SCENERY RECONSTRUCTION IN TWO DIMENSIONS WITH MANY COLORS

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ABSTRACT. In [6] Kesten observed that the known reconstruction methods of random sceneries seem to strongly depend on the one dimensional setting of the problem and asked whether a construction still is possible in two dimensions. In this paper we answer the above question in the affirmative under the condition that the number of colors in the scenery is large enough.

1. INTRODUCTION AND THE MAIN RESULT

The following problem has its roots in ergodic theory but may also be considered interesting in its own rights. Consider a graph (V, E) and color its vertices in an arbitrary way (so we do not only concentrate on proper colorings in the strict sense that any two adjacent vertices need to have a different color). This coloring will be called a scenery on (V, E). Then we run a random walk on (V, E) of which we only know the color record (i.e. the sequence of colors it reads at the vertices) but not where it actually reads them. The question then is: Can we still say anything about how V was colored?

This problem – which at first glance might seem a bit hopeless – was first investigated independently by Benjamini and den Hollander and Keane [5]. From here the problem splits into basically three branches:

1. Can we distinguish two (known) sceneries by their random walk record? or, more ambitiously:
2. Can we even reconstruct (unknown) sceneries by the observations we obtain from a random walk? and:
3. Are their sceneries which cannot be reconstructed or distinguished by the color record of a random walk?

Basic answers to all of these three question have been already given while other aspects are still wide open. For example Benjamini and Kesten [1] discovered the very strong result that almost surely any two given sceneries on the integer lattices $\mathbb{Z}$ or $\mathbb{Z}^2$ can be distinguished by a simple random walk on these lattices given that the colors are selected by an i.i.d. process. Matzinger [9] showed that on $\mathbb{Z}$ even more is true: Almost every i.i.d. two-color scenery can be reconstructed from the color record of a simple random walk (which even might have non zero probability to stand still). This implies Benjamini's and Kesten's result in one dimension as well as the earlier observation by the same author that the same holds true for three and more colors [8]. However, notice that Benjamini's and Kesten's techniques also work in a two dimensional situation or when the random walk is allowed to jump.

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A remarkable answer to Question 3 has been given by Lindenstrauss [7] who showed that there are still uncountably many sceneries on $\mathbb{Z}$ which cannot be distinguished from the color record of a simple random walk.

To be more specific: In what follows (V, E) will always be the integer lattice $\mathbb{Z}^2$ and a function $\xi : \mathbb{Z}^2 \to \mathbb{Z}$ will be called a two dimensional scenery. For a subset $D \subset \mathbb{Z}^2$ we call $\xi : D \to \mathbb{Z}$ a piece of scenery. If the range of ξ contains exactly m elements we will say that ξ has m colors or that it is an m-color scenery. Two sceneries ξ and $\overline{\xi}$ will be called equivalent, if there are $a \in \mathbb{Z}^2$ and

$$M \in \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

such that

$$\xi(x) = \overline{\xi}(Mx + a) \qquad \forall x \in \mathbb{Z}^2.$$

Similarly, we call two pieces of scenery $\xi : D \to \mathbb{Z}$ and $\overline{\xi} : \overline{D} \to \mathbb{Z}$ equivalent, if again

$$\xi(x) = \overline{\xi}(Mx + a) \qquad \forall x \in D$$

holds true (a and M as above) and moreover $M(D) + a = \overline{D}$.

In other words ξ and $\overline{\xi}$ are equivalent (in symbols $\xi \sim \overline{\xi}$) if they can be obtained by translation and reflection on the coordinate axes from each other. It is rather obvious that in general we cannot expect to distinguish equivalent sceneries by their color record and thus also reconstruction will work only up to equivalence. Throughout this paper we will consider ξ's that result from an unbiased i.i.d. random process with m colors (thus we will also say that ξ has m colors), that is the $\xi(v)$ are i.i.d. for all $v \in \mathbb{Z}^2$ and

$$P(\xi(0) = i) = \frac{1}{m}$$

for all colors $i \in \{0, \ldots, m-1\}$. Moreover, let $(S_k)_{k \in \mathbb{N}}$ be simple random walk in two dimensions starting at the origin.

The main result of this paper states that if m is large enough the color record of (S_k), i.e.

$$\chi := (\chi(k))_{k \in \mathbb{N}} := (\xi(S_k))_{k \in \mathbb{N}}$$

contains enough information to reconstruct ξ almost surely up to equivalence. Additionally, we will present a well defined algorithm that given the scenery on a finite set reconstructs the whole scenery with probability larger than one half. In the next section we will see why this actually suffices to prove the main theorem. This, in a more mathematical way, is expressed in the following theorem, which states that with sufficiently many colors reconstruction of ξ from χ (up to equivalence) is possible with probability one.

Theorem 1.1. *There exists $m_0 \in \mathbb{N}$ such that if $m \geq m_0$, there exists a measurable function (with respect to the canonical σ-fields)*

$$\mathcal{A} : \{0, \ldots, m-1\}^{\mathbb{N}} \to \{0, \ldots, m-1\}^{\mathbb{Z}^2}$$

such that

$$\mathbb{P}(\mathcal{A}(\chi) \sim \xi) = 1. \tag{1.1}$$

Here the measure $\mathbb{P}$ lives on the product space of the outcomes of ξ and the space of all random walk paths.

Remark 1.2. *We have not calculated any lower bound for m_0 yet. We are also convinced that the methods presented here, will lead to an m_0 which is terribly large and far off any reasonable number and, in particular, any of the "borderline"-cases $m = 4, 5$ for which we have as many colors (or one more color, respectively) as we have neighbors in $\mathbb{Z}^2$ or even $m = 2$ (for which we doubt that Theorem 1.1 is valid). This is basically so, since we decided to keep the present proof as simple and transparent as possible and to use as many colors as necessary to this end. The specification of a good bound on m_0 will be subject to further research of the authors.*

This note has two further sections. In Section 2 we present the basic ideas of the algorithm used to reconstruct a random scenery while Section 3 contains the rigorous proof of Theorem 1.1.

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2. The Main Ideas and Basic Notations

The proof of Theorem 1.1 crucially bases on an induction argument. Given that we already know the scenery on a finite set A (for a special choice of A) we show how to extend this knowledge to the points sitting next to A. The following three lemmas are the building blocks of this induction. First we see that it suffices to exhibit an algorithm that reconstructs the scenery with probability larger than $1/2$ in order to be able to reconstruct the scenery almost surely.

Lemma 2.1. *For all $m \geq 2$ (where m designates the number of colors in ξ), if there exists a measurable map*

$$\overline{\mathcal{A}} : \{0, \dots, m-1\}^{\mathbb{N}} \to \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

such that

$$\mathbb{P}(\overline{\mathcal{A}}(\chi) \sim \xi) > 1/2$$

then there also exists a measurable

$$\mathcal{A} : \{0, \dots, m-1\}^{N} \to \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

with

$$\mathbb{P}(\mathcal{A}(\chi) \sim \xi) = 1.$$

The proof of Lemma 2.1 will be given in Section 3.
Lemma 2.1 will be useful, since we will soon see that with sufficiently many colors we are able to reconstruct with large probability the scenery on finite regions of $\mathbb{Z}^2$ such as the integer circle of radius n denoted by

$$B^n := \{x \in \mathbb{Z}^2 : ||x|| \leq n\}.$$

Here $|| \cdot ||$ stands for the Hilbert norm in $\mathbb{Z}^2$. Moreover, in the following we will frequently use the following notation: we will write $f|B$ for the restriction of f to a subset B of the domain of definition of f, for example $\xi|B$ will be a piece of scenery (that is the scenery restricted to some subset B of $\mathbb{Z}^2$), while $\chi|B$ will be a part of the observations (here B will be a subset of $\mathbb{N}$).

The next two lemmas will basically contain the induction. Lemma 2.2 below is the start of the induction, while Lemma 2.3 contains the induction step. So, first we show that we can reconstruct $\xi|B^n$ for each finite n with arbitrary large probability, as long as the scenery contains sufficiently many colors.

Lemma 2.2. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $m_1 \in \mathbb{N}$ such that if $m \geq m_1$ there exists a measurable function*

$$\mathcal{A}^n : \{0, \dots, m-1\}^{\mathbb{N}} \to \{0, \dots, m-1\}^{B^n}$$

such that

$$P(\mathcal{A}^n(\chi) \sim \xi|B^n) \geq 1 - \varepsilon.$$

Also Lemma 2.2 will be proven in the next section.

The next lemma is the induction step in the sense that it states that we can reconstruct $\xi|B^{n+1}$ with large probability provided we know $\xi|B^n$ up to equivalence and the number of colors is large enough.

