The Cardinality and Combinatorics of Infinite Sets under Determinacy

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Cardinalities

Definition (Cantor)
Define a proper class equivalence relation on all sets by $X \approx Y$ if and only if there is a bijection between X and Y. A cardinality is a $\approx$-equivalence class. If X is a set, then let $|X| = [X]_\approx$ be the $\approx$-equivalence class of X.

Define $|X| \leq |Y|$ if and only if there exists an injection $\Phi : X \to Y$. $|X| < |Y|$ if and only if $(|X| \leq |Y|)$ and $\neg(|Y| \leq |X|)$.

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A cardinal is an ordinal which does not inject into any smaller ordinal.

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The axiom of choice tends to erase defining characteristics of sets or make them irrelevant in distinguishing sets by size. The axiom of choice is not the only setting for study mathematical size. Other robust frameworks exist which are motivated by combinatorics and descriptive set theory.

Descriptive Set Theory

We will be mostly concerned with cardinalities of very familiar sets. These tend to be surjective images of $\mathbb{R}$ or equivalently quotients of equivalence relations on $\mathbb{R}$. Results from descriptive set theory characterizes the injections between certain quotients which have simply definable liftings.

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Fact (ZF; Silver's dichotomy)
If E is a $\boldsymbol{\Pi}^1_1$ equivalence relation on $\mathbb{R}$, then exactly one of the following occurs.

- *$\mathbb{R}/E$ is countable.*
- *$= \leq_B E$, there is a Borel reduction $\Phi : \mathbb{R} \to \mathbb{R}$ so that $x = y$ if and only if $\Phi(x) \; E \; \Phi(y)$. (So Φ induces an injection of $\mathbb{R}$ into $\mathbb{R}/E$.)*

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Define E_0 on ${}^\omega 2$ by $x\ E_0\ y$ if and only if there exists an $m \in \omega$ so that for all $n \geq m$, $x(n) = y(n)$.

Fact (ZF; E_0-dichotomy; Harrington-Kechris-Louveau)
If E is a $\mathbf{\Delta}^1_1$ equivalence relation, then exactly one of the following occurs.

- $E \leq_B =$*. The reduction induces an injection of* $\mathbb{R}/E$ *into* $\mathbb{R}$ *or* $\mathscr{P}(\omega)$*.*
- $E_0 \leq_B E$*. The reduction induces an injection of* $\mathbb{R}/E_0$ *injects into* $\mathbb{R}/E$*.*

Determinacy

Determinacy provides a robust choiceless framework which extends the theory of "Borel cardinality" of quotients of equivalence relations to a theory for genuine cardinalities.

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The following is the most basic form of the axiom of determinacy, AD.

Let $A \subseteq {}^{\omega}\omega$.

	1	a_0		a_2		a_4		$\cdots$	
G_A									$\vec{a}$
	2		a_1		a_3		a_5	$\cdots$	

All the a_i are natural numbers. Player 1 wins G_A if and only if $\vec{a} \in A$. The axiom of determinacy, AD, is the assertion that for all $A \subseteq {}^{\omega}\omega$, one of the two players has a winning strategy in G_A.

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There are other variations of AD. One important technical extension is Woodin's theory of AD^+.

In the above setting, if each move comes from $\mathbb{R}$, then the determinacy of all such games is denoted $\mathrm{AD}_{\mathbb{R}}$. $\mathrm{AD}_{\mathbb{R}}$ implies Uniformization which is a choice principle for $\mathbb{R}$-indexed family of nonempty subsets of $\mathbb{R}$.

Classification Program

Determinacy assumptions influence the cardinalities and combinatorics of sets which are surjective images of $\mathbb{R}$.

- Completely classify the cardinalities and their injection relation below familiar sets which are surjective images of $\mathbb{R}$.
- For familiar sets, determines its global position or relation to all other sets which are surjective images of $\mathbb{R}$.

Toward the ambitious goal of complete classification, the following problems should be addressed.

- Gain a sufficient understanding of the cardinality of familiar sets to determine cofinality or regularity.

Regularity and Cofinality

Definition
A set X has Y-regular cardinality if and only if for all $\Phi : X \to Y$, there exists a $y \in Y$ so that $|\Phi^{-1}[\{y\}]| = |X|$.

X has locally regular cardinality if and only if for all Y with $|Y| < |X|$, X is Y-regular.

X has globally regular cardinality if and only if for all Y with $\neg(|X| \leq |Y|)$, X is Y-regular.

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Definition
The local cofinality of X is

$$\mathrm{lcof}(X) = \{Y : (\exists Z)(Z \subseteq X \wedge Z \approx Y \wedge X \text{ does not have } Y\text{-regular cardinality}\}.$$

If X is a set, let $\mathrm{Surj}(X)$ be the set of Y so that Y is a surjective image of X. The global cofinality of X is

$$\mathrm{gcof}(X) = \{Y \in \mathrm{Surj}(X) : X \text{ does not have } Y\text{-regular cardinality}\}.$$

Regularity and Cofinality

Fact (ZF)
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If X is locally regular, then $\mathrm{lcof}(X) = |X|$. If X is globally regular, then $\mathrm{gcof}(X) = \{Y \in \mathrm{Surj}(X) : |X| \leq |Y|\}$.

Cardinality of the Power Set of ω

Fact (AD**)**
If $X \subseteq \mathscr{P}(\omega)$ is uncountable, then $|X| = |\mathbb{R}| = |\mathscr{P}(\omega)|$. $\mathbb{R}$ has ω-regular cardinality and hence $\mathbb{R}$ has locally regular cardinality. $\mathrm{lcof}(\mathbb{R}) = |\mathbb{R}|$.

