INTERMITTENCY FOR BRANCHING RANDOM WALK IN PARETO ENVIRONMENT

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We consider a branching random walk on the lattice, where the branching rates are given by an i.i.d. Pareto random potential. We describe the process, including a detailed shape theorem, in terms of a system of growing lilypads. As an application we show that the branching random walk is intermittent, in the sense that most particles are concentrated on one very small island with large potential. Moreover, we compare the branching random walk to the parabolic Anderson model and observe that although the two systems show similarities, the mechanisms that control the growth are fundamentally different.

1. Introduction and main results.

1.1. *Introduction.* Branching processes in random environments are a classical subject going back to [25, 27]. We are interested in branching random walks (BRW), where particles branch but also have spatial positions and are allowed to migrate to other sites.

We consider a particular variant of the model defined on $\mathbb{Z}^d$. Start with a single particle at the origin. Each particle performs a continuous-time nearest-neighbor symmetric random walk on $\mathbb{Z}^d$. When at site $z \in \mathbb{Z}^d$, a particle splits into two new particles at rate $\xi(z)$, where the potential $(\xi(z), z \in \mathbb{Z}^d)$ is a collection of nonnegative i.i.d. random variables. The two new particles then repeat the stochastic behavior of their parent, started from z. This particular model was first described in [12], although until now analysis has concentrated on the expected number of particles: see the surveys [11, 18, 21]. In this article we show that while the study of the actual number of particles is more technically demanding, it is still tractable, and reveals surprising and interesting behavior which warrants further investigation.

We begin by recalling what is known about the expected number of particles. More precisely, we fix a realization of the environment $(\xi(z), z \in \mathbb{Z}^d)$ and take expectations over migration and branching. We denote the expected number of

particles by

$$u(z,t) = E^{\xi}\big[\#\{\text{particles at site } z \text{ at time } t\}\big].$$

The superscript ξ indicates that this expression is still random due to its dependence on the environment. By considering the different possibilities in a infinitesimal time step, one can easily see that $u(z,t)$ solves the following stochastic partial differential equation, known as the *parabolic Anderson model* (PAM):

$$\partial_t u(z,t) = \Delta u(z,t) + \xi(z)u(z,t) \qquad \text{for } z \in \mathbb{Z}^d, t \geq 0,$$
$$u(z,0) = \mathbb{1}_{\{z=0\}} \qquad \text{for } z \in \mathbb{Z}^d.$$

Here, Δ is the discrete Laplacian defined for any function $f : \mathbb{Z}^d \to \mathbb{R}$ as

$$\Delta f(z) = \sum_{y \sim z}\big(f(y) - f(z)\big), \qquad z \in \mathbb{Z}^d,$$

where we write $y \sim z$ if y is a neighbor of z on the lattice $\mathbb{Z}^d$. Starting with the seminal work of [12] the PAM has been intensively studied in the last twenty years. Much interest stems from the fact that it is one of the more tractable models to exhibit an effect called *intermittency*, which roughly means that the solution is concentrated in a few peaks that are spatially well separated. For the PAM this effect is well understood: see the surveys [11, 18, 21]. The size and the number of peaks depends essentially on the tail of ξ, that is, on the decay of $\mathbb{P}(\xi(0) > x)$ for large x. For a bounded potential the size of the relevant islands grows with t. In the intermediate regime, when the potential is double exponentially distributed, the size of the islands remains bounded. Finally, it is believed, and in a lot of cases proven, that for any potential with heavier tails, there is a single island consisting of a single point containing almost all of the mass. In the most extreme case when the potential is Pareto distributed, a very detailed understanding of the evolution of the solution has emerged; see [19, 22, 26].

While the expected number of particles, that is, the PAM, is well understood, a lot less is known for the actual number of particles in the branching random walk. The only results so far for this particular model concern higher moments of particles

$$E^{\xi}\big[\#\{\text{particles at site } z \text{ at time } t\}^n\big].$$

These were studied in a special case by [1] using analytic methods and for a larger class of potentials and providing finer asymptotics in [14] using spine methods for higher moments as developed in [15].

1.2. *Main result.* Motivated by the detailed understanding of the parabolic Anderson model in the case of Pareto potentials, we from now on assume that

$\{\xi(z), z \in \mathbb{Z}^d\}$ is a collection of independent and identically distributed Pareto random variables. Denoting the underlying probability measure on $(\Omega, \mathcal{F})$ by Prob, we have in particular that for a parameter $\alpha > 0$, and any $z \in \mathbb{Z}^d$,

$$\text{Prob}(\xi(z) > x) = x^{-\alpha} \qquad \text{for all } x \geq 1.$$

Moreover, we assume throughout that $\alpha > d$, which is a necessary condition for the total mass in the PAM to remain finite; see [12].

For a fixed environment ξ, we denote by P_y^ξ the law of the branching simple random walk in continuous time with binary branching and branching rates $\{\xi(z), z \in \mathbb{Z}^d\}$ started with a single particle at site y. Finally, for any measurable set $F \subset \Omega$, we define

$$\mathbb{P}_y(F \times \cdot) = \int_F P_y^\xi(\cdot) \, \text{Prob}(d\xi).$$

If we start with a single particle at the origin, we omit the subscript y and simply write P^ξ and $\mathbb{P}$ instead of P_0^ξ and $\mathbb{P}_0$.

We define $Y(z, t)$ to be the set of particles at the point z at time t. Moreover, we let $Y(t)$ be the set of all particles present at time t. We are interested in the number of particles

$$N(z, t) = \#Y(z, t) \quad \text{and} \quad N(t) = \#Y(t).$$

The aim of this paper is to understand the long-term evolution of the branching random walk, and we therefore introduce a rescaling of time by a parameter $T > 0$. We also rescale space and the potential. If $q = \frac{d}{\alpha - d}$, the right scaling factors for the potential, respectively, space, turn out to be

$$a(T) = \left(\frac{T}{\log T}\right)^q \quad \text{and} \quad r(T) = \left(\frac{T}{\log T}\right)^{q+1}.$$

This scaling is the same as that used in the parabolic Anderson model (cf. [19, 26]) and guarantees the right balance between the peaks of the potential and the cost of reaching the corresponding sites. We now define the rescaled lattice as

$$L_T = \{z \in \mathbb{R}^d : r(T)z \in \mathbb{Z}^d\},$$

and for $z \in \mathbb{R}^d$, $R \geq 0$ define $L_T(z, R) = L_T \cap B(z, R)$ where $B(z, R)$ is the open ball of radius R about z in $\mathbb{R}^d$. For $z \in L_T$, the rescaled potential is given by

$$\xi_T(z) = \frac{\xi(r(T)z)}{a(T)},$$

and we set $\xi_T(z) = 0$ for $z \in \mathbb{R}^d \setminus L_T$. The correct scaling for the number of particles at z is given by

$$M_T(z, t) = \frac{1}{a(T)T} \log_+ N(r(T)z, tT).$$

We will see that in order to bound the rescaled number of particles $M_T(z, t)$, we first have to understand at what time z is hit. We therefore introduce the hitting time of a point $z \in L_T$ as

$$H_T(z) = \inf\{t > 0 : Y(r(T)z, tT) \neq \varnothing\}.$$

Our main result states that we can predict the behavior of the branching random walk purely in terms of the potential. For this purpose we introduce the *lilypad model*.

For any site $z \in L_T$, we set

$$h_T(z) = \inf_{\substack{y_0, \ldots, y_n \in L_T : \\ y_0 = z, \, y_n = 0}} \left(\sum_{j=1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} \right),$$

where throughout $|\cdot|$ will denote the ℓ_1-norm on $\mathbb{R}^d$. We call $h_T(z)$ the first hitting time of z in the lilypad model. We think of each site y as being home to a lilypad, which grows at speed $\xi_T(y)/q$. Note that $h_T(0) = 0$, so that the lilypad at the origin begins growing at time 0, but other lilypads only begin to grow once they are touched by another lilypad. For convenience, we set $h_T(z) = h_T([z]_T)$ for any point $z \in \mathbb{R}^d \setminus L_T$, where

$$[z]_T = \left(\frac{\lfloor r(T)z_1 \rfloor}{r(T)}, \ldots, \frac{\lfloor r(T)z_d \rfloor}{r(T)} \right) \qquad \text{for any } z = (z_1, \ldots, z_d) \in \mathbb{Z}^d.$$

This system of hitting times is an interesting model in its own right, describing a first passage percolation model on $\mathbb{Z}^d$.

Although there are no "particles" in this system of growing lilypads, we define

$$m_T(z, t) = \sup_{y \in L_T} \{ \xi_T(y)(t - h_T(y))_+ - q|z - y| \},$$

which we think of as the rescaled number of particles in the lilypad model. We will show that with high probability its value matches very closely that of $M_T(z, t)$.

We will give a heuristic explanation for these definitions in Section 1.6. For an idea of how the system evolves, see Figures 1 to 6, where we plot the growth
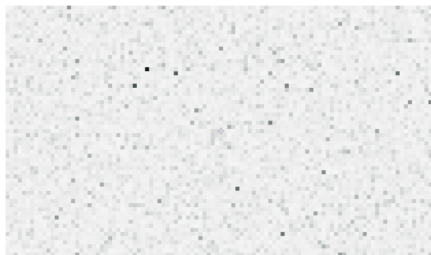

FIG. 1. *The grey squares represent the potential: dark means large potential. A lilypad starts to grow from near the origin.*
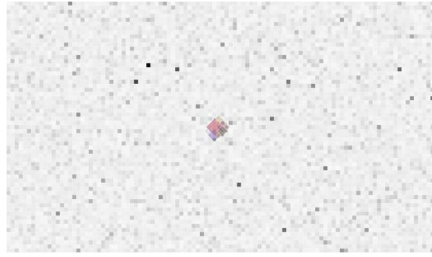
FIG. 2. *Some more points of reasonable potential are hit and a number of other visible lilypads are launched.*
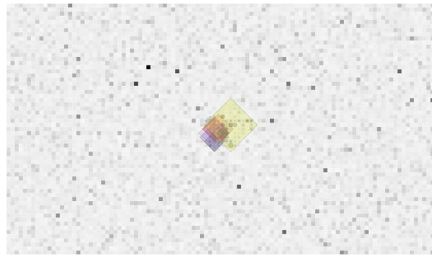

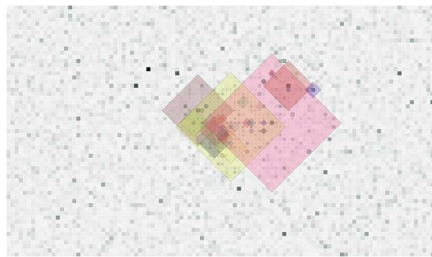FIG. 3. *A point of larger potential is hit and its lilypad, in yellow, grows faster.*

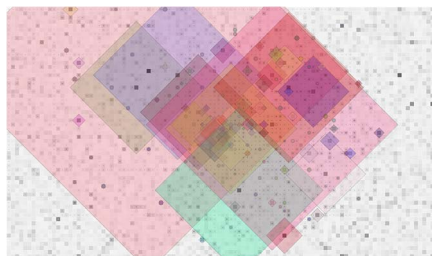FIG. 4. *Space is covered quickly and the dark spots in the top left will soon be hit.*

FIG. 5. *The darkest spot is hit and launches a fast-growing pink lilypad.*
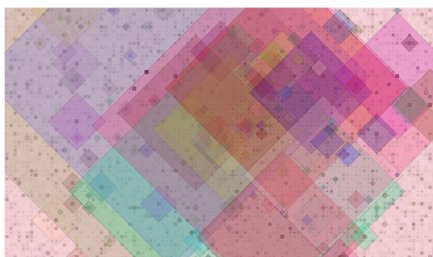
FIG. 6. *The whole visible region is covered.*

of the sites hit as time advances. A simulation of the process can be seen at
http://tiny.cc/lilypads.

THEOREM 1.1 (Approximation by lilypad model). *For any $t_\infty > 0$, as $T \to \infty$,*

$$\sup_{t \le t_\infty} \sup_{z \in L_T} |M_T(z,t) - m_T(z,t)| \to 0 \qquad \text{in } \mathbb{P}\text{-probability.}$$

Moreover, for any $R > 0$, as $T \to \infty$,

$$\sup_{z \in L_T(0,R)} |H_T(z) - h_T(z)| \to 0 \qquad \text{in } \mathbb{P}\text{-probability.}$$

REMARKS. (1) *The lilypad model is well defined.* It is not a priori clear that
the hitting times $\{h_T(z)\}$ are well defined. However, we will show in Lemma 3.4
that any finite ball gets covered eventually by the lilypad model, and in Lemma 3.6
that there is no explosion; that is, for any finite time t there exists $R > 0$ such that
the lilypad model is entirely contained within $B(0, R)$ at time t.

(2) *Interpretation as a first-passage percolation model.* As mentioned above it is
possible to interpret the lilypad hitting times as a first-passage percolation model.
We connect each pair of vertices in L_T via two directed edges. We associate to the
directed edge going from x to y the passage time $q \frac{|x-y|}{\xi_T(x)}$. Then $h_T(z)$ is the first
passage time from 0 to z.

(3) *Convergence to a Poissonian model.* From extreme value theory it can be
shown that in a suitable sense

$$\sum_{z \in L_T} \delta_{(z, \xi_T(z))} \Rightarrow \Gamma,$$

where Γ is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure $dz \times \alpha x^{-(\alpha+1)} dx$. See [26] for precise statements and an application to the PAM. This
suggests that the lilypad model and therefore also the hitting times and number of
particles in the branching random walk should converge in distribution to a version
of the lilypad model defined in terms of the Poisson point process. We will carry
out the details of this analysis in a further paper.

1.3. *Applications of the lilypad model.* Theorem 1.1 tells us that the BRW is well approximated by the lilypad model. We now describe some consequences of that approximation. As an easy first application, we describe the support of the branching random walk. For this we define

$$S_T(t) = \{z \in \mathbb{R}^d : H_T([z]_T) \le t\} \quad \text{and} \quad s_T(t) = \{z \in \mathbb{R}^d : h_T([z]_T) \le t\},$$

which we think of as the support of the BRW and the lilypad model, respectively.

THEOREM 1.2. *If d_H denotes the Hausdorff distance, for any $t_\infty > 0$, as $T \to \infty$,*

$$\sup_{t \le t_\infty} d_H(S_T(t), s_T(t)) \to 0 \qquad \text{in } \mathbb{P}\text{-probability}.$$

REMARK. Note that our definition of the support $S_T(t)$ is not the same as

$$\{z \in \mathbb{R}^d : Y(r(T)z, tT) \ne \varnothing\},$$

which is the set of sites that are occupied at time t. For example, $S_T(t)$ is by definition always a connected set, since the underlying random walk is nearest-neighbor, while the latter set may be disconnected since particles can jump away from the bulk. However, the two sets are almost the same in the following sense: Theorem 1.1 tells us that very shortly after a site has been visited by the BRW it will be occupied by many particles, which ensure that the site will be occupied from then onward.

A more striking application of our description is that the BRW shows intermittent behavior: all the mass is concentrated around a single peak of the potential.

THEOREM 1.3. *For $t > 0$ let $w_T(t)$ be the point in L_T that maximizes $\{m_T(z, t)\}$, where in the case of a tie we choose arbitrarily. Then, for any fixed $t > 0$, and $\varepsilon_T = \frac{3}{q} \log^{-1/4} T$,*

$$\frac{\sum_{z \in L_T(w_T(t), \varepsilon_T)} N(r(T)z, tT)}{\sum_{z \in L_T} N(r(T)z, tT)} \to 1 \qquad \text{in } \mathbb{P}\text{-probability}.$$

REMARK. *One point localization and further extensions.* The above theorem tells us that with high probability almost all of the mass is contained within a small ball about $w_T(t)$. In fact, with high probability almost all of the mass is contained actually at the single site $w_T(t)$. Proving this is more difficult and will be carried out in a further paper. Other, even more delicate results are known for the behavior of the PAM, including the almost sure fluctuations of the process (see [19]), and we plan to address the corresponding questions for the BRW in future work. We will also postpone to future work a detailed description of further properties that can be described by the lilypad model. These include a description of genealogies of particles as well as aging for the process.

1.4. *The parabolic Anderson model revisited.* We recall that the expected number of particles at a site z at time t is given by $u(z, t)$, the solution of the parabolic Anderson model. As pointed out in the Introduction, the parabolic Anderson model has been studied extensively, and a reader familiar with the literature will recognize that our predictions in terms of the lilypad model do not resemble those for the parabolic Anderson model. This raises the natural question of how different the actual number of particles is from the expected number.

We will make this comparison more transparent by first considering the support of the branching random walk. We already know from Theorem 1.2 that the support is described by the lilypad model. Without this description, a naive guess for the support of the BRW would be that a site gets hit roughly as soon as the expected number of particles, that is, the solution of the PAM, at that site becomes larger than 1. We show that this guess is dramatically wrong.

Previous work on the PAM has focused on showing, for example, one-point localization, but to understand the expected "support" we need information on the growth at every site, not just those with large potential. It turns out that by a simple version of our arguments for the BRW, we can also describe the profile of the PAM.

For this, we define the growth rate of particles and "hitting time" at a site $z \in L_T$ for the PAM as

$$\Lambda_T(z, t) = \frac{1}{a(T)T} \log_+ u(r(T)z, tT)$$

and

$$\mathcal{T}_T(z) = \inf\{t \geq 0 : u(r(T)z, tT) \geq 1\}.$$

In a similar fashion to the lilypad model for the BRW, we can define the *PAM lilypad model* by specifying the "number of particles" as

$$\lambda_T(z, t) = \sup_{y \in L_T} \{\xi_T(y)t - q|y| - q|z - y|\} \vee 0.$$

Moreover, the "hitting time" for the PAM lilypad model is given by

$$\tau_T(z) = \inf_{y \in L_T} \left\{ q \frac{|y|}{\xi_T(y)} + q \frac{|z - y|}{\xi_T(y)} \right\}.$$

We can also describe the support of the PAM and its lilypad model, which we define, respectively, as

$$S_T^{\text{PAM}}(t) = \{z \in \mathbb{R}^d : \mathcal{T}_T([z]_T) \leq t\} \quad \text{and} \quad s_T^{\text{PAM}}(t) = \{z \in \mathbb{R}^d : \tau_T([z]_T) \leq t\}.$$

THEOREM 1.4. *For any $R, t_\infty > 0$, the following hold as $T \to \infty$:*

(i) $\sup_{t \leq t_\infty} \sup_{z \in L_T} |\Lambda_T(z, t) - \lambda_T(z, t)| \to 0$ *in $\mathbb{P}$-probability.*

(ii) $\sup_{z \in L_T(0, R)} |\mathcal{T}_T(z) - \tau_T(z)| \to 0$ *in $\mathbb{P}$-probability.*

(iii) $\sup_{t \leq t_\infty} d_{\text{H}}(S_T^{\text{PAM}}(t), s_T^{\text{PAM}}(t)) \to 0$ *in $\mathbb{P}$-probabililty.*

REMARKS.   (1) One can show that

$$\lambda_T(z,t) = \sup_{y \in L_T} \{ \xi_T(y)(t - \tau_T(y))_+ - q|z - y| \},$$

which is very similar to the definition of

$$m_T(z,t) = \sup_{y \in L_T} \{ \xi_T(y)(t - h_T(y))_+ - q|z - y| \}.$$

(2) We stress that although Theorem 1.4 is a new result, and we provide a short and self-contained proof, it could be proved using existing PAM technology.

THEOREM 1.5.   (i) *The support of the branching random walk* $S_T(t)$ *is connected at all times, while the support of the PAM is disconnected in the sense that for any* $t > 0$,

$$\liminf_{T \to \infty} \mathbb{P}(S_T^{\mathrm{PAM}}(t) \text{ is disconnected}) > 0.$$

(ii) *Let* $W_T(t)$ *be the maximizer of the branching random walk and* $W_T^{\mathrm{PAM}}(t)$ *the maximizer of the parabolic Anderson model* (*where possible ties are resolved arbitrarily*). *Then for any* $t > 0$,

(1) $$\liminf_{T \to \infty} \mathbb{P}\left(|W_T(t) - W_T^{\mathrm{PAM}}(t)| \le \frac{3}{q} \log^{-1/4} T \right) > 0.$$

At the same time, for any $\kappa > 0$ *and* $t > 0$,

$$\liminf_{T \to \infty} \mathbb{P}(|W_T(t) - W_T^{\mathrm{PAM}}(t)| \ge \kappa) > 0.$$

The explanation for this behavior is that in the PAM a site z outside the current support can have such a high potential value $\xi_T(z)$ that in expectation it becomes optimal to go straight to that site despite the high cost. This leads to exponentially large values of the expectation in areas disconnected from the rest of the support; see Figure 7 for an illustration.

However, the branching random walk can only spread at a speed that depends on the values of the potential at sites that it has already visited. Therefore its support remains connected and particles cannot jump ahead to profit from larger values of the potential.

For the second part of the theorem we show that there are scenarios when the BRW can catch up with the PAM. On the other hand, we can show that there are times when the maximizers are spatially separated. See also Figures 8, 9, 10 and 11 for an illustration.
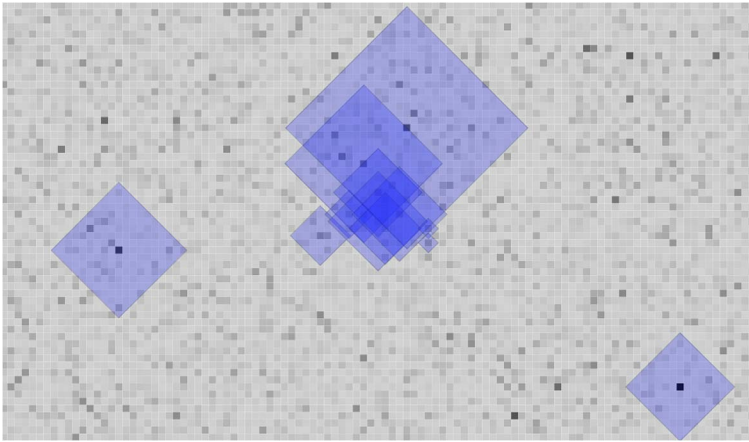
FIG. 7. *The blue regions show the support of the PAM at a particular time. Note that the support is disconnected.*
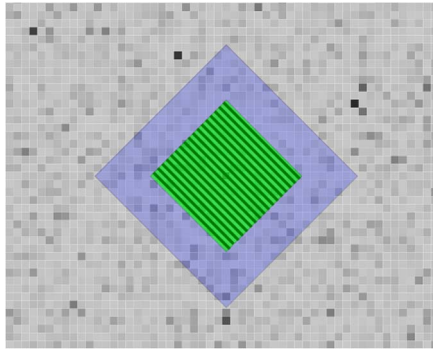

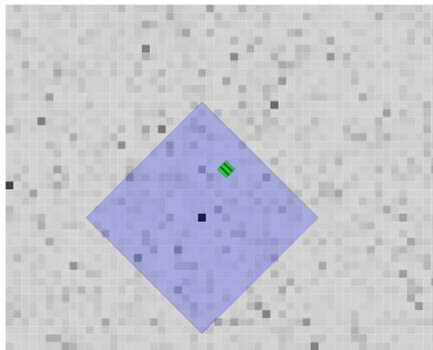FIG. 8. *Support of BRW (striped green) and PAM (blue) with same maximizer in $\mathbb{Z}^2$.*

FIG. 9. *Support of BRW (striped green) and PAM (blue) with different maximizers in $\mathbb{Z}^2$.*
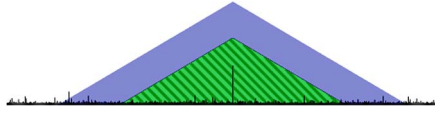
FIG. 10. *Number of particles in BRW (striped green) and PAM (blue) with same maximizer in $\mathbb{Z}$.*

1.5. *Related work.* There are several natural ways of letting a random environment influence the evolution of branching random walks. One possibility, introduced in [9], is to model spatial heterogeneity by associating to each site a randomly sampled offspring distribution. Alternatively, the offspring distribution can also vary in time as an space–time i.i.d. sequence (see, e.g., [4, 8, 16, 28]), or even both the motion of the particles and the offspring distribution can be influenced by the environment; see, for example, [6].

Closely related to our model is a branching random walk on $\mathbb{Z}^d$ in discrete time with a spatial i.i.d. offspring distribution. Here, much more is known about the number of particles. In their early work on this subject for $d = 1$, Greven and den Hollander [13] and Baillon et al. [2] start with an infinite population and describe the local and global growth rates in terms of a variational problem (depending on a drift in the underlying random walk). Many other authors address the question of survival (see, e.g., [3, 10]) and recurrence vs. transience; see, for example, [5, 6, 23, 24].

