Chapter 3
Immigration

3.1 The model

A natural extension of the population model considered so far is to incorporate the possibility of *immigration* which means that at any time individuals from outside may join the population and then produce offspring like the others. We set out with a precise definition of the model to be studied in this chapter.

Definition 3.1. For $k \in \mathbb{N}_0$, let $(Z_n(k))_{n\geq 0}$ be independent GWP's with the same offspring distribution $(p_n)_{n\geq 0}$ and offspring mean m. Further, let $Z_0(1), Z_0(2), \ldots$ be identically distributed with common distribution $(c_n)_{n\geq 0}$. Then

$$Z_n := \sum_{k=0}^{n} Z_{n-k}(k), \quad n \in \mathbb{N}_0 \tag{3.1}$$

is called *Galton-Watson process with immigration (GWPI)* with offspring distribution $(p_n)_{n\geq 0}$, immigration distribution $(c_n)_{n\geq 0}$ and $Z_0 = Z_0(0)$ ancestors. As before, it is further called *subcritical, critical*, or *supercritical)* depending on whether $\mathsf{m} < 1$, $\mathsf{m} = 1$, or $\mathsf{m} > 1$, respectively.

The interpretation of (3.1) is that any generation $n \in \mathbb{N}$ is formed by the progeny of the previous one (its endogenous part), viz.

$$Z_n' := \sum_{k=0}^{n-1} Z_{n-k}(k),$$

plus a random number $Z_0(n)$ of immigrants that join the population and then produce offspring like all other individuals. In other words, they spawn a subpopulation described by a simple GWP $(Z_k(n))_{k\geq 0}$ with the same offspring distribution $(p_n)_{n\geq 0}$. Naturally, its k^{th} generation forms a subset of the $(n+k)^{th}$ generation of the considered total population.

Let $\mathscr{F}_n$ be the σ-field generated by $\{Z_k(j) : 0 \le j+k \le n\}$ for $n \ge 0$. The assumptions stated in Def. 3.1 ensure that $Z_0(n)$ is independent of both, $\mathscr{F}_{n-1}$ and Z'_n, for each $n \ge 1$ and that

$$\mathbb{P}(Z'_n = j, Z_0(n) = k|\mathscr{F}_{n-1}) \;=\; \mathbb{P}(Z'_n = j\,|\mathscr{F}_{n-1})\,\mathbb{P}(Z_0(n) = k) \quad \text{a.s.} \tag{3.2}$$

As Z'_n gives the number of offspring of the $(n-1)^{th}$ generation, we further have

$$\mathbb{P}(Z'_n = j\,|\mathscr{F}_{n-1}, Z_{n-1} = i) \;=\; \mathbb{P}(Z'_n = j\,|Z_{n-1} = i) \;=\; p_j^{*i} \tag{3.3}$$

for all $n \ge 1$ and $i, j \ge 0$. We thus see that $(Z_n)_{n\ge0}$ constitutes a temporally homogeneous Markov chain on $\mathbb{N}_0$ with transition matrix $P = (p_{ij})_{i,j\ge0}$ given by

$$\begin{aligned} p_{ij} &= \mathbb{P}(Z_n = j\,|Z_{n-1} = i) \\ &= \sum_{k=0}^{j} \mathbb{P}(Z'_n = j-k, Z_0(n) = k\,|Z_{n-1} = i) \\ &= \sum_{k=0}^{j} \mathbb{P}(Z'_n = j-k\,|Z_{n-1} = i)\,\mathbb{P}(Z_0(n) = k) \\ &= \sum_{k=0}^{j} p_{j-k}^{*i} c_k \end{aligned}$$

for all $i, j \in \mathbb{N}_0$.

So far we have not specified the distribution of $Z_0 = Z_0(0)$. As in the non-immigration case, we could consider a standard model with probability measures $\mathbb{P}_i$, $i \in \mathbb{N}_0$, such that $(Z_n)_{n\ge0}$ is a GWPI as described under every $\mathbb{P}_i$ with $\mathbb{P}_i(Z_0 = i) = 1$. For ease of exposition, however, the subsequent analysis will be done under the assumption that $Z_0(0)$ has the same distribution $(c_n)_{n\ge0}$ as all other $Z_0(k)$, $k \ge 1$. In the afore-mentioned standard model this amounts to choosing $\mathbb{P} = \sum_{i\ge0} c_i \mathbb{P}_i$ as the underlying probability measure. The reader can readily verify that a deviation from this assumption does either not at all affect the results or just requires some minor adjustment [☞ e.g. Problem 3.5].

3.2 Generating functions

Since gf's are again heavily used in the subsequent analysis, we first collect some basic properties of them in the present setup. Let h_n denote the gf of Z_n, f as before the gf of $(p_n)_{n\ge0}$ with n-fold iteration f_n, and h the gf of the immigration distribution $(c_n)_{n\ge0}$. Note that $h_0 = h$ by our convention that $Z_0(0)$ has distribution $(c_n)_{n\ge0}$.

In order to provide a recursive relation for h_n, there are two distinct ways of decomposing Z_n at any fixed time n. Adopting the *forward view*, Z_n is split up into its endogenous part Z'_n and its exogenous or immigration part $Z_0(n)$ at time n, whereas under the *backward view* Z_n is decomposed into the subpopulation size $Z_n(0)$ of those individuals stemming from the $Z_0(0)$ ancestors at time 0 (the natives) and the total number of all other ones (having immigration background), viz. $\sum_{k=1}^{n} Z_{n-k}(k)$. Here are the formal details:

Using (3.2) and (3.3), we infer

$$\mathbb{E}(s^{Z_n}|\mathscr{F}_{n-1}) \;=\; \mathbb{E}(s^{Z_n'}|\mathscr{Z}_{n-1})\,\mathbb{E}(s^{Z_0(n)}) \;=\; f(s)^{Z_{n-1}}h(s) \quad \text{a.s.}$$

and thereupon the *forward equation*

$$h_n(s) \;=\; h(s)\,\mathbb{E}f(s)^{Z_{n-1}} \;=\; h(s)\,h_{n-1}\circ f(s) \tag{3.4}$$

for any $n\in\mathbb{N}$ and $s\in[0,1]$. The backward view embarks on (3.1) and provides us with

$$\begin{aligned} h_n(s) \;&=\; \prod_{k=0}^{n}\mathbb{E}\,s^{Z_{n-k}(k)} \\ &=\; \prod_{k=0}^{n}\left(\sum_{j\ge 0}\mathbb{E}(s^{Z_{n-k}(k)}|Z_0(k)=j)\,c_j\right) \\ &=\; \prod_{k=0}^{n}\left(\sum_{j\ge 0} f_{n-k}(s)^j c_j\right) \\ &=\; \prod_{k=0}^{n} h\circ f_k(s) \end{aligned} \tag{3.5}$$

and thus with the *backward equation*

$$h_n(s) \;=\; h\circ f_n(s)\,h_{n-1}(s) \tag{3.6}$$

for any $n\in\mathbb{N}$ and $s\in[0,1]$. As an immediate consequence of this equation, we find that h_n decreases to a limiting function h_∞ on $[0,1]$ satisfying $h_\infty(1)=1\ge h_\infty(1-)$ and $h_\infty(s)=\sum_{k\ge 0}d_k s^k$ for suitable $d_k\ge 0$. The forward equation then further implies

$$h_\infty(s) \;=\; h(s)\,h_\infty\circ f(s) \tag{3.7}$$

for $s\in[0,1]$, which in terms of random variables takes the form of another distributional equation, viz.

