FLUCTUATIONS OF THE OVERLAP IN THE HOPFIELD MODEL AT THE CRITICAL TEMPERATURE

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ABSTRACT. We investigate the behavior of the Hopfield model of spin glasses and neural networks with finitely many patterns at the critical (inverse) temperature $\beta_c = 1$. We investigate the measure-valued random variables given by the distribution of the scaled overlap under the Gibbs measure and show their weak convergence towards a limiting distribution which is non-Gaussian in the sense that it is not given by a Dirac measure concentrated in a Gaussian distribution.
Moreover, we analyze the behavior of the model with variable temperature converging to one and see that there is a threshold for the speed of convergence. If β_N converges to β_c much faster than $\frac{1}{\sqrt{N}}$ the non–Gaussian behavior of the overlap persists, whereas for a slower speed of convergence the order parameter (appropriately scaled) obeys a Central Limit Theorem.

1. Introduction and Description of the Model

The Hopfield model, together with the Sherrington–Kirkpatrick model of spin glasses, is one of the most popular models of disordered spin systems. The origin of the interest in it is at least twofold. On the one hand it can be regarded as a very simple model of a brain, hence as a neural network. This point of view has been taken e.g. in the original paper by Hopfield [Ho82], as well as in the papers [MPRV87], [N88], [KP88], [Lou97], [Lö96] and many others. On the other hand the Hopfield model may also be regarded as a spin glass with the additional feature that the complexity of the disorder can be adjusted by an additional parameter, the number of patterns. This has been the motive in a variety of papers including [FP77], where the Hopfield model has been introduced, and [ST92], [BGP94], [BGP95a], [BGP95b], [G96a], [G96c], [BG96a], [BG96b], [BG96c], [T96], and [BG97], to list some of the papers in this area (of course, this list is far from being complete).

To be more specific let us define the Hopfield model. First of all let us fix a number $M \in \mathbb{N}$ which will denote the number of so–called patterns, which other than in the most general set–up of the Hopfield model will not depend on the number of spins or "neurons", N, in this paper. By

$$H_N(\sigma) := -\frac{1}{2N} \sum_{\mu=1}^{M} \sum_{i,j=1}^{N} \sigma_i \sigma_j \xi_i^\mu \xi_j^\mu \tag{1}$$

we will denote the (random) Hamiltonian of the Hopfield model, which we will regard as a function of the spin configuration $\sigma \in \{-1,+1\}^N$. The variable $\xi_i^\mu \in \{-1,+1\}$ stands for the i'th component of the μ'th pattern. We will assume that the $\xi_i^\mu \in \{-1,+1\}$ are i.i.d. unbiased random variables, i.e. that

$$P(\xi_i^\mu = +1) = P(\xi_i^\mu = -1) = \frac{1}{2}$$

as well as

$$P(\sigma_i = +1) = P(\sigma_i = -1) = \frac{1}{2}$$

independent of all the other ξ_j^ν and σ_i. With ξ we will denote the matrix of the $(\xi_i^\mu)_{i,\mu}$, while $\xi_i = (\xi_i^1, \dots, \xi_i^M)$ and $\xi^\mu = (\xi_1^\mu, \dots, \xi_N^\mu)$, respectively, stand for the i'th row and μ'th column of this matrix, respectively. $\mathbb{P}$ and P_σ, resp., symbolize for the distributions with respect to ξ and σ, resp., $\mathbb{E}$ and E_σ are the corresponding expectations.
The Hopfield model may now either be identified with the Hamiltonian (1) or, equivalently, with the Gibbs measure

$$\varrho_{N,\beta}(\sigma) := \varrho_{N,\beta,\xi}(\sigma) := \frac{e^{-\beta H_N(\sigma)}}{Z_{N,\beta}} \tag{2}$$

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where the so–called partition function

$$Z_{N,\beta} := Z_{N,\beta,\xi} := \sum_{\sigma} e^{-\beta H_N(\sigma)}$$

is the appropriate normalization to make $\varrho_{N,\beta}$ a probability measure and $\beta > 0$ denotes the inverse temperature.
As we will have to deal with the case of an external magnetic field we also introduce the measures

$$\varrho_{N,\beta,\tilde{h}}(\sigma) := \varrho_{N,\beta,\xi,\tilde{h}}(\sigma) := \frac{e^{-\beta H_N(\sigma)+\langle \tilde{h}, m_N(\sigma)\rangle}}{Z_{N,\beta,\tilde{h}}} \tag{3}$$

where $\tilde{h} \in \mathbb{R}^M$, $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^M$ and $Z_{N,\beta,\tilde{h}}$ is chosen in such a way that $\varrho_{N,\beta,\tilde{h}}$ becomes a probability measure.

Note that the Hamiltonian (1) may be rewritten in the following very convenient form

$$H_N(\sigma) = -\frac{N}{2}||m_N(\sigma)||_2^2 \tag{4}$$

where

$$m_N(\sigma) := (m_N^\mu(\sigma))_{\mu=1,\dots,M} := (\sum_{i=1}^N \sigma_i \xi_i^\mu)_{\mu=1,\dots,M} \tag{5}$$

is the vector of overlaps or overlap vector and $||\cdot||_2$ is the Euclidean norm in $\mathbb{R}^M$. (4) shows that m_N is an important quantity for the study of the Hopfield model, a so–called order parameter, and motivates that its distribution under $\varrho_{N,\beta}$ has been of major interest in a couple of recent papers.

Nevertheless, the investigation of the thermodynamics of the Hopfield model originally started with the analysis of the free energy of this model, i.e. with the limits

$$\lim_{N\to\infty} \mathbb{E}\frac{1}{N}\log Z_{N,\beta,\xi}$$

and

$$\lim_{N\to\infty} \frac{1}{N}\log Z_{N,\beta,\xi},$$

(see [ATZ92] and the references given therein). In [BGP94] the distribution of the overlap under $\varrho_{N,\beta,\tilde{h}}$ was investigated. They showed that (even under the weaker assumption $M/N \to 0$) for almost all realizations of ξ the distribution of the overlap $m_N(\sigma)$ under the Gibbs measure ϱ_{N,β,he_1} converges weakly – as first $N \to \infty$ and then $h \to 0\pm$ – towards the Dirac measure $\delta_{a^\pm(\beta)e_1}$ concentrated in $a^\pm(\beta)e_1$, that means they proved a law of large numbers for these distributions. Here $h \in \mathbb{R}$, e_1 denotes the first unit vector of the canonical basis in $\mathbb{R}^M$ and $a^+(\beta)$ resp. $a^-(\beta)$ are the largest (resp. smallest) root of the Curie-Weiss equation

$$a = \tanh(\beta a)$$

(this especially implies that $a^+(\beta) = a^-(\beta) = 0$ for $\beta \le \beta_c = 1$).
This weak law also extends to the more general case that M depends on N in such a way that $M(N)/N \to 0$ as $N \to \infty$. Note that it strongly resembles the law of large numbers for the magnetization in the Curie–Weiss model (see [E85]). This is, of course, not accidental. Indeed, for $M = 1$ the Hopfield model and the Curie-Weiss model agree (by a simple change of variables) and the weak law for the Hopfield model may be summarized by saying that for M growing slowly enough (as a function of N) the Hopfield model almost surely behaves like an overlap of M orthogonal Curie-Weiss models.
However, on the scale of fluctuations, i.e. when analyzing the distribution of $\sqrt{N}m_N(\sigma)$ the character of the disorder becomes visible even when the number of patterns remains bounded (in N). Indeed, as has been shown by Gentz [G96a],[G96c] and Bovier and Gayrard [BG96c] for M not growing too fast ($M/N \to 0$ suffices) and $(\beta, h) \neq (1, 0)$ the overlap vector satisfies a Central Limit Theorem (CLT) with the covariance matrix that could be expected from the analogy with the Curie–Weiss model and a centring that for the case $\beta > \beta_c = 1$ or $h \neq 0$ has a random component depending on the realization of the matrix ξ.

In this paper we are going to investigate the fluctuations of the overlap vector at the critical temperature $\beta = 1$. Recall that in the Curie–Weiss model the criticality of $\beta = 1$ can be seen in various ways, one of them being the breakdown of the CLT. As a matter of fact at this temperature the magnetization of the Curie–Weiss model – scaled by a factor $N^{1/4}$ – converges weakly to a random variable with distribution proportional to $\exp(-\frac{1}{12}x^4)$ with respect to Lebesgue measure.

