Questions Münster

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1 George Willis

We will be looking at the continuity of the scale function with respect to the Braconnier topology on $\mathrm{Aut}(G)$ for a group G.

Note first that we can restrict at the scale to a function from G to $\mathbb{N}$ by embedding G into $\mathrm{Aut}G$ as inner automorphism. This function is always continuous, however, the whole scale function is not always continuous.

Example. For $G = (\mathbb{F}_p((t)), +)$ the scale is not continuous.

Question 1.1. For which group G is the scale continuous?

Note that if we have a series $\alpha_n \to \alpha$ then for every compact open subgroup there exists a n s.t. $\alpha_k(U) = \alpha(U)$ for any $k \geq n$.
If U is minimising for α then $s(\alpha_k) \leq s(\alpha)$ for all $k \geq n$.

Question 1.2. What if G is compactly generated?

Andre: George could you sketch the example we had in our discussion, which was c.g.? And adjust the question.

Question 1.3. Is it the case that:

$$\mathrm{Aut}(G) \text{ is locally compact} \iff \text{the scale is continuous?}$$

2 André Nies

Melnikov, Nies and Willis (see appendix of [5]) have built a computably tdlc group with a noncomputable scale function. The group can be made torsion-free by working with Laurent power series with coefficients in $\mathbb{Z}_p$ instead of F_p.

Question 2.1. For which computable tdlc groups G is the scale function $s : G \to \mathbb{N}$ computable?

The given example is not known to have a unique computable presentation. An affirmative answer to the following would provide a computably tdlc group whose scale function is intrinsically noncomputable.

Question 2.2. Is there a G with a unique computable presentation, whose scale is not computable?

For definitions and background see [5].

3 Colin Reid

Take G to be a totally disconnected locally compact group and define

$$\mathcal{O}(G) := \{H \le G \mid H \text{ open }\} \subset \mathrm{Sub}(G)$$

where we endow the $\mathrm{Sub}(G)$ with the Chabauty topology. Now we can compare $\mathcal{O}(G)$ with its closure in $\mathrm{Sub}(G)$ by looking at the following sets:

$$\mathcal{IO}(G) := \{\bigcap_{i\in I} H_i \mid \{H_i \mid i \in I\} \subseteq \mathcal{O}(G)\}$$
$$\mathcal{RIO}(G) := \{\overline{\bigcup^{\uparrow} H_i} \mid \{H_i \mid i \in I\} \subseteq \mathcal{IO}(G),\ H_i \le_o \overline{\bigcup^{\uparrow} H_i}\}$$

One can easily see that every intersection of open subgroups belongs to $\overline{\mathcal{O}(G)}$ and that in any closed subset X of $\mathrm{Sub}(G)$, the closure of an ascending union of elements of X belongs to X. Thus

$$\mathcal{O}(G) \subseteq \mathcal{IO}(G) \subseteq \mathcal{RIO}(G) \subseteq \overline{\mathcal{O}(G)}.$$

However, the class $\mathcal{RIO}(G)$ has several closure properties (see [7, Theorem 4.11]):

- An intersection of RIO subgroups is RIO.
- For any subgroup H of G

$$K \in \mathcal{RIO}(H), H \in \mathcal{RIO}(G) \implies K \in \mathcal{RIO}(G).$$

- The closure of an ascending union $\overline{\bigcup^{\uparrow}_{i\in I} H_i}$ is RIO if each H_i is RIO.

Question 3.1. Is $\mathcal{RIO}(G) = \overline{\mathcal{O}(G)}$

A potentially more answerable sub-question is:

Question 3.2. Given $H \in \overline{\mathcal{O}(G)}$ compactly generated, is $H \in \mathcal{IO}(G)$

Note: If G acts on a locally finite tree with Tits' independence property (P), then $\mathcal{RIO}(G) = \overline{\mathcal{O}(G)}$. This is not an interesting example because open subgroups are severely restricted, see for instance [8, Theorem 1.13]

4 Stephan Tornier

Reid–Smith [8] recently parameterised pairs (T, G) consisting of a tree T and a (P_1)-closed subgroup $G \le \mathrm{Aut}(T)$ in terms of a graph-based combinatorial structure. Efforts are being made to generalise this parameterisation to (P_k)-closed groups ($k \in \mathbb{N}$). Recall that a group $G \le \mathrm{Aut}(T)$ is (P_k)-closed if

$$G = G^{(P_k)} := \{h \in \mathrm{Aut}(T) \mid \forall v \in V(T)\ \exists g \in G:\ h|_{B(v,k)} = g|_{B(v,k)}\}.$$

The more complex nature of (P_k)-closed groups is reflected for example in the values that the order of edge inversions in such groups can take. Whereas every (P_1)-closed group that contains an inversion necessarily also contains one of order 2, this is not true for (P_k)-closed groups in general: by [3] there is a (P_2)-closed group acting on T_3 for which the smallest order of an inversion is 4.

Question 4.1. Let T be a tree and suppose $G \leq \mathrm{Aut}(T)$ is (P_k)-closed and contains an inversion. Does G necessarily contain an inversion of finite order?

Question 4.2. If so, is there an upper bound (in terms of k?) on the smallest order of such an inversion across all (P_k)-closed groups acting on a given tree?

