Chapter 2
Classical limit theorems

2.1 Supercritical case: The theorems by Heyde-Seneta and Kesten-Stigum

Throughout this section, $(Z_n)_{n\geq 0}$ will always denote a supercritical GWP in a standard model with nondegenerate offspring distribution $(p_n)_{n\geq 0}$ (i.e. $\mathbb{V}\mathrm{ar}\, Z_1 > 0$) and finite offspring mean m. We keep the notation of the previous chapter, thus f denotes the gf of $(p_n)_{n\geq 0}$, q the extinction probability under $\mathbb{P} = \mathbb{P}_1$, $W_n = \mathsf{m}^{-n} Z_n$ for $n \in \mathbb{N}_0$ and W its a.s. limit. As should be clear by now, it is enough to study the asymptotic growth behavior of Z_n under $\mathbb{P}$. The goal is to prove the following two famous results.

Theorem 2.1. [Heyde-Seneta] *Any supercritical GWP $(Z_n)_{n\geq 0}$ with finite offspring mean m admits a normalization $(k_n)_{n\geq 0}$ such that $k_n^{-1} k_{n+1} \to \mathsf{m}$ and $W_n^* := k_n^{-1} Z_n$ converges a.s. to a nondegenerate finite random variable W^* satisfying $\mathbb{P}(W^* > 0) = \mathbb{P}(Z_n \to \infty) = 1 - q$. Moreover,*

$$\lim_{n\to\infty} \frac{Z_n}{\mu^n} = \begin{cases} 0, & \text{if } \mu > \mathsf{m}, \\ \infty, & \text{if } 0 < \mu < \mathsf{m} \end{cases} \quad \text{a.s. on } \{Z_n \to \infty\}. \tag{2.1}$$

The second theorem addresses the question of a necessary and sufficient condition under which we may choose $k_n = \mathsf{m}^{-n}$, $n \geq 0$, as a normalizing sequence in the previous result.

Theorem 2.2. [Kesten-Stigum] *In the situation of the previous result, the following assertions are equivalent:*

$$\mathbb{P}(W > 0) > 0, \tag{2.2}$$

$$\mathbb{E}W = 1, \tag{2.3}$$

$$\lim_{n\to\infty} \mathbb{E}|W_n - W| \;=\; 0, \tag{2.4}$$

$$(W_n)_{n\geq 0} \textit{ is uniformly integrable (ui)}, \tag{2.5}$$

$$\mathbb{E}\sup_{n\geq 0} W_n \;<\; \infty, \tag{2.6}$$

$$\mathbb{E}Z_1 \log Z_1 \;=\; \sum_{n\geq 1} p_n n \log n \;<\; \infty. \tag{ZlogZ}$$

Remark 2.3. Due to their importance in the theory of branching processes, the results deserve some historical comments. It was LEVINSON [18] who first observed that W may vanish a.s. if $\mathbb{V}\mathrm{ar} Z_1 = \infty$. Later, SENETA [27] first proved that any supercritical GWP $(Z_n)_{n\geq 0}$ with finite offspring mean m admits a sequence $(k_n)_{n\geq 0}$ of normalizing constants such that $k_n^{-1} Z_n$ converges in distribution to a nontrivial limiting variable W^*. He further showed that $\mathbb{E}W^* < \infty$ iff $k_n \simeq C\mathsf{m}^n$ for some $C \in \mathbb{R}_>$ iff (ZlogZ). HEYDE [15] improved his result by showing that the convergence holds even almost surely and that $k_n^{-1} k_{n+1} \to \mathsf{m}$. This explains that nowadays any such sequence $(k_n)_{n\geq 0}$ is often called *Heyde-Seneta norming*. Finally, KESTEN & STIGUM [16], in the more general context of multitype processes, obtained Theorem 2.2.

Remark 2.4. With the help of the Heyde-Seneta theorem, is not difficult to verify and left as an exercise [☞ Problem 2.9] that the nondegenerate limit of a normalized supercritical GWP is unique up to a multiplicative constant. In particular, $W^* = cW$ a.s. for some $c > 0$ if (ZlogZ) holds true.

We begin with the proof of Theorem 2.1 which is based on a martingale argument due to HEYDE [15] and furnished by Lemma 2.5 below. Since f is increasing, convex and one-to-one on $[q,1]$, it possesses an inverse $g := f^{-1}$ on this subinterval which is increasing and concave with $g(q) = q$ as shown in Figure 2.1. Let $g_n = g^{\circ n}$ be the n-fold composition of g and notice that $g_n = f_n^{-1}$ for each $n \in \mathbb{N}_0$, where $g_0(s) = f_0(s) := s$. Then define

$$X_n(s) \;=\; g_n(s)^{Z_n} \;=\; e^{-Z_n/k_n(s)}, \quad n \geq 0 \tag{2.7}$$

for any $s \in [q,1)$, where $k_n(s) := (-\log g_n(s))^{-1} \in \mathbb{R}_>$. Let $(\mathscr{F}_n)_{n\geq 0}$ denote the filtration defined in Section 1.1.