Since our interest is in the effect of heavy-tailed environments, we assume that the branching rates are bounded away from zero and thus avoid the issue of recurrence and transience. Indeed we see that as soon as a site is occupied, there are almost immediately exponentially many particles, and we focus on analyzing the growth of the branching process by describing when sites are hit and how the number of particles evolves thereafter. We find that for our choice of potential the sites that are hit, as well as the local growth rates, are—even after rescaling appropriately—random. Furthermore, we will show that in our case the growth rates for the actual number of particles deviates dramatically from those for the expected number. This is in sharp contrast to existing shape theorems for branching random walks with spatial i.i.d. offspring distribution; see [6, 7]. More precisely, in [6] it is proved that under a uniform ellipticity condition (and only in the recurrent case) the rescaled set of visited sites is well approximated by a deterministic,
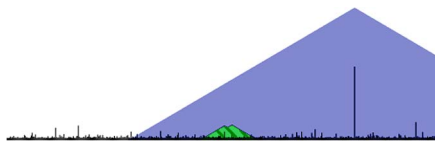

FIG. 11. *Number of particles in BRW (striped green) and PAM (blue) with different maximizers in $\mathbb{Z}$.*

convex and compact set almost surely. This is strengthened in [7] under the assumption that the environment is uniformly elliptic, and the number of immediate offspring is uniformly bounded. Here the authors show that the local growth rates of the quenched expectation (i.e., without averaging over the environment) as well as that of the actual number of particles are described by the same deterministic function.

Compared to work on the PAM, we see several similarities, most prominently intermittency. However, there are also stark differences, both in the results—Theorem 1.5 gives a snapshot, but this reflects just a small part of the contrast described more fully by the clearly distinct lilypad models—and in the methodology. Indeed, one of the main difficulties in studying intermittency in the PAM is that one must control all possible "good islands," whereas for the BRW we must control not just all possible good islands but all possible *paths*, or sequences of good islands. This increases the difficulty significantly and substantial technological innovation is required.

1.6. *Heuristics*. It is already known from work on the PAM that if we look at the expected number of particles at each site in $\mathbb{Z}^d$, the system essentially behaves as follows: the first particle chooses an optimal site z [which will be at distance of order $r(T)$ from the origin], runs there in a short time (order $\ll T$) and sits there for time of order T to take advantage of the large potential at z.

Our first question is whether the branching system follows the same tactic. The answer is no: the probability of one particle running distance $r(T)$ in time $\ll T$ is extremely small, and so the behavior outlined above is effectively impossible. In expectation there is no problem since the enormous reward more than compensates for the small probability of the event, but without taking expectations it is clear that in order to cover large distances, we need to have lots of particles already present in the system.

Suppose that we have some particles at a site z, and that $\xi(z) = Aa(T)$. How long does it take those particles to reach another site y? If $|z - y| = Rr(T)$, where $R \asymp 1$, then the probability that a single random walk started at z is at y at time tT is approximately $e^{-a(T)TqR}$. (The dependence on t is of smaller order, which explains why particles in the PAM run large distances in small times.) Thus we need of the order of $e^{a(T)TqR}$ particles at z before we can reach y. Particles breed at rate $\xi(z)$, so ignoring the motion for a moment, by time tT we should have of the order of $e^{Ata(T)T}$ particles at z. Thus we expect that it takes time $t \approx qR/A$ to reach y from z.

Given the calculations above, we are drawn to the idea that once a site z is hit, particles move outward from z at speed proportional to $\xi(z)$. We imagine a growing "lilypad" of particles centered at z and growing outward at a constant speed. Each site hit by z's lilypad then launches its own lilypad which grows at rate proportional to its potential. Of course if $\xi(z)$ is large, then most of the sites hit

by z's lilypad have smaller potential, so their lilypads grow more slowly and have no discernible effect. Only when z's lilypad touches a point of greater potential do we start uncovering new terrain at a faster rate.

In reality this does not accurately describe how particles behave because if $\xi(z)$ is large, then particles wait at z until the last possible second before running quickly to their desired destination. Besides this, our rough calculations required the potential at z and also the distance between y and z to be large. In particular we should worry about the system at small times, since when we start with one particle at the origin, there might be no points of large potential nearby. Nevertheless, this collection of deterministic (given the environment) growing lilypads does give a caricature of the dynamics of the system that is surprisingly accurate and useful.

Now suppose that we want to know when particles first hit a fixed site z. In order to hit z, we must find a point y_1 of large potential whose lilypad has touched z. We must then ensure that y_1 is hit sufficiently early, so we must find a suitable point y_2 whose lilypad has touched y_1: working backward in this way, we construct a sequence of points leading back toward the origin, and by looking at their potentials—together with their positions relative to one another—we can decide when z should be hit.

1.7. *Organization of the paper.* We begin with some simple estimates on random walks and branching processes in Section 2. In Section 3 we develop some initial estimates on the behavior of the system of lilypads outlined above. We then move on, in Section 4, to give upper bounds on the number of particles in the branching random walk, and then provide lower bounds in Section 5. These are tied together in Section 6 to prove Theorems 1.1, 1.2 and 1.3. The relatively straightforward proof of Theorem 1.4 is given in Section 7, and then in Section 8 we compare the BRW with the PAM by proving Theorem 1.5.

1.8. *Frequently used notation and terminology.* We suppose that under P^ξ, and under an auxiliary probability measure P, we have a simple random walk $(X(u))_{u \geq 0}$, started from 0, independent of the environment and of the branching random walk above.

We fix $c_d, C_d > 0$ such that for any $R, T > 0$ with $Rr(T) > 1$,

$$c_d R^d r(T) \leq \#L_T(0, R) \leq C_d R^d r(T).$$

Sometimes, for events A and B, we say "on A, $\mathbb{P}$-almost surely B occurs." By this we mean that $\mathbb{P}(A \cap B^c) = 0$.

Given $v \in Y(t)$, and $s \leq t$, we write $X_v(s)$ for the position of the unique ancestor of v that was present at time s.

At the end of the article we include a glossary of frequently used notation for reference.

2. Simple estimates on random walks and branching processes. We collect here a few basic results that will be needed later. Lemmas 2.1 and 2.2 will be easy results about the growth of branching processes, and Lemmas 2.3, 2.4 and 2.5 give us control over simple random walks. We also give a Chernoff bound in Lemma 2.6 and an estimate on the largest values of the potential in Lemma 2.7.

First we check that branching processes do not grow much slower than they should. The following result is very basic, but will still be useful occasionally.

LEMMA 2.1. *Let* $(\Upsilon_t)_{t \geq 0}$ *be a Yule process (a continuous-time Galton–Watson process with 2 children at every branch) branching at rate r under an auxiliary probability measure P. For any $r' < r$, there exists a constant c such that*

$$P(\Upsilon_t < \exp(r't)) \leq c \exp((r' - r)t/2) \qquad \text{for all } t \geq 0.$$

PROOF. Let $T_0 = 0$, and for $n \geq 1$ let T_n be the nth birth time of the process, and define $V_n = T_n - T_{n-1}$. Then the random variables $(V_n, n \geq 1)$ are independent, and V_n is exponentially distributed with parameter rn. Thus, using Markov's inequality,

$$P(T_n > t) = P\left(\sum_{j=1}^{n} V_j > t\right) \leq E\left[e^{(r/2)\sum_{j=1}^{n} V_j}\right] e^{-rt/2} \leq \prod_{j=1}^{n}\left(1 - \frac{1}{2j}\right)^{-1} e^{-rt/2}.$$

However,

$$\prod_{j=1}^{n}\left(1 - \frac{1}{2j}\right)^{-1} = \exp\left(-\sum_{j=1}^{n}\log\left(1 - \frac{1}{2j}\right)\right) \leq \exp\left(\sum_{j=1}^{n}\left(\frac{1}{2j} + \frac{1}{2j^2}\right)\right) \leq cn^{1/2}$$

for some constant c. Taking $n = \lfloor e^{r't}\rfloor$, we get

$$P(\Upsilon_t < e^{r't}) \leq P(T_n > t) \leq ce^{r't/2 - rt/2}. \qquad \square$$

Although the previous lemma is occasionally useful, we will need a slightly different estimate in other places. Since our particles can move around, it will often be more useful to be able to know that the number of particles at a single site does not grow much slower than it should.

LEMMA 2.2. *Suppose that $\xi(0) \geq 4d$. Then $P^\xi(N(0, t) < \frac{1}{2}e^{(\xi(0) - 2d)t}) \leq 15/16$ for all $t \geq 0$.*

PROOF. Let $\mathcal{N}(t)$ be the set of particles that have not left 0 by time t, so that $N(0, t) \geq \mathcal{N}(t)$. Clearly $\mathcal{N}$ is a birth–death process in which each particle breeds at rate $\xi(0)$ and dies at rate $2d$. Note that $E^\xi[\mathcal{N}(t)] = e^{(\xi(0) - 2d)t}$ and, by the Paley–Zygmund inequality,

$$P^\xi\left(\mathcal{N}(t) \geq \frac{1}{2}E^\xi[\mathcal{N}(t)]\right) \geq \frac{E^\xi[\mathcal{N}(t)]^2}{4E^\xi[\mathcal{N}(t)^2]}.$$

Thus it suffices to show that $E^\xi[\mathcal{N}(t)^2] \leq 4e^{2(\xi(0)-2d)t}$. By choosing $\delta > 0$ small and conditioning on what happens by time δ, if $\mathcal{N}^1(t)$ and $\mathcal{N}^2(t)$ are independent copies of $\mathcal{N}(t)$, then

$$E^\xi[\mathcal{N}(t+\delta)^2] = E^\xi[\mathcal{N}(t)^2](1 - \delta(\xi(0) + 2d))$$
$$+ E^\xi[(\mathcal{N}^1(t) + \mathcal{N}^2(t))^2]\delta\xi(0) + O(\delta^2)$$

so

$$\frac{d}{dt}E^\xi[\mathcal{N}(t)^2] = (\xi(0) - 2d)E^\xi[\mathcal{N}(t)^2] + 2\xi(0)e^{2(\xi(0)-2d)t}.$$

By solving this ODE we obtain

$$E^\xi[\mathcal{N}(t)^2] = \frac{2\xi(0)}{\xi(0) - 2d}e^{2(\xi(0)-2d)t} + \left(1 - \frac{2\xi(0)}{\xi(0) - 2d}\right)e^{(\xi(0)-2d)t},$$

which, when $\xi(0) \geq 4d$, is at most $4e^{2(\xi(0)-2d)t}$, as required. $\square$

Recall that $X(t), t \geq 0$ is a continuous-time random walk on $\mathbb{Z}^d$. We give a lower bound on the probability that $X(sT) = r(T)z$. Define

$$\mathcal{E}_T^1(s, R) = \frac{R}{\log T}(\log R - \log s) + \frac{2ds}{a(T)}.$$

LEMMA 2.3. *For $z \in L_T$ and $s > 0$, $T > e$,*

$$P(X(sT) = r(T)z) \geq \exp(-a(T)T(q|z| + \mathcal{E}_T^1(s, |z|))).$$

PROOF. Fix a path of length $r(T)|z|$ from 0 to $r(T)z$. To reach $r(T)z$, it suffices to make exactly $r(T)|z|$ jumps by time t, all along our chosen path. Thus

$$P(X(t) = z) \geq \frac{1}{(2d)^{r(T)|z|}}e^{-2dsT}\frac{(2dsT)^{r(T)|z|}}{(r(T)|z|)!}.$$

Using the fact that $n! \leq n^n = \exp(n \log n)$, the above is at least

$$\exp(-2dsT + r(T)|z|\log(sT) - r(T)|z|\log(r(T)|z|))$$
$$\geq \exp(-r(T)|z|\log(T^q|z|/s) - 2dsT)$$
$$= \exp\left(-a(T)T\left(q|z| + \frac{|z|}{\log T}(\log|z| - \log s) + \frac{2ds}{a(T)}\right)\right). \quad \square$$

Now we need an upper bound on the probability that $X(t) = z$. In order to reach z, a random walk must jump at least $|z|$ times, and this bound will be enough for us, so for $s \geq 0$ and $R > 0$, define

$$J_T(s, R) = P(X(u) \text{ jumps at least } Rr(T) \text{ times before time } sT),$$

and let

$$\mathcal{E}_T^2(s, R) = \frac{R}{\log T}(\log s - \log R + 1 + \log(2d) + (q+1)\log\log T).$$

LEMMA 2.4. *For any $R > 0$ and $s > 0$, $T > e$,*

$$J_T(s, R) \le \exp\{-a(T)T(qR - \mathcal{E}_T^2(s, R))\}.$$

PROOF. The number of jumps that $X(u)$ makes up to time sT is Poisson distributed with parameter $2\,ds T$, so that

$$J_T(s, R) \le \frac{(2\,ds T)^{Rr(T)}}{(Rr(T))!}.$$

By Stirling's formula $n! \ge \exp(n\log n - n)$, giving a new upper bound of

$$\exp\big(Rr(T)\big(-\log(Rr(T)) + 1 + \log(2\,ds T)\big)\big)$$
$$= \exp\big(-qr(T)R\log T$$
$$\qquad + r(T)R\big(1 + \log s + \log(2d) + (q+1)\log\log T - \log R\big)\big)$$
$$= \exp\big(-a(T)T(qR - \mathcal{E}_T^2(s, R))\big),$$

where we used the definitions of $r(T)$ and $a(T)$ as well as $\mathcal{E}_T^2$. □

Our third estimate on random walks is slightly different. Instead of looking at the probability that a random walk moves a long way in a relatively short time, we now want to ensure that the probability a random walk moves a short distance in a relatively long time is reasonably large. This is a consequence of a standard local central limit theorem; see, for example, Theorem 2.1.3 of [20].

LEMMA 2.5. *There exists a constant $c > 0$ such that provided $|z| \le \sqrt{t}$,*

$$P(X(t) = z) \ge ct^{-d/2}.$$

The following well-known version of the Chernoff bound will also be useful.

LEMMA 2.6. *Suppose that $Z_1, \ldots, Z_k$ are independent Bernoulli random variables, and let $Z = \sum_{i=1}^r Z_i$. Then*

$$P\left(Z \le \frac{E[Z]}{2}\right) \le \exp\left(-\frac{E[Z]}{8}\right).$$

Finally, we give some simple estimates on the maximum of the environment within a ball. For $R > 0$, let

$$\bar{\xi}_T(R) = \max_{z \in L_T(0, R)} \xi_T(z).$$

LEMMA 2.7. (i) *For any $T > e$, any $R > 0$ and $v > 0$,*

$$\mathbb{P}(\bar{\xi}_T(R) \leq v) \leq e^{-c_d R^d v^{-\alpha}}.$$

(ii) *Provided that $Rr(T) \geq 1$, for any $N \geq 1$ and any $v > 0$,*

$$\mathbb{P}(\#\{z \in L_T(0, R) : \xi_T(z) \geq v\} \geq N) \leq \left(\frac{C_d e R^d v^{-\alpha}}{N}\right)^N.$$

PROOF. (i) We may assume without loss of generality that $va(T) \geq 1$; otherwise all points satisfy $\xi_T(z) > v$. By independence,

$$\mathbb{P}(\bar{\xi}_T(R) \leq v) = \mathbb{P}(\xi_T(0) \leq v)^{\#L_T(0,R)}$$

$$\leq \left(1 - (va(T))^{-\alpha}\right)^{c_d R^d r(T)^d} \leq e^{-c_d R^d v^{-\alpha}},$$

where the last inequality follows from the inequality $1 - x \leq e^{-x}$ and the fact that $r(T)^d = a(T)^\alpha$.

(ii) The number of points in $L_T(0, R)$ with (rescaled) potential larger than v is dominated by a binomial random variable with $C_d e R^d r(T)^d$ trials of success probability $v^{-\alpha} a(T)^{-\alpha}$. Thus

$$\mathbb{P}(\#\{z \in L_T(0, R) : \xi_T(z) \geq v\} \geq N) \leq \binom{C_d R^d r(T)^d}{N} v^{-N\alpha} a(T)^{-N\alpha}$$

$$\leq \frac{(C_d R^d r(T)^d)^N}{N!} v^{-N\alpha} a(T)^{-N\alpha}.$$

Since $r(T)^d = a(T)^\alpha$, and $N! \geq N^N e^{-N}$, we get the result. $\square$

3. First properties of the lilypad model. There are several fairly simple facts about the environment that will be useful to us later. We begin with some almost self-evident observations in Section 3.1 that nonetheless take some time to prove rigorously: Lemma 3.1 tells us that the infimum in the definition of our lilypad hitting times $h_T(z)$ is attained, Lemma 3.2 proves that the infimum may be broken up into more manageable chunks, while Lemma 3.3 records properties of the potential along an optimal path. In Section 3.2 we bound the growth of our lilypad model: Lemma 3.4 gives upper bounds on the time required to cover a ball about the origin, and Corollary 3.5 gives a more explicit bound on the time to cover a particular small ball. Then Lemma 3.6 ensures that the lilypad model does not quickly exit large balls and thus does not explode in finite time. We will use many of these results often, usually without reference.

3.1. *An alternative formulation for hitting times in the lilypad model.* As mentioned earlier, we want to show that the hitting times in the lilypad model have two

equivalent formulations. We define

$$h_T(z) = \inf_{\substack{y_0,\ldots,y_n \in L_T: \\ y_0=z, y_n=0}} \left\{ \sum_{j=1}^n q \frac{|y_{j-1}-y_j|}{\xi_T(y_j)} \right\}$$

and claim that

$$h_T(z) = \inf_{y \neq z} \left\{ h_T(y) + q \frac{|z-y|}{\xi_T(y)} \right\}.$$

First we check that the infimum is attained.

LEMMA 3.1. *For any $T > 0$, the infimum in the definition of $h_T(z)$ is attained for some sequence $y_0, \ldots, y_n$, $\mathbb{P}$-almost surely.*

PROOF. Note that for $\lambda = \frac{1}{2}\frac{\alpha-d}{\alpha}$, by the definition of $a(T)$ and $r(T)$

$$\mathbb{P}\big(\exists y \in B(0, e^k): \xi_T(y) > e^{(1-\lambda)k}\big) \leq C_d\big(r(T)e^k\big)^d \big(a(T)c_k e^{(1-\lambda)k}\big)^{-\alpha}$$
$$= C_d e^{-(1/2)(\alpha-d)k}.$$

Therefore, by the Borel–Cantelli lemma, there exists K such that for all $k \geq K$, $\max_{y \in B(0,e^k)} \xi_T(y) \leq e^{(1-\lambda)k}$. By increasing K if necessary we may assume also that $\frac{h_T(z)+1}{q} \leq e^{\lambda K}$. Suppose for contradiction that there exists a sequence $(y_j)_{j=0,\ldots,n}$ with $y_0 = z$ and $y_n = 0$ such that for at least one j, $y_j \notin L_T(0, e^K)$ and

$$\sum_{j=0}^n q \frac{|y_{j-1}-y_j|}{\xi_T(y_j)} \leq h_T(z) + \frac{1}{2}.$$

Define $\ell = \max\{j: y_j \notin B(0, e^K)\}$, so that by assumption $\ell \in \{0, \ldots, n-1\}$. Then by the triangle inequality,

$$h_T(z) + \frac{1}{2} \geq \sum_{j=\ell+1}^n q \frac{|y_{j-1}-y_j|}{\xi_T(y_j)} \geq q \frac{|y_\ell|}{\max_{y \in L_T(0,e^K)} \xi_T(y)} \geq q \frac{e^K}{e^{(1-\lambda)K}}$$
$$= q e^{\lambda K}.$$

This contradicts our choice of K, and we deduce that

$$h_T(z) = \inf_{\substack{y_0,\ldots,y_n \in L_T(0,e^K): \\ y_0=z, y_n=0}} \left\{ \sum_{j=1}^n q \frac{|y_{j-1}-y_j|}{\xi_T(y_j)} \right\}.$$

This infimum is over a finite set, so the minimum is attained. □

We can now prove our alternative formulation of the hitting times.

LEMMA 3.2. $\mathbb{P}$-*almost surely, for any* $z \neq 0$ *and* $T > 0$,

$$h_T(z) = \inf_{y \neq z} \left\{ h_T(y) + q \frac{|z - y|}{\xi_T(y)} \right\}.$$

PROOF. Fix $z \neq 0$. First suppose there exists y such that

$$h_T(y) + \frac{|z - y|}{\xi_T(y)} < h_T(z).$$

Then by Lemma 3.1, there exist n and $y_0, \ldots, y_n$ such that $y_0 = y$, $y_n = 0$ and $h_T(y) = \sum_{j=1}^n q|y_{j-1} - y_j|/\xi_T(y_j)$. Defining $y'_0 = z$ and for $i = 0, \ldots, n$ letting $y'_{i+1} = y_i$, we have by definition of $h_T(z)$

$$h_T(z) \leq \sum_{j=1}^{n+1} q \frac{|y'_{j-1} - y'_j|}{\xi_T(y'_j)} = h_T(y) + q \frac{|z - y|}{\xi_T(y)} < h_T(z).$$

This is a contradiction, so we have established that

$$h_T(z) \leq \inf_{y \neq z} \left\{ h_T(y) + q \frac{|y - z|}{\xi_T(y)} \right\}.$$

For the opposite inequality, choose (by Lemma 3.1) n and distinct $y_0, \ldots, y_n$ such that $y_0 = z$, $y_n = 0$ and

$$h_T(z) = \sum_{j=1}^n q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)}.$$

We claim that $h_T(y_1) = \sum_{j=2}^n q|y_{j-1} - y_j|/\xi_T(y_j)$. If not, then there exist m and $x_0, \ldots, x_m$ such that $x_0 = y_1$, $x_m = 0$ and

$$h_T(y_1) = \sum_{j=1}^m q \frac{|x_{j-1} - x_j|}{\xi_T(x_j)} < \sum_{j=2}^n q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)}.$$

Let $y'_0 = y_0 = z$, $y'_1 = y_1$, and for $j = 2, \ldots, m + 1$, $y'_j = x_{j-1}$. Then

$$h_T(z) \leq \sum_{i=1}^{m+1} q \frac{|y'_{i-1} - y'_i|}{\xi_T(y'_i)} = q \frac{|y_0 - y_1|}{\xi_T(y_1)} + \sum_{j=1}^m q \frac{|x_{j-1} - x_j|}{\xi_T(x_j)}$$

$$< \sum_{j=1}^n q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} = h_T(z).$$

This is a contradiction, so our claim that $h_T(y_1) = \sum_{j=2}^n q|y_{j-1} - y_j|/\xi_T(y_j)$ holds. Then

$$h_T(z) = h_T(y_1) + q \frac{|z - y_1|}{\xi_T(y_1)},$$

which completes the proof. $\square$

LEMMA 3.3. *Let $T > 0$, $z \in L_T$, and suppose that*

$$h_T(z) = \sum_{j=1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)},$$

for distinct points y_j, $j = 1, \ldots, n$ with $y_0 = z$ and $y_n = 0$. Then, for any $k \in \{1, \ldots, n\}$,

$$h_T(y_k) = \sum_{j=k+1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)}.$$

Moreover, the sequence $(\xi_T(y_j), j \geq 1)$ is nonincreasing.

PROOF. We have already shown in the proof of Lemma 3.2 that

$$h_T(y_1) = \sum_{j=2}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)}.$$

Iterating the argument shows the first statement.

