Designing and managing unit-linked life insurance contracts with guarantees

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Abstract

We consider the insurance companies' problem of optimal management of unit-linked life insurance contracts with a guarantee (ULLIG). The premium inflow from the customer is invested to build a customer-specific portfolio. The minimal guarantee obliges the insurance company to pay the difference "shortfall" between guaranteed sum and the actual accumulated portfolio value. We formulate the optimal management problem as a multistage stochastic optimization problem with objective of expected portfolio maximization, with the inclusion of a high penalty term for the shortfall.

The model may also used for assessing the shortfall risk and for the optimal design of new contracts. In the latter case, the parameters of the contract are determined that they are most favorable for the customer, but under a risk constraint for the insurance company.

In a theoretical section, we study a continuous time model of optimal portfolio allocation under guarantee in the case of regular premium inflows and random benefit outflows.

Key words: unit-linked life insurance, financial guarantees, multi-stage stochastic programming, optimal investment strategies

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1 Introduction

Unit-linked life insurance contracts with guarantee (ULLIG) combine insurance and investment so that by sacrificing some of the upside gain, the client gets some guarantee on the downside. This feature makes such contracts attractive for a wide, mostly risk averse public.

Here are typical conditions for such a contract:

The client pays a deposit b at the beginning of the contract and an annual premium B at the beginning of the subsequent years $2, 3, \ldots, T$ (The premium may also be paid in monthly or quarterly installments). By setting b or B to zero, we get the special cases of regular annual payment resp. single installment. The premium inflow is split into an insurance part and an investment part. The latter is used to build a customer-specific portfolio Y_t consisting of shares of the reference fund as well as bonds, which is rebalanced at regular times.

If the client dies in year t before the maturity date of the contract (which is the end of year T), her/his legal successors get the death benefit D_t and the contract expires. If the client survives the maturity date, she/he gets a survival benefit S, which is the maximum of the actual portfolio value Y_T and some guaranteed sum G_T.

The death benefit as well as the guaranteed survival benefit may depend on the performance of the reference fund Z_t within the contract period. These values are determined by the *death benefit formula*

$$D_t = D_t(b, B, Z_0, Z_1, \ldots, Z_t) \tag{1}$$

and the *minimal guarantee formula*

$$G_T = G(b, B, Z_0, \ldots, Z_T). \tag{2}$$

Both formulas as well as the specification of the reference fund are part of the contract. The survival benefit S is

$$S = \max(Y_T, G_T).$$

Examples for death and survival benefit formulas are presented in section 2.2.

Some contracts give also the right to lapse the contract and specify the lapse times and the surrender values. In this paper, this option is not considered.

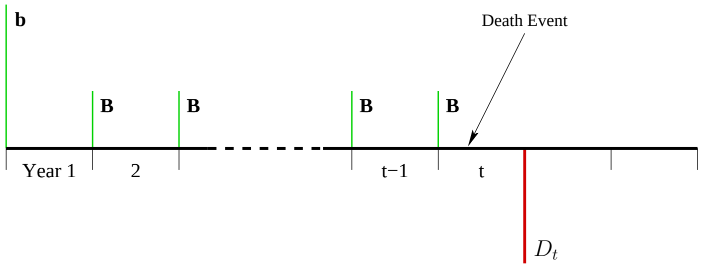

Fig. 1. The cash flows, if the customer dies in year t.

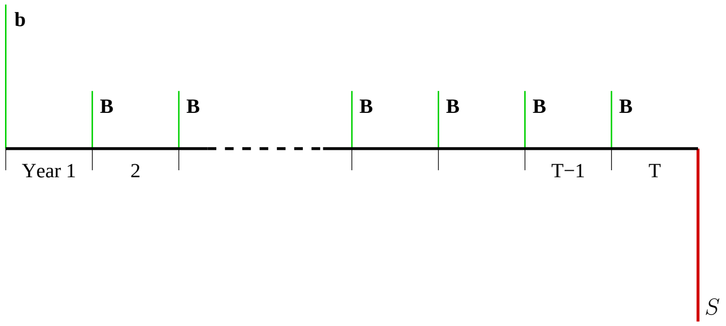

Fig. 2. The cash-flows, if the customer survives year T

The cash-flows originating from such a contract are shown in Figures 1 and 2. A bar in the up direction symbolizes a positive cash flow for the insurance company, a downwards bar is a negative cash-flow.

This paper discusses the methodology of optimally managing ULLIG contracts on the customer level, to find the optimal allocation of the insurance part and the investment part as well as the composition of the investment portfolio. As a byproduct, the model allows to calculate the profit margins for the insurance company as well as to design new ULLIG products.

The first papers about products with minimal guarantee concentrated on pricing such contracts, such as Brennan and Schwartz (1976) and Boyle and Schwartz (1977).

More recent papers about pricing are Miltersen and Persson (1999), Grosen and Jørgensen (2000), Jensen et al. (2001). Specific features of contracts due to legal pecularities were described by Bacinello (2001), Susinno and Giraldi (2000), Siglienti (2000), for the Italian market and Chadburn (1997) for the UK market.

The stochastic programming approach, which is very suitable for solving long-term Asset Liability Management models (see Ziemba and Mulvey (1998), Ziemba (2003) and e.g. Høyland and Wallace (2005) in this volume) was also successfully applied to handle guarantee-based products, see e.g. Consiglio et al. (2001) and Consiglio et al. (2005b) with application to the UK market in Consiglio et al. (2002) and to the Italian market in Consiglio et al. (2003).

In Chapter 5 of this volume (Consiglio et al. (2005a)) the Prometeia model for the Italian market is proposed. In this model, the initial capital, which consists of the initial equity captial plus the premium payments of the customers, is invested into a portfolio, which consists of stocks and bonds. The initial portfolio portfolio composition is kept constant for the full holding period. Only single payment contracts are considered. Neither, a premium inflow from customers to the company, nor a rebalancing due to bad performance takes place at later stages. Liabilities grow at market rate, but not slower than the minimal guaranteed rate. Lapse probabilities may depend on the market performance (variable lapse), mortality is only considered through its expectation. If the funds are not sufficient, new equity captial has to be put into the company to cover the liablities. The objective is to maximize the Certainty Equivalent Excess Return on Equity (CEexROE) for shareholders using the Log or Power Utility function. The excess return is defined as $\frac{\text{assets--liabilities}}{\text{equity capital}}$ at the terminal stage.

In contrast, our model is based on the following design principles:

- The model is on the level of single contracts. Specific features of single contracts like age and gender of the client, single installment or periodic installments are taken into account.
- Mortality risk is a separate risk factor independent of market risk and is modeled by fully coupling the event risk model with market risk model.
- A dynamic portfolio strategy is considered. The portfolio is rebalanced at predetermined stages.
- The portfolio decision does not only concern the allocation of assets, but also the amount invested in short term conventional life insurance.

Pricing of contingent claims and dynamic management of portfolios are just the two sides of the same coin. Since the survival and and death benefits are contingent on the tradable fund value, we may consider the problem as a pricing problem for contingent claims, which depend on an underlying fund and mortality.

Pricing and managing of contingent claims by stochastic optimization

Suppose that a claim S_T contingent to a stock price Z_T must be paid by some financial agent at time T.

The pricing problem for S_T finds the minimal initial capital v needed to superreplicate S_T with a portfolio of stocks (fund values) and bonds, i.e.

$$\text{Minimize in } x_s, y_s : v \tag{3a}$$
subject to
$$x_0 + y_0 Z_0 \leq v \tag{3b}$$
$$x_{s-1}(1+r) + y_{s-1} Z_s \geq x_s + y_s Z_s;\ 1 \leq s < T \tag{3c}$$
$$x_T + y_T Z_T \geq S_T. \tag{3d}$$

Here x_s denotes the amount invested in a risk-free asset (with return r) and y_s is the number of shares bought from the stock or fund in year s.

The correct price of a contingent claim does not apply the formula: "Price equals the expected, discounted cash-flow". Whenever replication (or super-replication) with already priced instruments is possible, then the price has to be determined by an optimal management problem of type (3).

The Black-Scholes formula (see e.g. Karatzas and Shreve (1998), p. 49) is the limit of the optimal values of optimization programs of type (3), when the number of time steps between 0 and the maturity time T tends to infinity and the process Z_s converges in distribution to the geometric Brownian Motion (GBM).

If the initial capital v_0 is given and is larger than the minimal capital v required by (3), the usual way of formulating an optimal management problem is to optimize the expected utility of excess over the claim i.e.

$$\text{Maximize in } x_s, y_s : \mathbb{E}(U(x_T + y_T Z_T - S_T)) \tag{4a}$$
subject to
$$x_0 + y_0 Z_0 \leq v_0 \tag{4b}$$
$$x_{s-1}(1+r) + y_{s-1} Z_s \geq x_s + y_s Z_s;\ 1 \leq s < T \tag{4c}$$
$$x_T + y_T Z_T \geq S_T \tag{4d}$$

where U is an utility function. In this formulation, there are no transaction costs, but by a slight extension, they may be built in.

2 A discrete time decision model

Pricing and management of ULLIG contracts follows the general pattern presented in the introduction. The management model for these contracts, which is developed in this paper, consists of the following building parts:

- a mortality model,
- a stochastic fund value model,
- a stochastic interest rate model.

Mortality depends on gender and age of the customer. Let τ be the residual lifetime variable of the customer at the beginning of the contract. If death occurs in year t, then $\tau = t$.

Let $\pi_t^{(D)} = \mathbb{P}\{\tau = t\}$ be the death probabilities and $\pi_t^{(S)} = \mathbb{P}\{\tau > t\}$ the survival probabilities, all conditional on the fact that the customer is alive at the beginning of the contract.