Due to the analogy described above one would expect a similar behavior for the Hopfield model. On the other hand observe that, other than the magnetization in the Curie–Weiss model, the quantities $m_N(\sigma)$ depend on an extra random parameter and that therefore limit theorems can only be stated with respect to this additional randomness. Indeed, it turns out that the right point of view seems to be to consider the random distributions of the overlap under the Gibbs measures at temperature one, i.e.

$$\varrho_{N,1,\xi}(N^{1/4}m_N(\sigma) = \cdot\,)$$

as random variables taking values in the set of probaility measures on $\mathbb{R}^M$ and to show their weak convergence (with respect to $\mathbb{P}$) to a random distribution with density (with respect to M-dimensional Lebesgue measure) proportional to

$$\exp(\sum_{\mu\neq\nu}\frac{1}{2}\eta_{\mu,\nu}x_\mu x_\nu - \frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 - \frac{1}{12}\sum_{\mu=1}^M x_\mu^4)$$

where η is distributed according to $\mathcal{N}(0,\Sigma)$, a $M(M-1)$–dimensional Gaussian distribution with mean zero and covariance matrix $\Sigma = (\sigma_{(\mu,\nu),(\mu',\nu')})$. Here

$$\sigma_{(\mu,\nu),(\mu',\nu')} = \begin{cases} 1 & \text{if } (\mu,\nu) = (\mu',\nu') \\ & \text{or } (\mu,\nu) = (\nu',\mu') \\ 0 & \text{otherwise.} \end{cases}$$

Especially, this means that also at $\beta = 1$ the fluctuations of the overlap show the influence of the disorder. The methods of proof will basically be a Hubbard–Stratonovich transform of the measure of interest together with a Taylor expansion of the resulting density.
Of course, a result valid for almost all realizations of ξ would be more in the spirit of the other results on the Hopfield model. On the other hand, some recent discussion for the Hopfield model (see e.g. [BG96b, Section 8] might indicate that an almost sure result is a stroke of luck and that usually all one can expect is convergence in distribution.

Moreover it will be interesting to ask what happens, if β depends on N in such a way that β_N converges to 1. Depending on the speed of convergence we will either have a limit of the above form or a Gaussian limit or a mixture of both.

This paper has two more sections. Section 2 contains explicit statements of the results described above. Section 3 is devoted to the proofs.

2. Statement of the Results

In this section we will give a precise statement of our results for the Hopfield model at the critical temperature.
The case which has always been in the center of our interest is the case where $\beta = 1$ is fixed.
Note that the measures of interest $\varrho_{N,1,\xi}\circ(N^{1/4}m_N(\sigma))^{-1}$ are random variables (in ξ) with values in the space of measures on $\mathbb{R}^M$, i.e. in $\mathcal{M}_1(\mathbb{R}^M)$ endowed with the Prohorov metric.

Theorem 2.1. *Let $\beta = \beta_c = 1$. Then*

$$\mathbb{P}Q_{N,\xi}^{-1} := \mathbb{P}\circ(\varrho_{N,1}\circ(N^{1/4}m_N(\sigma))^{-1})^{-1} \Rightarrow \mathbb{P}\circ Q_\eta^{-1}. \tag{6}$$

Here '$\Rightarrow$' denotes weak convergence and Q_η is a random probability measure on $\mathbb{R}^M$ with density with respect to M-dimensional Lebesgue measure

$$\frac{\exp(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu - \frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 - \frac{1}{12}\sum_{\mu=1}^M x_\mu^4)}{\Xi} \tag{7}$$

where

$$\Xi := \int_{\mathbb{R}^M}\int \exp(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu - \frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 - \frac{1}{12}\sum_{\mu=1}^M x_\mu^4)d^M(x).$$

and finally $\eta := (\eta_{\mu,\nu})_{\mu\neq\nu=1}^M$ is a random vector distributed according to $\mathbb{P} := \mathcal{N}(0,\Sigma)$ where $\mathcal{N}(0,\Sigma)$ is a $M(M-1)$–dimensional symmetric Gaussian distribution with mean zero and covariance matrix $\Sigma = (\sigma_{(\mu,\nu),(\mu',\nu')})$. The σ's are given by

$$\sigma_{(\mu,\nu),(\mu',\nu')} = \begin{cases} 1 & \textit{if } (\mu,\nu) = (\mu',\nu') \\ & \textit{or } (\mu,\nu) = (\nu',\mu') \\ 0 & \textit{otherwise.} \end{cases} \tag{8}$$

Remark 2.2. *Note that the scaling factor $N^{1/4}$ for the overlap vector is the same as the one for the magnetization in the Curie–Weiss model at the critical temperature. Notice that also similar to this case we have a non–Gaussian limit distribution.*

As it turns out Theorem 2.1 above can be regarded as a general case of the following theorem treating the case of variable β. To clarify notations for two sequences (a_N) and (b_N) let us write $a_N \ll b_N$ if $a_N/b_N \to 0$ and $a_N \gg b_N$ if $b_N \ll a_N$.

Theorem 2.3. *Let β_N depend on N such that $\beta_N \to 1$. Then:*

1. *If $|1-\beta_N| \ll 1/\sqrt{N}$, then again*
$$\mathbb{P}Q_{N,\xi}^{-1} \Rightarrow \mathbb{P}\circ Q_\eta^{-1}, \tag{9}$$
where η and Q_η are defined as in Theorem 2.1.
2. *If $1-\beta_N \gg 1/\sqrt{N}$ the measures*
$$\mathbb{P}\circ(\varrho_{N,\beta_N}\circ(\sqrt{N(1-\beta_N)}m_N(\sigma))^{-1})^{-1} \tag{10}$$
satisfy the CLT. More precisely the measures given by (10) converge weakly to the Dirac measure concentrated in the M-dimensional Normal Law with identity covariance matrix.
If moreover
$$\sum_{N=1}^{\infty}\frac{1}{N^2(1-\beta_N)^2}<\infty$$
we also have that the measures
$$\varrho_{N,\beta_N}(\sqrt{N(1-\beta_N)}m_N(\sigma)\in\cdot\,) \tag{11}$$
satisfy the CLT $\mathbb{P}$-almost-surely, i.e. they converge weakly to a M-dimensional Normal Law with identity covariance matrix.
3. *If $|(1-\beta_N)| = \frac{\gamma}{\sqrt{N}}$ then*
$$\mathbb{P}\circ Q_{N,\xi}^{-1} \Rightarrow \mathbb{P}\circ\tilde{Q}_\eta^{-1}$$
where η and $\mathbb{P}$ are as above and $\tilde{Q}_\eta$ is a random measure with a density with respect to M-dimensional Lebesgue measure given by
$$\frac{\exp(\frac{1}{2}\gamma\sum_{\mu=1}^M x_\mu^2+\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2-\frac{1}{12}\sum_{\mu=1}^M x_\mu^4)}{\Xi'}$$
where Ξ' is the corresponding normalizing factor
$$\Xi' := \int_{\mathbb{R}^M}\exp(\frac{1}{2}\gamma\sum_{\mu=1}^M x_\mu^2+\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2-\frac{1}{12}\sum_{\mu=1}^M x_\mu^4)d^M(x).$$
and the $\eta_{\mu,\nu}$ are Gaussian random variables as in Theorem 2.1.

3. Proofs

This section contains the proofs of the results presented in the previous section.
As already mentioned in the introduction the core of the proofs consists of an analysis of the Hubbard–Stratonovich transform [H59],[S58] of the measures in question. So let us first of all calculate these Hubbard–Stratonovich transforms. Similar calculations can also be found in [BGP94], Lemma 2.2, or [EN78], Lemma 3.3.

Lemma 3.1. *Let $\mathcal{N}(0,\frac{a_N}{\beta}Id)$ (to avoid unnecessary complications $1/N \le a_N \le 1$) denote an M–dimensional Gaussian distribution with mean zero and covariance matrix $\frac{a_N}{\beta}Id$ (Id being the $M\times M$ identity matrix) independent of all the ξ_i^μ and σ. For $\beta>0$ denote by $\chi_{N,\beta,a_N,\xi}$ the convolution of our measures of interest with an M-dimensional Gaussian measure. More precisely*
$$\chi_{N,\beta,a_N} := \chi_{N,\beta,a_N,\xi} := \varrho_{N,\beta,\xi}\circ(\sqrt{a_N N}m_N(\sigma))^{-1}*\mathcal{N}(0,\frac{a_N}{\beta}). \tag{12}$$
Then χ_{N,β,a_N} possesses a density $f_{N,a_N}(x)$ with respect to M–dimensional Lebesgue measure and
$$f_{N,a_N}(x)=\frac{\exp(-\beta N\Phi_{N,\beta}(\frac{x}{\sqrt{Na_N}}))}{\int\exp(-\beta N\Phi_{N,\beta}(\frac{x}{\sqrt{Na_N}}))d\lambda^M(x)} \tag{13}$$

where

$$\Phi_{N,\beta}(x) := \Phi_{N,\beta,\xi}(x) := \frac{1}{2}||x||_2^2 - \frac{1}{\beta N}\sum_{i=1}^N \log\cosh(\beta < \xi_i, x >). \tag{14}$$

Here $< \cdot, \cdot >$ *stands for the inner product in* $\mathbb{R}^M$.

