The h-Transformation and its Application to Mathematical Finance

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Abstract

In this paper the h-transformation is applied to optimal stopping problems for one-dimensional diffusions. We will establish sufficient conditions which lead to continuation regions in interval form for discounted reward functions $e^{-\lambda t}g(x)$. We will see that the approach can be applied to various stopping problems related to mathematical finance. In particular we can easily compute the price of an American perpetual put in an extended Black-Scholes model.

Key words: American option, optimal stopping, portfolio optimization, diffusion, change of measure ;

JEL classification G11, G12 ;

Mathematics Subject Classification (1991): 60G40, 62L15, 60J70, 60J60 ;

1 . Introduction

For the following we consider a one-dimensional, regular, conservative diffusion

(1) $$X = ((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$$

on an open interval $E=(r_1,r_2)$ with $-\infty \le r_1 < r_2 \le +\infty$ in canonical form, see Freedman(1971), Rogers,Williams (1987) p.271 for a definition. Thus Ω consists of the set of all continuous functions from $[0,\infty)$ to E, $X_t(\omega)=\omega(t)$, $\mathcal{F}_t^0=\sigma(X_s : s\le t)$, $\mathcal{F}_t=\mathcal{F}_{t+}^0$ for all $t\ge 0$.

We assume that X is generated by a second order elliptic differential operator

$$A=\frac{1}{2}\sigma^2(x)\partial_x^2+\mu(x)\partial_x \tag{2}$$

with strictly positive continuous σ and continuous μ. This means that

$$(f(X_t)-f(X_0)-\int_0^t Af(X_s)ds)_{t\ge 0}$$

defines a martingale w.r.t. P_x for each $f\in C_K^2(E)$ and each $x\in E$. Here $C_K^2(E)$ denotes the space of twice continuously differentiable functions with compact support contained in E. The speed measure and scale function are determined in this case by

$$m(dx)=\frac{2}{\sigma^2(x)}\exp(B(x))dx \quad , \quad s(x)=\int_c^x \exp(-B(y))dy \tag{3}$$

with $B(x)=\int_c^x \frac{2\mu(y)}{\sigma^2(y)}dy$ and $c\in E$ arbitrarily chosen. Note that the diffusion is conservative. Thus the boundary points of E are inacessible and the corresponding semigroup is uniquely determined by its differential generator A.

We are interested in the problem of optimal stopping for a reward function of the form

$$(t,x)\to e^{-\lambda t}g(x) \tag{4}$$

with $\lambda>0$ and a measurable non negative function g. Let us denote by $\mathcal{S}$ the set of all $(\mathcal{F}_t)_{t\ge 0}$ Markov times. Then our aim is to determine an optimal Markov time τ_x^* such that

$$E_x e^{-\lambda\tau_x^*}g(X_{\tau_x^*})1_{\{\tau_x^*<\infty\}}=\sup_{\tau\in\mathcal{S}} E_x e^{-\lambda\tau}g(X_\tau)1_{\{\tau<\infty\}}$$

and to compute the optimal value function

$$v(x)=\sup_{\tau\in\mathcal{S}} E_x e^{-\lambda\tau}g(X_\tau)1_{\{\tau<\infty\}} \tag{5}$$

for all $x\in E$. If X is a geometric Brownian motion v would denote the price of an American perpetual option corresponding to the payoff g, see section 3.

In section 4 we treat the perpetual put on a stock in an extended Black Scholes model with state dependent volatility. The calculation of the put price is a non trivial problem where the results of section 2 can be applied to.

Another interesting example related to mathematical finance is given by a diffusion with generator

$$A = \frac{1}{2}x^2(1-x)^2\partial_x^2 + (\hat{b}-x)x(1-x)\partial_x$$

on state space $E = (0,1)$, with $\hat{b} \in (0,1)$. It is related to portfolio optimization under consideration of transaction costs, see Morton,Pliska (1995). In section 5 we adapt their ideas to power utility functions.

The above described optimal stopping problem is well studied, see Shiryayev (1978), van Moerbeke (1974a),(1974b) and others and solutions are usually obtained by solving the corresponding free boundary value problem

$$\begin{array}{rcl} Av &=& \lambda v \quad \text{on } \mathcal{C} \\ v = g \quad , & & v' = g' \quad \text{on } \partial\mathcal{C} \quad . \end{array}$$

Here $\mathcal{C} = \{x : v(x) > g(x)\}$ denotes the continuation region and its compliment $\mathcal{E} = \{x : v(x) = g(x)\}$ is called stopping or early exercise region. Another approach was suggested by Salminen (1985) who applied the Choquet representation of λ-excessive functions to optimal stopping.

Recently Beibel and Lerche (1997) use in the case of Brownian motion a decomposition of the reward process into a positive martingale and a uniformly bounded process to give another view to optimal stopping problems. Their ideas combined with some aspects mentioned in Salminen are the starting point for this paper. We will point out the relations which stand behind the approach of Beibel and Lerche and how it can be extended to one dimensional diffusions. The basic technique we use is the h-transformation and an associated change of measure, see the following section.