The death probabilities can be found using mortality tables. Let $q_a^{(s)}$ be the yearly hazard rates in published such tables, where a is the age of the customer at the beginning of the contract and $s = m, f$ is his/her gender. Then $q_a^{(s)}$ and $\pi_t^{(D)}$ respectively $\pi_t^{(S)}$ are related by

$$\begin{aligned} \pi_t^{(D)} &= (1 - q_{a+1}^{(s)})(1 - q_{a+2}^{(s)}) \dots (1 - q_{a+t-1}^{(s)}) q_{a+t}^{(s)} \qquad & 1 \le t \le T \\ \pi_t^{(S)} &= (1 - q_{a+1}^{(s)})(1 - q_{a+2}^{(s)}) \dots (1 - q_{a+t-1}^{(s)})(1 - q_{a+t}^{(s)}) \qquad & 1 \le t \le T. \end{aligned}$$

For very long term contracts, cohort specific projected mortality tables must be used.

The fund value model and the interest rate model are typically estimated using historic data and/or expert opinion. Plethora of models have been developed in the past years to model financial time series, among them are simple random walk models, ARMA models, GARCH models with all its variants, diffusion processes, jump-diffusion processes and much more. Since we use a computational approach, there are no limitations for the type of model for the fund process and the interest rate process. In the second part of the paper, we follow an analytic approach and specialize the fund process to a geometric Brownian Motion and the interest rate process to a deterministic process.

Once the models for Z_t are determined, the contingent values for the death benefits D_t, using the death benefit formula (1) and the minimal guarantee G_T using the minimal guarantee formula (2) can be calculated.

At every time of decision the insurance company may decide to restructure the customers portfolio. The total capital at time t may be invested in three investment forms:

- in the reference fund Z_t,
- in bonds accruing a random interest R_t,
- in conventional death insurance with contract duration of one year. In case of death, the insurance pays a sum of α_t for each unit of premium in the subsequent year. We use the simple formula $\alpha_t = 0.95/q^{(s)}_{a+t-1}$, where a is the age and s is the gender of the customer at beginning of the contract. An extension to more complicated formulas can be easily made.

Let us first consider the cash-flow process of the company only in the survival case. This process consists in income of size b at the beginning of the contract and of B every subsequent year as well as a payment of S at the end of year T. The insurance company builds a portfolio of consisting of x_t invested in bonds and y_t pieces of the fund (Z_t). An amount of w_t goes to insurance. Suppose that, following its portfolio strategy, the insurance company has built a individual portfolio value of Y_T. The objective is to maximize

$$\mathbb{E}([Y_T - G_T]^+ - \delta[Y_T - G_T]^-)$$

under the financing constraints for the portfolio. Here $\delta > 1$ is a penalty for shortfall. The optimization problem is

$$\begin{aligned}
&\text{Maximize in } x_s, y_s, w_s: \ \mathbb{E}([Y_T - G_T]^+ - \delta[Y_T - G_T]^-) && \text{(5a)}\\
&\text{subject to} && \\
&Y_0 = \gamma_1 \cdot b && \text{(5b)}\\
&Y_0 = x_0 + y_0 Z_0 + w_0 && \text{(5c)}\\
&Y_s = (x_{s-1}(1 + R_{s-1}) + y_{s-1} Z_s + B)\gamma_2 \geq x_s + y_s Z_s + w_s; 1 \leq s < T && \text{(5d)}\\
&Y_T = x_{T-1}(1 + R_{T-1}) + y_{T-1} Z_T && \text{(5e)}\\
&x_t, y_t, w_t \geq 0. && \text{(5f)}
\end{aligned}$$

The constants $0 < \gamma_1, \gamma_2 \leq 1$ represent the net factors, after deduction of management fees, taxes, etc.

In model (5), the insurance part w_t does not play a role and the optimal choice is $w_t = 0$ for all t.

The full model considers the death events as well. If the residual lifetime of the customer is $\tau = t$, then the payment is D_t at the end of year t and the whole process stops. We use a technical discount rate r to compare payments at different times. The full model is

Maximize in x_s, y_s, w_s :

$$\sum_{t=1}^{T}(1+r)^{-t}\pi_t^{(D)}\mathbb{E}([Y_t+\alpha_t w_{t-1}-D_t]^+ - \delta[Y_t+\alpha_t w_{t-1}-D_t]^-|\tau=t)$$

$$+(1+r)^{-T}\pi_T^{(S)}\mathbb{E}([Y_T-G_T]^+ - \delta[Y_T-G_T]^-|\tau>T) \quad (6a)$$

subject to

$$Y_0=\gamma_1\cdot b \quad (6b)$$

$$Y_0=x_0+y_0Z_0+w_0 \quad (6c)$$

$$Y_s=(x_{s-1}(1+R_{s-1})+y_{s-1}Z_s+B)\gamma_2 \geq x_s+y_sZ_s+w_s; 1\leq s<T \quad (6d)$$

$$Y_T=x_{T-1}(1+R_{T-1})+y_{T-1}Z_T \quad (6e)$$

$$x_t, y_t, w_t \geq 0. \quad (6f)$$

Alternatively, one could use the interest rate process R_t itself to discount future payments.

Maximize in x_s, y_s, w_s :

$$\sum_{t=1}^{T}\pi_t^{(D)}\mathbb{E}((1+R_{t-1})^{-1}\cdots(1+R_0)^{-1}\Big([Y_t+\alpha_t w_{t-1}-D_t]^+$$

$$-\delta[Y_t+\alpha_t w_{t-1}-D_t]^-\Big)|\tau=t)$$

$$+(1+r)^{-T}\pi_T^{(S)}\mathbb{E}((1+R_{T-1})^{-1}\cdots(1+R_0)^{-1}\Big([Y_T-G_T]^+$$

$$-\delta[Y_T-G_T]^-\Big)|\tau>T) \quad (7a)$$

subject to

$$Y_0=\gamma_1\cdot b \quad (7b)$$

$$Y_0=x_0+y_0Z_0+w_0 \quad (7c)$$

$$Y_s=(x_{s-1}(1+R_{s-1})+y_{s-1}Z_s+B)\gamma_2 \geq x_s+y_sZ_s+w_s; 1\leq s<T \quad (7d)$$

$$Y_T=x_{T-1}(1+R_{T-1})+y_{T-1}Z_T \quad (7e)$$

$$x_t, y_t, w_t \geq 0. \quad (7f)$$

Both models (6) and its variant (7) are linear optimization problems in the decision functions x_s, y_s, w_s. In order to solve them numerically one has to use approximations in order to bring the decision functions x_t, y_t, w_t down to decision vectors. To do so, the processes Z_t and R_t are approximated by processes, which may only take a small finite number of values, i.e. by tree processes.

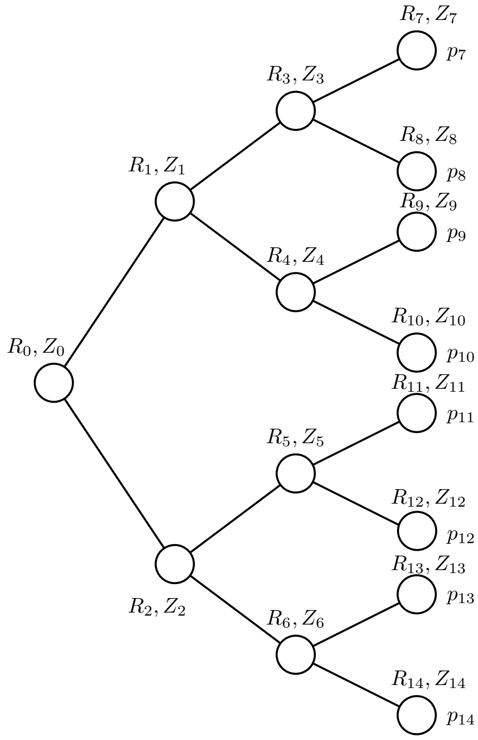

Fig. 3. An example tree of height 3, representing the processes R_t, Z_t.

2.1 Computational issues: The tree model

A discrete-time, non-recombining tree is a simple but flexible model for approximating all kinds of stochastic processes. If the process is Markovian, a lattice model (recombining tree) may suffice for the representation. However, since the decisions may depend on the whole history of the observed financial processes, it is on the complete history tree, where the decision variables have to be placed. Hence, we assume that a tree was estimated for the financial processes anyway.

Moreover, trees represent also processes, which are not first order Markovian, for example higher order Markovian or non-Markovian processes. We assume that the lifetime is independent of the finance processes (R_t, Z_t), we allow however a dependency between the fund process (Z_t) and the interest rate process (R_t), as shown in Figure 3.

First, a tree process representing the fund values and the interest rates is estimated using historical data and expert projections. A more detailed description of how to get from data to trees is contained in section 2.3.

The lifetime process is a binary process consisting of survival nodes (s), which branch in the next level and death nodes (d), which are terminal nodes. The path probabilities are given by $\pi_1^{(D)}, \ldots, \pi_T^{(D)}$ for the death paths and $\pi_T^{(S)}$ for

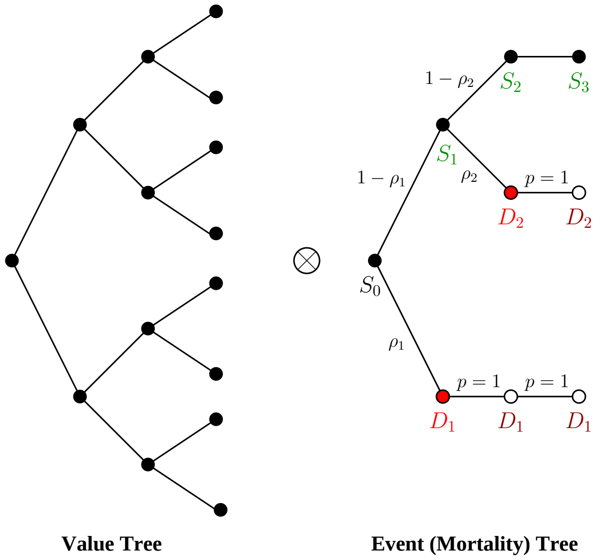

Fig. 4. The finance process tree and the lifetime tree will be combined by the tensor product operation. The lifetime tree contains the survival (s) and the death (d) events.

