Bounds for the American perpetual put on a stock index

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Abstract

Let us consider n stocks with dependent price processes each following a geometric Brownian motion. We want to investigate the American perpetual put on an index of those stocks. We will provide inner and outer boundaries for its early exercise region by using a decomposition technique for optimal stopping.

JEL classification G120, G130

1 . Introduction

Index options are commonly traded securities. Thus it is useful to provide reasonable pricing formulas. Assuming that the index itsself follows a geometric Brownian motion would lead to an ordinary Black-Scholes model of one single stock and their well known pricing formulas can be applied. As Lamperton, Lapeyre (1993) pointed out, this simplification is not consistent as far as the weighted sum of geometric Brownian motions behaves not like a geometric Brownian motion itsself. To be more precise we have to analyze the following more complex model. Let

$$S(t) = (S_1(t), \ldots, S_n(t)) \quad , t \geq 0$$

be the price process of n-stocks such that

$$dS_i(t) = S_i(t)(rdt + \sigma_i dW_i(t)) \quad \text{for all } 1 \leq i \leq n \quad . \tag{1}$$

The constants $r > 0, \sigma_1, \ldots, \sigma_n > 0$ denote the interest rate and the volatility of each stock. $W_1, \ldots, W_n$ are dependent standard Wiener-processes with $[W_i, W_j] = \rho_{ij} t$ as quadratic covariation process for each $1 \leq i, j \leq n$.
These price processes live on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$ with filtration generated by the n Wiener processes. The assumptions for the dynamics in (1) imply that P is a risk neutral probability measure, which in

general can be achieved by a Girsanov transformation. Hence the discounted price processes are martingales w.r.t. P and the fair price of contingent claims are their expected discounted payoff, see Karatzas, Shreve (1998).

The n- dimensional price process can also be considered as a Markov-process with state space $(0,\infty)^n$ and generator

$$(2) \qquad A = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2 \frac{\partial^2}{\partial x_i} + \sum_{i<j}\rho_{ij}\sigma_i\sigma_j x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + r\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \quad .$$

According to $a = (a_1, \ldots, a_n) \in (0,\infty)^n$ the index I_a of n stocks is defined by

$$I_a(t) = \sum_{i=1}^{n} a_i S_i(t) = a \cdot S(t) \quad \text{for all } t \geq 0 \quad .$$

Our goal is to consider a put on the index, which is an option with payoff $(K - a \cdot S(t))^+$, K denoting the strike-price. Let $P_x = P(\cdot|S(0) = x)$ for all $x \in (0,\infty)^n$. Then

$$p(x,T) = E_x\, e^{-rT}(K - a \cdot S(T))^+$$

is the fair price of the European put with running time T and can be computed explicitly, see Lamperton, Lapeyre (1993) .
In the case of an American put the early exercise is possible according to each stopping time τ. Hence the fair price is given by

$$(3) \qquad v_a(x,T) = \sup_{\tau \leq T} E_x e^{-r\tau}(K - a \cdot S(\tau))^+ \quad ,$$

see Karatzas, Shreve (1998). This computation respectively the determination of the optimal early exercise strategy concerns the theory of optimal stopping. In general, exact pricing formulas for American options with finite running time seem not to be obtainable. Even in the ordinary one stock model the price can only be computed numerically. Simpler is the analysis of the corresponding stopping problem with infinite time horizon which would lead to an American put with infinite running time. For this, so called American perpetual put, an exact pricing formula can be calculated in the ordinary one stock Black Scholes model. Thus, for the case of index options, it is natural to treat first this case of infinite running time. The hope is that this investigation yields new ideas to obtain approximations in the finite running time case too. Furthermore, the knowledge of the early exercise region can be exploited for numerical calculations in the finite running time case, as will be explained in section 4.

Let me denote by $\mathcal{S}$ the set of all stopping times and define

$$v_a(x) = \sup_{\tau \in \mathcal{S}} E_x e^{-r\tau}(K - a \cdot S(\tau))^+ \quad \text{for all } x \in (0,\infty)^n, \tag{4}$$

which is the fair price of an American perpetual put with initial stock price $x = (x_1 \ldots, x_n)$. Then the region $\mathcal{E}_a$ of early exercise is defined by

$$\mathcal{E}_a = \{x \in (0,\infty)^n : v_a(x) = (K - a \cdot x)^+\} \quad .$$

Its complement

$$\mathcal{C}_a = \{x \in (0,\infty)^n : v_a(x) > (K - a \cdot x)^+\}$$

is called the continuation region. Thus an initial stock price vector x lies in the early exercise region iff its price coincides with the payoff from immediate expiration and it is reasonable that the first entrance time into the early exercise region is an optimal exercise strategy.