For the second statement, suppose that there exists $k \in \{1, \ldots, n-1\}$ such that $\xi_T(y_k) < \xi_T(y_{k+1})$. We show that it is then faster to reach z without traveling via y_k. Indeed, by the triangle inequality

$$\sum_{j \in \{1,\ldots,n\}, j \neq k, k+1} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y_{k-1} - y_{k+1}|}{\xi_T(y_{k+1})}$$

$$\leq \sum_{j \in \{1,\ldots,n\}, j \neq k, k+1} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y_{k-1} - y_k| + |y_k - y_{k+1}|}{\xi_T(y_{k+1})}$$

$$< \sum_{j=1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} = h_T(z),$$

contradicting the definition of $h_T(z)$. □

3.2. *Bounding the lilypad model.* We want to make sure that the lilypad model behaves relatively sensibly: that small balls are covered quickly, but large balls are not. We begin with the former statement; more precisely, we show that for any time $t > 0$, we can find a radius $R > 0$ such that with high probability, $B(0, R)$ is covered by time t. For $R > 0$, let

$$\bar{h}_T(R) = \sup_{z \in B(0,R)} h_T(z).$$

LEMMA 3.4. *For all $k \in \mathbb{N}$, $T > e$ and $\gamma \in (d/\alpha, 1)$,*

$$\mathbb{P}\left(\bar{h}_T(2^{-k}) > \frac{4q}{1 - 2^{\gamma-1}} 2^{(\gamma-1)k} \right) \leq \sum_{j=k}^{\infty} e^{-c_d 2^{(\alpha\gamma - d)j}}.$$

Moreover, for any $R > 0$, $T > e$ and $\gamma \in (d/\alpha, 1)$,

$$\mathbb{P}\left(\bar{h}_T(R) > 2^{\gamma(k+1)}qR + \frac{4q}{1 - 2^{\gamma-1}}2^{(\gamma-1)k}\right) \leq \sum_{j=k}^{\infty} e^{-c_d 2^{(\alpha\gamma-d)j}}.$$

PROOF. Let $B_k = L_T(0, 2^{-k})$. Define $A_k = \{\exists z \in B_k : \xi_T(z) \geq 2^{-\gamma k}\}$. Then by Lemma 2.7(i), for any $T > e$,

$$\mathbb{P}(A_k^c) \leq e^{-c_d 2^{(\alpha\gamma-d)k}}.$$

Thus

$$\sum_{j=k+1}^{\infty} \mathbb{P}(A_j^c) \leq \sum_{j=k+1}^{\infty} e^{-c_d 2^{(\alpha\gamma-d)j}}.$$

However, if $A_{k+1}, A_{k+2}, \ldots$ all occur, then we may choose $y_j \in B(0, 2^{-(k+j)})$ such that $\xi_T(y_j) \geq 2^{-\gamma(k+j)}$ for each $j \geq 1$. Clearly there exists n such that $y_j = 0$ for all $j \geq n$. Take $z \in B(0, 2^{-k})$, and let $y_0 = z$. Then for $j \geq 1$,

$$q\frac{|y_{j-1} - y_j|}{\xi_T(y_j)} \leq q\frac{2 \cdot 2^{-(k+j-1)}}{2^{-\gamma(k+j)}} = 4q \cdot 2^{(\gamma-1)(k+j)},$$

so by the definition of $h_T(z)$,

$$h_T(z) \leq 4q \sum_{j=1}^{n} 2^{(\gamma-1)(k+j)} \leq \frac{4q}{1 - 2^{\gamma-1}}2^{(\gamma-1)k}$$

which proves our first claim. For the second, it suffices to observe that if $x \in B(0, R)$ and we can find the above sequence $y_1, y_2, \ldots$, then by Lemma 3.2

$$h_T(x) \leq h_T(y_1) + q\frac{|x - y_1|}{\xi_T(y_1)} \leq 4q \sum_{j=2}^{n} 2^{(\gamma-1)(k+j)} + q\frac{R + 2^{-(k+1)}}{2^{-\gamma(k+1)}}$$

$$\leq 2^{\gamma(k+1)}qR + \frac{4q}{1 - 2^{\gamma-1}}2^{(\gamma-1)k}. \qquad \square$$

By choosing $\gamma = (d/\alpha + 1)/2$ and $k = \frac{\psi+1}{(1-\gamma)\log 2}\log\log T$, we get the following corollary.

COROLLARY 3.5. *For large T and any $\psi > 0$,*

$$\mathbb{P}\left(\bar{h}_T\left(\log^{-2(\psi+1)(q+1)}T\right) > \frac{4q}{1 - 2^{\gamma-1}}\log^{-(\psi+1)}T\right) \leq T^{-1}.$$

Now we show that conversely, we can find a radius $R > 0$ such that with high probability the lilypad model does not exit $B(0, R)$ by time t.

LEMMA 3.6. *For any $t > 0$, provided that $Rr(T) \geq 1$,*

$$\{\exists z \in L_T \setminus B(0, R) : h_T(z) \leq t\} \subseteq \Big\{ \max_{y \in L_T(0, R)} \xi_T(y) \geq qR/t \Big\}.$$

As a result,

$$\mathbb{P}\big(\exists z \in L_T \setminus B(0, R) : h_T(z) \leq t\big) \leq C_d e q^{-\alpha} R^{d-\alpha} t^{\alpha}.$$

REMARK. In particular, the lilypad model does not explode in finite time.

PROOF OF LEMMA 3.6. Let $D_T = L_T(0, R + 1) \setminus L_T(0, R)$, the boundary of $L_T(0, R)$. Let $\tilde{z}$ be the point with smallest lilypad hitting time in D_T, that is, $h_T(\tilde{z}) = \min_{y \in D_T} h_T(y)$. Then from the definition of h_T, any point $z \in L_T \setminus B(0, R)$ satisfies $h_T(z) \geq h_T(\tilde{z})$. For the same reason, in the definition of $h_T(\tilde{z})$, we can restrict the infimum to points y_i within $L_T(0, R)$ so that if we set $y_0 = \tilde{z}$,

$$h_T(\tilde{z}) = \inf_{\substack{y_1, \ldots, y_n \in L_T(0, R) : \\ y_n = 0}} \sum_{j=1}^{n} q \frac{|y_{i-1} - y_i|}{\xi_T(y_i)} \geq q \frac{|\tilde{z}|}{\max_{y \in L_T(0, R)} \xi_T(y)}$$

$$\geq \frac{qR}{\max_{y \in L_T(0, R)} \xi_T(y)}.$$

Then, by the above estimate, we have that

$$\{\exists z \in L_T \setminus B(0, R) : h_T(z) \leq t\} \subseteq \{h_T(\tilde{z}) \leq t\} \subseteq \Big\{ \max_{y \in L_T(0, R)} \xi_T(y) \geq \frac{qR}{t} \Big\}.$$

Lemma 2.7(ii), with $N = 1$, then tells us that

$$\mathbb{P}\Big(\max_{y \in L_T(0, R)} \xi_T(y) \geq \frac{qR}{t} \Big) \leq C_d e q^{-\alpha} R^{d-\alpha} t^{\alpha}$$

which gives the desired bound. □

4. Upper bounds. In this section we come back to the branching random walk. We will check that particles do not arrive anywhere earlier than they should and—as a consequence—that the number of particles at any site is not too large. Our main tool will be the many-to-one lemma, also known as the Feynman–Kac formula, which we introduce in Section 4.1. We then apply this to bound the hitting times in terms of an object G_T, which we go on to study in Section 4.2. The tactic will be to give a recursive bound on G_T along any sequence of points of increasing potential, and then in Section 4.3 we fix a particular sequence and calculate the resulting estimate. Finally we apply this hard work in Sections 4.4 and 4.5 to show, respectively, that particles do not arrive early and that there are not too many particles.

4.1. *The many-to-one lemma.* We introduce a standard tool, sometimes called the many-to-one lemma (in the branching process literature) and sometimes the Feynman–Kac formula (in the Parabolic Anderson and statistical physics literature). It gives us a way of calculating expected numbers of particles in our branching random walk by considering the behavior of a single random walk. Recall that under P^ξ, $(X(u))_{u\geq 0}$ is a simple random walk on $\mathbb{Z}^d$ independent of our branching random walk.

LEMMA 4.1 (Many-to-one lemma/Feynman–Kac formula). *If f is measurable, then* Prob-*almost surely, for any $s > 0$,*

$$E^\xi\left[\sum_{v\in Y(s)} f\big((X_v(u))_{u\in[0,s]}\big)\right] = E^\xi\left[\exp\left(\int_0^s \xi(X(u))\,du\right) f\big((X(u))_{u\in[0,s]}\big)\right].$$

The interested reader may find a proof in [17], or for a more modern approach [15]. Both references give far more general versions of this lemma, and in fact we will need one such generalization. It is not too surprising, given that the many-to-one lemma involves the equality of two expectations, that there is a martingale hidden away here. For the more general version, essentially we want to stop the martingale at a stopping time, rather than at a fixed time s; but while the concept of a stopping time is simple enough for our single random walk $(X(u))_{u\geq 0}$, we need something a bit more general for our branching random walk. This is where the concept of a *stopping line* enters. There is a whole theory built around this idea, but we will need only the simplest part of it, which can be deduced rather easily, avoiding a detailed discussion. Indeed, fix $T > 1$ and a point $z \in L_T$, and imagine that any particle that hits $r(T)z$ is absorbed there, alive but no longer moving or breeding. When working with this alternative system, we will attach a superscript $\sim$ to our notation, so, for example, $\tilde{Y}(s)$ will be the set of particles present at time s in the alternative system.

We make two observations about the alternative system. First, the many-to-one lemma still holds, but since particles stop breeding as soon as they hit z, if we define

$$H_T^*(z) = \inf\{t \geq 0 : X(tT) = r(T)z\},$$

then we have

$$E^\xi\left[\sum_{v\in\tilde{Y}(tT)} f\big((\tilde{X}_v(u))_{u\in[0,tT]}\big)\right]$$

$$= E^\xi\left[\exp\left(T\int_0^{\tilde{H}_T^*(z)\wedge t} \xi\big(\tilde{X}(uT)\big)\,du\right) f\big((\tilde{X}(u))_{u\in[0,tT]}\big)\right],$$

where $(\tilde{X}(u))_{u\geq 0}$ is a simple random walk absorbed at $r(T)z$. Second, notice that we may take the obvious coupling so that the two systems are identical until $H_T(z)$; in particular $H_T(z) \leq t$ if and only if $\tilde{H}_T(z) \leq t$.

Before we apply our two observations to prove a key lemma, we will introduce some notation that we will use throughout this section. We will work with a fixed large T, distinct points $z_1, z_2, \ldots \in L_T$ and constants $t, t_1, t_2, \ldots \in \mathbb{R}^+$. We are interested in bounding the event

$$A_T(j, z, t) = \left\{ \begin{array}{c} H_T(z) \leq t, \, H_T(z_i) \geq H_T(z) \wedge t_i \; \forall i \leq j, \\ H_T(z_i) \geq H_T(z) \; \forall i > j \end{array} \right\}.$$

Similarly, we define $A_T^*(j, z, t)$ for the random walk by replacing H_T by H_T^* in the above definition. Informally, $A_T(j, z, t)$ [resp., $A_T^*(j, z, t)$] is the event that z is hit by time t by the rescaled branching random walk (resp., the rescaled random walk), none of the $z_i, i > j$ are hit before z and those $z_i, i \leq j$ that are hit before z are not hit before t_i.

We will bound the probability of the event $A_T(j, z, t)$ in terms of the following key quantity:

$$(2) \qquad G_T(j, z, s, t) = E^\xi \left[\exp\left(T \int_0^{H_T^*(z) \wedge s} \xi(X(uT)) \, du \right) \mathbb{1}_{A_T^*(j, z, t)} \right],$$

where the need for an extra parameter $s \geq 0$ becomes apparent later on.

LEMMA 4.2. $\mathbb{P}$-almost surely, for any $T > 0$, distinct points $z_1, z_2, \ldots \in L_T$, any $z \in L_T$, any $t, t_1, t_2, \ldots \in \mathbb{R}$ and any $j \geq 0$,

$$P^\xi(A_T(j, z, t)) \leq E^\xi \left[\exp\left(T \int_0^{H_T^*(z)} \xi(X(uT)) \, du \right) \mathbb{1}_{A_T^*(j, z, t)} \right] = G(j, z, t, t).$$

PROOF. All statements below hold $\mathbb{P}$-almost surely. By our second observation above,

$$P^\xi(H_T(z) \leq t, \, H_T(z_i) \geq H_T(z) \wedge t_i \; \forall i \leq j, \, H_T(z_i) \geq H_T(z) \; \forall i > j)$$
$$= P^\xi(\tilde{H}_T(z) \leq t, \, \tilde{H}_T(z_i) \geq \tilde{H}_T(z) \wedge t_i \; \forall i \leq j, \, \tilde{H}_T(z_i) \geq \tilde{H}_T(z) \; \forall i > j).$$

Now, if some particle is to hit z without hitting any of the z_i too early (where "too early" is interpreted appropriately depending on whether $i \leq j$), there must be a first particle to do so; so writing $H_T^v(z)$ for the (rescaled) first time that particle v—or one of its ancestors or descendants—hits z,

$$P^\xi(\tilde{H}_T(z) \leq t, \, \tilde{H}_T(z_i) \geq \tilde{H}_T(z) \wedge t_i \; \forall i \leq j, \, \tilde{H}_T(z_i) \geq \tilde{H}_T(z) \; \forall i > j)$$
$$\leq P^\xi(\exists v \in \tilde{Y}(tT) : \tilde{H}_T^v(z) \leq t, \, \tilde{H}_T^v(z_i) \geq \tilde{H}_T^v(z) \wedge t_i \; \forall i \leq j,$$
$$\tilde{H}_T^v(z_i) \geq \tilde{H}_T^v(z) \; \forall i > j)$$
$$\leq E^\xi \left[\sum_{v \in \tilde{Y}(tT)} \mathbb{1}_{\{\tilde{H}_T^v(z) \leq t, \tilde{H}_T^v(z_i) \geq \tilde{H}_T^v(z) \wedge t_i \; \forall i \leq j, \tilde{H}_T^v(z_i) \geq \tilde{H}_T^v(z) \; \forall i > j\}} \right].$$

We now apply the many-to-one lemma for the alternative system to the last expectation to see that it equals

$$E^{\xi}\left[\exp\left(T \int_0^{\tilde{H}_T^*(z)} \xi(X(uT))\,du\right)\mathbb{1}_{\left\{\begin{array}{c}\tilde{H}_T^*(z) \leq t,\, \tilde{H}_T^*(z_i) \geq \tilde{H}_T^*(z) \wedge t_i\ \forall i \leq j, \\ \tilde{H}_T^*(z_i) \geq \tilde{H}_T^*(z)\ \forall i > j\end{array}\right\}}\right].$$

But now we can use our second observation again to remove all the $\sim$ superscripts from the above statement and deduce the desired result. $\quad\square$

Our first aim is to show that the probability that we hit z early is small. We use Lemma 4.2, and our tactic is to let $z_1, z_2, \ldots$ be the set of points of large potential in increasing order of ξ, and work by induction on the largest j such that we hit z_j before z. It will then be important how long we spend at z_j, and to control this we will need to "decouple" the time in the $\exp(\cdot)$ part of Lemma 4.2 from the time in the indicator function, which explains the extra parameter s in the definition of G_T. We will concentrate on bounding G_T in the next section, but a clue as to how we will use it comes via the following easy corollary of Lemma 4.2.

COROLLARY 4.3. $\mathbb{P}$-almost surely, for any $z \in L_T$, any $s, t \geq 0$ and any $j \geq 0$,

$$P^{\xi}\big(H_T(z) \leq s \wedge t,\, H_T(z_i) \geq H_T(z) \wedge t_i\ \forall i \leq j,\, H_T(z_i) \geq H_T(z)\ \forall i > j\big)$$
$$\leq G_T(j, z, s, t).$$

PROOF. By Lemma 4.2,

$$P^{\xi}\big(H_T(z) \leq s \wedge t,\, H_T(z_i) \geq H_T(z) \wedge t_i\ \forall i \leq j,\, H_T(z_i) \geq H_T(z)\ \forall i > j\big)$$
$$\leq G_T(j, z, s \wedge t, s \wedge t),$$

but $G_T(j, z, s, t)$ is increasing in both s and t. $\quad\square$

4.2. *Bounding G_T.* The work above will allow us to reduce the problem of proving upper bounds to bounding $G_T(j, z, s, t)$. As mentioned above, we want to work by induction, and the following result allows us to bound $G_T(j, \cdot, \cdot, \cdot)$ in terms of $G_T(j - 1, \cdot, \cdot, \cdot)$. Recall that

$$J_T(t, R) = P\big(X(u) \text{ jumps at least } Rr(T) \text{ times before time } tT\big).$$

LEMMA 4.4. *Suppose that $j \geq 1$ and that $\xi_T(y) \leq \xi_T(z_1) \leq \xi_T(z_2) \leq \cdots \leq \xi_T(z_j)$ for all $y \notin \{z_1, z_2, \ldots\}$. Then $\mathbb{P}$-almost surely, for any z and any $s, t \geq 0$,*

$$G_T(j, z, s, t) \leq G_T(j - 1, z, s, t)$$
$$+ G_T(j - 1, z_j, t_j, t)e^{a(T)T\xi_T(z_j)(s - t_j)_+}J_T\big(t, |z - z_j|\big),$$

where for $x \in \mathbb{R}$, $x_+ = x \vee 0$.

PROOF. The main idea is that either we hit z_j before hitting z, or we do not. In the latter case, we reduce to $G_T(j-1, z, s, t)$, and in the former case, our best tactic is to get to z_j as quickly as possible (since it has larger potential than any other point we are allowed to visit) and stay there for as long as we can. Getting there as quickly as possible gives us $G_T(j-1, z_j, t_j, t)$, and staying there until time s gives us the exponential factor; then we must also at some point run to z, which costs us $J_T(t, |z - z_j|)$.

By default, all statements below hold Prob-almost surely. If $z = z_j$, then $G_T(j, z, s, t) \leq G_T(j-1, z, s, t)$, so the inequality trivially holds. We may therefore assume that $z \neq z_j$. Note then that either $H_T^*(z_j) > H_T^*(z)$ or $H_T^*(z_j) < H_T^*(z)$. In the former case $A_T^*(j-1, z, t)$ occurs, and therefore

$$G_T(j, z, s, t) \leq G_T(j-1, z, s, t)$$

$$+ E^\xi \left[\exp\left(T \int_0^{H_T^*(z) \wedge s} \xi(X(uT)) \, du \right) \mathbb{1}_{A_T^*(j, z, t)} \mathbb{1}_{\{H_T^*(z_j) < H_T^*(z)\}} \right].$$

On $A_T^*(j, z, t)$, since $\xi_T(z_j) \geq \xi_T(z_i)$ for all $i \leq j$ and $t_j \leq H_T^*(z_j) < H_T^*(z_i)$ for all $i > j$, we have

$$T \int_0^{H_T^*(z) \wedge s} \xi(X(uT)) \, du$$

$$\leq T \int_0^{H_T^*(z_j) \wedge t_j} \xi(X(uT)) \, du + a(T) T \xi_T(z_j)(s - t_j)_+.$$

Note also that

$$A_T^*(j, z, t) \cap \{ H_T^*(z_j) < H_T^*(z) \}$$

$$\subseteq A_T^*(j-1, z_j, t) \cap \{ t_j \leq H_T^*(z_j) < H_T^*(z) \leq t \}.$$

Thus

$$G_T(j, z, s, t) \leq G_T(j-1, z, s, t)$$

$$(3) \qquad + E^\xi \left[\exp\left(T \int_0^{H_T^*(z_j) \wedge t_j} \xi(X(uT)) \, du \right) \mathbb{1}_{A_T^*(j-1, z_j, t)} \right.$$

$$\left. \times \exp\left(a(T) T \xi_T(z_j)(s - t_j)_+ \right) \mathbb{1}_{\{H_T^*(z_j) < H_T^*(z) \leq t\}} \right].$$

If $(\mathcal{G}_u, u \geq 0)$ is the natural filtration for X, we observe that

$$P^\xi \left(H_T^*(z_j) < H_T^*(z) \leq t | \mathcal{G}_{H_T^*(z_j)T} \right) \leq J_T(t, |z - z_j|).$$

Inserting this into (3) (the first part inside the expectation is $\mathcal{G}_{H_T^*(z_j)T}$-measurable), we obtain

$$G_T(j, z, s, t) \leq G_T(j-1, z, s, t)$$

$$+ E^\xi \left[\exp\left(T \int_0^{H_T^*(z_j) \wedge t_j} \xi(X(uT)) \, du \right) \mathbb{1}_{A_T^*(j-1, z_j, t)} \right]$$

$$\times \exp\left(a(T) T \xi_T(z_j)(s - t_j)_+ \right) J_T(t, |z - z_j|).$$

We now recognize the expectation above as $G_T(j-1, z_j, t_j, t)$, which gives us exactly the expression required. $\square$

Now we have a way of reducing j until it hits 0, and so we need a bound on $G_T(0, z, s, t)$. Recall that $\bar{h}_T(R) = \max_{z \in L_T(0,R)} h_T(z)$. The lemma below gives a simple bound when s is slightly smaller than $h_T(z)$ and z is outside a ball about the origin. It may be useful to imagine applying it when R is small and z is a long way from $B(0, R)$, so that $\bar{h}_T(R) < \delta$ and $q(\gamma-1)|z| + q\gamma R < 0$. (This is exactly what we shall do later.)

LEMMA 4.5. *Let $\gamma \in (0, 1)$, $\delta > 0$ and $t > 0$. $\mathbb{P}$-almost surely, if $\xi_T(y) \le \bar{\xi}_T(R)$ for all $y \notin \{z_1, z_2, \ldots\}$, then for $z \notin B(0, R)$, if $\gamma h_T(z) - \delta \ge 0$,*

$$G_T(0, z, \gamma h_T(z) - \delta, t)$$
$$\le \exp\big(a(T)T\big(\bar{\xi}_T(R)(\gamma \bar{h}_T(R) - \delta) + q(\gamma-1)|z| + q\gamma R + \mathcal{E}_T^2(t, |z|)\big)\big).$$

PROOF. We again work $\mathbb{P}$-almost surely throughout. Whenever $y \notin \{z_1, z_2, \ldots\}$, we have $\xi_T(y) \le \bar{\xi}_T(R)$, and therefore on $A_T^*(0, z, t)$ we have

$$T \int_0^{H_T^*(z) \wedge (\gamma h_T(z) - \delta)} \xi\big(X(uT)\big)\, du \le a(T)T\bar{\xi}_T(R)(\gamma h_T(z) - \delta).$$

Also,

$$h_T(z) \le \min_{y \in B(0,R)} \left\{ h_T(y) + q \frac{|z - y|}{\xi_T(y)} \right\} \le \bar{h}_T(R) + q \frac{|z| + R}{\bar{\xi}_T(R)}.$$

Thus

$$G_T(0, z, \gamma h_T(z) - \delta, t)$$
$$\le E^\xi\Big[\exp\big(a(T)T\big(\bar{\xi}_T(R)(\gamma \bar{h}_T(R) - \delta) + q\gamma|z| + q\gamma R\big)\big)\mathbb{1}_{A_T^*(0,z,t)}\Big]$$
$$= \exp\big(a(T)T\big(\bar{\xi}_T(R)(\gamma \bar{h}_T(R) - \delta) + q\gamma|z| + q\gamma R\big)\big)P^\xi\big(A_T^*(0, z, t)\big).$$

However, $P^\xi(A_T^*(0, z, t))$ is at most the probability that our random walk jumps $|z|r(T)$ times by time tT, which is exactly $J_T(t, |z|)$. Applying Lemma 2.4 completes the proof. $\square$

4.3. *Fixing parameters.* Until now we have worked with general points z_i and times t_i. Now we want to specialize to our particular situation. We suppose that we are given a fixed time $t_\infty \ge 1$ and proceed to fix a variety of parameters which we will use to ensure that the probability that particles arrive at any point z substantially before $h_T(z) \wedge t_\infty$ is small. We choose:

- $\psi \in (\frac{1}{2}, 1)$;
- $\delta_T = 1/(3 \log^\psi T)$;

- $\gamma_T = 1 - 1/(3t_\infty \log^\psi T)$;
- $\theta_T = \log^{-2(\psi+1)(q+1)} T$, so that by Corollary 3.5, $\mathbb{P}(\bar{h}_T(\theta_T) > \delta_T/2) \to 0$;
- $\eta_T = (1 - \gamma_T)\theta_T/3 = \log^{-\psi-2(\psi+1)(q+1)} T/(9t_\infty)$;
- $\rho_T = \log\log T$, so that by Lemma 2.7(ii), $\mathbb{P}(\bar{\xi}_T(\rho_T) \geq \frac{q(\rho_T - 2)\gamma_T}{t_\infty + \delta_T}) \to 0$;
- $\nu_T = \log^{-2\psi d/\alpha - 2(\psi+1)q} T$ so that by Lemma 2.7(i),

$$\mathbb{P}(\bar{\bar{\xi}}_T(\eta_T) < \nu_T) \to 0;$$

- $K_T = \log^{3\psi d + 2(\psi+1)q\alpha} T$, so that by Lemma 2.7(ii),

$$\mathbb{P}(\#\{z \in B(0, \rho_T) : \xi_T(z) \geq \nu_T\} > K_T) \to 0;$$

- $\beta_T = \log^{-5\psi d - 4(\psi+1)(q+1)\alpha} T$.

We also define

$$\Gamma_T = \left\{ \bar{h}_T(\theta_T) \leq \delta_T/2, \bar{\bar{\xi}}_T(\eta_T) \geq \nu_T, \right.$$

$$\left. \#\{z \in B(0, \rho_T) : \xi_T(z) \geq \nu_T\} \leq K_T, \bar{\xi}_T(\rho_T) < \frac{q(\rho_T - 2)\gamma_T}{t_\infty + \delta_T} \right\}.$$

We think of Γ_T as a good event on which the environment behaves sensibly. Note from above that $\mathbb{P}(\Gamma_T) \to 1$.