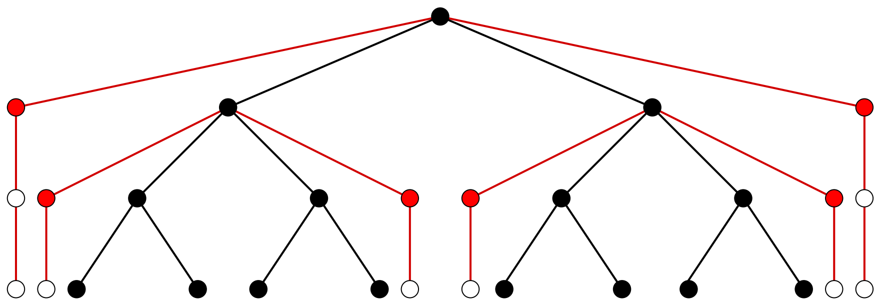

Fig. 5. The product tree.

the one survival path.

Since statistical independence of the financial processes and the mortality process is assumed, the two trees can be combined by constructing the product tree (tensor product of trees).

The size of the product tree mainly depends on the tree structure of the value tree. Consider the value tree and let n_t be the total number of nodes in stage t and N_t be the number of total nodes up to stage t, i.e. $\sum_{i=1}^{t} n_i$ (without root node). The total number of nodes of the product tree is shown in Table 1. E.g. coupling a binary value tree of height $T = 3$ with a mortality tree of the same height (see Figure 4) results in a $n = (1, 4, 10, 14)$ product tree, as shown in Figure 5.

Let $\mathcal{N}$ be the set of the nodes of the combined tree (except the root), let $\mathcal{N}_t = \mathcal{N}_t^s \cup \mathcal{N}_t^d$ be the set of all nodes at time t, where $\mathcal{N}_t^s$ is the set of all survival nodes and $\mathcal{N}_t^d$ is the set of all death nodes. Let $\mathcal{T} = \mathcal{N}_T$ the set of all

Stage	0	1	2	...	T-1	T
Value tree	1	n_1	n_2		n_{T-1}	n_T
Event tree	1	2	3		T-1	T-1
Product tree	1	$N_1 + n_1$	$N_2 + n_2$		$N_{T-1} + n_{T-1}$	$N_{T-1} + n_T = N_T$

Table 1
Number of nodes in the value, event and product tree

terminal nodes and $\mathcal{T}^s$ the subset of all terminal survival nodes. If n is a node except the root, then $n-$ denotes the predecessor of n. The set of successors of n is denoted by $n+$. The probability to reach node n is p_n. By $t(n)$ we denote the time of node n.

The fund values and the interest rate process live on the tree, i.e. they are defined as $Z_n, n \in \mathcal{N}$, resp. $R_n, n \in \mathcal{N}$. Using the contracted formulas for the death benefit (1) and the minimal guarantee (2), one may calculate the death benefits D_n for all death nodes and the guarantees G_n for the terminal survival nodes.

The optimization problem (6) in tree formulation is

$$\begin{aligned}
&\text{Maximize in } x_n, y_n, w_n: \\
&\sum_{t=1}^{T}(1+r)^{-t}\pi_t^{(D)} \sum_{n \in \mathcal{N}_t^d} p_n([Y_n + \alpha_t w_n - D_n]^+ - \delta[Y_n + \alpha_t w_n - D_n]^-) \\
&+(1+r)^{-T}\pi_T^{(S)} \sum_{n \in \mathcal{T}^s} p_n([Y_n - G_n]^+ - \delta[Y_n - G_n]^-) && \text{(8a)} \\
&\text{subject to} \\
&Y_0 = x_0 + y_0 Z_0 \le \gamma_1 \cdot b && \text{(8b)} \\
&(x_{n-}(1+R_{n-}) + y_{n-}Z_n + B)\gamma_2 \ge x_n + y_n Z_n + w_n \quad \text{for } n \in \mathcal{N} \setminus \mathcal{T} && \text{(8c)} \\
&Y_n = x_{n-}(1+R_{n-}) + y_{n-}Z_n \quad \text{for } n \in \mathcal{T}^s && \text{(8d)} \\
&x_n, y_n, w_n \ge 0. && \text{(8e)}
\end{aligned}$$

By solving this model, one finds not only the optimal risk management strategy (x_n, y_n, w_n) of the insurance company, but also the probability of shortfall i.e. the probability that $Y_n + \alpha_{t(n)} < D_n$ or $Y_n < G_n$.

The multiperiod stochastic optimization model (8) can be seen as a black box. Inputs are the characteristic features of the contract (survival benefit, death benefit, guarantee, but also the stochastic fund model etc.) and output is the risk associated with the optimal management of the contract, in particular the shortfall probability and the expected size of the shortfall.

The primary use of the model is to transform the characteristics of the contract in the characteristics of the optimal management and the associated risk. However, the model may also be used for the design of new products: By choosing a constraint for risk exposure (expressed e.g. in terms of the shortfall probability for instance), the maximal guarantee which does not lead to a violation of the risk constraint may be calculated.

The model can be extended easily. A variant of this model would include a lapse model. In that case, it is slightly more difficult to model the lapse events. Empirical evidence shows that the frequency of lapses depends on the performance of the fund and on the credit rating of the insurance company (see discussion in Consiglio et al. (2000). If lapse occurs, the surrender values are calculated by usual formulas employed by insurance companies. Thus the best way to model is to introduce an additional dependent risk factor in the tree. This would make the tree bigger, but allows for more flexibility.

2.2 Examples for contract specification

ULLIG contracts differ in the way how the death benefit and the guarantee is calculated.

Here are some examples for death benefit formulas:

Fixed death benefit. $D_t = f_1 \cdot b + f_2 \cdot B$, where f_1, f_2 are some factors, depending on age and gender of the customer

Death benefit depending on total contribution $D_t = f \cdot (b + B \cdot (\tau - 1))$, where f is some factor.

Portfolio dependent death benefit. The benefit is the maximum of a fixed sum and the actual portfolio value.

Fund value dependent death benefit. The benefit is the maximum of fixed sum and some percentage of the fund value increase.

Examples for guarantee formulas:

- **Guaranteed annual increase.**

$$G = b(1+g)^T + B \cdot \sum_{t=1}^{T-1} (1+g)^{T-t}$$

- **Guaranteed yield to maturity.**

$$G = f \cdot (b \max(Z_T/Z_0, (1+g)^T) + B \cdot \sum_{t=1}^{T-1} \max(Z_T/Z_t, (1+g)^{T-t}))$$

  where $0 < f < 1$ is some factor.

Age at time t	Male	Female
18-40	1.09	1.24
41-50	1.04	1.08
51-60	1.014	1.03
61-75	1.004	1.008
76-	1.0	1.0

Table 2
The factors f

Example: SU2001

A good example for an ULLIG contract with complex benefit formulas is the contract SU 2001 (Safe Unit 2001) issued by the Italian company CARIVITA (today called IntesaVITA).

SU 2001 was placed from January 15th to April 10th, 2001. The total volume was 215 million Euro. The conditions for this contract were given as follows:

The client pays fixed sum b (multiples of 2500 Euro) at the initial date and makes no more payments until maturity of the contract. SU2001 is based on the fund SUG2001, which values are denoted by Z_t. The quota of fund ownership is defined as $Q = b \cdot 0.98/Z_0$. (An underwriting fee of 2% is deducted at the beginning).

The contract matures at time T. The minimal guarantee is

$$G = Q \cdot \max(\max_{0 \leq t \leq T_1} Z_t, 0.8 \cdot \max_{T_1 \leq t \leq T} Z_t). \tag{9}$$

Here $T_1 < T$ is an intermediate observation date. The death benefit D_t at time t is

$$D_t = Q[Z_t + \min(f \cdot Z_t, 10)] \tag{10}$$

where f is a factor which depends on the gender and age at time t of death the customer (see Table 2).

2.3 Numerical results

The multi-stage stochastic optimization models (problems 5 and 8) were modeled with AMPL (Fourer et al. (2002)) and solved using the MOSEK solver.

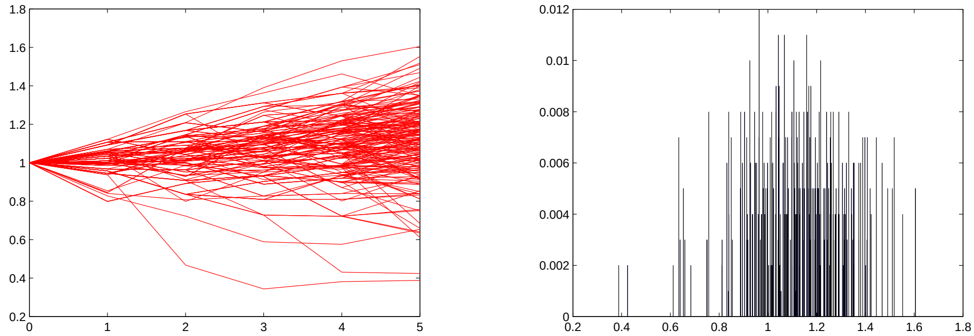

Fig. 6. Scenario tree: Values (left), probabilities of scenarios in final stage (right)

The workflow to conduct numerical evaluations was implemented in MatLab and additional parsers have been developed in the Python programming language.

The coupled value and mortality (product) scenario trees might get large, even with a moderate number of stages and number of succeeding nodes at each node of the tree, see Table 1 above. Hence, the scenario generation methodology for generating the value tree must be chosen carefully. To obtain the numerical results below, a multi-stage scenario generator was used, where the number of nodes per stage can be specified in advance, i.e. the scenario tree exhibits a stage-wise fixed structure and the generator calculates the optimal values of the respective number of nodes per stage as well as the optimal links between stages and assigns correct probabilities to these arcs. The scenario generation is based on a minimization of probability metrics (see Pflug (2001) for more details).