The case of a single stock was treated by several people and solved with different methods see Mc Kean (1965) , Beibel, Lerche (1997) . It turns out that the early exercise region is given by $\mathcal{E}_1 = (0, K\frac{\alpha}{1+\alpha}]$ with $\alpha = \frac{2r}{\sigma^2}$ and

$$v_1(x) = \begin{cases} x^{-\alpha}(K\frac{\alpha}{1+\alpha})^\alpha(K - K\frac{\alpha}{1+\alpha}) & \text{if } x \geq K\frac{\alpha}{1+\alpha} \\ K - x & \text{if } x \leq K\frac{\alpha}{1+\alpha} \end{cases} \tag{5}$$

Note that $v_a(x) = v_1(ax) = v_{ax}(1)$ for $a > 0$ and therefore $\mathcal{E}_a = (0, \frac{K}{a}\frac{\alpha}{1+\alpha}]$.

For an index of multiple stocks the corresponding optimal stopping problem is more difficult, since the payoff depends on a multidimensional diffusion process. The main result stated in Theorem 2.4 gives an appropriate inner approximation of the early exercise region. The used method relies basically on an approach introduced by Beibel and Lerche (1997) and gives new insights to optimal stopping problems for multidimensional Markov-process.

In section 3 a rough outer bound of the early exercise region can be given by constructing an appropriate subharmonic function.

Finally in section 4 we calculate the early exercise region in the finite running time case by backward induction.

2 . The inner approximation

Let $S = (S_1, \ldots, S_n)$ be the price process of a vector of n-stocks each following a geometric Brownian motion, i.e. satisfying the stochastic differential equation (1). Let $a \in (0,\infty)^n$ be a vector indicating the weight of each stock in the index $I_a(t) = a \cdot S(t) = \sum_{i=1}^n a_i S_i(t)$. Recall that

$$v_a(x) = \sup_{\tau \in \mathcal{S}} E_x e^{-r\tau}(K - a \cdot S(\tau)^+ = \sup_{\tau \in \mathcal{S}} E_x e^{-r\tau}(K - a \cdot S(\tau)^+ 1_{\{\tau < \infty\}}$$

for all $x \in (0,\infty)^n$ denotes the price of an American perpetual put with strike K, initial stock price x and weight vector a . Broadie, Detemple (1997) stated geometric properties of the early exercise region even in the case of finite running time. Let us briefly adjust their results to our context. We introduce the following notations for vectors $x, y \in (0,\infty)^n, \alpha \in I\!R^n$.

$$\begin{array}{rcl} xy &=& (x_1y_1, \ldots, x_ny_n) \quad , \\ x \leq y &\Longleftrightarrow& x_i \leq y_i \text{ for all } 1 \leq i \leq n \quad , \\ x^\alpha &=& x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \quad , \\ \mathbf{1} &=& (1, \ldots, 1) \quad , \\ T_a &=& \{x \in (0,\infty)^n : K - a \cdot x > 0\} \\ |x| &=& \sum_{i=1}^n x_i \end{array}$$

Since $(\frac{S_1(t)}{x_1}, \ldots, \frac{S_n(t)}{x_n})_{t\geq 0}$ has the same distribution w.r.t. P_x as $(S_1(t), \ldots, S_n(t))_{t\geq 0}$ according to $P_{\mathbf{1}}$ it holds

$$\sup_{\tau \in \mathcal{S}} E_x e^{-r\tau}(K - a \cdot S(\tau))^+ = \sup_{\tau \in \mathcal{S}} E_{\mathbf{1}} e^{-r\tau}(K - (ax) \cdot S(\tau))^+ \quad .$$

Thus $v_a(x) = v_{\mathbf{1}}(ax) = v_{ax}(\mathbf{1})$ for all $a, x \in (0,\infty)^n$.
A subset $M \subset (0,\infty)^n$ is called *south-west connected* , iff $y \in M$ implies $x \in M$ for all $x \in (0,\infty)^n$ such that $x \leq y$. We may state

2.1 Theorem: *The early exercise region $\mathcal{E}_a$ is a closed, convex, SW-connected subset of the simplex T_a.*
Furthermore

$$\mathcal{E}_a = \{(\frac{1}{a_1} x_1, \ldots, \frac{1}{a_n} x_n) : x \in \mathcal{E}_{\mathbf{1}}\}$$

is the image of a linear transformation of $\mathcal{E}_{\mathbf{1}}$.

Proof: That $\mathcal{E}_a$ is a closed set relative to $(0,\infty)^n$ follows from the theory of optimal stopping. It is contained in the simplex T_a, since we have zero payoff outside. It is the image of a linear transformation of $\mathcal{E}_{\mathbf{1}}$ since $v_a(x) = v_{ax}(\mathbf{1}) = v_{\mathbf{1}}(ax)$ for all $x \in (0,\infty)^n$.