Lemma 2.5. *For each $s \in [q,1)$, the sequence $(X_n(s), \mathscr{F}_n)_{n\geq 0}$ constitutes a martingale taking values in $[0,1]$ and thus converges $\mathbb{P}$-a.s. and in L^p for any $p \geq 1$ to a random variable $X_\infty(s)$ which is nondegenerate for any $s \in (q,1)$.*

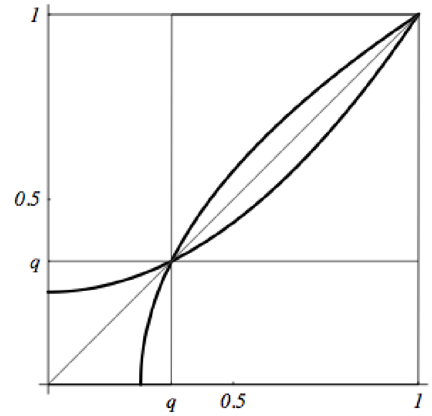

Fig. 2.1 The gf f and its inverse g.

Proof. The martingale property of $(X_n(s),\mathscr{F}_n)_{n\ge 0}$ is easily deduced from

$$\begin{aligned}\mathbb{E}(X_n(s)|\mathscr{F}_{n-1}) &= \mathbb{E}\left(g_n(s)^{\sum_{k=1}^{Z_{n-1}} X_{n,k}}\middle|Z_{n-1}\right) = \mathbb{E}\left(\prod_{k=1}^{Z_{n-1}} g_n(s)^{X_{n,k}}\middle|Z_{n-1}\right)\\ &= \big(\mathbb{E}(g_n(s)^{Z_1})\big)^{Z_{n-1}} = f\circ g_n(s)^{Z_{n-1}} = X_{n-1}(s) \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

for any $n\in\mathbb{N}$. Since the sequence is $[0,1]$-valued and thus bounded, it converges $\mathbb{P}$-a.s. as well as in L^p by the martingale convergence theorem and its L^p-extension [☞ [12, Cor. 2.2]]. Moreover, $(X_n^2(s),\mathscr{F}_n)_{n\ge 0}$ forms a submartingale and $\mathbb{V}\mathrm{ar}X_1(s) = \mathbb{V}\mathrm{ar}\,g(s)^{Z_1} > 0$ for any $s\in\{t\in[q,1): 0<g(t)<1\}\supset(q,1)$, the last fact being ensured by $\mathbb{V}\mathrm{ar}Z_1>0$ (a standing assumption). Using Jensen's inequality, we now infer that

$$\mathbb{E}X_\infty(s)^2 \;\ge\; \mathbb{E}X_1(s)^2 \;>\; (\mathbb{E}X_1(s))^2$$

and thereby $\mathbb{V}\mathrm{ar}X_\infty(s)\ge\mathbb{V}\mathrm{ar}X_1(s)>0$. □

Proof of Theorem 2.1. Turning to

$$W^*(s) \;:=\; -\log X_\infty(s) \quad \text{for } s\in(q,1),$$

the previous lemma implies that $W^*(s)$ is nonnegative and nondegenerate and that

$$W_n^*(s) \;:=\; -\log X_n(s) \;=\; \frac{Z_n}{k_n(s)} \;\to\; W^*(s) \quad \mathbb{P}\text{-a.s.}$$

However, we must still verify that $W^*(s)$ is a.s. finite (at least for one $s\in(q,1)$).

Towards this end, we first show that $k_n(s)^{-1}k_{n+1}(s)\to\mathsf{m}$ for each $s\in(q,1)$. As $g(s)\ge s$ for $s\in[q,1)$ [☞ Figure 2.1], we have that g_n increases to a limit g_∞ on $[q,1)$ satisfying

$$g_\infty(q) = q \quad \text{and} \quad g_\infty(s) = 1 \text{ for } s \in (q,1).$$

Note for the second assertion that $g_\infty(s) < 1$ for some $s \in (q,1)$ in combination with Corollary 1.8 would entail the contradiction

$$s \;=\; \lim_{n\to\infty} f_n \circ g_n(s) \;\le\; \lim_{n\to\infty} f_n \circ g_\infty(s) \;=\; q.$$

Using $-\log x \simeq 1-x$ as $x \uparrow 1$, we now infer

$$k_n(s) \;=\; -\frac{1}{\log(1-(1-g_n(s)))} \;\simeq\; \frac{1}{1-g_n(s)} \tag{2.8}$$

which together with $\lim_{s\uparrow 1} \frac{1-g(s)}{1-s} = g'(1) = \mathsf{m}^{-1}$ yields

$$\lim_{n\to\infty} \frac{k_{n+1}(s)}{k_n(s)} \;=\; \lim_{n\to\infty} \frac{1-g_n(s)}{1-g_{n+1}(s)} \;=\; \mathsf{m} \quad \text{for all } s \in (q,1) \tag{2.9}$$

as desired.

With (2.9) at hand, we may invoke Problem 1.27(b) to infer that, for any $s \in (q,1)$, $\mathbb{P}(W^*(s) = 0)$ and $\mathbb{P}(W^*(s) < \infty)$ are both fixed points of f and thus (using the nondegeneracy of $W^*(s)$)

$$q \;=\; \mathbb{P}(W^*(s) = 0) \quad \text{and} \quad \mathbb{P}(W^*(s) < \infty) \;\in\; \{q,1\}.$$

But we also have for any $s \in (q,1)$ that

$$s \;=\; \mathbb{E}X_\infty(s) \;=\; \mathbb{E}e^{-\log W^*(s)} \;\le\; \mathbb{P}(W^*(s) < \infty)$$

and therefore $\mathbb{P}(W^*(s) < \infty) = 1$. Hence, any $(k_n(s))_{n\ge 0}$ with $s \in (q,1)$ provides a Heyde-Seneta norming.