$$Z_\infty \;\stackrel{d}{=}\; Y+\sum_{n=1}^{Z_\infty} X_n \tag{3.8}$$

where all occurring random variables are independent with Z_∞ having gf h_∞, Y having gf h and thus distribution $(c_n)_{n\ge 0}$, and $X_1, X_2, \ldots$ having gf f and thus distribution $(p_n)_{n\ge 0}$. In the next section, we give a necessary and sufficient condition for h_∞ to be the gf of a proper distribution on $\mathbb{N}_0$, i.e. $h_\infty(1-)=\sum_{k\ge 0}d_k=1$. As for Z_∞, this means that this random variable is a.s. finite. Let us finally note that by (3.5), we find that

$$h_\infty(s) \;=\; \prod_{n\ge 0} h\circ f_n(s), \tag{3.9}$$

for $s\in[0,1]$.

3.3 Subcritical and critical case: a stability theorem and a gamma limit law

Knowing from the previous chapters that a simple GWP never stabilizes on a positive finite level, it is natural to ask whether this may be accomplished for a GWPI under a suitable immigration rate. Plainly, a positive probability for at least one immigrant per generation excludes almost certain extinction of the population, but 0 may now occur as an intermittent state, for it is no longer absorbing. On the other hand, if explosion of the population is to be excluded, any population member can only generate a finite subpopulation which means that its pertinent GWP must have extinction probability one and thus be subcritical or critical. The following theorem by FOSTER & WILLIAMSON [11] provides a necessary and sufficient condition for stability result of this kind.

Theorem 3.2. [Foster-Williamson] *Let $(Z_n)_{n\ge 0}$ be a GWPI with offspring mean $\mathsf{m}\in(0,1]$. Then Z_n converges in distribution to a finite random variable Z_∞ with gf h_∞ iff*

$$\int_0^1 \frac{1-h(s)}{f(s)-s}\,ds \;<\; \infty. \tag{3.10}$$

In this case, h_∞ is the unique solution to equation (3.7) *which is continuous on* $[0,1]$ *and satisfies* $h_\infty(1)=1$.

Denoting by $\pi=(\pi_j)_{j\ge 0}$ the distribution of Z_∞, the previous result implies that

$$\pi_j \;=\; \lim_{n\to\infty}\mathbb{P}(Z_{n+1}=j) \;=\; \lim_{n\to\infty}\sum_{i\ge 0}\mathbb{P}(Z_n=i)\,p_{ij} \;=\; \sum_{i\ge 0}\pi_i p_{ij}$$

for all $i\ge 0$ which means that π satisfies the invariance or balance equation $\pi=\pi P$ and is thus a stationary distribution of the Markov chain $(Z_n)_{n\ge 0}$, i.e.

$$\mathbb{P}_\pi((Z_n)_{n\ge 0}\in\cdot) \;=\; \mathbb{P}_\pi((Z_{k+n})_{n\ge 0}\in\cdot)$$

for all $k\in\mathbb{N}$. It can further be shown [☞ Problem 3.5] that every state $j\in\mathbb{N}_0$ with $\pi_j>0$ is positive recurrent and that Z_n has asymptotic distribution π under any initial distribution. Therefore, π is the unique nonnegative solution to the balance equation modulo positive scalars.

As a consequence of Thm. 3.2 and an analytic lemma similar to Lemma 2.6, we obtain the following older result due to HEATHCOTE [14]. The reader is asked to give a proof in Problem 3.7.

Corollary 3.3. [Heathcote] *Let $(Z_n)_{n\geq 0}$ be a subcritical GWPI with $p_0 < 1$. Then Z_n converges in distribution to a finite random variable Z_∞ with gf h_∞ iff*

$$\sum_{n\geq 2} c_n \log n \ < \ \infty. \tag{3.11}$$

For a critical GWPI with finite reproduction variance $\sigma^2 = f''(1)$ and finite *immigration mean* $\nu = \sum_{n\geq 1} nc_n = h'(1)$, the situation is different, for Thm. 3.2 easily implies that Z_n cannot converge in distribution to a finite random variable [☞ Problem 3.8]. Instead, a conditional limit theorem similar to the one obtained for critical ordinary GWP [☞ (2.33) in Thm. 2.24] holds true. The following heuristic argument may serve as an intuitive explanation. Per generation, an average number of ν individuals immigrate and create critical simple GWP's of which, by (2.31) in Thm. 2.24, approximately $2\nu/(n\sigma^2)$ survive the next n generations. In the long run we thus expect about $2\nu/\sigma^2$ processes to survive, and by (2.33) they are approximately exponentially distributed with parameter $2/\sigma^2$ after normalization. Consequently, $n^{-1}Z_n$ given $Z_n > 0$ is expected to have a gamma law with parameters $2\nu/\sigma^2$ and $2/\sigma^2$, as $n \to \infty$. The result was proved by FOSTER in his Ph.D. thesis [10] and also by SENETA [28] and PAKES [23].

Theorem 3.4. [Foster-Pakes-Seneta] *Let $(Z_n)_{n\geq 0}$ be a critical GWPI with finite offspring variance $\sigma^2 = f''(1)$ and finite immigration mean $\nu = h'(1)$. Then $n^{-1}Z_n$ converges in distribution to a random variable Z_∞ having a gamma law with parameters $\alpha := 2\nu/\sigma^2$ and $\beta := 2/\sigma^2$, i.e.*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z_n}{n} \leq t\right) \ = \ \int_0^t \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\beta x}\, dx$$

for all $t \in \mathbb{R}_>$.

It is worth pointing out that a critical GWPI with *infinite* offspring variance σ^2 may very well converge in distribution without normalization. An example is given in Problem 3.9, and we also refer to Sect. 3.5 for further studies of the Markov chain $(Z_n)_{n\geq 0}$ in the critical case.