To prove SW-connectedness let $y \in \mathcal{E}_a$ and $x \leq y$. It holds for each stopping time τ due to $z_1^+ - z_2^+ \leq (z_1 - z_2)^+$

$$\begin{array}{ll} E_{\mathbf{1}} e^{-r\tau}(& K - (ax) \cdot S(\tau) \ \ )^+ 1_{\{\tau < \infty\}} \\ \leq & E_{\mathbf{1}} e^{-r\tau} \left((K - (ax) \cdot S(\tau))^+ - (K - (ay) \cdot S(\tau))^+ \right) 1_{\{\tau < \infty\}} + v_{ay}(\mathbf{1}) \\ \leq & E_{\mathbf{1}} e^{-r\tau} ((ay - ax) \cdot S(\tau)) 1_{\{\tau < \infty\}} + v_a(y) \\ \leq & (ay - ax) \cdot \mathbf{1} + K - a \cdot y = K - a \cdot x \quad . \end{array}$$

The last inequality follows with Fatou's lemma since $(e^{-rt} S_i(t))_{t \geq 0}$ is a positive martingale for all $1 \leq i \leq n$. Thus all points southwest to y belong to the early exercise region $\mathcal{E}_a$.

The function v_a is as supremum of convex functions itsself convex. This yields the convexity of $\mathcal{E}_a$ due to

$$\begin{array}{rcl} v_a(\lambda x + (1-\lambda) y) & \leq & \lambda v_a(x) + (1-\lambda) v_a(y) \\ & = & \lambda (K - a \cdot x) + (1-\lambda)(K - a \cdot y) \\ & = & K - a \cdot (\lambda x + (1-\lambda) y) \end{array}$$

for all $\lambda \in (0,1), x, y \in \mathcal{E}_a$.

Considering a put of each single stock provides points

$$(6)\ x^{(i)} = (0, \ldots, 0, \frac{K}{a_i} \frac{\beta_i}{1+\beta_i}, 0, \ldots, 0) \quad , \beta_i = 2r/(\sigma_i^2) \text{ for all } 1 \leq i \leq n$$

such that the convex hull of $x^{(1)}, \ldots, x^{(n)}$ is contained in the early exercise region $\mathcal{E}_a$. This is the hyperplane determined by these points intersected with $(0,\infty)^n$. The SW-connectedness yields that all points below this hyperplane belong to $\mathcal{E}_a$. Hence as a first consequence of Theorem (2.1) we may conclude

$$(7) \quad G = \{x \in (0,\infty)^n : x \leq y \text{ for some } \ y \in conv(x^{(1)}, \ldots, x^{(n)})\} \quad .$$

is contained in the early exercise region $\mathcal{E}_a$.

A more advanced investigation is necessary to improve this result. The method to apply relies basically on an approach introduced by Beibel, Lerche (1997). The main task is a construction of appropriate martingales. Recall the notations at the beginning of the section.

2.2 Lemma: *Let p denote the following polynomial in n variables:*

$$p(\alpha) = \frac{1}{2} \sum_{i=1}^n \alpha_i (\alpha_i + 1) \sigma_i^2 + \sum_{i<j} \alpha_i \alpha_j \rho_{ij} \sigma_i \sigma_j - r \sum_{i=1}^n \alpha_i - r$$

Then for each $\alpha \in I\!R^n$

$$M_\alpha(t) = e^{-rt} S(t)^{-\alpha} \quad , \quad t \geq 0$$

defines a positive sub(super)martingale if $p(\alpha) \geq (\leq) 0$*. In particular* M_α *is a positive martingale for* $p(\alpha) = 0$*.*

Proof: From the stochastic differential equations (1) satisfied by $S(t)$ Ito's formula yields

$$(8) \qquad dS(t)^{-\alpha} = S(t)^{-\alpha}\left((p(\alpha)+r)dt + \sum_{i=1}^{n} \sigma_i \alpha_i dW_i(t)\right) \quad ,$$

due to $d[S_i, S_i]_t = S_i(t)^2 \sigma_i^2 dt, d[S_i, S_j]_t = S_i(t) S_j(t) \rho_{ij} \sigma_i \sigma_j dt.$
Let $N(t) = \sum_{i=1}^n \int_0^t \sigma_i \alpha_i dW_i(s)$. Then $X_\alpha(t) = e^{-(p(\alpha)+r)t} S(t)^{-\alpha}$ is a local martingale, since

$$dX_\alpha(t) = X_\alpha(t) dN(t) \quad , X_\alpha(0) = x^{-\alpha} \quad P_x \text{ a.s.} \quad .$$

Thus X_α is the exponential local martingale w.r.t. N and can be written as

$$X_\alpha(t) = x^{-\alpha} \exp(N(t) - \frac{1}{2}[N]_t) \quad , t \geq 0 \quad .$$

From this the martingale property follows with Novikov's criterion since $[N]_t$ is proportional to t . Due to $M_\alpha(t) = e^{p(\alpha)t} X_\alpha(t)$ the assertion follows .

