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Stochastic Analysis

Exercise Sheet 0

You do not have to submit the solutions of this paper, but to prepare it for the tutorial on 10/18/2017

Exercise 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$, $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ and $X, Y, X_1, X_2, ...$ are integrable. Prove the following properties of the conditional expectation:

1 Linearity
$$\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha \mathbb{E}[X|\mathcal{G}] + \beta \mathbb{E}[Y|\mathcal{G}] \quad \mathbb{P} - \text{a.s.} \ \forall\, \alpha, \beta \in \mathbb{R}$$

2 Positivity
$$X \geq 0 \quad \mathbb{P} - \text{a.s.} \ \Rightarrow \mathbb{E}[X|\mathcal{G}] \geq 0 \quad \mathbb{P} - \text{a.s.}$$

3 Tower law
$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]|\mathcal{G}_1] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_1] \quad \mathbb{P} - \text{a.s.}$$

4 Law of total expectation
$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X] \quad \mathbb{P} - \text{a.s.}$$

5 Taking out what is known
X, Y are non-negative, then
$$Y \text{ is } \mathcal{G} - \text{measurable} \ \Rightarrow \mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}] \quad \mathbb{P} - \text{a.s.}$$

6 Independence
$$X \text{ is independent of } \mathcal{G} \Rightarrow \mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X] \quad \mathbb{P} - \text{a.s.}$$

7 Monotone convergence
$$X_n \nearrow X \quad \mathbb{P} - \text{a.s.} \ \Rightarrow \mathbb{E}[X_n|\mathcal{G}] \nearrow \mathbb{E}[X|\mathcal{G}] \quad \mathbb{P} - \text{a.s.}$$

8 Dominated convergence
$$X_n \to X \quad \mathbb{P} - \text{a.s.} \ \text{ and } \sup_{n \geq 1} |X_n| \in L^1(\mathcal{F}) \Rightarrow \mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}] \quad \mathbb{P} - \text{a.s.}$$

Hint: Use for 5:
Let X, Y be non-negative and Y be $\mathcal{G}$-measurable, then Y is a version of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ if and only if $\mathbb{E}[XZ] = \mathbb{E}[YZ]$ for all non-negative $\mathcal{G}$-measurable Z.

Exercise 2

Show that the following statements are equivalent:

1 $\mu \ll \nu$ on some probability space $(\Omega, \mathcal{F})$

2 $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall A \in \mathcal{F}$

$$\nu(A) < \delta \Rightarrow \mu(A) < \varepsilon$$

Exercise 3

Suppose $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}[X|Y] = Y$ a.s. and $\mathbb{E}[Y|X] = X$ a.s.
Prove that $X = Y$ a.s.