The system offers a rich set of parameters to modify to allow for an optimal adoption of the model to the needs of the company issuing the pension fund as well as including contract specific details. Hence, only a subset of all possible results is shown below.

For the future development of the underlying fund, the Standard and Poors 500 Index was taken as a reference for the simulation and scenario generation of the forecast. Daily closing values of 8 years (January 1996 to January 2004) have been used to fit an ARMA(1,1)/GJR(1,1) time series model, from which 1000 paths have been simulated for a possible fund development of the next 5 years. It is obvious, that the choice of the underlying fund development process has a quite large impact on the results.

A scenario tree with a stage-wise fixed structure (25/50/75/100/200 nodes per stage) was generated. This scenario tree as well as the probabilities of each scenario in the final stage are shown in Figure 6.

Age	Gender	Year 1	2	3	4	5
30	Female	0.29	0.37	0.37	0.47	0.53
30	Male	0.88	0.81	1	0.99	1.12
50	Female	2.60	2.75	3.15	3.31	3.49
50	Male	4.73	5.59	6.04	6.66	7.32

Table 3
Example: Death probabilities $q^{(s)}_{x+t}$ $(\cdot 10^{-3})$ for five subsequent years

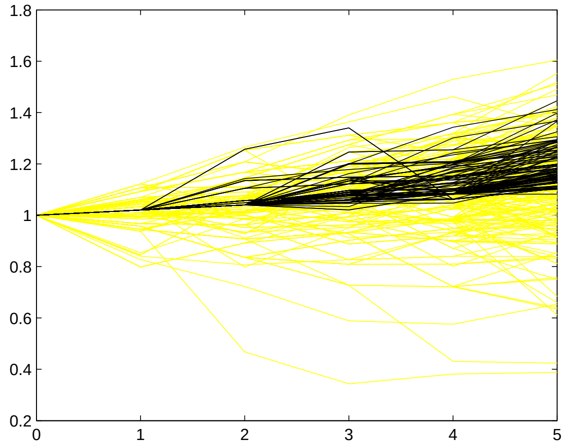

Fig. 7. Example: Underlying fund tree and wealth development

Furthermore, Austrian mortality tables (years 2000/2001) have been used to calculate survival and death probabilities for different age and gender classes. Table 3 summarizes death probabilities for selected age and gender classes. As discussed above, cohort specific projected mortality tables have to be used, if long term models are considered.

The investment problem is to decide on the investment into three basic asset categories: the underlying fund, a risk-free bond and the yearly re-insurance. The time horizon in the above example is $T = 5$ yearly stages, and there is a single installment $B = 1000$ at the beginning of the contract.

The underlying fund tree and one possible wealth development is shown in Figure 7. In this case, the calculation was done for a 30-year old woman, with a fixed Euro 80000 death benefit, an annual guaranteed survival benefit of 2 percent on the initial installment. A risk-free rate of 4 percent per year was used.

Two different numerical examples have been conducted to show the variety of possible report generation from the model for the management of the pension fund. Section 2.3.1 shows the shortfall probability as well as the expected short-

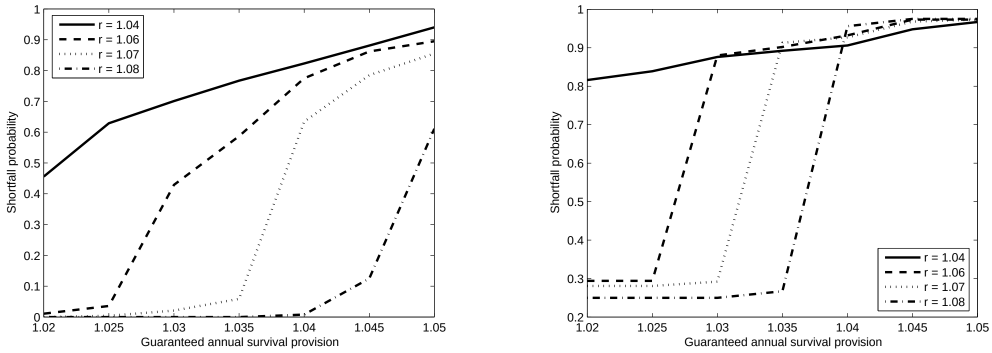

Fig. 8. Shortfall probability for different levels of guaranteed survival benefit rates.

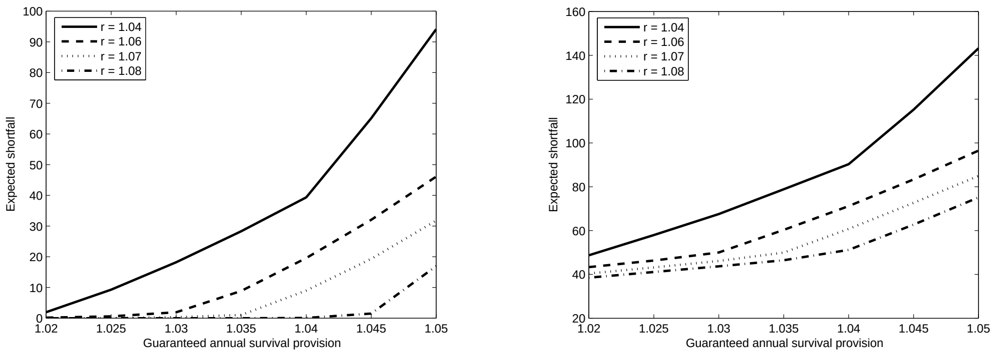

Fig. 9. Expected shortfall for different levels of guaranteed survival benefit rates.

fall over a range of guaranteed annual survival benefit rates. These diagrams support the decision on which level of this rate should be granted. Section 2.3.2 shows the suggested amount of wealth invested in the re-insurance asset per stage.

2.3.1 Shortfall probability and expected shortfall

Figure 8 shows the shortfall probability of a 30-year old woman (left) and a 50-year old man (right). A fixed Euro 80000 death benefit was used. Figure 9 shows the expected shortfall in these two cases. The calculations have been conducted for 4 different risk free rates $r = 1.04, 1.06, 1.07, 1.08$ over a a range of guaranteed annual survival benefit rates $s = 1.02, 1.025, 1.03, 1.035, 1.04, 1.045, 1.05$.

2.3.2 Amount of insurance per stage

Figure 10 shows the amount invested into the conventional (one-year) life insurance per stage for four person classes (Female/Male, Age 30/50). Nothing will be invested in the last stage, as the current model is designed to pay the survival benefit, even if the client dies in the last stage. As above a fixed Euro

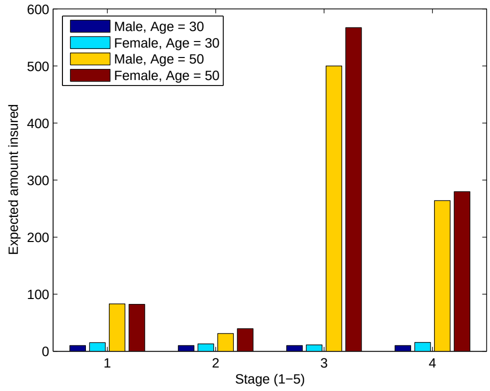

Fig. 10. Insurance per stage

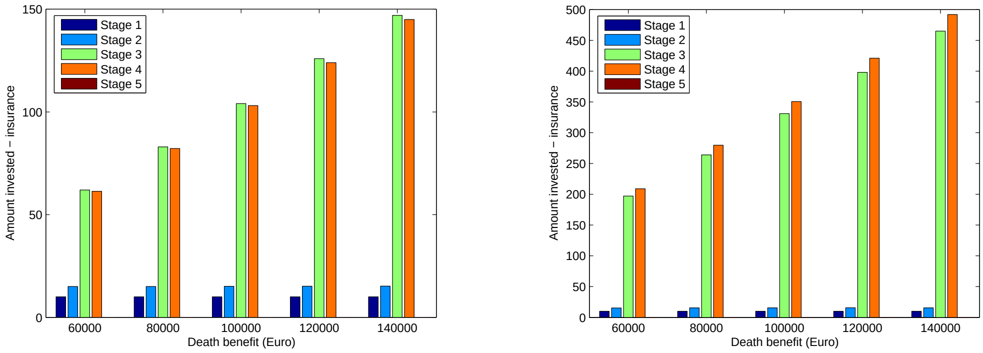

Fig. 11. Insurance per stage with different guaranteed death benefits

80000 death benefit and an annual guaranteed survival benefit of 2 percent on the initial installment was assumed. The risk-free rate was set to 3 percent per year.

Figure 11 summarizes the effects of higher guaranteed death benefits on the amount invested in insurance. The two examples were calculated for a 30-year old man (left) and a 50-year old woman (right) with the same assumptions on the survival benefit and the risk-free rate as above.

3 A continuous time model

In this section, we present a continuous time variant of our decision model and derive – on a pure analytical basis – some of its properties. We assume that the insurance company does investment and trading on a continuous time basis, while the premium inflow as well as the outflow of death resp. survival benefits occurs only at discrete times

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T.$$

To allow for slightly more flexibility, we allow that these fixed dates may also be fractions of years. Thus the customer pays an initial installment b and at times $t_i; i = 1, \ldots, n$ the premium B. In case of death in the interval $(t_i, t_{i+1}]$, the death benefit D is paid. In case of survival to maturity T, the survival benefit S is paid, and this sum is guaranteed. As before, denote by τ the residual lifetime variable of the customer.

The fundamental difference between the continuous time model and the discrete model introduced in Section 2 is that the guarantee – given that it is feasible by pure bond investments – may be reached with probability 1. Therefore, the objective in this section is no longer to penalize the shortfall, but to maximize the utility of the surplus, under the constraint that the shortfall is zero. In case of a linear utility, the previous discrete model is just a penalty cost formulation of the continuous time model.