2 . The h-transformation applied to optimal stopping

Let X be a one-dimensional diffusion on E with generator (2) . We fix a discount factor $\lambda > 0$ and consider a solution of

$$\frac{1}{2}\sigma^2(x)h''(x) + \mu(x)h'(x) = \lambda h(x) \quad \text{on E.} \tag{6}$$

Then $(e^{-\lambda t}h(X_t))_{t\geq 0}$ defines a local martingale w.r.t. P_x for all $x \in E$. We assume that it is in fact a martingale. Then for each starting point $x \in E$ a probability measure $\bar{P}_x$ can be defined on $(\Omega, \mathcal{F}_\infty)$ by

$$\bar{P}_x(A) = E_x e^{-\lambda t} \frac{h(X_t)}{h(x)} 1_A \quad \text{f.a. } A \in \mathcal{F}_t, t \geq 0. \tag{7}$$

We denote by $(T_t)_{t\geq 0}$ the semigroup corresponding to $(P_x)_{x\in E}$ and as usual by $b\mathcal{B}$ the space of bounded measurable functions, $\mathcal{B} = \mathcal{B}(E)$ the Borel $\sigma-$ algbra on E.

It is well known, see Borodin, Salminen (1996) p.33 or Sharpe (1988) p.298 that the family of probability measures $(\bar{P}_x)_{x\in E}$ defines a regular diffusion on E with semigroup $(\bar{T}_t)_{t\geq 0}$ given by

$$\bar{T}_t f = \frac{1}{h} e^{\lambda t} T_t(fh) \quad \text{for all } f \in b\mathcal{B}, t \geq 0. \tag{8}$$

Furthermore its speed measure $\bar{m}$ and scale function $\bar{s}$ are determined by

$$\bar{m}(dx) = h^2(x) m(dx) \quad , \quad \bar{s}(dy) = \frac{1}{h^2(y)} s(dy) \quad , \tag{9}$$

From this relation it is easily verified that the generator $\bar{A}$ of $(\bar{P}_x)_{x\in E}$ coincides with

$$\frac{1}{2}\sigma^2(x)\partial_x^2 + (\mu(x) + \sigma^2(x)\frac{h'(x)}{h(x)})\partial_x \tag{10}$$

on $C_K^2(E)$. Furthermore

$$\bar{A}f = \frac{1}{h}(A - \lambda)(fh) \tag{11}$$

for all $f \in b\mathcal{E}$ such that $fh \in D(A)$.

We want to exploit this h-transformation for optimal stopping to derive continuation regions in interval form. In order to do this we have to choose an appropriate λ-harmonic function h satisfying (6). For this purpose we recall the construction of positive, decreasing respectively increasing solutions of (6).

Let V be the unique solution of

$$\frac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) = 1 \quad , \quad V(c) = 0, V'(c) = 0. \tag{12}$$

It can be written as

$$V(x) = \int_c^x s'(y) \int_c^y \frac{2}{s'(z)\sigma^2(z)} dy$$

for all $x \in E$. Since the boundaries of E are inaccessible, Feller's test of explosion provides

$$V(r_1) = +\infty = V(r_2) \quad .$$

Following Mandl pp.25 the unique λ-excessive function u satisfying (6) with $u(c) = 1, u'(c) = 0$ fulfills

$$1 + V(x)\lambda \leq u(x) \leq \exp(\lambda V(x)) \quad .$$

We define for all $x \in E$

$$(13) \qquad u_1(x) = u(x) \int_x^{r_2} \frac{p'(y)}{u(y)^2} dy \quad , \quad u_2(x) = u(x) \int_{r_1}^{x} \frac{p'(y)}{u(y)^2} dy \quad .$$

Then compare to Mandl (1968) u_1, u_2 are positive decreasing respectively increasing solutions of (6) and they satisfy

$$(14) \qquad u_1(r_1) = +\infty \quad , \quad u_2(r_2) = +\infty \quad .$$

These λ-harmonic functions u_1, u_2 play an important role for optimal stopping by using their transformed measures. It turns out with Lai (1973) that the martingale property of $(e^{-\lambda t} u_i(X_t))_{t \geq 0}$ depends on whether r_i is a natural or entrance boundary.

2.1 Lemma:

(i) r_1 is a natural boundary iff $(e^{-\lambda t} u_2(X_t))$ is a martingale.

(ii) r_2 is a natural boundary iff $(e^{-\lambda t} u_1(X_t))$ is a martingale.

(iii) Let r_1, r_2 be natural boundaries and let h be a positive solution of (6). Then $(e^{-\lambda t} h(X_t))$ defines a positive martingale.

Proof: The assertions follow immediately from Lai (1973) p.434 and p.428.

From Salminen (1985) we know that $u_i, i = 1, 2$ are minimal λ-excessive functions. This means that

$$(15) \qquad \lim_{t \to \infty} X_t = r_1 \quad P_x^{(1)} - a.s. \quad , \quad \lim_{t \to \infty} X_t = r_2 \quad P_x^{(2)} - a.s.$$

with $(P_x^{(1)})_{x \in E}, (P_x^{(2)})_{x \in E}$ denoting the u_1-respectively u_2-transformed family of probability measures.

These facts can be exploited to derive continuation regions in interval form. At first we will give sufficient conditions which lead to one-sided regions.

2.2 Theorem: *Let r_2 be a natural boundary and $g : E \to [0, \infty)$ be a non negative measurable reward function that fulfills*

(A1) $x \to \frac{g(x)}{u_1(x)}$ *is uniformly bounded with a unique maximum at $b \in E$.*

(A2) g is continuously twice differentiable on $(r_1, b+\epsilon)$ for some $\epsilon > 0$ and it holds $Ag \leq \lambda g$ on (r_1, b).