Lemma 2.3. *There exists $m_2 \in \mathbb{N}$ such that for $m \geq m_2$ there is a sequence of measurable functions $(\tilde{\mathcal{A}}^n)_{n \in \mathbb{N}}$,*

$$\tilde{\mathcal{A}}^n : \{0, \dots, m-1\}^{B^n} \times \{0, \dots, m-1\}^{\mathbb{N}} \to \{0, \dots, m-1\}^{B^{n+1}}$$

such that given a sequence $(\psi^n)_{n \in \mathbb{N}}$ of pieces of scenery with

$$P(\xi|B^n \sim \psi^n) = 1,$$

then $\mathbb{P}$-a.s.

$$\tilde{\mathcal{A}}^n(\psi^n, \chi) \sim \xi|B^{n+1} \tag{2.1}$$

for all but finitely many n.

Roughly speaking, Lemma 2.3 means that the algorithm obtained by concatenating the different $\tilde{\mathcal{A}}^n$'s works well, in the sense that given $\xi|B^n$ and the observations χ it almost surely fails to reconstruct $\xi|B^{n+1}$ only for finitely many n.

To explain the proof of the induction step, which is crucial to the whole proof of Theorem 1.1, observe that the main difficulty in the reconstruction of sceneries is, of course, that we do not exactly know where the random walk precisely is. This is even more a problem in two dimensions than it is in one dimension as the random walk in one dimensions by time N has returned to the origin about $\sqrt{N}$ times, and therefore produces a lot of information about the neighborhood of the origin. In two dimensions the local time of the origin at time N is only about $\log N$. Thus we have to find an accurate method for guessing when the random walk is close to the origin from the observations χ it produces. This will be achieved by using a set of signal words, i.e. sequences of subsequent colors in B^n. Their frequent appearance in the observations will indicate that we really are in a neighborhood of B^n.

This "guessing that the random walk is inside B^n" is the first step of the reconstruction algorithm. More accurately, these words which will indicate that we are inside B^n (the so called *signal words*) are horizontal, non-overlapping words inside B^n of length proportional to $\log n$. The set of these words will be called $\mathcal{S}^n$. Whenever we read more than n^β words during a time interval of length n^2 whose endpoint is inside $[0, e^{n^\alpha}]$ (α and β some numbers to be specified later), we will "guess" that the walk is inside B^{n^2+n}. The union of these time intervals will be called τ^n and the reconstruction will only take place during τ^n. Note that τ^n designates a random set.

More formally in the sequel let $c_1, c_2, c_3 > 0$ be positive constants (not depending on n) which we will specify later. For convenience we will assume that $c_i \log n \in \mathbb{N}$ for each $i = 1, 2, 3$ (which of course means the c_i slightly depend on n but this dependence is irrelevant). Let

$$\begin{aligned}\mathcal{S}^n \;:=\; & \Big\{w = (w_1, \dots, w_{c_1 \log n}) \Big| \exists k \in \mathbb{Z} \text{ and } (x, y) \in \mathbb{Z}^2 : x = k c_1 \log n \\ & (x+s, y) \in B^n \text{ and } w_s = \xi((x+s, y)), \quad \forall 0 \leq s \leq c_1 \log n - 1\Big\}.\end{aligned}$$

In other words $\mathcal{S}^n$ "partitions" $\xi|B^n$ into disjoint horizontal words of length $c_1 \log n$. Moreover let $1 < \alpha < \beta < 2$ be two real numbers close to two to be specified later,

$$\begin{aligned}\mathcal{I}_{\alpha,\beta} \;:=\; & \{I = [t, t+n^2] \big| t \leq e^{n^\alpha} - n^2, \\ & \chi|I \text{ contains more than } n^\beta \text{ different words from } \mathcal{S}^n\},\end{aligned}$$

and

$$\tau^n := \tau^n_{\alpha,\beta} := \bigcup_{I \in \mathcal{I}_{\alpha,\beta}} I.$$

As sketched above, the point is that during the times $k \in \tau^n$ we can be pretty sure that the random walk is "close to B^n", more precisely that it is inside B^{n+n^2}. This will ensure that the reconstruction takes place at the boundary of B^n and not anywhere else.

As a matter of fact, the probability for the random walk to go right through a given signal word is equal to $(1/4)^{c_1 \log n}$. Thus for c_1 very small the random walk when being inside B^n typically reads $n^{2-\varepsilon_1}$ signal words during a time interval of length n^2. Here $\varepsilon_1 > 0$ can be made arbitrarily small. This is basically so, because the random walk typically visits about $n^2/\log n$ distinct points in a time window of length n^2, and thus during these time steps it would roughly visit about $n^2/\log n \times (1/4)^{c_1 \log n} \geq n^{2-\varepsilon_1}$ (for c_1 small enough) signal words.
Now, if the number of colors m is large enough we can choose c_1 small and still the signal words will be typical of B^n (that is, the probability to read them in a given ball $B_y^{n^2}$ – the ball of radius n^2 centered in y – is small, as long as the ball does not touch B^n). Indeed, there are less than $\pi n^4 4^{c_1 \log n}$ different paths of length $c_1 \log n$ inside $B_y^{n^2}$. Thus by independence the probability for a given signal word to appear in $B_y^{n^2} \setminus B^n$ is less than $\pi n^4 (4/m)^{c_1 \log n}$, which is as small as we want to, if only m is large enough. As a matter of fact, exploiting the independence of the signals in a large deviations argument we will be able to show, that up to time e^{n^α} the random walk in a time interval of length n^2 will only be able to read more than n^β $(\alpha, \beta$ as above) signal words if it spends this time in B^{n^2+n} and that the probability of reading so many signals elsewhere is about e^{-n^α}. So, our test, to check when we are back in B^n will not fail until time roughly $e^{n\alpha}$. But by that time we will have returned to the origin about $n^{\alpha-\varepsilon_2}$ times ($\varepsilon_2 > 0$, small). If now m were so large that there were only different colors inside B^n this would suffice to reconstruct ξ on the boundary of B^n. We simply would have to follow the walk until it exits B^n and read the first color outside as the color of a boundary point. If all colors were different, we would clearly know where this boundary point was. Moreover, there are order n points in ∂B^n, so $n^{1+\varepsilon_3}$ ($\varepsilon_3 > 0$) returns to the origin would suffice to

reconstruct the scenery on the boundary of B^n. As we already saw that we have about n^α of such returns, we would be done.
However, we are not allowed to choose m growing with n, so we cannot assume that all colors inside B^n are different. So we have to employ more subtle methods to reconstruct ξ on the boundary of B^n.

To describe this reconstruction part we have to introduce some more notations. Let

$$\underline{\partial} B^n := \{z \in B^n | \, \exists y \in \mathbb{Z}^2 \setminus B^n \text{ such that } z \text{ and } y \text{ are neighbors } \}$$

be the inner boundary of B^n and

$$\overline{\partial} B^n := B^{n+1} \setminus B^n$$

be its outer boundary (observe that $\overline{\partial} B^n$ may differ from the outer boundary of B^n in the lattice topology). Since by definition $B^n \cup \overline{\partial} B^n = B^{n+1}$ it clearly suffices to reconstruct $\overline{\partial} B^n$ with large enough probability.
The strategy will be to guess the color of a point v in $\overline{\partial} B^n$ by extending a walk to a neighboring point in $\underline{\partial} B^n$ by two further steps. Of course, we have to be very careful of both, to walk to $v \in \underline{\partial} B^n$ and to extend the walk into the right direction.

The principal idea behind this reconstruction can be described quite easily. Draw a straight (horizontal or vertical) line through v and suppose we knew already the the colors of a line segment of length approximately $\log n$ *inside* B^n and containing v as well as the colors of a line segment of about the same length *outside* B^n at distance 2 from v. Then we could figure out the two missing colors between these two segments by just waiting until the random walk first reads the colors of the segment inside B^n (in the right order) and then after a waiting time of 2 the colors of the segment outside B^n. Except, if the walk is far away from v (which we can exclude by the above arguments) the walk must have followed the straight line supporting the two segments at least partially and thus the missing two colors are the colors read between reading the colors of the two segments. Indeed, the "following partially" part above needs a little more technical work. In fact we could deviate from the above line segment and just accidentally read the right colors. We will get rid of this nuisance by characterising the missing two points as the shortest distance between two cones rather than between two line segments. This idea will be made more precise below.

Now a major difficulty is that we do not know the colors outside B^n. Thus we have to think of another characterisation of the segment outside B^n (supported by the same line as the inner segment). It will turn out that it is useful to think of it as the segment whose colors can be read in shorter time by starting with the inner segment than by starting with any segment parallel to it.

