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Stochastic Analysis

Exercise Sheet 13

Submission: 01/24/2018 2 p.m.

Exercise 1 (4 points)

Let $x(t)$ be a $\mathcal{F}$-adapted, continuous process taking values in $\mathbb{R}^d$. For $x(0) = 0$ the following statements are equivalent:

(i) $(x(t))_{t\geq 0}$ is SBM

(ii) $(x(t))_{t\geq 0}$ is a continuous martingale with $\langle x^i, x^j \rangle = \delta_{ij} t$ for $i, j \in \{1, ..., d\}$, where $\langle x^i, x^j \rangle$ denotes the covariation.

Hint: For (i) $\Rightarrow$ (ii) you can use that $x^i(t)x^j(t) - \langle x^i, x^j \rangle$ has to be a martingale. For the other direction you can use that (ii) implies
(iii) $x(t \wedge T)$ is for every T a continuous martingale and for $f = (f^1, ..., f^d)$ the process

$$\mathcal{E}_t^{if} = \exp\left(i \sum_k \int_0^t f_s^k \mathrm{d}x^k(t) + \frac{1}{2} \sum_k \int_0^t (f_s^k)^2 \mathrm{d}s \right)$$

is a $\mathbb{C}$-valued martingale. Then use a proper f and uniqueness of the characteristic function.

Exercise 2 (8 points)

Calculate by using Itô's formula:

(i) $\int_0^t s^2 B_s \mathrm{d}B_s$

(ii) $\int_0^t s B_s^2 \mathrm{d}B_s$

(iii) $\int_0^t B_s^3 \exp(B_s^2) \mathrm{d}B_s$

(iv) $\int_0^t s B_s^3 \exp(B_s^2) \mathrm{d}B_s$

where B_s is always SBM.

Exercise 3 (8 points)

We consider the time homogeneous case, where $x(t) = x(t; x)$ is a family of solutions that solve

$$x(t) = x + \int_0^t \sigma(x(s)) \mathrm{d}\beta(s) + \int_0^t b(x(s)) \mathrm{d}s.$$

Show

(i) If a smooth function $u(x)$ on $[a, b]$ solves the equation

$$(Lu)(x) \equiv 0; \text{ for } x \in (a, b)$$

then

$$u(x) = P_x(x(\tau) = a)u(a) + P_x(x(\tau) = b)u(b)$$

where $\tau = \inf\{t : x(t) \notin (a,b)\}$. $u(x(t))$ is a martingale and stop it at τ.

(ii) For $\lambda > 0$, the quantity

$$E^{P_x}\left[e^{-\lambda\tau}\right] = u(x)$$

where again $\tau = \inf\{t : x(t) \notin (a,b)\}$ can be obtained by solving

$$Lu = \lambda u \text{ in } (a,b); u(a) = u(b) = 1.$$

The process $u(x(t))e^{-\lambda t}$ is a martingale.

(iii) For $\lambda > 0$ and bounded f, the quantity

$$E^{P_x}\left[\int_0^\infty e^{-\lambda t} f(x(t))\mathrm{d}t\right] = u(x)$$

can be obtained as the unique bounded solution of

$$\lambda u - Lu = f.$$

Use the fact that

$$e^{-\lambda t}u(x(t)) + \int_0^t e^{-\lambda s} f(x(s))\mathrm{d}s$$

is a martingale. Equate expectations at $t = 0$ and $t = \infty$.

(iv) The Feynman-Kac formula is valid and one can evaluate

$$u(t,x) = E^{P_x}\left[\exp\left(\int_0^t V(x(s))\mathrm{d}s\right) f(x(t))\right]$$

as the solution of

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + b(x)\frac{\partial u(t,x)}{\partial x} + V(x)u(t,x)$$

with $u(0,x) = f(x)$.