It remains to verify (2.1) which is easy. First, $\mu^{-n} Z_n \to 0$ a.s. for $\mu > \mathsf{m}$ follows directly from the a.s. convergence of $W_n = \mathsf{m}^{-n} Z_n$. Turning to the case $\mu \in (0,\mathsf{m})$, observe that

$$\lim_{n\to\infty} \frac{k_{n+1}/\mu^{n+1}}{k_n/\mu^n} \;=\; \frac{\mathsf{m}}{\mu} \;>\; 1$$

implies $\mu^{-n} k_n \to \infty$ and therefore on $\{Z_n \to \infty\} = \{W^* > 0\}$

$$\lim_{n\to\infty} \frac{Z_n}{\mu^n} \;=\; \lim_{n\to\infty} \frac{Z_n}{k_n} \cdot \frac{k_n}{\mu^n} \;=\; \infty \quad \mathbb{P}\text{-a.s.}$$

which completes the proof of Theorem 2.1. □

A second, purely analytical lemma provides the key to the crucial part of the proof of the Kesten-Stigum theorem and will in fact be used also in the subcritical case [☞ Section 2.2]. It is therefore derived first after some preparations.

A first order expansion of f at 1 has the form

$$f(s) \;=\; 1-\mathsf{m}(1-s)+r(s)(1-s) \quad \text{for } s\in(-1,1), \tag{2.10}$$

where

$$r(s) \;:=\; \mathsf{m}-\frac{1-f(s)}{1-s} \tag{2.11}$$

has the properties

$$r(q)=\mathsf{m}-1, \quad r(1):=\lim_{s\uparrow 1} r(s)=0 \quad \text{and} \quad r'(s)\le 0 \quad \text{for } s\in[0,1) \tag{2.12}$$

as one can easily verify.

Replacing s with $g(s)$ in (2.10), we obtain after a simple calculation that

$$\frac{1-g(s)}{1-s} \;=\; \frac{1}{\mathsf{m}}\left(1-\frac{r\circ g(s)}{\mathsf{m}}\right)^{-1} \quad \text{for } s\in(q,1)$$

and then for general $n\in\mathbb{N}$ (replace s with $g_{n-1}(s)$)

$$\frac{1-g_n(s)}{1-g_{n-1}(s)} \;=\; \frac{1}{\mathsf{m}}\left(1-\frac{r\circ g_n(s)}{\mathsf{m}}\right)^{-1} \quad \text{for } s\in(q,1).$$

Consequently, for each $n\in\mathbb{N}$ and $s\in(q,1)$,

$$\frac{1-g_n(s)}{1-s} \;=\; \prod_{k=1}^{n}\frac{1-g_k(s)}{1-g_{k-1}(s)} \;=\; \frac{1}{\mathsf{m}^n}\left[\prod_{k=1}^{n}\left(1-\frac{r\circ g_n(s)}{\mathsf{m}}\right)\right]^{-1}. \tag{2.13}$$

The announced lemma provides an equivalent condition for (ZlogZ) in terms of the function r just introduced. We write $\sum x_n \asymp \sum y_n$ as shorthand for $c_1\sum x_n \le \sum y_n \le c_2\sum x_n$ for suitable $0<c_1<c_2<\infty$.

Lemma 2.6. *For each $\delta\in(0,1)$, the condition $\sum_{n\ge 1} r(1-\delta^n)<\infty$ is equivalent to* (ZlogZ).

Proof. Put $a_n=\mathbb{P}(Z_1>n)=\sum_{k>n}p_k$ for $n\in\mathbb{N}_0$ and recall that $\mathsf{m}=\sum_{n\ge 0}a_n$. Then, by embarking on the definition of r,

$$\begin{aligned} r(s) \;&=\; \mathsf{m}-\sum_{k\ge 0}\left(1-\sum_{n\ge 0}p_n s^n\right)s^k \\ &=\; \mathsf{m}-\sum_{n\ge 0}s^n+\sum_{k\ge 0}\sum_{n\ge k}p_{n-k}s^n \\ &=\; \mathsf{m}-\sum_{n\ge 0}s^n+\sum_{n\ge 0}(1-a_n)s^n \\ &=\; \mathsf{m}-\sum_{n\ge 0}a_n s^n \end{aligned}$$

for $s \in (0,1)$. Putting $\alpha = -\log\delta$, it follows for $n \geq 2$ that

$$r(1-\delta) + \int_1^n r(1-e^{-\alpha x})\,dx \;\geq\; \sum_{k=1}^{n} r(1-\delta^k)$$
$$\geq\; \int_1^n r(1-e^{-\alpha x})\,dx \;=\; \frac{1}{\alpha}\int_{1-\delta}^{1-\delta^n} \frac{r(s)}{1-s}\,ds.$$

