SOLVING PROBLEMS OF OPTIMAL STOPPING WITH LINEAR COSTS OF OBSERVATIONS

A. Irle, V. Paulsen

University of Kiel , Department of Mathematics, D-24098 Kiel Germany

Abstract

In sequential analysis and starting with the seminal treatment of the optimality of the SPRT due to Wald and Wolfowitz (1950) various problems of optimal stopping with linear costs of observation arise. Here we shall describe a new method for solving such problems. For a payoff $g(X_t) - ct$ we propose a linear representation of the form

$$g(X_t) - ct = h(X_t) + M_t$$

where M_t is a local martingale, and the function h and the local martingale depend on a parameter λ. In the case of a diffusion process X_t we shall show that, for a proper choice of λ, the boundary points of the optimal stopping region can be obtained from those points of the state space where the maxima of h are located. This method is inspired by a method of Beibel and Lerche (1997,2001) who, for optimal stopping problems with discounted payoff, used a multiplicative representation $e^{-rt}g(X_t) = h(X_t)M_t$ with a suitable martingale.

Key words: Optimal stopping, diffusion, martingale, sequential analysis;

Mathematics Subject Classification (2000):
60G40, 60J60, 62L15, 62L10

1 . INTRODUCTION

Starting from a basic filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ we consider the continuous time optimal stopping problem for an adapted payoff process $(Z_t)_{t\geq 0}$. Hence we want to maximize EZ_τ among all stopping times τ with respect to the filtration for which EZ_τ exists. In the notation of this paper stopping times are, by definition, finite a.s. We consider a problem with linear costs of observation and assume that, for some constant $c > 0$, the payoff process takes the form

$$Z_t = g(X_t) - c\,t,\ t \geq 0.$$

Here $(X_t)_{t\geq 0}$ is another stochastic process with state space E and g is a real-valued measurable mapping on the state space.

Our treatment of this problem is inspired by a method of Beibel and Lerche (1997) where the following approach is used. Assume that the payoff $(Z_t)_{t\geq 0}$ has a multiplicative decomposition of the form $Z_t = h(X_t)M_t$ with a continuous stochastic process $(X_t)_{t\geq 0}$ and a positive martingale $(M_t)_{t\geq 0}$. If h has a maximum uniquely located at some x^* then the first time X_t reaches x^* is an optimal stopping time under suitable conditions. Beibel and Lerche (1997) show that this approach can be applied successfully to a variety of stopping problems with discounting for payoff of the form $e^{-rt}g(X_t)$. In (2001) they extend their results to the case of one-dimensional diffusions under random discounting. Within this framework, i.e. stopping problems for one-dimensional diffusions with discounting, Dayanik, Karatzas (2003) and Dayanik (2003) present a new approach by characterizing excessive functions via generalized concavity. This allows to determine the value function as smallest concave majorant of the reward function and leads to explicit solutions in a variety of problems.

In problems with linear costs of observation, a linear decomposition seems to be more appropriate than one of multiplicative type. Starting point for deriving such a linear decomposition is the following proposition formulated for a general payoff process $(Z_t)_{t\geq 0}$.

1.1 Proposition *If there exist a finite constant B and a stochastic process $(M_t)_{t\geq 0}$ such that*

$$\sup_{t\geq 0}(Z_t - M_t) = B < \infty \qquad a.s.$$

and

$$\tau^* = \inf\{t \geq 0 : Z_t - M_t = B\}$$

is a.s. finite with $EM_{\tau^*} = EM_0$, *then*

$$\begin{aligned} EZ_{\tau^*} &= \sup\{EZ_\tau : \tau \textit{ stopping time with } EM_\tau = EM_0\} \\ &= B + EM_0 \, . \end{aligned}$$

Proof. For each stopping time τ with $EM_\tau = EM_0$ we get

$$EZ_\tau = E(Z_\tau - M_\tau) + EM_0 \le \sup_{t\ge 0}(Z_t - M_t) + EM_0 \le B + EM_0 \, .$$

But the right hand bound is attained by using τ^*. Hence the assertion follows.