Considering the function $h(x) = x^{-\alpha}$ the condition $p(\alpha) = 0$ implies $Ah = rh$, where A denotes the generator of the n-dimensional stock price process. Hence $e^{-rt}h(x)$ is a harmonic function for the associated space-time process, which provides another argument for the martingale property of M_α. Considering the discounted payoff in the light of the martingale M_α yields the following important decompostion:

$$(9) \qquad e^{-rt}(K - \sum_{i=1}^{n} S_i(t))^+ = M_\alpha(t) g_\alpha(S(t))$$

with $g_\alpha(x) = x^\alpha (K - \sum_{i=1}^n x_i)^+$. This decomposition will provide an upper bound for the option price.

2.3 Lemma: *Let* $\alpha \in (0,\infty)^n$*. Then* $g_\alpha(x) = x^\alpha (K - \sum_{i=1}^n x_i)^+$ *is uniformly bounded in* $(0,\infty)^n$ *with maximum at* $m_\alpha = K \frac{\alpha}{1+|\alpha|}, |\alpha| = \sum_{i=1}^n \alpha_i$.

Proof: For $0 < s < K$ we consider g_α on the hyperplane $E_s = \{x \in (0,\infty)^n : \sum_{i=1}^n x_i = s\}$. Maximisation of x^α over E_s with the method of Lagrange yields a unique maximum at $s\frac{\alpha}{|\alpha|}$. Hence we have to maximize $g_\alpha(s\frac{\alpha}{|\alpha|})$ on $0 < s < K$, providing $s = K\frac{|\alpha|}{1+|\alpha|}$ as unique maximum . Hence the assertion follows.

After these preliminaries the announced inner approximation of the early exercise region can be given. The main point is that the convex set

$$\Sigma = \{\alpha \in (0,\infty)^n : p(\alpha) \leq 0\}$$

can be mapped onto a subset $\mathcal{E}'_a$ of the early exercise region which contains the set G of all points below the convex hull of $x^{(1)}, \ldots, x^{(n)}$, see (6),(7). Let us introduce the transformation

$$\psi : (0,\infty)^n \longrightarrow T_{\mathbf{1}}; \alpha \mapsto K\frac{\alpha}{1+\alpha}$$

with inverse

$$\phi : T_{\mathbf{1}} \longrightarrow (0,\infty)^n; x \mapsto \frac{1}{K}\frac{1}{1-|x|}x \quad .$$

These transformations map straight lines onto straight lines and preserves therefore convexity. Recall the definition of the points $x^{(1)}, \ldots, x^{(n)}$ given in (6). We define vectors $\beta^{(i)}$ for all $1 \leq i \leq n$ by

$$\beta^{(i)} = (0, \ldots, 0, \frac{2r}{\sigma_i^2}, 0, \ldots, 0) \quad . \tag{10}$$

and a set

$$\Gamma = \{\alpha \in (0,\infty)^n : \alpha \leq \beta \text{ for some } \ \beta \in conv(\beta^{(1)}, \ldots, \beta^{(n)}\}$$

off all points below the hyperplane generated by $\beta^{(1)}, \ldots, \beta^{(n)}$. Then we may state

2.4 Theorem: *Let $a \in (0,\infty)^n$ be a weight vector. Then*

(i) The set

$$\mathcal{E}'_a = \{x \in T_a : p(\phi(\frac{1}{a_1}x_1, \ldots, \frac{1}{a_n}x_n)) \leq 0\}$$

is contained in the early exercise region $\mathcal{E}_a$.

(ii) $\mathcal{E}'_a$ is a convex set that contains G.

(iii) The option price $v_a(x)$ satisfies

$$v_a(x) \leq \inf_{\alpha \in \Theta} h_\alpha(ax) \quad ,$$

with $h_\alpha(x) = (K\frac{\alpha}{1+|\alpha|})^\alpha (K - K\frac{|\alpha|}{1+|\alpha|}) x^{-\alpha}$, $\Theta = \{\alpha \in (0,\infty)^n : p(\alpha) = 0\}$.