We now let

$$Z = \{z \in L_T(0, \rho_T) : \xi_T(z) > \bar{\bar{\xi}}_T(\eta_T)\}, \qquad \kappa(T) = \#Z,$$

and

$$Z' = \{z \notin L_T(0, \rho_T) : \xi_T(z) > \bar{\bar{\xi}}_T(\eta_T)\}.$$

We label the elements of Z as $z_1, \ldots, z_{\kappa(T)}$ such that $\xi_T(z_1) \leq \cdots \leq \xi_T(z_{\kappa(T)})$, and the elements of Z' arbitrarily as $z_{\kappa(T)+1}, z_{\kappa(T)+2}, \ldots$. Let $t_i = (\gamma_T h_T(z_i) - \delta_T)_+$ for each i. Note that z_i and t_i only depend on the environment ξ so that we are allowed to apply the results in Sections 4.1 and 4.2. (Of course z_i and t_i also depend on T, but keeping track of this would make our notation unwieldy.)

We can now translate our general results about G_T from the previous section to get bounds for our particular choice of z_i and t_i.

LEMMA 4.6. *On Γ_T, $\mathbb{P}$-almost surely, for any $z \notin B(0, \eta_T)$ and any $t > 0$,*

$$G_T\big(0, z, (\gamma_T h_T(z) - \delta_T)_+, t\big) \leq \exp\big(a(T)T\big(\tfrac{2}{3}q(\gamma_T - 1)|z| + \mathcal{E}_T^2(t, |z|)\big)\big).$$

PROOF.   Note that if $\gamma_T h_T(z) - \delta_T < 0$, then

$$G_T\big(0, z, (\gamma_T h_T(z) - \delta_T)_+, t\big) \leq P^\xi\big(A_T^*(0, z, t)\big)$$

$$\leq \exp\big(a(T)T\big(-q|z| + \mathcal{E}_T^2(t, |z|)\big)\big),$$

where the first inequality comes directly from the definition of G_T and the second is from Lemma 2.4. In particular, on Γ_T this bound applies if $z \in B(0, \theta_T) \setminus B(0, \eta_T)$. Thus it remains to consider $z \notin B(0, \theta_T)$ such that $\gamma_T h_T(z) - \delta_T \geq 0$. Since on Γ_T we have $\bar{h}_T(\eta_T) \leq \bar{h}_T(\theta_T) < \delta_T$, by Lemma 4.5 (with $R = \eta_T$),

$$G_T\big(0, z, (\gamma_T h_T(z) - \delta_T)_+, t\big)$$
$$\leq \exp\big(a(T)T\big(q(\gamma_T - 1)|z| + q\gamma_T \eta_T + \mathcal{E}_T^2(t, |z|)\big)\big).$$

However, we chose $\eta_T = (1 - \gamma_T)\theta_T/3 \leq (1 - \gamma_T)|z|/3$, and the result follows. $\square$

Now we want to apply Lemma 4.4 to bound $G_T(j, \cdot, \cdot, \cdot)$ for $j \geq 1$. Note that we cannot induct directly on G_T since Lemma 4.4 relates $G_T(j, z, \cdot, \cdot)$ to $G_T(j - 1, z_j, \cdot, \cdot)$ rather than $G_T(j - 1, z, \cdot, \cdot)$. However, we can work with

$$\bar{G}_T = \max_{k \leq \kappa(T)} G_T\big(\kappa(T), z_k, t_k, t_\infty\big).$$

Since $\psi < 1$, we can choose T_1 such that

(4) $$\mathcal{E}_T^2(t_\infty, R) \leq q(1 - \gamma_T)R/3 \qquad \forall R \geq \beta_T, T \geq T_1$$

and

(5) $$1 + \exp\big(a(T)T \cdot q\beta_T\big) \leq \exp\big(a(T)T \cdot 2q\beta_T\big) \qquad \forall T \geq T_1.$$

LEMMA 4.7. *On Γ_T, for all $T \geq T_1$, $\mathbb{P}$-almost surely,*
$$\bar{G}_T \leq e^{a(T)T(q(\gamma_T - 1)\eta_T/3 + 2K_T q\beta_T)}.$$

PROOF. By Lemma 4.4, $\mathbb{P}$-almost surely, for $j, k \leq \kappa(T)$,

$$G_T(j, z_k, t_k, t_\infty)$$
$$\leq G_T(j - 1, z_k, t_k, t_\infty)$$
$$\quad + G_T(j - 1, z_j, t_j, t_\infty)e^{a(T)T\xi_T(z_j)(t_k - t_j)_+} J_T\big(t_\infty, |z_k - z_j|\big)$$
$$\leq G_T(j - 1, z_k, t_k, t_\infty)$$
$$\quad + G_T(j - 1, z_j, t_j, t_\infty)e^{a(T)T\gamma_T q|z_k - z_j|} J_T\big(t_\infty, |z_k - z_j|\big),$$

where we use that if $t_k - t_j > 0$, then by Lemma 3.2 and since $t_j \geq \gamma_T h_T(z_j) - \delta$,

$$(t_k - t_j)_+ \leq \gamma_T\big(h_T(z_k) - h_T(z_j)\big) \leq \gamma_T q \frac{|z_k - z_j|}{\xi_T(z_j)}.$$

Now, if $|z_k - z_j| < \beta_T$, then [since trivially $J_T(t_\infty, |z_k - z_j|) \leq 1$]
$$e^{a(T)T\gamma_T q|z_k - z_j|} J_T\big(t_\infty, |z_k - z_j|\big) \leq e^{a(T)Tq\beta_T};$$

on the other hand, if $|z_k - z_j| \geq \beta_T$, then by Lemma 2.4

$$e^{a(T)T\gamma_T q |z_k - z_j|} J_T\big(t_\infty, |z_k - z_j|\big) \leq e^{a(T)T(\gamma_T q |z_k - z_j| - q |z_k - z_j| + \mathcal{E}^2(t_\infty, |z_k - z_j|))}$$

so that if $T \geq T_1$ by (4),

$$e^{a(T)T\gamma_T q |z_k - z_j|} J_T\big(t_\infty, |z_k - z_j|\big) \leq 1 \leq e^{a(T)Tq\beta_T}.$$

Either way, we can conclude that for any $j \leq \kappa(T)$ and $T \geq T_1$ by (5),

$$\max_{k \leq \kappa(T)} G_T(j, z_k, t_k, t_\infty) \leq \max_{k \leq \kappa(T)} G_T(j - 1, z_k, t_k, t_\infty)\big(1 + e^{a(T)Tq\beta_T}\big)$$

$$\leq \max_{k \leq \kappa(T)} G_T(j - 1, z_k, t_k, t_\infty) e^{a(T)T2q\beta_T}.$$

Iterating this inequality $\kappa(T)$ times beginning with $\max_{k \leq \kappa(T)} G_T(\kappa(T), z_k, t_k, t_\infty)$ gives

(6) $$\bar{G}_T \leq \max_{k \leq \kappa(T)} G_T(0, z_k, t_k, t_\infty) e^{a(T)T2\kappa(T)q\beta_T},$$

but on Γ_T, $\kappa(T) \leq K_T$. Then applying Lemma 4.6 gives the result. $\square$

Very similar arguments allow us to get an estimate on G_T for *any* point outside $B(0, \eta_T)$.

LEMMA 4.8. *On Γ_T, $\mathbb{P}$-almost surely, for any $z \notin B(0, \eta_T)$ and any $T \geq T_1$,*

$$G_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big)$$

$$\leq e^{a(T)Tq(\gamma_T - 1)\eta_T/3} + K_T e^{a(T)T(q(\gamma_T - 1)\eta_T/3 + (2K_T + 1)q\beta_T)}.$$

PROOF. Essentially we just apply Lemma 4.4 again to relate $G_T(j, z, \cdot, \cdot)$ to $G_T(j - 1, z_j, \cdot, \cdot)$, which we can now control using Lemma 4.7. Indeed, by Lemma 4.4, for any $j \leq \kappa(T)$ and $s \geq 0$,

$$G_T(j, z, s, t_\infty)$$

$$\leq G_T(j - 1, z, s, t_\infty) + \bar{G}_T \max_{k \leq \kappa(T)} e^{a(T)T\xi_T(z_k)(s - t_k)_+} J_T\big(t_\infty, |z - z_k|\big).$$

When $s = (\gamma_T h_T(z) - \delta_T)_+$, we get $(s - t_k)_+ \leq \gamma_T q \frac{|z - z_k|}{\xi_T(z_k)}$, so

$$G(j, z, s, t_\infty) \leq G_T(j - 1, z, s, t_\infty) + \bar{G}_T \max_{k \leq \kappa(T)} e^{a(T)T\gamma_T q |z - z_k|} J_T\big(t, |z - z_k|\big).$$

As in the proof of Lemma 4.7, considering two cases (when $|z - z_k| < \beta_T$ and when $|z - z_k| \geq \beta_T$), we get

$$e^{a(T)T\gamma_T q |z - z_k|} J_T\big(t_\infty, |z - z_k|\big) \leq e^{a(T)Tq\beta_T}.$$

Thus

$$G\big(j, z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big)$$

$$\leq G_T\big(j - 1, z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big) + \bar{G}_T e^{a(T)Tq\beta_T}.$$

Iterating $\kappa(T)$ times gives

$$G\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big)$$

$$\leq G_T\big(0, z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big) + \kappa(T)\bar{G}_T e^{a(T)Tq\beta_T};$$

then applying Lemmas 4.6 and 4.7 [together with the fact that on Γ_T, $\kappa(T) \leq K_T$] completes the proof. □

4.4. *Particles do not arrive too early.* We are finally in a position to prove our first real result, that $H_T(z)$ does not occur significantly before $h_T(z)$ for any z.

PROPOSITION 4.9. *For any $t_\infty > 0$, there exists T_2 such that for all $T \geq T_2$,*

$$\mathbb{P}\big(\exists z : H_T(z) \leq (\gamma_T h_T(z) - \delta_T) \wedge t_\infty\big) \leq \mathbb{P}\big(\Gamma_T^c\big) + e^{-T} \to 0.$$

PROOF. All that remains is to tie together the threads developed above. Note that by Lemma 3.6,

(7)
$$\big\{\exists z \in L_T \setminus B(0, \rho_T - 2) : \gamma_T h_T(z) - \delta_T \leq t_\infty\big\}$$

$$\subseteq \left\{\bar{\bar{\xi}}_T(\rho_T - 2) \geq \frac{q(\rho_T - 2)\gamma_T}{t_\infty + \delta_T}\right\} \subseteq \Gamma_T^c.$$

Since our random walks only make nearest neighbor steps, particles must enter $L_T(0, \rho_T) \setminus L_T(0, \rho_T - 2)$ before they can exit $B(0, \rho_T)$. Thus if there exists z outside $B(0, \rho_T)$ such that $H_T(z) \leq t_\infty$, then there must exist z within $L_T(0, \rho_T) \setminus L_T(0, \rho_T - 2)$ such that $H_T(z) \leq t_\infty$. Thus on Γ_T,

$$\big\{\exists z : H_T(z) \leq (\gamma_T h_T(z) - \delta_T) \wedge t_\infty\big\}$$

$$\subseteq \big\{\exists z \in L_T(0, \rho_T) : H_T(z) \leq (\gamma_T h_T(z) - \delta_T) \wedge t_\infty\big\}.$$

If a point is hit early, then there must be a *first* point that is hit early; thus, recalling that

$$A_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty\big)$$

$$= \big\{H_T(z) \leq (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty, H_T(z_i) \geq t_i \wedge H_T(z) \ \forall i \leq \kappa(T),$$

$$H_T(z_i) \geq H_T(z) \ \forall i > \kappa(T)\big\},$$

we have reduced the problem to showing that on Γ_T,

$$P^\xi\bigg(\bigcup_{z \in L_T(0, \rho_T)} A_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty\big)\bigg) \leq e^{-T}.$$

However, by Corollary 4.3, for any $z \in L_T$,

$$P^\xi\big(A_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty\big)\big) \leq G_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big),$$

and by Lemma 4.8, for any $z \notin B(0, \eta_T)$, on Γ_T, $\mathbb{P}$-almost surely

$$G_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+, t_\infty\big)$$
$$\leq e^{a(T)Tq(\gamma_T - 1)\eta_T/3} + K_T e^{a(T)T(q(\gamma_T-1)\eta_T/3 + (2K_T+1)q\beta_T)}.$$

Now,

$$\frac{q(\gamma_T - 1)\eta_T}{3} = -\frac{q}{81t_\infty^2 \log^{2\psi + 2(\psi+1)(q+1)} T}$$

and

$$(2K_T + 1)q\beta_T \leq \frac{3q}{\log^{2\psi d + 2(\psi+1)(q+1)\alpha} T},$$

so by taking T large (not depending on the environment ξ), we can certainly ensure that

$$e^{a(T)T(q(\gamma_T-1)\eta_T/3 + (2K_T+1)q\beta_T)} \leq e^{-2T}.$$

Thus for large T, for any $z \notin B(0, \eta_T)$, on Γ_T, $\mathbb{P}$-almost surely

$$G_T\big(\kappa(T), z, \gamma_T h_T(z) - \delta_T, t_\infty\big) \leq (K_T + 1)e^{-2T}.$$

Also, on Γ_T, if $z \in B(0, \eta_T)$, then $h_T(z) < \delta_T$ and hence $P^\xi(A_T(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty)) = 0$. We deduce that for large T,

$$P^\xi\Bigg(\bigcup_{z \in L_T(0, \rho_T)} A_T\big(\kappa(T), z, (\gamma_T h_T(z) - \delta_T)_+ \wedge t_\infty\big)\Bigg)$$
$$\leq C_d \rho_T^d r(T)^d (K_T + 1)e^{-2T}e^{-T},$$

which converges to 0 as $T \to \infty$. $\quad\square$

4.5. *There are not too many particles.* Now that we have bounded the probability that any of our points is hit early, we check that the number of particles at any site cannot be too large (given that no point is hit early). We work with the same parameters as above, and the same choice of $z_i, t_i, i \geq 1$. Indeed our whole tactic will be very similar, except that instead of looking at the expected number of particles at z at time $s \wedge H_T(z)$, we will instead just look at time s [conditional on not having hit z substantially before $h_T(z)$].

Define the events

$$\mathcal{H}_T = \big\{H_T(y) > (\gamma_T h_T(y) - \delta_T) \wedge t_\infty \ \forall y\big\}$$

and for $s > 0$

$$\mathcal{H}_T^*(s) = \{H_T^*(y) > (\gamma_T h_T(y) - \delta_T) \wedge s \; \forall y\}.$$

We know from Proposition 4.9 that $\mathbb{P}(\mathcal{H}_T) \to 1$ as $T \to \infty$.

We begin with a lemma that allows us to control the number of particles at z by linking to something we already know a lot about: G_T.

LEMMA 4.10. *On Γ_T, $\mathbb{P}$-almost surely, for any $s \leq t_\infty$ and any z,*

$$E^\xi \left[\sup_{u \in (s-\delta_T, s]} \#\{v \in Y(uT) : X_v(uT) = r(T)z\} \mathbb{1}_{\mathcal{H}_T} \right]$$

$$\leq e^{a(T)T\bar{\xi}_T(\eta_T)s} J_T(s, |z|)$$

$$+ \sum_{j=1}^{\kappa(T)} G_T(j, z_j, t_j, t_\infty) e^{a(T)T\xi_T(z_j)(s-t_j)+a(T)T\xi_T(z_{\kappa(T)})\delta_T} J_T(s, |z - z_j|)$$

$$\times \mathbb{1}_{\{t_j \leq s\}}.$$

PROOF. The plan is as follows: we apply the many-to-one lemma to turn our expectation over the branching random walk into an expectation involving only one random walk. Then either we do not hit any z_j before time s, in which case our potential is small, or there is a last z_j that we hit. For each j we then use similar calculations to those in the proof of Lemma 4.4.

For $s > 0$, let

$$\tilde{A}_T^*(0, z, s) = \{\exists u \in (s - \delta_T, s] : X(uT) = r(T)z, H_T^*(z_i) > s - \delta_T \; \forall i\} \cap \mathcal{H}_T^*(s),$$

and for $j \geq 1$, let

$$\tilde{A}_T^*(j, z, s) = \left\{ \begin{array}{c} \exists u \in (s - \delta_T, s] : X(uT) = r(T)z, \\ H_T^*(z_j) < s - \delta_T, H_T^*(z_i) > s - \delta_T \; \forall i > j \end{array} \right\} \cap \mathcal{H}_T^*(s).$$

Informally, $\tilde{A}_T^*(j, z, s)$ says that we are at z around time s, we traveled via z_j (unless $j = 0$) and not via z_i for $i > j$, and no-one was hit early.

First note that

$$E^\xi \left[\sup_{u \in (s-\delta_T, s]} \#\{v \in Y(uT) : X_v(uT) = r(T)z\} \mathbb{1}_{\mathcal{H}_T} \right]$$

$$\leq E^\xi \left[\sum_{v \in Y(sT)} \mathbb{1}_{\{\exists u \in (s-\delta_T, s] : X_v(uT) = r(T)z, H_T^v(y) > (\gamma_T h_T(y) - \delta_T) \wedge s \; \forall y\}} \right].$$

By the many-to-one lemma, this equals

$$E^\xi \left[\exp\left(T \int_0^s \xi(X(uT)) \, du \right) \mathbb{1}_{\{\exists u \in (s-\delta_T, s] : X(uT) = r(T)z\} \cap \mathcal{H}_T^*(s)} \right].$$

However, either we do not hit any z_j, or there is a last j such that we hit z_j before time $s - \delta_T$, so the above is at most

$$\sum_{j=0}^{\infty} E^{\xi}\left[\exp\left(T\int_0^s \xi(X(uT))\,du\right)\mathbb{1}_{\tilde{A}_T^*(j,z,s)}\right].$$

If $t_j = \gamma_T h_T(z_j) - \delta_T > s$, then $\tilde{A}_T^*(j,z,s) = \varnothing$. As in (7), on Γ_T we have $\gamma_T h_T(z_j) - \delta_T > t_{\infty}$ for all $j > \kappa(T)$, so $\tilde{A}_T^*(j,z,s) = \varnothing$ for all $j > \kappa(T)$. Thus we may restrict the sum above to $j \leq \kappa(T)$ such that $h_T(z_j) \leq (s + \delta_T)/\gamma_T$. We then know that on $\tilde{A}_T^*(j,z,s)$, between times $t_j T$ and $(s - \delta_T)T$ our potential is at most $a(T)\xi_T(z_j)$, and between times $(s - \delta_T)T$ and sT our potential is at most $a(T)\xi_T(z_{\kappa(T)})$. This tells us that

$$E^{\xi}\left[\exp\left(T\int_0^s \xi(X(uT))\,du\right)\mathbb{1}_{\tilde{A}_T^*(j,z,s)}\right]$$

$$\leq E^{\xi}\left[\exp\left(T\int_0^{t_j}\xi(X(uT))\,du + a(T)T\xi_T(z_j)(s - t_j)\right.\right.$$

$$\left.\left. + a(T)T\xi_T(z_{\kappa(T)})\delta_T\right)\mathbb{1}_{\tilde{A}_T^*(j,z,s)}\right].$$

Recall that we defined

$$A_T^*(j,z,t) = \left\{\begin{array}{c} H_T^*(z) \leq t,\ H_T^*(z_i) \geq H_T^*(z) \wedge t_i\ \forall i \leq j, \\ H_T^*(z_i) \geq H_T^*(z)\ \forall i > j \end{array}\right\}.$$

Note that for $j \geq 1$,

$$\tilde{A}_T^*(j,z,s) \subseteq A_T^*(j,z_j,s)$$

$$\cap \{H_T^*(z_j) \in (t_j, s - \delta_T], \exists u \in (s - \delta_T, s]: X(uT) = r(T)z\}.$$

Further, recalling that $\mathcal{G}_u$, $u \geq 0$ is the natural filtration of our random walk X, we have

$$P^{\xi}\left(H_T^*(z_j) \leq s - \delta_T, \exists u \in l(s - \delta_T, s]: X(uT) = r(T)z \,|\, \mathcal{G}_{(H_T^*(z_j)\wedge(s-\delta))T}\right)$$

$$\leq J_T(s, |z - z_j|).$$

Thus since $e^{T\int_0^{t_j}\xi(X(uT))\,du}\mathbb{1}_{A_T^*(j,z_j,s)}\mathbb{1}_{\{H_T^*(z_j)>t_j\}}$ is $\mathcal{G}_{H_T^*(z_j)}$-measurable,

$$E^{\xi}\left[\exp\left(T\int_0^{t_j}\xi(X(uT))\,du + a(T)T\xi_T(z_j)(s - t_j)\right)\right.$$

$$\left. \times \exp\left(a(T)T\xi_T(z_{\kappa}(T))\delta_T\right)\mathbb{1}_{\tilde{A}_T^*(j,z,s)}\right]$$

$$\leq E^{\xi}\left[\exp\left(T\int_0^{t_j}\xi(X(uT))\,du\right)\mathbb{1}_{A_T^*(j,z_j,s)}\mathbb{1}_{\{H_T^*(z_j)>t_j\}}\right]$$

$$\times \exp\left(a(T)T\xi_T(z_j)(s - t_j) + a(T)T\xi_T(z_{\kappa(T)})\delta_T\right)J_T(s, |z - z_j|).$$

However,

$$E^\xi\left[\exp\left(T\int_0^{t_j\wedge(s-\delta_T)}\xi(X(uT))\,du\right)\mathbb{1}_{A_T^*(j,z_j,s)}\mathbb{1}_{\{H_T^*(z_j)>t_j\}}\right]$$
$$\leq G_T(j,z_j,t_j,t_\infty),$$

so putting all of this together,

$$E^\xi\big[\#\{v\in Y(sT):X_v(sT)=r(T)z\}\mathbb{1}_{\mathcal{H}_T}\big]$$

$$\leq E^\xi\left[\exp\left(T\int_0^s\xi(X(uT))\,du\right)\mathbb{1}_{\tilde{A}_T^*(0,z,s)}\right]$$

$$+\sum_{j=1}^{\kappa(T)}G_T(j,z_j,t_j,t_\infty)e^{a(T)T\xi_T(z_j)(s-t_j)+a(T)T\xi_T(z_{\kappa(T)})\delta_T}J_T\big(s,|z-z_j|\big)$$

$$\times\mathbb{1}_{\{t_j\leq s\}}.$$

Finally, since $\xi_T(y)\leq\bar{\xi}_T(\eta_T)$ for all $y\notin\{z_1,z_2,\ldots\}$, we have

$$E^\xi\left[\exp\left(T\int_0^s\xi(X(uT))\,du\right)\mathbb{1}_{\tilde{A}_T^*(0,z,s)}\right]\leq e^{a(T)T\bar{\xi}_T(\eta_T)s}J_T\big(s,|z|\big),$$

and the result follows. □

We now use our knowledge of G_T to get a bound in terms of $m_T(z,s)$.

LEMMA 4.11. *There exists T_3 such that for any $T\geq T_3$, on Γ_T, $\mathbb{P}$-almost surely, for any $s\leq t_\infty$ and any z,*

$$E^\xi\left[\sup_{u\in(s-\delta_T,s]}\#\{v\in Y(uT):X_v(uT)=r(T)z\}\mathbb{1}_{\mathcal{H}_T}\right]$$

$$\leq\exp\left(a(T)T\left(m_T(z,s)+\frac{1}{2}\log^{-1/2}T\right)\right).$$

PROOF. Note that, as in (7), we may assume that $z\in B(0,\rho_T)$; otherwise the expectation is 0. Clearly our starting point is Lemma 4.10. First we show that

$$e^{a(T)T\bar{\xi}_T(\eta_T)s}J_T\big(s,|z|\big)\leq\exp\big(a(T)T\big(m_T(z,s)+\tfrac{1}{4}\log^{-1/2}T\big)\big).$$

To do this, choose $y\in B(0,\eta_T)$ such that $\xi_T(y)=\bar{\xi}_T(\eta_T)$. Then applying Lemma 2.4,

$$e^{a(T)T\bar{\xi}_T(\eta_T)s}J_T\big(s,|z|\big)$$

$$\leq\exp\big(a(T)T\big(\xi_T(y)s-q|z|+\mathcal{E}_T^2(s,|z|)\big)\big)$$

$$\leq\exp\big(a(T)T\big(\xi_T(y)\big(s-h_T(y)\big)-q|z-y|$$

$$+\xi_T(y)h_T(y)+q|y|+\mathcal{E}_T^2(s,|z|)\big)\big)$$

$$\leq\exp\big(a(T)T\big(m_T(z,s)+\xi_T(y)h_T(y)+q|y|+\mathcal{E}_T^2(s,|z|)\big)\big).$$

However, on Γ_T, we have $h_T(y) \leq \delta_T/2 = 1/(6\log^\psi T)$, and $\psi > \frac{1}{2}$, so on Γ_T,

$$\xi_T(y)h_T(y) \leq \max_{y' \in L_T(0,\rho_T)} \xi_T(y')\delta_T/2 \leq \frac{q}{2t_\infty}\rho_T\delta_T \leq \frac{1}{8}\log^{-1/2} T,$$

for T large. Also $|y| \leq \eta_T \leq \log^{-\psi} T$, and since $|z| < \rho_T$, for large T we have $\mathcal{E}_T^2(s,|z|) \leq (q+2)(\log\log T)^2/\log T$. Thus

$$e^{a(T)T\bar{\xi}_T(\eta_T)s} J_T(s,|z|) \leq \exp(a(T)T(m_T(z,s) + \tfrac{1}{4}\log^{-1/2} T)),$$

as claimed.