To formalise this idea for $v \in \underline{\partial} B^n$ we define a segment $\sigma(v)$ (the segment associated with v) in the following way: Let $\sigma(v)$ be the horizontal or vertical segment of length $(c_2 + c_3) \log n$ with endpoints v and $\sigma_0(v) \in B^n$, such that the angle between this segment and the tangent to the circle of radius $|v|$ centered in 0 in the point v is at least 45 degrees (the latter is needed to ensure that the objects below are well defined).
The first $c_2 \log n$ lattice points (starting from $\sigma_0(v)$) will be called the root segment of v and abbreviated by $\hat{\sigma}(v)$, the rest of $\sigma(v)$ is called second root segment and

will be denoted by the symbol $\overline{\sigma}(v)$, while the left and right neighboring segments of $\hat{\sigma}(v)$ of the same length $c_2 \log n$ as $\hat{\sigma}(v)$ (or the lower and upper segment next to the root segment of v, if $\sigma(v)$ is a horizontal segment, respectively) are named the side segments of v. For these we reserve the symbols $\lambda(v)$ and $\rho(v)$, and their starting points (next to $\sigma_0(v)$) are denoted by $\lambda_0(v)$ and $\rho_0(v)$, respectively. Finally the segment of length $c_2 \log n$ following $\sigma(v)$ after one step when we keep following the line supporting $\sigma(v)$ will be called the invisible segment associated with v and denoted by $\varphi(v)$. Its endpoints are called v_2 and $\varphi_0(v)$. The words associated with these segments will be called the root word, second root word, side words, and invisible words, respectively. Finally the lattice points we want to guess the color of, that is the points on $\varphi(v)$ of distance one and two to v are named v_1 and v_2. All this is illustrated in Figure 1 below.

Let us now describe how this reconstruction works.

The idea behind the above setup is that in order to read the color of v_1 and v_2 we take a neighboring vertex $v \in \underline{\partial} B^n$ and read the color of v_1 and v_2 as the next colors when we have read $\sigma(v)$ from $\sigma_0(v)$ to v. To guarantee that indeed we read the color of the right points we require that the algorithm picks a word w of length $c_2 \log n$ satisfying the following conditions

1. w appears in $\chi|\tau^n$ directly (one step) after the word supported by $\sigma(v)$.
2. In $\chi|\tau^n$ the shortest time for w to appear after the root word of v is exactly equal to $c_3 \log n + 1$.
3. In $\chi|\tau^n$ the shortest time for w to appear after the side word of v is exactly $c_3 \log n + 2$.

Condition 2 assures that we do not run backwards after having read the word supported by $\sigma(v)$ while Condition 3 guarantees that we have not deviated from the segment from $\sigma_0(v)$ to v while reading the scenery.
Thus we estimate $\xi(v_2)$ to be the first color of w. The estimate for $\xi(v_1)$ will be the the color between $\sigma(v)$ and w, when they appear in $\chi|[0, e^{n^\alpha}]$ one step apart from each other. If there is no word w satisfying the above conditions we let the algorithm terminate (our conditions imply that this will happen only with extremely small probability).

To realize this idea, that is to actually prove Theorem 1.1, we need some more definitions, which we will give now. For $v \in \underline{\partial} B^n$ the half space associated to v – which will be denoted by $\mathcal{H}(v)$ – is the half space separating $\hat{\sigma}(v)$ from $\overline{\sigma}(v)$ orthogonal to $\sigma(v)$ and with $\overline{\sigma}(v)$ in $\mathcal{H}(v)$. The first quart-space $Q_1(v)$ associated with v will be the right-angular cone based in v_2 with bisecting line along $\varphi(v)$ such that the major part of $\varphi(v)$ is inside this cone. The second quart-space $Q_2(v)$ associated with v is the right-angular cone based on the line separating $\mathcal{H}(v)$ from its complement such that $\hat{\sigma}(v)$ is on its bisecting line and $\hat{\sigma}(v)$ is in this cone. The third quart-space $Q_3(v)$ associated with v will be defined as the right-angular cone based on the line separating $\mathcal{H}(v)$ from its compliment such that $\lambda(v)$ is on its bisecting line and $\lambda(v)$ is in this cone. Finally, the fourth quart-space $Q_4(v)$ associated with v will be the right-angular cone based on the line separating $\mathcal{H}(v)$ from its compliment such that $\rho(v)$ is on its bisecting line and $\rho(v)$ is in this cone. The base points of Q_3, Q_2 and Q_4, respectively, are denoted by a, b, and c, respectively.

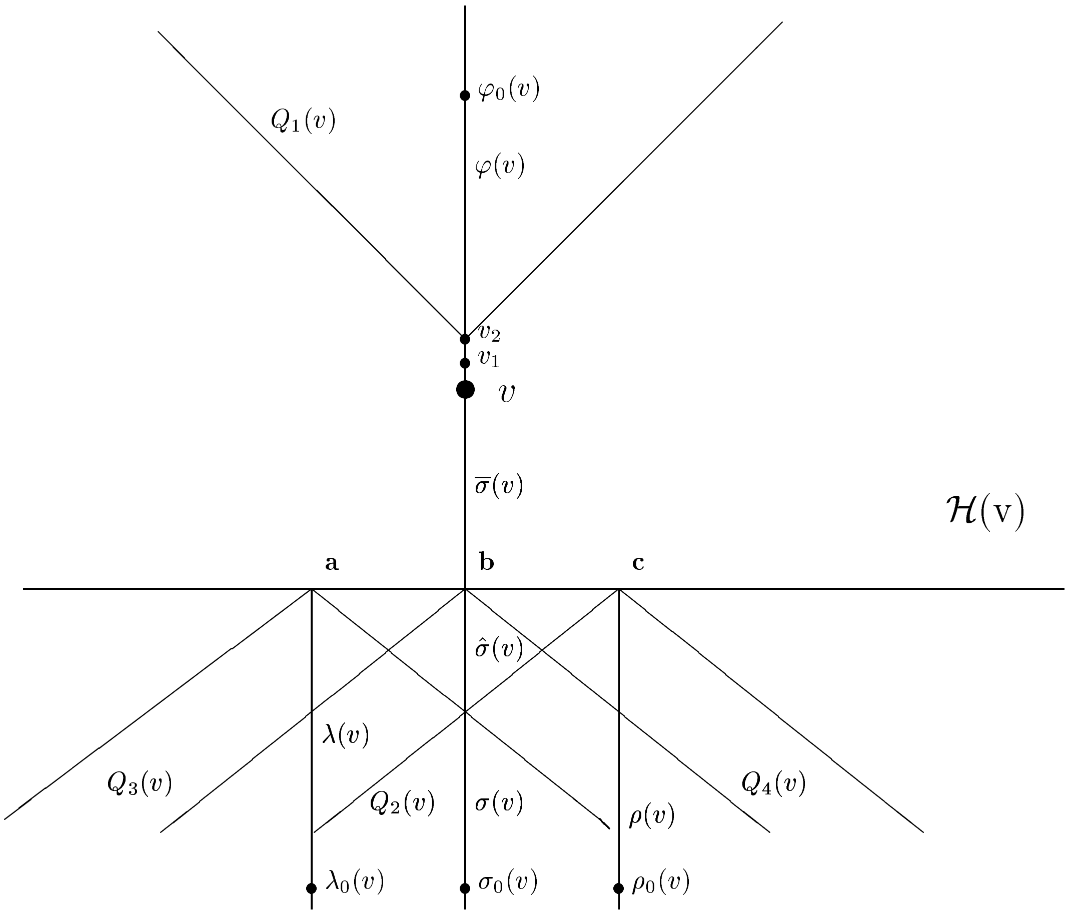

FIGURE 1

All this is illustrated in Figure 1. In this figure the points $v, a, b, c, \lambda_0(v), \sigma_0(v)$, and $\rho_0(v)$ are supposed to be inside B^n, whilst v_1, v_2, and $\rho_0(v)$ are supposed to be outside B^n.

3. PROOFS

In this section we give the proofs of Theorem 1.1 and Lemma 2.1, Lemma 2.2, and Lemma 2.3. Let us start with the proof of Lemma 2.1.

Proof of Lemma 2.1: Let Θ denote the right shift on $\mathbb{N}_0$ such that, if $\chi = (\chi(0), \chi(1), \chi(2), \dots)$,

$$\Theta^l(\chi) = (\chi(l), \chi(l+1), \dots)$$

for each $l \in \mathbb{N}$. Moreover, let $X(l)$ be the indicator for the event that the reconstruction algorithm $\overline{\mathcal{A}}$ applied to the observations shifted by l give rise to a scenery which is equivalent to the actual scenery, that is $X(l) = 1$ if $\overline{\mathcal{A}}(\Theta^l(\chi)) \sim \xi$ and $X(l) = 0$ otherwise. Obviously, $(X(l), l \in \mathbb{N})$ is stationary with

$$\mathbb{P}(X(l) = 1) = \mathbb{P}(\overline{\mathcal{A}}(\chi) \sim \xi) > \frac{1}{2}$$

for all l.

