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Stochastic Analysis

Exercise Sheet 8

Submission: 12/6/2017 2 p.m.

Exercise 1 (6 points)

(a) Show that if $X \sim \mathrm{N}(0,\sigma^2)$, then $\mathbb{E}[X^k] = 0$ for all k odd and $\mathbb{E}[X^{2k}] = c_k\left(\mathbb{E}[X^2]\right)^k = c_k\sigma^{2k}$, where $c_k = 1 \cdot 3 \cdots (2k-1)$

(b) Let $x(t)$ be BM. Follow the proof of Kolmogorov's Continuity theorem to show that the map $t \mapsto x(t)$ is a.s. Hölder Continuous with exponent $\gamma \in (0, 1/2)$.
Hint: (i) Note that by (a),

$$\mathbb{E}[(x(t)-x(s))^4] = 3\mathbb{E}[(x(t)-x(s))^2]^2 = 3|t-s|^2 = 3|t-s|^{1+1}$$

(ii) Now follow the proof of Kolmogorov to deduce that BM is γ-Hölder Cont. with parameter $\gamma < 1/4$.
(iii) But now improve (i) to larger power k s.t.

$$\mathbb{E}[(x(t)-x(s))^{2k}] = c_k|t-s|^k = c_k|t-s|^{1+(k-1)}.$$

Then Kolmogorov implies $t \mapsto x(t)$ is γ-Hölder with $\gamma \in (0, \frac{k-1}{2k})$.
Now choose k large to conclude $x(t) \in C^\gamma$ for $\gamma \in (0, 1/2)$.

(c) Show that $t \mapsto x(t)$ is not 1/2-Hölder Cont. by proving that for any const. A,

$$\mathbb{P}(x(\cdot) : |x(t)-x(s)| \le A|t-s|^{1/2} \text{ for all } s,t) = 0$$

Hint: Observe for any $A > 0$

$$A \ge \sup_{0\le s\le t\le T} \frac{|x(t)-x(s)|}{|t-s|^{1/2}} \ge \sup_j \sqrt{n}\left|x\left(\frac{j+1}{n}\right) - x\left(\frac{j}{n}\right)\right|.$$

If A is larger than the maximum of n independent Standard Gaussian RV's, and since n can be arbitrarily large, and a Gaussian RV is unbounded, A must be infinity.

Exercise 2 (4 points)

Let $x(t)$ be a BM start at 0. Then define $Y(t) = \sigma x(t) + x + mt$ for some $x \in \mathbb{R}$. This is a BM with variance σ^2, starting at x with drift m. Show that the generator of $(Y(t))_t$ is given by

$$(Lf)(x) = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) + m\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Exercise 3 (6 points)

(a) Let $X \sim \mathrm{N}(\mu,\sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$. For $t \in \mathbb{R}$, calculate the characteristic function $\phi_X(t) = \mathbb{E}[e^{itX}]$.

(b) Let $(X_1, ..., X_n)^T$ be a n-dimensional Gaussian RV. Show by using characteristic functions that the RVs $X_1, ..., X_n$ are independent iff they are uncorrelated.
Hint: Use the uniqueness of the characteristic function to show that $\mathbb{P}^{(X_1,...,X_n)} = \bigotimes_{i=1}^n \mathbb{P}^{X_i}$.

(c) Let X, Y be iid $\mathrm{N}(\mu, \sigma^2)$ distributed RVs with $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Show by using characteristic functions that $X + Y$ is independent of $X - Y$.

Exercise 4 (4 points)

Let $X_n \sim \mathrm{Bin}(n, p_n)$, $Y \sim \mathrm{Poi}(\lambda)$ and $np_n \to \lambda$ as $n \to \infty$. Calculate the characteristic functions of X_n and Y and show $X_n \xrightarrow{d} Y$ by using Lévy's continuity theorem.