Here the question arises whether such processes $(M_t)_{t\ge 0}$ which would typically be local martingales can be found for problems of optimal stopping problems in nontrivial situations. It is the the aim of this paper to show that, as for the Beibel and Lerche (1997) multiplicative decomposition for discounted payoffs, this linear decomposition can successfully be applied to payoffs with linear costs, hence in the realm of sequential analysis. So this method yields an alternative to the method of obtaining optimal continuation regions via free boundary problems. We shall consider the situation where the process

$$X = ((X_t)_{t\ge 0}, (\mathcal{F}_t)_{t\ge 0}, (P_x)_{x\in E}) \, ,$$

is a one-dimensional diffusion on an open interval $E = (r_1, r_2)$ with $-\infty \le r_1 < r_2 \le +\infty$. We refer to Karatzas and Shreve (1988) for the standard facts on diffusions which will be used in this paper. We assume that X is generated by a second order elliptic differential operator

$$A = \frac{1}{2}\sigma^2(x)\partial_x^2 + \mu(x)\partial_x$$

with strictly positive continuous σ and continuous μ. This implies that

$$(f(X_t) - f(x) - \int_0^t Af(X_s)ds)_{t\ge 0}$$

defines a martingale with respect to P_x for each twice continuously differentiable function f on E with compact support and each starting point $x \in E$.

This result, often refered to as Dynkin's identity, will provide the local martingales for our linear decomposition.

The speed measure and scale function of the diffusion are given by

$$m(dx) = \frac{2}{\sigma^2(x)}\exp(\psi(x))dx \, , \; s(x) = \int_a^x \exp(-\psi(y))dy \, ,$$

where $\psi(x) = \int_a^x 2\mu(y)/\sigma^2(y)dy$ and $a \in E$ may be chosen arbitrarily. Note that the diffusion is assumed to be conservative. Thus the boundary points of E are inaccessible and the corresponding semigroup is uniquely determined by its differential generator A.

The scale function s is the unique solution of the equation

$$As = 0 \tag{1}$$

subject to the boundary conditions $s(a) = 0,\ s'(a) = 1$. By Dynkin's identity,

$$(s(X_t))_{t\geq 0}$$

defines a local martingale with respect to P_x for any x.

For $c > 0$ the unique solution of the equation

$$Au = c$$

subject to the boundary conditions $u(a) = 0 = u'(a)$ is given by

$$u(x) = c\int_a^x s'(y)\int_a^y \frac{2}{s'(z)\sigma^2(z)}dzdy \tag{2}$$

for all $x \in E$. Again by Dynkin's identity,

$$(u(X_t) - c\,t)_{t\geq 0}$$

defines a local martingale with respect to P_x for any x.

As the diffusion is conservative, $u(x)$ tends to infinity when x tends to a boundary point. Let us call a stopping time τ regular if there exists a first exit time $\sigma_I = \inf\{t:\ X_t \notin I\}$ from some bounded interval I of E such that

$$\tau \leq \sigma_I.$$

Note that $E_x\sigma_I < \infty$ for bounded I. By Dynkin's identity and optional sampling we obtain for any regular stopping time τ

$$E_x s(X_\tau) = s(x) \text{ and } E_x(u(X_\tau) - c\tau) = u(x) \text{ for all } x.$$

To compare the growth of s and u near the boundaries we look at s/u. Since $u(x) \to \infty$ for x tending to a boundary we obtain by L'Hospital's rule

$$\lim_{x\to r_i} \frac{|s(x)|}{u(x)} = \lim_{x\to r_i} \frac{|s'(x)|}{u'(x)} = 1/\int_a^{r_i} \frac{2}{s'(z)\sigma^2(z)}\, dz = \gamma_i, \text{ say.}$$

Our later considerations will be most clearcut if the scale function s increases slower than u near a boundary. So we formulate the assumption

(A1) $\gamma_i = 0$ for both boundary points r_1, r_2.

We point out the following consequence of this assumption although this will not be used in the sequel: The inaccessible boundary points r_1, r_2 may be classified as being natural boundaries or entrance boundaries. For an entrance boundary r_i, Lemma 10 of Lai (1973) shows that $\gamma_i > 0$. Hence under (A1) both boundaries are natural. It is shown in Lai (1973), Theorem 5, that this holds if and only if $(s(X_t))_{t\geq 0}$ and $(u(X_t) - ct)_{t\geq 0}$ are martingales. So validity of (A1) implies that these processes are not only local martingales, but martingales.

2 . OPTIMAL STOPPING WITH LINEAR COSTS

Let $(X_t)_{t\geq 0}$ be a diffusion as described in the Introduction. Using the filtration generated by the diffusion we consider the optimal stopping problem for the payoff

$$Z_t = g(X_t) - c\,t\,, t \geq 0$$

for a continuous g which is bounded from below and fixed $c > 0$. Stopping the diffusion at time point t in state x gives the reward $g(x)$ whereas taking the observations will cost $c\,t$ leading to $g(x) - c\,t$ as payoff. We are faced with the problem of maximizing the expected payoff

$$E_x(g(X_\tau) - c\,\tau)$$

among all stopping times τ such that this expectation exists. The following assumption ensures that the expected payoff exists for all stopping times and that the optimal value is finite.