Furthermore, we show that $(X(l), l \in \mathbb{N})$ is also mixing. To this end observe that each $X(l)$ is measurable with respect to

$$\sigma(\xi(i - S_k), S_k, i \in \mathbb{Z}^2, k \geq l)$$

(as, of course, $(\xi(i))_{i \in \mathbb{Z}^2}$ is stationary). Consider the stationary sequence of couples $(Z(0), Z(1), Z(2), \ldots)$ where $Z(i) = (S_i, \xi(S_i))$. Finally for each

$$A \in \sigma(Z(0), Z(1), Z(2), \ldots)$$

denote by

$$A^s := \Theta^s(A).$$

Recall that we have to verify that

$$\lim_{s \to \infty} |\mathbb{P}(B \cap A^s) - \mathbb{P}(B)\mathbb{P}(A^s)| = 0$$

holds true for each each pair of measurable sets $A, B \in \sigma(\xi(i), S_k, i \in \mathbb{Z}^2, k \geq l) = \sigma(Z(0), Z(1), Z(2), \ldots)$.

Indeed, let $\varepsilon > 0$ be given. By a standard exercise in measure theory there exists an $n \in \mathbb{N}$ and an event

$$A_0 \in \sigma(\xi(i), S_k, i \in W_{2n}, 0 \leq k \leq n)$$

(where W_{2n} is the discrete cube of side length $2n$ in $\mathbb{Z}^2$ centered in the origin 0), such that

$$\mathbb{P}(A \Delta A_0) < \varepsilon$$

(where for two sets A, B we denote by $A \Delta B$ the symmteric difference between A and B).

Hence by stationarity there also exists

$$A_0^s \in \sigma(\xi(i), S_k, i \in W_{2n}^{S_s}, s \leq k \leq s + n)$$

(where $W_{2n}^{S_s}$ is the discrete cube of side length $2n$ in $\mathbb{Z}^2$ centered in S_s) such that

$$\mathbb{P}(A^s \Delta A_0^s) < \varepsilon.$$

By the same arguments, for a given set $B \in \sigma(\xi(i), S_k, i \in \mathbb{Z}^2, k \geq l)$ there exists $B_\varepsilon \in \sigma(\xi(i), S_k, i \in W_{2n'}, 0 \leq k \leq n')$ such that

$$\mathbb{P}(B \Delta B_\varepsilon) < \varepsilon.$$

Now observe that for any fixed measurable $B \in \sigma(\xi(i), S_k, i \in \mathbb{Z}^2, k \geq l)$

$$\begin{aligned} & \lim_{s \to \infty} |\mathbb{P}(A^s \cap B) - \mathbb{P}(A^s)\mathbb{P}(B)| \\ \leq \quad & \lim_{s \to \infty} |\mathbb{P}(A_0^s \cap B_\varepsilon) - \mathbb{P}((A_0^s)\mathbb{P}(B_\varepsilon)| + 4\varepsilon \\ \leq \quad & \lim_{s \to \infty} 2 \sum_{k=0}^{n} P(S_{s+k} \in W_{2n'}) + 4\varepsilon \end{aligned} \tag{3.1}$$

since the events A_0^s and B_ε are independent, if $S_{s+k} \notin W_{2n'}$ for all $k = 0, \ldots, n$. Now observe that the probability on the right hand side above converges to zero, that is

$$\lim_{s\to\infty} \sum_{k=0}^{n} P(S_{s+k} \in W_{2n'}) = 0$$

yielding that

$$\lim_{s\to\infty} |\mathbb{P}(A^s \cap B) - \mathbb{P}(A^s)\mathbb{P}(B)| \le 4\varepsilon.$$

As this is true for all $\varepsilon > 0$ we indeed obtain the mixing property

$$\lim_{s\to\infty} |\mathbb{P}(A^s \cap B) - \mathbb{P}(A^s)\mathbb{P}(B)| = 0$$

for the random variables $X(l)$.

Hence by the ergodic theorem

$$\frac{X(1) + X(2) + \ldots + X(l)}{l}$$

converges to a limit larger than 1/2 almost surely. Thus under the assumption that

$$P(\overline{\mathcal{A}}(\chi) \sim \xi) > \frac{1}{2}.$$

we can identify the equivalence class of ξ as the only equivalence class which eventually is equivalent to the majority of the $\overline{\mathcal{A}}(\Theta^l(\chi))$'s.

$\square$

Let us now prove Lemma 2.2.

Proof of Lemma 2.2: The principal idea behind the proof of Lemma 2.2 is that with enough colors within a large area a certain color is typical of the point underlying it. This will help us to reconstruct the scenery on two basic shapes, which will help to reconstruct the scenery on the points of a three by three square and hence also on any other square. In a final step we will see this already suffices to reconstruct the scenery within a large ball.
To be more precise let

$$E_{01}^n := \bigcap_{x\neq y\in B^n} \{\xi(x) \neq \xi(y)\},$$

and

$$E_{02}^n := \bigcap_{\substack{x,y\in B^n,\\ ||x-y||=1}} \quad \bigcap_{z\in B^n} \bigcap_{\substack{v\notin B^n:\\ \xi(v)=\xi(z)}} \{(S_k)_k \text{ passes from } x \text{ to } y \text{ in one step before visiting } v\}$$

In words the event E_{01}^n says that all colors inside B^n are different, while E_{02}^n states that all edges inside B^n are crossed by $(S_k)_{k\in\mathbb{N}}$ before it visits a point outside B^n having the same color as one of the points inside B^n.
We now show that under the condition that E_{01}^n and E_{02}^n hold true, we can reconstruct the scenery $\xi|B^n$. The reconstruction will be based on the following two important cases.

Case I: Let $x, y, z, v \in B^n$ be the corners of a unit square with x and z (and as well y and v) across the diagonal. Then, if E_{01}^n and E_{02}^n hold, and we know the colors of of x, y and z, we can figure out the color of v. As a matter of fact, the color of v is the first color appearing, neighboring both the color of x and the color of z, and different from the color of y. (Here and in the following we call two colors neighboring if they are read at subsequent times).

Case II: Let $x_1, x_2, x_3, x_4, y \in B^n$ be a "cross" with center y , that is x_1, x_2x_3, x_4, y are pairwise different and

$$|x_1 - y| = |x_2 - y| = |x_3 - y| = |x_4 - y| = 1.$$

Knowing that E^n_{01} and E^n_{02} hold as well as the colors of x_1, x_2, x_3 and y we can find out the color of x_4 as the only color neighboring $\xi(y)$ different from $\xi(x_1), \xi(x_2)$, and $\xi(x_3)$.

We will now see that these two basic techniques suffice to reconstruct $\xi|B^n$, if E^n_{01} and E^n_{02} hold. Indeed, denoting by Q_j the $2j+1$ by $2j+1$ square with center zero, we can first reconstruct $\xi|Q_1$. To this end we first recover the color of the origin (which is, or course, trivial) and the the colors of $(1,0), (0,1), (-1,0)$, and $(0,-1)$. Indeed, the colors themselves are known from the observations. Their relative position to each other (which is all we need, because we only want to reconstruct up to equivalence) can be detected from the fact that the color of y can be read with distance two of the color of x without reading the color of the origin in the meantime (and the same for the colors of v and z, for example), while this is not true for the colors of x and z, or the color of y and v, respectively, where we have to pass zero in between.
Once we know the $\xi|\{(1,0),(0,1),(-1,0),(0,-1)\}$ up to equivalence we can reconstruct the scenery on Q_1 by applying Case I to the four corner points of Q_1.
Now we can proceed inductively. Knowing the $\xi|Q_j \cap B^n$, we want to reconstruct $\xi|Q_{j+1} \cap B^n$, that is we want to find out the color of the boundary points of Q_{j+1} (as far as they are inside B^n). For all points with at least one coordinate different from $2j+1, 2j, -2j-1$, or $-2j$, this can be done by applying the technique of Case II. Then the color of the points with one coordinate equal to $2j$ or $-2j$ can be reconstructed by applying the technique of Case I. Finally the same technique yields the color of the corner points of Q_{j+1}.