Then the continuation region is given by $\mathcal{C} = (b, r_2)$ and $\tau^* = \inf\{t \geq 0 : X_t \leq b\}$ is an optimal Markov time. Furthermore the optimal value function v fulfills

$$v(x) = \begin{cases} u_1(x)\frac{g(b)}{u_1(b)} & \text{if } x > b \\ g(x) & \text{if } x \leq b \end{cases} .$$

Proof: Since r_2 is natural, $(e^{-\lambda t}u_1(X_t))$ defines a positive martingale and we can exploit the u_1-transformed familiy $P_x^{(1)}$ of probability measures. Depending on $x \in E$ we introduce the function f by $f(y) = u_1(x)\frac{g(y)}{u_1(y)}$ for all $y \in E$. Then we take the decomposition

$$e^{-\lambda t}g(X_t) = e^{-\lambda t}\frac{u_1(X_t)}{u_1(x)}f(X_t)$$

of the reward process and obtain for each Markov time τ

$$E_x e^{-\lambda\tau}g(X_\tau)1_{\{\tau<\infty\}} = E_x^{(1)}f(X_\tau)1_{\{\tau<\infty\}} \leq u_1(x)\frac{g(b)}{u_1(b)} .$$

Thus the right side is an upper bound for the optimal value $v(x)$. If $x > b$ this upper bound can be attained by the hitting time $\tau_b = \inf\{t \geq 0 : X_t = b\}$, since X_t tends to the lower boundary r_1 $P_x^{(1)}$ a.s. and

$$(16) \qquad E_x e^{-\lambda\tau_b}g(X_{\tau_b})1_{\{\tau_b<\infty\}} = E_x^{(1)}f(X_{\tau_b})1_{\{\tau<\infty\}} = u_1(x)\frac{g(b)}{u_1(b)} .$$

This implies that (b, r_2) is contained in the continuation region. It remains to examine that immediate stopping is optimal for $x \in (r_1, b)$. This can be deduced in the following way. For an arbitrary Markov time τ it holds

$$(17) \quad E_x e^{-\lambda\tau}g(X_\tau)1_{\{\tau<\infty\}} = E_x^{(1)}f(X_\tau)1_{\{\tau<\infty\}} \leq E_x^{(1)}f(X_{\tau\wedge\tau_b})1_{\{\tau<\infty\}}$$

since f has a maximum at b. To treat the right side we use a localisation argument with a sequence of hitting times $(\tau_{a_n}), a_n \downarrow r_1$. Since f is bounded dominated convergence implies

$$(18) \qquad E_x^{(1)}f(X_{\tau\wedge\tau_b})1_{\{\tau<\infty\}} = \lim_{n\to\infty} E_x^{(1)}f(X_{\tau\wedge\tau_{a_n}\wedge\tau_b})1_{\{\tau<\infty\}}$$

The function f is C^2 on $[a_n, b]$. Thus with optional sampling or Dynkin's formula we obtain with $\tau_{a_n,b} = \tau_{a_n} \wedge \tau_b$

(19) $$E_x^{(1)} f(X_{\tau\wedge\tau_{a_n,b}}) = f(x) + E_x^{(1)} \int_0^{\tau\wedge\tau_{a_n,b}} \bar{A}f(X_s)ds \quad .$$

Due to

$$\bar{A}f = \frac{1}{u_1}(A-\lambda)(fu_1) = \frac{u_1(x)}{u_1}(A-\lambda)g$$

on (r_1, b) assumption (ii) implies $\bar{A}f \leq 0$ on (r_1, b) and therefore

$$E_x e^{-\lambda\tau} g(X_\tau) 1_{\{\tau<\infty\}} \leq f(x) = g(x) \quad .$$

Thus the payoff from immediate stopping cannot be improved by any Markov time and the Theorem is proved.

If we consider the λ-harmonic u_2 instead of u_1 the same conclusions can be drawn and we obtain an analogous Theorem.

2.3 Theorem: *Let r_1 be a natural boundary and $g : E \to [0, \infty)$ be a measurable reward function that fulfills*

(B1) $x \to \frac{g(x)}{u_2(x)}$ is uniformly bounded with a unique maximum at $b \in E$.

(B2) g is continuously twice differentiable on $(b - \epsilon, r_2)$ for some $\epsilon > 0$ and it holds $Ag \leq \lambda g$ on (b, r_2).

Then the continuation region is given by $\mathcal{C} = (b, r_2)$ and $\tau^* = \inf\{t \geq 0 : X_t \geq b\}$ defines an optimal Markov time. Furthermore

$$v(x) = \begin{cases} u_2(x)\frac{g(b)}{u_2(b)} & \text{if } x < b \\ g(x) & \text{if } x \geq b \end{cases} \quad .$$

To obtain a two sided stopping region we have to consider a positiv linear combination of u_1 and u_2.

2.4 Theorem: *Let r_1, r_2 be natural boundaries and $g : E \to [0,\infty)$ be a measurable function that satisfies*

(C1) there exist $q_1, q_2 > 0$ such that the λ-harmonic function h defined by $h(x) = q_1 u_1(x) + q_2 u_2(x)$ fulfills $x \to \frac{g(x)}{h(x)}$ is uniformly bounded with maximum attained at exactly two points $b_1 < b_2$.

(C2) g is continuously twice differentiable on $(r_1, b_1 + \epsilon) \cup (b_2 - \epsilon, r_2)$ for some $\epsilon > 0$ and $Ag \leq \lambda g$ on $(r_1, b_1) \cup (b_2, r_2)$.