Proof (of Thm. 3.2). We have already mentioned at the end of the previous section that h_n converges pointwise on $[0,1]$ to a function h_∞, namely [☞ (3.9)]

$$h_\infty \;=\; \prod_{n\ge 0} h\circ f_n \;=\; \prod_{n\ge 1}\big(1-(1-h\circ f_n)\big)$$

for $s\in[0,1]$. It is the gf of a distribution Q on $\mathbb{N}_0\cup\{\infty\}$ which satisfies $Q(\mathbb{N}_0)=1$ iff $\lim_{s\uparrow 1} h_\infty(s)=h_\infty(1-)=1$. In the following we will consider $\log h_\infty$ on $[0,1]$. It follows from

$$0 \;\le\; -\sum_{k\ge n}\log h\circ f_k(s) \;\le\; -\sum_{k\ge n}\log h\circ f_k(0)$$

for all $s\in[0,1]$ and $n\in\mathbb{N}_0$ that $\log h_\infty$ is the uniform limit of the continuous functions $\sum_{k=0}^n \log h\circ f_k$ and thus continuous as well on $[0,1]$ iff

$$-\sum_{k\ge 0}\log h\circ f_k(0) \;\asymp\; \sum_{k\ge 0}\big(1-h\circ f_k(0)\big) \;<\; \infty. \tag{3.12}$$

Therefore it suffices to show the equivalence of (3.10) and (3.12) hereafter. By the convexity of f,h, the functions

$$\frac{1-h(s)}{1-s} \quad\text{and}\quad \frac{1-s}{f(s)-s} \;=\; \left(1-\frac{1-f(s)}{1-s}\right)^{-1}$$

are both nondecreasing in s, which in combination with $f_k(0)\uparrow 1$ implies

$$\begin{aligned}
\sum_{k\ge 0}\big(1-h\circ f_k(0)\big) \;&=\; \sum_{k\ge 0}\frac{1-h\circ f_k(0)}{f\circ f_k(0)-f_k(0)}\big(f_{k+1}(0)-f_k(0)\big)\\
&\le\; \int_0^1 \frac{1-h(s)}{f(s)-s}\,ds\\
&\le\; \sum_{k\ge 0}\frac{1-h\circ f_{k+1}(0)}{f\circ f_{k+1}(0)-f_{k+1}(0)}\big(f_{k+1}(0)-f_k(0)\big)\\
&=\; \sum_{k\ge 0}\big(1-h\circ f_{k+1}(0)\big)\frac{f_{k+1}(0)-f_k(0)}{f\circ f_{k+1}(0)-f\circ f_k(0)}.
\end{aligned}$$

By the mean value theorem,

$$\frac{f_{k+1}(0)-f_k(0)}{f\circ f_{k+1}(0)-f\circ f_k(0)} \;=\; \frac{1}{f'(s_k)}$$

for some $s_k\in(f_k(0),f_{k+1}(0))$ and each $k\in\mathbb{N}_0$. Since $f'(s_k)\to f'(1)=\mathsf{m}\in(0,1]$, the first and the last sum in the above estimation are always finite together, that is

$$\sum_{k\ge 0}\big(1-h\circ f_k(0)\big) \;\asymp\; \sum_{k\ge 0}\big(1-h\circ f_{k+1}(0)\big)\frac{f_{k+1}(0)-f_k(0)}{f\circ f_{k+1}(0)-f\circ f_k(0)}$$

and this proves the desired equivalence.

It remains to show the uniqueness assertion for which we assume that $\widehat{h}_\infty$ is another continuous solution to equation (3.7) on [0,1] with $\widehat{h}_\infty(1)=1$. By the standard iteration argument, we infer that on $[0,1)$

$$\begin{aligned}|h_\infty - \widehat{h}_\infty| \;&=\; |h|\cdot|h_\infty\circ f - \widehat{h}_\infty\circ f| \;\le\; |h_\infty\circ f - \widehat{h}_\infty\circ f|\\ &\le \ldots \le\; |h_\infty\circ f_n - \widehat{h}_\infty\circ f_n| \;\to\; |h_\infty(1) - \widehat{h}_\infty(1)| \;=\; 0,\end{aligned}$$

as $n\to\infty$. The last convergence is guaranteed by the continuity of $h_\infty,\widehat{h}_\infty$ at 1 and the fact that $f_n \nearrow 1$ on $[0,1)$ by Cor. 1.8. □

Proof (of Thm. 3.4). First note that $n^{-1}Z_n$ and Z_∞ have LT's $h_n(e^{-t/n})$ and $(1+\beta t)^{-\alpha}$, respectively, so that

$$\lim_{n\to\infty}\log h_n(e^{-t/n}) \;=\; -\alpha\,\log(1+\beta t) \tag{3.13}$$

for $t\in\mathbb{R}_>$ must be verified. By (3.5),

$$\log h_n(e^{-t/n}) \;=\; \sum_{k=0}^{n}\log h\circ f_k(e^{-t/n}) \tag{3.14}$$

for $t\in\mathbb{R}_>$ and $n\in\mathbb{N}_0$. Further ingredients to the proof of (3.13) are a Taylor-like expansion of $-\log h$ at 1, viz.

$$-\log h(s) \;=\; (\nu+\rho(s))(1-s) \tag{3.15}$$

with $\lim_{s\uparrow 1}\rho(s)=0$, and Lemma 2.26 which provides us with

$$\frac{1}{1-f_n(s)} - \frac{1}{1-s} \;=\; n(\beta + r_n(s)) \tag{3.16}$$

where $\lim_{n\to\infty} r_n(s) = 0$ uniformly in $s\in[0,1]$. The last equation may be rewritten as

$$1-f_n(s) \;=\; \frac{1-s}{1+n(\beta+r_n(s))(1-s)}. \tag{3.17}$$

A combination of (3.14)–(3.17) leads to

$$\begin{aligned}-\log h_n(e^{-t/n}) \;&=\; \sum_{k=0}^{n}\Big(\nu+\rho\circ f_k(e^{-t/n})\Big)(1-f_k(e^{-t/n}))\\ &=\; \sum_{k=0}^{n}\frac{\big(\nu+\rho\circ f_k(e^{-t/n})\big)(1-e^{-t/n})}{1+k(\beta+r_k(e^{-t/n}))(1-e^{-t/n})} \;=:\; S_n(e^{-t/n})\end{aligned}$$

for all $t\in\mathbb{R}_>$. Fixing any t and $\varepsilon\in(0,\beta)$, let $N\in\mathbb{N}$ be so large that

$$\sup_{n\ge N}\sup_{k\ge 1}|\rho\circ f_k(e^{-t/n})| < \varepsilon \quad\text{and}\quad \sup_{n\ge N}\sup_{k\ge 1}|r_k(e^{-t/n})| < \varepsilon,$$

where $f(e^{-t/N})\le f(e^{-t/n})\le 1$ for all $n\ge N$ and $k\ge 1$ should be observed. Obviously, $\lim_{n\to\infty} S_N(e^{-t/n}) = 0$ and, setting $\Delta_{N,n}(e^{-t/n}) = S_n(e^{-t/n}) - S_{N-1}(e^{-t/n})$,

$$\sum_{k=N}^{n} \frac{(\nu-\varepsilon)(1-e^{-t/n})}{1+k(\beta+\varepsilon)(1-e^{-t/n})} \;\le\; \Delta_{N,n}(e^{-t/n}) \;\le\; \sum_{k=N}^{n} \frac{(\nu+\varepsilon)(1-e^{-t/n})}{1+k(\beta-\varepsilon)(1-e^{-t/n})}$$

for $n \ge N$. By using these facts together with the simple inequality

$$\frac{a}{b}\log\left(\frac{1+b(n-1)}{1+bN}\right) \;\le\; \sum_{k=N}^{n} \frac{a}{1+bk} \;\le\; \frac{a}{b}\log\left(\frac{1+bn}{1+b(N-1)}\right)$$

for any $a,b>0$ we further obtain

$$\frac{\nu-\varepsilon}{\beta+\varepsilon}\log\left(\frac{1+(\beta+\varepsilon)(1-e^{-t/n})(n-1)}{1+(\beta+\varepsilon)(1-e^{-t/n})N}\right)$$
$$\le\; \Delta_{N,n}(e^{-t/n}) \;\le\; \frac{\nu+\varepsilon}{\beta-\varepsilon}\log\left(\frac{1+(\beta-\varepsilon)(1-e^{-t/n})n}{1+(\beta-\varepsilon)(1-e^{-t/n})(N-1)}\right).$$