Proof. The proof follows the ideas of the proof Lemma 2.2 in [BGP94] which is a special case of our Lemma 3.1 (the case $a_N = \frac{1}{N}$). So let A be a Borel subset of $\mathbb{R}^M$. Then

$$\begin{aligned}
&\chi_{N,\beta,a_N}(A)\\
= \;& \varrho_{N,\beta}(\sqrt{a_N N} m_N(\sigma))^{-1} * \mathcal{N}(0, \frac{a_N}{\beta} Id)(A)\\
= \;& \sum_{\sigma\in\{0,1\}^N} \mathcal{N}(0, \frac{a_N}{\beta} Id)(A - \sqrt{a_N N} m_N(\sigma))\varrho_{N,\beta}(\sigma)\\
= \;& \sum_{\sigma\in\{0,1\}^N} \left(\frac{\beta}{2\pi a_N}\right)^{M/2} \int_A \exp\left(-\frac{\beta}{2a_N}||x-\sqrt{a_N N}m_N(\sigma)||^2\right)\varrho_{N,\beta}(\sigma) d\lambda^M(x)\\
= \;& \frac{1}{2^N}\sum_{\sigma\in\{0,1\}^N}\left(\frac{\beta}{2\pi a_N}\right)^{M/2}\frac{1}{Z_{N,\beta}}\int_A \exp\left(-\frac{\beta}{2a_N}\sum_{\mu=1}^M (x_\mu - \sqrt{a_N N} m_N^\mu(\sigma))^2\right)\\
& \times \exp\left(\frac{\beta N}{2}\sum_{\mu=1}^M m_N^\mu(\sigma))^2\right) d\lambda^M(x)\\
= \;& \frac{1}{Z_{N,\beta}}\left(\frac{\beta}{2\pi a_N}\right)^{M/2}\int_A \exp\left(-\frac{\beta}{2a_N}\sum_{\mu=1}^M (x_\mu)^2\right)\\
& \times \frac{1}{2^N}\sum_{\sigma\in\{0,1\}^N}\exp\left(\frac{\beta\sqrt{N}}{\sqrt{a}_N}\sum_{\mu=1}^M m_N^\mu(\sigma)x_\mu\right) d\lambda^M(x)\\
= \;& \frac{1}{Z_{N,\beta}}\left(\frac{\beta}{2\pi a_N}\right)^{M/2}\int_A \exp\left(-\frac{\beta}{2a_N}\sum_{\mu=1}^M (x_\mu)^2\right)\\
& \times \frac{1}{2^N}\sum_{\sigma\in\{0,1\}^N}\exp\left(\frac{\beta}{\sqrt{a_N N}}\sum_{\mu=1}^M\sum_{i=1}^N \xi_i^\mu \sigma_i x_\mu\right) d\lambda^M(x)\\
= \;& \frac{1}{Z_{N,\beta}}\left(\frac{\beta}{2\pi a_N}\right)^{M/2}\int \mathbb{I}_A(x)\exp\left(-\frac{\beta}{2a_N}\sum_{\mu=1}^M (x_\mu)^2\right)\\
& \quad \exp\left(\sum_{i=1}^N \log\cosh\left(\frac{\beta}{\sqrt{a_N N}}\sum_{\mu=1}^M \xi_i^\mu x_\mu\right)\right) d\lambda^M(x)\\
= \;& \frac{1}{Z_{N,\beta}}\left(\frac{\beta}{2\pi a_N}\right)^{M/2}\int \mathbb{I}_A(x)\exp\left(-\beta N \Phi_{N,\beta}(\frac{x}{\sqrt{N a_N}})\right) d\lambda^M(x).
\end{aligned}$$

The fact that χ_{N,β,a_N} is a probability measure implies that

$$Z_{N,\beta} = \left(\frac{\beta}{2\pi a_N}\right)^{M/2}\int_{\mathbb{R}^M}\exp\left(-\beta N\Phi_{N,\beta}(\frac{x}{\sqrt{N a_N}})\right) d\lambda^M(x)$$

and proves the lemma. ☐

Let us next analyze the relation between the measures of interest and the corresponding Hubbard–Stratonovich transforms.

Lemma 3.2. *If we denote by* w-lim *the weak limit of a sequence of measures then*

1. *If* $a_N \downarrow 0$ *and* β_N *is bounded away form zero, then* $\text{w-lim}_{N\to\infty}\varrho_{N,\beta_N}\circ(\sqrt{a_N N}m_N(\sigma))^{-1}$ *exists if and only if* $\text{w-lim}_{N\to\infty}\chi_{N,\beta_N,a_N}$ *exists and then*

$$\underset{N\to\infty}{\text{w-lim}}\,\chi_{N,\beta_N,a_N} = \underset{N\to\infty}{\text{w-lim}}\,\varrho_{N,\beta_N}\circ(\sqrt{a_N N}m_N(\sigma))^{-1} \tag{15}$$

2. *Assume $a_N/\beta_N \downarrow 0$. Then* $\text{w-lim}_{N\to\infty} \mathbb{P}(\varrho_{N,\beta_N,\xi}(\sqrt{a_N N} m_N(\sigma))^{-1})^{-1}$ *exists if and only if* $\text{w-lim}_{N\to\infty} \mathbb{P}\chi^{-1}_{N,\beta_N,a_N}$ *exists, and if one of the limits exists, then*
$$\underset{N\to\infty}{\text{w-lim}}\, \mathbb{P}(\varrho_{N,\beta_N,\xi}(\sqrt{a_N N} m_N(\sigma))^{-1})^{-1} = \underset{N\to\infty}{\text{w-lim}}\, \mathbb{P}\chi^{-1}_{N,\beta_N,a_N}. \tag{16}$$

Proof. 1. is standard by Fourier transforms, since the measures $\mathcal{N}(0, \frac{a_N}{\beta})$ weakly converge to the Dirac measure concentrated in zero, if $a_N \to 0$.

2. By Lemma 3.3 below it is sufficient to show
$$\Big|\int g\Big(\int f\, d\varrho_{N,\beta_N,\xi}(\sqrt{a_N N} m_N(\sigma))^{-1}\Big) d\mathbb{P} - \int g\Big(\int f\, d\chi_{N,\beta_N,a_N}\Big) d\mathbb{P}\Big| \to 0 \tag{17}$$
as $N \to \infty$ for all Lipschitz continuous $f : \mathbb{R}^M \to \mathbb{R}^k$ and all Lipschitz continuous $g : \mathbb{R}^k \to \mathbb{R}$ and all $k \in \mathbb{N}$. Denote by K_f and K_g the Lipschitz constants of f and g. As
$$\int f\, d\chi_{N,\beta_N,a_N} = \iint f(x+y)\, \varrho_{N,\beta_N,\xi}(\sqrt{a_N N} m_N(\sigma))^{-1} \otimes \mathcal{N}\Big(0, \frac{a_N}{\beta_N}\,\mathrm{Id}\Big)(d(x,y)),$$
we find that the left-hand side of (17) is bounded above by
$$\begin{aligned} &K_g \int \Big(\iint \|f(x) - f(x+y)\|\, \varrho_{N,\beta_N,\xi}(\sqrt{a_N N} m_N(\sigma))^{-1} \otimes \mathcal{N}\Big(0, \frac{a_N}{\beta_N}\,\mathrm{Id}\Big)(d(x,y))\Big) d\mathbb{P} \\ &\le K_f K_g \int \|y\|\, \mathcal{N}\Big(0, \frac{a_N}{\beta_N}\,\mathrm{Id}\Big)(dy) \to 0 \end{aligned}$$
as $a_N/\beta_N \to 0$. (Recall that M is fixed.)

□

The next lemma provides a checkable condition for the weak convergence in the set $\mathcal{M}(\mathcal{M}(\mathbb{R}^M))$ of all probability measures on the probability measures on $\mathbb{R}^M$ (which is what we need to prove Theorem 2.3.)

Lemma 3.3. *Let $M \in \mathbb{N}$ be any finite number. Let $(Q_{N,\xi})_{N\in\mathbb{N}}$ be a sequence of random probability measures on $\mathbb{R}^M$ and Q_ξ be another random probability measure on $\mathbb{R}^M$ (where ξ indicates the randomness which is assumed to be triggered by a probability $\mathbb{P}$). Denote by*
$$\mu_N := \mathbb{P} \circ Q^{-1}_{N,\xi} \in \mathcal{M}(\mathcal{M}(\mathbb{R}^M))$$
the distribution of Q_N and similarly by
$$\mu := \mathbb{P} \circ Q^{-1}_{\xi} \in \mathcal{M}(\mathcal{M}(\mathbb{R}^M)).$$
the distribution of Q. Consider the following condition
$$\int g(\int f dQ_{N,\xi}) d\mathbb{P} \to_{N\to\infty} \int g(\int f dQ_\xi) \tag{18}$$
for all Lipschitz continuous functions $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support, all Lipschitz continuous functions $g : \mathbb{R}^k \to \mathbb{R}$ with compact support, and all k. Then μ_N converges weakly to μ if and only if (18) holds.

Proof. We show that (18) implies
$$\liminf_{N\to\infty} \mu_N(G) \ge \mu(G) \tag{19}$$
for all sets $G \subseteq \mathcal{M}(\mathbb{R}^M)$ which are open in the weak topology (note that (19) is equivalent to the assertion of weak convergence of the μ_N).

First of all note that
$$\int g(\int f dQ_{N,\xi}) d\mathbb{P} = \int g(\int f d\nu) d\mu_N(\nu)$$
as well as
$$\int g(\int f dQ_\xi) d\mathbb{P} = \int g(\int f d\nu) d\mu(\nu).$$
Furthermore for $F = \{f_1, \dots, f_k\} \subset C^L_c(\mathbb{R}^M, \mathbb{R})$, $k \in \mathbb{N}$ and $\tilde\nu \in \mathcal{M}(\mathbb{R}^M)$ let $h_{F,\tilde\nu} : \mathcal{M}(\mathbb{R}^M) \to \mathbb{R}$ be defined by
$$h_{F,\tilde\nu}(\nu) := \max_{f\in F} |\int f d\nu - \int f d\tilde\nu|.$$

Here $C_c^L(\mathbb{R}^M,\mathbb{R})$ denotes the set of all Lipschitz continuous functions with compact support from $\mathbb{R}^M$ to $\mathbb{R}$. The the set $\mathcal{B}$ of all "cylinder sets"

$$B(\tilde{\nu},F,\varepsilon):=\{\nu\in\mathcal{M}(\mathbb{R}^M):h_{F,\tilde{\nu}}(\nu)<\varepsilon\}$$

defined by

$$\mathcal{B}:=\{B(\tilde{\nu},F,\varepsilon):\tilde{\nu}\in\mathcal{M}(\mathbb{R}^M),F\subset C_b(\mathbb{R}^M,\mathbb{R})\text{ finite},\ \varepsilon>0\}$$

forms a basis of the vague topology on $\mathcal{M}(\mathbb{R}^M)$. Since $\mathcal{M}(\mathbb{R}^M)$ only consists of probability measures the vague topology and the weak topology agree, and thus $\mathcal{B}$ also forms a basis of the weak topology.