This shows that under the condition that E^n_{01} and E^n_{02} hold true we can reconstruct $\xi|B^n$ up to equivalence. It remains to understand that both, E^n_{01} and E^n_{02} hold true with arbitrary large probability for fixed n and large enough m. Indeed, this is not very hard to see. For E^n_{01}, note that

$$\mathbb{P}\big((E^n_{01})^c\big) \le \text{const}\, n^2 \frac{1}{m},$$

which can be made arbitrarily small by choosing m large.
Similar techniques apply to E^n_{02}. Note that by taking T large enough the random walk $(S_k)_{k \le T}$ up to time T has visited each point in B^n, at least L times (L some number to be chosen soon, cf. [10] for similar results). Then the probability that there is an edge in B^n the random walk does not visit up to time T is bounded by

$$\text{const}\, n^2 \left(\frac{3}{4}\right)^L,$$

which is arbitrarily small for L large enough. If we now first choose L, the take T as above, and finally choose m so large that also the probability that all colors in

B^T are distinct (by the same techniques as above) is as large as we want to, we see that

$$\mathbb{P}\big((E_{02}^n)^c\big) \leq \varepsilon$$

for each $\varepsilon > 0$ if only m is large enough. This finishes the proof of Lemma 2.2. $\square$

Next we will prove Lemma 2.3, which is indeed the key ingredient to the proof of Theorem 1.1.

Proof of Lemma 2.3: Let E^n denote the event that given a piece of scenery ψ with $\psi \sim \xi|B^n$ the "reconstruction algorithm at step n" $\overline{\mathcal{A}}^n$ produces a piece of scenery $\overline{\mathcal{A}}^n(\psi,\chi)$ with

$$\overline{\mathcal{A}}^n(\psi,\chi) \sim \xi|B^{n+1}.$$

We need to show that with probability one E^n holds for all but a finite number of n's (in the following we will also say that an event holds for almost all n if it holds for all n but a finitely many).
To do so see we decompose E^n for $n \in \mathbb{N}$ in such a way that

$$E^n \supset E_1^n \cap E_2^n \cap E_3^n.$$

We will then show that each of E_i^n, $i = 1, 2, 3$ holds for all but finitely many n's.

Whenever in the sequel we will say about a piece of scenery ψ that "ψ appears in A with starting point x", or "ψ appears in A with endpoint y", respectively, where $\psi \in \{0, \ldots, m-1\}^l$ for some l, $A \subseteq \mathbb{Z}^2$, and $x, y \in \mathbb{Z}^2$, we will mean that

$$\chi|T = \psi$$

for some realisation of the random walk S_n, some discrete time interval $T = [t_0, t_0 + l - 1]$ such that $S_{t_0} = x$ (or $S_{t_0+l-1} = y$, respectively) and $S|T \subset A$. In other words ψ appears in A with starting point x (or endpoint y) if it can be read inside of A by a nearest neighbor walk starting in x (ending in y). Moreover if, for one of the line segments $\sigma(v), \hat{\sigma}(v), \overline{\sigma}(v), \varphi(v)$ or $\lambda(v)$, we refer to $\xi|\mathcal{L}$ ($\mathcal{L} \in \{\sigma(v), \hat{\sigma}(v), \overline{\sigma}(v), \varphi(v), \lambda(v)\}$) we mean the observations obtained by reading ξ along $\mathcal{L}$ from the center of B^n to the outside of B^n
Now let

$$E_1^n := \bigcap_{x \in B^{\exp(n^\alpha)}} \{\text{There are less than } n^\beta \text{ different word from } \mathcal{S}^n \text{ appearing in } \xi|(B_x^{n^2} \setminus B^n)\},$$

where $B_x^{n^2}$ stands for the discrete ball of radius n^2 centered in x.
Observe that the definition of τ^n implies that on E_1^n we have that $S_k \in B^{n+n^2}$ for all $k \in \tau^n$.
Moreover let

$$E_2^n = E_{21}^n \cap E_{22}^n \cap E_{23}^n \cap E_{24}^n \cap E_{25}^n$$

with

$$E_{21}^n := \bigcap_{v \in \underline{\partial} B^n} \bigcap_{\overline{\sigma}(v)} \{\xi|\overline{\sigma}(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with starting pointinside } \mathcal{H}(v)\},$$

$$E_{22}^n := \bigcap_{v \in \underline{\partial} B^n} \bigcap_{\hat{\sigma}(v)} \{\xi|\hat{\sigma}(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_2(v)\},$$

$$E^n_{23} := \bigcap_{v\in\underline{\partial}B^n}\bigcap_{\lambda(v)}\{\xi|\lambda(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_3(v)\},$$

$$E^n_{24} := \bigcap_{v\in\underline{\partial}B^n}\bigcap_{\rho(v)}\{\xi|\rho(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_4(v)\},$$

and

$$E^n_{25} := \bigcap_{v\in\underline{\partial}B^n}\bigcap_{\varphi(v)}\{\xi|\varphi(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with starting point } x \in Q_1(v)\}.$$

Finally let

$$E^n_3 = E^n_{3,\sigma} \cap E^n_{3,\lambda} \cap E^n_{3,\rho},$$

where

$$\begin{aligned} E^n_{3,\sigma} \;&:=\; \bigcap_{v\in\underline{\partial}B^n}\{ \text{ All nearest neighbor walks of length } (2c_2+c_3)\log n + 1 \\ &\qquad\qquad \text{initially reading } \xi|\hat{\sigma}(v) \text{ are realized at least once during } \tau(n)\}, \\ E^n_{3,\lambda} \;&:=\; \bigcap_{v\in\underline{\partial}B^n}\{ \text{ All nearest neighbor walks of length } (2c_2+c_3)\log n + 1 \\ &\qquad\qquad \text{initially reading } \xi|\lambda(v) \text{ are realized at least once during } \tau(n)\},\end{aligned}$$

and

$$\begin{aligned} E^n_{3,\rho} \;&:=\; \bigcap_{v\in\underline{\partial}B^n}\{ \text{ All nearest neighbor walks of length } (2c_2+c_3)\log n + 1 \\ &\qquad\qquad \text{initially reading } \xi|\rho(v) \text{ are realized at least once during } \tau(n)\}.\end{aligned}$$

Before we show that $E^n_1 \cap E^n_2 \cap E^n_3$ indeed happens for all but a finite number of n's, let us see that this will actually imply the desired result, that is, let us see, that

$$E^n \supset E^n_1 \cap E^n_2 \cap E^n_3.$$

As a matter of fact, for each event in E^n_1 we know that during $\tau(n)$ we must be close to B^n, more precisely, we know, that during $\tau(n)$ the walk is inside B^{n^2+n}. Then E^n_3 ensures that in this time $\tau(n)$ we read each sequence of length $(2c_2 + c_3)\log n + 1$ beginning with either $\xi|\hat{\sigma}(v)$, $\xi|\rho(v)$, or $\xi|\lambda(v)$ for each $v \in \underline{\partial}B^n$ at least once. E^n_2 now guarantees that during these times the walk is close to the points a, b, and c (of the appropriate v). Finally E^n_2 together with E^n_3 ensures some of the walks actually pass the points a, b, and c, correspondingly. Therefore, we are able to read the color of the vertices v_1 and v_2 next to v in direction of $\sigma(v)$.

To be more specific: For fixed v we divide the words of length $(2c_2 + c_3)\log n + 1$ beginning with one of the 3 sequences $\xi|\hat{\sigma}(v)$, $\xi|\lambda(v)$ and $\xi|\rho(v)$ into two categories, the words Σ which begin with $\xi|\hat{\sigma}(v)$, and the set $\overline{\Sigma}$ which don't. The words from Σ are the candidates for revealing the color of v_1 and v_2, while with the help of the words from $\overline{\Sigma}$ we keep control over actually reading the right colors. First we delete from Σ those words which do not continue with $\xi|\overline{\sigma}(v)$ at the $c_2 \log n + 1$'st to $(c_2 + c_3)\log n$'th step. For these words the walk obviously deviates from the straight line that supports $\sigma(v)$. Of course, we might also deviate from this line without producing a color record different from $\xi|\sigma(v)$. However, notice that such walks that deviate from $\sigma(v)$ have the property that after having $\xi|\hat{\sigma}(v)$ at the $c_2 \log n + 1$'st to $(c_2 + c_3)\log n$'th step it continues with a word which could have been read earlier when starting along $\lambda(v)$ or $\rho(v)$. To exclude these walks we cancel all walks from Σ which after the first $(c_2+c_3)\log n$ steps read a word which also can be read earlier by starting along $\rho(v)$ or $\lambda(v)$, that is by starting with a word from $\overline{\Sigma}$. Conditions $E^n_{3,\lambda}$ and $E^n_{3,\rho}$ ensure that $\overline{\Sigma}$ contains all such words we need for comparison. This obviously excludes all possibilities to deviate from the line supporting $\sigma(v)$. (In practise this procedure might be a bit tedious but it evidently works). Note that we also may delete some words from Σ which actually do not describe a deviation from the straight line underneath $\sigma(v)$ (since it might happen that the same sequence of colors occurs twice, once as a continuation of a walk starting from $\rho(v)$ or $\lambda(v)$ and once as a walk that indeed followed $\sigma(v)$. However, the condition associated with E^n_{25} guarantees that after the deletion there are still some walks left.