Now let n tend to infinity to conclude that

$$\frac{\nu-\varepsilon}{\beta+\varepsilon}\log(1+(\beta+\varepsilon)t) \;\le\; \liminf_{n\to\infty} S_n(e^{-t/n})$$
$$\le\; \limsup_{n\to\infty} S_n(e^{-t/n}) \;\le\; \frac{\nu+\varepsilon}{\beta-\varepsilon}\log(1+(\beta-\varepsilon)t),$$

which proves (3.13) because ε can be made arbitrarily small. □

Problems

Problem 3.5. Let $(Z_n)_{n\ge 0}$ be a GWPI in a standard model with offspring mean $\mathsf{m} \in (0,1]$. Show that, under every $\mathbb{P}_i$, Z_n converges in distribution to the law π on $\mathbb{R}_> \cup \{\infty\}$ having gf h_∞. Argue that this implies the uniqueness of π modulo positive scalars as a nonzero solution to the invariance equation $\pi = \pi P$, $P = (p_{ij})_{i,j\in\mathbb{N}_0}$, and that all states j with $\pi_j = 0$ are transient.

Problem 3.6. In the situation of Thm. 3.2, show that $\min_{n\ge 0} Z_n = j_0$ $\mathbb{P}$-a.s., where $j_0 = \min\{j \ge 0 : c_j > 0\}$ and $\mathbb{P} = \mathbb{P}_1$.

Problem 3.7. (a) Show the equivalence of (3.11) in Heathcote's result [Cor. 3.3] to each of

$$\sum_{n\ge 0}(1-h(1-\delta^n)) < \infty \quad \text{for all (some) } \delta \in (0,1) \tag{3.18}$$

and

$$\int_0^1 \frac{1-h(s)}{1-s}\,ds \;<\; \infty. \tag{3.19}$$

[Hint: Show first the equivalence of (3.18) and (3.19) via a similar argument as in the proof of Lemma 2.6.]

(b) Prove Heathcote's result with the help of part (a) and Thm. 3.2.

Problem 3.8. Show with the help of Thm. 3.2, but *without* using Thm. 3.4, that a critical GWPI with finite reproduction variance and finite immigration mean never converges in distribution to a finite random variable.

Problem 3.9. (a) Show that $f(s) := s+\beta(1-s)^{\alpha}$ for $\alpha \in (1,2)$ and $\beta \in (0,1/\alpha)$ constitutes the gf of a distribution on $\mathbb{N}_0$ with mean one and infinite variance.

(b) [SENETA] Let $(Z_n)_{n\geq 0}$ be a GWPI with offspring gf f of the form in (a) and finite immigration mean. Prove that Z_n converges in distribution to a finite random variable Z_∞.

Problem 3.10. Let $(Z_n)_{n\geq 0}$ be a GWPI with offspring mean $\mathsf{m} \in (0,1]$ and stationary distribution $\pi = (\pi_n)_{n\geq 0}$ with mean η. Prove that $\eta < \infty$ iff $\mathsf{m} < 1$ and $\nu < \infty$, and that in this case $\eta = (1-\mathsf{m})^{-1}\nu$ holds true.

Problem 3.11. Let f be a subcritical gf with $\mathsf{m} = f'(1)$ and then g the gf of the pertinent Yaglom limit in Thm. 2.14, so that $g(s) = \mathsf{m}\, g\circ f(s) + (1-\mathsf{m})$. Define

$$h_\infty(s) \;=\; \frac{1-g(s)}{1-s} \quad \text{and} \quad h(s) \;=\; \frac{1-f(s)}{\mathsf{m}(1-s)}$$

and show that h_∞ is a solution to (3.7) for the pair (f,h). [Note: This is an intriguing connection between the functional equation (2.24) valid for g and equation (3.7). On the other hand, as $h_\infty(1) = g'(1) \geq 1$, h_∞ is usually not a gf of a distribution, but if $g'(1) < \infty$, one can take $h_\infty/g'(1)$ which clearly also solves (3.7).]

3.4 The supercritical case: a counterpart of the Heyde-Seneta theorem

Let us turn to the supercritical case and assume $1 < \mathsf{m} < \infty$. Looking back at our considerations for simple supercritical GWP's, it appears to be natural to set out with a study of the asymptotic behavior of $W_n = \mathsf{m}^{-n} Z_n$. If the immigration mean ν is finite, one can indeed readily verify that

$$\frac{1}{\mathsf{m}^n}\left(Z_n - \frac{\nu(\mathsf{m}^{n+1}-1)}{\mathsf{m}-1}\right) \;=\; W_n - \frac{\nu(\mathsf{m}^{n+1}-1)}{\mathsf{m}^n(\mathsf{m}-1)}, \quad n \geq 0 \tag{3.20}$$

forms an L^1-bounded martingale which thus converges a.s. to an integrable random variable $W - (\mathsf{m}-1)^{-1}\nu\mathsf{m}$ [☞ Problem 3.14]. However, Thm. 3.12 below will provide us with a more general result by pursuing the same plan that led us to the Heyde-Seneta theorem in Sect. 2.1.

Recall from (3.1) that $Z_n = \sum_{k=0}^{n} Z_{n-k}(k)$ where the $(Z_n(k))_{n\geq 0}$ are iid simple GWP's with ancestor distribution $(c_n)_{n\geq 0}$. As in Sect. 2.1, let g_n denote the inverse of f_n on $[q,1]$, q as usual the smallest fixed point of f in $[0,1]$, and $k_n = -1/\log g_n(s)$ for an arbitrary fixed $s \in (q,1)$. Thm. 2.1 then implies that

$$\lim_{n\to\infty} \frac{Z_n(k)}{k_n} \ = \ V^*(k) \quad \text{a.s.}$$

for all $k \in \mathbb{N}_0$, and the $V^*(k)$ are clearly iid random variables. Defining

$$\varphi(t) \ = \ \mathbb{E}\left(e^{-tV^*(k)}\Big|Z_0(k)=1\right), \quad t \geq 0,$$

the unconditional LT of $V^*(k)$ is easily calculated as

$$\phi(t) \ = \ \sum_{n\geq 0} c_n \mathbb{E}\left(e^{-tV^*(k)}\Big|Z_0(k)=n\right) \ = \ \sum_{n\geq 0} c_n \varphi(t)^n \ = \ h\circ\varphi(t) \tag{3.21}$$

for $t \geq 0$, having once again utilized Lemma 1.2. Recall from Thm. 2.8 that φ satisfies the functional equation

$$\varphi(\mathsf{m}t) \ = \ f\circ\varphi(t), \quad t\geq 0. \tag{3.22}$$

We are now ready to formulate the announced counterpart of the Heyde-Seneta theorem which is again a result owing to SENETA [29].

