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Winter Semester 2017/18
12/06/2017

Stochastic Analysis

Exercise Sheet 9

Submission: 12/13/2017 2 p.m.

Exercise 1 (6 points)

Let $(X_n)_{n\geq 0}$ be a Markov chain (MC) taking values in a statespace S with transition probability $\pi(x, \mathrm{d}y)$. Let $\mathbb{P}_x$ denote the law of the MC starting at $x \in S$. Let $u : S \to \mathbb{R}$ be a function which is bounded above and below with a uniformly positive lower bound. Show that for all $n \in \mathbb{N}$

$$\mathbb{E}^{\mathbb{P}_x}\left[e^{\sum_{j=0}^{n-1}\log\left(\frac{u(X_j)}{\pi u(X_j)}\right)}\right] \leq \sigma < \infty$$

where $(\pi u)(y) = \int_S u(z)\pi(y, \mathrm{d}z)$.

Exercise 2 (6 points)

We know BM in $\mathbb{R}$ has generator $L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Let $u : \mathbb{R} \to \mathbb{R}$ s.t. u is bounded above and below with a uniformly positive lower bound. Show that for all $t > 0$ and all $x \in \mathbb{R}$

$$\mathbb{E}^{\mathbb{P}_x}\left[e^{\int_0^t\left(\frac{-Lu}{u}\right)(x(s))\mathrm{d}s}\right] \leq C < \infty.$$

Exercise 3 (3 points)

Show that BM in $\mathbb{R}$ has unbounded variation.

Exercise 4 (5 points)

Let $\tau_L = \inf\{t > 0 : x(t) \notin (-L, L)\}$ for some $L > 0$ that is fixed, where $x(\cdot)$ is BM starting at 0. Show that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_0(\tau_L > t) < 0.$$

Determine the value of the limit.