We now move on to bounding

$$\sum_{j=1}^{\kappa(T)} G_T(j,z_j,t_j,t_\infty)e^{a(T)T\xi_T(z_j)(s-t_j)+a(T)T\xi_T(z_{\kappa(T)})\delta_T} J_T(s,|z-z_j|)\mathbb{1}_{\{t_j \leq s\}}.$$

By Lemma 2.4, $J_T(s,|z-z_j|) \leq \exp(-a(T)T(q|z-z_j| - \mathcal{E}_T^2(s,|z-z_j|)))$, so the above is at most

$$\sum_{j=1}^{\kappa(T)} G_T(j,z_j,t_j,t_\infty)\mathbb{1}_{\{h_T(z_j) \leq (s+\delta_T)/\gamma_T\}}$$

$$\times e^{a(T)T(\xi_T(z_j)(s-h_T(z_j)-q|z-z_j|)+\xi_T(z_j)(1-\gamma_T)h_T(z_j)+2\xi_T(z_{\kappa(T)})\delta_T+\mathcal{E}_T^2(s,|z-z_j|))}.$$

Since we are assuming $z \in B(0,\rho_T)$, and $j \leq \kappa(T)$ so $z_j \in B(0,\rho_T)$, we have $|z - z_j| \leq 2\rho_T$ so as above for large T we have $\mathcal{E}_T^2(s,|z-z_j|) \leq (q+2)(\log\log T)^2/\log T$. Also, if $h_T(z_j) \leq s \leq t_\infty$, then on Γ_T we have $\xi_T(z_j)(1-\gamma_T)h_T(z_j) + 2\xi_T(z_{\kappa(T)})\delta_T \leq (\log\log T)/\log^\psi T$. Thus, as $\psi > \frac{1}{2}$, the above is at most

$$\sum_{j=1}^{\kappa(T)} G_T(j,z_j,t_j,t_\infty)e^{a(T)T(\xi_T(z_j)(s-h_T(z_j)-q|z-z_j|)+(1/8)\log^{-1/2} T)}.$$

However, $\xi_T(z_j)(s - h_T(z_j)) - q|z-z_j| \leq m_T(z,s)$, and on Γ_T, $\kappa(T) \leq K_T$, and Lemma 4.7 tells us that for each $j \leq \kappa(T)$,

$$G_T(j,z_j,t_j,t_\infty) \leq \exp(a(T)T(q(\gamma_T-1)\eta_T/3 + 2K_Tq\beta_T)).$$

As in the proof of Proposition 4.9, it is easy to check that this is at most $e^{-2T} \leq 1$. Putting all of this together, we get

$$\sum_{j=1}^{\kappa(T)} G_T(j,z_j,t_j,t_\infty)e^{a(T)T\xi_T(z_j)(s-t_j+\delta_T)} J_T(s,|z-z_j|)\mathbb{1}_{\{t_j \leq s\}}$$

$$\leq K_T \exp\left(a(T)T\left(m_T(z,s) + \frac{1}{8}\log^{-1/2} T\right)\right)$$

$$\leq \exp\left(a(T)T\left(m_T(z,s) + \frac{1}{4}\log^{-1/2} T\right)\right).$$

Finally, by Lemma 4.10 and the above two calculations,

$$E^\xi \Big[\sup_{u \in (s-\delta_T, s]} \#\{v \in Y(uT) : X_v(uT) = r(T)z\} \mathbb{1}_{\mathcal{H}_T} \Big]$$

$$\leq \exp\Big(a(T)T\Big(m_T(z, s) + \frac{1}{4} \log^{-1/2} T \Big) \Big)$$

$$+ \exp\Big(a(T)T\Big(m_T(z, s) + \frac{1}{4} \log^{-1/2} T \Big) \Big)$$

$$\leq \exp\Big(a(T)T\Big(m_T(z, s) + \frac{1}{2} \log^{-1/2} T \Big) \Big). \qquad \square$$

We have now done the hard work, so we can show that the number of particles at each point z behaves more or less as it should.

PROPOSITION 4.12. *There exists T_4 such that for all $T \geq T_4$,*

$$\mathbb{P}(\exists u \in (0, t_\infty], \exists z \in L_T : M_T(z, u) > m_T(z, u) + \log^{-1/2} T)$$

$$\leq 2\mathbb{P}(\Gamma_T^c) + 2e^{-T} \to 0.$$

PROOF. Since, for any z, $m_T(z, u)$ is increasing in u, Markov's inequality and Lemma 4.11 tell us that if $s \leq t_\infty$ and $T \geq T_3$, then on Γ_T

$$P^\xi\big(\exists u \in (s-\delta_T, s] : N\big(r(T)z, uT\big) > e^{a(T)T(m_T(z,u)+\log^{-1/2} T)}, \mathcal{H}_T\big)$$

$$\leq E^\xi\Big[\sup_{u \in (s-\delta_T, s]} N\big(r(T)z, uT\big) \mathbb{1}_{\mathcal{H}_T} \Big] e^{-a(T)T(m_T(z, s-\delta_T)+\log^{-1/2} T)}$$

$$\leq e^{a(T)T(m_T(z,s)+(1/2)\log^{-1/2} T - m_T(z, s-\delta_T) - \log^{-1/2} T)}.$$

By the definition of m_T, on Γ_T we have since $\psi > \frac{1}{2}$

$$m_T(z, s) \leq m_T(z, s-\delta_T) + \sup_{y : h_T(y) \leq s} \xi_T(y)\delta_T$$

$$\leq m_T(z, s-\delta_T) + (\log \log T)/(3 \log^\psi T)$$

$$\leq m_T(z, s-\delta_T) + \frac{1}{4} \log^{-1/2} T,$$

so

$$P^\xi\big(\exists u \in (s-\delta_T, s] : N\big(r(T)z, uT\big) > e^{a(T)T(m_T(z,u)+\log^{-1/2} T)}, \mathcal{H}_T\big)$$

$$\leq e^{-(1/4)a(T)T \log^{-1/2} T}.$$

Written in terms of M_T, this is (on Γ_T)

$$P^\xi\Big(\exists u \in (s-\delta_T, s] : M_T(z, u) > m_T(z, u) + \frac{1}{\log^{1/2} T}, \mathcal{H}_T \Big)$$

$$\leq e^{-(1/4)a(T)T \log^{-1/2} T}.$$

Taking a union over $s = \delta_T, 2\delta_T, \ldots, \lceil t_\infty / \delta_T \rceil \delta_T$ and $z \in L_T(\rho_T)$, on Γ_T we have

$$P^\xi\big(\exists u \in (0, t], \exists z \in L_T(\rho_T) : M_T(z, u) > m_T(z, u) + \log^{-1/2} T, \mathcal{H}_T\big)$$

$$\leq \frac{t_\infty + 1}{\delta_T} C_d r(T)^d \rho_T^d e^{-(1/4)a(T)T \log^{-1/2} T} \leq e^{-T}.$$

On Γ_T, we have $\gamma_T h_T(z) - \delta_T \geq t_\infty$ for all $z \in L_T \setminus L_T(\rho_T)$ [see (7)], and therefore on $\mathcal{H}_T$ there cannot exist $z \in L_T \setminus L_T(\rho_T)$ such that $M_T(z, u) > 0$ for any $u \leq t_\infty$. This allows us to change $z \in L_T(\rho_T)$ to $z \in L_T$ in the estimate above. Thus, applying Proposition 4.9, for large T,

$$\mathbb{P}\big(\exists u \in (0, t_\infty], \exists z \in L_T : M_T(z, u) > m_T(z, u) + \log^{-1/2} T\big)$$

$$\leq \mathbb{P}\big(\Gamma_T^c\big) + \mathbb{P}\big(\mathcal{H}_T^c\big) + e^{-T} \leq 2\mathbb{P}\big(\Gamma_T^c\big) + 2e^{-T} \to 0. \qquad \square$$

5. Lower bounds. We now turn our attention to lower bounds on hitting times and the number of particles. The key is to check that if y has reasonably large potential, and we start with lots of particles at y, then we can travel to z in time roughly $q|z - y|/\xi_T(y)$. We do this in Lemma 5.2. Since we are not likely to start from a site with large potential, we must also check that things behave well near the origin, which is carried out in Lemmas 5.3 and 5.4. These results are then applied in Section 5.2 to check that $H_T(z)$ is not too much larger than $h_T(z)$, and in Section 5.3 to ensure that there are never too few particles at a site.

Let $\mu_T = \log^{1/4} T$. Then for $z \in L_T$, define

$$H'_T(z) = \inf\{t > 0 : N(r(T)z, tT) > \exp(\mu_T)\}.$$

To avoid the randomness that occurs when we only have a few particles, we work with $H'_T(z)$ instead of $H_T(z)$.

5.1. *Preliminary estimates for the lower bounds.* We start by checking that if we start with lots of particles at a site y, the number of particles grows as expected.

LEMMA 5.1. *Take $y \in L_T$ such that $\xi_T(y) \geq \frac{\mu_T^3}{a(T)}$ and choose $p \in [1/2, 1]$. Then for any $s \geq 0$, for large T (depending only on d), for any $s \geq 0$,*

$$P^\xi\left(N\left(r(T)y, H'_T(y)T + \left(1 + \frac{p}{\mu_T^2}\right)sT\right) \leq \frac{1}{64}e^{a(T)T\xi_T(y)s(1+p/(2\mu_T^2))+\mu_T}\right)$$

$$< \frac{1}{2}e^{-\log^2 T}.$$

PROOF. By definition there are e^{μ_T} particles at $r(T)y$ at time $H'_T(y)T$. By Lemma 2.2, we expect at least $\frac{1}{16}\exp(\mu_T)$ of these to have at least

$$\frac{1}{2}\exp\left((\xi_T(y)a(T) - 2d)\left(1 + \frac{p}{\mu_T^2}\right)sT\right)$$

descendants at $r(T)y$ at time $H'_T(y)T + (1 + \frac{p}{\mu_T^2})sT$. Note that when T is large, since $\xi_T(y) \geq \frac{\mu_T^3}{a(T)}$, we have

$$(\xi_T(y)a(T) - 2d)\left(1 + \frac{p}{\mu_T^2}\right)$$

$$\geq \xi_T(y)a(T) + \xi_T(y)a(T)\frac{p}{2\mu_T^2} + p\mu_T - 2d\left(1 + \frac{p}{\mu_T^2}\right)$$

$$\geq \xi_T(y)a(T)\left(1 + \frac{p}{2\mu_T^2}\right).$$

The result now follows from Lemma 2.6 since $\exp(-\frac{1}{8} \cdot \frac{1}{16}\exp(\mu_T)) \leq \frac{1}{2}\exp(-\log^2 T)$ when T is large. $\quad\square$

Now that we know that the number of particles at y grows as expected, we can check that particles move from y to z in time roughly $q|z - y|/\xi_T(y)$.

LEMMA 5.2. *Take $y, z \in L_T$ such that $y \neq z$ and $\frac{\mu_T^3}{a(T)} \leq \xi_T(y) \leq \exp(\mu_T)$. Suppose that $s \geq q|z - y|/\xi_T(y)$. For large T (depending only on d and q),*

$$P^\xi\left(N\left(r(T)z, H'_T(y)T + \left(1 + \frac{1}{\mu_T^2}\right)sT\right) \leq e^{a(T)T(\xi_T(y)s - q|z-y|) + \mu_T}\right) < e^{-\log^2 T}.$$

Note in particular that if we apply this result at time $s = q|z - y|/\xi_T(y)$, then we already see $\exp(\mu_T)$ particles.

PROOF OF LEMMA 5.2. By Lemma 5.1, for large T,

$$P^\xi\left(N\left(r(T)y, H'_T(y)T + \left(1 + \frac{1}{2\mu_T^2}\right)sT\right) \leq \frac{1}{64}e^{a(T)T\xi_T(y)s(1+1/(4\mu_T^2)) + \mu_T}\right)$$

$$< \frac{1}{2}e^{-\log^2 T}.$$

By Lemma 2.3,

$$P\left(X\left(\frac{sT}{2\mu_T^2}\right) = r(T)(z - y)\right)$$

$$\geq \exp\left(-a(T)T\left(q|z - y| + \mathcal{E}_T^1\left(\frac{s}{2\mu_T^2}, |z - y|\right)\right)\right).$$

In words, we have a large number of particles at $r(T)y$ just before the time we are interested in, and each has a reasonable probability of being at $r(T)z$ at the time

we are interested in. Applying Lemma 2.6, we are done, provided that

$$\frac{1}{128}e^{a(T)T\xi_T(y)s(1+1/(4\mu_T^2))+\mu_T-a(T)T(q|z-y|+\mathcal{E}_T^1(s/(2\mu_T^2),|z-y|))}$$

$$\geq e^{a(T)T(\xi_T(y)s-q|z-y|)+\mu_T},$$

which reduces to showing that

$$\frac{1}{128}e^{a(T)T(\xi_T(y)s/(4\mu_T^2)-\mathcal{E}_T^1(s/(2\mu_T^2),|z-y|))} \geq 1.$$

However, since $|z-y| \leq \xi_T(y)s/q \leq s\exp(\mu_T)/q$ and $\xi_T(y) \geq \frac{\mu_T^3}{a(T)}$, for large T,

$$\mathcal{E}_T^1\left(\frac{s}{2\mu_T^2},|z-y|\right) = \frac{|z-y|}{\log T}\left(\log|z-y|-\log\left(\frac{s}{2\mu_T^2}\right)\right) + \frac{ds}{\mu_T^2 a(T)}$$

$$\leq \frac{\xi_T(y)s}{q\log T}(\mu_T - \log q + \log 2 + 2\log\mu_T) + \frac{\xi_T(y)s}{q\log T}\cdot\frac{dq}{\mu_T}$$

$$\leq \frac{2\xi_T(y)s}{q\mu_T^3}.$$

Thus for large T,

$$\frac{1}{128}e^{a(T)T(\xi_T(y)s/(4\mu_T^2)-\mathcal{E}_T^1(s/(2\mu_T^2),|z-y|))} \geq \frac{1}{128}e^{a(T)T\xi_T(y)s/(8\mu_T^2)},$$

but since $y \neq z$, we have $|y-z| \geq 1/r(T)$ so that

$$a(T)T\frac{\xi_T(y)s}{8\mu_T^2} \geq a(T)T\frac{q|z-y|}{8\mu_T^2} \geq \frac{q\mu_T^2}{8} \geq \log(128). \qquad \square$$

We now know that if there are lots of particles at y and y has reasonable potential, then we can travel from y to any other point z in a suitable time. We now make sure that there are some points—indeed, all points close enough to the origin—with lots of particles. We make no attempt to optimize our argument and use only simple estimates.

LEMMA 5.3. *For any $\phi > 0$, $\mathbb{P}$-almost surely, for all $T > e$,*

$$P^\xi\left(\exists z \in B\left(0, \log^\phi T\right): N\left(z, 5\log^{2\phi} T\right) < e^{\lambda\log^\phi T}\right) < ce^{-\lambda\log^\phi T},$$

where $\lambda = \frac{\log 2}{8(d+1)}$ and c is a constant depending only on d and ϕ.

PROOF. We write P^1 to mean the law of a BRW with branching rate 1 everywhere. Since $\mathbb{P}(\xi(z) \geq 1) = 1$, if we can prove the lemma under P^1, then by an easy coupling it must hold for $\mathbb{P}$-almost every environment ξ. Also by adjusting c it suffices to consider large T.

Note that by Lemma 2.1,

$$P^1\big(N\big(4\lambda\log^\phi T\big) < e^{2\lambda\log^\phi T}\big) \le ce^{-\lambda\log^\phi T}.$$

However, since $|X(4\lambda\log^\phi T)|$ is stochastically dominated by a Poisson random variable of parameter $8d\lambda\log^\phi T$, we have

$$P\big(\big|X\big(4\lambda\log^\phi T\big)\big| > \log^\phi T\big) \le E\big[e^{|X(4\lambda\log^\phi T)|\log 2}\big]e^{-(\log 2)\log^\phi T}$$
$$= e^{8d\lambda\log^\phi T-(\log 2)\log^\phi T} = e^{-8\lambda\log^\phi T},$$

and applying the many-to-one lemma,

$$P^1\big(\exists v \in Y\big(4\lambda\log^\phi T\big) : \big|X_v\big(4\lambda\log^\phi T\big)\big| > \log^\phi T\big)$$
$$\le e^{4\lambda\log^\phi T}P\big(\big|X\big(4\lambda\log^\phi T\big)\big| > \log^\phi T\big) \le e^{-4\lambda\log^\phi T}.$$

So (adjusting c as necessary) with probability at least $1 - ce^{-\lambda\log^\phi T}$ we have at least $e^{2\lambda\log^\phi T}$ particles spread over $B(0, \log^\phi T)$ at time $4\lambda\log^\phi T$.

Take one such particle v. We now wait a further time $5\log^{2\phi} T - 4\lambda\log^\phi T$, which is at least $(2\log^\phi T)^2$ when T is large. For any $z \in B(0, \log^\phi T)$, by Lemma 2.5 the probability that v has a descendant at z at this time is at least $c'\log^{-d\phi} T$ for some constant c'. Thus, by Lemma 2.6,

$$P^1\big(N\big(z, 5\log^{2\phi} T\big) < (c'/2)e^{2\lambda\log^\phi T}\log^{-d\phi} T\big)$$
$$< \exp\big(-(c'/8)e^{2\lambda\log^\phi T}\log^{-d\phi} T\big).$$

Taking a union over all $z \in B(0, \log^\phi T) \cap \mathbb{Z}^d$, we get the desired result. $\square$

We have now established that with high probability every site within $B(0, \log^\phi T)$ has lots of particles by time $5\log^{2\phi} T$. Using Lemma 2.7 we can ensure that at least one of these sites—call it z_0—has reasonably large potential. The small problem we face is that in the definition of h_T, our trail of points starts from 0 and not from z_0. The following lemma helps us to get around this fact, essentially by stating that travelling via z_0 does not cost much.

LEMMA 5.4. *Fix $T > e$. Suppose that $z \in L_T$ and $h_T(z) = \sum_{j=1}^n q|y_{j-1} - y_j|/\xi_T(y_j)$ where $y_0 = z$ and $y_n = 0$. Let*

$$n' = \begin{cases} 0, & \text{if } \xi_T(y_j) < \mu_T^3/a(T) \ \forall j \ge 1, \\ \max\{j \ge 1 : \xi_T(y_j) \ge \mu_T^3/a(T)\}, & \text{otherwise.} \end{cases}$$

For any $z_0 \in L_T$ such that $\xi_T(z_0) \ge 2\mu_T^3/a(T)$, we have

$$q\frac{|y_{n'}|}{\xi_T(z_0)} \le h_T(z_0) + q\frac{|z_0|}{\xi_T(z_0)}.$$

PROOF. First note that since $\max_{j>n'} \xi_T(y_j) \leq \frac{1}{2}\xi_T(z_0)$, by Lemma 3.3 and the triangle inequality,

$$h_T(y_{n'}) = \sum_{j=n'+1}^{n} q\frac{|y_{j-1} - y_j|}{\xi_T(y_j)} \geq 2q\frac{|y_{n'}|}{\xi_T(z_0)},$$

but also,

$$h_T(y_{n'}) \leq h_T(z_0) + q\frac{|y_{n'} - z_0|}{\xi_T(z_0)} \leq h_T(z_0) + q\frac{|y_{n'}|}{\xi_T(z_0)} + q\frac{|z_0|}{\xi_T(z_0)}.$$

Combining these two statements, we see that

$$q\frac{|y_{n'}|}{\xi_T(z_0)} \leq h_T(z_0) + q\frac{|z_0|}{\xi_T(z_0)}. \qquad \square$$

5.2. *Particles do not arrive too late.* We are now ready to prove our main result for this section, namely that the probability that anyone arrives late is small. As we hinted earlier, we will apply Lemma 5.2 at time $q|z - y|/\xi_T(y)$ to check that we move from y to z in the time allotted. (The extra work to consider more general s was not wasted, however: it will be used when we check that the *number* of particles grows as claimed.) The rest of the proof simply involves tying up some loose ends.

PROPOSITION 5.5. *As $T \to \infty$,*

$$\mathbb{P}\big(\exists z : H_T'(z) \wedge t_\infty > h_T(z) + (t_\infty + 1)\log^{-1/2} T\big) \to 0.$$

PROOF. Let $\phi = \frac{3\alpha}{2d}$ and $\lambda = \frac{\log 2}{8(d+1)}$. We consider the following four events:

- Γ_T: in particular [see (7)], $h_T(z) > t_\infty$ for all $z \notin B(0, \rho_T)$;
- $N(r(T)z, 5\log^{2\phi} T) \geq \exp(\lambda \log^\phi T)$ for all $z \in L_T(0, r(T)^{-1}\log^\phi T)$;
- there exists $z_0 \in L_T(0, r(T)^{-1}\log^\phi T)$ such that $\xi_T(z_0) > \frac{2\mu_T^3}{a(T)}$ and $h_T(z_0) \leq \frac{1}{2\mu_T^2}$;
- for all $y \neq z$ in $L_T(0, \rho_T)$ such that $\xi_T(y) \geq \frac{\mu_T^3}{a(T)}$, we have $H_T'(z) \leq H_T'(y) + (1 + \mu_T^{-2})q\frac{|z-y|}{\xi_T(y)}$.

We recall that $\mathbb{P}(\Gamma_T) \to 1$ as $T \to \infty$. By Lemma 5.3, the probability of the second event also tends to 1 as $T \to \infty$. By Lemma 2.7 and our choice of ϕ, together with Corollary 3.5, the probability of the third event also tends to 1 as $T \to \infty$. Finally, by applying Lemma 5.2 when $s = q|z - y|/\xi_T(y)$ [note that on Γ_T, there exists c such that $\xi_T(y) \leq c\log\log T$ for all $y \in L_T(0, \rho_T)$] together with the fact that there are at most $c_d^2 r(T)^{2d}\rho_T^{2d} \ll \exp(\log^2 T)$ pairs of points $y, z \in L_T(0, \rho_T)$, the probability of the fourth event tends to 1 as $T \to \infty$. Thus it suffices to prove that on the intersection of these four events, for every z, we have $H_T'(z) \wedge t_\infty \leq h_T(z) + (t_\infty + 1)/\mu_T^2$. In particular, we can assume that $h_t(z) \leq t_\infty$.