3.1 The market model

We suppose that the market consists of $d + 1$ assets. One asset is a standard bond and has no systematic risk. Its price process is $(\beta(t))_{0 \le t \le T}$. The other d assets are risky and their prices are

$$Z(t) = (Z_1(t), \cdots, Z_d(t)), \quad t \in [0, T].$$

We assume a Black-Scholes (geometric Brownian motion) model with random coefficients. Thus we have the dynamics

$$d\beta(t) = \beta(t) r(t) dt, \quad \beta(0) = 1 \tag{11}$$

for the bond process and

$$dZ_i(t) = Z_i(t)\left[\mu_i(t)\,\mathrm{d}t + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t)\right] \quad Z_i(0) = z_i \in (0,\infty) \tag{12}$$

$i = 1, \cdots, d$ for the equities. Here $W(t)$ is a d dimensional Wiener process $W(t) = (W_1(t), \cdots, W_d(t)), 0 \le t \le T$ which generates after augmentation a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. The interest rate $(r(t))_{0\le t\le T}$, drift vector $\mu(t) = (\mu_1(t), \cdots, \mu_d(t))$ and the volatility matrix processes $\sigma_{i,j}(t)_{1\le i,j\le d}, 0 \le t \le T$ are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$ and satisfy the mild integrability condition

$$\int_0^T (|r(t)| + ||\mu(t)|| + ||\sigma(t)||^2)\,\mathrm{d}t < \infty, \quad \text{a.s.} \tag{13}$$

To have an arbitrage-free model of a complete financial market we assume that the volatility matrix $\sigma(t)$ is invertible at each $0 \le t \le T$. Furthermore the *market price of risk* process

$$\theta(t) = \sigma(t)^{-1}(\mu(t) - r(t)\mathbb{1}) \quad , \quad 0 \le t \le T, \quad \mathbb{1} = (1, \cdots, 1) \tag{14}$$

is supposed to fulfill

$$\int_0^T ||\theta(t)||^2\,\mathrm{d}t < \infty \tag{15}$$

and

$$\mathbb{E}\left[\exp(-\int_0^T \theta(t)dW_t - \frac{1}{2}\int_0^T ||\theta(t)||^2\,\mathrm{d}t)\right] = 1. \tag{16}$$

For this market model the process

$$L(t) = \exp(-\int_0^t \theta(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2\,\mathrm{d}s), \quad 0 \le t \le T \tag{17}$$

is a martingale and the unique martingale measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ can be defined by

$$\mathbb{P}^*(A) = \mathbb{E}L(T)1_A, \quad A \in \mathcal{F}_T. \tag{18}$$

An application of Girsanov's Theorem provides that

$$\bar{W}(t) = W(t) + \int_0^t \theta(s)\,\mathrm{d}s \quad ,0 \leq t \leq T \tag{19}$$

is a Wiener process w.r.t. $\mathbb{P}^*$ and the stock price processes fulfill w.r.t. $\bar{W}$ the dynamics

$$dZ_i(t) = Z_i(t)\left[r(t)\,\mathrm{d}t + \sum_{j=1}^{d} \sigma_{i,j}(t)\,d\bar{W}_j(t)\right], \quad i = 1, \cdots, d. \tag{20}$$

The martingale measure $\mathbb{P}^*$ can be used for pricing contingent t-claims, which are $\mathcal{F}_t$ measurable random variables S payable at time t. The unique arbitrage-free initial price $V_0(S)$ of such a contract in our complete financial market is the expected discounted payoff w.r.t. to the martingale measure $\mathbb{P}^*$:

$$V_0(S) = \mathbb{E}^*\beta(t)^{-1}S. \tag{21}$$

Trading strategies are progressively measurable R^d-valued processes

$$\xi(t) = (\xi_1(t), \cdots, \xi_d(t)), \quad 0 \leq t \leq T$$

with

$$\int_0^T ||\xi(t)\sigma(t)||^2\,\mathrm{d}t + \int_0^T |\xi(t)^\top(\mu(t) - r(t)\mathbb{1})|\,\mathrm{d}t < \infty, \quad \text{a.s.} \tag{22}$$

The real number $\xi_i(t)$ denotes the amount of money the investor holds in the i-th stock at time t. If we denote by $Y^{b,\xi}(t)$ the wealth of the investor at t, assuming that the initial budget was b and the trading strategy ξ is followed, then $Y(t) - \sum_{i=1}^d \xi_i(t)$ is the amount invested in bonds. If no additional investment or consumption is allowed we are in a self-financing setting and the wealth process $Y^{b,\xi}(t)$ satisfies the dynamics

$$\begin{aligned} dY^{b,\xi}(t) &= (Y^{b,\xi}(t) - \xi^\top(t)\mathbb{1})\,r(t)\,\mathrm{d}t + \xi(t)'[\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)], \\ Y^{b,\xi}(0) &= b \end{aligned} \tag{23}$$

A solution of the above equation for the discounted wealth can be given in the following form

$$\beta(t)^{-1}Y^{b,\xi}(t) = b + \int_0^t \beta(s)^{-1}\xi(s)^\top[\sigma(s)\,\mathrm{d}W(s) + (\mu(s) - r(s)\mathbb{1})\,\mathrm{d}s] \quad (24)$$

for $0 \le t \le T$.

We emphasize that each contingent t-claim S can be replicated from initial wealth $b = V_0(S)$ by a self-financing trading strategy ξ followed up to time t, i.e. the associated wealth process satisfies

$$Y^{V_0(S),\xi}(t) = S, \quad (25)$$

i.e. coincides at maturity t with S.

The optimal management problem may now formulated as follows: The insurance company receives initial premiums and regular installments B, given that the customer is alive. On the other hand, it has to pay the death benefit in case of death and the survival benefit in case of survival. Such strategies are no longer self financing strategies, but allow money inflow until a stopping criterion (the death event or the maturity) is met. In addition, some constraints may be put on the set of feasible strategies.

The goal is to maximize the expected final utility under the given constraints. In general, this problem is a dynamic optimization problem. However, the powerful martingale method suited for the Black-Scholes model allows to reduce the problem to a static variational problem. Given the solution of the static problem, the optimal strategy may be derived in a second step.

We review this technique briefly here. The books of Karatzas (1997) or Korn (1997) serve as a standard reference.

3.2 Maximizing expected utility

Let U be strictly increasing, continuously differentiable and concave utility function defined on the positive reals fulfilling the Inada condition (see Inada (1963))

$$\lim_{v\to\infty} U'(v) = 0 \quad , \quad \lim_{v\to 0} U'(v) = +\infty. \quad (26)$$

The most important examples are $U(v) = \log(v)$ and $U(v) = \frac{1}{\alpha}v^\alpha, \alpha \in (0,1)$.

The problem of an investor who would like to maximize its expected utility of terminal wealth by investing in a self-financing trading strategy ξ and who has initial capital $b > 0$, which will never fall negative is the so called Merton problem

$$\max_{\xi \in \mathcal{A}(b)} U(Y^{b,\xi}(T)). \tag{27}$$

where $\mathcal{A}(b)$ the set of all self-financing trading strategies ξ such that its associated wealth process $Y^{b,\xi}(t)$ becomes never negative and satisfies $\mathbb{E}U(Y^{b,\xi}(T))^- < \infty$.

This dynamic optimization problem was firstly solved by Merton (1971) using the standard tools of control theory. The so called martingale method, was introduced by Pliska (1986), Cox and Huang (1989), Karatzas et al. (1991) and others.

We briefly would like to recall the main ideas in the following. Because of the fundamental relationship between strategies and correctly prices contingent claims (25, every attainable terminal wealth from an initial capital not exceeding b can be seen as a contingent T-claim with initial price not larger than b and vice versa. Hence to determine an optimal terminal wealth one has to look (A) for an optimal T-claim financeable from an initial capital not exceeding b and (B) a trading strategy replicating this optimal claim.

The static problem (A) can be easily solved by a pointwise Lagrange approach. Denote by $\mathcal{B}(b)$ the set of contingent non negative T-claims S such that $\mathbb{E}U(S)^- < \infty$ and $V_0(S) \leq b$. Then the static problem is given by

$$\max_{S \in \mathcal{B}(b)} \mathbb{E}U(S). \tag{28}$$

For to solve this by a Lagrange approach we note that the constraint can be expressed as

$$V_0(S) = \mathbb{E}^* \beta(T)^{-1} S = \mathbb{E}\beta(T)^{-1} L(T) S = \mathbb{E}H(T)S \tag{29}$$

with $H(t) = \beta(t)^{-1} L(t), 0 \leq t \leq T$ denoting the discounted martingale process. Introducing a Lagrange multiplier λ the pertaining unconstrained static problem is

$$\max_{S \geq 0} \mathbb{E} \left(U(S) - \lambda H(T) S \right). \tag{30}$$

If the utility function satisfies the Inada conditions (26), the solution is given by

$$S = I(\lambda H(T)) \tag{31}$$

where I is the inverse of U'.

To determine finally a solution to (28) we calculate a Lagrange multiplier λ satisfying the constraint, hence

$$\mathbb{E}H(T)I(\lambda H(T)) = b. \tag{32}$$

If U' does not vanish at infinity, then there is no finite solution, since $H(T)$ is unbounded. This fact distinguishes the continuous time geometric Brownian motion model from the discrete model discussed in the previous sections. In fact maximizing expected wealth is a reasonable objective in discrete models, but not in the Black-Scholes situation.