(A2) For each $x \in E$ there exists some $\epsilon > 0$ such that

$$E_x \sup_{t\geq 0}(g(X_t) - (c-\epsilon)t) < \infty\,.$$

2.1 Proposition *Assume (A2). Then the optimal value*

$$v(x) = \sup_{\tau} E_x(g(X_\tau) - c\tau)$$

is finite for all $x \in E$ and can be obtained by maximizing over all regular stopping times. Furthermore the first exit time

$$\tau^* = \inf\{t \geq 0 : X_t \notin \mathcal{C}\}$$

from the continuation region

$$\mathcal{C} = \{x : v(x) > g(x)\}$$

is an optimal stopping time.

Proof. Obviously (A2) implies finiteness of v. The optimality of τ^* is supplied by the general theory of optimal stopping, see Shiryayev (1978). Writing

$$E_x Z_\tau = E_x(g(X_\tau) - (c - \epsilon)\tau) - \epsilon E_x \tau$$

it is clear from (A2) that $E_x Z_\tau = -\infty$ if and only if $E_x \tau = \infty$. Hence it is sufficient to maximize over all stopping times with finite expectation. Consider any such stopping time τ.

Choose sequences $(l_n)_n, (s_n)_n$ in the state space with $l_n \downarrow r_1, s_n \uparrow r_2$. Let

$$\sigma_n = \inf\{t : \ t \notin (l_n, s_n)\} \text{ and } \rho_n = \min\{\sigma_n, \tau\}.$$

Then $\rho_n \uparrow \tau$, and Fatou's lemma implies

$$E_x g(X_\tau) \leq \liminf E_x g(X_{\rho_n}).$$

Furthermore

$$\lim E_x \rho_n = E_x \tau < \infty.$$

This yields

$$E_x(g(X_\tau) - c\tau) \leq \liminf E_x(g(X_{\rho_n}) - c\rho_n),$$

which shows that it suffices to maximize over all regular stopping times.

In the following we describe the basic idea how to apply Proposition 1.1 to determine the continuation region $\mathcal{C}$. We use the stochastic process

$$M_t^\lambda = u(X_t) - c\,t - \lambda s(X_t), \tag{3}$$

which is a local martingale for every $\lambda \in I\!R$. The optional sampling property holds for any regular stopping time which, by Proposition 2.1, is sufficient for our purposes. We may choose any λ and with the right choice of λ will be able to apply Proposition 1.1. Then we have the linear decomposition $Z_t = (Z_t - M_t^\lambda) + M_t^\lambda$, and the modified payoff takes the form

$$Z_t - M_t^\lambda = g(X_t) - u(X_t) - \lambda s(X_t) \ ,$$

so that we have removed the cost of observation term. By considering the function

$$h(\lambda, x) = g(x) - u(x) - \lambda s(x)$$

we are left to determine a λ^* such that the supremum of $h(\lambda^*, \cdot)$ is attained at two points $b_l < b_r$, since then, starting from $x \in (b_l, b_r)$, the maximum will be reached in finite time by a regular stopping time. We shall now show how to apply this approach and start with a smooth reward function g.

3 . THE SMOOTH CASE

Here we assume that g is twice continously differentiable. To examine whether the continuation region is non-empty we apply the following proposition. We remark that the continuation region is an open set hence the union of open intervals. Let us call any such non-empty open interval a generating interval of $\mathcal{C}$.

3.1 Proposition *Let (A2) be valid. Then*

(i) The continuation region is empty if and only if

$$Ag(x) \leq c$$

for all $x \in E$.

(ii) Each generating interval of the continuation region contains a point a such that $Ag(a) > c$.

Proof. (i) Assume first $Ag(x) \leq c$ for all $x \in E$. For any regular stopping time τ and any $x \in E$ Dynkin's identity implies

$$\begin{aligned} E_x g(X_\tau) &= g(x) + \int_0^\tau Ag(X_s) ds \\ &\leq g(x) + c\, E_x \tau \quad . \end{aligned}$$

from which $\mathcal{C} = \emptyset$ follows.

Conversely, consider a starting point a such that $Ag(a) > c$. Then there exists an interval $I = (a - \epsilon, a + \epsilon)$ with $r_1 < a - \epsilon, a + \epsilon < r_2$ such that $Ag(x) > c$ for all $x \in I$. Using the first exit time σ_I yields for every $x \in I$

$$v(x) \geq E_x g(X_{\sigma_I}) - c E_x(\sigma_I) = g(x) + \int_0^{\sigma_I} Ag(X_s) ds - c\, E_x \sigma_I > g(x) \ ,$$

hence $I \subset \mathcal{C}$.