Choose n and distinct $y_0, \ldots, y_n$ such that $y_0 = z$, $y_n = 0$ and $h_T(z) = \sum_{j=1}^{n} q|y_{j-1} - y_j|/\xi_T(y_j)$. Let n' be as in Lemma 5.4, and note that by Lemma 3.3, $\xi_T(y_j) \geq \xi_T(y_{n'}) \geq \mu_T^3/a(T)$ for $1 \leq j \leq n'$. Since we are working on Γ_T, we may assume that $y_j \in B(0, \rho_T)$ for all j. Then from the fourth event above,

$$
\begin{aligned}
H_T'(z) &= \sum_{j=1}^{n'} (H_T'(y_{j-1}) - H_T'(y_j)) + H_T'(y_{n'}) - H_T'(z_0) + H_T'(z_0) \\
&\leq \sum_{j=1}^{n'} (1 + \mu_T^{-2}) q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + (1 + \mu_T^{-2}) q \frac{|y_{n'} - z_0|}{\xi_T(z_0)} + H_T'(z_0) \\
&\leq (1 + \mu_T^{-2}) h_T(z) + (1 + \mu_T^{-2}) q \frac{|y_{n'}| + |z_0|}{\xi_T(z_0)} + H_T'(z_0).
\end{aligned}
$$

By Lemma 5.4 we have $q|y_{n'}|/\xi_T(z_0) \leq h_T(z_0) + q|z_0|/\xi_T(z_0)$. We know from the second event above that $H_T'(z_0) \leq \frac{5 \log^{2\phi} T}{T}$, and from the third event that z_0 is chosen such that $|z_0| \leq \frac{\log^\phi T}{r(T)}$, $h_T(z_0) \leq \frac{1}{2\mu_T^2}$ and $\xi_T(z_0) > \frac{2\mu_T^3}{a(T)}$. Thus when T is large,

$$
\begin{aligned}
H_T'(z) &\leq (1 + \mu_T^{-2}) \left(h_T(z) + h_T(z_0) + 2q \frac{|z_0|}{\xi_T(z_0)} \right) + H_T'(z_0) \\
&\leq (1 + \mu_T^{-2}) h_T(z) + (1 + \mu_T^{-2}) \left(\frac{1}{2\mu_T^2} + 2q \frac{(\log^\phi T)}{r(T)} \frac{a(T)}{2\mu_T^3} \right) + 5 \frac{\log^{2\phi} T}{T} \\
&\leq (1 + \mu_T^{-2}) h_T(z) + \mu_T^{-2} \leq h_T(z) + (t_\infty + 1)/\mu_T^2,
\end{aligned}
$$

where we recall that $a(T)/r(T) = \log T/T$ and $\mu_T = \log^{1/4} T$. □

5.3. *There are not too few particles.* We now want to show that with high probability, there are at least about $m_T(z, s)$ particles at each site z, for all $s \leq t_\infty$. Again our main tool will be Lemmas 5.1 and 5.2. Since these lemmas apply only at fixed times, and we want to be sure that there is *never* a time when the number of particles is too small, we start by translating into continuous time.

LEMMA 5.6.   *Take $y, z \in L_T$ such that $\frac{\mu_T^3}{a(T)} \leq \xi_T(y) \leq \mu_T$. Then for large T,*

$$
\begin{aligned}
P^\xi \Bigg(\exists s \in \left[q \frac{|z - y|}{\xi_T(y)}, t_\infty \right] : \\
N\big(r(T)z, H_T'(y)T + (1 + \mu_T^{-2})sT\big) < e^{a(T)T(\xi_T(y)s - q|z-y|)} \Bigg) \\
< \exp\left(-\frac{1}{2} \log^2 T \right).
\end{aligned}
$$

PROOF. Within this proof only, we will use the shorthand

$$N_s = N(r(T)z, H_T'(y)T + (1 + \mu_T^{-2})sT)$$

and

$$E_{s,p} = \exp(a(T)T(\xi_T(y)s - q|z - y|) + p\mu_T).$$

For each $j \geq 0$, let

$$s_j = q\frac{|z - y|}{\xi_T(y)} + \frac{j}{4a(T)T}.$$

Our plan is to apply Lemmas 5.1 and 5.2 with $s = s_j$ for each j, and then to show that the number of particles cannot drop suddenly between s_j and s_{j+1} for any j.

For any j,

$$a(T)T\xi_T(y)(s_{j+1} - s_j) = \xi_T(y)/4 \leq \mu_T/4.$$

Thus if $k = \lfloor 4a(T)Tt_\infty \rfloor$,

$$P^\xi\big(\exists u \in [q|z - y|/\xi_T(y), t_\infty] : N_u < E_{u,0}\big)$$

$$\leq \sum_{j=0}^{k} P^\xi\big(\exists u \in [s_j, s_{j+1}) : N_u < E_{s_j,1/4}\big)$$

$$\leq \sum_{j=0}^{k} P^\xi(N_{s_j} < E_{s_j,1/2})$$

$$+ \sum_{j=0}^{k} P^\xi\big(\exists u \in [s_j, s_{j+1}) : N_u < E_{s_j,1/4}|N_{s_j} \geq E_{s_j,1/2}\big).$$

A simple application of Lemma 5.1 (if $y = z$) or Lemma 5.2 (if $y \neq z$) tells us that the first sum is at most $(4a(T)Tt_\infty + 1)\exp(-\log^2 T)$ when T is large, and so it suffices to prove the same for the second sum.

Let

$$I_j = [H_T'(y)T + (1 + \mu_T^{-2})s_jT, H_T'(y)T + (1 + \mu_T^{-2})s_{j+1}T).$$

Note that when T is large, $|I_j| \leq \frac{1}{2a(T)}$ for each j. Given that $N_{s_j} \geq E_{s_j,1/2}$, and the probability that a particle does not move during an interval of length $\frac{1}{2a(T)}$ is $\exp(-d/a(T))$, we expect at least $E_{s_j,1/2}\exp(-d/a(T))$ particles to remain at $r(T)z$ throughout the interval I_j. By Lemma 2.6, the event that the number of particles that actually stay is less than $\frac{1}{2}E_{s_j,1/2}\exp(-d/a(T))$ has probability at most $\exp(-\frac{1}{8}E_{s_j,1/2}\exp(-d/a(T))) < \exp(-\log^2 T)$. Since $\frac{1}{2}E_{s_j,1/2}\exp(-d/a(T)) > E_{s_j,1/4}$ when T is large, this completes the proof. $\square$

PROPOSITION 5.7. *As $T \to \infty$,*

$$\mathbb{P}\big(\exists s \in [0, t_\infty], z : M_T(z, s) < m_T(z, s) - \log^{-1/4} T\big) \to 0.$$

PROOF. We may assume without loss of generality that $t_\infty \geq 1$. We work on Γ_T and assume that the event

$$\mathcal{H}'_T = \big\{H'_T(y) \wedge t_\infty \leq h_T(y) + (t_\infty + 1)\mu_T^{-2} \text{ for all } y\big\}$$

holds. We know that $\mathbb{P}(\Gamma_T) \to 1$, and Proposition 5.5 tells us that $\mathbb{P}(\mathcal{H}'_T) \to 1$, so it suffices to prove our result under these conditions. In particular we may restrict to $z \in L_T(0, \rho_T)$.

Fix such a site z. Note that on Γ_T,

$$P^\xi \big(\exists s \in [0, t_\infty] : M_T(z, s) < m_T(z, s) - \mu_T^{-1}, \mathcal{H}'_T\big)$$

$$\leq \sum_{y \in L_T(0, \rho_T)} P^\xi \big(\exists s \in [0, t_\infty] : N\big(r(T)z, sT\big) \vee 1$$

$$< e^{a(T)T(\xi_T(y)(s - h_T(y))_+ - q|z-y| - \mu_T^{-1})}, \mathcal{H}'_T\big).$$

Observe that if $\xi_T(y) < \frac{\mu_T^3}{a(T)}$, then $\xi_T(y)(t_\infty - h_T(y)) - q|z - y| - \mu_T^{-1} < 0$ for large T, so the probability above is zero. Recall also that on Γ_T, there exists a constant c such that $\xi_T(y) \leq c \log \log T$ for all $y \in L_T(0, \rho_T)$. We deduce that we may restrict the sum above to y such that $\frac{\log^{3/4} T}{a(T)} \leq \xi_T(y) \leq c \log \log T$.

Fix such a site $y \in L_T$. If $s \leq H'_T(y) \wedge t_\infty + (1 + \mu_T^{-2})q\frac{|z-y|}{\xi_T(y)}$, then on $\mathcal{H}'_T$ we have

$$s \leq h_T(y) + (t_\infty + 1)\mu_T^{-2} + (1 + \mu_T^{-2})q\frac{|z - y|}{\xi_T(y)},$$

so

$$\xi_T(y)\big(s - h_T(y)\big) - q|z - y| - \mu_T^{-1}$$
$$\leq (t_\infty + 1)\mu_T^{-2}\xi_T(y) + \mu_T^{-2}q|z - y| - \mu_T^{-1}$$
$$\leq (t_\infty + 1)\mu_T^{-2}c \log \log T + 2\mu_T^{-2}q \log \log T - \mu_T^{-1}$$

which is negative for large T. Thus

$$P^\xi \bigg(\exists s \in \Big[0, H'_T(y) \wedge t_\infty + (1 + \mu_T^{-2})q\frac{|z - y|}{\xi_T(y)}\Big]:$$

$$N\big(r(T)z, sT\big) \vee 1 < e^{a(T)T(\xi_T(y)(s - h_T(y))_+ - q|z-y| - \mu_T^{-1})}, \mathcal{H}'_T\bigg) = 0.$$

As a result, it suffices to look at $s \geq H_T'(y) + (1 + \mu_T^{-2})q\frac{|z-y|}{\xi_T(y)}$, and by substituting in $u = \frac{s - H_T'(y)}{1 + \mu_T^{-2}}$ we get that

$$P^\xi\left(\exists s \in \left[H_T'(y) + (1 + \mu_T^{-2})q\frac{|z-y|}{\xi_T(y)}, t_\infty\right]:\right.$$

$$\left. N(r(T)z, sT) \vee 1 < e^{a(T)T(\xi_T(y)(s - h_T(y))_+ - q|z-y| - \mu_T^{-1})}, \mathcal{H}_T'\right)$$

$$\leq P^\xi\left(\exists u \in \left[q\frac{|z-y|}{\xi_T(y)}, t_\infty\right]: N(r(T)z, H_T'(y)T + (1 + \mu_T^{-2})uT)\right.$$

$$\left. < e^{a(T)T(\xi_T(y)(u + u\mu_T^{-2} + H_T'(y) - h_T(y)) - q|z-y| - \mu_T^{-1})}, \mathcal{H}_T', H_T'(y) < t_\infty\right)$$

$$\leq P^\xi\left(\exists u \in \left[q\frac{|z-y|}{\xi_T(y)}, t_\infty\right]:\right.$$

$$\left. N(r(T)z, H_T'(y)T + (1 + \mu_T^{-2})uT) < e^{a(T)T(\xi_T(y)u - q|z-y|)}\right).$$

By Lemma 5.6, this is at most $\exp(-\frac{1}{2}\log^2 T)$, and since there are at most $c_d^2 r(T)^2 \rho_T^2 \ll \exp(\frac{1}{2}\log^2 T)$ suitable pairs of points y, z, the result follows. $\quad\square$

6. Proofs of Theorems 1.1, 1.2 and 1.3. It now remains to draw the results of the previous sections together.

PROOF OF THEOREM 1.1. The fact that

$$\sup_{t \leq t_\infty} \sup_{z \in L_T} |M_T(z, t) - m_T(z, t)| \to 0$$

in $\mathbb{P}$-probability follows immediately from Propositions 4.12 and 5.7. We therefore concentrate on showing that $\sup_{z \in L_T(0, R)} |H_T(z) - h_T(z)| \to 0$ in $\mathbb{P}$-probability.

Fix any $R, \delta, \varepsilon > 0$. Clearly it suffices to prove the theorem when t_∞ is large, and by Lemma 3.4 by making t_∞ large we may ensure that $\mathbb{P}(\exists z \in L_T(0, R): h_T(z) \geq t_\infty - \delta) < \varepsilon/2$. Now, by Proposition 5.5, we may choose T_∞ large enough such that for any $T \geq T_\infty$, we have

$$\mathbb{P}(\exists z: H_T'(z) \wedge t_\infty > h_T(z) + \delta) < \varepsilon/2.$$

Then for $T \geq T_\infty$,

$$\mathbb{P}(\exists z \in L_T(0, R): H_T(z) > h_T(z) + \delta)$$

$$\leq \mathbb{P}(\exists z \in L_T(0, R): H_T'(z) \wedge t_\infty > h_T(z) + \delta)$$

$$+ \mathbb{P}(\exists z \in L_T(0, R): h_T(z) \geq t_\infty - \delta) < \varepsilon.$$

For the lower bound on $H_T(z)$, by increasing T_∞ if necessary we may assume that for any $T \geq T_\infty$ we have $(1 - \gamma_T)t_\infty + \delta_T \leq \delta$, where γ_T and δ_T are as in Section 4.3. By Proposition 4.9 we may also ensure that for $T \geq T_\infty$ we have

$$\mathbb{P}\big(\exists z : H_T(z) < (\gamma_T h_T(z) - \delta_T) \wedge t_\infty\big) < \varepsilon/2.$$

Then for $T \geq T_\infty$,

$$\begin{aligned}
\mathbb{P}\big(&\exists z \in L_T(0, R) : H_T(z) < h_T(z) - \delta\big) \\
&\leq \mathbb{P}\big(\exists z \in L_T(0, R) : h_T(z) > t_\infty\big) \\
&\quad + \mathbb{P}\big(\exists z : H_T(z) < \gamma_T h_T(z) + (1 - \gamma_T)t_\infty - \delta, h_T(z) \leq t_\infty\big) \\
&\leq \varepsilon/2 + \mathbb{P}\big(\exists z : H_T(z) < (\gamma_T h_T(z) - \delta_T) \wedge t_\infty\big) \\
&< \varepsilon. \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square
\end{aligned}$$

In order to prove Theorem 1.2 we first show that, with high probability, the lilypad model does not change rapidly over small time intervals.

LEMMA 6.1. *For any $t_\infty, \delta, \varepsilon > 0$, there exists $\eta > 0$ such that for all large T,*

$$\mathbb{P}\left(s_T(t + \eta) \subseteq \bigcup_{y \in s_T(t)} B(y, \delta) \ \forall t \leq t_\infty\right) \geq 1 - \varepsilon.$$

PROOF. By Lemma 3.6 we may choose R such that for any $T > e$,

$$\mathbb{P}\big(\exists z \in L_T \setminus B(0, R) : h_T(z) \leq t_\infty + 1\big) < \varepsilon/2.$$

Then by Lemma 2.7 we may choose $\Upsilon > 0$ such that for any $T > e$,

$$\mathbb{P}\Big(\max_{z \in L_T(0,R)} \xi_T(z) > \Upsilon\Big) < \varepsilon/2.$$

Now choose $T_\infty > e$ such that $1/r(T_\infty) < \delta/4$, and choose $\eta < (q\delta/2\Upsilon) \wedge 1$. As a result of the above bounds, for any $T \geq T_\infty$ we have $\mathbb{P}(\exists z \in s_T(t_\infty + \eta) : \xi_T(z) > \Upsilon) < \varepsilon$.

Fix $T > T_\infty$ and $t \leq t_\infty$, and take $x \in s_T(t + \eta) \setminus s_T(t)$. We will show that if $\xi_T(z) \leq \Upsilon$ for all $z \in s_T(t + \eta)$, then $d(x, s_T(t)) \leq \delta$. Let $w = [x]_T$, so that $w \in L_T$ and $t \leq h_T(w) \leq t + \eta$. Take $y_0, \ldots, y_n \in L_T$ such that $y_0 = w$, $y_n = 0$, $h_T(y_j) \leq t + \eta$ for all j, and

$$h_T(w) = \sum_{j=1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)}.$$

Let $k = \min\{j : y_j \in s_T(t)\}$. Then choose $y \in L_T$ such that $|y_{k-1} - y_k| = |y_{k-1} - y| + |y - y_k|$ and $y \notin s_T(t)$, but $d(y, s_T(t)) \leq 1/r(T)$. That is, y is the first point

in L_T on the geodesic between y_k and y_{k-1} that is outside $s_T(t)$. (There may be more than one such point, but any will do.) Now,

$$
\begin{aligned}
h_T(w) &= \sum_{j=1}^{k-1} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y_{k-1} - y_k|}{\xi_T(y_k)} + \sum_{j=k+1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} \\
&= \sum_{j=1}^{k-1} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y_{k-1} - y|}{\xi_T(y_k)} + \sum_{j=k+1}^{n} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y - y_k|}{\xi_T(y_k)} \\
&\geq \sum_{j=1}^{k-1} q \frac{|y_{j-1} - y_j|}{\xi_T(y_j)} + q \frac{|y_{k-1} - y|}{\xi_T(y_k)} + h_T(y) \\
&\geq q \frac{|w - y|}{\Upsilon} + h_T(y),
\end{aligned}
$$

where the last line followed from the triangle inequality plus the assumption that $\xi_T(z) \leq \Upsilon$ for all $z \in s_T(t + \eta)$. Thus $|w - y| \leq (h_T(w) - h_T(y))\Upsilon/q$, and since $t \leq h_T(y)$ and $h_T(w) \leq t + \eta$, we have $|w - y| \leq \eta\Upsilon/q < \delta/2$. But now $d(x, s_T(t)) \leq |x - w| + |w - y| + d(y, s_T(t)) \leq 2/r(T) + \delta/2 < \delta$ which proves our claim. $\square$

PROOF OF THEOREM 1.2. Fix $t_\infty, \delta, \varepsilon > 0$, and apply Lemma 6.1 to choose $\eta > 0$ such that for all large T,

$$
\mathbb{P}\left(s_T(t + \eta) \subseteq \bigcup_{y \in s_T(t)} B(y, \delta) \,\forall t \leq t_\infty\right) \geq 1 - \varepsilon/3.
$$

Fix $t_\infty > 0$, and choose T_∞ large enough so that the above holds and, using Proposition 4.9,

$$
\mathbb{P}\big(\exists z : H_T(z) \leq (h_T(z) - \eta) \wedge (t_\infty + 1)\big) < \varepsilon/3 \qquad \text{for all } T \geq T_\infty.
$$

On the event $\{H_T(z) > (h_T(z) - \eta) \wedge (t_\infty + 1) \,\forall z\}$, if $H_T(z) \leq t \leq t_\infty$, we must have $h_T(z) < H_T(z) + \eta$ and thus $S_T(t) \subseteq s_T(t + \eta)$.

Increasing T_∞ again as required, we can ensure that for all $T \geq T_\infty$, by Proposition 5.5,

$$
\mathbb{P}\big(\exists z : H_T(z) \wedge t_\infty > h_T(z) + \eta\big) < \varepsilon/3.
$$

On the event $\{H_T(z) \wedge t_\infty \leq h_T(z) + \eta \,\forall z\}$, if $h_T(z) \leq t - \eta$, then $H_T(z) \leq t$, and thus $s_T(t - \eta) \subset S_T(t)$.

We have therefore established that with probability at least $1 - \varepsilon$, for any $t \leq t_\infty$,

$$
S_T(t) \subseteq s_T(t + \eta) \subseteq \bigcup_{y \in s_T(t)} B(y, \delta)
$$

and

$$s_T(t) \subset \bigcup_{y \in s_T(t-\eta)} B(y, \delta) \subset \bigcup_{y \in S_T(t)} B(y, \delta),$$

which proves the theorem. $\square$

Before proving Theorem 1.3, we do most of the work in the following lemma. For $z \in L_T$ and $t \geq 0$, let

$$\tilde{m}(z, t) = \xi_T(z)(t - h_T(z))_+.$$

Intuitively, $\tilde{m}(z, t)$ is a rescaled count of how many particles should be born at z by time t.

LEMMA 6.2. *For any $t, \varepsilon > 0$ there exists $\delta > 0$ such that for all large T,*

$$\mathbb{P}\left(\exists z_1, z_2 \in L_T : \begin{matrix} \tilde{m}_T(z_1, t) \geq \tilde{m}_T(z_2, t) \geq \tilde{m}_T(y, t) \ \forall y \neq z_1, \\ |\tilde{m}_T(z_1, t) - \tilde{m}_T(z_2, t)| < \delta \end{matrix}\right) < \varepsilon.$$

PROOF. Fix $t, \varepsilon > 0$, and assume that T is large. By Lemma 3.6 we may choose $R > 0$ such that

$$\mathbb{P}(\exists z \notin B(0, R) : \tilde{m}_T(z, t) > 0) < \varepsilon/4.$$

By Lemma 2.7 we may then choose $\Upsilon > 0$ such that

$$\mathbb{P}(\exists z \in L_T(0, R) : \xi_T(z) > \Upsilon) < \varepsilon/4.$$

We can also find $\eta > 0$ such that, by Lemmas 2.7 and 3.4,

$$\mathbb{P}(\tilde{m}_T(z, t) \leq \eta \ \forall z) < \varepsilon/4.$$

For $z_1, z_2 \in B(0, R)$, we are interested in the event

$$\mathcal{M}_T(z_1, z_2) = \{|m_T(z_1, t) - m_T(z_2, t)| < \delta, \xi_T(z_1) > \eta/2t, \xi_T(z_2) > \eta/2t,$$
$$h_T(z_1) < t - \eta/2\Upsilon, h_T(z_2) < t - \eta/2\Upsilon\}.$$

This is because, provided $\delta < \frac{\eta}{2t} \wedge \frac{\eta}{2\Upsilon} \wedge \frac{\eta}{2}$,

$$\mathbb{P}\left(\exists z_1, z_2 \in L_T : \begin{matrix} \tilde{m}_T(z_1, t) \geq \tilde{m}_T(z_2, t) \geq \tilde{m}_T(y, t) \ \forall y \neq z_1, \\ |\tilde{m}_T(z_1, t) - \tilde{m}_T(z_2, t)| < \delta \end{matrix}\right)$$

(8)
$$\leq \mathbb{P}(\exists z \notin B(0, R) : \tilde{m}_T(z, t) > 0) + \mathbb{P}(\exists z \in L_T(0, R) : \xi_T(z) > \Upsilon)$$

$$+ \mathbb{P}(\tilde{m}_T(z, t) \leq \eta \ \forall z) + \sum_{z_1, z_2 \in L_T(0, R)} \mathbb{P}(\mathcal{M}_T(z_1, z_2)).$$

We now estimate $\mathbb{P}(\mathcal{M}_T(z_1, z_2))$. Suppose first that $h_T(z_1) \leq h_T(z_2)$. Then

$$\mathcal{M}_T(z_1, z_2)$$
$$\subseteq \left\{ \xi_T(z_2) \in \left[\frac{\xi_T(z_1)(t - h_T(z_1)) - \delta}{t - h_T(z_2)} \vee \frac{\eta}{2t}, \frac{\xi_T(z_1)(t - h_T(z_1)) + \delta}{t - h_T(z_2)} \right], \right.$$
$$\left. \xi_T(z_1) > \eta/2t, t - h_T(z_2) > \eta/2\Upsilon \right\}.$$

Now, $\xi_T(z_2)$ is independent of $\{h_T(z_1) \leq h_T(z_2)\}$, $h_T(z_2)$ and $\xi_T(z_1)$. Further, given $\{h_T(z_1) \leq h_T(z_2)\}$, $\xi_T(z_2)$ is conditionally independent of $h_T(z_1)$. Also $\mathbb{P}(\xi_T(z_2) \in [\mu, \mu + \nu])$ is decreasing in μ for $\mu \geq 1$ and increasing in ν. Thus

$$\mathbb{P}\big(\mathcal{M}_T(z_1, z_2) \cap \{h_T(z_1) \leq h_T(z_2)\}\big)$$
$$\leq \mathbb{P}\big(\xi_T(z_2) \in [\eta/2t, \eta/2t + 4\delta\Upsilon/\eta]\big)\mathbb{P}\big(\xi_T(z_1) \geq \eta/2t\big)$$
$$= (\eta/2t)^{-2\alpha} a(T)^{-2\alpha} \left(1 - \left(1 + \frac{8t\Upsilon\delta}{\eta^2} \right)^{-\alpha} \right)$$
$$\leq \frac{2^{2\alpha+3} t^{2\alpha+1} \alpha \Upsilon \delta}{a(T)^{2\alpha} \eta^{2\alpha+2}}.$$

By symmetry we also have

$$\mathbb{P}\big(\mathcal{M}_T(z_1, z_2) \cap \{h_T(z_2) \leq h_T(z_1)\}\big) \leq \frac{2^{2\alpha+3} t^{2\alpha+1} \alpha \Upsilon \delta}{a(T)^{2\alpha} \eta^{2\alpha+2}},$$

and therefore

$$\mathbb{P}\big(\mathcal{M}_T(z_1, z_2)\big) \leq \frac{2^{2\alpha+4} t^{2\alpha+1} \alpha \Upsilon \delta}{a(T)^{2\alpha} \eta^{2\alpha+2}}.$$

Plugging this and our previous estimates into (8), we see that

$$\mathbb{P}\left(\exists z_1, z_2 \in L_T : \begin{array}{c} \tilde{m}_T(z_1, t) \geq \tilde{m}_T(z_2, t) \geq \tilde{m}_T(y, t) \; \forall y \neq z_1, \\ |\tilde{m}_T(z_1, t) - \tilde{m}_T(z_2, t)| < \delta \end{array} \right)$$
$$< 3\varepsilon/4 + C_d^2 R^{2d} r(T)^{2d} \frac{2^{2\alpha+4} t^{2\alpha+1} \alpha \Upsilon \delta}{a(T)^{2\alpha} \eta^{2\alpha+2}},$$

which, by choosing δ sufficiently small, we may ensure is at most ε. $\quad\square$

PROOF OF THEOREM 1.3.   By Proposition 5.7, for large T we have

$$\mathbb{P}\big(M_T(z, t) \geq m_T(z, t) - \log^{-1/4} T \; \forall z\big) > 1 - \varepsilon/5.$$

By Proposition 4.12, for large T we have

$$\mathbb{P}\big(M_T(z, t) \leq m_T(z, t) + \log^{-1/4} T \; \forall z\big) > 1 - \varepsilon/5.$$

By Lemma 3.6, we may choose $R > 0$ such that for $T > e$ we have

$$\mathbb{P}\big(h_T(z) > t + 1 \ \forall z \notin B(0, R)\big) > 1 - \varepsilon/5,$$

and then by Proposition 4.9, since if $H_T(z) > t$, then $M_T(z, t) = 0$, for large T we have

$$\mathbb{P}\big(h_T(z) > t + 1 \text{ and } M_T(z, t) = 0 \ \forall z \notin B(0, R)\big) > 1 - 2\varepsilon/5.$$

Finally, by Lemma 6.2, there exists $\delta > 0$ such that for large T we have

$$\mathbb{P}\big(\exists z_1 \in L_T : \tilde{m}_T(z_1, t) \geq \tilde{m}_T(z, t) + \delta \ \forall z \in L_T\big) > 1 - \varepsilon/5.$$

Therefore, with probability at least $1 - \varepsilon$, all of the above events hold. Assume that they do all hold, and fix $z \in L_T(0, R)$. Note that

$$m_T(z, t) = \sup_y \{\tilde{m}_T(y, t) - q|z - y|\},$$

so either $m_T(z, t) \leq \tilde{m}_T(z_1, t) - \delta$ or $m_T(z, t) = \tilde{m}_T(z_1, t) - q|z - z_1|$. Thus if $|z - z_1| > \frac{3}{q} \log^{-1/4} T$ and T is large, then

$$M_T(z, t) \leq m_T(z, t) + \log^{-1/4} T \leq m_T(z_1, t) - 2 \log^{-1/4} T$$
$$\leq M_T(z_1, t) - \log^{-1/4} T.$$

We deduce that if $|z - z_1| > \frac{3}{q} \log^{-1/4} T$ and T is large, then

$$N\big(r(T)z, tT\big) \leq e^{a(T)T M_T(z_1,t) - a(T)T \log^{-1/4} T} = N\big(r(T)z_1, tT\big) e^{-a(T)T \log^{-1/4} T}.$$

Summing up, with probability at least $1 - \varepsilon$ we have a point $z_1 \in L_T$ such that

$$\sum_{z \in L_T} N\big(r(T)z, tT\big) \leq \sum_{z \in L_T(z_1, (3/q) \log^{-1/4} T)} N\big(r(T)z, tT\big)$$
$$+ C_d R^d r(T)^d e^{-a(T)T \log^{-1/4} T} N\big(r(T)z_1, tT\big).$$

The result follows. □

7. **The parabolic Anderson model.**   Our aim in this section is to prove Theorem 1.4. Recall that we defined

$$\Lambda_T(z, t) = \frac{1}{a(T)T} \log_+ E^\xi\big[N(r(T)z, tT)\big]$$

and

$$\lambda_T(z, t) = \sup_y \{\xi_T(y)t - q|y| - q|z - y|\} \vee 0,$$

and that we claimed that these two objects are similar in size for all z when T is large. The idea is that (in the rescaled picture) the size of the population at z is

dominated by particles that look for the site y that maximizes $\xi_T(y)t - q|y| - q|z - y|$, run quickly to y at cost $q|y|$, sit there breeding until just before time t and thus gain a reward of $\xi_T(y)t$, and then run quickly to z at cost $q|z - y|$.