Then $\mathcal{C} = (b_1, b_2)$ and $\tau^* = \inf\{t \geq 0 : X_t \in (b_1, b_2)\}$ is an optimal Markov time. Furthermore

$$v(x) = \begin{cases} h(x)\frac{g(b_1)}{h(b_1)} = h(x)\frac{g(b_2)}{h(b_2)} & \text{if } b_1 < x < b_2 \\ g(x) & \text{if } x \notin (b_1, b_2) \end{cases} .$$

Proof: Since r_1, r_2 are natural we can consider the h-transformed family $(\bar{P}_x)_{x \in E}$ of probability measures defined by (7). For $x \in E$ we introduce $f(y) = \frac{g(y)}{h(y)} h(x)$ and obtain as in Theorem (2.2) that the optimal value $v(x)$ is bounded by the maximum $f(b_1) = f(b_2) = \frac{g(b_1)}{h(b_1)} h(x)$ due to

$$E_x e^{-\lambda\tau} g(X_\tau) 1_{\{\tau < \infty\}} = \bar{E}_x f(X_\tau) 1_{\{\tau < \infty\}}$$

for all Markov times τ. If $x \in (b_1, b_2)$ the first exit time of (b_1, b_2) coincides with τ^* , is $\bar{P}_x$ a.s. finite, and attains the upper bound $f(b_1) = f(b_2)$ as expected payoff. Thus τ^* is optimal and (b_1, b_2) is contained in the continuation region.
If $x < b_1$ the expected payoff of an arbitrary Markov time τ can be improved by $\tau \wedge \tau_{b_1}$ and as in Theorem (2.2) assumption (C2) leads to $v(x) \leq g(x)$. Thus $(r_1, b_1]$ lies in the stopping region. The same holds true for (b_2, r_2) by an analogous argument.

Contrary to the preceding Theorems the condition (C1) cannot be easily examined. For the case of Brownian motion Beibel, Lerche (1997) give sufficient conditions which can easily adapted to general diffusions. Simpler is the symmetric case, i.e. there exists an $m \in E$ such that $\sigma(m + x) = \sigma(m - x)$, $\mu(m + x) = -\mu(m - x)$ for all $m + x \in E$. For a symmetric reward function g with $g(m+x) = g(m-x)$ we can choose $q_1 = q_2$ and have to examine whether $y \to g(m + y)/h(m + y)$ is uniformly bounded at a point β. Then (C1) is fulfilled with $b_1 = m - \beta, b_2 = m + \beta$.

In the following we will apply these results to stopping problems coming from mathematical finance.

3 . American perpetual options in the Black-Scholes model

In the Black-Scholes model the price process $(S_t)_{t\geq 0}$ of a stock is given by a geometric Brownian motion

$$S_t = S_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t) \quad , \quad t \geq 0. \tag{20}$$

This is a diffusion with state space $(0,\infty)$, natural boundaries and differential generator $\frac{1}{2}x^2\sigma^2\partial_x^2 + \mu x\partial_x$ with $\mu \in I\!R, \sigma > 0$. We assume that money on a bank account grows with a constant interest rate $\lambda > 0$. To calculate prices for European or American claims one has to consider the risk neutral probability measure which leads to a change of drift of the stock price process. Its differential generator is then given by

$$A = \frac{1}{2}x^2\sigma^2\partial_x^2 + \lambda x\partial_x \quad . \tag{21}$$

Decreasing respectively increasing λ-harmonic functions u_1, u_2 which satisfy $Au_i = \lambda u_i$ are given by

$$u_1(x) = x^{-\alpha} \quad , \quad u_2(x) = x \tag{22}$$

with $\alpha = 2\lambda/\sigma^2$. Then the conditions (A1),(A2) of Theorem (2.2) for a reward function g are fulfilled if $y^\alpha g(y)$ is bounded with a unique maximum at $b \in (0,\infty)$ and $Ag \leq \lambda g$ on $(0,b)$. As a generalization of the ordinary put we consider reward functions g of the form $g(x) = f((K-x)^+)$ with $K > 0$.

3.1 Theorem: *If f is strictly increasing , twice differentiable and concave with $f(0) = 0$ then $x \to x^\alpha f((K-x)^+)$ is bounded at a unique $b \in (0,K)$ and the early exercise region corresponding to the reward function $g(x) = f((K-x)^+)$ is given by $\mathcal{E} = (0,b]$. Furthermore the price of the American perpetual option satisfies*

$$v(x) = \begin{cases} x^{-\alpha}b^\alpha f((K-b)^+) & \text{if } x > b \\ g(x) & \text{if } x \leq b \end{cases} \quad .$$

Proof: At first we examine that the function $\phi(y) = y^\alpha g(y)$ has a unique maximum b in $(0,K)$. Due to

$$\phi'(y) = 0 \quad \Leftrightarrow \quad \alpha f(K-y) - yf'(K-y) = 0$$

this follows from $\psi(y) = f(K-y) - yf'(K-y)$ is strictly decreasing with $\psi(0) > 0, \psi(K) < 0$. Secondly it holds $Ag \leq \lambda g$ on $(0,K)$ since

$$Ag(x) = \frac{1}{2}\sigma^2 x^2 f''(K-x) - \lambda x f'(K-x) \leq 0 \quad .$$

Thus Theorem (2.2) can be applied and yields the assertion.

For $f(x) = x$ we obtain the American put with early exercise region $\mathcal{E} = (0, K\frac{\alpha}{1+\alpha}]$. In the case $f(x) = x^\beta$ with $0 < \beta \leq 1$ the unique maximum of $x \to x^\alpha f((K-x)^+)$ is attained at $K\frac{\alpha}{\alpha+\beta}$ and therefore $\mathcal{E} = (0, K\frac{\alpha}{\alpha+\beta}]$.