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Assuming AD, well ordered unions of meager sets are meagers.

Fact (AD)
$\mathbb{R}$ is not wellorderable. Thus $\neg(|\mathbb{R}| \leq |\omega_1|)$ and $\neg(|\omega_1| \leq |\mathbb{R}|)$. $\mathbb{R}$ has ON-regular cardinality.

Cardinality of the Power Set of ω

Woodin generalized Harrington's proof of the Silver's dichotomy to show that $|\mathbb{R}|$ has a very special global relationship to all other cardinalities which are surjective images of $\mathbb{R}$.

Fact (Woodin; AD^+**)**
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *X is well orderable.*
- $|\mathbb{R}| \leq |X|$.

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If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *X is well orderable.*
- $|\mathbb{R}| \leq |X|$.

Theorem (AD^+; Chan-Jackson-Trang)
$\mathbb{R}$ has globally regular cardinality. $\mathrm{gcof}(\mathbb{R}) = \{X \in \mathrm{surj}(X) : |\mathbb{R}| \leq |X|\} = \{X \in \mathrm{surj}(\mathbb{R}) : X \text{ is not wellorderable}\}$.

Cardinality of $\mathbb{R}/E_0$

Hjorth generalized the E_0-dichotomy of H-K-L to establish the global position of $|\mathbb{R}/E_0|$ among all other cardinalities which are surjecive images of $\mathbb{R}$.

Fact (Hjorth; AD^+**)**
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *There exists a $\delta \in \mathrm{ON}$, $|X| \leq |\mathscr{P}(\delta)|$ (X is linearly orderable) (Tame)*
- $|\mathbb{R}/E_0| \leq |X|$. *($X$ is not linearly orderable) (Untame)*

The cardinality structure of $\mathbb{R}/E_0$ is completely classified by the following picture.

$$\begin{array}{c} \mathbb{R}/E_0 \\ \big\uparrow \\ \mathbb{R} \end{array}$$

Cardinality of $\mathbb{R}/E_0$

Fact (C-J-T; AD**)**
Let Y be a linearly orderable set. Then $\mathbb{R}/E_0$ is Y-regular.

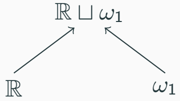
Fact (Hjorth; AD^+**)**
If X is a surjective image of $\mathbb{R}$, then exactly one of the following holds.

- *There exists a $\delta \in \mathrm{ON}$, $|X| \leq |\mathscr{P}(\delta)|$ (X is linearly orderable) (Tame)*
- *$|\mathbb{R}/E_0| \leq |X|$. ($X$ is not linearly orderable) (Untame)*

Theorem (C-J-T; AD^+**)**
$\mathbb{R}/E_0$ has globally regular cardinality. $\mathrm{gcof}(\mathbb{R}/E_0) = \{X \in \mathrm{Surj}(\mathbb{R}) : |\mathbb{R}/E_0| \leq X\} = \{X \in \mathrm{Surj}(\mathbb{R}) : X \text{ is not linearly orderable}\}$.

Cardinality of $\mathbb{R} \sqcup \omega_1$

Fact (AD**)**
The structure of the cardinality of $\mathbb{R} \sqcup \omega_1$ is given by the following picture.



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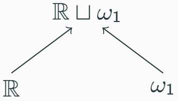


Fact (C-J-T; AD)
$\mathbb{R} \sqcup \omega_1$ does not have 2-regular cardinality.

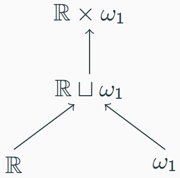
$\mathrm{gcof}(\mathbb{R} \sqcup \omega_1) = \{X \in \mathrm{Surj}(\mathbb{R}) : |X| > 1\}.$

Let $\Phi : \mathbb{R} \sqcup \omega_1 \to 2$ be

$$\Phi(x) = \begin{cases} 0 & x \in \mathbb{R} \\ 1 & x \in \omega_1 \end{cases}$$

Cardinality of $\mathbb{R} \times \omega_1$

The four natural subsets of $\mathbb{R} \times \omega_1$ have the following cardinality relation.



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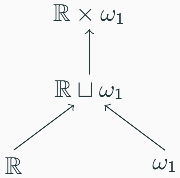

Assuming AD + Uniformization (which following from $\mathrm{AD}_{\mathbb{R}}$), the above picture is the complete local classification of the cardinalities below $|\mathbb{R} \times \omega_1|$.

Main idea: Let $X \subseteq \mathbb{R} \times \omega_1$. Let $X_r = \{\alpha : (r, \alpha) \in X\}$ is either countable or size ω_1. When X_r is countable, one would like a bijection of ω with $\mathrm{ot}(X_r)$. This amounts to uniformizing a suitable relation $R \subseteq \mathbb{R} \times \mathrm{WO}$.

Cardinality of $\mathbb{R} \times \omega_1$

Fact (C-J-T)
(AD) $\mathbb{R} \times \omega_1$ *does not have* $\mathbb{R}$*-regular cardinality or* ω_1*-regular cardinality.*

(AD^+) $\mathrm{gcof}(\mathbb{R} \times \omega_1) = \{X \in \mathrm{Surj}(\mathbb{R}) : |\mathbb{R}| \leq |X| \vee |\omega_1| \leq |X|\} = \{X \in \mathrm{Surj}(\mathbb{R}) : X$ *is uncountable*$\}$.