We first rule out unfriendly environments. Define the event

$$\Delta_T(t) = \left\{ \exists k \geq 0, \exists z \in B(0, 2^{k+1} \log \log T) : \xi_T(z) \geq \frac{q}{2t} 2^k \log \log T \right\}.$$

By Lemma 2.7 we know that

$$\mathbb{P}\left(\exists z \in B(0, 2^{k+1} \log \log T) : \xi_T(z) \geq \frac{q}{2t} 2^k \log \log T \right)$$

$$\leq C_d e 2^{(k+1)d} (\log \log T)^d \left(\frac{q}{2t} \right)^{-\alpha} 2^{-\alpha k} (\log \log T)^{-\alpha}$$

$$\leq C t^\alpha 2^{k(d-\alpha)} (\log \log T)^{d-\alpha}$$

for some constant C, and thus for any fixed t, $\mathbb{P}(\Delta_T(t)) \to 0$ as $T \to \infty$. We view $\Delta_T(t)$ as a bad event and work on the complement, $\Delta_T(t)^c$.

7.1. *Upper bound.*

LEMMA 7.1. *For any $t_\infty > 0$, there exists $T_0 > 0$ such that for any $T \geq T_0$, for all $z \in L_T$ and all $t \leq t_\infty$, on $\Delta_T(t_\infty)^c$,*

$$E^\xi[N(r(T)z, tT)]$$

$$\leq \frac{1}{2} + \sum_{y \in L_T(0, \log \log T)} \exp\left(a(T)T \left(\xi_T(y)t - q|y| - q|z - y| \right. \right.$$

$$\left. \left. + \frac{(\log \log T)^3}{\log T} \right) \right).$$

PROOF. Take $z \in L_T$ and $t \leq t_\infty$. We apply the Feynman–Kac formula and split the probability space according to the supremum of $|X(s)|$.

$$E^\xi[N(r(T)z, tT)] = E^\xi[e^{\int_0^{tT} \xi(X(s))\, ds} \mathbb{1}_{\{X(tT)=r(T)z\}}]$$

$$= E^\xi[e^{\int_0^{tT} \xi(X(s))\, ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| \geq r(T) \log \log T\}}]$$

$$+ E^\xi[e^{\int_0^{tT} \xi(X(s))\, ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| < r(T) \log \log T\}}].$$

We check first that the term in which $\sup_{s \leq tT} |X(s)| \geq r(T) \log \log T$ is small.

$$E^\xi[e^{\int_0^{tT} \xi(X(s))\, ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| \geq r(T) \log \log T\}}]$$

$$= \sum_{k=0}^\infty E^\xi[e^{\int_0^{tT} \xi(X(s))\, ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)|/(r(T) \log \log T) \in [2^k, 2^{k+1})\}}]$$

$$\leq \sum_{k=0}^{\infty} \exp\Big(tT \max_{x \in B(0, 2^{k+1}r(T)\log\log T)} \xi(x)\Big)$$

$$\times P^{\xi}\Big(\sup_{s \leq tT} |X(s)| \geq 2^k r(T)\log\log T\Big)$$

$$\leq \sum_{k=0}^{\infty} \exp\Big(tT \max_{x \in B(0, 2^{k+1}r(T)\log\log T)} \xi(x)\Big) J_T\big(t, 2^k \log\log T\big).$$

Now, we know from Lemma 2.4 that

$$J_T\big(t, 2^k \log\log T\big) \leq \exp\big(-a(T)T\big(2^k q \log\log T - \mathcal{E}_T^2(t, 2^k \log\log T)\big)\big)$$

and thus on $\Delta_T(t_\infty)^c$,

$$E^{\xi}\big[e^{\int_0^{tT} \xi(X(s))\,ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| \geq r(T)\log\log T\}}\big]$$

$$\leq \sum_{k=0}^{\infty} \exp\Big(a(T)T\Big(2^k \frac{q}{2} \log\log T - 2^k q \log\log T + \mathcal{E}_T^2(t, 2^k \log\log T)\Big)\Big).$$

However,

$$\mathcal{E}_T^2\big(t, 2^k \log\log T\big) \leq \frac{2^k \log\log T}{\log T}\big(\log t + 1 + \log(2d) + (q+1)\log\log T\big),$$

so

$$E^{\xi}\big[e^{\int_0^{tT} \xi(X(s))\,ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| \geq r(T)\log\log T\}}\big] \leq \tfrac{1}{2}$$

for large T. In particular this shows that on $\Delta_T(t_\infty)^c$, if $|z| \geq \log\log T$, then $\mathbb{E}[N(r(T)z, tT)] \leq \frac{1}{2}$. We may therefore assume that $|z| < \log\log T$.

We are now left with the term when $\sup_{s \leq tT} |X(s)| < r(T)\log\log T$. We further split our probability space depending on the site of maximal potential that we visit before time t.

$$E^{\xi}\big[e^{\int_0^{tT} \xi(X(s))\,ds} \mathbb{1}_{\{X(tT)=r(T)z, \sup_{s \leq tT} |X(s)| < r(T)\log\log T\}}\big]$$

$$\leq \sum_{y \in L_T(0, \log\log T)} E^{\xi}\big[e^{\int_0^{tT} \xi(X(s))\,ds} \mathbb{1}_{\left\{\begin{smallmatrix} X(tT) = r(T)z,\, \exists s \leq t : X(sT) = r(T)y, \\ \sup_{s \leq t} \xi(X(sT)) = \xi(r(T)y) \end{smallmatrix}\right\}}\big]$$

$$\leq \sum_{y \in L_T(0, \log\log T)} \exp\big(a(T)T\xi_T(y)t\big)$$

$$\times P^{\xi}\big(\exists s \leq t : X(sT) = r(T)y, X(tT) = r(T)z\big)$$

$$\leq \sum_{y \in L_T(0, \log\log T)} \exp\big(a(T)T\xi_T(y)t\big) J_T\big(t, |y|\big) J_T\big(t, |z-y|\big)$$

$$\leq \sum_{y \in L_T(0, \log\log T)} \exp\big(a(T)T\big(\xi_T(y)t - q|y| - q|z-y|\big)$$

$$\times \exp\big(\mathcal{E}_T^2(t, |y|) + \mathcal{E}_T^2(t, |z-y|)\big)\big),$$

where the last inequality uses Lemma 2.4. For large T and $y \in L_T(0, \log\log T)$, we have $\mathcal{E}_T^2(t, |y|) \leq (\log\log T)^3/(2\log T)$ and similarly for $\mathcal{E}_T^2(t, |z - y|)$ since we are assuming that $|z| < \log\log T$. This gives the result. $\square$

7.2. *Lower bound.*

LEMMA 7.2. *For any $t_\infty > 0$, there exists $T_0 > 0$ such that for any $T \geq T_0$, for all $y, z \in L_T$ and all $t \leq t_\infty$, on $\Delta_T(t_\infty)^c$,*

$$E^\xi\big[N\big(r(T)z, tT\big)\big] \vee 1$$

$$\geq \exp\!\Big(a(T)T\Big(\xi_T(y)t - q|y| - q|z - y| - 6\frac{(\log\log T)^2}{\log T}\Big)\Big).$$

PROOF. Fix $t \leq t_\infty$. On $\Delta_T(t_\infty)^c$, if $|y| \geq \log\log T$ or if $|y| \leq \log\log T$ and either $|z| \geq \log\log T$ or $t \leq 2/\log T$, then for large T,

$$\xi_T(y)t - q|y| - q|z - y| - 6(\log\log T)^2/\log T \leq 0,$$

so there is nothing to prove. We may therefore assume that $|y| < \log\log T$, $|z| < \log\log T$ and $t > 2/\log T$. Then by the Feynman–Kac formula,

$$E^\xi\big[N\big(r(T)z, tT\big)\big]$$

$$= E^\xi\big[e^{\int_0^{tT} \xi(X(s))\,ds}\mathbb{1}_{\{X(tT)=r(T)z\}}\big]$$

$$\geq E^\xi\big[e^{\int_0^{tT} \xi(X(s))\,ds}\mathbb{1}_{\{X(sT)=r(T)y\ \forall s\in[1/\log T, t-1/\log T], X(tT)=r(T)z\}}\big]$$

$$\geq e^{a(T)T\xi_T(y)(t-2/\log T)}$$

$$P^\xi\big(X(sT)=r(T)y\ \forall s\in[1/\log T, t-1/\log T], X(tT)=r(T)z\big).$$

By the Markov property,

$$P^\xi\big(X(sT)=r(T)y\ \forall s\in[1/\log T, t-1/\log T], X(tT)=r(T)z\big)$$

$$= P^\xi\big(X(T/\log T)=r(T)y\big)P^\xi\big(X(sT)=0\ \forall s\in[0, t-2/\log T]\big)$$

$$\times P^\xi\big(X(T/\log T)=r(T)(z-y)\big).$$

By Lemma 2.3 and the fact that the probability our random walk remains at its current location for time s is the probability that a Poisson random variable of parameter $2\,ds$ is zero, this is at least

$$\exp\!\big(-a(T)T\big(q|y| + \mathcal{E}_T^1(1/\log T, |y|)\big)\big)$$

$$\times \exp\!\big(-2d(t - 2/\log T)T\big)$$

$$\times \exp\!\big(-a(T)T\big(q|z-y| + \mathcal{E}_T^1(1/\log T, |z-y|)\big)\big).$$

It is easy to check that since $y, z \in B(0, \log \log T)$, we have

$$\mathcal{E}_T^1(1/\log T, |y|) \leq 2(\log \log T)^2/\log T$$

and

$$\mathcal{E}_T^1(1/\log T, |z - y|) \leq 3(\log \log T)^2/\log T$$

for large T. Thus if T is large,

$$E^\xi[N(r(T)z, tT)]$$
$$\geq \exp(a(T)T(\xi_T(y)t - q|y| - q|z - y| - 6(\log \log T)^2/\log T). \quad \square$$

7.3. *Proof of Theorem* 1.4.   The two estimates given by the previous two lemmas are the tools we need to complete the proof of Theorem 1.4.

PROOF OF THEOREM 1.4.   We begin with part (i). Fix $t_\infty > 0$. For an upper bound, we know from Lemma 7.1 that for all large T, for any $z \in L_T$ and $t \leq t_\infty$, on $\Delta_T(t_\infty)^c$,

$$E^\xi[N(r(T)z, tT)]$$
$$\leq \frac{1}{2} + \sum_{y \in L_T(0, \log \log T)} \exp\Big(a(T)T\Big(\xi_T(y)t - q|y| - q|z - y|$$
$$+ \frac{(\log \log T)^3}{\log T}\Big)\Big).$$

Since $\lambda_T(z, t) \geq \sup_{y \in L_T}\{\xi_T(y)t - q|y| - q|z - y|\}$, we immediately see that

$$E^\xi[N(r(T)z, tT)]$$
$$\leq \frac{1}{2} + C_d(\log \log T)^d r(T)^d \exp\Big(a(T)T\Big(\lambda_T(z, t) + \frac{(\log \log T)^3}{\log T}\Big)\Big),$$

and thus

$$\Lambda_T(z, t) := \frac{1}{a(T)T}\log_+ E^\xi[N(r(T)z, tT)] \leq \lambda_T(z, t) + \frac{(\log \log T)^3}{\log T} + \frac{1}{T}.$$

For a lower bound, we know from Lemma 7.2 that on $\Delta_T(t_\infty)^c$, for any $y, z \in L_T$ and $t \leq t_\infty$,

$$E^\xi[N(r(T)z, tT)] \vee 1$$
$$\geq \exp(a(T)T(\xi_T(y)t - q|y| - q|z - y| - 6(\log \log T)^2/\log T)).$$

Without loss of generality, we can assume $\lambda_T(z, t) > 0$. Then choosing y such that $\lambda_T(z, t) = \xi_T(y)t - q|y| - q|z - y|$, we have

$$E^\xi[N(r(T)z, tT)] \vee 1 \geq \exp(a(T)T(\lambda_T(z, t) - 6(\log \log T)^2/\log T))$$

and thus

$$\Lambda_T(z, t) \geq \lambda_T(z, t) - 6\frac{(\log\log T)^2}{\log T}.$$

Since $\mathbb{P}(\Delta_T(t_\infty)) \to 0$ as $T \to \infty$, we have the desired result.

We now move on to part (ii). Fix $R, \varepsilon, \delta > 0$. It is easy to see by the triangle inequality and Lemma 3.3 that $\tau_T(z) \leq h_T(z)$ for all $z \in L_T$ and $T > e$, so by Lemma 3.4, there exists t_∞ such that

$$\mathbb{P}\big(\tau_T(z) < t_\infty - \delta \ \forall z \in L_T(0, R)\big) > 1 - \varepsilon/4.$$

Moreover, by Lemma 3.4, again since $\tau_T(z) \leq h_T(z)$,

$$\mathbb{P}\big(\tau_T(z) \leq \delta \ \forall z \in L_T\big(0, (\log\log T)^4/\log T\big)\big) > 1 - \varepsilon/4.$$

By Lemma 2.7, we can find a large K such that

(9) $$\mathbb{P}\big(\exists y_0 \in L_T\big(0, 2^{-K}\big) : \xi_T(y_0) \geq 4q2^{-K}/\delta\big) > 1 - \varepsilon/4.$$

Also we can choose T large enough such that $\mathbb{P}(\Delta_T(t_\infty)^c) > 1 - \varepsilon/4$. Then, with probability at least $1 - \varepsilon$, we may assume that all of the above events hold, and it suffices to show that then for any z with $\tau_T(z) < t_\infty - \delta$, we have

(10) $$\Lambda_T(z, t) = 0 \qquad \forall t \leq \tau_T(z) - \delta \quad \text{and}$$

(11) $$\Lambda_T\big(z, \tau_T(z) + \delta\big) > 0.$$

To show (10), note that if $|z| \leq (\log\log T)^4/\log T$, then $\tau_T(z) \leq \delta$ by assumption, and the statement is trivial. Therefore we may assume that $|z| > (\log\log T)^4/\log T$. By Lemma 7.1, on $\Delta_T(t_\infty)^c$ we have for any $t \leq t_\infty$,

$$E^\xi\big[N\big(r(T)z, tT\big)\big]$$
$$\leq \frac{1}{2} + \sum_{y \in L_T(0, \log\log T)} \exp\Big(a(T)T\Big(\xi_T(y)t - q|y| - q|z - y|$$
$$+ \frac{(\log\log T)^3}{\log T}\Big)\Big).$$

We claim that for every y,

$$\xi_T(y)\big(\tau_T(z) - \delta\big) - q|y| - q|z - y| \leq -3(\log\log T)^3/\log T,$$

which is enough to guarantee that $\Lambda_T(z, t) = 0$ for all $t \leq \tau_T(z) - \delta$. Indeed, if $\xi_T(y) \geq 3(\log\log T)^3/(\delta \log T)$, then by the definition of $\tau_T(z)$ we have

$$\xi_T(y)\big(\tau_T(z) - \delta\big) - q|y| - q|z - y| \leq -\xi_T(y)\delta \leq -3(\log\log T)^3/\log T.$$

On the other hand if $\xi_T(y) < 3(\log\log T)^3/(\delta\log T)$, since $\tau_T(z) \leq t_\infty$, we have by the triangle inequality

$$\xi_T(y)(\tau_T(z) - \delta) - q|y| - q|z - y| \leq \xi_T(y)\tau_T(z) - q|z|$$
$$\leq 3t_\infty(\log\log T)^3/(\delta\log T) - q(\log\log T)^4/\log T,$$

which is smaller than $-3(\log\log T)^3/\log T$ when T is large.

For (11), by Lemma 7.2 we have that on $\Delta_T(t_\infty)^c$, for any $y \in L_T$,

$$E^\xi\big[N\big(r(T)z, (\tau_T(z) + \delta)T\big)\big] \vee 1$$
$$\geq \exp\big(a(T)T\big(\xi_T(y)(\tau_T(z) + \delta) - q|y| - q|z - y| - 6(\log\log T)^2/\log T\big)\big).$$

If $|z| < 2^{-K}$, then choose $y = y_0$ from (9), and note that

$$\xi_T(y_0)(\tau_T(z) + \delta) - q|y_0| - q|z - y_0| \geq 4q2^{-K} - q2^{-K} - 2q2^{-K} = q2^{-K},$$

so $\Lambda_T(z, \tau_T(z) + \delta) \geq q2^{-K} - 6\frac{(\log\log T)^2}{\log T}$. On the other hand if $|z| \geq 2^{-K}$, choose y such that $\tau_T(z) = \frac{q}{\xi_T(y)}(|y| + |z - y|)$. By assumption we know that $\tau_T(z) \leq t_\infty$, and since by the triangle inequality $\tau_T(z) \geq q|z|/\xi_T(y)$, we deduce that $\xi_T(y) \geq q2^{-K}/t_\infty$. Then

$$\xi_T(y)(\tau_T(z) + \delta) - q|y| - q|z - y| = \xi_T(y)\delta \geq q\delta 2^{-K}/t_\infty,$$

and thus $\Lambda_T(z, (\tau_T(z) + \delta)T) \geq \frac{q\delta 2^{-K}}{\log\log T} - 6\frac{(\log\log T)^2}{\log T} > 0$ provided T is large.

Finally, the proof of part (iii) of the theorem is essentially the same as the proof of Theorem 1.2, checking that with high probability the PAM lilypad model does not grow too fast, as in Lemma 6.1, and combining this with our knowledge of the hitting times from part (ii) above. □

8. Comparing the BRW with the PAM.
In this section we prove Theorem 1.5. We start by showing the corresponding statements for the maximizers of the lilypad models. As a first step, we construct conditions on the potential under which we can control the maximum of the BRW lilypad. We will see in Lemma 8.4 that these conditions occur simultaneously with positive probability uniformly in T.

BRW Setup. Fix $T, t, \kappa > 0$. Suppose that $r > 0$ and $\eta \geq 8qr/t$ (we will choose r and η later) and that $R > (\frac{2\eta t}{q}) \vee 3\kappa$. Assume that the potential $(\xi_T(z), z \in L_T)$ satisfies the following conditions:

(A) there exists a site x in $L_T(0, r)$ such that $\xi_T(x) \in [\eta, 2\eta)$;
(B) for all sites $y \in L_T(0, R) \setminus \{x\}$, we have that $\xi_T(y) \leq \eta/2$;
(C) $\max_{z \in L_T(0,r)} h_T(z) \leq t/8$.

PROPOSITION 8.1. *Under assumptions* (A), (B), (C) *and for* T *sufficiently large, if* $z \in L_T$ *is such that* $h_T(z) \le t$, *then*

$$m_T(z, t) = m_T(x, t) - q|z - x| = \xi_T(x)(t - h_T(x)) - q|z - x|,$$

and otherwise $m_T(z, t) = 0$. *In particular,* x *is the unique maximizer of* $m_T(\cdot, t)$.

PROOF. The idea of the proof is first to check that all sites outside the ball $L_T(0, R)$ are hit after time t. Then we make sure that the site x with large potential is hit so early that by time t the lilypad has grown far enough to "overtake" all other lilypads.

We first show that any site outside $L_T(0, R)$ is hit after time t. Indeed, by Lemma 3.6 we have that

$$\{\exists z \in L_T \setminus B(0, R) : h_T(z) \le t\} \subseteq \left\{ \max_{y \in L_T(0, R)} \xi_T(y) \ge qR/t \right\}.$$

However, $qR/t > 2\eta$, so we have $h_T(z) > t$ for all $z \in L_T \setminus B(0, R)$.

For our next task, first suppose that $|z| < \frac{\eta t}{8q} + r$. Then since $x \in L_T(0, r)$,

$$\xi_T(x)(t - h_T(x)) - q|z - x| > \xi_T(x)t - \xi_T(x)t/8 - \eta t/8 - 2qr$$

$$\ge 7\xi_T(x)t/8 - 3\eta t/8 \ge \xi_T(x)t/2.$$

Since $\xi_T(x) \ge 2\xi_T(y)$ for any $y \in L_T(0, R) \setminus \{x\}$, we therefore have

$$\xi_T(x)(t - h_T(x)) - q|z - x| > \sup_{y \ne x} \{\xi_T(y)(t - h_T(y))\}$$

$$\ge \sup_{y \ne x} \{\xi_T(y)(t - h_T(y)) - q|z - y|\}$$

and hence

$$m_T(z, t) = \xi_T(x)(t - h_T(x)) - q|z - x| = m_T(x, t) - q|z - x|.$$

Now suppose that $|z| \ge \frac{\eta t}{8q} + r$. We claim that $h_T(z) = h_T(x) + q|z - x|/\xi_T(x)$. Indeed,

$$h_T(x) + q\frac{|z - x|}{\xi_T(x)} \le \frac{t}{8} + q\frac{r}{\eta} + q\frac{|z|}{\eta},$$

and for any $y_0, \ldots, y_n \in L_T(0, R) \setminus \{x\}$ with $y_0 = z$ and $y_n = 0$, by the triangle inequality we have

$$\sum_{j=1}^{n} q\frac{|y_j - y_{j-1}|}{\xi_T(y_{j-1})} \ge q\frac{2}{\eta}|z| \ge q\frac{|z|}{\eta} + \frac{t}{8} + q\frac{r}{\eta},$$

which proves the claim. As a result we have that for any $y \in L_T \setminus \{z\}$,

$$h_T(x) + q\frac{|x - z|}{\xi_T(x)} = h_T(z) \le h_T(y) + q\frac{|y - z|}{\xi_T(y)}.$$

Thus, for any $y \in L_T(0, R) \setminus \{x\}$ such that $h_T(y) \leq t$, we have that since $\xi_T(y) \leq \frac{1}{2}\xi_T(x)$,

$$\xi_T(y)(t - h_T(y)) - q|y - z| \leq \xi_T(y)\left(t - h_T(x) - q\frac{|x - z|}{\xi_T(x)}\right)$$

$$\leq \frac{1}{2}(\xi_T(x)(t - h_T(x)) - q|x - z|).$$

We deduce that in this case too we have

$$m_T(z, t) = \xi_T(x)(t - h_T(x)) - q|x - z| = m_T(x, t) - q|z - x|. \qquad \square$$

On top of conditions (A), (B) and (C) outlined above, we now construct two further scenarios in which the maximizers of the BRW and PAM lilypad models agree, respectively, do not agree, at time t.