(ii) Consider a generating intervall I^*. Then for any starting point $x \in I^*$

$$v(x) = E_x(g(X_{\sigma^*}) - c\sigma^*) ,$$

where σ^* is the first exit time from I^*. Necessarily we have $E_x\sigma^* < \infty$. But if $Ag(x) \leq c$ for all $x \in I^*$ we obtain a contradiction as in (i).

We may conclude that in the case of a non-empty continuation region there exists an $a \in E$ such that $Ag(a) > c$. This a is the starting point for the following considerations and is also used to define u and s, compare with (1), (2).

To obtain suitable maxima the following assumption is needed.

(S1) $\lim_{x \to r_i} \frac{g(x)}{u(x)} = 0$ for both boundary points r_1, r_2.

This means that g increases slower than u near the boundaries. We recall that this is also true for the scale function s under assumption (A1).

The following arguments will provide the maxima we need.

3.2 Lemma *If $\lambda = g'(a)$, then $x \mapsto h(\lambda, x)$ has a strict local minimum at $x = a$.*

Proof. This follows from

$$g'(a) - u'(a) - \lambda s'(a) = g'(a) - \lambda = 0,$$

$$Ag(a) - Au(a) - \lambda As(a) = Ag(a) - Au(a) = Ag(a) - c > 0 .$$

(A1) and (S1) imply that for each $\lambda \in \mathbb{R}$ the function

$$h(\lambda, x) = g(x) - u(x) - \lambda s(x)$$

is bounded above and attains its supremum. Under these assumptions we deduce that for $\lambda = g'(a)$ the function $h(\lambda, \cdot)$ has at least two local maxima, one located at a point smaller than a and the other at a point larger than a. We will now change λ in such a way that we obtain global maxima to the left and to the right of a whose values of course must coincide. Therefore we introduce the functions β_r, β_l by

$$\beta_r(\lambda) = \sup_{x>a} h(\lambda, x) = \sup_{x>a}(g(x) - u(x) - \lambda s(x)) \tag{4}$$

and similarly

$$\beta_r(\lambda) = \sup_{x<a} h(\lambda, x) \ . \tag{5}$$

3.3 Proposition *Assume (A1) and (S1). Then there exist λ^* and $b_l(\lambda^*) < a < b_r(\lambda^*)$ such that*

$$h(\lambda^*, b_l(\lambda^*)) = \beta_l(\lambda^*) = \beta_r(\lambda^*) = h(\lambda^*, b_r(\lambda^*)) \ .$$

Proof. As suprema of linear functions, β_l and β_r are convex hence continuous. Due to $s(x) > 0$ for $x > a$ and $s(x) < 0$ for $x < a$, the function β_r is decreasing from $+\infty$ to $-\infty$ whereas β_l is increasing from $-\infty$ to $+\infty$. Thus there exists λ^* with

$$\beta_l(\lambda^*) = \beta_r(\lambda^*) \quad .$$

Since the suprema are attained there exists some $b_l(\lambda^*) \leq a \leq b_r(\lambda^*)$ such that

$$h(\lambda^*, b_l(\lambda^*)) = \beta_l(\lambda^*) = \beta_r(\lambda^*) = h(\lambda^*, b_r(\lambda^*)) \quad .$$

It follows

$$b_l(\lambda^*) < a < b_r(\lambda^*)$$

since otherwise $h(\lambda^*, \cdot)$ would have a global maximum in a. This would imply $\lambda^* = g'(a)$ contradicting Lemma 3.2.

The above points $b_l(\lambda^*), b_r(\lambda^*)$ might not be unique but we choose those with minimal distance to a and denote them by

$$b_l = b_l(\lambda^*), \ b_r = b_r(\lambda^*). \tag{6}$$

These considerations result in the following theorem describing the continuation region.

3.4 Theorem *Assume (A1),(A2),(S1). Let g be twice continuously differentiable. Let I be a generating interval of the continuation region. Choose $a \in I$ with $Ag(a) > c$. Define λ^*, b_l, b_r with respect to a as in (6). Then*

$$I = (b_l, b_r)$$

and

$$v(x) = \beta_l(\lambda^*) + u(x) - \lambda^* s(x) \text{ for all } x \in (b_l, b_r).$$

Proof. Due to Proposition 3.1, I contains a point a such that $Ag(a) > c$. Starting with a, we determine as explained above λ^*, b_l, b_r such that $x \mapsto g(x) - u(x) - \lambda^* s(x)$ attains its maximum at the points b_l, b_r.