Let $\Phi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ be $\Phi_1(r, \alpha) = r$ and $\Phi_2 : \mathbb{R} \times \omega_1 \to \omega_1$ be $\Phi_2(r, \alpha) = \alpha$.

Cardinality of $\mathbb{R} \times \omega_1$

Woodin observed that without Uniformization, there are many other cardinalities below $\mathbb{R} \times \omega_1$ than the four listed above. Work in $L(\mathbb{R}) \models \mathsf{AD}$. Let $\mathbb{X} = {}_\omega\mathbb{O}$ be the direct limit of the Vopěnka forcing on $\mathbb{R}^n$ for all $n \in \omega$.

Let $x \leq_{\mathbb{X}} y$ if and only if $x \in L[\mathbb{X}, y]$. $x \equiv_{\mathbb{X}} y$ if and only if $x \leq_{\mathbb{X}} y$ and $y \leq_{\mathbb{X}} x$. A $\mathbb{X}$-degree is a $\equiv_{\mathbb{X}}$-equivalence classes. $\mathcal{D}_{\mathbb{X}}$ is the collection of all $\mathbb{X}$-degrees. $\mu_{\mathbb{X}}$ is a measure on $\mathcal{D}_{\mathbb{X}}$ defined by $A \in \mu_{\mathbb{X}}$ if and only if A contains a $\mathbb{X}$-cone. Woodin showed that $\prod_{\mathcal{D}_{\mathbb{X}}} \omega_1/\mu_{\mathbb{X}}$ is a wellordering under AD^+.

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Fact (Woodin; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$)
Let $W_2 = \bigsqcup_{r \in \mathbb{R}} \omega_2^{L[\mathbb{X},r]}$. Then ω_1 does not inject into W_2 and $|\mathbb{R}| < |W_2| < |\mathbb{R} \times \omega_1|$.

Proof.
Suppose $\Phi : W_2 \to \mathbb{R}$ is an injection. On a $\mathbb{X}$-cone of $e \in \mathbb{R}$, $\Phi \restriction W_2 \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$. Woodin showed that on a $\mathbb{X}$-cone of e, $L[\mathbb{X}, e] \models \mathsf{CH}$. Thus for a suitable e, $L[\mathbb{X}, e] \models \Phi \restriction (\{e\} \times \omega_2^{L[\mathbb{X},e]}) : \{e\} \times \omega_2^{L[\mathbb{X},e]} \to \mathbb{R}^{L[\mathbb{X},e]}$ is an injection which violates CH. $\square$

Cardinality of $\mathbb{R} \times \omega_1$

There are no cardinalities between $|\mathbb{R}|$ and $|W_2|$.

Theorem (C-J; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$**)**
If $X \subseteq W_2$, then $|X| \leq |\mathbb{R}|$ or $|W_2| = |X|$.

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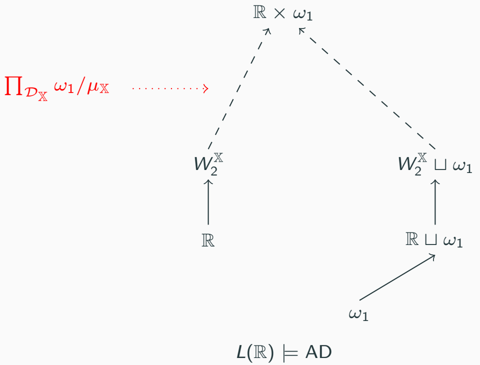
If $X \subseteq W_2$, then $|X| \leq |\mathbb{R}|$ or $|W_2| = |X|$.

Definition
Let $F : \mathbb{R} \to \omega_1$ be an $\mathbb{X}$-invariant function with respect to $\mathbb{X}$-constructibility degree. Let $W_F = \bigsqcup_{r \in \mathbb{R}} \omega_{F(r)}^{L[\mathbb{X}, r]}$. For each $\alpha \in \prod_{\mathcal{D}_{\mathbb{X}}} \omega_1 / \mu_{\mathbb{X}}$, let $Y_\alpha = |W_F^{\mathbb{X}}|$ for any F such that $[F]_{\mu_{\mathbb{X}}} = \alpha$.

Cardinality of $\mathbb{R} \times \omega_1$

Fact (C-J; $\mathsf{AD} + V = L(\mathbb{R})$)
$\langle Y_\alpha : \alpha \in \prod_{\mathcal{D}_{\mathbb{X}}} \omega_1/\mu_{\mathbb{X}} \rangle$, under the injection relation, is isomorphic to the ultrapower ordering. It is cofinal among the cardinalities below $\mathbb{R} \times \omega_1$ which do not possess a copy of ω_1. For any $\alpha \in \prod_{\mathcal{D}_{\mathbb{X}}} \omega_1/\mu_{\mathbb{X}}$, there are no cardinalities between Y_α and $Y_{\alpha+1}$. The first ω_1 many cardinalities below $|\mathbb{R} \times \omega_1|$ is exactly $\{Y_\alpha : \alpha < \omega_1\}$.



Conjecture: This is the complete classification of the cardinalities below $\mathbb{R} \times \omega_1$ in $L(\mathbb{R})$.

Definition (Correct type functions)
Let $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$. f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \restriction \alpha) < f(\alpha)$. f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n + 1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$. f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

If $X \subseteq \mathrm{ON}$, then $[X]^{\epsilon}_{*}$ is the collection of increasing functions $f : \epsilon \to X$ of the correct type.