(S1) Suppose that for all $y \notin L_T(0, R)$, we have that

$$\xi_T(y) < \eta + q(|y| - r)/t.$$

(S2) Suppose that there exists a point $x' \in L_T(0, R + 1) \setminus L_T(0, R)$ such that

$$\xi_T(x') > 2\eta + q(R + 1)/t,$$

and that for all $y \neq x$ such that $y \notin L_T(0, R)$, we have that

$$\xi_T(y) < \eta + q(|y| - r)/t.$$

Recall that

$$\lambda_T(z, t) = \sup_{y \in L_T} \{\xi_T(y)t - q|y| - q|y - z|\} \vee 0.$$

LEMMA 8.2. *Suppose* (A), (B), (C) *hold*.

(i) *If in addition* (S1) *holds, then* x *[as defined in* (A)*] is the unique maximizer of* $\lambda_T(\cdot, t)$ *and of* $m_T(\cdot, t)$.

(ii) *If in addition* (S2) *holds, then* x *[as defined in* (A)*] is the unique maximizer of* $m_T(\cdot, t)$, *whereas* x' *is the unique maximizer of* $\lambda_T(\cdot, t)$ *and* $|x - x'| > 2\kappa$.

PROOF. Under (A), (B), (C) we already know from Proposition 8.1 that x is the unique maximizer of the BRW lilypad $m_T(\cdot, t)$. Moreover, $|x| < r$ and $|x'| \geq R > (16r) \vee 3\kappa$ so $|x - x'| > 2\kappa$. Thus it suffices to prove the statements about $\lambda_T(\cdot, t)$. Note from the definition of $\lambda_T(\cdot, t)$ that in particular

$$\sup_{z \in L_T} \lambda_T(z, t) = \sup_{z \in L_T} \{\xi_T(z)t - q|z|\}.$$

(i) By the above it suffices to show that $t\xi_T(x) - q|x| > t\xi_T(y) - q|y|$ for all $y \in L_T \setminus \{x\}$. Take $y \in L_T \setminus \{x\}$. If $y \in L_T(0, R)$, then $\xi_T(y) \leq \eta/2$ and thus

$$t\xi_T(y) - q|y| \leq \eta t/2 \leq t\xi_T(x) - \eta t/2 \leq t\xi_T(x) - 4qr < t\xi_T(x) - q|x|.$$

On the other hand, if $|y| \geq R$, then by assumption (S1),

$$\xi_T(y)t - q|y| < \eta t + q(|y| - r) - q|y| \leq \xi_T(x)t - q|x|.$$

Thus x is the unique maximizer of $\lambda_T(\cdot, t)$.

(ii) Arguing as in part (i), we already know that $\lambda_T(y, t) \leq \lambda_T(x, t)$ for all $y \neq x, x'$. Thus we only need to show that $\lambda_T(x', t) > \lambda_T(x, t)$. Indeed, by the assumption on $\xi_T(x')$, we have

$$\xi_T(x')t - q|x'| > 2\eta t + q(R + 1) - q|x'| \geq \xi_T(x)t - q|x|.$$

Thus x' is the unique maximizer of $\lambda_T(\cdot, t)$. □

Next we construct a scenario in which the support of the PAM lilypad is disconnected.

(S3) For R as above, suppose there exists $x' \in L_T$ with $2R \leq |x'| \leq 2R + 1$ such that $\xi_T(x') \in ((2R + 1)q/t, 5Rq/2t)$. Moreover, assume that for any $y \notin L_T(0, R) \cup \{x'\}$ we have $\xi_T(y) < q|y|/t$.

LEMMA 8.3. *If events* (A), (B) *and* (S3) *hold, then the support of the PAM lilypad model is not connected at time* t.

PROOF. Recall that for $z \in L_T$, we defined

$$\tau_T(z) = \inf_{y \in L_T} \left\{ \frac{q}{\xi_T(y)}(|y| + |z - y|) \right\}.$$

Suppose that $\tau(z) = q(|x'| + |z - x'|)/\xi_T(x') \leq t$. Then

$$|x' - z| \leq \frac{\xi_T(x')t}{q} - |x'| < \frac{5R}{2} - |x'| \leq \frac{R}{2}.$$

In particular if $\tau(z) = q(|x'| + |z - x'|)/\xi_T(x') \leq t$, then $z \notin B(0, 3R/2)$. Moreover,

$$\tau_T(x') \leq \frac{q|x'|}{\xi_T(x')} < q|x'| \frac{t}{(2R + 1)q} \leq t.$$

Now suppose that $\tau_T(z) = q(|y| + |z - y|)/\xi_T(y) \leq t$ for some site $y \in L_T(0, R)$. By assumptions (A) and (B) we have $\xi_T(y) \leq 2\eta$, which combined with the triangle inequality yields

$$\tau_T(z) \geq \frac{q|z|}{2\eta}.$$

We deduce that for such z we have $|z| \leq 2\eta t/q < R$, and thus $z \in B(0, R)$.

Finally, since under (S3) for any $y \notin L_T(0, R) \cup \{x'\}$, $\frac{q|y|}{\xi_T(y)} > t$, we can conclude that

$$
\begin{aligned}
\{z \in L_T : \tau_T(z) &\leq t\} \\
&= \left\{ z \in L_T : \tau_T(z) = \frac{q}{\xi_T(x')}(|x'| + |z - x'|) \leq t \right\} \\
&\cup \left\{ z \in L_T : \tau_T(z) = \inf_{y \in L_T(0, R)} \left\{ \frac{q}{\xi_T(y)}(|y| + |z - y|) \right\} \leq t \right\}.
\end{aligned}
$$

However, by the above, the two sets on the right-hand side are nonempty and are separated by distance at least $R/2$, which immediately implies that the support of the PAM lilypad model at time t is disconnected. $\square$

Finally, we show that our conditions on the potential are fulfilled with positive probability.

LEMMA 8.4. *For any $t > 0$, there exist $r > 0$, $\eta \geq 8qr/t$ and $R > (\frac{2\eta t}{q}) \vee 3\kappa$ such that for any $i = 1, 2, 3$, the probability that events* (A), (B), (C) *and* (Si) *occur simultaneously is bounded away from 0 for all large T.*

PROOF. To show that there exist r, η, R such that events (A), (B) and (C) occur simultaneously with probability bounded away from 0, we use similar tactics to the proof of Lemma 3.4. As in Lemma 3.4 we take $\gamma \in (d/\alpha, 1)$, let $B_k = L_T(0, 2^{-k})$ for $k \geq 1$ and set

$$
A_k = \{\exists z \in B_k : \xi_T(z) \geq 2^{-\gamma k}\}.
$$

Then by Lemma 2.7,

$$
\mathbb{P}(A_k^c) \leq e^{-c_d 2^{(\alpha\gamma - d)k}}.
$$

Thus we may choose K such that

$$
\mathbb{P}\left(\bigcap_{k=K}^{\infty} A_k \right) > 1 - \frac{c_d}{8} e^{-C_d 2^{\alpha}} \quad \text{and} \quad 2^{-K} \leq \left(\frac{t}{8q} \right)^{q+1}.
$$

As in the proof of Lemma 3.4, on the event $\bigcap_{k=K}^{\infty} A_k$ we have $\bar{h}_T(2^{-K}) \leq \frac{4q}{1-2^{\gamma-1}} 2^{(\gamma-1)(K-1)}$, which by increasing K if necessary we may assume is at most $t/8$. Now let $r = 2^{-K}$ and $\eta = r^{d/\alpha}$. By the second condition on K it is easy to check that $\eta \geq 8qr/t$ as required. Define

$$
A'_K = \{\exists x \in B_K : \xi_T(x) \in [\eta, 2\eta), \xi_T(y) \leq \eta/2 \ \forall y \in B_K \setminus \{x\}\}.
$$

Then

$$\mathbb{P}(A'_K) \geq c_d 2^{-dK} r(T)^d \big(\eta^{-\alpha} a(T)^{-\alpha} - (2\eta)^{-\alpha} a(T)^{-\alpha}\big)$$

$$\times \big(1 - (\eta/2)^{-\alpha} a(T)^{-\alpha}\big)^{C_d 2^{-dK} r(T)^d},$$

which for large T is at least

$$\frac{c_d}{4} r^d \eta^{-\alpha} \exp(-C_d 2^{\alpha} r^d \eta^{-\alpha}) = \frac{c_d}{4} e^{-C_d 2^{\alpha}},$$

by our choice of η. Thus

$$\mathbb{P}\left(A'_K \cap \bigcap_{k=K}^{\infty} A_k\right) > \frac{c_d}{8} e^{-C_d 2^{\alpha}}.$$

On the event $A'_K \cap \bigcap_{k=K}^{\infty} A_k$, conditions (A) and (C) are satisfied. Since the potential on $L_T(0, R) \setminus L_T(0, r)$ is independent of that on $L_T(0, r)$, and

$$\mathbb{P}\left(\xi_T(y) \leq \frac{\eta}{2} \ \forall y \in L_T(0, R) \setminus L_T(0, r)\right) \geq \left(1 - \left(\frac{\eta}{2}\right)^{-\alpha} a(T)^{-\alpha}\right)^{C_d R^d r(T)^d}$$

$$\geq c_{R,\eta}$$

for some constant $c_{R,\eta}$ depending on R and η, conditions (A), (B) and (C) occur simultaneously with probability at least $c_{R,\eta} c_d e^{-C_d 2^{\alpha}}/8$.

Since (S1), (S2) and (S3) only involve sites outside $L_T(0, R)$, they are independent of the events above, and so it suffices to show that for some $R > (\frac{2\eta t}{q}) \vee 3\kappa$, each occurs with positive probability. Note that for any $k \geq 1$,

$$\mathbb{P}\big(\exists y \in L_T\big(kR, (k+1)R\big) : \xi_T(y) \geq q(kR - r)/t\big)$$

$$\leq C_d\big((k+1)^d - k^d\big) R^d r(T)^d \big(q(kR - r)/t\big)^{-\alpha} a(T)^{-\alpha}$$

$$\leq C_d R^{d-\alpha} t^{\alpha} q^{-\alpha} \frac{((k+1)^d - k^d)}{(k - r/R)^{\alpha}}$$

$$\leq C_d d 2^{d+\alpha} R^{d-\alpha} t^{\alpha} q^{-\alpha} k^{d-1-\alpha}.$$

Thus

$$\mathbb{P}\big(\exists y \notin L_T(0, R) : \xi_T(y) \geq q(|y| - r)/t\big) \leq C_d d 2^d R^{d-\alpha} t^{\alpha} q^{-\alpha} \sum_{k=1}^{\infty} k^{d-1-\alpha},$$

which we can make arbitrarily small by choosing R large. This in particular establishes that (S1) occurs with positive probability. The fact that (S2) and (S3) each occurs with positive probability then follows by essentially repeating the calculation of $\mathbb{P}(A'_K)$ above. $\quad\square$

REMARK.   We could have proved Lemma 8.4 in a more elegant way by introducing a scaling limit for the potential as in [26], Section 2.2. We chose the more hands-on route in order to avoid introducing a new tool at the very end of the article.

PROOF OF THEOREM 1.5.   (i) Combining Lemmas 8.3 and 8.4 we know that with positive probability the support $s_T^{\mathrm{PAM}}(t)$ of the PAM lilypad model at time t is contained in two disjoint sets that are separated by distance at least $\frac{R}{2}$. Together with Theorem 1.4(iii) this implies that the actual support $S_T^{\mathrm{PAM}}(t)$ is also disconnected with positive probability.

(ii) By Lemma 8.4, there exists $\varepsilon > 0$ such that the probability that (A), (B), (C) and either (S1) or (S2) occurs is bounded below by ε.

From [19], Theorem 1.3, we know that with probability at least $1 - \varepsilon/4$, the PAM is concentrated in a single site which they call Z_{tT}, in the sense that

$$\frac{u(Z_{tT}, tT)}{\sum_{z \in \mathbb{Z}^d} u(z, tT)} \geq \frac{3}{4}.$$

This immediately implies that $u(\cdot, tT)$ is maximal in Z_{tT}, that is, in our notation that $W_T^{\mathrm{PAM}}(t) = Z_{tT}/r(T)$.

The site Z_{tT} is the maximizer of a functional $\Phi_{tT}(z)$ [defined in terms of $|z|$, the potential $\xi(z)$ and the number of paths leading to z]. Rather than stating the explicit definition, we recall the following simplfication. By [22], Lemma 3.3, we can choose N large enough such that with probability at least $1 - \varepsilon/4$, $\frac{Z_{tT}}{r(T)} \in L_T(0, N)$ and $\frac{\phi_{tT}(Z_{tT})}{a_T} \in [\frac{1}{N}, N]$. In particular, by [22], Lemma 3.2, we know that there exists a constant $C = C(N, q, t)$, such that for $z = \frac{Z_{tT}}{r(T)}$,

$$(12) \qquad \left| \frac{\Phi_{tT}(r_T z)}{a(T)} - \left(\xi_T(z) - \frac{q}{t}|z| \right) \right| \leq \frac{C \log \log T}{\log T}.$$

On the scenarios (A), (B), (C) and either (S1) or (S2), by the same argument (12) also holds for $z = w_T^{\mathrm{PAM}}(t)$. However, we have already seen in Lemma 8.2 that there exists $\delta > 0$ such that on either event (S1) or (S2), we have that for all $y \in L_T$,

$$t\xi_T(y) - q|y| \leq t\xi_T\big(w_T^{\mathrm{PAM}}(t)\big) - q\big|w_T^{\mathrm{PAM}}(t)\big| - \delta.$$

Comparing with (12) and using that Z_{tT} is the maximizer of Φ_{tT} shows that necessarily $w_T^{\mathrm{PAM}}(t) = Z_{tT}/r(T)$.

Moreover, the proof of Theorem 1.3 shows that with probability at least $1 - \varepsilon/4$, the maximizers of the lilypad model and the BRW are close, that is, $|w_T(t) - W_T(t)| \leq \frac{3}{q} \log^{-1/4}(T)$.

Hence, by combining all of the above, with probability at least $\varepsilon/4$ we have that $w_T^{\mathrm{PAM}}(t) = W_T^{\mathrm{PAM}}$ and $|w_T(t) - W_T(t)| \leq \frac{3}{q} \log^{-1/4}(T)$, while at the same time (A), (B), (C) and either (S1) or (S2) hold. The proof of statement (ii) is then completed by Lemma 8.2, which implies that on (S1), $|W_T(t) - W_T^{\mathrm{PAM}}(t)| \leq \frac{3}{q} \log^{-1/4} T$, and on (S2), $|W_T(t) - W_T^{\mathrm{PAM}}(t)| \geq \kappa$. □

Glossary of notation.

Notation	Definition/description
α	Constant such that $\mathbb{P}(\xi(0) > x) = x^{-\alpha}$ for $x \geq 1$
d	Dimension; we work in $\mathbb{Z}^d$ and $\mathbb{R}^d$
q	$d/(\alpha - d)$
$a(T)$	$(T/\log T)^q$ (rescaling of potential ξ)
$r(T)$	$(T/\log T)^{q+1}$ (spatial rescaling)
$\xi_T(z)$	$\xi(r(T)z)/a(T)$ (rescaled potential at z)
L_T	$\{z \in \mathbb{R}^d : r(T)z \in \mathbb{Z}^d\}$
$L_T(y, R)$	$L_T \cap B(y, R)$
$\lvert \cdot \rvert$	L^1 norm
$X(t)$	Random walk, independent of BRW and environment
$Y(t)$	Set of all particles at time t
$Y(z, t)$	Set of particles at z at time t
$N(t)$	#Y(t)
$N(z, t)$	#Y(z,t)
$H_T(z)$	Rescaled first time a particle hits z, $\inf\{t > 0 : N(r(T)z, tT) \geq 1\}$
$H_T^*(z)$	Rescaled first time X hits z, $\inf\{t > 0 : X(tT) = r(T)z\}$
$H_T'(z)$	Rescaled first time there are $\geq \exp(\log^{1/4})$ particles at z
$h_T(z)$	$\inf_{y_0, \ldots, y_n : y_0 = z, y_n = 0}(\sum_{j=1}^n q \frac{\lvert y_{j-1} - y_j \rvert}{\xi_T(y_j)}) \overset{z \neq 0}{=} \inf_{y \neq z}\{h_T(y) + q\frac{\lvert z - y \rvert}{\xi_T(y)}\}$
$M_T(z, t)$	Rescaled # particles at z at time t, $M_T(z, t) = \frac{1}{a(T)T} \log_+ N(r(T)z, tT)$
$m_T(z, t)$	$\sup_{y \in \mathbb{R}^d}\{\xi_T(y)(t - h_T(y)) - \lvert z - y \rvert\}$
$J_T(t, R)$	Probability X jumps $\geq Rr(T)$ times before tT
$\mathcal{E}_T^1(t, R)$	$\frac{R}{\log T}(\log R - \log t) + \frac{2dt}{a(T)}$ (usually small)
$\mathcal{E}_T^2(t, R)$	$\frac{R}{\log T}(\log t - \log R + 1 + \log(2d) + (q + 1)\log\log T)$ (usually small)
$\bar{\xi}_T(R)$	$\max_{y \in B(0, R)} \xi_T(y)$
$\bar{h}_T(R)$	$\max_{y \in B(0, R)} h_T(y)$
t_∞	Fixed time (we are usually only interested up to time t_∞)
Z	$\{z \in L_T(0, \rho_T) : \xi_T(z) > \bar{\xi}_T(\eta_T)\}$, where η_T is small
$\kappa(T)$	#Z
$z_1, \ldots, z_{\kappa(T)}$	Elements of Z in increasing order of ξ
Z'	$\{z \in L_T \setminus L_T(0, \rho_T) : \xi_T(z) > \bar{\xi}_T(\eta_T)\}$, where η_T is small
$z_{\kappa(T)+1}, \ldots$	Elements of Z' in arbitrary order
$t_1, t_2, \ldots$	Usually $t_i = \gamma_T h_T(z_i) - \delta_T$ where $\gamma_T \approx 1$ and δ_T is small
$A_T^*(j, z, t)$	$\left\{ \begin{matrix} H_T^*(z) \leq t, H_T^*(z_i) \geq H_T^*(z) \wedge t_i \ \forall i \leq j, \\ H_T^*(z_i) \geq H_T^*(z) \ \forall i > j \end{matrix} \right\}$
$G_T(j, z, s, t)$	$E^\xi[\exp(T \int_0^{H_T^*(z) \wedge s} \xi(X(uT))\, du)\mathbb{1}_{A_T^*(j,z,t)}]$
$\bar{G}_T$	$\max_{k \leq \kappa(T)} G_T(\kappa(T), z_k, t_k, t_\infty)$
μ_T	$\log^{1/4} T$

REFERENCES

[1] ALBEVERIO, S., BOGACHEV, L. V., MOLCHANOV, S. A. and YAROVAYA, E. B. (2000). Annealed moment Lyapunov exponents for a branching random walk in a homogeneous random branching environment. *Markov Process. Related Fields* **6** 473–516. MR1805091

[2] BAILLON, J.-B., CLÉMENT, PH., GREVEN, A. and DEN HOLLANDER, F. (1993). A variational approach to branching random walk in random environment. *Ann. Probab.* **21** 290–317. MR1207227

[3] BARTSCH, C., GANTERT, N. and KOCHLER, M. (2009). Survival and growth of a branching random walk in random environment. *Markov Process. Related Fields* **15** 525–548. MR2598127

[4] BIRKNER, M., GEIGER, J. and KERSTING, G. (2005). Branching processes in random environment—a view on critical and subcritical cases. In *Interacting Stochastic Systems* 269–291. Springer, Berlin. MR2118578

[5] COMETS, F., MENSHIKOV, M. V. and POPOV, S. YU. (1998). One-dimensional branching random walk in a random environment: A classification. *Markov Process. Related Fields* **4** 465–477. MR1677053

[6] COMETS, F. and POPOV, S. (2007). On multidimensional branching random walks in random environment. *Ann. Probab.* **35** 68–114. MR2303944

[7] COMETS, F. and POPOV, S. (2007). Shape and local growth for multidimensional branching random walks in random environment. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 273–299. MR2365644

[8] COMETS, F. and YOSHIDA, N. (2011). Branching random walks in space–time random environment: Survival probability, global and local growth rates. *J. Theoret. Probab.* **24** 657–687. MR2822477

[9] DAWSON, D. and FLEISCHMANN, K. (1983). On spatially homogeneous branching processes in a random environment. *Math. Nachr.* **113** 249–257. MR0725493

[10] GANTERT, N., MÜLLER, S., POPOV, S. and VACHKOVSKAIA, M. (2010). Survival of branching random walks in random environment. *J. Theoret. Probab.* **23** 1002–1014. MR2735734

[11] GÄRTNER, J. and KÖNIG, W. (2005). The parabolic Anderson model. In *Interacting Stochastic Systems* 153–179. Springer, Berlin. MR2118574

[12] GÄRTNER, J. and MOLCHANOV, S. A. (1990). Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.* **132** 613–655. MR1069840

[13] GREVEN, A. and DEN HOLLANDER, F. (1992). Branching random walk in random environment: Phase transitions for local and global growth rates. *Probab. Theory Related Fields* **91** 195–249. MR1147615

[14] GÜN, O., KÖNIG, W. and SEKULOVIĆ, O. (2013). Moment asymptotics for branching random walks in random environment. *Electron. J. Probab.* **18** no. 63, 18. MR3078022

[15] HARRIS, S. C. and ROBERTS, M. I. (2011). The many-to-few lemma and multiple spines. Preprint. Available at arXiv:1106.4761.

[16] HU, Y. and YOSHIDA, N. (2009). Localization for branching random walks in random environment. *Stochastic Process. Appl.* **119** 1632–1651. MR2513122

[17] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1969). Branching Markov processes. III. *J. Math. Kyoto Univ.* **9** 95–160. MR0246376

[18] KÖNIG, W. (2015). The parabolic Anderson model. Preprint. Available at www.wias-berlin.de/people/koenig/.

[19] KÖNIG, W., LACOIN, H., MÖRTERS, P. and SIDOROVA, N. (2009). A two cities theorem for the parabolic Anderson model. *Ann. Probab.* **37** 347–392. MR2489168

[20] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk*: *A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157

[21] MÖRTERS, P. (2011). The parabolic Anderson model with heavy-tailed potential. In *Surveys in Stochastic Processes* 67–85. Eur. Math. Soc., Zürich. MR2883854

[22] MÖRTERS, P., ORTGIESE, M. and SIDOROVA, N. (2011). Ageing in the parabolic Anderson model. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 969–1000. MR2884220

[23] MÜLLER, S. (2008). A criterion for transience of multidimensional branching random walk in random environment. *Electron. J. Probab.* **13** 1189–1202. MR2430704

[24] MÜLLER, S. (2008). Recurrence and transience for branching random walks in an I.I.D. random environment. *Markov Process. Related Fields* **14** 115–130. MR2433298

[25] SMITH, W. L. and WILKINSON, W. E. (1969). On branching processes in random environments. *Ann. Math. Statist.* **40** 814–827. MR0246380

[26] VAN DER HOFSTAD, R., MÖRTERS, P. and SIDOROVA, N. (2008). Weak and almost sure limits for the parabolic Anderson model with heavy tailed potentials. *Ann. Appl. Probab.* **18** 2450–2494. MR2474543

[27] WILKINSON, W. E. (1968). Branching processes in stochastic environments. PhD thesis, Univ. of North Carolina at Chapel Hill, 1967. MR2617450

[28] YOSHIDA, N. (2008). Central limit theorem for branching random walks in random environment. *Ann. Appl. Probab.* **18** 1619–1635. MR2434183

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