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Stochastic Analysis

Exercise Sheet 6

Submission: 11/22/2017 2 p.m.

Exercise 1 (8 points)

(a) Let τ be an exponential RV with parameter λ, i.e. $\mathbb{P}(\tau > t) = e^{-\lambda t}$. If N is a RV with geometric distribution, i.e. $\mathbb{P}(N = k) = p(1-p)^{k-1}$, then show that the distribution of $S = \tau_1 + ... + \tau_N$, where $\{\tau_i\}_{i=1}^n$ are independent copies of τ and are independent of N, is again exponential with parameter λp.

(b) Let $L = \{a_{i,j}\}_{i,j=1}^k$ be any matrix such that

- $a_{i,j} \geq 0 \quad \forall i \neq j$
- $a_{i,i} \leq 0 \quad \forall i = 1, ..., k$
- $\sum_{j=1}^k a_{i,j} = 0 \quad \forall i = 1, ..., k$

Show that the maximum principle is valid for such a matrix L. In other words, show that if $f : \{1, ..., k\} \to \mathbb{R}$ is any function which has a maximum at some x_0, then $(Lf)(x_0) \leq 0$.

(c) For any $t \geq 0$, let $p(t) = \{p_{i,j}(t)\}_{i,j=1}^k$ be a matrix so that $p_{i,j}(t) \geq 0$ for all i, j and all t and the entries on every row sums to 1. Such a matrix $p(t)$ is called a stochastic matrix. Show that if $p(t)$ is any stochastic matrix satisfying $p(t+s) = p(t)p(s)$ for all $t, s > 0$ and $p(t) \to I$ as $t \to 0$, then the limit

$$L = \lim_{t \searrow 0} \frac{p(t) - I}{t}$$

exists and $L = \{a_{i,j}\}_{i,j=1}^k$ so that

- $a_{i,j} \geq 0 \quad \forall i \neq j$
- $a_{i,i} \leq 0 \quad \forall i = 1, ..., k$
- $\sum_{j=1}^k a_{i,j} = 0 \quad \forall i = 1, ..., k$

(d) Let χ be the state space $\chi = \{1, 2\}$ and

$$L = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then calculate the transition matrix $p(t) = \{p_{i,j}^{(t)}\}_{i,j=1}^2$ at time t.

Exercise 2 (4 points)

Consider a branching process in continuous time, i.e. assume that every individual in a population renews itself at rate λ, i.e. at a random time τ with $\mathbb{P}(\tau > t) = e^{-\lambda t}$. At the renewal time the individual is replaced by a random number $Z \geq 0$ of offsprings. Let $\mathbb{P}(Z = k) = p_k$. Write down the matrix $L = \{a_{i,j}\}$ for this model.

Exercise 3 (8 points)

(a) Fix $\lambda > 0$. Show that the unique solution to the boundary value problem

$$\begin{cases} \lambda u - Lu = g & \text{on } A \\ u = f & \text{on } A^{\mathsf{c}} \end{cases}$$

is given by

$$u(x) = \mathbb{E}^{\mathbb{P}_x}\left[e^{-\lambda \tau_A} f(x_{\tau_A}) + \int_0^{\tau_A} e^{-\lambda s} g(x(s)) \mathrm{d}s\right]$$

where

$$(Lh)(x) = a(x) \int_{\chi} (h(y) - h(x)) \pi(x, \mathrm{d}y)$$

is the generator of a Markov chain with law $\mathbb{P}_x$ starting at $x \in \chi$ and

$$\tau_A = \inf\{t > 0 : x_t \notin A\}$$

is the first exit time from a measurable set A.

(b) Under the same hypotheses as in part (a) and assuming that $\mathbb{P}_x(\tau_A < \infty) = 1$ show that the unique solution u for u being harmonic in A with prescribed boundary value f on A^{c} is given by

$$u(x) = \mathbb{E}^{\mathbb{P}_x}\left[f(x_{\tau_A})\right].$$

That is, the above expression of u uniquely solves

$$\begin{cases} Lu = 0 & \text{on } A \\ u = f & \text{on } A^{\mathsf{c}}. \end{cases}$$

(c) Under the same hypotheses as in part (a), and assuming

$$\mathbb{E}^{\mathbb{P}_x}[\tau_A] < \infty,$$

show that for any bounded function $g : \chi \to \mathbb{R}$ the unique solution u for

$$\begin{cases} Lu = g & \text{on } A \\ u = f & \text{on } A^{\mathsf{c}} \end{cases}$$

is given by

$$u(x) = \mathbb{E}^{\mathbb{P}_x}\left[f(x_{\tau_A}) - \int_0^{\tau_A} g(x_{\tau_A})\right].$$