In the convex case $\beta > 1$ the Theorem can not be applied but it is easy to examine that conditions (A1),(A2) are fulfilled if $\beta \leq 2\alpha/(1+\alpha)$.
The reward function $g(x) = e^{-\gamma x}$ with $\gamma > 0$ leads to an early exercise region $\mathcal{E} = (0, \alpha/\gamma]$.

An example of an unbounded reward function g which leads to an early exercise region $(0, b]$ is given by $g(x) = x^{-\beta}e^{-\gamma x}$ with $0 < \beta < \alpha$. Then $x^\alpha g(x)$ is bounded with maximum at $b = \frac{\alpha-\beta}{\gamma}$. Furthermore $Ag \leq \lambda g$ on $(0, b)$, since $Ag(x) = g(x)P(x)$ with

$$P(x) = \frac{1}{2}\sigma^2\beta(1+\beta) + \gamma\beta\sigma^2 x + \frac{1}{2}\gamma^2\sigma^2x^2 - \lambda\beta - \lambda\gamma$$

and $P(0) \leq \lambda, P(b) \leq \lambda$. Hence $(0, \frac{\alpha-\beta}{\gamma}]$ denotes the early exercise region.

For call-type reward functions g conditions (B1),(B2) of Theorem (2.3) read $x \to g(x)/x$ is bounded at a unique maximum b and $Ag \leq \lambda g$ on $(b, +\infty)$.

This of course does not hold in the case of an ordinary call $g(x) = (x-K)^+$. Reward functions of the form $g(x) = ((x-K)^+)^\beta$ with $0 < \beta < 1$ satisfy the above conditions. The unique maximum of $g(x)/x$ is attained at $b = K\frac{1}{1-\beta}$ and $Ag \leq \lambda g$ on (b, ∞) since

$$Ag(x) \leq \lambda g(x) \quad \Leftrightarrow \quad P(x) \leq 0$$

with $P(x) = \frac{1}{2}\sigma^2x^2\beta(\beta-1) + \lambda x\beta(x-K) - \lambda(x-K)^2$ and $P(K\frac{1}{1-\beta}) \leq 0$.

4 . The perpetual put in an extended Black-Scholes model

A generalization of the classical Black.Scholes model of section 3 can be obtained by assuming a state dedending volatility. This means that the stock price process fulfills the dynamics

$$dS_t = S_t(\mu dt + \sigma(S_t)dW_t) \quad . \tag{23}$$

For pricing options we have to consider, as in the preceding section, the evolution under the risk neutral probability measure. Hence with λ denoting the constant interest rate we have to consider a diffusion with generator

$$A = \frac{1}{2}\sigma^2(x)x^2\partial_x^2 + \lambda x\partial_x \tag{24}$$

on $(0,\infty)$. We assume that $\sigma : (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$0 < \inf_{x>0} \sigma(x) \leq \sup_{x>0} \sigma(x) < \infty \tag{25}$$

Then ,for each starting point $x \in (0,\infty)$, a unique weak solution of the stochastic differential equation (23) with $S_0 = x$ can be defined up to an explosion time, which is the first exit time of $(0,\infty)$. Due to Feller's test the condition (25) yields that an explosion cannot occur amlost surely. Thus the familiy of laws of weak solutions of (23), which depend on the starting point $x \in (0,\infty)$, forms a conservative regular diffusion with generator A and natural boundaries. Note that X denotes its coordinate process.

We want to apply the results of section 2 for calculating the price

$$v(x) = \sup_{\tau} E_x e^{-\lambda\tau}(K - X_\tau)^+ \tag{26}$$

of an American perpetual put with strike $K > 0$. Therefore we have to determine the decreasing respectively increasing solutions u_1, u_2 of

$$Au = \lambda u \quad \text{on} \quad (0,\infty). \tag{27}$$

Since $(e^{-\lambda t} X_t)_{t\geq 0}$ defines a P_x-martingale for all $x \in (0,\infty)$

$$u_2(x) = x$$

is a non negative increasing solution of (27).

To determine a decreasing one the reduction procedure of d'Alembert for ordinary linear differential equations can be applied. We define

$$\zeta(x) = \exp\left(-\int_1^x \frac{2\lambda}{\sigma^2(y)y} dy\right) \quad , \quad \eta(x) = -\int_x^\infty \frac{2\lambda}{\sigma^2(y)y^2}\zeta(y)dy \tag{28}$$

for all $x \in (0,\infty)$. Then

$$u_1(x) = x\eta(x) + \zeta(x) \quad , \quad x \in (0,\infty)$$

is a decreasing solution of (27) since

$$u_1'(x) = \eta(x) < 0 \quad \text{for all } x \in (0,\infty).$$

Furthermore an easy calculation yields, due to (25),

$$\lim_{x\to 0} u_1(x) = \infty \quad , \quad \lim_{x\to\infty} u_1(x) = 0 \quad . \tag{29}$$

An application of Theorem 2.2 provides

4.1 Theorem: *The function* $x \to \frac{(K-x)^+}{u_1(x)}$ *has a unique maximum at a point* $b \in (0, K)$. *The price of the American perpetual put with strike* K *and initial stock price* x *satisfies*

$$v(x) = \begin{cases} (K-b)\frac{u_1(x)}{u_1(b)} & \text{if } x > b \\ K - x & \text{if } x \le b \end{cases} \quad . \tag{30}$$

The Markov time $\tau = \inf\{t \ge 0 : X_t \le b\}$ *defines an optimal exercise strategy.*