Finally, we also have to take care of not walking backwards after having read $\sigma(v)$. To this end consider the color record of the walk only from time $(c_2+c_3)\log n+2$ to time $(2c_2+c_3)\log n$. If these colors (in the same order) can be read *earlier* by starting with $\xi|\hat{\sigma}(v)$ (note that by condition E^n_3 we keep record of all such walks) we take this as an index for having stepped backwards. Indeed, all walks that after having read $\xi|\sigma(v)$ follow the line segment in the opposite direction for a while have this property. Again, we might also be deleting some admissible walks (that is, walks that actually go in the right direction) but condition E^n_3 together with E^n_{25} guarantees that we will not delete all walks form Σ. After these cancellations Σ will obviously only contain walks that follow $\sigma(v)$ and then step to v_1. In particular, their color record will be the same at the first $(c_2+c_3)\log+1$ positions and we will be able to read the color of v_1 as the $(c_2+c_3)\log+1$'st color of these records.

With the knowledge of $\xi(v_1)$ we may similarly delete all walks from Σ which do not step to v_2 in the $(c_2+c_3)\log+2$'nd step and thus we will also be able to obtain information about $\xi(v_2)$.

As this works for all $v\in\underline{\partial}B^n$ we are indeed able to reconstruct the scenery on B^{n+1} proving that

$$E^n\supset E^n_1\cap E^n_2\cup E^n_3.$$

It remains to show that $E^n_1\cap E^n_2\cap E^n_3$ is true for all but finitely many n, if we choose α and β in the correct manner.

E^n_1 **holds for all but finitely many** n: Let $\omega\in\mathcal{S}^n$ be any fixed signal word in B^n. By this we mean that ω is the signal word between two fixed starting points; so note that ω although being fixed in this sense, will still be random. Let $y\notin B^n$ be any potential starting point for ω outside B^n. By independence of the colors

$$\mathbb{P}(\omega\text{ appears in }\xi|(\mathbb{Z}^2\setminus B^n)\text{ with starting point }y)\le\left(\frac{4}{m}\right)^{c_1\log n}$$

as there are $4^{c_1\log n}$ different walks of length $c_1\log n$ starting in y. Thus for any y

$$\mathbb{P}(\omega\text{ appears in }\xi|(B^{n^2}_y\setminus B^n))\le\pi n^4\left(\frac{4}{m}\right)^{c_1\log n}=\pi n^{4+c_1(\log 4-\log m)}$$

as there are πn^4 different points inside B^{n^2}.

Now the indicators I_w for the event that the word $w \in \mathcal{S}^n$ appears in $B_y^{n^2} \setminus B^n$ are conditionally independent (for different w) under $\mathbb{P}$ given $\xi|(B_y^{n^2} \setminus B^n)$ as the different words have mutually disjoint support and therefore are independent. To understand this point correctly it is of utmost importance to recall that $\mathcal{S}^n$ is a random set (under $\mathbb{P}$). The independence claimed above would not be true for any fixed set of words or if we did not condition on knowing $\xi|(B_y^{n^2} \setminus B^n)$.

Hence the number of $w \in \mathcal{S}^n$ appearing in $B_y^{n^2} \setminus B^n$ is stochastically bounded by a Binomial random variable with $N = n^2/c_1 \log n$ different trials and success probability $p = \pi n^{2+c_1(\log 4 - \log k)}$. By concentration of measure, (or the simple fact the the rate function of a large or moderate deviation principle for i.i.d. Bernoullis can be quadratically bounded), cf. [12], for N i.i.d. Bernoullis X_i with success probability p we have that

$$P(\sum_{i=1}^{N} X_i \geq Np + \Delta) \leq e^{-const.\frac{\Delta^2}{N}}$$

for each N and each $\Delta > 0$. Applying this to our situation and moreover choosing m in such a way that

$$\pi n^{2+c_1(\log 4 - \log m)} \leq 1/n$$

(which is possible for every fixed c_1) yields

$$\begin{aligned} &\mathbb{P}(\text{There are more than } n^\beta \text{ different words from } \mathcal{S}^n \text{ in } \xi|B_y^{n^2} \setminus B_n) \\ \leq\ & e^{-const.\frac{n^2}{c_1 \log n}(\frac{n^\beta c_1 \log n}{n^2} - \frac{1}{n})^2} = e^{-const.\frac{(n^\beta - n)^2}{c_1 n^2 \log n}} \\ \leq\ & \mathrm{e}^{-n^{2(\beta-1)-\varepsilon}}, \end{aligned}$$

for each positive $\varepsilon > 0$ and n large enough. Hence

$$\mathbb{P}((E_1^n)^c) \leq \sum_{x \in B^{\exp(n^\alpha)}} e^{-n^{2(\beta-1)-\varepsilon}} = \pi e^{2n^\alpha} e^{-n^{2(\beta-1)-\varepsilon}}. \tag{3.2}$$

As $\alpha < \beta$, and we can choose them such that $2\beta - 2 > \alpha + \varepsilon$ the right hand side of (3.2) clearly is summable in n, which by the Borel–Cantelli Lemma yields that E_1^n holds for all but finitely many n.

E_2^n **holds for all but a finite number of** n**:** Since the proofs of that E_{2i}^n holds for almost all n are very similar for each i, we just show it for E_{22}^n and leave the other proofs to the reader.

To this end consider any $v \in \underline{\partial} B^n$ and any oriented connected segment s in $\mathbb{Z}^2$ of length $c_2 \log n$. Note that if the endpoint of s is not in Q_2 the i'th point of $\hat{\sigma}(v)$ is different from all the j'th points of s, $j \leq i$, and thus $\xi(\hat{\sigma}_i(v))$ is a "fresh random variable. Thus by conditional independence the probability of reading $\hat{\sigma}(v)$ along $\xi|s$ is bounded by

$$\mathbb{P}(\xi|s = \hat{\sigma}(v)) \leq \left(\frac{1}{m}\right)^{c_2 \log n},$$

and therefore for every fixed $x \in B^{n^2+n} \setminus Q_2(v)$

$$\mathbb{P}(\hat{\sigma}(v) \text{ appears with endpoint } x) \leq \left(\frac{4}{m}\right)^{c_2 \log n}.$$

As there are at most $\pi(n^2+n)^2$ points in B^{n^2+n} and there are at most $const \times n$ points $v \in \underline{\partial} B^n$, we obtain

$$\mathbb{P}((E_{22}^n)^c) \leq \left(\frac{4}{m}\right)^{c_2 \log n} const \times n\pi(n^2+n)^2 = n^{c_2(\log 4 - \log m)}$$

The right hand side of this inequality becomes summable if we choose m large enough (depending on c_2 – or c_3 when we want to prove that another E_{2i}^n holds). This choice of m will basically be the proof of Theorem 1.1. Thus (again by a Borel–Cantelli argument) E_{22}^n holds true for all but finitely many n.

Note that until now we are still free to choose c_1, c_2, c_3.

E_3^n holds for all but finitely many n: Again we only give the proof in detail for one of the events, which will be $E_{3,\sigma}^n$. The proof for the other two events follows the same lines.

We split this proof into several parts.
First let us prove that in a certain (stricter than usual) sense the random walk by time e^{n^α} has returned to the origin more than n^γ times, where $\gamma < \alpha < \beta$. A result like this seems to be very much in the spirit of a result by Erdös and Taylor [2], who showed that almost surely a random walk at time e^n has returned to the origin between $n/(\log n)^{1+\varepsilon}$ and $(1+\varepsilon)n \log\log n$ times for all but finitely many n's and every positive $\varepsilon > 0$. The reason why we cannot simply refer to this result is that we also want these returns to the origin to be apart at least n^2 from each other. So, more precisely let us introduce a sequence ϑ_i^n of stopping times such that $\vartheta_0^n = 0$ for all n and ϑ_{i+1}^n is the time of the first return of the random walk S_k to the origin after time $\vartheta_i + n^2$. This will ensure that in the meantime the random walk is able to hit one of the boundary points of B^n. So we want to check that for $\gamma < \alpha < \beta$ (γ appropriately chosen afterwards) the event

$$E_{31}^n := \{\vartheta_{n^\gamma}^n \leq e^{n^\alpha}\}$$

happens for all but finitely many n's. Indeed, choosing $\delta = \frac{\alpha-\gamma}{2}$ the result by Erdös and Taylor [2] quoted above states that the event

$$E_{311}^n := \{ \text{ Up to time } e^{n^\alpha} \text{ there are more than } n^{\gamma+\delta} \text{ returns to the origin } \}$$

holds true almost surely for all but a finite number of n's. Next we will show that the same is true for the event

$$E_{312}^n := \bigcap_{i=1}^{n^\gamma} \{\text{In the interval } [\vartheta_i^n, \vartheta_i^n + n^2] \text{ there are less than } n^\delta \text{ returns to the origin } \}.$$

As a matter of fact the probability for a simple random walk to start in the origin and not to return to it for t steps is bounded below by $\frac{2\pi}{\log t}$ for t large enough [11, p.167], [3]. Applying this yields

$$\begin{aligned} &\mathbb{P}(\text{In the interval } [\vartheta_i^n, \vartheta_i^n + n^2] \text{ there are more than } n^\delta \text{ returns to the origin }) \\ \leq\ & \left(1 - \frac{\pi}{\log n}\right)^{n^\delta} \leq e^{-n^{\delta/2}} \end{aligned}$$

for each $i = 1, \dots, n^\gamma$ and n large enough. Hence by bounding the probability of a union by the sum of the probabilities

$$\mathbb{P}((E^n_{312})^c) \le n^\gamma e^{-n^{\delta/2}}$$

which is finitely summable. Therefore E^n_{312} holds for all but a finite number of n's. As E^n_{311} and E^n_{312} together imply E^n_{31} also E^n_{31} holds for almost all n.