Theorem 3.12. [Seneta] *Let $(Z_n)_{n\geq 0}$ be a supercritical GWPI with finite reproduction mean* m *and $c_0 < 1$. Then $W^* := \lim_{n\to\infty} k_n^{-1} Z_n$ exists a.s. and satisfies*

$$\sum_{n\geq 2} c_n \log n \ = \ \infty \quad \Rightarrow \quad W^* = \infty \quad \textit{a.s.}, \tag{3.23}$$

$$\sum_{n\geq 2} c_n \log n \ < \ \infty \quad \Rightarrow \quad 0 < W^* < \infty \quad \textit{a.s.} \tag{3.24}$$

In the last case, the LT ψ of W^* satisfies the functional equation

$$\psi(t) \ = \ \prod_{n\geq 0} h\circ\psi\left(\frac{t}{\mathsf{m}^n}\right) \tag{3.25}$$

for $t \geq 0$. One may choose $k_n = \mathsf{m}^n$ in the above result if (ZlogZ) *holds true for the offspring distribution $(p_n)_{n\geq 0}$.*

A result similar to Heyde's Lemma 2.5 will be given first as a prerequisite for the proof of this theorem. Recall that $\mathscr{F}_n = \sigma(Z_k(j) : 0 \leq j+k \leq n)$ and put $X_n(s) = g_n(s)^{Z_n}$ for $n \in \mathbb{N}_0$ and $s \in [q,1]$.

Lemma 3.13. *For each $s \in [q,1]$, $(X_n(s),\mathscr{F}_n)_{n\ge 0}$ forms a nonnegative supermartingale taking values in $[0,1]$. It converges a.s. and in L^p for any $p \ge 1$ to a random variable $X(s)$.*

Proof. We must only show the supermartingale property, for then all further properties follow immediately because the sequence is bounded. By using the basic model properties stated in Sect. 3.1, we infer for any $n \in \mathbb{N}_0$ and $s \in [q,1]$ that

$$\begin{aligned}
\mathbb{E}\left(g_{n+1}(s)^{Z_{n+1}}|\mathscr{F}_n\right) \ &= \ \mathbb{E}\left(g_{n+1}(s)^{Z'_{n+1}}g_{n+1}(s)^{Z_0(n+1)}|Z_n\right)\\
&\le \ \mathbb{E}\left(g_{n+1}(s)^{Z'_{n+1}}|Z_n\right)\\
&= \ g_n(s)^{Z_n} \quad \text{a.s.}
\end{aligned}$$

which is the desired conclusion. □

Proof (of Thm. 3.12). With $k_n = -1/\log g_n(s)$, $W_n^* = -\log X_n(s) = k_n^{-1}Z_n$ and $W^* = -\log X(s)$ for any fixed $s \in (q,1)$, we infer from the previous lemma that W_n^* converges a.s. to W^*, where W^* may be infinite with positive probability. Now consider the LT's ψ_n and ψ, say, of W_n^* and W^*, respectively. Then $\lim_{n\to\infty}\psi_n(t) = \psi(t)$ for all $t \ge 0$. By the definition of the k_n and the backward equation (3.6) for the h_n we obtain

$$\begin{aligned}
\psi_n(t) \ &= \ h_n(e^{t\log g_n(s)}) \ = \ h_n(g_n(s)^t)\\
&= \ h\circ f_n(g_n(s)^t)\cdot h_{n-1}(g_n(s)^t)\\
&= \ h\left(\mathbb{E}\left(g_n(s)^{tZ_n(0)}\Big|Z_0(0)=1\right)\right)\cdot\mathbb{E}\left(g_n(s)^{tZ_{n-1}}\right)\\
&= \ \mathbb{E}\left(g_n(s)^{tZ_n(0)}\right)\cdot\mathbb{E}\left(g_n(s)^{tZ_{n-1}}\right)\\
&= \ \mathbb{E}\left(e^{-tZ_n(0)/k_n}\right)\cdot\psi_{n-1}\left(\frac{tk_{n-1}}{k_n}\right).
\end{aligned} \tag{3.26}$$

Since $\lim_{n\to\infty}k_n^{-1}Z_n(0) = V^*(0)$ a.s., we infer with the help of (3.21) that

$$\lim_{n\to\infty}\mathbb{E}\left(e^{-tZ_n(0)/k_n}\right) \ = \ \phi(t) \ = \ h\circ\varphi(t). \tag{3.27}$$

Moreover, $k_{n-1}^{-1}k_n \to \mathsf{m}$ by Thm. 2.1, so that

$$\lim_{n\to\infty}\psi_{n-1}\left(\frac{tk_{n-1}}{k_n}\right) \ = \ \lim_{n\to\infty}\mathbb{E}\left(e^{-t(k_{n-1}/k_n)W_{n-1}^*}\right) \ = \ \psi\left(\frac{t}{\mathsf{m}}\right). \tag{3.28}$$

A combination of (3.26)–(3.28) provides us with the functional equation

$$\psi(t) \ = \ h\circ\varphi(t)\cdot\psi\left(\frac{t}{\mathsf{m}}\right)$$

for $t > 0$ and then via iteration

$$\psi(t) \;=\; \psi(0+)\prod_{n\ge 0} h\circ\varphi\left(\frac{t}{\mathsf{m}^n}\right) \;=\; \psi(0+)\prod_{n\ge 0} h\circ g_n\circ\varphi(t) \tag{3.29}$$

for $t > 0$, where as usual $\psi(0+) = \lim_{t\downarrow 0}\psi(t)$. Note that $\psi(0+) = \psi(0) = 1$ iff $W^* < \infty$ a.s. For the second equality in (3.29) it is further to be noted that $\varphi(t) > \mathbb{P}(V^*(0) = 0|Z_0(0) = 1) = q$ by Thm. 2.1, whence (3.22) implies $g\circ\varphi(t) = \varphi(t/\mathsf{m})$ and upon iteration $g_n\circ\varphi(t) = \varphi(t/\mathsf{m}^n)$ for all $t > 0$ and $n\in\mathbb{N}$. For $t = 1$, we find

$$\begin{aligned}
\psi(1) \;&=\; \lim_{n\to\infty}\mathbb{E}(e^{-tW_n^*})\\
&=\; \lim_{n\to\infty}\mathbb{E}(g_n(s)^{Z_n})\\
&=\; \lim_{n\to\infty} h_n\circ g_n(s)\\
&=\; \lim_{n\to\infty}\prod_{k=0}^{n} h\circ f_k\circ g_n(s) && \text{(by (3.5))}\\
&=\; \lim_{n\to\infty}\prod_{k=0}^{n} h\circ g_{n-k}(s)\\
&=\; \prod_{k\ge 0} h\circ g_k(s). && (3.30)
\end{aligned}$$