Proof: Note that $v_a(x) = v_{ax}(\mathbf{1}) = v_{\mathbf{1}}(ax)$ for all $x \in (0,\infty)^n$. Thus we may assume $a = \mathbf{1}$, this means that we consider a put on the sum of n-stocks. Recall for $p(\alpha) = 0$ the martingale M_α from Lemma 2.2 and the function g_α from Lemma 2.3. For each stopping time τ it holds

$$\begin{aligned} E_x e^{-r\tau}(K - \sum_{i=1}^n S_i(\tau))^+ &= E_x M_\alpha(\tau) g_\alpha(S(\tau)) 1_{\{\tau<\infty\}} \\ &\leq \sup_{y \in (0,\infty)^n} g_\alpha(y) E_x M_\alpha(\tau) 1_{\{\tau<\infty\}} \\ &\leq g_\alpha(K\frac{\alpha}{1+|\alpha|}) x^{-\alpha} \quad , \end{aligned}$$

since M_α is a positive martingale with $E_x M_\alpha(t) = x^{-\alpha}$. Hence

$$v_{\mathbf{1}}(x) \leq h_\alpha(x) \quad \text{for all } x \in (0,\infty)^n \quad . \tag{11}$$

The latter is a majorant of the payoff $(K - \sum_{i=1}^n x_i)^+$ which touches it at $K\frac{\alpha}{1+|\alpha|}$. Hence, if the inital price of the stock vector is given by $K\frac{\alpha}{1+|\alpha|}$, the expected payoff from any early exercise strategie τ does not exceed the immediate payoff from expiration. Thus $K\frac{\alpha}{1+|\alpha|}$ is contained in the early exercise region. The collection of all these points is a $n-1$ dimensional manifold

$$B = \{K\frac{\alpha}{1+|\alpha|} : \alpha \in (0,\infty)^n, p(\alpha) = 0\}$$

which is the image of the manifold $\theta = \{\alpha \in (0,\infty)^n : p(\alpha) = 0\}$ under ψ. Due to $p(\beta^{(i)}) = 0, \psi(\beta^{(i)}) = x^{(i)}$ for all $1 \leq i \leq n$ it holds

$$\Sigma = \{\alpha \in (0,\infty)^n : p(\alpha) \leq 0\} = conv(\Theta) \cup \Gamma$$

and

$$\mathcal{E}'_{\mathbf{1}} = \psi(\Sigma) = conv(B) \cup G \quad .$$

The first set of the union is contained in the early exercise region $\mathcal{E}_{\mathbf{1}}$ due to its convexity and the second one due to its SW-connectedness. Thus assertion (i) follows. The set $\mathcal{E}'_{\mathbf{1}}$ is convex since it is the image of the convex set Σ under the transformation ψ . The last part of the theorem finally follows immediately

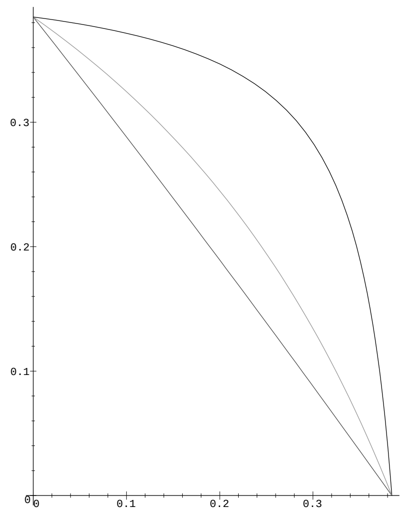

Figure 1: early exercise region under dependence

from (11).

To illustrate the theorem we determine approximately the early exercise region for the sum of two stocks. Therefore we have to consider the polynomial

$$p(\alpha_1, \alpha_2) = \frac{1}{2}\alpha_1(\alpha_1 + 1)\sigma_1^2 + \frac{1}{2}\alpha_2(\alpha_2 + 1)\sigma_2^2 + \sigma_1\sigma_2\alpha_1\alpha_2\rho - r(\alpha_1 + \alpha_2) - r$$

and solutions of the equation $p(\alpha_1, \alpha_2) = 0$. Introducing polar coordinates $\alpha_1 = R\cos\phi, \alpha_2 = R\sin\phi$ the above equation has for each $\phi \in [0, \frac{\pi}{2}]$ a positive solution $R(\phi)$ which can be determined by solving the corresponding quadratic polynomial in R. Thus all points of the curve $\gamma(\phi) = (R(\phi)\cos\phi, R(\phi)\sin\phi)$, $\phi \in [0, \pi/2]$ solve the above equation and are to be transformed by ψ providing

$$\frac{K}{1 + \gamma_1(\phi) + \gamma_2(\phi)}(\gamma_1(\phi), \gamma_2(\phi)) \quad , \quad \phi \in [0, \frac{\pi}{2}]$$

as boundary of $\mathcal{E}_1'$.

The first figure shows the influence of the correlation ρ. For $\sigma_1 = 0.4 = \sigma_2, r = 0.05, K = 1$ the boundaries are plotted for $\rho \in \{-0.9, 0, 0.9\}$.