Next we will show that many of the intervals $[\vartheta^n_i, \vartheta^n_i + n^2]$ above are indeed signal times, that is we will show that we read more than n^β different signals in all of these time intervals. To this end introduce random variables Y_i which are indicators for the event that the interval $[\vartheta^n_i, \vartheta^n_i + n^2]$ is a signal time, that is for the event that there are more than n^β signal words read in $[\vartheta^n_i, \vartheta^n_i + n^2]$. To avoid the dependence among reading different signal words we only concentrate on such words which are "far apart" form each other. To this end we partition the inner part of B^n, that is $B^n \setminus \partial_{(\log n)^3} B^n$ where

$$\partial_{(\log n)^3} B^n := \{x \in B^n, d(x, \underline{\partial} B^n) \le (\log n)^3\}$$

and $d(\cdot,\cdot)$ is the lattice distance in $\mathbb{Z}^2$, into boxes of lengths $c_1 \log n$ and $(\log n)^3$. Let

$$W^n_{k,l} := \{(x,y) \in B^n : k \log n \le x \le (k+1)\log n, l\,(\log n)^3 \le y < (l+1)\,(\log n)^3\},$$

$(k, l \in \mathbb{Z})$.

Now consider the following indicators: Let $\mathbb{I}^{1,n}(i)$ be the indicator for the event that $S_{\vartheta^n_i + n^{\frac{4+\beta}{3}}} \in B^{n/\log n}$. $\mathbb{I}^{2,n}(i)$ denotes the indicator for the event that the whole trajectory $(S_k)_{k=\vartheta^n_i,\dots,\vartheta^n_i + n^{\frac{4+\beta}{3}}}$ is contained in B^n. Furthermore, let $\mathbb{I}^{3,n}(i)$ be one if the random walk visits more than $n^{\frac{2(1+\beta)}{3}}/(\frac{2(1+\beta)}{3}\log n)$ distinct points in $[\vartheta^n_i, \vartheta^n_i + n^{\frac{4+\beta}{3}}]$ and zero otherwise.
Moreover let $\mathbb{I}^{4,n}_{k,l}(i)$ be the indicator for the event that in the time interval $[\vartheta^n_i, \vartheta^n_i + n^{\frac{4+\beta}{3}}]$ the walk enters $W^n_{k,l}$ and within $(\log n)^3$ steps after the first entrance time touches one of the lines $x = k \log n$ or $x = (k+1)\log n$, and finally follows the straight line supporting the the word associated with the starting point it touched.

First consider the event $\{\mathbb{I}^{1,n}(i) = 0\}$. By concentration of measure (cf. [12]) we have for every fixed i

$$\mathbb{P}(\mathbb{I}^{1,n}(i) = 0) = \mathbb{P}(||S_{\vartheta^n_i + n^{\frac{4+\beta}{3}}}|| \ge \frac{n}{\log n}) \le e^{-const. \frac{n^{\frac{2-\beta}{3}}}{(\log n)^2}}.$$

Therefore, as $\beta < 2$

$$\mathbb{P}\left(\Big(\bigcap_i \{\mathbb{I}^{1,n}(i) = 1\}\Big)^c \right) \le n^2 e^{-const. \frac{n^{\frac{2-\beta}{3}}}{(\log n)^2}}$$

which is finitely summable and thus $\bigcap_i \{\mathbb{I}^{2,n}(i) = 1\}$ holds true for almost all n.

By the same argument

$$\begin{aligned} \mathbb{P}(\mathbb{I}^{2,n}(i) = 0) &= \mathbb{P}(\exists\, t \in [\vartheta^n_i, \vartheta^n_i + n^{\frac{4+\beta}{3}}] : ||S_t|| \ge n) \\ &\le n^2 \mathbb{P}(||S_{\vartheta^n_i + n^{\frac{4+\beta}{3}}}|| \ge n) \le n^2 e^{-const. \frac{n^{2-\beta}}{3}}. \end{aligned}$$

Thus, also $\bigcap_i \{\mathbb{I}^{2,n}(i) = 0\}$ holds true for all but finitely many n.

To bound the probability that $\mathbb{I}^{3,n}(i)$ is equal to zero, first observe that the number of distinct points D_t visited by a simple symmetric random walk starting in the origin by time t satisfies (cf.[4], [3])

$$\mathbb{E}D_t \geq \frac{2t}{\log t}$$

for all t large enough. Moreover such a random walk clearly can only have visited at most t points (i.e. $D_t \leq t$) up to time t. Together this implies

$$P(D_t \geq \frac{t}{\log t}) \leq \frac{1}{\log t}. \tag{3.3}$$

Partitioning the interval $[\vartheta_i^n, \vartheta_i + n^{\frac{4+\beta}{3}}]$ into $n^{\frac{2-\beta}{3}}$ intervals of length $n^{2\frac{1+\beta}{3}}$ and applying (3.3) with $t = n^{2\frac{1+\beta}{3}}$ (observe that $\log t = 2\frac{1+\beta}{3}\log n$) yields for any fixed i:

$$P(\mathbb{I}^{3,n}(i) = 0) \leq \left(1 - \frac{const.}{\log n}\right)^{n^{\frac{2-\beta}{3}}} \leq e^{-const.\frac{n^{\frac{2-\beta}{3}}}{\log n}}.$$

Hence by the same summability argument as above $\bigcap_i \{\mathbb{I}^{3,n}(i) = 1\}$ holds for almost all n.

Next let us have a closer look at $\{\mathbb{I}^{4,n}_{k,l}(i) = 1\}$. Suppose that we already know that S_n enters the sector $W^n_{k,l}$ within $[\vartheta_i^n, \vartheta_i^n + n^{\frac{4+\beta}{3}}]$. Considering just the projection of the walk to the x-axes, we see a nearest neighbor random walk on $\mathbb{Z}$ with holding probability 1/2. The points $k \log n$ and $(k+1)\log n$ obtained by projecting the vertical limiting lines of $W^n_{k,l}$ may be considered absorbing barriers for this random walk. As the expected hitting time of one of these barriers is of order $(\log n)^2$, after time $(\log n)^3$ we will have hit one of the boundaries with a probability bounded away from zero (in n). In other words that is to say, that S_n conditioned on that it will visit $W^n_{k,l}$ at all, will touch one of its left and right boundary lines within $(\log n)^3$ after the first entrance time into this sector with probability bounded away from zero. As the word associated to this boundary point has length $c_1 \log n$ the probability that the walk touches a boundary point and then follows the walk associated to it is bounded by $const.(1/4)^{c_1 \log n}$.

Note that the events $\{\mathbb{I}^{4,n}_{k,l}(i) = 1\}$ are not independent for different choices of (k,l) and the same i and n. First due to the fact that (S_k) is a Markov chain the event $\{\mathbb{I}^{4,n}_{k,l}(i) = 1\}$ increases the chances that we also hit a square close to $W^n_{k,l}$. However, also given that we visit both $W^n_{k,l}$ and $W^n_{(k+1),l}$ for example, the events $\{\mathbb{I}^{4,n}_{k,l}(i) = 1\}$ and $\{\mathbb{I}^{4,n}_{k+1,l}(i) = 1\}$ are dependent since reading a word associated with a boundary point of $W^n_{k,n}$ might easily coincide with touching a boundary point of $W^n_{k+1,n}$ less than $(\log n)^3$ steps after the first entrance time. To cope with this effect we disregard every other square, that is we consider the indicators

$$\hat{\mathbb{I}}^{4,n}_{k,l}(i) := \mathbb{I}^{4,n}_{k,l}(i)\mathbb{I}(k,l)$$

where $\mathbb{I}(k,l)$ is $+1$ if k and l are even and 0 otherwise, instead.