Consequently,

$$\sum_{n\geq 1} r(1-\delta^n) \;<\; \infty \quad \text{iff} \quad \int_0^1 \frac{r(s)}{1-s}\,ds \;<\; \infty.$$

The next thing to observe is that

$$\begin{aligned}
0 \;\leq\; \int_0^1 \frac{r(s)}{1-s}\,ds \;&=\; \int_0^1 \sum_{k\geq 0}\left(\mathsf{m} - \sum_{n\geq 0} a_n s^n\right) s^k\,ds \\
&=\; \int_0^1 \sum_{n\geq 1} a_n \sum_{k\geq 0}\left(s^k - s^{n+k}\right)\,ds \\
&=\; \int_0^1 \sum_{n\geq 1} a_n \sum_{k=0}^{n-1} s^k\,ds \\
&=\; \sum_{n\geq 1} a_n \sum_{k=1}^{n} \frac{1}{k} \\
&\asymp\; \sum_{n\geq 1} a_n \log n.
\end{aligned}$$

In order to finally conclude that the last sum converges iff (ZlogZ) holds, it suffices to note that

$$\mathbb{E}Z_1(\log Z_1 - 1) \;=\; \int_0^\infty \log t\;\mathbb{P}(Z_1 > t)\,dt \;\asymp\; \sum_{n\geq 1} a_n \log n$$

holds true. □

Proof of Theorem 2.2. The implications "(2.6)⇒(2.5)⇒(2.4)⇒(2.3)⇒(2.2)" follow from standard facts in probability theory, which leaves us with the proof of "(2.2) ⇔ (ZlogZ)" and "(2.2) ⇒ (2.6)".

Beginning with the equivalence assertion, we first show that (ZlogZ) holds iff

$$\lim_{n\to\infty} \frac{\mathsf{m}^n}{k_n(s)} \;<\; \infty \quad \text{for } s \in (q,1). \tag{2.14}$$

which in view of $k_n(s) \simeq (1-g_n(s))^{-1}$ and (2.13) is equivalent to the assertion

$$\sum_{n\geq 1}\log\left(1-\frac{r\circ g_n(s)}{\mathsf{m}}\right) \asymp \sum_{n\geq 1} r\circ g_n(s) \;<\; \infty \quad \text{for } s\in(q,1). \tag{2.15}$$

Fix any $s_0 \in (q,1)$ such that $\mathsf{m}_0 := g'(s_0)^{-1} > 1$. The concavtity of g provides us with

$$\frac{1}{\mathsf{m}} \;\leq\; \frac{1-g(s)}{1-s} \;\leq\; \frac{1}{\mathsf{m}_0} \quad \text{for all } s\in[s_0,1).$$

Moreover, $g_n(s_0) > s_0$ and $g_n \uparrow$ imply $g_n([s_0,1]) \subset [s_0,1]$ for all n, so that upon iteration and some simple algebra

$$1-\frac{1-s}{\mathsf{m}_0^n} \;\leq\; g_n(s) \;\leq\; 1-\frac{1-s}{\mathsf{m}^n} \tag{2.16}$$

for all $s \in [s_0,1)$ and $n \in \mathbb{N}_0$. For $s \in (q,s_0)$, one can always find $N = N_s \geq 1$ such that $g_n(s) \geq s_0$ for all $n \geq N$. Consequently, (2.16) implies that

$$1-\frac{1-g_N(s)}{\mathsf{m}_0^{n-N}} \;\leq\; g_n(s) \;\leq\; 1-\frac{1-g_N(s)}{\mathsf{m}^{n-N}}. \tag{2.17}$$

for $s \in (q,s_0)$ and $n \geq N$. A combination of (2.16) and (2.17) obviously shows that, for each $s \in (q,1)$ there exist $a,b \in (0,1)$ and $N = N_s \in \mathbb{N}$ such that

$$1-a^n \;\leq\; g_n(s) \;\leq\; 1-b^n \tag{2.18}$$

for all $n \geq N$. This finally proves the equivalence of (ZlogZ) and (2.15), and thus of (ZlogZ) and (2.14). But the latter condition is also equivalent to (2.2) because

$$W_n \;=\; \frac{k_n(s)}{\mathsf{m}^n}\cdot\frac{Z_n}{k_n(s)}$$

obviously converges a.s. to a nondegenerate limit iff $\lim_{n\to\infty}\mathsf{m}^{-n}k_n(s) > 0$.

The proof of ”(2.2) $\Rightarrow$ (2.6)” embarks on the fact that, if W is nondegenerate and thus $\mathbb{E}W \in \mathbb{R}_>$, the SLLN ensures, for any fixed $a \in (0,\mathbb{E}W)$, the existence of $N \in \mathbb{N}$ and $b \in \mathbb{R}_>$ such that

$$\mathbb{P}\left(\sum_{k=1}^{t\mathsf{m}^n} W(k) > at\mathsf{m}^n\right) \;\geq\; b \quad \text{for all } t\geq 1 \text{ and } n\geq N.$$