The second figure shows the dependence of the volatility. For $\sigma_2 = 0.4, \rho = 0.0, K = 1, r = 0.05$ the boundaries are plotted for $\sigma_1 \in \{0.2, 0.4, 0.6\}$.

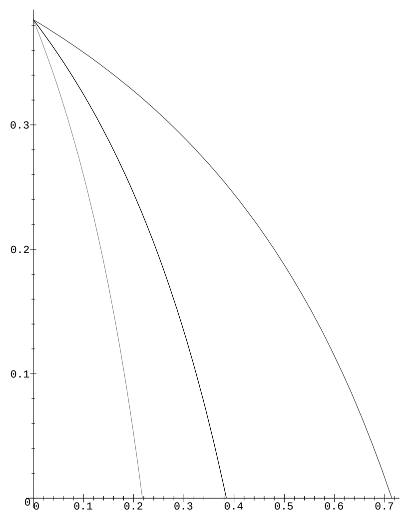

Figure 2: early exercise region when the volatilities differ

It turns out that the early exercise region shrinks with increasing correlation ρ. Heuristicallly we may argue as follows. Positive correlation leads to a run of both prices in the same direction. Thus the chance to get a lower sum is larger. Hence the price of the put is higher and the region of early exercise smaller than in the case of negative correlation.

The second plot shows that the early exercise region increases with decreasing volatility. Shortly we may argue: the higher is the volatility the higher is the price of the put and thus the smaller is the region of early exercise.

We may state the following conclusion. The early exercise region of a put on an index is a convex set with different shapes depending on the volatility and correlation parameters. Assuming a one stock model would lead to an early exercise region of the form $\{x \in (0, \infty)^n : a \cdot x \leq b\}$ which can not take into account this influence of the parameters. As was shown in the above figures the shape of the inner approximation varies sensitively with a change of the parameters. Thus it is even more reasonable to use the approximation of the early exercise region in the complex model than the exact solution of the one stock model.

3 . An outer approximation

Let us fix a weight vector $a \in (0,\infty)^n$ for the rest of this section. We want to provide an outer approximation, a set $\mathcal{E}_a''$, such that $\mathcal{E}_a \subset \mathcal{E}_a''$. To do this we first construct an appropriate r-subharmonic function of the form

$$\phi(x) = f(a \cdot x) \quad , \quad x \in (0,\infty)^n.$$

Let us denote by

$$Q_{ij} = \begin{cases} \sigma_i^2 & , \text{if } i = j \\ \rho_{ij}\sigma_i\sigma_j & , \text{if } i \neq j \end{cases} \quad , \quad i,j = 1,\cdots,n \tag{12}$$

the covariance matrix of the n-dimensional Wiener-process. The matrix Q is symmetric, positive definite with positive eigenvalues. Let λ denote the smallest eigenvalue of Q. Then the following lemma can be shown.

3.1 Lemma: *Let* $\delta = \frac{2nr}{\lambda}, b = K\frac{\delta}{1+\delta}$. *The function*

$$\phi : (0,\infty)^n \to (0,\infty), x \to (\frac{a \cdot x}{b})^{-\delta}(K - b)$$

fulfills

(i) $A\phi \geq r\phi$,

(ii) $\phi(x) \geq (K - a \cdot x)^+ \quad , \quad x \in (0,\infty)^n$,

(iii) $\phi(x) = (K - a \cdot x)^+ \quad$ *on* $\quad \{x : a \cdot x = b\}$.

Proof: Recall the definition of the generator A in (2). Since $\phi(x) = f(a \cdot x)$ with $f : (0,\infty) \to (0,\infty);\ y \to (\frac{y}{b})^{-\delta}(K - b)$, an easy calculation yields

$$\begin{aligned} A\phi(x) &= \frac{1}{2}Q(ax)\cdot(ax)\, f''(a\cdot x) + r(a\cdot x)f'(a\cdot x) \\ &\geq \frac{1}{2}\frac{\lambda}{n}(a\cdot x)^2 f''(a\cdot x) + r(a\cdot x)\, f'(a\cdot x). \end{aligned} \tag{13}$$

The last inequality follows from

$$Q(ax)\cdot(ax) \geq \lambda||ax||_2^2 \geq \frac{\lambda}{n}(\sum_{i=1}^n a_i x_i)^2 = \frac{\lambda}{n}(a\cdot x)^2.$$

Recall that the function f conicides with the price of an ordinary American perpetual put on one stock with volatility $\sqrt{\frac{\lambda}{n}}$. Thus f fulfills

$$\begin{aligned}\frac{1}{2}\frac{\lambda}{n}y^2 f''(y) + ryf'(y) &= rf(y) \quad , \quad y \in (0,\infty), \\ f(y) &\geq (K-y)^+ \quad , y \in (0,\infty), \\ f(b) &= K - b \end{aligned} \tag{14}$$

and the assertion follows together with (13).