Now observe that on $\{\mathbb{I}^{2,n}(i) = 1\} \cap \{\mathbb{I}^{3,n}(i) = 1\}$ the random walk visits more than $n^{2\frac{1+\beta}{3}} / \frac{2(1+\beta)}{3} \log n$ distinct points within $[\vartheta_i^n, \vartheta_i^n + n^{\frac{4+\beta}{3}}]$ – all of them lying in B^n –

and therefore, as each of the $W^n_{k,l}$ has $c_1(\log n)^4$ points, also $n^{2\frac{1+\beta}{3}}/(2c_1\frac{1+\beta}{3}(\log n)^5)$ distinct $W^n_{k,l}$'s. As one fourth of them will have both k and l even $\hat{\mathbb{I}}^{4,n}_{k,l}(i)$ has a chance to become $+1$ for $n^{2\frac{1+\beta}{3}}/(8c_1\frac{1+\beta}{3}(\log n)^5)$ different choices of (k,l). Given the indices (k,l) for which this is true the events $\{\hat{\mathbb{I}}^{4,n}_{k,l}(i)=1\}$ indeed are independent and have probability at least $const.(1/4)^{c_1\log n}$. Hence again by moderate deviations or concentration of measure on $\{\mathbb{I}^{2,n}(i)=1\}\cap\{\mathbb{I}^{3,n}(i)=1\}$

$$\mathbb{P}(\sum_{k,l}\hat{\mathbb{I}}^{4,n}_{k,l}(i)\le n^\beta)\le\exp\left(-const.\frac{n^{\frac{1}{3}(2-\beta)-c_1\log 4}}{(\log n)^{10}}\right)\le e^{-n^\varepsilon}$$

for some small ε, if c_1 is small enough (depending on how large we have chosen β before). As e^{-n^ε} is finitely summable even after multiplication with the number of different ϑ^n_i we obtain that on the event $\bigcap_i\{\{\mathbb{I}^{2,n}(i)=1\}\cap\{\mathbb{I}^{3,n}(i)=1\}\}$ we have $\sum_{k,l}\hat{\mathbb{I}}^{4,n}_{k,l}(i)\ge n^\beta$ for all i and all but finitely many n's. As also $\bigcap_i\{\{\mathbb{I}^{2,n}(i)=1\}\cap\{\mathbb{I}^{3,n}(i)=1\}\}$ holds for almost all n

$$\sum_{k,l}\hat{\mathbb{I}}^{4,n}_{k,l}(i)\ge n^\beta$$

also is true for almost all n. As finally also $\bigcap_i\{\mathbb{I}^{1,n}(i)=1\}$ for all but a finite number of n's, we arrive at

$$\bigcap_i\{\{Y_i=1\}\cap\{\mathbb{I}^{1,n}(i)=1\}\}$$

for all n but finitely many.

Let us summarise what we know already. For almost all n the following holds true: Until time e^{n^α} we have more than n^γ (γ smaller than α) different intervals of length n^2 of signal times. The signals are read in the first $n^{\frac{4+\beta}{3}}$ steps, after which the random walks stops in a distance at most $n/\log n$ from the origin.

Finally we have to show that in these time intervals $[\vartheta^n_i,\vartheta^n_i+n^2]$ we also read all words of length $(2c_2+c_3)\log n$ beginning with either a root word or a side word associated to any of the boundary points. To avoid trouble with independence we will only concentrate on events where this happens in one of the time intervals $J^n_i:=[\vartheta^n_i+n^{\frac{4+\beta}{3}},\vartheta^n_i+n^2], i=1,2,\ldots$.
To this end, first observe that (S_k) on J^n_i has variance

$$\sqrt{n^2-n^{\frac{4+\beta}{3}}}\ge\frac{n}{2}$$

for n large enough. Therefore, and since "in the worst case" $S_{n^{\frac{4+\beta}{3}}}=0$ with positive probability bounded away from zero (S_k) exits B^n during J_i. This bound will be used to estimate the probability to hit the beginning $\sigma_0(v)$ of a root word for a boundary point $v\in\underline{\partial}B^n$ or the beginning of one of its side words. This probability can be computed as the probability of hitting this point conditioned on that we hit the (discrete) sphere it is contained in, times the probability that we hit this sphere at all. The latter probability is bounded below by a constant away from zero, by the above considerations. On the other hand the probability to hit a certain point in $\underline{\partial}B^n$ conditioned on that we leave B^n is bounded below by $\frac{\varkappa}{n}$ for some constant $\varkappa>0$ no matter where in $B^{n/\log n}$ we started. Of course, it suffices to understand that this is true for large n. But observing that under the scaling $\mathbb{Z}^2 \to \frac{1}{n}\mathbb{Z}^2$, the boundary $\underline{\partial} B^n$ converges to the unit sphere, $B^{n/\log n}$ shrinks to the origin and (S_k) converges (after rescaling also the time axes which is irrelevant for our argument) to Brownian motion $W^0(t)$ starting in the origin and moreover taking into account that the harmonic measure on the unit sphere (any sphere centered in zero) with respect to $W^0(t)$ is the uniform distribution on it, shows that the above bound indeed holds. So, as all starting points of root words and side words lie in $B^{n^2} \setminus B^{n^2-(c_2+c_3)\log n}$ we see that the probability of hitting any fixed starting point is bounded from below by $\frac{\varkappa'}{n}$ for some $\varkappa' > 0$ ($\varkappa'$ results from multiplying $\varkappa$ with the probability of exiting B^{n^2} in a certain J_i).

Now the probability of reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2+c_3)\log n$ given that we first read $\sigma_0(v)$ has (for any fixed $v \in \underline{\partial} B^n$) probability

$$\left(\frac{1}{4}\right)^{(2c_2+c_3)\log n} = n^{-(2c_2+c_3)\log 4}.$$

So the (unconditioned) probability of reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2+c_3)\log n$ is bounded below by

$$\frac{\varkappa}{n}\left(\frac{1}{4}\right)^{(2c_2+c_3)\log n} = \varkappa n^{-1-(2c_2+c_3)\log 4}.$$

On the other hand there are n^γ different time intervals where we can read such a word. So the probability of not reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2+c_3)\log n$ in all of these intervals behaves like

$$\left(1 - \varkappa n^{-1-(2c_2+c_3)\log 4}\right)^{n^\gamma} \leq \exp\left(-\varkappa n^{\gamma-1-(2c_2+c_3)\log 4}\right).$$

As we can choose c_2 and c_3 as small as we want to and $\gamma > 1$ (and still γ, α) this probability is smaller than e^{-n^ε} for some $\varepsilon > 0$. The same holds true for the probability of reading a side word and then any fixed continuation of length $(c_2+c_3)\log n$ given that we read its first letter. As for fixed n there are only polynomially many of such words (more precisely, as there less than

$$6\pi n 4^{(c_2+c_3)\log n} = 6\pi n^{1+(c_2+c_3)\log 4}$$

such words) the probability of not reading all of them is bounded by

$$6\pi n^{1+(c_2+c_3)\log 4} e^{-n^\varepsilon}$$

which is finitely summable in n. Therefore, by the Borel-Cantelli Lemma, also E_n^3 holds for all but finitely many n's. This finishes the proof of Lemma 2.3. □

The proof of the main theorem now only consists of choosing the constants in the correct order.

Proof of Theorem 1.1: To finish the proof we finally specify the order in which we choose the constants. So first we choose α, β, γ with $2\beta - 2 > \alpha$ (such that right hand side in (3.2) is finitely summable), and $1 < \gamma < \alpha$. Then we choose c_1, c_2 and c_3 to make the last part of the above proof of Lemma 2.3 work (note that this part does not depend on the number of colors m). If we now choose m larger than a certain number m_1 (coming from the arguments which guarantee that E_1^n and E_2^n hold), this procedure ensures that the reconstruction in Lemma 2.3 works with probability one for almost all n. Call the largest n for which it does not work N. Then according to Lemma 2.2 for each $\varepsilon > 0$ there is a number m_2 and a reconstruction algorithm $\mathcal{A}^N$ such that if $m \geq m_2$ we can reconstruct $\xi|B^N$ with probability larger than $1-\varepsilon$. If we now choose $m \geq \max\{m_1, m_2\}$ and concatenate $\mathcal{A}^N$ from Lemma 2.2 with $\mathcal{A}^n$ for $n \geq N+1$ from Lemma 2.3, we obtain an algorithm $\mathcal{A}$ which reconstructs ξ with probability larger than $1-\varepsilon$.

In view cf Lemma 2.1 this suffices to prove Theorem 1.1. □

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