The above mentioned arguments can be made rigorous and we refer to Theorem 2.3.2 in Karatzas (1997):

3.3 Proposition: *In order to solve the dynamic problem (27) it is sufficient to solve the static problem (28) and then find the strategy which replicates this solution. Suppose the static problem (28) has a finite optimal value for each initial capital $b > 0$, then the optimal attainable terminal wealth is given by (31) with λ the unique solution of (32).*

The second step (B) is to determine the optimal trading strategy ξ that replicates the optimal terminal wealth $I(\lambda H(T))$. Note that its associated wealth process $Y(t), 0 \le t \le T$ fulfills

$$Y(t) = \frac{1}{H(t)}\mathbb{E}[H(T)I(\lambda H(T))|\mathcal{F}_t], \quad 0 \le t \le T. \tag{33}$$

Due the martingale representation theorem there exists a progressively measurable process ψ such that

$$H(t)Y(t) = b + \int_0^T \psi(s)\,\mathrm{d}W_s, \quad 0 \le t \le T. \tag{34}$$

Together with (24) and integration by parts we obtain the optimal trading strategy ξ by solving

$$\sigma'(t)\xi(t) = \frac{\psi(t)}{H(t)} + Y(t)\theta(t), \quad 0 \le t \le T. \tag{35}$$

For $U(v) = \log(v)$ the previous approach leads to the optimal terminal wealth

$$S = \frac{b}{H(T)}. \tag{36}$$

Its associated wealth process fulfills $Y(t) = \frac{b}{H(t)}$, $0 \le t \le T$ and can be obtained by using the so called Merton strategy

$$\xi(t) = (\sigma(t)\sigma^\top(t))^{-1}(\mu(t) - r(t)\mathbb{1})Y(t), \quad 0 \le t \le T, \tag{37}$$

which is an optimal trading strategy. For power utility an explicit solution can be obtained in a model of deterministic coefficients, see page 39, Example 2.2.5 in Karatzas (1997) for further details.

3.4 A guarantee constraint

For guaranteed products, we now add the constraint that the terminal wealth must exceed an initially set benchmark $G > 0$. For simplicity we assume that G is a positive constant but our arguments will also work for a contingent T-claim. We only need to adapt the previous approach to obtain a solution.

Let $\mathcal{A}_1(b)$ be the set of trading strategies $\xi \in \mathcal{A}(b)$ such that its terminal wealth a.s. exceeds G and $\mathcal{B}_1(b)$ the set of those contingent claims $S \in \mathcal{B}(b)$ such that $S \ge G$ almost surely. Then the optimization problem with guaranteed payoff G at terminal time T is

$$\max_{\xi \in \mathcal{A}_1(b)} \mathbb{E}U(Y^{b,\xi}(T)). \tag{38}$$

Its static counterpart is

$$\max_{S \in \mathcal{B}_1(b)} \mathbb{E}U(S). \tag{39}$$

To ensure a guaranteed payoff G at terminal time at least an initial capital $G \cdot P(0,T)$ is necessary. Here we denote with $P(t,T)$ the price at t of a zero coupon bond which pays 1 at maturity T. We have that

$$P(t,T) = \mathbb{E}^*(\frac{\beta_t}{\beta_T}|\mathcal{F}_t). \tag{40}$$

We assume $b > G \cdot P(0,T)$ for our initial capital b and a pointwise Lagrange maximization yields an optimal terminal wealth $S = S(\lambda)$ given by

$$S(\lambda) = \max\{G, I(\lambda H(T))\}. \tag{41}$$

The Lagrange multiplier λ can be computed by solving

$$V_0(S(\lambda)) = G \cdot P(0,T) + \mathcal{C}(I(\lambda H(T)), S) = b \tag{42}$$

with $\mathcal{C}(I(\lambda H(T)), S)$ denoting the initial price of a call with strike G on an asset with payoff $I(\lambda H(T))$ at T. Since the call price tends to ∞ for $\lambda \to 0$ respectively 0 for $\lambda \to \infty$ the above equation can be solved for each $b > G \cdot P(0,T)$ and we get an analogous statement to proposition 3.3.

3.5 Proposition: *Let the assumptions of proposition 3.3 be fulfilled and let $b > G \cdot P(0,T)$. Then a solution of (39) is given by (41) with λ being the unique solution of (42). The optimal trading strategy, a solution of (38), can be determined by*

(i) buying G zero coupon bonds with maturity T for a price of $P(0,T)$ each and

(ii) investing the remaining initial capital of $b - G \cdot P(0,T)$ in a strategy that replicates a call on $I(\lambda H(T))$ which is the optimal wealth in the unconstrained portfolio optimization problem w.r.t. a modified initial capital.

Example. Consider the Black Scholes model for a bond with constant interest rate r, one stock with constant coefficients (μ, σ) and log-utility. In this model, the Black-Scholes call price has volatility $|\theta| = |\frac{\mu - r}{\sigma}|$. Thus the equation (42) for a $G > 0$ such that $Ge^{-rT} < b$ is

$$Ge^{-rT} + \mathcal{C}(\frac{1}{\lambda}H(T), S) = b \tag{43}$$

where

$$\mathcal{C}(\frac{1}{\lambda}H(T), S) = \frac{1}{\lambda}\phi(g_1(\frac{1}{\lambda}, T)) - Se^{-rT}\phi(g_2(\frac{1}{\lambda}, T)) \tag{44}$$

with

$$g_1(y,T) = \frac{\log(\frac{y}{K}) + (r + \frac{1}{2}\theta 2)T}{\sqrt{\theta 2T}}, g_2(y,T) = \frac{\log(\frac{y}{K}) + (r - \frac{1}{2}\theta 2)T}{\sqrt{\theta 2T}}. \quad (45)$$

We may solve (43) to obtain the optimal expected utility of terminal wealth

$$w(\lambda) = \mathbb{E}\,\log(\max\{G, \frac{1}{\lambda}H(T)\})$$

.

A straightforward calculation yields

$$w(\lambda) = \log\frac{1}{\lambda} + \log(\lambda G)\Phi(h_1(\lambda)) + (r + \frac{1}{2}\theta 2)T\Phi(h_2(\lambda))) + |\theta|\sqrt{T}\varphi(h_1(\lambda)) \quad (46)$$

with functions

$$h_1(\lambda) = \frac{\log(\lambda G) - (r + \frac{1}{2}\theta 2)T}{\sqrt{\theta 2T}}, h_2(\lambda) = \frac{\log(\lambda G) + (r + \frac{1}{2}\theta 2)T}{\sqrt{\theta 2T}}, \quad (47)$$

and φ, Φ denoting the density respectively distribution function of a standard normal distribution. The optimal expected return is

$$\bar{R}(\lambda) = \frac{1}{T}(w(\lambda) - \log b) \quad (48)$$

with λ depending on the initial capital Ge^{-rT} needed to ensure the terminal wealth guarantee.

We have plotted the above defined function $\bar{R}$ when $r = \log(1 + 0.03)$, initial capital $b = 1000$ and $|\theta| = 0.1$, resp. $|\theta| = 0.3$. The more money we need to reserve for the terminal wealth guarantee, the less is the optimal expected return and this effect is more intensive for higher $|\theta|$. At the limit we obtain a return from a pure Bond respectively Merton strategy. In the plot we have included the bounds on $\bar{R}(\lambda)$ given by

$$\max\{r, \frac{1}{T}(\log(\frac{1}{\lambda}) - \log b) + (r + \frac{1}{2}\theta 2)\} \le \bar{R}(\lambda) \le r + \frac{1}{2}\theta 2. \quad (49)$$

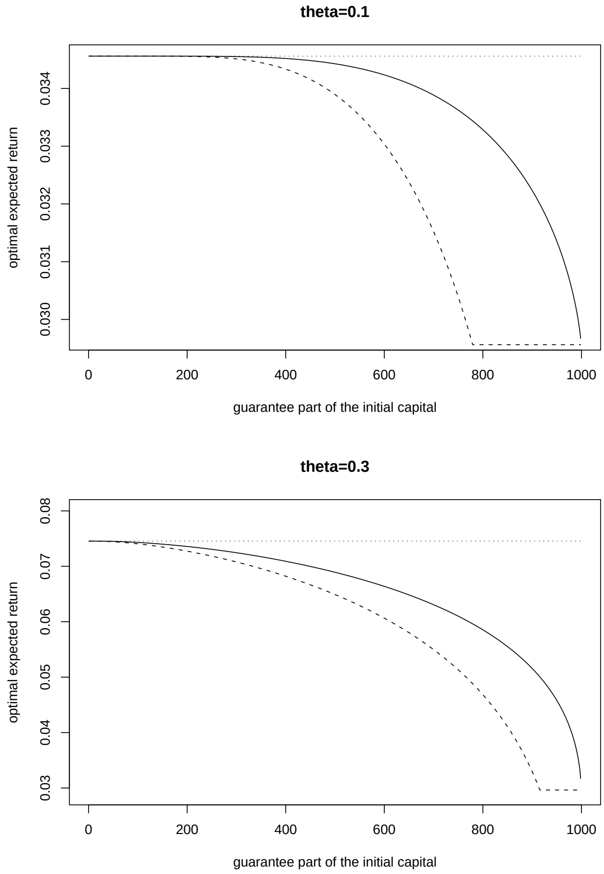

Fig. 12. Optimal expected return in a Black-Scholes model

The upper bound follows from the fact that investing money without a constraint on terminal wealth must lead to a higher return than trading subject to a terminal constraint. The lower bound is true since

$$\mathbb{E}\ \log(\max\{G, \frac{1}{\lambda}H(T)\}) \geq \mathbb{E}\ \log(\frac{1}{\lambda}H(T)) = \log(\frac{1}{\lambda}) + (r + \frac{1}{2}\theta 2)T. \quad (50)$$

3.6 Periodical investment and consumption

The next step is to introduce the periodical inflows and outflows of the portfolio. Assume that the investor additionally to its initial capital b receives nonnegative amounts $B(t_1), \cdots, B(t_n)$ and has to pay nonnegative amounts $C(t_1), \ldots C(T_n)$ at time points $0 < t_1 < \cdots < t_n < t_{n+1} = T$. Introduce the net values $A(t_i) = B(t_i) - C(t_i)$. These values may be random.