Property (i) of Lemma 3.1 shows that

$$(x,t) \to e^{-rt}\phi(x)$$

defines a subharmonic function for the space time process. This can be used to derive an outer approximation

3.2 Theorem: *Let $a \in (0,\infty)^n$ be a weight vector. Then, the early exercise region $\mathcal{E}_a$ is contained in the set*

$$\mathcal{E}''_a = \{x \in (0,\infty)^n : K - a \cdot x \geq b\}$$

with $b = K\frac{\delta}{1+\delta}, \delta = \frac{2nr}{\lambda}$.

Proof: Let $H = \{x \in (0,\infty)^n : K - a \cdot x = b\}$ and

$$\tau = \inf\{t \geq 0 : S(t) \in H\} = \inf\{t \geq 0 : K - a \cdot S(t) = b\} \quad .$$

For $x \in (0,\infty)^n$ such that $K - a \cdot x < b$ we have to show that the price of the American put exceeds the immediate payoff, i.e. $v_a(x) > (K - a \cdot x)^+$.

We consider the r-subharmonic function ϕ of Lemma 3.1. Since it is bounded on $\{z \in (0,\infty)^n : K - a \cdot z \leq b\}$ a uniformly integrable bounded submartingale is defined by

$$(e^{-r(\tau\wedge t)}\phi(S(\tau \wedge t)))_{t\geq 0}$$

w.r.t. P_x. Hence,

$$\begin{aligned} v_a(x) &\geq E_x e^{-r\tau}(K - a \cdot S(\tau))1_{\{\tau<\infty\}} \\ &= E_x e^{-r\tau}\phi(S(\tau))1_{\{\tau<\infty\}} \\ &\geq \phi(x) > (K - a \cdot x)^+ \end{aligned}$$

The preceding theorem improves the trivial fact that the early exercise region $\mathcal{E}_a$ is contained in the simplex $T_a = \{x \in (0,\infty)^n : K - a \cdot x = 0\}$. The obtained outer approximation determines a non obvious simplex covering the early exercise region. The disadvantage is that it does not take into account the influence of the different parameters. Covariance matrices with same smallest eigenvalue lead to the same outer approximation, whereas the early exercise region may vary strongly.

4 . Numerical computations

To get an impression of the relation between the early exercise region and its inner and outer approximation some numerical calculations for the case of two stocks will be presented. The results will underline the conjecture that the shape of the inner approximation almost coincides with that of the early exercise region.

For numerical computations we choose a discrete Cox Ross Rubinstein model, analogous to Musiela, Rutkowski (1997) p. 41, as an approximation of the continuous time model. Since we use the backward induction algorithm only optimal stopping problems with finite time horizon can be treated. Thus, in principle a numerical evaluation of prices of American index options are possible. But, one has to admit that the computational effort is very large and only small dimensions can be treated.

Let us fix a maximal time to expiration $T > 0$. We divide the time interval $[0, T]$ into N periods of equal length T/N. At the time points $(k\frac{T}{N})_{0 \le k \le N}$ we define a n-dimensional geometric random walk in the following way. Let

$$u_i = \exp(\sigma_i \sqrt{\frac{T}{N}}) \quad , \quad d_i = \exp(-\sigma_i \sqrt{\frac{T}{N}}) \tag{15}$$

denote the possible percentual changes of the i-th stock during one period. Then

$$S_i^N(k\frac{T}{N}) = x_i u_i^{\sum_{j=1}^{k} Z_i(j)} \quad , 0 \le k \le N, 1 \le i \le n \tag{16}$$

denotes the evolution of the i-th stock with initial price $x_i > 0$ in a CRR model at the discrete time points $(k\frac{T}{N})_{0 \le k \le N}$. We assume that $(Z(k))_{1 \le k \le N}$ is a sequence of iid $\{-1,1\}^n$-valued random vectors. We choose the probabilities for an upward move such that the model is risk neutral. Thus we put

$$p_i^* = prob(Z_i(1) = 1) = \frac{(1 + r^{(N)}) - d_i}{u_i - d_i} \quad , \quad 1 \le i \le N \tag{17}$$

with $r^{(N)} = \exp(r\frac{T}{N}) - 1$ denoting the one period interest rate. Furthermore we adjust the dependence structure of the continous time model by assuming

$$\text{Corr}(Z_i(1), Z_j(1)) = \rho_{ij} \quad \text{for all } i \neq j. \tag{18}$$

By linear interpolation we can think of S^N as an n-dimensional continuous time process on the time interval $[0,T]$. Donsker's invariance principle yields that the law of S^N converges weakly to that of an n-dimensional stock price process with dynamics as in (1).

