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Stochastic Analysis

Exercise Sheet 10

Submission: 12/20/2017 2 p.m.

Exercise 1 (4 points)

Let v be a bounded, continuous function and let $u(t,x)$ be a solution of

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}u + v(x)u(t,x) = 0.$$

Then show that

$$z(t) = u(t, x(t)) \exp\left(\int_0^t v(x(s))\mathrm{d}s\right)$$

is a martingale with respect to $\mathbb{P}_x$. What is the connection to Feynman-Kac formula?

Exercise 2 (5 points)

Let $b(x) = x^2$ and $x(0) = x = 1$. For $\beta(t)$ being a standard Brownian motion (SBM) starting at 0, show that the solution of $x(t) = 1 + \beta(t) + \int_0^t |x(s)|^2\mathrm{d}s$ blows up with positive probability before $t = 2$ by comparing it to the solution of

$$x(t) = \int_0^t |x(s)|^2\mathrm{d}s$$

Hint: Note that if $\beta(t) \geq -1$ for $0 \leq t \leq T$, then $x(t) \geq \int_0^t |x(s)|^2\mathrm{d}s$.

Exercise 3 (3 points)

Show that $L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$ is the generator of the process $x(t) = x + \beta(t) + \int_0^t b(x(s))\mathrm{d}s$ where $\beta(\cdot)$ is SBM and $b(\cdot)$ is a uniformly Lipschitz function.

Exercise 4 (3 points)

Calculate by using Itô's formula:

(i) $\int_0^t \mathrm{d}x(s)$

(ii) $\int_0^t x(s)\mathrm{d}x(s)$

(iii) $\int_0^t x(s)^2\mathrm{d}x(s)$

(iv) $\int_0^t (s + e^{x(s)})\mathrm{d}x(s)$

(v) $\int_0^t \frac{1}{1+|x(s)|}\mathrm{d}x(s)$

(vi) $\int_0^t x(s)e^{x(s)^2}\mathrm{d}x(s)$

where $x(s)$ is always SBM.

Exercise 5 (5 points)

Let $x_1(t), ..., x_d(t)$ be independent Brownian motions. Then we call the vector $(x_1(t), ..., x_d(t))$ a d-dimensional Brownian motion.

(i) If $d = 2$ and $r < |x| < R$. Then prove that

$$\mathbb{P}_x(\tau_{B(0,r)} < \tau_{B(0,R)}) = \frac{\log R - \log |x|}{\log R - \log r}$$

where x is the starting point of the d-dimensional BM and $\tau_{B(0,r)}$ is the first time the BM visits the ball with radius r and center 0.
Then show $\mathbb{P}_x(\tau_{B(0,r)} < \infty) = 1$ and $\mathbb{P}_x(x(t) \in B(0, r) \text{ infinitely often}) = 1$.

(ii) Now let $d \geq 3$, $r < |x| < R$. Show that

$$\mathbb{P}_x(\tau_{B(0,r)} < \tau_{B(0,R)}) = \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}$$

and

$$\mathbb{P}_x(\tau_{B(0,r)} < \infty) = \left(\frac{|x|}{r}\right)^{2-d}.$$