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Stochastic Analysis

Exercise Sheet 3

Submission: 11/2/2017 10 a.m.

Exercise 1 (4 points)

(a) Give an example of a martingale which is not convergent almost surely.

(b) Find two martingales whose sum is not a martingale.

Exercise 2 (2 points)

Let $(S_n)_{n\in\mathbb{N}_0}$ be a simple random walk (SRW) and $\mathcal{F}_n = \sigma(X_0, ..., X_n)$. Let $\tau = \inf\{n < 100 : S_n = \max\{S_k : k = 1, ..., 99\}\}$. Is τ a stopping time? Justify your answer.

Exercise 3 (6 points)

Flip a sequence of independent fair coins $X_1, .., X_n$ such that $\mathbb{P}(X_i = \pm 1) = 1/2$ for all i. Note that the random variable $S_n = X_1 + ... + X_n$ is a SRW.

(a) Show that if τ is a stopping time with $\mathbb{E}[\tau] < \infty$, then

$$S^\star(\omega) = \sup_{1\leq n\leq \tau} |S_n(\omega)|$$

is square integrable. Conclude that $\mathbb{E}[S_\tau] = 0$.
Hint: Use that $S_n^2 - n$ is a martingale.

(b) Let τ_1 be the first time that you get a "HT". Show that τ_1 is a stopping time with $\mathbb{E}[\tau_1] < \infty$. What is $\mathbb{E}[\tau_1]$?

(c) What is the expectation of the first time we get a "HH"?

(d) What is the expectation of the first time we get a "HTH"?

Exercise 4 (Polya's urn) (2 points)

An urn contains 1 red ball and 1 green ball. At each time we draw one ball out (uniformly from all balls), then put it back with an extra ball of the same color. Let

$$X_n = \frac{\#\text{red balls at time n}}{\#\text{balls at time n}}.$$

Prove that X_n is a martingale.

Exercise 5 (6 points)

(a) Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$. If $X \in L^1$, show that

$$\lim_{n\to\infty} \mathbb{E}[X|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_\infty]$$

a.s. and in L^1.

Hint:
Step 1: We say that a collection of sets $\mathcal{D}$ is a Π-system if it is closed under intersection, i.e. $A, B \in \mathcal{D} \Rightarrow A \cap B \in \mathcal{D}$. You can assume the following

Theorem. Let $\mathcal{D}$ be a Π-system and let F be a set of bounded functions that satisfies

- $1_D \in F \quad \forall D \in \mathcal{D}$
- $\forall f_1, f_2 \in F$, $c_1 f_1 + c_2 f_2 \in F \quad \forall c_1, c_2 \in \mathbb{R}$
- if a monotonically increasing sequence $(f_n)_{n\geq 1}$ in F converges to a bounded function, then $f \in F$.

Then F contains all bounded $\sigma(\mathcal{D})$-measurable functions.

Step 2: Now show that if $(X_n)_n$ is a martingale that converges in L^1 to a random variable X, then $X_n = \mathbb{E}[X|\mathcal{F}_n]$.

(b) Conclude that for any $A \in \mathcal{F}_\infty$

$$\mathbb{E}[1_A|\mathcal{F}_n] \xrightarrow{\text{a.s.}} 1_A$$

(this is called Levy's $0-1$ law).

(c) Let $X_1, X_2, ...$ be iid and let

$$\mathcal{F}_T = \bigcap_{n=1}^{\infty} \sigma\left(\bigcup_{k\geq n} \mathcal{F}_k\right)$$

be the "tail-σ-algebra". Use the last two parts to show

$$\mathbb{P}(A) \in \{0, 1\} \quad \forall A \in \mathcal{F}_T$$

(Kolmogorov's $0-1$ law).