It follows for all $t \geq 1$ that

$$\begin{aligned}
\mathbb{P}(W > at) \;&\geq\; \mathbb{P}\left(W > at, \sup_{j\geq N} W_j > t\right) \\
&=\; \sum_{n\geq N} \mathbb{P}\left(W > at, Z_n > t\mathsf{m}^n, \sup_{N\leq j<n} W_j \leq t\right) \\
&=\; \sum_{n\geq N} \sum_{k>t\mathsf{m}^n} \mathbb{P}_k(W > at\mathsf{m}^n)\,\mathbb{P}\left(Z_n = k, \sup_{N\leq j<n} W_j > t\right) \\
&\geq\; \sum_{n\geq N} \sum_{k>t\mathsf{m}^n} \mathbb{P}\left(\sum_{j=1}^{t\mathsf{m}^n} W(j) > at\mathsf{m}^n\right) \mathbb{P}\left(Z_n = k, \sup_{N\leq j<n} W_j > t\right) \\
&\geq\; b\,\mathbb{P}\left(\sup_{j\geq N} W_j > t\right),
\end{aligned}$$

where (1.31), (1.32) have been utilized for the third line and (1.22) for the fourth line. The obtained tail estimate may now be used to infer

$$\begin{aligned}
\mathbb{E}\sup_{n\geq 0} W_n \;&\leq\; N \,+\, \mathbb{E}\sup_{n\geq N} W_n \;=\; N \,+\, \int_0^\infty \mathbb{P}\left(\sup_{n\geq N} W_n > t\right) dt \\
&\leq\; (N+1) \,+\, \frac{1}{b}\int_1^\infty \mathbb{P}(W > at)\, dt \;=\; (N+1) \,+\, \frac{\mathbb{E}W}{ab} \;<\; \infty,
\end{aligned}$$

which is the desired conclusion. □

At least at first glance, the two theorems just proved have an intriguing aspect. If (ZlogZ) holds, then the population grows like m^n on the event of survival, whereas otherwise the growth is *slower* by Lemma 1.24. On the other hand, a violation of (ZlogZ) means that the offspring distribution has heavier tails $a_n = \sum_{k>n} p_k$ which in turn entails large numbers of offspring per individual with higher probabilities. As a consequence, we would expect the population to grow faster than m^n on the event of survival if (ZlogZ) fails to hold. The following lemma, the simple proof of which we leave to the reader as an exercise [☞ Problem 2.10], elucidates the phenomenon.

Lemma 2.7. [Antipodal lemma] *Let $(Z_n)_{n\geq 0}$ and $(\hat{Z}_n)_{n\geq 0}$ be two GWP with one ancestor, the same finite offspring mean m, reproduction variances σ^2 and $\hat{\sigma}^2$, extinction probabilities q and $\hat{q}$, and offspring distributions $(p_n)_{n\geq 0}$ and $(\hat{p}_n)_{n\geq 0}$ with gf's f and $\hat{f}$, respectively. Suppose that*

$$\mathbb{P}(Z_1 > n) \;=\; \sum_{k>n} p_k \;\leq\; \sum_{n>k} \hat{p}_k \;=\; \mathbb{P}(\hat{Z}_1 > n)$$

for all $n \in \mathbb{N}$ with strict inequality for at least one n. Then

$$f(s) < \hat{f}(s) \quad \textit{for all } s \in [0,1),$$

in particular $p_0 < \hat{p}_0$, *and* $q < \hat{q}$ *if* $\mathsf{m} > 1$. *Furthermore,* $p_1 > \hat{p}_1$, *and* $\sigma^2 \le \hat{\sigma}^2$ *if* $\sigma^2 < \infty$.

What the lemma shows is that, if among two supercritical populations with identical offspring mean one has a higher chance of producing n or more children per individual for any $n \ge 2$, then this must be compensated by a higher probability per individual to have no offspring and thus by a higher extinction probability of the population. Bearing in mind that each population is exposed to the antipodal forces extinction and explosion, the lemma further tells us that an increase of one force, while keeping the reproduction mean fixed, must come along with an increase of its antipode and that, as a result of a thus increased reproduction variance, extinction becomes the more attractive force owing to the fact that it is absorbing while explosion is not. In the context of these observations, the two theorems by Heyde-Seneta and Kesten-Stigum now tell us that (ZlogZ) marks a phase transition in the sense that, a supercritical population for which this condition fails not only has a higher chance of extinction but also a smaller growth rate as a population with the same offspring mean and satisfying (ZlogZ).

Striving for further information on the distribution Q, say, of the *Heyde-Seneta limit* W^*, that is of W if (ZlogZ) holds, leads back to the distributional equation (1.25) it satisfies [☞ Problem 1.28]. Although an explicit computation of Q is usually impossible [☞ Problem 2.11 for an exception], the equation may be used to derive interesting properties of Q like absolute continuity with respect to Lebesgue measure (when restricted to $\mathbb{R}_>$) or density properties. This will indeed be accomplished in Section **??**, but the following result provides a first taste of the procedure.

Theorem 2.8. *The Laplace transform (LT)* φ *of the Heyde-Seneta limit* W^* *in Theorem 2.1 satisfies the functional equation*

$$\varphi(\mathsf{m}t) \;=\; f \circ \varphi(t) \tag{2.19}$$

for all $t \in \mathbb{R}_\ge$. *Furthermore,* $\mathbb{E}W^* = |\varphi'(0)| := \lim_{t\downarrow 0} |\varphi'(t)|$ *is finite iff* (ZlogZ) *holds true in which case* φ *forms the unique solution to* (2.19) *with given right derivative at* 0.

