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Stochastic Analysis

Exercise Sheet 2

Submission: 10/25/2017 2 p.m.

Exercise 1 (4 points)

(a) In exercise 4 of exercise sheet 1 we introduced a special branching process, the Galton-Watson process. Let $M_n = \frac{S_n}{\mu^n}$ be again some branching process, where μ is the mean offspring. Let $\mu \leq 1$ and $\mathbb{P}(Y_{n,k} = 1) \neq 1$. We call this case the Extinction, as $M_\infty = 0$ a.s. On the other hand we have $\mathbb{E}[M_n] = 1$ for all $n \in \mathbb{N}$. Show by using this branching process that

$$M_n \xrightarrow{\text{a.s.}} M_\infty \not\Rightarrow M_n \longrightarrow M_\infty \text{ in } L^1.$$

You can use that $(M_n)_n$ is a martingale.

(b) Construct a non-negative martingale X_n, different from that one in (a), s.t. $\mathbb{E}[X_n] = 1$ for all $n \in \mathbb{N}$ but $X^\star \notin L^1$ when $X^\star = \sup_{n\in\mathbb{N}} X_n$.

Hint: Use for (b) the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, $\mathbb{P} =$Unif$(0,1)$ with filtration $\mathcal{F}_n = \sigma$-alg. generated by intervals ending with $j/2^n$ for some positive integer j and the random variables

$$X_n = \begin{cases} 2^n, & \text{if } 0 \leq x \leq 2^{-n} \\ 0, & \text{if } 2^{-n} \leq x \leq 1 \end{cases}$$

Exercise 2 (4 points)

Let $(X_k)_{k\in\mathbb{N}}$ be iid random variables with $\mathbb{E}[X_1] = 0$ and let $S_n = X_1 + ... + X_n$. Show that

$$\mathbb{P}(\sup_{1\leq k\leq n} S_k > l) \leq \frac{\sum_{k=1}^n \sigma_k^2}{l^2},$$

where $\sigma_k^2 = \mathrm{Var}(X_k) = \mathbb{E}[X_k^2]$.

Exercise 3 (4 points)

Let $X \sim$ Normal$(0,1)$ and $a > 0$. Show

$$\frac{1}{\sqrt{2\pi}} \frac{a}{1+a^2} e^{-\frac{a^2}{2}} \leq \mathbb{P}(X > a) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-\frac{a^2}{2}}. \tag{1}$$

Note that the inequalities in (1) are very useful and you should keep them in mind, as we will need them repeatedly in this course.

Exercise 4 (8 points)

Let $X_1, ..., X_n$ be iid with $X_1 \sim$ Normal$(0,1)$. The moment generating function of $S_n = X_1 + ... + X_n$ is $\mathbb{E}[e^{\theta S_n}] = e^{\frac{\theta^2 n}{2}}$.

(a) Show $\mathbb{P}(\sup_{1\leq k\leq n} S_k \geq c) \leq e^{\frac{\theta^2 n}{2}-\theta c}$ for any $\theta > 0$.

(b) Maximize over θ to show $\mathbb{P}(\sup_{1\leq k\leq n} S_k \geq c) \leq e^{-\frac{c^2}{2n}}$.

(c) For $h(n) = \sqrt{2n\log\log n}$, apply (b) for a sequence, $C_n = rh(r^{n-1})$ for some $r > 1$ and use Borel-Cantelli-Lemma 1 to conclude

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} \leq 1 \quad \text{a.s.}$$

(d) Apply the estimate

$$\mathbb{P}(X > a) \geq \frac{1}{\sqrt{2\pi}}\frac{a}{1+a^2}e^{-\frac{a^2}{2}}$$

and use independence and Borel-Cantelli-Lemma 2, to conclude

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} \geq 1 \quad \text{a.s.}$$

Conclude that $\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n\log\log n}} = 1$ a.s.