, if and only if, there exists a C*-algebra morphism

$$\varphi \colon C_0((0,1], M_n) = C_0((0,1]) \otimes M_n \mapsto A$$

with the property that $a = \gamma \cdot \varphi(f_0 \otimes 1_n)$ for some $\gamma \in (0, \infty)$ and $f_0(t) := t$, for $t \in (0,1]$.

H_n is the smallest hereditary C*-subalgebra of A that is generated by the convex combinations of the n-homogenous elements in A_+. Since the set of n-homogenous is invariant under automorphisms of A, – especially by inner automophisms –, it follows that H_n is a closed ideal of A.

This ideal H_n has the property that A/H_n has only irreducible representations on Hilbert spaces of dimensions $\leq (n-1)$.

(as ideal and not necessarily fixing ???)

GO ON NOW !!!!!!

10. On order in A_+ for C*-algebras A:

It is very likely that the result in the following proposition is well-known, but we could not find a reference for it in text-books.

Proposition:
If two elements $a, b \in A_+$ satisfy $a \leq b$, then for each $\gamma > 1$ there exists an operator $d \in C*(a,b) \subseteq A$ with $d^*bd = a^\gamma$.

What happens ????

What we really later need is that:

If two elements $a, b \in A_+$ satisfy $a \leq b$ then there exists for each $\varepsilon \in (0, \|a\|)$ an element $dinA$ with the property $d^*bd = (a-\varepsilon)_+$.

Similar problem: If $\|a-b\| < \gamma < \min(\|a\|, \|b\|)$ Does there exists $d \in A$ with $(a-\gamma)_+ = d^*bd$?

Proof: ...

It suffices to consider the case $a \neq 0$ (by rescaling later a, b and d with non-negative real numbers). Moreover, it suffices to consider only the case where $A := C*(a,b)$.

We consider the multiplier algebra $\mathcal{M}(A)$ and write γ (and δ) in place of $\gamma \cdot 1_{\mathcal{M}(A)}$ (and in place of $\delta \cdot 1_{\mathcal{M}(A)}$), and consider mainly $\delta = 2^{-n}$ for $n := 1, 2, \ldots$.

Let $e_n := (b+2^{-n})^{-1/2}a^{1/2}$ and $d_n := e_n \cdot a^\lambda$ with $\lambda = (\gamma-1)/2$, i.e., $\gamma = 1+2\lambda$.

We (want to ???) show that the d_n converge to an element of $d \in C*(a,b)$ with $d^*bd = a^\gamma$ and $\|b\| \leq 1$.

??? is really

Use that $\|e_n\|^2 = \|e_n e_n^*\| = \|e_n^* e_n\|$

and then that $\|d_n\| \leq \|a\|^\lambda \cdot \|e_n\|$ and estimate by

$$e_n e_n^* = (b+2^{-n})^{-1/2}a(b+2^{-n})^{-1/2} \leq (b+2^{-n})^{-1/2}b(b+2^{-n})^{-1/2} = (b+2^{-n})^{-1}b\,.$$

In particular $\|e_n\| \leq 1$ for all $n = 1, 2, \ldots$.

Recall that $e_n = (b+2^{-n})^{-1/2}a^{1/2}$.

The definition of e_n shows that $e_n^*(b+2^{-n})e_n = a$. It is equivalent to $e_n^* b e_n = a - 2^{-n}e_n^* e_n$, If the (e_n) converges in A to an element $e \in C*(a,b) \subseteq A$ then $e^*be = a$ and $\|e\| \leq 1$.

The problem is that this e_n do not converge in A (itself)?

$d_n := e_n \cdot a^\lambda$ with $\lambda = (\gamma-1)/2$, i.e., $\gamma = 1+2\lambda$.

All is now proven up to the estimate of $\|d_n - d_m\|$:

$\|d_n - d_m\| \leq \|a\|^\lambda \|e_n - e_m\|$ If $\|a\|^{\leq}1$ then we get same estimates ... ???