Partition Properties

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If $X \subseteq \mathrm{ON}$, then $[X]^\epsilon_*$ is the collection of increasing functions $f : \epsilon \to X$ of the correct type.

Definition (Correct type partition relation)
Let $\epsilon \leq \kappa$ and $\gamma < \kappa$. $\kappa \to_* (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon_* \to \gamma$, there is a $\delta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \delta$.

$\kappa \to_* (\kappa)^{<\epsilon}_\gamma$ and $\kappa \to_* (\kappa)^\epsilon_{<\gamma}$ are given the obvious meanings.

If $\kappa \to_* (\kappa)^{<\kappa}_2$ (which implies $\kappa \to_* (\kappa)^{<\kappa}_{<\kappa}$), then κ is called a weak partition cardinal. If $\kappa \to_* (\kappa)^\kappa_2$, then κ is called a strong partition cardinal. If $\kappa \to_* (\kappa)^\kappa_{<\kappa}$, then κ is called a very strong partition cardinal.

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Definition (Partition measures)
Let μ^ϵ_κ be a filter on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\epsilon_\kappa$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$. $\kappa \to_* (\kappa)^\epsilon_2$ implies μ^ϵ_κ is an ultrafilter. $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ implies μ^ϵ_κ is κ-complete. $\kappa \to_* (\kappa)^2_2$ implies the ω-club filter μ^1_κ is normal.

AD implies there are many partition cardinals.

Theorem

- *(Martin)* $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$.
- *(Martin-Paris)* $\omega_2 \to_* (\omega_2)^{<\omega_2}_2$ *but* $\neg(\omega_2 \to_* (\omega_2)^{\omega_2}_2)$.
- *(Jackson) For all* $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1}$ *is a very strong partition cardinal.*
- *(Jackson) For all* $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2}$ *is a weak partition cardinal but not a strong partition cardinal.*
- *(Kechris-Kleinberg-Moschovakis-Woodin)* $\boldsymbol{\delta}^2_1$ *and* Σ_1*-stable ordinals* $\boldsymbol{\delta}_A$ *of* $L(A, \mathbb{R})$ *for any* $A \in \mathscr{P}(\mathbb{R})$ *are very strong partition cardinals. There are cofinally many very strong partition cardinals below* Θ *(the supremum of the ordinals which are surjective images of* $\mathbb{R}$*).*

The Boldface GCH

Definition
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Fact
If there is a κ-complete ultrafilter on κ, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

If particular if $\kappa \to_* (\kappa)^2_2$ holds, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

So $\omega_1 \to_* (\omega_1)^2_2$ implies the boldface GCH *at ω.*

Definition
The boldface GCH holds at κ if and only if there is no injection of κ^+ into $\mathscr{P}(\kappa)$.

Fact
If there is a κ-complete ultrafilter on κ, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

If particular if $\kappa \to_* (\kappa)^2_2$ holds, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.

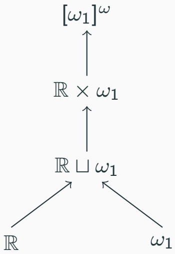
So $\omega_1 \to_* (\omega_1)^2_2$ implies the boldface GCH *at ω.*

Theorem (AD$^+$; Woodin, Steel)
The boldface GCH *holds below Θ.*

Martin showed that for $n \in \omega$, $\omega_{n+1} = \prod_{[\omega_1]^n_*} \omega_1/\mu^n_{\omega_1}$. With Jackson and Trang, we have a combinatorial proof of the boldface GCH below $\omega_{\omega+1}$. This type of argument should hold below the supremum of the projective ordinals using Jackson's description analysis.

Cardinality of $[\omega_1]^\omega$

Using the $\omega_1 \to_* (\omega_1)^\omega_{<\omega_1}$, one has the following relation among the five familiar cardinalities below $|[\omega_1]^\omega|$.



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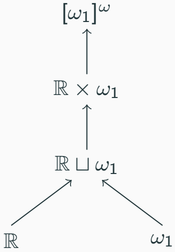
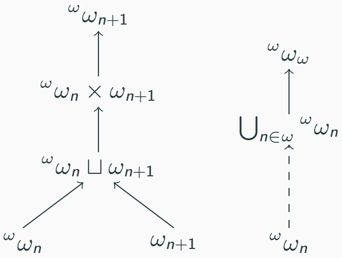


Theorem (Woodin; $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$)
If $X \subseteq [\omega_1]^\omega$, then either $|X| \leq |\mathbb{R} \times \omega_1|$ or $|[\omega_1]^\omega| = |X|$.

Thus under $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, this is the complete classification of the cardinalities below $|[\omega_1]^\omega|$.

Cardinality of $[\omega_1]^\omega$

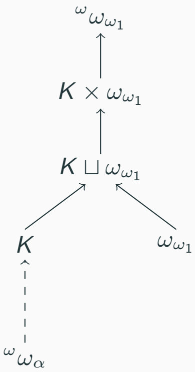
There is a proof of the Woodin $[\omega_1]^\omega$-dichotomy under AD and Uniformization that can be adapted to ω-sequence through some higher cardinals.



A similar classification should hold for ${}^\omega\omega_\alpha$ when $\alpha < \omega_1$.

Cardinality of $[\omega_1]^\omega$

At ${}^\omega\omega_{\omega_1}$, new behaviors appear. Let $K = \bigsqcup_{w \in \mathrm{WO}} {}^\omega\omega_{\mathrm{ot}(w)}$. The following summarizes some of the structure below ${}^\omega(\omega_{\omega_1})$.