In order to show (19) let $G\subset\mathcal{M}(\mathbb{R}^M)$ be any open set. Since $\mathcal{B}$ is a basis there is an index set I such that

$$G=\cup_{\kappa\in I}B_\kappa$$

where

$$B_\kappa:=B(\nu_\kappa,F_\kappa,\varepsilon_\kappa)\in\mathcal{B}.$$

Now $\mathcal{M}(\mathbb{R}^M)$ is a Polish space (in the weak topology), and therefore μ is regular, i.e. for given $\delta>0$ there is a compact set $K\subset G$ such that

$$\mu(G)\le\mu(K)+\delta.$$

By the compactness of K there are finitely many indices, say $\kappa_1,\dots,\kappa_l$ such that $K\subset\cup_{i=1}^l B_{\kappa_i}$. Therefore

$$\mu(G)\le\mu(K)+\delta\le\mu(\cup_{i=1}^l B_{\kappa_i})+\delta.$$

and

$$\mu_N(G)\ge\mu_N(\cup_{i=1}^l B_{\kappa_i})$$

as well as

$$\mu(\cup_{i=1}^l B_{\kappa_i})\ge\mu(G)-\delta$$

and thus it suffices to prove (19) for sets of the form $G=\cup_{i=1}^l B_{\kappa_i}$.

So let $G=\cup_{i=1}^l B_{\kappa_i}$ with $B_{\kappa_i}=B(\tilde{\nu}_{\kappa_i},F_{\kappa_i},\varepsilon_{\kappa_i})$ and $F_{\kappa_i}=\{f_1^i,\dots,f_{k_i}^i\}$. Put

$$c_i:=\left(\int f_j^i d\tilde{\nu}_{\kappa_i}\right)_{j=1,\dots,k_i}\in\mathbb{R}^{k_i}.$$

For $L\in\mathbb{N}$ define the Lipschitz continuous function $g_L:\mathbb{R}^{\sum_{i=1}^l k_i}\to\mathbb{R}$ by

$$g_L(x_1,\dots,x_l):=\min\left[1,\max(0,L[\max_{i=1,\dots,l}(1-\frac{||x_i-c_i||_\infty}{\varepsilon_{\kappa_i}})])\right].$$

which has compact support. Then

$$\begin{aligned}
&\nu\in\cup_{i=1}^l B_{\kappa_i}\\
\Leftrightarrow\ &\exists i\in\{1,\dots,l\}\text{ with }h_{F_{\kappa_i},\tilde{\nu}_{\kappa_i}}(\nu)<\varepsilon_{\kappa_i}\\
\Leftrightarrow\ &\exists i\in\{1,\dots,l\}\text{ with }1-\frac{||\left(\int f_j^i d\nu\right)_{j=1,\dots,k_i}-c_i||_\infty}{\varepsilon_{\kappa_i}}>0\\
\Leftrightarrow\ &\max_{i=1,\dots,l}\left[1-\frac{||\left(\int f_j^i d\nu\right)_{j=1,\dots,k_i}-c_i||_\infty}{\varepsilon_{\kappa_i}}\right]>0\\
\Leftrightarrow\ &g_L\left(((\int f_j^i d\nu)_{j=1,\dots,k_i})_{i=1,\dots,l}\right)\uparrow 1\qquad(L\uparrow\infty)
\end{aligned}$$

Now suppose (18) holds, then

$$\begin{aligned}
&\liminf_{N\to\infty}\mu_N(\cup_{i=1}^l B_{\kappa_i})\\
\ge\ &\liminf_{N\to\infty}\int g_L\Big(\big((\int f_j^i d\nu_{\kappa_i}\big)_{j=1,\dots,k_i}\big)_{i=1,\dots,l}\Big)\mu_N(d\nu)\quad\forall L\in\mathbb{R}^+\\
\ge\ &\liminf_{N\to\infty}\int g_L\Big(\big((\int f_j^i d\nu_{\kappa_i}\big)_{j=1,\dots,k_i}\big)_{i=1,\dots,l}\Big)\mu(d\nu)\quad\forall L\in\mathbb{R}^+
\end{aligned}\tag{20}$$

where the first inequality follows from the fact that the indicator on $\cup_{i=1}^l B_{\kappa_i}$ majorizes the integrand on the right hand side (by the above considerations) and the second inequality is just our assumption

(18). Now for $L \to \infty$ the integrand on the right hand side of (20) converges monotonely to the indicator function on $\cup_{i=1}^{l} B_{\kappa_i}$ and thus the right hand of (20) converges to $\mu(\cup_{i=1}^{l} B_{\kappa_i})$.
As the other direction is trivial this proves the lemma. $\square$

Before we finally start to prove Theorem 2.3 we state and prove another Lemma which will be of help in the course of the proof of Theorem 2.3.

Lemma 3.4. *Put $c : \sqrt{20 f_{CW}(4)}$ and let B_r denote the ball with radius r centered in zero. Moreover assume that $1/2 \le \beta_N \le 2$. Then for any $r > 0$ small enough*

$$\limsup_{N\to\infty} \sup_{x \in B_{c\sqrt{Na_N}} \setminus B_{r\sqrt{Na_N}}} -\beta_N \Phi_{N,\beta_N}\left(\frac{x}{\sqrt{Na_N}}\right) < 0. \tag{21}$$

Proof. Using the self-averaging property of the ξ_i we obtain

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{x \in B_{c\sqrt{Na_N}} \setminus B_{r\sqrt{Na_N}}} -\beta_N \Phi_{N,\beta_N}\left(\frac{x}{\sqrt{Na_N}}\right) \\
= &\limsup_{N\to\infty} \sup_{x \in B_c \setminus B_r} -\frac{\beta_N}{2}||x||^2 + \frac{1}{N}\sum_{i=1}^{N} \log\cosh\langle \xi_i, \beta_N x\rangle \\
\le &\limsup_{N\to\infty} \sup_{x \in B_c \setminus B_r} \mathbb{E}\big[-\frac{\beta_N}{2}\langle \xi_1, x\rangle^2 + \log\cosh\langle \xi_1, \beta_N x\rangle\big] \\
\le &\limsup_{N\to\infty} \sup_{x \in B_{c\beta_N} \setminus B_{r\beta_N}} \mathbb{E}\big[-\frac{1}{2\beta_N}\langle \xi_1, y\rangle^2 + \log\cosh\langle \xi_1, y\rangle\big] \\
\le &\limsup_{N\to\infty} \sup_{x \in B_{2c} \setminus B_{r/2}} \mathbb{E}\big[-\frac{1}{2}\langle \xi_1, y\rangle^2 + \log\cosh\langle \xi_1, y\rangle\big] \\
&+ \limsup_{N\to\infty} \sup_{x \in B_{2c} \setminus B_{r/2}} \frac{1}{2}\frac{\beta_N - 1}{\beta_M}||y||^2.
\end{aligned}$$

Observe that $\langle \xi_1, y\rangle$ abbreviates a sum of Rademachers and thus (cf. [LT91])

$$\mathbb{P}(|\langle \xi_1, y\rangle| \ge \frac{1}{8}||y||) > \frac{1}{3}$$

Since $||y|| \ge \frac{r}{2}$ this implies

$$\mathbb{P}(|\langle \xi_1, y\rangle| \ge \frac{1}{16}r) > \frac{1}{3}$$

and thus

$$\mathbb{E}\big[-\frac{1}{2}\langle \xi_1, y\rangle^2 + \log\cosh\langle \xi_1, y\rangle\big] < f_{CW}(0) = 0$$

(where $f_{CW}(0)$ denotes the free energy of the Curie-Weiss model at $\beta = 0$). $\square$

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. 1.) The core of the proof consists of the analysis of the argument $\Phi_{N,\beta}$ of the exponential function occuring in the density f_{N,β,a_N} of the Hubbard–Stratonovich transform of the measures $\varrho_{N,\beta}(\sqrt{a_N N} m_N(\sigma))$. Recall that

$$\Phi_{N,\beta}(x) := \frac{1}{2}||x||_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \log\cosh(\beta < \xi_i, x >).$$

