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Winter Semester 2017/18
12/20/2017

Stochastic Analysis

Exercise Sheet 11

Submission: 01/10/2018 2 p.m.

Exercise 1 (6 points)

Let $\mathbb{P}_x$ be the measure of a BM starting in x and $\mathbb{Q}_x$ the measure of the Ornstein-Uhlenbeck Process. Remember that the O-U process is BM with drift $b(x) = -cx$ and so corresponds to the operator

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - cx\frac{\mathrm{d}}{\mathrm{d}x}.$$

Calculate the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}_x}{\mathrm{d}\mathbb{Q}_x}\Big|_{\mathcal{F}_{[0,t]}}$.

The following exercises let you train some earlier topics of this course.

Exercise 2 (continuous martingales and OS) (4 points)

(i) Let X be a right-continuous martingale. Show that

$$\lambda\mathbb{P}(\sup_{0\le s\le t} X_s \ge \lambda) \le \mathbb{E}[X_t^+], \quad \lambda > 0,$$

where $X_t^+ = \max\{0, X_t\}$.
Hint: Remember the proof of Doob's inequality.

(ii) Let $(X_t)_{t\ge 0}$ be a continuous, non-negative supermartingale and $\tau = \inf\{t \ge 0 : X_t = 0\}$. Show that

$$X_{\tau+t} = 0 \quad \text{for all } t \ge 0$$

a.s. on the set $\{\tau < \infty\}$.

Exercise 3 (quadratic variation of BM) (3 points)

Let B be SBM and $\mathcal{P}_n$ be the partition of $[0, t]$ given by $t_j = j2^{-n}t$, $j = 1, ..., 2^n$. Show

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} (B_{t_i} - B_{t_{i-1}})^2 \to t \quad \text{a.s.}$$

by calculating mean and variance of the sum and using Borel-Cantelli.

Exercise 4 (BM as martingales) (3 points)

Let $B, B^{(1)}, B^{(2)}$ be SBM. Show that

(i) $X_t = B_t^3 - 3tB_t$ is a martingale

(ii) $Y_t = B_t^4 - 6tB_t^2 + 3t^2$ is a martingale

(iii) X, Y are martingales, where $(X_t, Y_t)^T = A(B_t^{(1)}, B_t^{(2)})^T$ for a 2×2 matrix A and where $B^{(1)}, B^{(2)}$ were independent.

Exercise 5 (Itô formula) (4 points)

Let B be SBM and f some function s.t. $\int_0^t f_s^2 \mathrm{d}s < \infty$ a.s. for all t. Define Z_t to be

$$Z_t = \exp\left\{ \int_0^t f_s \mathrm{d}B_s - \frac{1}{2} \int_0^t f_s^2 \mathrm{d}s \right\}.$$

Use Itô to show

(i) $Z_t = 1 + \int_0^t Z_s f_s \mathrm{d}B_s$.

(ii) If $Y_t = 1/Z_t$, then $Y_t = 1 + \int_0^t Y_s f_s^2 \mathrm{d}s - \int_0^t Y_s f_s \mathrm{d}B_s$.

We wish a merry Christmas and a happy new year.