Proof. The functional equation is a direct consequence of (1.25) and Problem 1.6. Rewriting it in the form $\varphi(t) = f \circ \varphi(t/\mathsf{m})$, we see that $\varphi(t/\mathsf{m}) = g \circ \varphi(t)$, for $\varphi(t) > \varphi(\infty) = \mathbb{P}(W^* = 0) = q$. Then, by iteration,

$$\varphi\left(\frac{t}{\mathsf{m}^n}\right) \;=\; g_n \circ \varphi(t)$$

for all $t \in \mathbb{R}_\geq$ and $n \in \mathbb{N}$, thus

$$1-\varphi\left(\frac{t}{\mathsf{m}^n}\right) \;=\; 1-g_n\circ\varphi(t) \;\simeq\; \frac{1}{k_n(\varphi(t))} \quad (n\to\infty)$$

by (2.8). It follows that

$$|\varphi'(0)| \;=\; \lim_{n\to\infty}\frac{1-\varphi(\mathsf{m}^{-n}t)}{\mathsf{m}^{-n}t} \;=\; \frac{1}{t}\lim_{n\to\infty}\frac{\mathsf{m}^n}{k_n(\varphi(t))},$$

which is finite iff (ZlogZ) holds true [☞ (2.14) in the proof of Theorem 2.2].

Suppose now that ψ is another solution to (2.19) with the same (finite) derivative as φ at 0. By convexity, $(s-r)^{-1}(f(s)-f(r)) \le f'(1) = \mathsf{m}$ for all $0 \le r < s \le 1$ which via iteration yields

$$\begin{aligned}
|\varphi(t)-\psi(t)| \;&=\; \left|f\circ\varphi\left(\frac{t}{\mathsf{m}}\right)-f\circ\psi\left(\frac{t}{\mathsf{m}}\right)\right|\\
&\le\; \mathsf{m}\left|\varphi\left(\frac{t}{\mathsf{m}}\right)-\psi\left(\frac{t}{\mathsf{m}}\right)\right|\\
&\;\;\vdots\\
&\le\; \mathsf{m}^n\left|\varphi\left(\frac{t}{\mathsf{m}^n}\right)-\psi\left(\frac{t}{\mathsf{m}^n}\right)\right| \xrightarrow{n\to\infty} t\,|\varphi'(0)-\psi'(0)| \;=\; 0.
\end{aligned}$$

for all $t \ge 0$, i.e. $\varphi = \psi$. □

Problems

Problem 2.9. Let $(Z_n)_{n\ge0}$ be a supercritical GWP satisfying $k_{j,n}^{-1}Z_n \to Y_j$ a.s. for two sequences $(k_{j,n})_{n\ge0}$ of positive reals and nondegenerate random variables Y_j ($j \in \{0,1\}$). Show that $Y_1 = cY_0$ a.s. for some $c > 0$.

Problem 2.10. Let the assumptions of the Antipodal Lemma 2.7 be given.

(a) Prove the lemma and furthermore $f_n(s) \le \hat{f}_n(s)$ for all $n \ge 1$ and $s \in [0,1]$.
(b) If $\mathsf{m} > 1$ and $\varphi, \hat{\varphi}$ denote the LT's of the a.s. limits of $\mathsf{m}^{-n}Z_n$ and $\mathsf{m}^{-n}\hat{Z}_n$, respectively, then $\varphi(t) \le \hat{\varphi}(t)$ for all $t \in \mathbb{R}_\geq$.

[Hint: Show first that

$$f(s) \;=\; 1-(1-s)\sum_{n\ge0}a_n s^n \quad \text{for all } s\in[0,1], \tag{2.20}$$

where $a_n = \sum_{k>n} p_k = \mathbb{P}(Z_1 > n)$ for $n \in \mathbb{N}_0$.]

Problem 2.11. Use (2.19) to show that $(Z_n)_{n\ge0}$ has linear fractional offspring distribution if $\mathbb{P}(W \in \cdot|W > 0)$ is an exponential distribution, necessarily with parameter $1-q$, for $\mathbb{E}W = 1$ and $\mathbb{P}(W = 0) = q$.

Problem 2.12. The following parts provide an alternative proof of the Kesten-Stigum theorem, taken from [2, Ch. 2]. We make the assumptions stated at the beginning of this section and further define (in the usual notation)

$$W_n \;=\; \frac{1}{\mathsf{m}^n}\sum_{k=1}^{Z_{n-1}} X_{n,k}\mathbf{1}_{\{X_{n,k}\le \mathsf{m}^n\}} \quad \text{and} \quad R_{n-1} \;=\; \mathbb{E}(W_{n-1}-W_n'|\mathscr{F}_n)$$

for $n\in\mathbb{N}$. Then show:

(a) $W_n - W_{n-1} = \mathsf{m}^{-n}\sum_{k=1}^{Z_{n-1}}(X_{n,k}-\mathsf{m})$ for each $n\in\mathbb{N}$.
(b) $R_n = \mathbb{E}(W_{n+1}-W_{n+1}'|\mathscr{F}_n) = \mathsf{m}^{-1}W_n\mathbb{E}Z_1\mathbf{1}_{\{Z_1>\mathsf{m}^n\}}$ for each $n\in\mathbb{N}_0$.
(c) $(W_n'-W_{n-1}+R_{n-1})_{n\ge 1}$ forms a L^2-bounded martingale difference sequence.
(d) $\sum_{n\ge 1}\mathbb{P}(W_n\ne W_n')<\infty$, thus $\mathbb{P}(W_n=W_n'\text{ eventually})=1$ by the Borel-Cantelli lemma, and $\sum_{n\ge 1}\mathbb{E}(W_n'-W_{n-1}+R_{n-1})^2<\infty$.
(e) $\sum_{n\ge 1}(W_n'-W_{n-1})$ and $\sum_{n\ge 0}R_n$ are both a.s. convergent.
(f) $\sum_{n\ge 1}(W_n'-W_{n-1}+R_{n-1})$ exists a.s. and in L^1.
(g) $\sum_{n\ge 0}\mathbb{E}R_n<\infty$ and (ZlogZ) are equivalent conditions.
(h) Proof of "(ZlogZ) $\Rightarrow$ (2.3)": (ZlogZ) implies that both, $\sum_{n\ge 0}R_n$ and $\sum_{n\ge 1}(W_n'-W_{n-1})$ exist in L^1, and

$$\mathbb{E}W \;\ge\; 1+\mathbb{E}\left(\sum_{n\ge N}(W_n'-W_{n-1})\right)$$

for all $N\ge 1$. This yields $\mathbb{E}W=1$ (why?).
(i) Proof of "(2.2) $\Rightarrow$ (ZlogZ)": Put $W'=\inf_{n\ge 0}W_n$. Then (2.2) implies $\mathbb{P}(W'>0)\ge\mathbb{P}(W>0)>0$ as well as

$$\frac{W'}{\mathsf{m}}\sum_{n\ge 0}\mathbb{E}Z_1\mathbf{1}_{\{Z_1>\mathsf{m}^n\}} \;\le\; \sum_{n\ge 0}R_n \;<\; \infty \quad \text{a.s.}$$

und thus also (ZlogZ) (why?).

2.2 Subcritical case: Two theorems by Kolmogorov and Yaglom and the expected extinction time

Keeping the notation from the previous section, we now consider the subcritical case when $\mathsf{m}<1$. Since ultimate extinction occurs with probability one, two questions of interest are how fast $\mathbb{P}(Z_n>0)$ decays to 0 and about the behavior of Z_n if the population has survived n generations. These are addressed by the subsequent two theorems due to KOLMOGOROV [17] and YAGLOM [33]. Defining the extinction epoch

$$T \;:=\; \inf\{n\ge 1: Z_n=0\},$$

we further prove by means of Kolmogorov's theorem a result on its expectation $\mathbb{E}_k T$ as $k \to \infty$.

Theorem 2.13. [Kolmogorov] *For a subcritical GWP* $(Z_n)_{n\geq 0}$ *with* $p_0 < 1$ *and offspring mean* m

$$c \ := \ \lim_{n\to\infty} \mathsf{m}^{-n}\,\mathbb{P}(Z_n > 0) \ \begin{cases} > 0, & \textit{if } (\mathrm{ZlogZ}) \textit{ holds true}, \\ = 0, & \textit{otherwise}. \end{cases}$$

With regard to the extinction epoch, Kolmogorov's theorem states that

$$\mathbb{P}(T > n) \ \simeq \ c\mathsf{m}^{-n}, \quad \text{as } n \to \infty, \tag{2.21}$$

provided that $p_0 < 1$ and (ZlogZ) is valid. In other words, T has exponentially decreasing tails of order m. An analytic definition of the constant c will be given in the proof of the theorem.

Theorem 2.14. [Yaglom] *Let* $(Z_n)_{n\geq 0}$ *be a subcritical GWP with* $p_0 < 1$ *and* c *as in the previous result. Then*

$$\lambda_k \ := \ \lim_{n\to\infty} \mathbb{P}(Z_n = k | Z_n > 0)$$

exists for any $k \in \mathbb{N}$ *and defines a probability law on* $\mathbb{N}$*, i.e.* $\sum_{k\geq 1} \lambda_k = 1$*, with mean*

$$\sum_{k\geq 1} k\lambda_k \ = \ \frac{1}{c} \tag{2.22}$$

to be defined as ∞ *if* $c = 0$*. This law satisfies the distributional equation*

$$\sum_{k=1}^{Y} X_k \ \stackrel{d}{=} \ \delta Y \tag{2.23}$$

where all occurring random variables are independent with

$$Y \stackrel{d}{=} (\lambda_k)_{k\geq 1}, \quad X_1, X_2, \ldots \stackrel{d}{=} (p_k)_{k\geq 0} \quad \textit{and} \quad \delta \stackrel{d}{=} \mathrm{Bern}(\mathsf{m}).$$

In terms of gf's, (2.23) *takes the equivalent form*

$$h \circ f(s) \ = \ \mathsf{m}\,h(s) \ + \ (1 - \mathsf{m}) \tag{2.24}$$

for all $s \in [0, 1]$*, where* h *denotes the gf of* $(\lambda_k)_{k\geq 1}$.

