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Stochastic Analysis

Exercise Sheet 7

Submission: 11/29/2017 2 p.m.

Exercise 1 (6 points)

Let $(x_t)_{t\geq 0}$ be a birth-death process taking values in $\mathbb{Z}_+ = \{0, 1, ...\}$ with transition rates $i \mapsto i+1$ with rate λ_i $\forall i = 0, 1, 2, ..$ and $i \mapsto i-1$ with rate μ_i $\forall i = 1, 2, ...$ and $\mu_0 = 0$.
Show that there is a bounded solution $u : \mathbb{Z}_+ \to \mathbb{R}_+$ to the equation

$$(Lu)(i) = \sigma u(i) \tag{1}$$

for some $\sigma > 0$ where $(Lu)(i) = \lambda_i(u(i+1) - u(i)) + \mu_i(u(i-1) - u(i))$ is the generator of the B-D process if and only if

$$\begin{cases} \sum_{n=1}^{\infty} \left(\frac{\mu_1\mu_2\cdots\mu_n}{\lambda_1\lambda_2\cdots\lambda_n} \left(\sum_{j=1}^{n} \frac{\lambda_1\cdots\lambda_{j-1}}{\mu_1\cdots\mu_{j-1}} \frac{1}{\mu_j} \right) \right) & < \infty \\ \sum_{n=1}^{\infty} \frac{\mu_1\cdots\mu_n}{\lambda_1\cdots\lambda_n} & < \infty \end{cases}$$

Hint: (i) Show that (1) has a bounded solution iff the solution to

$$\begin{cases} u(i+1) - u(i) &= \frac{\mu_i}{\lambda_i}\left[u(i) - u(i-1)\right] + \frac{\sigma}{\lambda_i} u(i) \quad \forall i \geq 1 \\ u(0) &= 1 \\ u(1) &= 1 + \frac{\sigma}{\lambda_0} \end{cases} \tag{2}$$

is bounded.

(ii) Show that the solution to (2) is bounded iff the solution to

$$\begin{cases} u(i+1) - u(i) &= \frac{\mu_i}{\lambda_i}\left[u(i) - u(i-1)\right] + \frac{1}{\lambda_i} \quad \forall i \geq 2 \\ u(0) &= 0 \\ u(1) &= 1 \end{cases}$$

is bounded.

Exercise 2 (4 points)

Show that for the B-D process the probability of explosion in a finite time is either identically 1 or identically 0 for all initial states.

Exercise 3 (5 points)

(a) For any convex function $\phi : \mathbb{R} \to \mathbb{R}$, prove that

$$(L\phi(u))(x) \geq \phi'(u(x))(Lu(x))$$

where $(Lf)(x) = a(x) \int (f(y) - f(x))\pi(x, \mathrm{d}y)$.
Hint: Jensen's inequality.

(b) Prove that for the B-D process, the solution to $(Lu)(i) = \sigma u(i)$ being bounded for some $\sigma > 0$ means that the solution to the same equation being bounded for any $\sigma > 0$.
Hint: Let u_σ solve

$$\begin{cases} Lu &= \sigma u \quad \text{on } A \\ u &= 1 \quad \text{on } A^{\mathsf{c}} \end{cases}$$

for some $\sigma > 0$. Take $v = u^\alpha$ for some $\alpha > 1$ and use Ex. 3(a).

Exercise 4 (5 points)

For a birth-death process with birth rate $(\lambda_n)_{n\geq 0}$ and date rate $(\mu_n)_{n\geq 1}$ with $\mu_0 = 0$, assume that

$$\lambda_n - \mu_n \leq C(n+1) \quad \forall n \geq 1$$

for some $C > 0$. Determine whether this B-D process explodes in finite time.
Hint:

(i) Use the Theorem regarding non-explosion discussed in class.

(ii) Take $u(n) = \sqrt{n+1}$ and note that $u(\cdot)$ is concave. Then use Ex. 3(a) in the opposite direction.