Conjecture: This is the complete classification below ${}^\omega\omega_{\omega_1}$ under $\mathrm{AD}_{\mathbb{R}}$.

Cardinality of $[\omega_1]^\omega$

Fact
If $\kappa \to_* (\kappa)^2_2$, then for all $\epsilon < \kappa$, $[\kappa]^\epsilon$ is not κ-regular.

For example, $[\omega_1]^\omega = \bigcup_{\delta < \omega_1} [\delta]^\omega$ and $|[\delta]^\omega| = |\mathbb{R}| < |[\omega_1]^\omega|$ where $\delta < \omega_1$.
Thus $[\omega_1]^\omega$ is not locally regular.

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Fact
If $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$, then $[\kappa]^\epsilon$ is ${}^\mu\delta$ regular for all $\delta, \mu < \kappa$.

For example, $[\omega_1]^\omega$ is $\mathbb{R}$-regular.

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Combined with Woodin's classification of $|[\omega_1]^\omega|$, one has the following local cofinality.

Fact ($\mathsf{AD}_{\mathbb{R}}$**)**
$\mathrm{lcof}([\omega_1]^\omega) = \{X : (\exists Z)(Z \subseteq [\omega_1]^\omega \wedge |Z| = |X| \wedge |\omega_1| \leq |X|\}$
$= \{X : |X| = |\omega_1| \vee |X| = |\mathbb{R} \sqcup \omega_1| \vee |X| = |\mathbb{R} \times \omega_1| \vee |X| = |[\omega_1]^\omega|\}.$

Cardinality of $[\omega_1]^\omega$

Definition
A set X is prime if and only if for all set Y and Z, if $|X| \leq |Y \times Z|$, then $|X| \leq |Y|$ or $|X| \leq |Z|$.

Theorem (C-J-T; AD)
$\mathbb{R}$ *is prime.*

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Theorem (C-J-T)
Assume $\kappa \to_* (\kappa)_2^{\omega+\omega}$. Then $[\kappa]^\omega$ is prime.

Theorem (AD; C-J-T)
For all $\kappa \leq \omega_{\omega+1}$, $[\kappa]^\omega$ is prime.

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For all $\kappa \leq \omega_{\omega+1}$, $[\kappa]^\omega$ is prime.

Theorem (AD; C-J-T)
For all $n \in \omega$, $\mathscr{P}(\omega_{n+1})$ does not injection into $\mathscr{P}(\omega_n) \times \mathrm{ON}$.

Proof.
Suppose $\Phi : \mathscr{P}(\omega_3) \rightarrow \mathscr{P}(\omega_2) \times \mathrm{ON}$ is an injection. Then $\Phi : [\omega_3]^\omega \rightarrow \mathscr{P}(\omega_2) \times \mathrm{ON}$ is an injection. Since $[\omega_3]^\omega$ is prime, $|[\omega_3]^\omega| \leq |\mathscr{P}(\omega_2)|$ (impossible by boldface GCH at ω_2) or $|[\omega_3]^\omega|$ injects into an ordinal (which is impossible since $[\omega_3]^\omega$ is not wellorderable). $\square$

Cardinality of $[\omega_1]^\omega$

Theorem (Woodin; $\mathsf{AD}_\mathbb{R}$**)**
If $X \subseteq [\omega_1]^\omega$*, then either* $|X| \leq |\mathbb{R} \times \omega_1|$ *or* $|[\omega_1]^\omega| = |X|$.

Woodin classification fails without uniformization.

Theorem (C.; $\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$**)**
Let $E_2 = \bigsqcup_{r \in \mathbb{R}} [\omega_2^{L[\mathbb{X}, r]}]^\omega$. $\neg(|\omega_1| \leq |E_2|)$ *and* $\neg(|E_2| \leq |\mathbb{R} \times \omega_1|)$.

Proof.
Suppose $\Phi : E_2 \to \mathbb{R} \times \omega_1$ is an injection. There is a $\mathbb{X}$-cone of $e \in \mathbb{R}$ so that any inner model M with $\mathbb{X}, e \in M$, $\Phi \restriction E_2 \cap M, \Phi^{-1} \restriction \mathbb{R} \times \omega_1 \cap M \in M$.

Let $f : \omega \to \omega_2^{L[\mathbb{X}, e]}$ be the Namba generic function over $L[\mathbb{X}, e]$. Since $\Phi \restriction E_2 \cap L[\mathbb{X}, e, f] \in L[\mathbb{X}, e, f]$, $\Phi(e, f) \in L[\mathbb{X}, e, f]$ and thus $\Phi(e, f) \in L[\mathbb{X}, e]$ since Namba forcing adds no new reals. Since $\Phi^{-1} \restriction \mathbb{R} \times \omega_1 \cap L[\mathbb{X}, e]$, $(e, f) = \Phi^{-1}(\Phi(e, f)) \in L[\mathbb{X}, e]$. However, it is impossible that the Namba generic belongs to the ground model. □

Cardinality of $[\omega_1]^{<\omega_1}$

Woodin showed that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ under $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ indirectly using a certain subset of $[\omega_1]^{<\omega_1}$.

Definition (Woodin)
Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

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Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

Fact (AD^+; **Woodin)**
S_1 does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals. Thus $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Proof.
Suppose $\Phi : S_1 \to {}^\omega\mathrm{ON}$ is an injection. Using the fact that all sets of reals have ∞-Borel codes, there is a set of ordinals J so that any inner model M with $J \in M$, one has $\Phi \restriction S_1 \cap M \in M$.