We assume that each $A(t_i)$ is an $\mathcal{F}_{t_i}$ measurable contingent t_i-claim which is uniformly bounded. Thus it admits a unique price process given by

$$V_t(A(t_i)) = \beta(t)\mathbb{E}^*(\beta(t_i)^{-1}A(t_i)|\mathcal{F}_t), \quad 0 \le t \le t_i \tag{51}$$

and the capitalized initial value of the income-outcome stream A is

$$V_0(A) = \sum_{i=1}^{n} V_0(A(t_i)). \tag{52}$$

We might have initial negative wealth but assume that $V_0(A) + b > 0$. The investor may choose a trading strategy ξ that is financed by the net income stream A. This means that its associated wealth process $Y^\xi(t)$ fulfills (23) with initial condition b and additionally

$$Y^\xi(t_i) - Y^\xi(t_i-) = A(t_i) \tag{53}$$

for each $1 \le i \le n$. Thus starting from an initial wealth b the trading strategy is self-financing between the t_i and the income $B(t_i)$ at t_i provides a jump for the wealth process which will be invested immediately according to $\xi(t_i)$. In this setting the maximization of expected utility of terminal wealth has been treated by Karatzas et al. (1991), El Karoui and Jeanblanc-Picque (1998).

Denote by $\bar{\mathcal{A}}(b, A)$ the set of all trading strategies ξ financed by A such that its associated wealth process Y^ξ has initial value b, is allowed to become negative but the terminal wealth must be non negative and fulfills $\mathbb{E}U(Y^\xi(T))^- < \infty$. Then the following borrow strategy becomes optimal (compare El Karoui and Jeanblanc-Picque (1998)):

- sell for each t_i a contract that delivers a payoff $B(t_i)$ at t_i,
- buy for each t_i a contract that delivers $C(t_i)$ at t_i,
- invest $b' = b + V_0(A)$ in an optimal trading strategy ξ as defined in Proposition 3.3 and trade until end,
- at each t_i use the income $B(t_i)$ to deliver the payoff w.r.t. our initially sold contract,
- at each t_i take the payoff from our initially bought contract to satisfy our consumption $C(t_i)$.

If we include the constraint that terminal wealth must exceed a predetermined benchmark $G > 0$ we have to assume that the initial net capitalized value $b + V_0(A) > G \cdot P(0, T)$. Then an optimal trading strategy can be defined as above only replacing the optimization step by its guaranteed terminal wealth counterpart, see Proposition 3.5,

- invest $b' = b + V_0(A)$ in an optimal trading strategy ξ as defined in

Proposition 3.5 and trade until end.

The above strategy is indeed optimal as can be seen by the following proposition.

3.7 Proposition: *Let $V_0(A) = \sum_{i=1}^{n} V_0(A_{t_i})$ be the capitalized value of the net income stream $A = (A(t_1), \cdots, A(t_n))$ such that $b' = b + V_0(A) > 0$. Then for each trading strategy $\phi \in \bar{\mathcal{A}}(b, A)$ there exists a trading strategy $\xi \in \mathcal{A}(b')$ such that their terminal wealth coincide, hence $Y^{\xi}(T) = Y^{\phi}(T)$ and vice versa. In particular, the optimal value of both optimization problems coincide, i.e.*

$$\max_{\phi \in \bar{\mathcal{A}}(b,A)} \mathbb{E}U(Y^{\phi}(T)) = \max_{\xi \in \mathcal{A}(b')} \mathbb{E}U(Y^{\xi}(T)). \tag{54}$$

Proof:

Let $\phi \in \bar{\mathcal{A}}(b, A)$. Since the financial market is complete we may consider for each t_i a B-contract that delivers payoffs of $B(t_i)$ and a C-contract that delivers payoffs of $C(t_i)$ at t_i. These contracts have unique fair price processes $V_0(B(t_i))$ and $V_0(C(t_i))$ with $V_0(A(t_i)) = V_0(B(t_i)) - V_0(C(t_i))$. The corresponding trading strategy ξ can be defined in the following way: Use the initial wealth $b' > 0$

- to go long in the B-contract for each t_i, $1 \leq i \leq n$,
- to go short in the C-contracts for each t_i, $1 \leq i \leq n$
- to invest b w.r.t. the trading strategy ϕ and trade according to ϕ until end,
- use at each t_i the payoff $A(t_i) = B(t_i) - C(t_)$ of the contracts to invest and trade until end.

Then the associated wealth process of ξ fulfills for each $0 \leq k \leq n$

$$Y^{\phi}(t) + \sum_{i=k+1}^{n} V_t(A(t_i)) = Y^{\xi}(t) \quad \text{for } t_k \leq t \leq t_{k+1}. \tag{55}$$

In the last trading interval from t_n to T both wealth processes coincide, hence also at terminal time. Due to our requirements $Y^{\phi}(t)$ and therefore $Y^{\xi}(t)$ stay always above a fixed lower bound and have non negative terminal wealth. Since ξ is self-financing, arbitrage arguments provide that $Y^{\xi}(t)$ is non negative for all $0 \leq t \leq T$. Hence $\xi \in \mathcal{A}(b')$.

To prove the other direction we define according to $\xi \in \mathcal{A}(b')$ the trading strategy $\phi \in \bar{\mathcal{A}}(b, A)$ in the following way:

- go short in the B-contracts for each t_i,
- go long in the C-contracts for each t_i,
- invest the obtained capital b' to invest w.r.t. the trading strategy ξ and trade until end,
- use at each t_i the net income $B(t_i)$ to deliver the payoff at t_i from our short position in the B-contract and pay $C(t_i)$ from our long position in C-contract.

Then we get the same evolution of wealth as in (55), hence the terminal wealth of both strategies coincide. That ϕ is indeed contained in $\bar{\mathcal{A}}(b, A)$ can be seen from (55) due to the fact that $Y^{\xi}(t) \geq 0$, $\sum_{i=k+1}^{n} V_t(A(t_i))$ stays uniformly bounded for all $0 \leq t \leq T$.

Remark. Suppose that there is only inflow B and no outflow ($C = 0$). The following trading strategy seems to be intuitively reasonable: Invest each received income $B(t_i)$ at t_i in an optimal trading strategy for the remaining trading interval $[t_i, T]$. It turns out, that the optimal strategy described above has a better performance. We illustrate this in a Black-Scholes model with constant coefficients and log-utility. Consider the case where we have initial capital b and one additional investment the same amount of b at $t_1 \in (0, T)$. Then the first strategy without borrowing leads to an evolution of wealth given by

$$\frac{b}{H(t)} \quad \text{for } 0 \leq t < t_1 \quad \text{and} \quad \frac{b}{H(t)} + \frac{bH(t_1)}{H(t)} \quad \text{for } t_1 \leq t < T \tag{56}$$

with $H(t) = e^{-rt} \exp(-\theta W_t - \frac{1}{2}\theta 2t), \theta = \frac{\mu - r}{\sigma}$. Its terminal wealth Y has the expected utility

$$\begin{aligned}
\mathbb{E} \log Y &= \mathbb{E} \log(\frac{b}{H(T)}(1 + H(t_1))) \\
&= \log b + \mathbb{E} \log \frac{1}{H(T)} + \mathbb{E} \log(1 + H(t_1)) \\
&= \log b + (r + \frac{1}{2}\theta 2)T + \mathbb{E} \log(1 + H(t_1)) \\
&< \log x + (r + \frac{1}{2}\theta 2)T + \log(1 + \mathbb{E} H(t_1)) \\
&= \log(b + be^{-rt_1}) + (r + \frac{1}{2}\theta 2)T
\end{aligned} \tag{57}$$

where the last term on the right is the expected utility of terminal wealth from the optimal borrow strategy described above, i.e. which invests the enlarged initial capital $b + b\exp(-rt_1)$ on the whole trading interval $[0, T]$ into the optimal Merton strategy.

If we include the guarantee constraint $Y(T) \geq G$ a.s., we have to assume that the initial net capitalized value is large enough $b + V_0(A) > G \cdot P(0, T)$, to make the problem feasible. Then an optimal trading strategy can be defined as above only replacing the optimal investment step by its guaranteed terminal wealth counterpart, as in Proposition 3.5,

- invest $b' = b + V_0(A)$ in an optimal trading strategy ξ as defined in Proposition 3.5 and trade until end.

3.8 Mortality risks

So far we have investigated how to optimally invest, if periodical in- and outflows from the portfolio may happen. In this section we clarify how the preceding notions can be applied in an insurance setting.

We take the view of an insurance company that has to manage a large portfolio of insurance contracts. In such a case the fluctuations average out and one might work with expected flows instead of random flows. Alternatively, one could consider an insurer who strictly separates his insurance portfolio from his investment portfolio. While the market risk is modeled in the Black-Scholes model, the event risk (mortality risk) is replaced by the expectations only. Extra costs in risk capital provision for mortality fluctuations are not taken into consideration.

First we investigate what amount of money an insurance company must periodically consume to cover its obligation from mortality risk. As before we consider a contract running time $[0, T]$ which is divided into $n+1$ periods with endpoints

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$$

and investigate from an actuarial point of view the obligation from a death insurance contract. Ingredients of such a contract are

- the distribution of the random residual life time τ of an individual of age a,
- the death benefit $D > 0$ which has to be delivered at the end of that time period at that a death would occur.