By using Taylor expansion for the $\log\cosh$-terms we obtain

$$\begin{aligned}
\Phi_{N,\beta}(x) \;=\; & \frac{1}{2}||x||_2^2 - \beta\frac{1}{2}||x||_2^2 - \beta\frac{1}{2N}\sum_{i=1}^{N}\sum_{\mu\neq\nu} \xi_i^\mu x_\mu \xi_i^\nu x_\nu + \beta^3\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 \\
+ \; & \beta^3\frac{1}{4}\sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 + \frac{\beta^3}{3N}\sum_{i=1}^{N}\sum_{\mu\neq\nu} \xi_i^\mu x_\mu^3 \xi_i^\nu x_\nu + \beta^3\frac{1}{2N}\sum_{i=1}^{N}{\sum}'_{\mu_1,\mu_2,\mu_3} x_{\mu_1}^2 \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \\
+ \; & \beta^3\frac{1}{12N}\sum_{i=1}^{N}{\sum}''_{\mu_1,\mu_2,\mu_3,\mu_4} \xi_i^{\mu_1} x_{\mu_1} \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \xi_i^{\mu_4} x_{\mu_4} + \mathcal{O}(\beta^5||x||^6)
\end{aligned}$$

(where $\sum'$ and $\sum''$, resp., denote the summation over pairwise different indices μ_1, μ_2, μ_3 and $\mu_1, \mu_2, \mu_3, \mu_4$, resp.). Note that the constant contained in the $\mathcal{O}$-term can be chosen independent of ξ (but depends on M). Indeed we have that the remainder of the Taylor-expansion is bounded from above by

$$||C_1 \beta^5 \frac{1}{N} \sum_{i=1}^N < \xi_i, x >^6 || \le C_1 \beta^5 \frac{1}{N} \sum_{i=1}^N (||\xi_i|| \, ||x||)^6 \le C_1 M^3 \beta^5 ||x||^6$$

(where C_1 is a constant). Hence

$$\begin{aligned}
-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt{N a_N}}) &= -\frac{1}{2}\beta_N \frac{1}{a_N}||x||_2^2(1-\beta_N) + \beta_N^2 \frac{1}{a_N}\frac{1}{2N}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu \xi_i^\nu x_\nu \\
&\quad -\beta_N^4 \frac{1}{N a_N^2}\frac{1}{12}||x||_4^4 - \beta_N^4 \frac{1}{N a_N^2}\frac{1}{4}\sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \\
&\quad -\beta_N^4 \frac{1}{3N a_N^2}\frac{1}{N}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu^3 \xi_i^\nu x_\nu \\
&\quad -\beta_N^4 \frac{1}{2N a_N^2}\frac{1}{N}\sum_{i=1}^N {\sum}'_{\mu_1,\mu_2,\mu_3} x_{\mu_1}^2 \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \\
&\quad -\beta_N^4 \frac{1}{12 N a_N^2}\frac{1}{N}\sum_{i=1}^N {\sum}''_{\mu_1,\mu_2,\mu_3,\mu_4} \xi_i^{\mu_1} x_{\mu_1} \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \xi_i^{\mu_4} x_{\mu_4} \\
&\quad +\mathcal{O}(\beta_N^6 \frac{1}{N^2 a_N^3}||x||^6)
\end{aligned} \tag{22}$$

(again with $\mathcal{O}$ not depending on ξ). Thus putting $a_N = N^{-1/2}$ yields

$$\begin{aligned}
-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt[4]{N}}) &= -\frac{1}{2}\beta_N \sqrt{N}||x||_2^2(1-\beta_N) + \beta_N^2 \frac{1}{2\sqrt{N}}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu \xi_i^\nu x_\nu \\
&\quad -\beta_N^4 \frac{1}{12}||x||_4^4 - \beta_N^4 \frac{1}{4}\sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 - \beta_N^4 \frac{1}{3N}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu^3 \xi_i^\nu x_\nu \\
&\quad -\beta_N^4 \frac{1}{2N}\sum_{i=1}^N {\sum}'_{\mu_1,\mu_2,\mu_3} x_{\mu_1}^2 \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \\
&\quad -\beta_N^4 \frac{1}{12N}\sum_{i=1}^N {\sum}''_{\mu_1,\mu_2,\mu_3,\mu_4} \xi_i^{\mu_1} x_{\mu_1} \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \xi_i^{\mu_4} x_{\mu_4} \\
&\quad +\mathcal{O}(\beta_N^6 \frac{1}{\sqrt{N}}||x||^6)
\end{aligned} \tag{23}$$

$$\begin{aligned}
&= \beta_N^2 \frac{1}{2\sqrt{N}}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu \xi_i^\nu x_\nu - \beta_N^4 \frac{1}{12}||x||_4^4 - \beta_N^4 \frac{1}{4}\sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \\
&\quad -\beta_N^4 \frac{1}{3N}\sum_{i=1}^N \sum_{\mu\neq\nu} \xi_i^\mu x_\mu^3 \xi_i^\nu x_\nu - \beta_N^4 \frac{1}{2N}\sum_{i=1}^N \sum_{\mu=1}^M {\sum}'_{\mu_1,\mu_2,\mu_3} x_{\mu_1}^2 \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \\
&\quad -\beta_N^4 \frac{1}{12N}\sum_{i=1}^N \sum_{i=1}^N {\sum}''_{\mu_1,\mu_2,\mu_3,\mu_4} \xi_i^{\mu_1} x_{\mu_1} \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \xi_i^{\mu_4} x_{\mu_4} \\
&\quad +\mathcal{O}(\beta_N^6 \frac{1}{\sqrt{N}}||x||^6) + \mathcal{O}(\beta_N \sqrt{N}|1-\beta_N| \, ||x||^2)
\end{aligned} \tag{24}$$

To analyze the weak convergence of

$$\mathbb{P} \circ (\varrho_{N,\beta_N} \circ (N^{1/4} m_N(\sigma))^{-1})^{-1}$$

in view of Lemma 3.2 it suffices to analyze the weak convergence of

$$\mathbb{P} \circ \chi_{N,\beta_N,N^{-1/2}}^{-1}.$$

In view of Lemma 3.3 to prove part one of Theorem 2.3 we have to check that

$$\int g(\int f d\chi_{N,\beta_N,N^{-1/2}})d\mathbb{P} \to_{N\to\infty} \int g(\int f dQ_\eta)d\mathbb{P} \tag{25}$$

for all Lipschitz continuous functions $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support, all Lipschitz continuous functions $g : \mathbb{R}^k \to \mathbb{R}$ with compact support, and all k, where Q_η, and P_η, resp. are defined as in Theorem 2.1.
So let $f : \mathbb{R}^M \to \mathbb{R}^k$ and $g : \mathbb{R}^k :\to \mathbb{R}$ be as above.
Note that by Lemma 3.1

$$\int f d\chi_{N,\beta,N^{-1/2}} = \frac{\int f(x)\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}{\int \exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}$$

$$= \frac{\int_{K_1} f(x)\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}{\int \exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx} \tag{26}$$

$$\tag{27}$$

where K_1 is the compact support of f.
Now we would be in a much better situation, if also the integral in the denominator could be taken over a compact set. To justify such a change let us remark that for each compact set K_2

$$|\int g(\int f d\chi_{N,\beta,N^{-1/2}}) - \int g\Big(\frac{\int_{K_1} f(x)\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}{\int_{K_2} \exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}\Big)| \le 2L_g||f||_\infty \mathbb{E}(\chi_{N,\beta,N^{-1/2}}(K_2^c))$$

if K_2 is at least so large that $\mathbb{E}(\chi_{N,\beta,N^{-1/2}}(K_2)) \ge 1/2$. Here L_g denotes the Lipshitz constant for the function g. So the replacement of the integral in the denominator on the right hand side of (26) by the corresponding integral over a compact set can be justified if for all $\varepsilon > 0$ we can find a number N_0 and a set compact set K_2 such that for all $N \ge N_0$

$$\mathbb{E}(\chi_{N,\beta,N^{-1/2}}(K_2^c)) \le \varepsilon.$$

So let $\varepsilon > 0$ be given. For $L > 0, \delta > 0$ define

$$\begin{aligned}
A_{N,L}^{\mu,\nu} &:= \{|\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^\mu \xi_i^\nu| > L\} \quad \text{for } \mu \ne \nu \\
B_{N,\delta}^{\mu,\nu} &:= \{|\frac{1}{N}\sum_{i=1}^N \xi_i^\mu \xi_i^\nu| > \delta\} \quad \text{for } \mu \ne \nu \\
C_{N,\delta}^{\mu_1,\mu_2,\mu_3,\mu_4} &:= \{|\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4}| > \delta\} \quad \text{for } \mu_1 \dots \mu_4 \text{ pairwise different}
\end{aligned}$$

and

$$D_{N,L,\delta} := \left(\bigcup_{\mu\ne\nu} A_{N,L}^{\mu,\nu} \cup \bigcup_{\mu\ne\nu} B_{N,\delta}^{\mu,\nu} \cup {\bigcup}'_{\mu_1,\mu_2,\mu_3,\mu_4} C_{N,\delta}^{\mu_1,\mu_2,\mu_3,\mu_4}\right)^c$$

where ${\bigcup}'_{\mu_1,\mu_2,\mu_3,\mu_4}$ denotes the union over pairwise different indices. Then by Chebyshev's and the CLT with the Berry-Esseen correction we can estimate

$$\begin{aligned}
\mathbb{P}(D_{N,L,\delta}^c) &\le M^2\mathbb{P}(A_{N,L}^{1,2}) + M^2\mathbb{P}(B_{N,\delta}^{1,2}) + M^4\mathbb{P}(C_{N,\delta}^{1,2,3,4}) \\
&\le M^2(1-\mathbb{P}(\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^\mu\xi_i^\nu \le L) + \mathbb{P}(\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^\mu\xi_i^\nu \le -L)) + M^2\frac{1}{N\delta^2} + M^4\frac{1}{N\delta^2} \\
&\le M^2(2(1-\Phi(L)) + Const.\frac{1}{\sqrt{N}}) + \frac{2M^4}{N\delta^2}
\end{aligned}$$

where Φ denotes the distribution function of the Standard Normal Distribution and $Const.$ is the Berry-Esseen constant.
Hence for given $\varepsilon > 0, \delta > 0$, there exist $L > 0$ and $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}(D_{N,L,\delta}^c) \le \varepsilon \quad \text{for } N \ge N_0.$$