Known examples of (P_k)-closed groups include:

- The generalised universal groups $U_k(F)$ acting on regular trees, introduced in [9] following Burger–Mozes' construction [2]. All of these contain an inversion of order 2 and in fact can be used to classify all locally transitive (P_k)-closed groups containing an inversion of order 2.
- The boundary 2-transitive Radu groups acting on biregular trees, introduced in [6]. In the vertex-transitive case, Radu groups contain inversions.
- Discrete groups acting on trees, such as the ones introduced in [3].

5 João Vitor Pinto e Silva

We be looking at totally disconnected locally compact second countable(tdlcsc) groups.

Definition 1. The discrete residual of a group G is defined as

$$\mathrm{Res}(G) := \bigcap_{O \trianglelefteq_{open} G} O$$

Andre: this is 1, something missing here?

using this we can define the decomposition rank ξ of an elementary group G, by transfinite recursion as follows

- $\xi(\{1\}) := 1$
- If G is compactly generated

$$\xi(G) = \xi(\mathrm{Res}(G)) + 1$$

- If G is not compactly generated we can write $G = \bigcup^{\uparrow}_{n \in \mathbb{N}} O_n$ where the O_n are compactly generated open subgroups

$$\xi(G) = \sup\{\xi(\mathrm{Res}(O_n))\} + 1$$

Note that the rank will always be countable and thus we get a function $\xi : \mathcal{E} \to \omega_1$.

Question 5.1. What is $\mathrm{Im}(\xi)$?

We already know that $[0, \omega^{\omega}+1] \subseteq \mathfrak{I}(\xi)$. Furthermore we have a construction analogous to the wreath product that increases the rank by 1.

6 Matteo Vannacci

A classical question asked before 1970[1] is

Question 6.1. Is every compact torsion topological group of finite exponent?

If we only restrict to finitely generated abstract groups, this is equivalent to the Restricted Burnside Problem and the answer to Question 6.1 in this case is positive by work of Hall-Higman and Zelmanov.

In the general case of compact topological groups, it is fairly easy to reduce to the totally disconnected case [4, pg. 69]. Hence, every compact torsion group is profinite. Moreover, Wilson [10, Theorem 1] has proved that every compact torsion group has a finite series of closed characteristic subgroups of the form:

$$\{1\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots G_{n-1} \triangleleft G_n = G$$

where the factors G_{i+1}/G_i are either a pro-p group for some prime p or are isomorphic (as a topological group) to a Cartesian product of isomorphic finite simple groups.

Since the Cartesian products appearing above have finite exponent, Question 6.1 reduces to the class of pro-p groups.

Question 6.2. Is every torsion pro-p group of finite exponent?

The last substantial progress on this question was made by Wilson in 1983. Here we try to connect Question 6.1 and 6.2 to the theory of locally compact and totally disconnected groups.

For any profinite group G, a virtual automorphism is an isomorphism among open subgroups of G. We define the set of virtual isomorphisms $\mathrm{VAut}(G)$. Addittionally, we define an equivalence relation $\sim$ on virtual automorphisms by declaring two virtual automorphisms equivalent if they agree on some open subgroup of G. It is easy to show that the set of equivalence classes of virtual automorphisms modulo $\sim$ is a group, called the *abstract commensurator* $\mathrm{Comm}(G)$ of G, with respect to composition on "common open subgroups". In [1], the authors define a topology on $\mathrm{Comm}(G)$ (the strong topology), that makes $\mathrm{Comm}(G)$ into a tdlc topological group. Finally, define the virtual center of G by $\mathrm{VZ}(G) = \{g \in G \mid C_G(g) \leq_o G\}$. If $\mathrm{VZ}(G) = 1$, then G embeds as an open compact subgroup of $\mathrm{Comm}(G)$.

First, it would be interesting to answer the following intermediate question:

Question 6.3. Let G be a torsion pro-p group. Must G have trivial virtual center?

If the answer to Question 6.3 was positive, we could always embed a torsion pro-p group G in its commensurator.

Let G be a torsion pro-p group and suppose that $\mathrm{VZ}(G) = 1$. Then, G is a torsion compact open subgroup in the tdlc group $\mathrm{Comm}(G)$. If such a group existed, this would give a new exotic example of tdlc group. On the other hand, if such a tdlc group could not exist, then this would give a positive answer to Question 6.1 (modulo Question 6.3).

[1] According to personal communication with G. Willis, Kaplanski was think about this question already in 1949.

7 Thomas Weigel

Question 7.1. How efficient is the scale?

Let G be a tdlc group, we will call it a bad guy(or uniscalar) if the scale $s : G \to \mathbb{Q}^*$ is equal to 1 everywhere.

Question 7.2. Are there classes of compactly generated groups $\mathcal{X}$ without bad guys?

Question 7.3. What classes $\mathcal{X}$ have the property where

$$G \in \mathcal{X} \text{ is a bad guy} \implies G \text{ is compact by discrete?}$$

$\mathcal{X} = \{\text{nilpotent groups}\}$ has the property. It is still unclear whether the class of soluble, topologically simple groups or the groups acting on a tree also have the property.

Additionally the class of groups with Cayley-Abels graphs of degree at most 3 has the property.

References

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