At of view that at the beginning of each time period a manager in the insurance company must announce what amount of money is needed for each contract to cover its mortality risk for the next period. If the injured individual is alive at t_{i-1} the risk sum D is payable at t_i in the case of death during $(t_{i-1}, t_i]$. Hence the insurance company has to reserve respectively consume for each individual alive at t_{i-1} the expected payoff

$$D \cdot P(t_{i-1}, t_i)\mathbb{P}(\tau \leq t_i | \tau > t_{i-1}) \tag{58}$$

But only with probability $\mathbb{P}(\tau > t_{i-1})$ such a contract is alive at our portfolio of contracts at t_{i-1}. Hence at beginning, at t_0, the manager must announce a consumption of

$$C(t_{i-1}) = D \cdot P(t_{i-1}, t_i)\mathbb{P}(t_{i-1} < \tau \leq t_i) \tag{59}$$

at t_{i-1}. This leads to an initial consumption of $C(0) = D \cdot P(0, t_1)\mathbb{P}(\tau \leq t_1)$ for covering the mortality risk of the first time period and a future consumption stream $C = (C(t_1), \cdots, C(t_n))$ for the remaining future time periods. The initial consumption together with the capitalized value of the consumption stream C can be seen from an actuarial point of view as the initial price of such a death insurance contract. Its value is

$$C(0) + V_0(C) = \sum_{k=0}^{n} D \cdot P(0, t_{k+1})\mathbb{P}(t_k < \tau \leq t_{k+1}) \tag{60}$$

A death insurance contract will be financed by a premium income stream. The insurer demands at each t_i a constant premium $b > 0$ from each injured individual alive at t_i. Thus a portfolio manager in the insurance company may initially calculate the expected income at t_i

$$B(t_i) = b\,\mathbb{P}(\tau > t_i) \quad \text{for } 0 \leq i \leq n.$$

Hence we receive an initial income $x = b$ and a future income stream

$$B = (B(t_1), \cdots, B(t_n)).$$

The capitalized value of all incomes are

$$b + V_0(B) = b + \sum_{k=1}^{n} B \cdot P(0, t_k)\mathbb{P}(\tau > t_k). \tag{61}$$

If this value coincide with that of the initial price determined by the consumption stream, i.e.

$$b + V_0(B) = C(0) + V_0(C), \tag{62}$$

then the equivalence principle holds. The premium income stream just suffices to cover the mortality risks. No additional capital can be invested in financial markets.

3.9 Managing unit-linked life insurance contracts

If

$$b + V_0(B) > C(0) + V_0(C)$$

the insurance company may not only cover its obligations from mortality risk but also has capital left for investing in a financial market. Hence it can be seen as an investor that would like to maximize his terminal wealth by taking into consideration his income-outcome stream defined by B respectively C and his initial wealth given by $b - C(0)$.

If there is no guaranteed survival payoff at terminal time, then it is a death insurance contract which allows the insurance company to invest a part of the received premium in risky assets of a financial market. Ingredients of such a contract are

- a running time interval $[0, T]$ divided into periods with endpoints

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T,$$

- a mortality risk sum D payable at the end of that time period at that a death will occur,
- constant premium income B at each t_i from each injured individual alive at t_i,
- investment of a part of the premium in financial markets.

The portfolio manager in an insurance company may see the above investment problem as a portfolio optimization problem with income-outcome stream $A = (A(t_1), \cdots, A(t_n))$ defined by

$$A(t_i) = B(t_i) - C(t_i) = B\,\mathbb{P}(\tau > t_i) - D\,P(t_i, t_{i+1})\,\mathbb{P}(t_i < \tau \le t_{i+1}) \tag{63}$$

for each $1 \le i \le n$. Furthermore the initial wealth of such a contract is

$$b' = b - D\,P(0, t_1)\,\mathbb{P}(\tau \le t_1), \tag{64}$$

i.e. the first premium b minus the the insurance premium for mortality risk during the first period. In this setting we may apply the results of our preceding sections and refer to the optimal strategy defined in continuation to Proposition 3.7.

Include now in addition to the preceding unit-linked life insurance contract a guaranteed survival payoff $S \geq G$ payable at terminal time T if the insured person is alive at maturity. This leads to a portfolio optimization problem with income-outcome stream A, initial wealth b and constraint on terminal wealth given by $S\,\mathbb{P}(\tau > T)$. To finance this constraint our capitalized net value $b + V_0(A)$ must exceed the initial capital $G\,\mathbb{P}(\tau > T)P(0,T)$ needed for ensuring the terminal wealth constraint. By applying the modified Proposition 3.7 we obtain the following optimal strategy

- sell for each t_i a contract that delivers a payoff $B(t_i) = B\,\mathbb{P}(\tau > t_i)$ at t_i,
- buy for each t_i a contract that delivers at t_i

$$C(t_i) = D\,P(t_i, t_{i+1})\,\mathbb{P}(t_i < \tau \leq t_{i+1}),$$

- buy $G\,\mathbb{P}(\tau > T)$ zero coupon bonds with maturity T
- invest $b + V_0(B) - G\,P(0,T)\mathbb{P}(\tau > T)$ in a Call on $I(\lambda H(T))$ with strike $G\,\mathbb{P}(\tau > T)$ and maturity T with λ solving the equation

$$b + V_0(A) = G\,\mathbb{P}(\tau > T)\,P(0,T) + V_0\left((I(\lambda H(T)) - G\,\mathbb{P}(\tau > T))^+\right),$$

- at each t_i use the income $B(t_i)$ to deliver the payoff w.r.t. our initially sold contract,
- at each t_i take the payoff from our initially bought contract to satisfy our consumption $C(t_i)$.

With this strategy we obtain the terminal wealth

$$G\,\mathbb{P}(\tau > T) + (I(\lambda H(T)) - G\,\mathbb{P}(\tau > T))^+ .$$

We give an application of this procedure for the log-utility case in a one stock Black-Scholes model with constant coefficients as has been treated in the first example with $\theta = 0.3$. We consider a 30 year old male individual who pays a yearly premium of 1500 Euro for a time period of 30 years. This income stream is initially valued via

$$b + V_0(B) = 29290 \text{ Euro.}$$

He is injured against mortality risk with a risk sum of 100000 Euro which leads to the initial price of the consumption stream

$$C(0) + V_0(C) = 7826 \text{ Euro.}$$

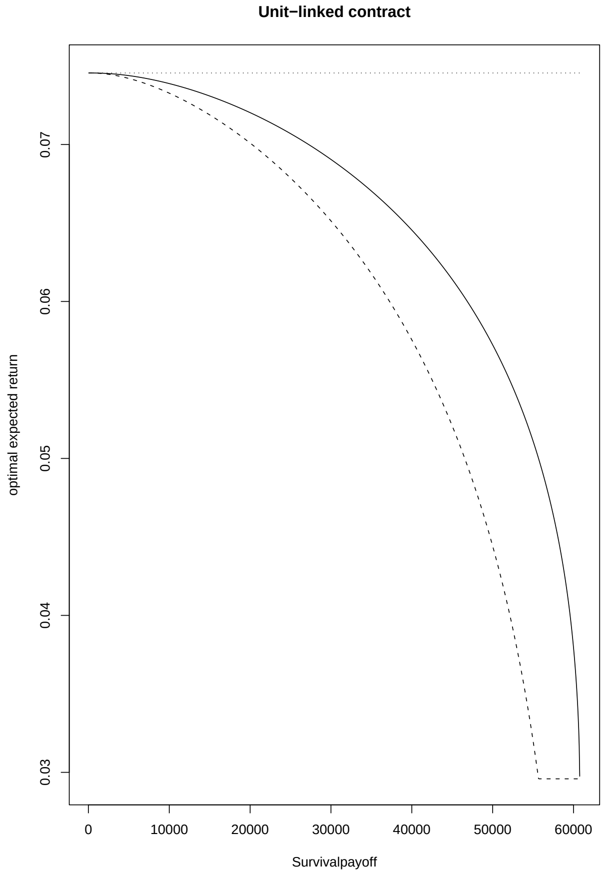

Fig. 13. Optimal expected return for a unit-linked contract with survival payoff

Dependent on the survival payoff we have calculated the optimal expected return from managing a unit-linked contract w.r.t.

$$b' = b + V_0(B) - C(0) + V_0(C) = 21463 \text{ Euro},$$

the initial net capital that can be invested. We may obtain the plot as in our first example and observe that a survival payoff less than 60000 Euro can be financed by the income stream. Furthermore the higher the survival payoff the less is the optimal expected return. In the limits we get the return from a pure Bond respectively optimal Merton strategy. As in the previous plot we have included the theoretical lower and upper bound on the optimal expected return.

4 Conclusion

We presented a model, which allows for pricing, managing and designing UL-LIG products subject to various legal circumstances. The basic model is a multi-stage stochastic optimization model for managing assets to balance liabilities emerging from such a contract. The model may be used to find the fair price of such a product. However, the model also finds risks, in particular the shortfall risk associated with such a contract and allows for limiting and controlling this source of risk. Extensions in many directions such as to include transaction costs, taxes, legal constraints, lapse risk etc. may be added easily to this basic model. The availability of extremely efficient mathematical programming solvers based on interior point methods (such as e.g. MOSEK) makes the described programs amenable to fast computation.

The main idea is to view insurance as an additional investment category. The return of such an investment is contingent to the death event and is sharply distinct from the return of market investments. It turns out that the fraction invested in insurance depends on the mortality risk as well as the fund performance. The rule is that the higher the funds and the lower the mortality risk, the smaller is the amount invested in insurance. Numerical results were presented to substantiate the usability of this model for management purposes.

In a second part, an analytic model for optimal investment under guarantee was studied. It is based on the Geometric Brownian Motion model for prices of all investment categories. Therefore, in this model the investment in insurance cannot be optimized and we only considered the situation of a fixed predetermined insurance part. For such a simplified model, we were able to demonstrate – adopting the log utility – that the optimal management under guarantee can be found by rules of going short or long in term contracts, buying options and using the delta hedge. This is in accordance with the findings of Brennan and Schwartz (1976). However, it turns out that the optimal dynamic strategy is to capitalize the expected premium inflow right from the beginning, and not to wait until the inflows happens.

While the detailed stochastic optimization model allows for considering all types costs and constraints and comes up with a optimal multi-stage solution, the analytic model shows how conventional instruments have to be composed for hedging the contract.

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