Let $\xi < \omega_1^V$ be an inaccessible cardinal of $L[J]$. Let $G \subseteq \mathrm{Coll}(\omega, <\xi)$ be generic over $L[J]$ with $G \in V$. Let $f \in S_1$ be the generic function. Since $\Phi \restriction S_1 \cap L[J][G] \in L[J][G]$, $\Phi(f) \in L[J][G] \cap {}^\omega\mathrm{ON}$. Since every ω-sequence belongs to an initial segment of G, there is a $\delta < \xi$ so that $\Phi(f) \in L[J][G \restriction \delta]$. Since $\Phi^{-1} \restriction {}^\omega\mathrm{ON} \cap L[J][G \restriction \delta] \in L[J][G \restriction \delta]$, $f = \Phi^{-1}(\Phi(f)) \in L[J][G \restriction \delta]$ which is impossible. □

Cardinality of $[\omega_1]^{<\omega_1}$

To prove $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ in a manner that could generalize to higher cardinals, we investigate almost everywhere behavior or continuity properties of functions.

Let $\Phi : [\omega_1]^\omega \to \omega_1$ be defined by $\Phi(f) = \sup(f) + f(13) + f(7)$. Φ depends only on $\sup(f)$ and the 7^{th} and 13^{th}-value of f. The next results states all function have this behavior almost everywhere.

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Theorem (AD**; C-J-T)**
Let $\epsilon < \omega_1$ and $\Phi : [\omega_1]^\epsilon \to \omega_1$.

(Short length continuity) There is a club $C \subseteq \omega_1$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \upharpoonright \delta = g \upharpoonright \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

(Strong short length continuity) There is a club $C \subseteq \omega_1$ and finitely many $\delta_0 < ... < \delta_{k-1} \leq \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if for all $i < k$, $\sup(f \upharpoonright \delta_i) = \sup(g \upharpoonright \delta_i)$, then $\Phi(f) = \Phi(g)$.

Cardinality of $[\omega_1]^{<\omega_1}$

Continuity yields very well controlled failure of injectiveness.

Theorem (C-J-T)
(AD) $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. $\neg(|[\omega_1]^{<\omega_1}| \leq |^\omega(\omega_\omega)|)$.

($\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) $[\omega_1]^{<\omega_1}$ *does not inject into* $^\omega\mathrm{ON}$, *the class of* ω*-sequences of ordinals.*

Cardinality of $[\omega_1]^{<\omega_1}$

To extend these results to the familiar weak and strong partition cardinals of determinacy above ω_1 requires continuity results proved by pure partition properties.

Theorem (C-J-T)
(Countable cofinality short length continuity) Suppose $\epsilon < \kappa$, $\mathrm{cof}(\epsilon) = \omega$, $\kappa \to_* (\kappa)^{\epsilon \cdot \epsilon}_2$, and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$. Then there is a $\delta < \epsilon$ and a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\mathrm{sup}(f) = \mathrm{sup}(g)$, then $\Phi(f) = \Phi(g)$.

Theorem (C-J-T)
Suppose $\kappa \to_* (\kappa)^{<\kappa}_2$. Then for all $\lambda < \kappa$, $[\kappa]^{<\kappa}$ does not inject into ${}^\lambda\mathrm{ON}$, the class of λ-sequences of ordinals. Thus $|[\kappa]^\lambda| < |[\kappa]^{<\kappa}|$ for all $\lambda < \kappa$.

Cardinality of $[\omega_1]^{<\omega_1}$

Fact (C-J-T)
If $\kappa \to_* (\kappa)^{<\kappa}_2$, then $[\kappa]^{<\kappa}$ does not have κ-regular cardinality. Thus $[\kappa]^{<\kappa}$ does not have locally regular cardinality.

For example, $[\omega_1]^{<\omega_1} = \bigcup_{\epsilon<\omega_1}[\omega_1]^\epsilon$ and the previous theorem showed that $|[\omega_1|^\epsilon| < |[\omega_1]^{<\omega_1}|$.

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Fact (C-J-T)
Let $\kappa \to_* (\kappa)^{\kappa}_2$. For all $\lambda < \kappa$, $[\kappa]^{<\kappa}$ has λ-regular cardinality.

Fact (C-J-T)
Let $\kappa \to_* (\kappa)^{\kappa}_{<\kappa}$. For all $\mu, \lambda < \kappa$, $[\kappa]^{<\kappa}$ is ${}^\mu\lambda$-regular.

Cardinality of $\mathscr{P}(\omega_1)$

We have yet to encounter a candidate for another locally or even globally regular cardinality. The study of the cardinality of $\mathscr{P}(\omega_1)$ is motivated by the following conjecture.

Conjecture: $\mathscr{P}(\omega_1)$ has locally regular cardinality and even globally regular cardinality.

We will provide emperical evidence for this conjecture by searching for cofinality of $\mathscr{P}(\omega_1)$.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD)
$|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$.