Note that $r_1 < b_l, b_r < r_2$. To apply Proposition 1.1 we consider the martingale M_{λ^*} defined in (3). For each starting point x, the first exit time from (b_l, b_r)

$$\tau^* = \inf\{t \geq 0 : X_t \notin (b_l, b_r)\}$$

is a.s. finite and, being regular, fulfills

$$E_x M_{\tau^*}^{\lambda^*} = E_x M_0^{\lambda^*} = u(x) - \lambda^* s(x) \ .$$

Now

$$B = \sup_{t \geq 0}(Z_t - M_t^{\lambda}) = \sup_x (g(x) - u(x) - \lambda^* s(x)) = \beta_l(\lambda^*),$$

and for every $x \in (b_l, b_r)$ we have

$$\tau^* = \inf\{t \geq 0 : Z_t - M_{\lambda^*}(t) = B\}.$$

For every regular stopping time $E_x M_\tau^{\lambda^*} = E_x M_0^{\lambda}$ and by Proposition 2.1

$$v(x) = \sup\{E_x Z_\tau : \ \tau \text{ regular } \}.$$

Hence by Proposition 1.1

$$v(x) = E_x Z_{\tau^*} = \beta_l(\lambda^*) + u(x) - \lambda^* s(x).$$

So we have optimality of τ^* when starting from $x \in (b_l, b_r)$.

Clearly (b_l, b_r) is contained in I since the immediate reward $g(x)$ is strictly less than $\beta_l(\lambda^*)$ for $x \in (b_l, b_r)$. To prove $I \subset (b_l, b_r)$ we note that both the first exit times from I and from (b_l, b_r) are optimal when starting from $x \in (b_l, b_r)$. But it is well known that the first exit time from the continuation region is the minimal optimal stopping time. Thus $I \subset (b_l, b_r)$ also holds.

This result leads to the following recursive procedure to determine the whole continuation region $\mathcal{C}$.

Step 1: Initialize $\mathcal{C} = \emptyset$

Step 2: Determine an $a \in E \setminus \mathcal{C}$ such that $Ag(a) > c$.
If this is not possible, stop; otherwise continue with step 3.

Step 3: Determine, with respect to a, the unique λ^*, b_l, b_r and put $\mathcal{C} = \mathcal{C} \cup (b_l, b_r)$. Goto step 2.

If the continuation region is connected then we obtain the optimal stopping region in the first round.

4 . THE NON-SMOOTH CASE

In many applications, the reward function g is continuously differentiable except for one point $a \in E$. This is e.g. the case if it has the form $g(x) = G(|x-a|)$ or $g(x) = G((x-a)^+)$ for a continuously differentiable function G. In the following we will give sufficient conditions under which a is contained in an open interval I of the continuation region and we will determine such an I explicitly. In order to adjust the considerations of the smooth case we suppose the following condition on g in a.

(S2) $g'(a-) = \lim_{x \uparrow a} g'(x) < \lim_{x \downarrow a} g'(x) = g'(a+)$

Note that a is again used to define the functions u and s. As in the smooth case there exists a unique λ^* and $b_l(\lambda^*) \le a \le b_r(\lambda^*)$ such that

$$\sup_{x<a} h(\lambda^*, x) = h(\lambda^*, b_l(\lambda^*)) = h(\lambda^*, b_r(\lambda^*)) = \sup_{x>a} h(\lambda^*, x) \ .$$

Due to

$$h'(\lambda^*, a-) = g'(a-) - \lambda^* < g'(a+) - \lambda^* = h'(\lambda^*, a+)$$

(S2) shows that the points $b_l(\lambda^*), b_r(\lambda^*)$ where the maxima are located do not coincide with a and again we may choose $b_l < a < b_r$ which are points of maxima with minimal distance to a. Thus, as in the smooth case, we can apply Proposition 1.1 to obtain the following theorem.

4.1 Theorem *Let g be continuously differentiable except for the point $a \in E$. Assume (A1), (A2), (S1), (S2). Define λ^*, b_l, b_r with respect to a as explained above. Then a is contained in a generating interval I of the continuation region and*

$$I = (b_l, b_r).$$

Furthermore

$$v(x) = \beta_l(\lambda^*) + u(x) - \lambda^* s(x) \text{ for all } x \in (b_l, b_r).$$

In various applications the function G is linear to the left and to the right of a. If the diffusion has drift term $\mu(x) = 0$ then $Ag(x) = 0$ for $x \neq a$ hence the continuation region is given by $I = (b_l, b_r)$. In other cases we may repeat the procedure on (r_1, b_l) and (b_r, r_2) as explained in the sequel of Theorem 3.4.