Moreover choose

$$E_N := \{||\frac{1}{N}\sum_{i=1}^{N}\xi\xi^T - Id||_{op} \le \varepsilon_N\}$$

– where $||\cdot||_{op}$ denotes the operator norm and $\varepsilon_N := 2\sqrt{\frac{M}{N}\log\log N}$ – and

$$F_{N,L,\delta} = D_{N,L,\delta} \cap E_N.$$

Then, as $\mathbb{P}(D^c_{N,L,\delta} \le vep$ for the right choice of the parameters as well as $\mathbb{P}(E^c_N) \to 0$ as $N \to \infty$, for given $\varepsilon > 0, \delta > 0$, there exist $L > 0$ a nd $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}(F^c_{N,L,\delta}) \le \varepsilon \quad \text{for } N \ge N_0.$$

Thus is suffices to show that

$$\mathbb{E}(\mathbb{I}_{F_{N,L,\delta}}\chi_{N,\beta,N^{-1/2}}(||x|| > K)) \le \varepsilon$$

for some number $K > 0$, where $\mathbb{I}$ the indicator function. Obviously

$$\mathbb{I}_{F_{N,L,\delta}}\chi_{N,\beta,N^{-1/2}}(||x|| > K) \le \mathbb{I}_{F_{N,L,\delta}}(\frac{\int_{||x||>K}\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}{\int_{||x||\le K}\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx} \tag{28}$$

To show that the denominator of the right hand side of (28) is bounded away from zero, we use that on $F_{N,L,\delta}$ we have that

$$\begin{aligned}
&\int_{||x||\le K}\exp(-\beta_N N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx \\
\ge\ &\int_{||x||\le K}\exp\Big(-\frac{\beta_N^2 L}{2}\sum_{\mu\neq\nu}|x_\mu||x_\nu| - \beta_N^4\frac{1}{12}||x||_4^4 - \beta_N^4\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 - \frac{\beta_N^4\delta}{3}\sum_{\mu\neq\nu}|x_\mu^3||x_\nu|\Big) \\
&\times\exp\Big(-\frac{\beta_N^4\delta}{2}{\sum}'_{\mu_1,\mu_2,\mu_3}x_{\mu_1}^2|x_{\mu_2}||x_{\mu_3}| - \frac{\beta_N^4\delta}{12}{\sum}'_{\mu_1,\mu_2,\mu_3,\mu_4}|x_{\mu_1}||x_{\mu_2}||x_{\mu_3}||x_{\mu_4}|\Big) \\
&\times\exp\Big(C(\frac{\beta_N^6K^6}{\sqrt{N}} + \beta_N\sqrt{N}|1-\beta_N|K^2)\Big)dx \\
\ge\ &\int_{||x||\le K}\exp\Big(-\frac{\beta_N^2M^2K^2L}{2} - 2\beta_N^4M^4K^4 + o(1)\Big)dx \\
\ge\ &const.(K,L,\delta) > 0
\end{aligned}$$

To treat the nominator in (28) we assume that $1/2 \le \beta_N \le 2$ and split the integral into three parts, an intermediate region, an outer region, and a region very far out (the inner region does not contribute due to our choice of the set we are integrating over):

<u>The region very far out:</u>

Denote by $f_{CW}(\beta)$ the free energy of the Curie-Weiss model at temperature β. Then

$$\begin{aligned}
&-N\beta_N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}) \\
=\ &-\frac{N\beta_N}{2}||\frac{x}{\sqrt{Na_N}}||^2 + \sum_{i=1}^{N}\log\cosh\langle\xi_i,\frac{\beta_N x}{\sqrt{Na_N}}\rangle \\
\le\ &-\frac{\beta_N}{2a_N}||x||^2 + \frac{\beta_N}{4a_N}\frac{1}{N}\sum_{i=1}^{N}\langle\xi_i,x\rangle^2 + Nf_{CW}(2\beta_N) \\
=\ &-\frac{\beta_N}{2a_N}||x||^2 + \frac{\beta_N}{4a_N}||x||^2(1+\varepsilon_N) + Nf_{CW}(2\beta_N) \\
\le\ &-\frac{\beta_N}{a_N}||x||^2(\frac{1}{2} - \frac{1}{4} - \frac{1}{2}\varepsilon_N) + Nf_{CW}(4) \\
\le\ &-\frac{\beta_N}{10a_N}||x||^2 + Nf_{CW}(4)
\end{aligned}$$

for N large enough. Here we have made use of the definition of the set E_N. Thus

$$-N\beta_N\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}} \le -\frac{1}{20a_N}||x||^2 = \frac{\sqrt{N}}{2o}||x||^2$$

for $||x||^2 \geq 20Na_N f_{CW}(4)$ (the region very far out) and therefore

$$\begin{aligned}
& \int_{||x||^2 \geq 20Na_N f_{CW}(4)} \exp(-N\beta_N \Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx \\
\leq & \int_{||x||^2 \geq 20Na_N f_{CW}(4)} \exp(-\frac{1}{20a_N}||x||^2)dx \\
\leq & \exp(-\frac{1}{40a_N} 20 f_{CW}(4) N a_N) \int_{\mathbb{R}^M} e^{-\frac{1}{40a_N}||x||^2} \\
= & e^{-\frac{N}{2}f_{CW}(4)} (\frac{\pi}{10a_N})^{M/2}
\end{aligned}$$

which converges to zero.

The outer region

Put $c : \sqrt{20 f_{CW}(4)}$. We have already shown that the integral over $||x|| > c\sqrt{Na_N}$ becomes negligible for large N. The aim of this section is to show that the same holds true if we replace c by any number $0 < r \leq c$.

Indeed Lemma 3.4 implies that for each $r \leq c$ and $\mathbb{P}$-almost all ξ there exist $N_1(\xi)$ and $C(\xi)$ such that for all $N \geq N_1(\xi)$ and all $x \in B_{c\sqrt{Na_N}} \backslash B_{r\sqrt{Na_N}}$

$$-N\beta_N \Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}) \leq -NC(\xi)$$

holds. Thus

$$\begin{aligned}
g_N(\xi) & := \int_{B_{c\sqrt{Na_N}} \backslash B_{r\sqrt{Na_N}}} \exp\bigl(-N\beta_N \Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}})\bigr)dx \\
& \leq \exp\bigl(-NC(\xi)\bigr)\, Vol(B_{c\sqrt{Na_N}}) \leq \exp\bigl(-\frac{NC(\xi)}{2}\bigr)
\end{aligned}$$

Now let us restrict the set $F_{N,L,\delta}$ once more and introduce the set

$$G_{N,L,\delta,r} := F_{N,L,\delta} \cap \bigcap_{\tilde{N} \geq N} \{\xi : g_N(\xi) \leq \exp\bigl(-\frac{NC(\xi)}{2}\bigr).$$

As by Lemma 3.4

$$\mathbb{P}\Bigl(\bigl(\bigcap_{\tilde{N} \geq N} \{\xi : g_N(\xi) \leq \exp\bigl(-\frac{NC(\xi)}{2}\bigr)\bigr)^c Bigr) \to\bigr) \quad \text{as } N \to \infty$$

it suffices to concentrate on the set $G_{N,L,\delta,r}$. But on $G_{N,L,\delta,r}$

$$\mathbb{E}(\mathbb{I}_{G_{N,L,\delta,r}} g_N(\xi)) \leq \mathbb{E}\bigl(\exp\bigl(-\frac{NC(\xi)}{2}\bigr)\bigr) \to 0$$

the latter by dominated convergence.