For the case of two stocks we have applied the above method to compute numerically the early exercise region for a put on a sum with strike $K = 1$. For three cases we present the results and compare it to the bounds for the perpetual put. In each figure the straight line determines the outer boundary

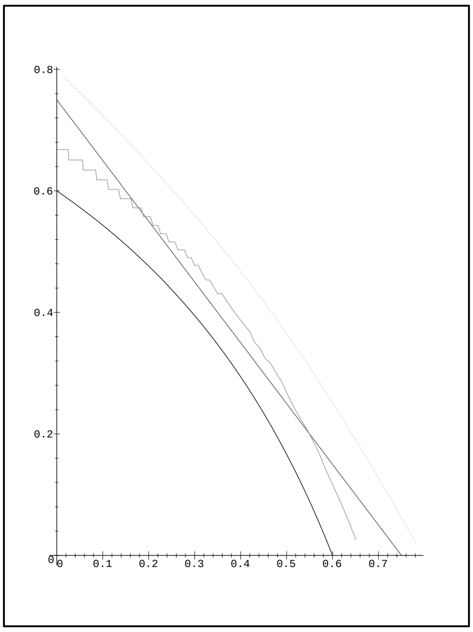

Figure 3: early exercise regions for different time maturities and equal volatility

of the early exercise region for the perpetual put. Furthermore, three curves indicate the boundary of the early exercise region for different maximal running times. The innermost curve shows the boundary of the inner approximation for the perpetual put. The next inner one is the boundary for a put with running time $T = 20/3$ and finally the outermost one is that for running time $T = 2/3$. For all treated cases we have assumed an interest rate $r = 0.03$. The first figure shows the results for equal volatilities $\sigma_1 = 0.2 = \sigma_2$ and the

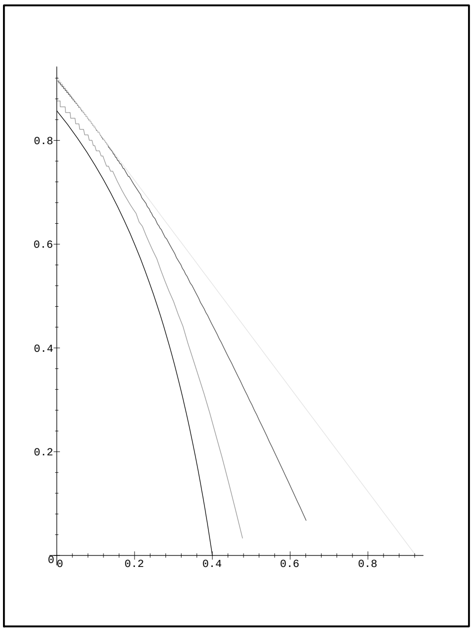

Figure 4: early exercise regions for different time maturities and different volatilities

second one those for different volatilities $\sigma_1 = 0.3$, $\sigma_2 = 0.1$. In both cases no correlation is assumed. In the third figure the results for correlation $\rho = -0.9$ with volatilities $\sigma_1 = 0.2 = \sigma_2$ are presented. It turns out that in all three cases the boundary according to the larger running time moves towards that from the inner approximation of the perpetual put. In particular the shape of both curves almost conicide. This indicates that the inner approximation is quite reasonable since it takes into account the influence of all parameters.

To compute numerically the early exercise region according to a put with finite running time needs a large effort which can be reduced in the following way. Whenever in the backward induction algorithm a stock price falls into the inner approximation set we know the price of the put which coincides with the

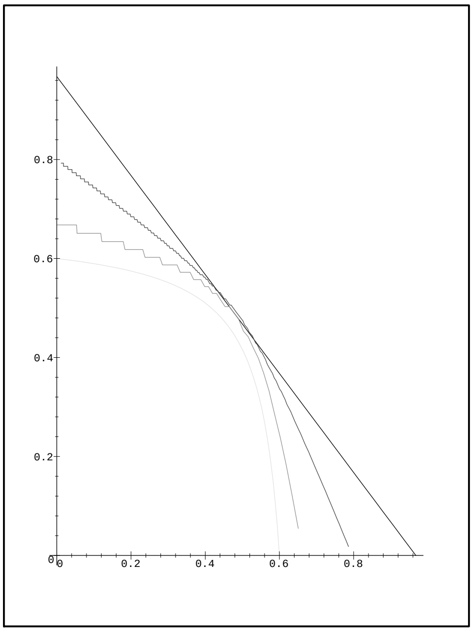

Figure 5: early exercise regions under dependence for different time maturities

immediate payoff. Thus in each step of the algorithm only computations for stock prices outside of the inner approximation are neccessary. This provides a further argument why the obtained inner approximation for the perpetual put is important, in particular for the computation of prices according to finite running time.

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