Jun.-Prof. Dr. C. Mukherjee
Yannic Bröker

Winter Semester 2017/18
10/11/2017

Stochastic Analysis

Exercise Sheet 1

Submission: 10/18/2017 2 p.m.

Exercise 1 (5 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. Assume that $\sigma, \tau, \tau_1, \tau_2, ...$ are stopping times with respect to $(\mathcal{F}_n)$. Show that the following expressions are also stopping times:

(a) $\min\{\sigma, \tau\}$ and $\max\{\sigma, \tau\}$

(b) $\sigma + \tau$

(c) $\liminf_{k\to\infty} \tau_k$ and $\limsup_{k\to\infty} \tau_k$

Exercise 2 (6 points)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of iid random variables, where $X_1 \sim \text{Normal}(0, \sigma^2)$ for some $\sigma^2 > 0$. For $\lambda \in \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ we define

$$M_n := \exp\left(\lambda \sum_{i=1}^{n} X_i - n\phi(\lambda)\right).$$

Compute ϕ such that the sequence $(M_n)_{n\in\mathbb{N}_0}$ is for every λ a martingale with respect to the canonical filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$.

We write $X \sim$ Unif(a, b) if X has the uniform distribution on the interval (a, b).

Exercise 3 (4 points)

We denote the base of a right-angled triangle by X and the perpendicular by Y. Compute the expected area of the triangle if we assume that $X \sim \text{Unif}(0, 1)$ and $\mathbb{P}(Y|X = x) =\text{Unif}(x, 2x)$ for $x \in (0, 1)$.

Exercise 4 (5 points)

The following model describes the evolution of a population:

Let $(Y_{n,k})_{n\in\mathbb{N}_0,k\in\mathbb{N}}$ be iid random variables in $\mathbb{N}_0$, where $Y_{n,k}$ is the number of children of the k-th individual in the n-th generation. We assume $\mathbb{E}[Y_{n,k}] < \infty$ for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. After one step every individual of the last generation dies such that we can define the number of living individuals by

$$S_0 = 1 \quad S_n = \sum_{k=1}^{S_{n-1}} Y_{n-1,k}, \quad n \geq 1.$$

Prove that

$$Z_n := \frac{S_n}{\mu^n}, \quad n \geq 1$$

is a martingale with respect to $\mathcal{F}_n = \sigma(S_0, ..., S_n)$.