Proof: The function $f(x) = \frac{K-x}{u_1(x)}$ fulfills on $(0, K)$

$$f'(x) = \frac{-\zeta(x) - K\eta(x)}{u_1(x)^2} \quad . \tag{31}$$

Thus we have to show that the function $h(x) = -K\eta(x) - \zeta(x)$ equals zero at a unique point in $(0, K)$. Due to

$$\lim_{x\to 0} \frac{\eta(x)}{\zeta(x)} = \lim_{x\to 0} \frac{\eta'(x)}{\zeta'(x)} = \lim_{x\to 0} -\frac{1}{x} = -\infty$$

and $h(K) = -\eta(K) < 0$ there exists a point $b \in (0, K)$ such that $h(b) = 0$. This point is uniquely defined since

$$h'(x) = \zeta(x)\left(\frac{2\lambda}{\sigma^2(x)x}(1 - K\frac{1}{x})\right) < 0$$

on $(0, K)$. An application of Theorem 2.2 yields the remaining assertion.

Usually the integrals in (28) cannot be determined explicitly. But by using numerical methods the perpetual put price function can easily be computed. As an example we have considered the put for the case

$$\sigma(x) = \gamma\sqrt{c(\frac{x-m}{x+m})^2 + 1} \quad , \quad x > 0 \tag{32}$$

with $\gamma, m, c > 0$. Thus the state dependent volatility coincides in m with γ and increases to $\sqrt{c+1}\gamma$ as x increases to infinity or decreases to zero. The parameter c determines the variability of volatility.

For the case $m = 100, \gamma = 0.3, K = 100$ and interest rate $\lambda = 0.03$ we have calculated the put price for $c \in \{0.1, 1, 10\}$. The resulting plot, see the following figure, shows that the price functions w.r.t. $c = 0.1, 1$ almost coincide. For $c = 10$ the variation of volatility is large enough. This provides a significantly higher price on the continuation region compared to that of the

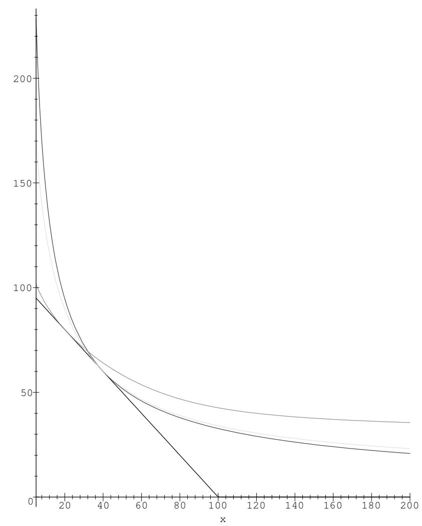

Figure 1: put prices

first two cases. Note that in the figure the payoff function $(K-x)^+$ and the function $x \to (K-b)\frac{u_1(x)}{u_1(b)}$ is plotted for the three choices of c.

Other choices for σ like decreasing or increasing volatilities were also investigated and we emphasize that for given σ the early exercise region and put price function can be determined easily with numerical methods by using Theorem 4.1.

5 . Portfolio optimization

In this section we will consider stopping problems for a diffusion with state space $(0,1)$ and generator

$$A = \frac{1}{2}\sigma^2 x^2 (1-x)^2 \partial_x^2 + \sigma^2 x(1-x)(\hat{b}-x)\partial_x \tag{33}$$

with $\hat{b} \in (0,1)$. At first we will explain its relation to mathematical finance in particular portfolio optimization.

An obvious question for a capital investor is how to divide his money into a risk free bank account and a risky asset like a stock. We assume that the risky asset follows a geometric Brownian motion with dynamics

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad , \quad t \geq 0 \tag{34}$$

volatility σ and rate of expected return μ which is larger than the constant interest rate r of the bankaccount $\beta_t = e^{rt}$. We denote as in Kartzas (1997) p.3 by $(\pi_t)_{t\geq 0}$ a self financing trading strategy. Thus π_t is the fraction of wealth invested in the risky asset at time t. Associated to a portfolio strategy π is its wealth process V that satisfies

$$dV_t = V_t((1-\pi_t)rdt + \pi_t(\mu dt + \sigma dW_t)) \quad , \quad V_0 = x \tag{35}$$

with initial capital $x > 0$, see Morton, Pliska (1995) p.339.

It is well known that for a given finite time horizon $T > 0$ and a utility function U a portfolio strategy can be determined that maximizes the expected utility of terminal wealth

$$EU(V_T) \tag{36}$$

among all self-financing portfolio strategies, see Karatzas (1997) p.41 Th.2.3.2. In particular for $U_1(x) = \log(x)$ and $U_\alpha(x) = \frac{1}{\alpha}x^\alpha$ with $0 < \alpha < 1$ the optimal portfolio strategy consists of holding a constant fraction of wealth in the risky asset over time see Example 2.2.4, 2.2.5 in Karatzas (1997).

The practical problem is that continuously an investor has to change the number of shares of the risky asset to hold this optimal balance point. This causes transaction and management costs which may not be neglegible.