Proof.
Suppose $\Phi : \mathscr{P}(\omega_1) \to [\omega_1]^{<\omega_1}$ is an injection. $L[\Phi] \models \mathrm{ZFC}$ and $L[\Phi] \models |\mathscr{P}(\omega_1^V)| > \omega_1^V \wedge |[\omega_1{}^V]^{<\omega_1{}^V}| = \omega_1^V$ (by boldface GCH at ω). Also $L[\Phi] \models \Phi : \mathscr{P}(\omega_1^V) \to [\omega_1^V]^{<\omega_1^V}$ is an injection, which is impossible. □

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This proof is not ideal with respect to the regularity conjecture as its does verify any instance of regularity for $\mathscr{P}(\omega_1)$. The regularity conjecture should be used as a standard for all cardinality computation relative to $\mathscr{P}(\omega_1)$.

Theorem (AD**; C-J)**
(Almost everywhere continuity) Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$*. There is a club* $C \subseteq \omega_1$ *so that for all* $f \in [C]^{\omega_1}_*$*, there exists an* $\alpha < \omega_1$ *so that for all* $g \in [C]^{\omega_1}_*$*, if* $g \restriction \alpha = f \restriction \alpha$*, then* $\Phi(f) = \Phi(g)$*.*

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Theorem (AD**; C-J)**
$\mathscr{P}(\omega_1)$ *has* ω_1*-regular cardinality and in fact, has* $[\omega_1]^{<\omega_1}$*-regular cardinality. Thus* $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$*.*

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (C)
Assume $\kappa \to_* (\kappa)^\kappa_2$*. Then* $\mathscr{P}(\kappa)$ *has* ON*-regular cardinality.*

Theorem (C)
Assume $\kappa \to_* (\kappa)^\kappa_2$. Then $\mathscr{P}(\kappa)$ has ON-regular cardinality.

Consider $\Psi : [\omega_1]^{\omega_1}_* \to \omega_2$ defined by $\Psi(f) = [f]_{\mu^1_{\omega_1}}$, which is the ultrapower of f under the club measure on ω_1. Note that if $A \subseteq \omega_1$ has measure 0 according to the club measure, then for any $f, g \in [\omega_1]^{\omega_1}_*$ with $f \restriction \kappa \setminus A = g \restriction \kappa \setminus A$, then $\Psi(f) = \Psi(g)$. This motivates a weak continuity result.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (C-J-T)
Assume $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. *Let* $\langle A_\alpha : \alpha < \kappa \rangle$ *be a sequence of disjoint subsets of* κ *so that each* $\kappa \setminus A_\alpha$ *is unbounded in* κ. *Let* $\Phi : [\kappa]^\kappa \to \mathrm{ON}$. *Then there is a finite set* $F \subseteq \kappa$ *and a club* $C \subseteq \kappa$ *so that for all* $\alpha \notin F$, *there is a* $\xi < \kappa$ *so that for all* $f, g \in [C]^\kappa_*$, *if* $f(0) > \xi$, $g(0) > \xi$ *and* $f \restriction \kappa \setminus A_\alpha = g \restriction \kappa \setminus A_\alpha$, *then* $\Phi(f) = \Phi(g)$.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (C-J-T)
Assume $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be a sequence of disjoint subsets of κ so that each $\kappa \setminus A_\alpha$ is unbounded in κ. Let $\Phi : [\kappa]^\kappa \to \mathrm{ON}$. Then there is a finite set $F \subseteq \kappa$ and a club $C \subseteq \kappa$ so that for all $\alpha \notin F$, there is a $\xi < \kappa$ so that for all $f, g \in [C]^\kappa_*$, if $f(0) > \xi$, $g(0) > \xi$ and $f \restriction \kappa \setminus A_\alpha = g \restriction \kappa \setminus A_\alpha$, then $\Phi(f) = \Phi(g)$.

Theorem (C-J-T)
Suppose $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. Then $\mathscr{P}(\kappa)$ is ${}^{<\kappa}\mathrm{ON}$-regular.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD; **C-J**)
$\mathscr{P}(\omega_1)$ has ω_1-regular cardinality and in fact, has $[\omega_1]^{<\omega_1}$-regular cardinality. Thus $|[\omega_1]^{<\omega_1}| < |\mathscr{P}(\omega_1)|$.

Thus $\mathscr{P}(\omega_1)$ would be locally regular if the following conjecture has a positive answer.

Conjecture: Assume $\mathsf{AD}_{\mathbb{R}}$. Suppose $X \subseteq \mathscr{P}(\omega_1)$. Then $|X| \leq |[\omega_1]^{<\omega_1}|$ or $|X| = |\mathscr{P}(\omega_1)|$.

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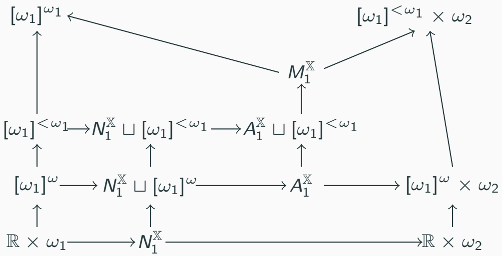
This conjecture cannot be true if the assumption of $\mathsf{AD}_{\mathbb{R}}$ is dropped.

Work in $L(\mathbb{R})$. Let $N_1^{\mathbb{X}} = \bigsqcup_{r \in \mathbb{R}} (\omega_1^V)^{+L[\mathbb{X},r]}$, $A_1^{\mathbb{X}} = \bigsqcup_{f \in [\omega_1]^{\omega}} (\omega_1^V)^{+L[\mathbb{X},f]}$, and $M_1^{\mathbb{X}} = \bigsqcup_{f \in [\omega_1]^{<\omega_1}} (\omega_1^V)^{+L[\mathbb{X},f]}$.



In this setting, the local regularity of $\mathscr{P}(\omega_1)$ would follow from the following conjecture.