5 . APPLICATIONS

We will show how our approach may be used in various examples, the first two stemming from sequential analysis.

5.1 Example

Consider the problem of determining a locally best sequential test for the drift of a Wiener process. As described in Irle (1981) this leads to the optimal stopping problem for a driftless Wiener process with payoff

$$f(x,t) = (x-a)^+ - ct\,.$$

The functions s, u subject to the boundary conditions at $a \in \mathbb{R}$ are given by

$$s(x) = x - a\,,\ u(x) = c(x-a)^2\,.$$

We may apply Theorem 4.1, its assumptions being clearly fulfilled. The functions β_l, β_r attain their maxima at

$$b_l(\lambda) = a - \frac{\lambda}{2c}\,,\ b_r(\lambda) = a + \frac{1-\lambda}{2c}$$

for $\lambda \in (0,1)$. Their maximal values coincide if

$$-c(\frac{\lambda}{2c})^2 + \lambda\frac{\lambda}{2c} = \frac{1-\lambda}{2c} - (\frac{1-\lambda}{2c})^2 c - \lambda\frac{1-\lambda}{2c},\ \text{hence for } \lambda^* = \frac{1}{2}\,.$$

It follows that the continuation region is given by

$$\mathcal{C} = (a - \frac{1}{4c}, a + \frac{1}{4c})$$

due to $Ag(x) = 0$ for $x \neq a$.

5.2 Example

We look at the problem of sequentally testing simple hypotheses for the drift of a Wiener process, see Shiryayev (1978), pp. 180. For a given prior probability $\pi \in (0,1)$ the process of posterior probabilities (X_t^π) fulfills the stochastic differential equation

$$dX_t^\pi = X_t^\pi(1 - X_t^\pi)dW_t\,,\; X_0^\pi = \pi\,.$$

This defines a diffusion with state space $(0,1)$ and generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2\,.$$

For determining an optimal Bayes test we have to consider the optimal stopping problem for this diffusion with payoff

$$f(x,t) = \max\{\delta(a-x), \gamma(x-a)\} - ct\,.$$

The positive constants δ, γ denote the loss for a wrong decision and $a = \frac{\gamma}{\gamma+\delta}$. Subject to a the functions s, u are given by

$$\begin{aligned} s(x) &= x - a, u(x) = cw(x) \quad \text{with} \\ w(x) &= 2(2x-1)\log\frac{x}{1-x} - (4\log\frac{a}{1-a} + \frac{2(2a-1)}{a(1-a)}(x-a)) \\ &\quad -2(2a-1)\log\frac{a}{1-a}. \end{aligned}$$

Note that

$$w'(x) = 4\log\frac{x}{1-x} + \frac{2(2x-1)}{x(1-x)} - (4\log\frac{a}{1-a} - \frac{2(2a-1)}{a(1-a)})$$

and that for $\lambda \le -\delta, \lambda \ge \gamma$ $\sup_{x<a} h(x,\lambda) \ne \sup_{x>a} h(x,\lambda)$. Hence we only have to consider $-\delta < \lambda < \gamma$.

The equation

$$w'(x) = \frac{-\lambda - \delta}{c}$$

has a unique solution $x = b_l(\lambda) < a$ if $\lambda > -\delta$ whereas

$$w'(x) = \frac{-\lambda + \gamma}{c}$$

can be uniquely solved by some $b_r(\lambda) > a$ if $\lambda < \gamma$. Hence to determine λ^* we solve numerically

$$h(\lambda, b_l(\lambda)) = h(\lambda, b_r(\lambda)) \quad \text{on } (-\delta, \gamma).$$

Then the continuation region is given by $(b_l(\lambda^*), b_r(\lambda^*))$ due to $Ag(x) = 0$ for $x \ne a$.

5.3 Example

Consider the following problem arising in mathematical finance. Investigating portfolio optimization under transaction costs, Morton and Pliska (1995) were led to the optimal stopping problem for the diffusion with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + x(1-x)(\frac{1}{2}-x)\partial_x$$

and payoff

$$f(x,t) = \log\frac{1}{1-x} - c\,t\,.$$

Obviously for $g(x) = \log 1/(1-x)$ the function $Ag(x)$ has its unique maximum at $x = 1/2$. So we assume $Ag(1/2) > c$ as otherwise the continuation region will be empty.