Morton and Pliska (1995) investigated the log-utility under consideration of transaction costs and suggested the following procedure. Starting with a fraction b of wealth in the risky asset we do not trade until a random stopping time τ_1. Then we rebalance our changed fraction to b, do not trade until a random stopping time τ_2, rebalance the portfolio etc.. The sequence of inter-transaction times $(\tau_{n+1} - \tau_n)_{n\in I\!N}$ should be chosen according to the solution of a stopping problem corresponding to the payoff process

$$\log(V_t/V_0) - (R-r)t \quad , \quad t \geq 0 \tag{37}$$

with $r < R < r + \frac{1}{2}\hat{b}(\mu - r)$. By assuming wealth proportional transaction costs they determined an optimal fraction b and growth rate R. Compare to section 3 and 4 of Morton, Pliska (1995).

In the following we will adapt this idea to the power utility function $U(x) = \frac{1}{\alpha}x^\alpha$. Note that allowing costless trading would lead to a growth of the optimal expected utility as

$$\frac{1}{\alpha}V_0^{\alpha}e^{\hat{R}T} \quad , \quad \hat{R} = r\alpha + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{\alpha}{1-\alpha} \quad . \tag{38}$$

Thus with transaction costs the expected utility of wealth would grow slowlier than $e^{\hat{R}T}$. Hence adapting the procedure of Morton and Pliska to power utility leads to a stopping problem corresponding to the payoff process

$$e^{-Rt}V_t^{\alpha}, t \geq 0 \tag{39}$$

with $r\alpha < R < \hat{R}$. Helpful for the treatment of this stopping problem is the following important observation. Starting with a fraction b of wealth in stock and holding the number $c = bV_0/S_0$ of stocks constant over time correspond to a portfolio strategy that satisfies $\pi_t = c\mathcal{S}_t/V_t$. Due to (35) it fulfills

$$d\pi_t = \pi_t(1-\pi_t)(\sigma^2(\hat{b}-\pi_t)dt + \sigma dW_t) \quad , \pi_0 = b. \tag{40}$$

Thus (π_t) evolves like a Markov process with generator A. Since in a trading strategy without transactions the number of shares of the bond is constant over time it holds $(1-\pi_t)V_t = (1-b)V_0e^{rt}$ and therefore

$$e^{-Rt}V_t^{\alpha} = ((V_0(1-b))^{\alpha}(\frac{1}{1-\pi_t})^{\alpha}e^{-(R-r\alpha)t} \quad \text{for all } t \geq 0 \quad . \tag{41}$$

Thus we have introduced a stopping stopping problem w.r.t. a Markov process with generator A and reward function $g(x) = \frac{1}{1-x}^{\alpha}$ and we can apply the results of the preceding sections. At first we examine that the boundary conditions are fulfilled. A scale function s is defined by

$$s(x) = \begin{cases} (x/1-x)^{1-2\hat{b}} & \text{if } \hat{b} < \frac{1}{2} \\ \log\frac{x}{1-x} & \text{if } \hat{b} = \frac{1}{2} \\ -(\frac{x}{1-x})^{1-2\hat{b}} & \text{if } \hat{b} > \frac{1}{2} \end{cases} \quad .$$

Thus for $\hat{b} = \frac{1}{2}$ the diffusion is non-exploding and recurrent on $(0,1)$. In the other cases it tends to the left respectively right boundary as $\hat{b}$ is less or larger than $\frac{1}{2}$. From the general form

$$f(x) = \frac{2}{(2\hat{b}-1)^2\sigma^2}((2\hat{b}-1)\log\frac{x}{1-x} - 1) + c1 + c2(\frac{x}{1-x})^{1-2\hat{b}}$$

for solutions of $Af = 1$ we can easily deduce that the diffusion is non exploding for $\hat{b} \neq \frac{1}{2}$ too. To apply the forgoing results we have to consider positive decreasing respectively increasing solutions of $Af = \lambda f$. They are given by

$$u_1(x) = (\frac{1-x}{x})^{\hat{b}-\frac{1}{2}+\frac{1}{2}\frac{\gamma}{\sigma}} \quad , \quad u_2(x) = (\frac{1-x}{x})^{\hat{b}-\frac{1}{2}-\frac{1}{2}\frac{\gamma}{\sigma}} \tag{42}$$

with $\gamma = \sqrt{\sigma^2(1-2\hat{b})^2 + 8\lambda}$. Note that

$$(43) \qquad \alpha_1 = \hat{b} - \frac{1}{2} + \frac{1}{2}\frac{\gamma}{\sigma} > 0 \quad , \quad \alpha_2 = \hat{b} - \frac{1}{2} - \frac{1}{2}\frac{\gamma}{\sigma} < 0$$

and that $0, 1$ are natural boundaries.

In the following we assume

$$(44) \qquad \tilde{b} = \frac{\mu - r}{\sigma^2}\frac{1}{1-\alpha} \in (0,1) \quad .$$

Note that $\tilde{b}$ denotes the fraction of wealth to be held constant over time when maximizing the expected utility of terminal wealth, see Karatzas (1997), Ex. 2.2.5.

In a first step we want to show that the continuation region of the reward g is non empty if the discount factor λ satisfies

$$(45) \qquad \max\{0, \frac{1}{2}\sigma^2\alpha(1-\alpha)(2\tilde{b}-1)\} < \lambda < \frac{1}{2}\tilde{b}(\mu - r)\alpha \quad .$$

5.1 Proposition *For λ satisfying the above inequalities we define*

$$\beta_{low}(\lambda) = \tilde{b} - \sqrt{\tilde{b}^2 - \frac{2\lambda}{\alpha(1-\alpha)\sigma^2}} \quad , \quad \beta_{up}(\lambda) = \tilde{b} + \sqrt{\tilde{b}^2 - \frac{2\lambda}{\alpha(1-\alpha)\sigma^2}} \quad .$$

Then $0 < \beta_{low}(\lambda) < \tilde{b} < \beta_{up}(\lambda) < 1$ and the continuation region contains $(\beta_{low}(\lambda), \beta_{up}(\lambda))$.