Conjecture: $(\mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R}))$ If $X \subseteq \mathscr{P}(\omega_1)$, then $|X| \leq |M_1^{\mathbb{X}}|$ or $|X| = |\mathscr{P}(\omega_1)|$.

Cardinality of $\mathscr{P}(\omega_1)$

For further evidence of global regularity for $\mathscr{P}(\omega_1)$, we should look at sets from the nontame side of Hjorth's dichotomy. Many familiar sets which are surjective images of $\mathbb{R}$ are naturally presented as quotients of equivalence relations on $\mathbb{R}$.

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For further evidence of global regularity for $\mathscr{P}(\omega_1)$, we should look at sets from the nontame side of Hjorth's dichotomy. Many familiar sets which are surjective images of $\mathbb{R}$ are naturally presented as quotients of equivalence relations on $\mathbb{R}$.

- Define E_1 on ${}^\omega\mathbb{R}$ by $x \; E_1 \; y$ if and only if there exists an $m \in \omega$ so that for all $n \geq m$, $x(n) = y(n)$.
- Define E_2 on ${}^\omega 2$ by $x \; E_2 \; y$ if and only if $\sum\{\frac{1}{n} : n \in x \triangle y\} < \infty$.

Cardinality of $\mathscr{P}(\omega_1)$

Theorem (AD; C-J-T)
Suppose $\epsilon \leq \omega_1$. Let E be one of the following equivalence relations.

- *E is an equivalence relation with all countable classes.*
- *E is E_0, E_1, E_2, countable, essentially countable, hyperfinite, smooth, or hypersmooth.*

Then $[\omega_1]^\epsilon_*$ has $\mathbb{R}/E$-regular cardinality.

Theorem (AD; C-J-T)
$\mathscr{P}(\omega_1)$ has $\mathbb{R}/E$-regular cardinality when E is one of the above equivalence relations.

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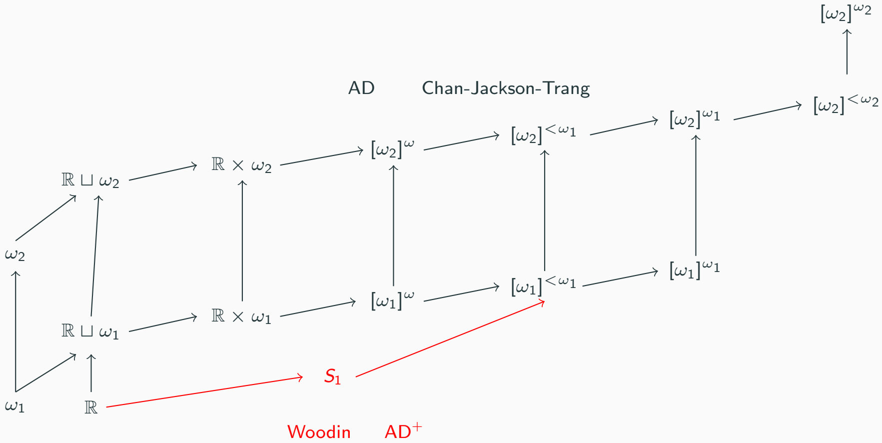
$=^+$ does not fall into this setting. Note $\mathbb{R}/=^+ \approx \mathscr{P}_{\omega_1}(\mathbb{R})$. However, we can still show the following.

Theorem (AD**; C-J-T)**
$\mathscr{P}(\omega_1)$ has $\mathscr{P}_{\omega_1}(\mathbb{R})$-regular cardinality.

Cardinality of $\mathscr{P}(\omega_2)$

We have some knowledge concerning the cardinality and cofinality of $[\omega_2]^\epsilon$ for $\epsilon < \omega_2$ and $[\omega_2]^{<\omega_2}$. The following diagram summarizes the known structure of the cardinality below $\mathscr{P}(\omega_2)$.



Cardinality of $\mathscr{P}(\omega_2)$

We know very little about the cofinality of $\mathscr{P}(\omega_2)$.

Does $\mathscr{P}(\omega_2)$ have 2-regular cardinality under any determinacy assumptions?

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Does $\mathscr{P}(\omega_2)$ have 2-regular cardinality under any determinacy assumptions?

The following is only known weak regularity result known about $\mathscr{P}(\omega_2)$.

Theorem (C-J-T; AD^+)
Suppose X is a surjective image of $\mathbb{R}$. Let $\delta \leq \lambda < \Theta$ be two cardinals so that $\mathrm{cof}(\delta) > \omega$. Suppose $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a sequence of subsets of X so that for all $\alpha < \omega_1$, $|A_\alpha| \leq |[\lambda]^{<\delta}|$. Then $|\bigcup_{\alpha<\omega_1} A_\alpha| \leq |[\lambda]^{<\delta}|$.

Theorem (C-J-T; AD^+)
If $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a sequence such that $\bigcup_{\alpha<\omega_1} A_\alpha = \mathscr{P}(\omega_2)$, then there is some $\alpha < \omega_1$ so that $\neg(|A_\alpha| \leq |[\omega_2]^{<\omega_2}|)$.

Questions

- Find another locally regular or globally regular cardinality. Is local regularity and global regularity always the same concept?
- Calibrate the cofinality of $\mathscr{P}(\omega_1)$. Is $\mathscr{P}(\omega_1)$ locally regular or globally regular?
- Is $\mathscr{P}(\omega_2)$ even 2-regular under any determinacy assumption?

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Thanks for listening!