Next, recall from (2.18) that

$$1-a^n \;\le\; g_n(s) \;\le\; 1-b^n$$

for all $n\ge N$ and suitable $a,b\in(0,1)$, $N\in\mathbb{N}$ which can be used to show that the infinite product in (3.30) and thus $\psi(1)$ is positive iff $\sum_{n\ge 2} c_n\log n < \infty$ [☞ Problem 3.15]. Assuming the last condition, use Lemma 2.5 to infer

$$\varphi(1) \;=\; \lim_{n\to\infty}\mathbb{E}\left(e^{-Z_n(0)/k_n}\middle|Z_0(0)=1\right) \;=\; \lim_{n\to\infty}\mathbb{E}\left(g_n(s)^{Z_n(0)}\middle|Z_0(0)=1\right) \;=\; s$$

and combine this with (3.29) (for $t = 1$) and (3.30) to obtain

$$0 \;<\; \prod_{n\ge 0} h\circ g_n(s) \;=\; \psi(1) \;=\; \psi(0+)\prod_{n\ge 0} h\circ gn(s),$$

giving $\psi(0+) = \mathbb{P}(W^* < \infty) = 1$. The functional equation (3.25) then follows directly from (3.29). Finally, the almost sure positivity of W^* is left to the reader as an exercise [☞ Problem 3.16] and may e.g. be deduced from $\psi(\infty) = \lim_{t\to\infty}\psi(t) = 0$.

Assuming (ZlogZ) for $(p_n)_{n\ge 0}$, we know from the proof of the Kesten-Stigum theorem 2.2 that $\lim_{n\to\infty}\mathsf{m}^{-n}k_n\in\mathbb{R}_>$ which finally proves the last assertion of the theorem. □

Problems

Problem 3.14. Let $(Z_n)_{n\geq 0}$ be a supercritical GWPI with finite reproduction mean m and finite immigration mean ν. Prove the following assertions:

(a) The sequence defined in (3.20) defines an L^1-bounded martingale and is thus a.s. convergent to an integrable random variable.

(b) If W and $V(k)$ denote the almost sure limits of $W_n := \mathsf{m}^{-n}Z_n$ [which exists by part (a)] and of $\mathsf{m}^{-n}Z_n(k)$, respectively, then $W = \sum_{n\geq 0}\mathsf{m}^{-n}V(n)$ a.s.

(c) The following assertions [☞ also Thm. 2.2] are each equivalent to condition (ZlogZ) for $(p_n)_{n\geq 0}$:

$$\mathbb{P}(W > 0) > 0, \tag{3.31}$$

$$\mathbb{E}W = \frac{\nu\mathsf{m}}{\mathsf{m}-1}, \tag{3.32}$$

$$\lim_{n\to\infty}\mathbb{E}|W_n - W| \;=\; 0, \tag{3.33}$$

$$(W_n)_{n\geq 0} \text{ is uniformly integrable (ui)}, \tag{3.34}$$

$$\mathbb{E}\sup_{n\geq 0} W_n \;<\; \infty. \tag{3.35}$$

(d) If (ZlogZ) fails to hold, then $W = 0$ a.s.

Problem 3.15. Let $(c_n)_{n\geq 0}$ be a distribution on $\mathbb{N}_0$ with gf h. Prove that, for any $\delta \in (0,1)$,

$$\prod_{n\geq 0} h(1-\delta^n) > 0 \quad \text{iff} \quad \sum_{n\geq 2} c_n \log n < \infty.$$

[Hint: Lemma 2.6]

Problem 3.16. Given the assumptions and the notation of Thm. 3.12 and further $\sum_{n\geq 2} c_n \log n < \infty$, prove the following assertions:

(a) $W^* = \sum_{n\geq 0}\mathsf{m}^{-n}V^*(n)$ a.s.

(b) W^* is a.s. positive and not constant.

(c) $\mathbb{E}W^* < \infty$ iff $\nu < \infty$ and (ZlogZ) holds true.

3.5 The critical case revisited: criteria for null recurrence and transience

In this section, we return to the critical case and deal with the question under which conditions a critical GWPI $(Z_n)_{n\geq 0}$ with finite offspring variance σ^2 and finite immigration mean ν is null recurrent. We have already pointed out that criterion (3.10) then rules out distributional convergence to a finite limit and thus the existence of positive recurrent states. On the other hand, null recurrence is still an open question, and the following result owing to PAKES [23], [25] provides almost exhaustive information.

For simplicity, it is assumed hereafter that $(Z_n)_{n\ge 0}$ is an irreducible Markov chain, for which $c_0 = h(0) > 0$ and $\sigma^2, \nu > 0$ are necessary conditions [☞ also Problem 3.6]. Irreducibility allows us to reduce our analysis to the null recurrence of 0 (by solidarity). Let a standard model be given with $\mathbb{P}_i(Z_0 = i) = 1$, $\mathbb{P} = \sum_{i\ge 0} c_i \mathbb{P}_i$ and $p_{ij}^n = \mathbb{P}_i(Z_n = j)$ for $i, j \in \mathbb{N}_0$. With positive recurrence ruled out, standard Markov chain theory tells us that $p_{00}^n \to 0$ and

$$0 \text{ null recurrent} \quad \text{iff} \quad \sum_{n\ge 0} p_{00}^n \;=\; \infty. \tag{3.36}$$

Before stating Pakes' result, we finally define

$$h_{in}(s) \;=\; \mathbb{E}_i s^{Z_n} \;=\; \sum_{j\ge 0} p_{ij}^n s^j$$

for $i, n \in \mathbb{N}_0$.

Theorem 3.17. [Pakes] *Let $(Z_n)_{n\ge 0}$ be an irreducible critical GWPI with offspring variance $\sigma^2 \in \mathbb{R}_>$ and immigration mean $\nu \in \mathbb{R}_>$. Put further $\gamma = \frac{2\nu}{\sigma^2} = \frac{2h'(1)}{f''(1)}$. Then the following assertions hold true:*

(a) $(Z_n)_{n\ge 0}$ *is null recurrent if $\gamma < 1$, and transient if $\gamma > 1$.*

(b) *If, furthermore, $\sum_{n\ge 2} p_n n^2 \log n < \infty$ and $\sum_{n\ge 2} c_n n \log n < \infty$, then there exists a positive and finite function $H(s) = \sum_{n\ge 0} a_n s^n$ on $[0,1)$ with nonnegative coefficients such that H forms a solution to (3.7), i.e. $H = h \cdot H \circ f$, and*

$$\lim_{n\to\infty} n^{\gamma} h_{in}(s) \;=\; H(s) \tag{3.37}$$

for all $i \in \mathbb{N}_0$ and $s \in [0,1)$. In particular, $(Z_n)_{n\ge 0}$ is null recurrent iff $\gamma \le 1$.

Remark 3.18. Recalling the discussion after Thm. 3.2 by Foster & Williamson it is clear that the measure $\pi = (\pi_i)_{i\ge 0}$ with gf H must be stationary for $(Z_n)_{n\ge 0}$ and thus infinite, i.e. $H(1) = \sum_{i\ge 0} \pi_i = \infty$, for otherwise $(Z_n)_{n\ge 0}$ would be positive recurrent.