Recall that $e_n := (b+2^{-n})^{-1/2}a^{1/2}$ and $d_n := e_n \cdot a^\lambda$,

$X_n := (b+2^{-n})^{-1/2}b^{1/2}$ converges in the point norm ???

Let $f_{n,m} := (b+2^{-n})^{-1/2} - (b+2^{-m})^{-1/2}$. Since $0 \leq x \leq y$ implies

Then $f_{n,m}^* = f_{n,m}$ and $(d_n - d_m)\cdot(d_n - d_m)^* = f_{n,m} a f_{n,m}$. Now $a \leq b$ implies that

$$f_{n,m} a f_{n,m} \leq f_{n,m} b f_{n,m} = [b^{1/2}(b+2^{-n})^{-1/2} - b^{1/2}(b+2^{-m})^{-1/2})]^2 .$$

??? It follows that (d_n) is a Cauchy sequence in A if ??? (??)

Then $\|d_n - d_m\|^2 = \|(d_n - d_m)(d_n - d_m)^*\|$, and $a \leq b$ implies that ?????

$$(d_n - d_m)(d_n - d_m)^* \leq ((b+2^{-n})^{-1/2} - (b+2^{-m})^{-1/2})b((b+2^{-n})^{-1/2} - (b+2^{-m})^{-1/2})$$

11. On dense (algebraic) ideals of C*-algebras:

Let A a C*-algebra and let $J \subseteq A$ an *algebraic* ideal of A that is dense in A, i.e., for every $a \in A$ there exists a sequence (c_n) in J with $\|a - c_n\| < 1/n$. Here we do not require that J is invariant under the involution $c \mapsto c^*$, i.e. we do not here require that $c \in J$ implies that $c^* \in J$.

Precisely expressed: J is a norm-dense linear subspace of A with the properties $b \cdot a, a \cdot b \in J$ for all $b \in J$ and $a \in A$.

One can see from this defining properties that $J_* := \{c^*; c \in J\} \subseteq A$ is again a dense algebraic ideal in A, because the involution $x \to x^*$ is isometric in A and the inclusions $a^* \cdot J \subseteq J$ and $J \cdot a^* \subseteq J$ for $a \in A$ imply that $J_* \cdot a \subseteq J_*$ and $a \cdot J_* \subseteq J_*$ for all $a \in A$.

It follows that the sets of products $J_* \cdot J$ and $J \cdot J_*$ are contained in $J_* \cap J$ and are both dense sets in A, that are invariant under multiplication by elements from A.

Thus, the intersection $J_* \cap J \subseteq J$ is again a dense ideal of A because $(J_* \cdot J) \cup (J \cdot J_*) \subseteq J_* \cap J$.

If J is a dense algebraic ideal of A, then J has following properties:

(1) for every $a \in A_+$ and $0 < \varepsilon < 1$ there exists $b \in J_+ := A_+ \cap J$ with $\|a - b\| < \varepsilon$.

Proof of property (1): It is enough to consider the case where $a \in A_+$ has norm $\|a\| = 1$. Then $\|a^{1/2}\| = \|a\|^{1/2} = 1$. Let $\beta := (1+\varepsilon)^{1/2} - 1$, i.e., $(1+\beta)^2 = 1+\varepsilon$ and $\varepsilon = \beta^2 + 2\beta$.

Since J is dense in A, there exists $c \in J$ with $\|a^{1/2} - c\| < \beta$. Then $c^*c \in A_+ \cap J$ and it follows that $\|a - c^*c\| < \beta(2+\beta) = \varepsilon$ by using the following general estimate in C*-algebras A:

If $x, y \in A$ and $\|x - y\| < \beta$, then

$$\|x^*x - y^*y\| < \beta \cdot (\|x\| + \|y\|) \leq \beta \cdot (\beta + 2\|x\|)$$

because $\|y\| \leq \|x\| + \|y - x\|$. Then take here $x := a^{1/2}$ and $y := c$.

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