The intermediate region

Now let us consider the region $K \leq ||x|| \leq r\sqrt{Na_N}$ for some $r > 0$. Since

$$\frac{1}{\sqrt{N}}||x||_2^6 \leq r^2 M ||x||_4^4$$

as well as

$$||x||_2^2 \leq \sqrt{M}||x||_4^2$$

we have on $F_{N,L,\delta}$ (or on $G_{N,L,\delta,r}$)

$$\begin{aligned}
&-N\beta_N\Phi_{N,\beta_N}(\frac{x}{\sqrt{N}a_N})\\
\leq\ & \beta_N^2\frac{1}{2\sqrt{N}}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu\xi_i^\nu x_\nu-\beta_N^4\frac{1}{12}||x||_4^4-\beta_N^4\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\\
&+\beta_N^4\frac{\delta}{3N}\sum_{\mu\neq\nu}|x_\mu|^3|x_\nu|+\frac{\beta_N^4\delta}{2}{\sum}'_{\mu_1,\mu_2,\mu_3}x_{\mu_1}^2|x_{\mu_2}||x_{\mu_3}|\\
&+\beta_N^4\frac{1}{12N}{\sum}''_{\mu_1,\mu_2,\mu_3,\mu_4}|x_{\mu_1}||x_{\mu_2}||x_{\mu_3}||x_{\mu_4}|\\
&+const.(\beta_N^6\frac{1}{\sqrt{N}}||x||_2^6+o(1)||x||_2^2\\
\leq\ & \beta_N^2\frac{1}{2}L\sum_{\mu\neq\nu}|x_\mu||x_\nu|-(\beta_N^4\frac{1}{12}-\beta_N^4\delta M-const.r^2M)||x||_4^4-\frac{\beta_N^4}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2+o(1)||x||_2^2))\\
\leq\ & -(\beta_N^4\frac{1}{12}-\beta_N^4\delta M-const.r^2M)||x||_4^4-\frac{\beta_N^4}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2+(\beta_N^2LM\frac{1}{2}o(1))||x||_2^2.
\end{aligned}$$

If we choose δ and r small enough and still $\beta_N\geq\frac{1}{2}$, we obtain that

$$-(\beta_N^4\frac{1}{12}-\beta_N^4\delta M-const.r^2M)\leq-\frac{1}{200}.$$

If moreover N is large enough

$$(\beta_N^2LM\frac{1}{2}o(1))\leq(1+2LM)\sqrt{M}||x||_4^2\leq\sup\{\frac{1}{400}||x||_4^4,400M(1+2LM)^2\}=:\sup\{\frac{1}{400}||x||_4^4,C_M\}.$$

Thus for δ and r small enough, and L and N large enough

$$\begin{aligned}
&\mathbb{E}\Big(\mathbb{I}_{G_{N,L,\delta,r}}\int_{K\leq||x||\leq r\sqrt{Na_N}}\exp\big(\Phi_{N,\beta_N}(\frac{x}{\sqrt{N}a_N}dx)\big)\Big)\\
\leq\ & \int_{||x||>K}\exp(-\frac{1}{200}||x||_4^4)\exp(\frac{1}{400}||x||_4^4+C_M)dx\\
=\ & \int_{||x||>K}\exp(-\frac{1}{400}||x||_4^4+C_M)dx\to0\quad\text{ as }K\to\infty
\end{aligned}$$

Thus, indeed, it suffices to consider the convergence of

$$\frac{\int_K f(x)\exp(-\beta_NN\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}{\int_K\exp(-\beta_NN\Phi_{N,\beta_N}(\frac{x}{\sqrt{Na_N}}))dx}$$

for Lipschitz continuous functions f and compact sets K. By the compactness of K for each $\varepsilon>0$ and all N large enough

$$\begin{aligned}
&(1-\varepsilon)\frac{\int_K f(x)\exp(-\beta_NN\Psi_{N,\beta_N}(x))dx}{\int_K\exp(-\beta_NN\Psi_{N,\beta_N}(x))dx}\leq\frac{\int_K f(x)\exp(-\beta_NN\Phi_{N,\beta_N}(x))dx}{\int_K\exp(-\beta_NN\Phi_{N,\beta_N}(x))dx}\\
\leq\ & (1+\varepsilon)\frac{\int_K f(x)\exp(-\beta_NN\Psi_{N,\beta_N}(x))dx}{\int_K\exp(-\beta_NN\Psi_{N,\beta_N}(x))dx}. \qquad (29)
\end{aligned}$$

Here

$$\begin{aligned}
\Psi_{N,\beta_N}(x):=\ & \beta_N^2\frac{1}{2\sqrt{N}}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu\xi_i^\nu x_\nu-\beta_N^4\frac{1}{12}||x||_4^4-\beta_N^4\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2-\beta_N^4\frac{1}{3N}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu^3\xi_i^\nu x_\nu\\
-\ & \beta_N^4\frac{1}{2N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,\mu_3\ p.d.}x_{\mu_1}^2\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3}\\
-\ & \beta_N^4\frac{1}{12N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,\mu_3,\mu_4}\xi_i^{\mu_1}x_{\mu_1}\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3}\xi_i^{\mu_4}x_{\mu_4}\\
+\ & C(\beta_N^6\frac{1}{\sqrt{N}}||x||^6+\beta_N\sqrt{N}|1-\beta_N|||x||^2)
\end{aligned}$$

and C is a constant depending on the compact set K only.
Note that $\Psi_{N,\beta_N}(x)$ and hence the integrands on the right hand side of (29) are continuous functions of

$$X'_N := \Big((\frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i^\mu \xi_i^\nu)_{\mu\neq\nu}, (\frac{1}{N} \sum_{i=1}^N \xi_i^\mu \xi_i^\nu)_{\mu\neq\nu}, (\frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1} \xi_i^{\mu_2} \xi_i^{\mu_3} \xi_i^{\mu_4})_{\mu_1,\mu_2,\mu_3,\mu_4 \ p.d.}, \beta_N, \frac{\beta_N^6}{\sqrt{N}}, \beta_N \sqrt{N} |1-\beta_N| \Big) \tag{30}$$

(which is a vector in $\mathbb{R}^{2M(M-1)+M(M-1)(M-2)(M-3)+3}$).
Thus the integrals in (29) are continuous functions of X'_N and so is their quotient. Moreover

$$f_{\min} \leq \frac{\int_K f(x) \exp(-\beta_N N \Psi_{N,\beta_N}(x)) dx}{\int_K \exp(-\beta_N N \Psi_{N,\beta_N}(x)) dx} \leq f_{\max}$$

where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum value of f, respectively. Thus the quotient of the integrals is also a bounded function.
Finally

$$P \circ (X'_N)^{-1} \Rightarrow ((\frac{1}{2}\eta_{\mu,\nu}), 0, 0, 1, 0, 0)$$

where the vector $(\eta_{\mu,\nu})$ obeys a $\mathcal{N}(0,\Sigma)$-distribution (since

$$E \frac{1}{N} \sum_{i,j=1}^N \xi_i^{\mu_1} \xi_i^{\mu_2} \xi_j^{\mu_3} \xi_j^{\mu_4} = \sigma_{(\mu_1,\mu_2),(\mu_3,\mu_4)}).$$

. By the continuity and boundedness of the quotient of the integrals in (29) this implies

$$\mathbb{P} \circ \left(\frac{\int_K f(x) \exp(-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt{N a_N}})) dx}{\int_K \exp(-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt{N a_N}})) dx} \right) \Rightarrow \frac{\int_K f(x) \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx}{\int_K \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx}$$

and therefore also that

$$\int g \left(\frac{\int_K f(x) \exp(-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt{N a_N}})) dx}{\int_K \exp(-\beta_N N \Phi_{N,\beta_N}(\frac{x}{\sqrt{N a_N}})) dx} \right) d\mathbb{P} \to_{N\to\infty} \int g \left(\frac{\int_K f(x) \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx}{\int_K \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx} \right) d\mathcal{N}(0,\Sigma)(\eta) \tag{31}$$

for all continuous and bounded functions g.

Finally we have to understand that also the right hand side of (31) is close to

$$\int g \left(\frac{\int_K f(x) \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx}{\int_K \exp\left(\frac{1}{2} \sum_{\mu\neq\nu} \eta_{\mu,\nu} x_\mu x_\nu - \frac{1}{12} ||x||_4^4 - \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 \right) dx} \right) d\mathcal{N}(0,\Sigma)(\eta).$$

This, in principle, follows the same ideas as the analogous proof for the distribution χ_{N,β_N,a_N}. So let f and g be as above. Then for any number K large enough

$$\int g(\int f dQ_\eta) d\mathbb{P} \leq \int g(\int_K f dQ_\eta) d\mathbb{P} + 2L_g ||f||_\infty \mathbb{E}(Q_\eta(||x|| \, K))$$

as well as

$$\int g(\int f dQ_\eta) d\mathbb{P} \geq \int g(\int_K f dQ_\eta) d\mathbb{P} - 2L_g ||f||_\infty \mathbb{E}(Q_\eta(||x|| \, K))$$

where again L_g denotes the Lipshitz constant for g. Hence we have to show that for all $\varepsilon > 0$ and there exists a K such that

$$\mathbb{E} Q_\eta(||x|| > K) \leq \varepsilon.$$

Now

$$
\begin{aligned}
& \mathbb{E}Q_\eta(||x||>K) \\
= \ & \mathbb{E}\left(\frac{\int_{||x||>K}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}{\int_{\mathbb{R}^M}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}\right) \\
\leq \ & \int_{\eta_{\mu,\nu}\leq L\forall\mu,\nu}\left(\frac{\int_{||x||>K}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}{\int_{\mathbb{R}^M}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}\right) \\
& +\mathbb{P}(\eta_{\mu,\nu}>L \text{ for one pair } \mu\neq\nu)
\end{aligned}
$$

Now $\mathbb{P}(\eta_{\mu,\nu}>L$ for one pair $\mu\neq\nu)$ can be made small if we make L large enough and on the other hand

$$
\int_{\mathbb{R}^M}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)
$$

clearly exists if all the $\eta_{\mu,\nu}$ are smaller than L and hence

$$
\frac{\int_{||x||>K}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}{\int_{\mathbb{R}^M}\exp\left(\frac{1}{2}\sum_{\mu\neq\nu}\eta_{\mu,\nu}x_\mu x_\nu-\frac{1}{12}||x||_4^4-\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2\right)}
$$

becomes small if K gets large.
Thus

$$
\mathbb{E}Q_\eta(||x||>K)\leq\varepsilon.
$$

for L and K large enough proving part one of Theorem 2.3.

