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Stochastic Analysis

Exercise Sheet 4

Submission: 11/8/2017 2 p.m.

Exercise 1 (5 points)

We consider the SRW ($S_n = X_1 + ... + X_n$, $\mathbb{P}(X_i = \pm 1) = 1/2$) on $\mathbb{Z}$ starting at $x \in (-R, R)$. We define the stopping time $\tau_R = \inf\{n \geq 0 : S_n \notin (-R, R)\}$. Take $\lambda < \frac{\pi}{2R}$ and $\sigma = -\log(\cos\lambda) > 0$.

(a) Show that

$$\mathbb{E}^{\mathbb{P}_x}[e^{\sigma\tau_R}] \leq \frac{\cos\lambda x}{\cos\lambda R}.$$

(b) Do we have equality in the above estimate? If so, what is the range of validity of the equality?

(c) Is it true that $\mathbb{E}^{\mathbb{P}_x}[e^{\sigma\tau_R}] < \infty$ for all $\sigma > 0$?

Exercise 2 (5 points)

Take the SRW in $d \geq 3$, with transition kernel $\pi(x, y)$ and $(\Pi f)(x) = \sum_y f(y)\pi(x, y)$.

(a) Construct a function $v : \mathbb{Z}^d \to (0, \infty)$ such that
   - $(\Pi v)(x) \leq v(x)$ for $|x| > L$ for some L.
   - v is strictly positive everywhere
   - $v(x) \to 0$ as $|x| \to \infty$.

(b) Use optional stopping (OST) to show that

$$\mathbb{P}_x[\tau_L < \infty] \leq \frac{v(x)}{\inf\limits_{|y| \leq L} v(y)} \tag{1}$$

(c) Now for fixed L, let $|x| \to \infty$ to show that (1) goes to 0. Hence $\mathbb{P}_x[\tau_L < \infty] \overset{|x|\to\infty}{\longrightarrow} 0$. Therefore transience in $\mathbb{Z}^d$ in $d \geq 3$.

Exercise 3 (5 points)

Use a similar strategy as in Exercise 2 (i.e. using OST) to show that the SRW is recurrent in $d = 2$.

Exercise 4 (5 points)

Consider the SRW on $\mathbb{Z}$ (see Ex 1).

(a) Show that the SRW is null-recurrent.

(b) Consider the SRW on $\mathbb{Z}$ with a slight drift towards 0:

- $\pi(x, x+1) - \pi(x, x-1) \geq \frac{a}{|x|}$ if $x \leq -l$
- $\pi(x, x-1) - \pi(x, x+1) \geq \frac{a}{|x|}$ if $x > l$.

Show that the RW with these transition probabilities is positive recurrent.

Hint for (b): Take X to be a countable set. Suppose you can find a function $v \geq 0$ a finite set F (which is the set of real values between $-l$ and l) and a constant $C \geq 0$ such that

$$\Pi v(x) - v(x) \leq \begin{cases} -1, & \text{if } x \notin F \\ C, & \text{if } x \in F. \end{cases}$$

Then show that for any $n \in \mathbb{N}$

$$-v(x) \leq -n + (1+C) \sum_{j=1}^{n} \sum_{y \in F} \pi^{(j-1)}(x, y). \tag{2}$$

If a RW with transition operator Π was null-recurrent, then show that (2) implies a contradiction to the assumptions imposed on v.

Exercise 5 (Bonus 3 points)

In exercise 4 of Exercise Sheet 2 you had to prove the identity

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1.$$

Now assume that the identity

$$\liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1$$

holds true. Both identities together were called the "Law of the iterated logarithm".

Now let $(X_n)_{n \geq 1}$ be a sequence of iid Normal$(0,1)$-distributed random variables and $S_n = X_1 + ... + X_n$.

(a) Prove that $\frac{S_n}{\sqrt{2n \log \log n}} - \frac{S_{n+1}}{\sqrt{2(n+1) \log \log (n+1)}}$ converges to 0 a.s.

(b) Use the Law of the iterated logarithm and (a) to prove that the limit points of $\frac{S_n}{\sqrt{2n \log \log n}}$ were given by the whole intervall $[-1, 1]$.

Exercise 6 (Bonus 3 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the canonical filtration $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$. Let $X = (X_n)_{n \in \mathbb{N}_0}$ be a process adapted to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ and $X_i \in L^1$ for all $i = 1, 2, ...$

(a) Show that there is always a representation

$$X = M + A$$

where $M = (M_n)_{n \in \mathbb{N}_0}$ is a $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$-martingale and $A = (A_n)_{n \in \mathbb{N}_0}$ is a $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$-previsible process with $A_0 = 0$.

(b) Show that the decomposition is unique.