Proof (of Thm. 3.17(a)). In view of (3.36) we must show that $\sum_{n\ge 0} p_{00}^n = (<)\infty$ if $\gamma < (>)1$. By Raabe's test [☞ e.g. [30, (7.16)],

$$\lim_{n\to\infty} n\left(1 - \frac{p_{00}^{n+1}}{p_{00}^n}\right) \begin{array}{c} > \\ < \end{array} 1 \quad \Rightarrow \quad \sum_{n\ge 0} p_{00}^n \begin{array}{c} = \\ < \end{array} \infty.$$

As one can easily see, (3.5) implies

$$p_{00}^{n+1} \;=\; \mathbb{P}_0(Z_{n+1} = 0) \;=\; \mathbb{P}(Z_n = 0) \;=\; h_n(0) \;=\; \prod_{k=0}^{n} h \circ f_k(0)$$

and therefore

$$n\left(1-\frac{p_{00}^{n+1}}{p_{00}^{n}}\right) \;=\; n(1-h\circ f_n(0))$$

for each $n\in\mathbb{N}_0$. Now use $h(s)=1-\nu(1-s)+o(1-s)$ as $s\uparrow 1$ and $n(1-f_n(0))\to 2/\sigma^2$ [Lemma 2.26] to finally conclude

$$n(1-h\circ f_n(0)) \;=\; \nu n(1-f_n(0))+n\,o(1-f_n(0)) \;\to\; \gamma$$

as $n\to\infty$ and thus the desired result. □

The proof of Thm. 3.17(b) will be furnished by the two subsequent lemmata, the first of which is of similar type as Lemma 2.6, while the second one sharpens Lemma 2.26. Given an arbitrary gf $g(s)=\sum_{n\ge 0}a_n s^n$ with $g'(1)<\infty$, we define

$$g^{(1)}(s) \;=\; \frac{1-g(s)}{1-s} \quad\text{and}\quad g^{(2)}(s) \;=\; \frac{g^{(1)}(1)-g^{(1)}(s)}{1-s} \tag{3.38}$$

for $s\in[0,1)$ and further $g^{(1)}(1)=g'(1)$, $g^{(2)}(1)=g''(1)/2$. By the convexity of g and g', these functions are nondecreasing and [☞ (2.20) in Problem 2.10]

$$\begin{aligned} g^{(1)}(s) \;&=\; \sum_{n\ge 0}a_{n+1}^{(1)}s^n, \quad a_n^{(1)} \;:=\; \sum_{k\ge n}a_k,\\ g^{(2)}(s) \;&=\; \sum_{n\ge 0}a_{n+1}^{(2)}s^n, \quad a_n^{(2)} \;:=\; \sum_{k\ge n}a_{k+1}^{(1)} \end{aligned} \tag{3.39}$$

for $s\in[0,1)$.

Lemma 3.19. *Let f be the gf of an arbitrary distribution $(p_n)_{n\ge 0}$ on $\mathbb{N}_0$. For $n\ge 1$, let $\delta_n\in(0,1)$ be such that $c:=\lim_{n\to\infty}n(1-\delta_n)$ is positive and finite. Then $\sum_{n\ge 1}n^{-1}(1-f(\delta_n))<\infty$ holds iff $\sum_{n\ge 2}p_n\log n<\infty$.*

Proof. It suffices to prove the lemma for the case $1-\delta_n=cn^{-1}$ for some $c\in(0,1)$, for there always exist $m,N\in\mathbb{N}$ and $0<c_1<c_2<1$ such that $c_1(n+m)^{-1}\le\delta_n\le c_2(n-m)^{-1}$ for all $n\ge N$. Then it follows that

$$\begin{aligned} \sum_{n\ge 1}\frac{1}{n}(1-f(c/n)) \;&=\; \sum_{n\ge 1}\frac{c}{n^2}f^{(1)}(1-c/n)\\ &\asymp\; \sum_{n\ge 1}\int_{1-c/n}^{1-c/(n+1)} f^{(1)}(s)\,ds\\ &=\; \int_{1-c}^{1} f^{(1)}(s)\,ds\\ &\asymp\; \sum_{n\ge 1}\frac{p_n^{(1)}}{n} \;=\; \sum_{n\ge 1}p_n\sum_{k=1}^{n}\frac{1}{k} \;\asymp\; \sum_{n\ge 1}p_n\log n \end{aligned}$$

and the lemma is proved. □

Lemma 3.20. *Let f be the gf of a distribution $(p_n)_{n\ge 0}$ on $\mathbb{N}_0$ with mean one and finite variance $\sigma^2 = f''(1)$. Then*

$$\sum_{n\ge 1}\frac{1}{n}\left|\frac{\sigma^2}{2}-\frac{1}{n}\left(\frac{1}{1-f_n(s)}-\frac{1}{1-s}\right)\right| \;<\; \infty \tag{3.40}$$

holds iff $\sum_{n\ge 2} p_n n^2 \log n < \infty$.

Proof. Put $g(s) = 2\sigma^{-2} f^{(2)}(s)$ so that $g(1) = 1$. Using $(1-s)^{-1}(f(s)-s) = 1 - f^{(1)}(s)$ for $s \in [0,1)$, we obtain

$$\begin{aligned}
\frac{\sigma^2}{2}-\frac{1}{n}\left(\frac{1}{1-f_n(s)}-\frac{1}{1-s}\right) \;&=\; \frac{\sigma^2}{2}-\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1-f_k(s)}-\frac{1}{1-f_{k-1}(s)}\right)\\
&=\; \frac{\sigma^2}{2}-\frac{1}{n}\sum_{k=1}^{n}\frac{f\circ f_{k-1}(s)-f_{k-1}(s)}{(1-f_{k-1}(s))(1-f_k(s))}\\
&=\; \frac{\sigma^2}{2}-\frac{1}{n}\sum_{k=1}^{n}\frac{1-f^{(1)}\circ f_{k-1}(s)}{1-f_k(s)}\\
&=\; \frac{\sigma^2}{2}-\frac{1}{n}\sum_{k=1}^{n}\frac{1-f^{(1)}\circ f_{k-1}(s)}{1-f_{k-1}(s)} \;+\; O\left(\frac{\log n}{n}\right)\\
&=\; \frac{\sigma^2}{2}\left(1-\frac{1}{n}\sum_{k=1}^{n} g\circ f_{k-1}(s)\right) \;+\; O\left(\frac{\log n}{n}\right)\\
&=\; \frac{\sigma^2}{2}\sum_{k=1}^{n}(1-g\circ f_{k-1}(s)) \;+\; O\left(\frac{\log n}{n}\right),
\end{aligned}$$

where we have used for the third line from below that, as $n\to\infty$,

$$1-f^{(1)}\circ f_{n-1}(s) = O\left(\frac{1}{n}\right),\; 1-f_n(s) = O\left(\frac{1}{n}\right) \text{ and } f_n(s)-f_{n-1}(s) = O\left(\frac{1}{n^2}\right)$$