For $a = 1/2$ the functions s, u are given by

$$s(x) = \log\frac{x}{1-x}\,,\quad u(x) = c\,(\log\frac{x}{1-x})^2\,.$$

We note that $g(x) - s(x)/2$ and u are even functions with respect to $a = 1/2$ which gives

$$\lambda^* = -\frac{1}{2}\,.$$

We obtain b_l, b_r by solving

$$\frac{1}{1-x} - u'(x) - \lambda^* s'(x) = 0\,.$$

Since $Ag(x) \le c$ for x outside (b_l, b_r) we obtain

$$\mathcal{C} = (b_l, b_r)\,.$$

5.4 Example

In this example we can compute an infinite sequence of generating intervalls of the continuation set. Consider driftless Brownian motion with $g(x) = -\cos(x)$ as reward function. Then $Ag(x) \le \frac{1}{2} = Ag(0) = Ag(2\pi n)$ for all $n \in \mathbb{Z}$. Hence the continuation region is empty if $c \ge \frac{1}{2}$. To apply our approach to $c < \frac{1}{2}$ we consider $a = 2n\pi$ for some $n \in \mathbb{Z}$ as starting point. Then

$$u(x) = c(x-a)^2 \quad , \quad g(x) = -\cos x = -\cos(x-a)$$

are even functions with respect to a. If we put $\lambda^* = 0$ and denote by b the unique solution of $\sin x = 2cx$ in $(0, \pi)$ then

$$g'(x) - u'(x) = \sin(x-a) - 2c(x-a) = 0$$

has solutions $a-b, a+b$ in $(a-\pi, a+\pi)$.

Due to $Ag(x) \leq c$ for $x \in (a-\pi, a+\pi) \setminus (a-b, a+b)$ we obtain the continuation set

$$\mathcal{C} = \bigcup_{n \in \mathbb{Z}} (2n\pi - b, 2n\pi + b) \quad .$$

6 . FURTHER EXTENSIONS

We shall show how to extend the forgoing approach.

6.1 Extensions to other diffusions

Some important examples like Brownian motion with drift or geometric Brownian motion do not fulfill condition (A1). A slight refinement is necessary to cover these examples. This will be explained in the following. In the Introduction we have noted that $\frac{|s(x)|}{u(x)}$ tends to $\gamma_1, \gamma_2 \in [0, \infty)$ as x tends to the boundaries r_1, r_2. For $\alpha_1 = \frac{1}{\gamma_1}, \alpha_2 = \frac{1}{\gamma_2}$ we obtain

$$\lim_{x \to r_1} \frac{u(x)}{|s(x)|} = \alpha_1, \lim_{x \to r_2} \frac{u(x)}{|s(x)|} = \alpha_2. \tag{7}$$

We keep the assumptions $(A2)$ and $(S1)$ on the reward function g and consider the smooth and non-smooth case simultanously. From a chosen starting point $a \in E$, we define the functions β_l, β_r as in (4), (5) and can conclude:

$$\begin{aligned} \beta_r(\lambda) &< \infty \quad \text{if} \quad \lambda > -\alpha_2, \\ \beta_l(\lambda) &< \infty \quad \text{if} \quad \lambda < \alpha_1, \\ \beta_r(\lambda) &= +\infty \quad \text{if} \quad \lambda < -\alpha_2, \\ \beta_l(\lambda) &= +\infty \quad \text{if} \quad \lambda > \alpha_1. \end{aligned}$$

Thus, β_l, β_r are strictly decreasing respectively increasing functions on $(-\alpha_2, \alpha_1)$. To get a unique interception point we assume

(A3) $\beta_r(\lambda_1) > \beta_l(\lambda_1)$, $\beta_r(\lambda_2) < \beta_l(\lambda_2)$ for some $\lambda_1 < \lambda_2 \in (-\alpha_2, \alpha_1)$.

Then there exists a unique point $\lambda^* \in (-\alpha_2, \alpha_1)$ such that

$$\beta_l(\lambda^*) = \beta_r(\lambda^*).$$

Due to $-\alpha_2 < \lambda^* < \alpha_1$ it follows

$$\lim_{x \to r_2} g(x) - u(x) - \lambda^* s(x) \quad = \quad -\infty \; ,$$

$$\lim_{x\to r_1} g(x) - u(x) - \lambda^* s(x) \quad = \quad -\infty.$$

Thus the functions β_l, β_r attain there maxima at finite points $b_l(\lambda^*) < a < b_r(\lambda^*)$ and as in the preceding section we may conclude that $(b_l(\lambda^*), b_r(\lambda^*))$ coincides with that generating interval of the continuation region that contains a.

We may summarize that our approach remains valid for the general case if the assumption (A1) is replaced by (A3). It also applies to diffusions with entrance boundaries.