Proof: Due to $Ag(x) = P(x)g(x)$ with

$$P(x) = \frac{1}{2}\sigma^2\alpha(\alpha-1)x^2 + \sigma^2\alpha\tilde{b}(1-\alpha)x$$

it follows

$$(46) \qquad Ag(x) > \lambda g(x) \Leftrightarrow P(x) > \lambda \Leftrightarrow x \in (\beta_{low}(\lambda), \beta_{up}(\lambda))$$

Due to (45) the rightside interval is contained in $(0,1)$. Furthermore g is λ subharmonic on it. Thus starting from $x \in (\beta_{low}(\lambda), \beta_{up}(\lambda))$ the first exit time τ from $(\beta_{low}(\lambda), \beta_{up}(\lambda))$ fulfills

$$E_x e^{-\lambda\tau} g(X_\tau) = g(x) + E_x \int_0^\tau (Ag(X_s) - \lambda g(X_s))ds > g(x) \quad .$$

Hence x is contained in the continuation region.

To apply Theorem 2.4 we have to consider λ-harmonic functions of the form

$$u(x,c) = cu_1(x) + (1-c)u_2(x)$$

with $0 < c < 1$. Due to (45) the function $x \to \frac{g(x)}{u(x,c)}$ is bounded on $(0,1)$ since

$$\begin{aligned} -\alpha_2 &= \sqrt{(\frac{1}{2} - (1-\alpha)\tilde{b})^2 + \frac{2\lambda}{\sigma^2}} + \frac{1}{2} - (1-\alpha)\tilde{b} \\ &> \sqrt{(\frac{1}{2} - (1-\alpha)\tilde{b})^2 + \alpha(1-\alpha)(2\tilde{b}-1)} + \frac{1}{2} - (1-\alpha)\tilde{b} = \alpha \quad . \end{aligned}$$

An analogous argument as in Beibel, Lerche (1997) p.98 yields a unique $0 < c^* < 1$ such that $x \to \frac{g(x)}{u(x,c^*)}$ has a maximum attained at two points $\beta_l^*(\lambda) < \beta_u^*(\lambda)$. This can be exploited to verify

5.2 Theorem: *Let λ fulfill condition (45). Then the continuation region $\mathcal{C}(\lambda)$ w.r.t. the reward function $g(x) = (\frac{1}{1-x})^\alpha$ satisfies*

$$\mathcal{C}(\lambda) = (\beta_l^*(\lambda), \beta_u^*(\lambda))$$

with $0 < \beta_l^*(\lambda) < \beta_{low}(\lambda) < \tilde{b} < \beta_{up}(\lambda) < \beta_u^*(\lambda)$. Furthermore the first exit time from $\mathcal{C}(\lambda)$ is an optimal Markov time.

Proof: The above considerations imply the condition (C1) of Theorem 2.4 and it remains to prove $\beta_l^*(\lambda) < \beta_{low}(\lambda) < \tilde{b} < \beta_{up}(\lambda) < \beta_u^*(\lambda)$. Then condition (C2) is satisfied and the assertion follows with Theorem 2.4. Since the function $\frac{g}{u(\cdot,c^*)}$ attains its maximum at $\beta \in \{\beta_l^*(\lambda), \beta_u^*(\lambda)\}$ it follows

$$\frac{g'(\beta)}{g(\beta)} = \frac{u'(\beta)}{u(\beta)} \quad , \quad \frac{g''(\beta)}{g(\beta)} < \frac{u''(\beta)}{u(\beta)}$$

Thus $\frac{Ag(\beta)}{g(\beta)} < \frac{Au(\beta)}{u(\beta)} = \lambda$ and the desired inequalities follow from (46).

The boundaries $\beta_l^*(\lambda), \beta_u^*(\lambda)$ can easily be computed numerically. For the case $\mu = 0.09$, $r = 0.03$, $\sigma = 0.5$, $\alpha = 0.6$ we have plotted the continuation region with its inner approximations $\beta_{low}(\lambda), \beta_{up}(\lambda)$ for

$$\lambda \in (\frac{1}{2}\sigma^2\alpha(1-\alpha)(2\tilde{b}-1), \frac{1}{2}\tilde{b}(\mu - r)\alpha) \quad .$$

The obtained figure demonstrate the following property: the larger λ the smaller is the continuation region. Thus, to obtain a larger growth rate of expected utility of wealth, more frequent transactions are necessary. For $\lambda = \frac{1}{2}\tilde{b}(\mu - r)\alpha$ the continuation region is empty, since continously rebalancing is necessary to obtain the optimal growth rate when trading is costless.

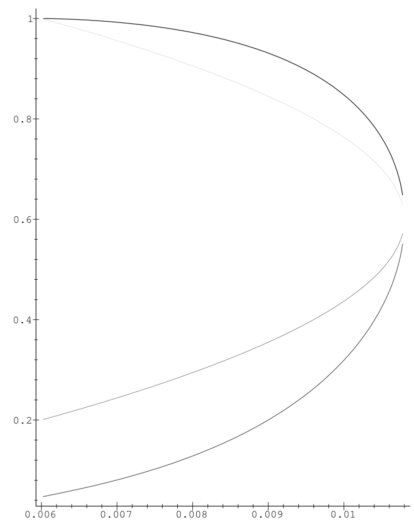

Figure 2: continuation regions in dependence of λ

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