[☞ Problem 2.35(a) for the last assertion]. Summation over $n \ge 1$ in the above estimation provides us with

$$\begin{aligned}
\sum_{n\ge 1}\frac{1}{n}\left|\frac{\sigma^2}{2}-\frac{1}{n}\left(\frac{1}{1-f_n(s)}-\frac{1}{1-s}\right)\right| \;&\asymp\; \sum_{n\ge 1}\frac{1}{n^2}\sum_{k=1}^{n}(1-g\circ f_{k-1}(s))\\
&\asymp\; \sum_{k\ge 1}\frac{1}{k}(1-g\circ f_{k-1}(s)).
\end{aligned}$$

Since $n(1-f_{n-1}(s)) \to 2/\sigma^2$, we can now invoke Lemma 3.19 to infer that the last sum converges iff $\sum_{n\ge 2} p_n^{(2)} \log n < \infty$ which in turn is easily seen to be equivalent to the condition $\sum_{n\ge 2} p_n n^2 \log n < \infty$ [☞ Problem 3.21]. □

Proof (of Thm. 3.17(b)). We start by noting that (in the usual notation of this chapter)

$$h_{in}(s) \;=\; \mathbb{E}_i\left(s^{\sum_{k=0}^n Z_{n-k}(k)}\right) \;=\; \mathbb{E}_i\, s^{Z_n(0)} \prod_{k=0}^{n-1} \mathbb{E}\, s^{Z_k(n-k)} \;=\; f_n(s)^i \prod_{k=0}^{n-1} h\circ f_k(s)$$

for all $i,n\in\mathbb{N}_0$ and $s\in[0,1]$. This in combination with $\lim_{n\to\infty} f_n(s)=1$ shows that it suffices to show (3.37) for $i=0$. Obviously,

$$n^\gamma h_{0n}(s) \;=\; h(s)\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^\gamma h\circ f_k(s) \tag{3.41}$$

for $n\in\mathbb{N}$ and $s\in[0,1]$, and this sequence converges to a finite, positive limit iff the sum $\sum_{n\ge 1}(d_n(s)-1)$ with $d_n(s) := (1+\frac{1}{n})^\gamma h\circ f_n(s)$ converges absolutely. By using the expansions

$$\left(1+\frac{1}{n}\right)^\gamma \;=\; 1+\frac{\gamma}{n}+O\left(\frac{1}{n^2}\right) \quad \text{and} \quad 1-h\circ f_n(s) \;=\; O\left(\frac{1}{n}\right)$$

as $n\to\infty$, we have

$$\begin{aligned} d_n(s)-1 \;&=\; \frac{\gamma}{n}-(1-h\circ f_n(s))-\frac{\gamma}{n}(1-h\circ f_n(s))+O\left(\frac{1}{n^2}\right)\\ &=\; \frac{\gamma}{n}-\nu(1-f_n(s))-(\nu-h^{(1)}\circ f_n(s))(1-f_n(s))+O\left(\frac{1}{n^2}\right) \end{aligned} \tag{3.42}$$

By Lemma 3.19 applied to $h^* := \nu^{-1}h^{(1)}$, we have

$$\sum_{n\ge 1}(\nu-h^{(1)}\circ f_n(s))(1-f_n(s)) \;\asymp\; \sum_{n\ge 1}\frac{1}{n}(1-h^*\circ f_n(s)) \;<\; \infty$$

iff $\sum_{n\ge 1} c_n^{(1)}\log n<\infty$ which in turn holds iff $\sum_{n\ge 2} c_n n\log n<\infty$.

This leaves us with the first and second term in (3.42). Defining

$$\Delta_n(s) \;:=\; \frac{\sigma^2}{2}-\frac{1}{n}\left(\frac{1}{1-f_n(s)}-\frac{1}{1-s}\right),$$

it is easily seen that

$$\nu(1-f_n(s)) \;=\; \left(\frac{n}{\nu}+\frac{1}{\nu(1-s)}-\frac{n}{\nu}\,\Delta_n(s)\right)^{-1}$$

and thereupon, using $\lim_{n\to\infty}\Delta_n(s)=0$ for each $s\in[0,1)$, that

$$\frac{\gamma}{n}-\nu(1-f_n(s)) \;=\; \frac{\dfrac{\nu}{1-s}-n\nu\Delta_n(s)}{n^2\left(1+\dfrac{1}{n(1-s)}-\Delta_n(s)\right)} \;=\; \frac{-\nu\Delta_n(s)}{n}+O\left(\frac{1}{n^2}\right).$$

Since $\sum_{n\ge 2}c_n n^2\log n<\infty$, Lemma 3.20 ensures that $\sum_{n\ge 1}n^{-1}|\Delta_n(s)|<\infty$ and thus, by putting the previous facts together, $\sum_{n\ge 1}|d_n(s)-1|<\infty$ for all $s\in[0,1)$ or, equivalently,

$$\lim_{n\to\infty}n^{\gamma}h_{0n}(s) \;=:\; H(s) \;\in\; \mathbb{R}_>$$

for all $s\in[0,1)$. A look at (3.41) shows that $n^{\gamma}h_{0n}(s)$, $n\ge 1$, satisfies the recursion

$$(n+1)^{\gamma}h_{0,n+1}(s) \;=\; h(s)\left(1+\frac{1}{n}\right)^{\gamma}n^{\gamma}h_{0n}(s)$$

for $s\in[0,1]$ which upon letting $n\to\infty$ immediately yields the asserted functional equation for H. As a particular consequence, we finally conclude that

$$n^{\gamma}p_{00}^{n} \;=\; n^{\gamma}h_{0n}(0) \;\to\; H(0) \;>\; 0,$$

as $n\to\infty$ and therefore $\sum_{n\ge 0}p_{00}^{n}=\infty$ iff $\gamma\le 1$. □

Problems

Problem 3.21. Given an arbitrary gf $f(s)=\sum_{n\ge 0}p_n s^n$, prove that

$$\sum_{n\ge 2}p_n^{(2)}\log n<\infty \quad\Leftrightarrow\quad \sum_{n\ge 2}p_n^{(1)}n\log n<\infty \quad\Leftrightarrow\quad \sum_{n\ge 2}p_n n^2\log n<\infty,$$

where the $p_n^{(i)}$ for $i\in\{1,2\}$ are as defined in (3.39).

Problem 3.22. In the situation of Pakes' Thm. 3.17, suppose that $(Z_n)_{n\ge 0}$ has linear fractional gf $f(s)=(2-s)^{-1}$. Prove the following assertions:

(a) If $h(s)=(2-s)^{-\alpha}$ for some $\alpha>0$, then $\lim_{n\to\infty}n^{-\alpha}p_{00}^{n}=1$.

(b) If the immigration distribution is Poissonian, i.e. $h(s)=e^{-\beta(1-s)}$ for some $\beta>0$, then $\lim_{n\to\infty}n^{-\beta}p_{00}^{n}=e^{-\kappa\beta}$, where $\kappa=0.5772...$ denotes Euler's constant.

[Hint: (3.41) and (1.21).]