2.) Putting $a_N=\frac{1}{\beta_N(1-\beta_N)}$ in (22) yields

$$
\begin{aligned}
& -\beta_N N\Phi_{N,\beta_N}\left(\frac{x}{\sqrt{N(\beta_N(1-\beta_N))}}\right)=-\frac{1}{2}||x||_2^2+\beta_N^2\frac{1}{2(\beta_N(1-\beta_N))}\frac{1}{N}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu\xi_i^\nu x_\nu \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{12}||x||_4^4-\beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{2N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,mu_3\ p.d.}x_{\mu_1}^2\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3} \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{12N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,\mu_3,\mu_4\ p.d.}\xi_i^{\mu_1}x_{\mu_1}\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3}\xi_i^{\mu_4}x_{\mu_4} \\
- \ & \beta_N^4\frac{1}{3N(\beta_N(1-\beta_N))^2}\frac{1}{N}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu^3\xi_i^\nu x_\nu+\mathcal{O}\left(\beta_N^6\frac{1}{N^2(\beta_N(1-\beta_N))^3}||x||^6\right)
\end{aligned}
$$

Now $1-\beta_N\ll\frac{1}{\sqrt{N}}$ implies that $\frac{1}{1-\beta_N}=\sqrt{N}\times o(N)$ and therefore that

$$
\begin{aligned}
& \beta_N^2\frac{1}{2(\beta_N(1-\beta_N))}\frac{1}{N}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu\xi_i^\nu x_\nu \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{12}||x||_4^4-\beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{4}\sum_{\mu\neq\nu}x_\mu^2x_\nu^2 \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{2N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,\mu_3\ p.d.}x_{\mu_1}^2\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3} \\
- \ & \beta_N^4\frac{1}{N(\beta_N(1-\beta_N))^2}\frac{1}{12N}\sum_{i=1}^N\sum_{\mu_1,\mu_2,\mu_3,\mu_4\ p.d.}\xi_i^{\mu_1}x_{\mu_1}\xi_i^{\mu_2}x_{\mu_2}\xi_i^{\mu_3}x_{\mu_3}\xi_i^{\mu_4}x_{\mu_4} \\
- \ & \beta_N^4\frac{1}{3N(\beta_N(1-\beta_N))^2}\frac{1}{N}\sum_{i=1}^N\sum_{\mu\neq\nu}\xi_i^\mu x_\mu^3\xi_i^\nu x_\nu+\mathcal{O}\left(\beta_N^6\frac{1}{N^2(\beta_N(1-\beta_N))^3}||x||^6\right)
\end{aligned}
\tag{32}
$$

weakly converges to zero. Thus the weak convergence of the second part of the Theorem can be proved following the ideas of the proof of part one.

The almost sure part of the assertion follows since under the condition

$$\sum_{N=1}^{\infty} \frac{1}{N^2(1-\beta_N)^2} < \infty$$

the random vector in (32) converges to zero, for almost all realizations of ξ.

3.) Taking into account that in this case (again with $a_N = N^{-1/2}$)

$$\begin{aligned} -\beta_N N \Phi_{N,\beta_N}\left(\frac{x}{\sqrt[4]{N}}\right) &= -\frac{1}{2}\gamma\beta_N ||x||_2^2 + \beta_N^2 \frac{1}{2\sqrt{N}} \sum_{i=1}^{N} \sum_{\mu\neq\nu} \xi_i^\mu x_\mu \xi_i^\nu x_\nu \\ &- \beta_N^4 \frac{1}{12}||x||_4^4 - \beta_N^4 \frac{1}{4} \sum_{\mu\neq\nu} x_\mu^2 x_\nu^2 - \beta_N^4 \frac{1}{3N} \sum_{i=1}^{N} \sum_{\mu\neq\nu} \xi_i^\mu x_\mu^3 \xi_i^\nu x_\nu \\ &- \beta_N^4 \frac{1}{2N} \sum_{i=1}^{N} \sum_{\mu_1,\mu_2,\mu_3 \ p.d.} x_{\mu_1}^2 \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \\ &- \beta_N^4 \frac{1}{N} \sum_{i=1}^{N} \sum_{\mu_1,\mu_2,\mu_3,\mu_4 \ p.d.} \xi_i^{\mu_1} x_{\mu_1} \xi_i^{\mu_2} x_{\mu_2} \xi_i^{\mu_3} x_{\mu_3} \xi_i^{\mu_4} x_{\mu_4} \\ &+ \mathcal{O}(\beta_N^6 \frac{1}{\sqrt{N}} ||x||^6) \end{aligned}$$

the proof of 3.) follows the lines of the proof of the first part. □

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References

[AGS85] D.J. Amit, G. Gutfreund, H. Sompolinsky; Spin-glass models of neural; Phys. Rev. A 32, 1007-1018 (1985)

[AGS87] D.J. Amit, G. Gutfreund, H. Sompolinsky; Statistical mechanics of neural networks near saturation; Ann. Phys. 173, 30-67 (1987)

[ATZ92] S. Albeverio, B. Tirozzi, and B. Zergalinski; Rigorous results for the free energy in the Hopfield model, Comm. Math. Phys. 150, 337-373 (1992)

[BG92] A. Bovier, V. Gayrard; Rigorous bounds on the storage capacity of the dilute Hopfield model; J. Stat. Phys. 69, 597-627 (1992)

[BG96a] A.Bovier, V.Gayrard; An almost sure large deviation principle for the Hopfield model, in Ann. Prob. 24, 1444-1476 (1996)

[BG96b] A.Bovier, V.Gayrard; Hopfield models as a generalized mean field model, Preprint, to appear in "Mathematics of spin glasses and neural networks", A. Bovier, P.Picco (eds.), "Progress in Probability", Birkhäuser (1997)

[BG96c] A.Bovier, V.Gayrard; An almost sure Central Limit Theorem for the Hopfiled Model, preprint

[BG97] A.Bovier, V.Gayrard; The retrieval phase of the Hopfield model, Prob. Th. Rel. Fields 107, 61-96 (1997)

[BGP94] A.Bovier, V.Gayrard, P.Picco; Gibbs states for the Hopfield model in the regime of perfect memory; Prob. Th. Rel. Fields 100, 329-363 (1994)

[BGP95a] A.Bovier, V.Gayrard, P.Picco; Large deviation principles for the Hopfield model and the Kac-Hopfield model; Prob. Th. Rel. Fields 101, 511-546 (1995)

[BGP95b] A.Bovier, V.Gayrard, P.Picco; Gibbs states for the Hopfield model with extensively many patterns; J. Stat. Phys. 79, 395-414 (1995)

[E85] R.S.Ellis; Entropy, large deviations and statistical mechanics; Grundlehren der Mathematischen Wissenschaften vol.271, Springer (1985).

[EN78] R.S.Ellis, C.M.Newman; Limit Theorems for Sums of Dependent Random Variables occuring in Statistical Mechanics , Z. Wahrscheinlichkeitsth, verw. Geb. 44, 117-139 (1978)

[FP77] L.A. Pastur, A.L. Figotin; Exactly soluble model of a spin-glas; Sov. J. of Low Temprature Phys. 3(6), 378-383 (1977)

[G96a] B. Gentz; An almost sure central limit theorem for the overlap parameters in the Hopfield model; St. Proc. Appl. 62, 243-262 (1996)

[G96b] B. Gentz; A central limit theorem for the overlap parameter in the Hopfield model; Ph. D. thesis; Universität Zürich (1996),

[G96c] B. Gentz; A central limit theorem for the overlap parameter in the Hopfield model; Ann. Prob. 26, 1809-1841 (1996)

[G97] B. Gentz; On the central limit theorem for the overlap parameter in the Hopfield model; to appear in "Mathematics of spin glasses and neural networks", A. Bovier, P.Picco (eds.), "Progress in Probability", Birkhäuser (1997)

[Ho82] J.J. Hopfield; Neural networks and physical systems with emergent collective computational abilities; Proc. nat. Acad. Sci. USA 79, 2554-2558 (1982)

[H59] J. Hubbard; Calculation of partition functions, Phys. rev. Lett. 3, 77-78 (1959)

[K88] J. Komlos, R. Paturi; Convergence results in an associative memory model, Neural Networks 1, 239-250 (1988)

[Lou97] D. Loukianova; Lower bounds on the restitution error in the Hopfield model; Prob. Th. Rel. Fields 107, 161-176 (1997)

[Lö96] M. Löwe; On the storage capacity of Hopfield models with weakly correlated patterns, Preprint, Universität Bielefeld, submitted (1996)

[MPRV87] R. McEliece, E. Posner, E. Rodemich. S. Venkatesh; The capacity of the Hopfield associative memory; IEEE Trans. inf. Th. 33, 461-482 (1987)

[N88] C. Newman; Memory capacity in neural networks; Neural Networks 1, 223-238 (1988)

[S58] R.L. Stratonovich; On a method of calculating quantum distribution functions, Sov. Phys. Doklady 2, 416-419 (1958)

[ST92] E. Scacciatelli, B. Tirozzi; Fluctuations of the free nergy in the Hopfield model, J.Stat. Phys. 67, 981-1008 (1992)

[T95] M. Talagrand; Résultats rigoureux pour le modèle de Hopfield, C.R. Acad. Sci. 321, I, 309-312,(1995)

[T96] M. Talagrand; Rigorous Results of the Hopfield Model with Many Patterns; Preprint, Paris (1996)

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