6.2 Example

Consider Brownian motion with drift as one example where our extension yields a solution. For simplicity we assume drift and variance equal to one. Thus, the generator is given by

$$A = \frac{1}{2}\partial_x^2 + \partial_x \, .$$

With respect to $a = 0$, the functions u, s are given by

$$\begin{aligned} u(x) &= c\,(-\frac{1}{2} + x + \frac{1}{2}e^{-2x}), \\ s(x) &= \frac{1}{2} - \frac{1}{2}e^{-2x}. \end{aligned}$$

Hence

$$\alpha_1 = \frac{1}{c}, \alpha_2 = +\infty.$$

We consider the reward function

$$g(x) = \max\{-x, 0\}.$$

To obtain the points of maximum for $h(\lambda, x) = g(x) - u(x) - \lambda^* s(x)$ we have to solve

$$-1 = c - ce^{-2x} + \lambda e^{-2x}$$

and

$$0 = c - ce^{-2x} + \lambda e^{-2x}$$

respectively. The solutions are

$$b_l(\lambda) = \frac{1}{2}\log(\frac{1-\frac{\lambda}{c}}{1+\frac{1}{c}}), b_r(\lambda) = \frac{1}{2}\log(1 - \frac{\lambda}{c}).$$

To compute the interception point λ^* we solve

$$g(b_l(\lambda)) - u(b_l(\lambda)) - \lambda^* s(b_l(\lambda)) = g(b_r(\lambda)) - u(b_r(\lambda)) - \lambda^* s(b_r(\lambda))$$

obtaining

$$\lambda^* = c - \frac{1}{e}(1+c)(\frac{1+c}{c})^c < 0.$$

By inserting into the avove equations we obtain, after some further calculations

$$b_l(\lambda^*) = \frac{1}{2}c\log(\frac{1+c}{c}) - \frac{1}{2}, b_r(\lambda^*) = \frac{1}{2}(1+c)\log(\frac{1+c}{c}) - \frac{1}{2}.$$

Due to $Ag(x) \leq 0$ for $x \neq 0$ the continuation region coincides with $(b_l(\lambda^*), b_r(\lambda^*))$.

6.3 Extension to unbounded g

In the forgoing we have assumed that g is bounded from below. This allows us to restrict the maximization to regular stopping times by applying Fatou's lemma. Now consider a general g, of course keeping the other assumptions.

Assume that for some a with $Ag(a) > c$ we have obtained λ^*, b_l and b_r. Now switch from g to g_1 such that $g_1 \geq g, g_1$ bounded below, $g_1|_{[b_l,b_r]} = g|_{[b_l,b_r]}$ and g_1 has the differentiability properties of g. Then for

$$Z_t^{(1)} = g_1(X_t) - ct, t \geq 0,$$

with optimal value v_1 we obtain for

$$\tau^* = \inf\{t : X_t \notin (b_l, b_r)\}$$

$$v(x) \leq v_1(x) = E_x(Z_{\tau^*}^{(1)}) = E_x(Z_{\tau^*}).$$

Hence optimality of τ^* holds also for g which is not necessarily bounded below.

Acknowledgement.

We thank the refree for pointing out the papers by Dayanik and Karatzas (2003) and Dayanik (2003) to us.

References

1. Beibel, M; Lerche, H.R. A new look at optimal stopping problems related to mathematical finance. Statistica Sinica **1997** *7, 93-108.*

2. Beibel, M; Lerche, H.R. Optimal stopping of regular diffusions under random discounting. Theory Probab. Appl. **2001** *45(4), 547-557.*

3. Dayanik, S; Karatzas, I. On the optimal stopping problem for one-dimensional diffusions. Stochastic Process. Appl. **2003** To appear.

4. Dayanik, S. Optimal stopping of linear diffusions with random discounting. **2003** Preprint

5. Irle, A. Locally most powerful sequential tests for stochastic processes; Stochastic Processes and their Applications. **1981** *11, 285-291.*

6. Karatzas, I; Shreve S.E. *Brownian Motion and Stochastic Calculus.* **1988** Second Edition Springer.

7. Lai, T.L. Space-time processes, parabolic functions and one dimensional diffusions. Trans. Amer. Math. Soc. **1973** *175, 409-438* .

8. Morton, A.J. ; Pliska, S. Optimal portfolio management with fixed transaction costs. Mathematical Finance. **1995** *5, 337-356.*

9. Shiryaev, A.N. *Optimal Stopping Rules.* **1978** Springer.

10. Wald, A.; Wolfowitz J. Bayes solutions of sequential decision problems Ann. Math. Statist. **1950** *21, 82-99.*