Note that a combination of the previous two theorems shows that, as $n \to \infty$,

$$\mathbb{P}(Z_n = 0) \simeq 1 - c\mathsf{m}^n \quad \text{and} \quad \mathbb{P}(Z_n = k) \simeq c\lambda_k \mathsf{m}^n \quad \text{for } k \geq 1$$

if $p_0 < 1$ and (ZlogZ) are valid.

The distribution $\lambda = (\lambda_k)_{k\geq 1}$ is often called *Yaglom distribution* or *Yaglom limit* associated with $(Z_n)_{n\geq 0}$ or $(p_n)_{n\geq 0}$. It has the notable property that, if $Z_0 \stackrel{d}{=} \lambda$, then the conditional law of any Z_n given $Z_n > 0$ is also λ, i.e.

$$\mathbb{P}_\lambda(Z_n \in \cdot | Z_n > 0) = \lambda \quad \text{for all } n \in \mathbb{N}, \tag{2.25}$$

where $\mathbb{P}_\lambda := \sum_{k\geq 1} \lambda_k \mathbb{P}_k$. This is the defining property of a so-called *quasi-stationary distribution* for the Markov chain $(Z_n)_{n\geq 0}$ and follows from

$$\begin{aligned}
\lambda_j &= \lim_{k\to\infty} \mathbb{P}(Z_{k+n} = j | Z_{k+n} > 0) \\
&= \lim_{k\to\infty} \frac{\sum_{i\geq 1} \mathbb{P}(Z_k = i)\,\mathbb{P}_i(Z_n = j)}{\sum_{i\geq 1} \mathbb{P}(Z_k = i)\,\mathbb{P}_i(Z_n > 0)} \\
&= \lim_{k\to\infty} \frac{\sum_{i\geq 1} \mathbb{P}(Z_k = i | Z_k > 0)\,\mathbb{P}_i(Z_n = j)}{\sum_{i\geq 1} \mathbb{P}(Z_k = i | Z_k > 0)\,\mathbb{P}_i(Z_n > 0)} \\
&= \frac{\sum_{i\geq 1} \lambda_i \,\mathbb{P}_i(Z_n = j)}{\sum_{i\geq 1} \lambda_i \,\mathbb{P}_i(Z_n > 0)} \\
&= \frac{\mathbb{P}_\lambda(Z_n = j)}{\mathbb{P}_\lambda(Z_n > 0)}
\end{aligned}$$

for all $j, n \in \mathbb{N}$.

Turning to the extinction epoch T, our final result determines the asymptotic behavior of its mean value when the number of ancestors goes to infinity.

Theorem 2.15. *Let $(Z_n)_{n\geq 0}$ be a subcritical GWP with $p_0 < 1$. Then*

$$\lim_{k\to\infty} \frac{\mathbb{E}_k T}{\log_{1/\mathsf{m}} k} = 1$$

if (ZlogZ) *is valid.*

Proof of Theorem 2.13. Let $r(s)$ be given by (2.11), so $\mathsf{m} - r(s) = (1-s)^{-1}(1-f(s))$ for $s \in [0,1)$. Replacing s with $f_k(s)$ for any $k \in \mathbb{N}$ and taking products, we obtain

$$\frac{1 - f_n(s)}{1 - s} = \prod_{k=0}^{n-1} \frac{1 - f_{k+1}(s)}{1 - f_k(s)} = \mathsf{m}^n \prod_{k=0}^{n-1} \left(1 - \frac{r \circ f_k(s)}{\mathsf{m}}\right) \tag{2.26}$$

for $s\in[0,1)$ and $n\in\mathbb{N}$. Since $0\le \mathsf{m}^{-1}r(s)<1$ for $s\in[0,1)$, we then infer that $\mathsf{m}^{-n}(1-s)^{-1}(1-f_n(s))$ decreases to a nonnegative limit $\psi(s)$, say, as $n\to\infty$, giving in particular

$$c \;=\; \lim_{n\to\infty}\frac{\mathbb{P}(Z_n>0)}{\mathsf{m}^n} \;=\; \lim_{n\to\infty}\frac{1-f_n(0)}{\mathsf{m}^n} \;=\; \psi(0) \tag{2.27}$$

and

$$\psi(0) \;=\; \lim_{n\to\infty}\prod_{k=0}^{n-1}\left(1-\frac{r\circ f_k(0)}{\mathsf{m}}\right) \;>\; 0 \quad \text{iff} \quad \sum_{k\ge 0} r\circ f_k(0) \;<\; \infty. \tag{2.28}$$

By convexity of f, we further have $1-f(s)\le \mathsf{m}(1-s)$ for $s\in[0,1]$ as well as $1-f(s)\ge f'(p_0)(1-s)$ for $s\in[p_0,1]$. When combined with $f(s)\ge f(0)=p_0$ for all $s\in[0,1]$, this yields upon iteration

$$f'(p_0)^{k-1}(1-f(s)) \;\le\; 1-f_k(s) \;\le\; \mathsf{m}^k(1-s)$$

for all $s\in[0,1]$ and $k\ge 1$, in particular

$$1-\mathsf{m}^k \;\le\; f_k(0) \;\le\; 1-a^k$$

for all $k\in\mathbb{N}$, where $a:=f'(p_0)\wedge(1-p_0)$. The latter constant is positive iff $p_0<1$, whence we infer from Lemma 2.6 that the equivalent assertions in (2.28) are actually valid iff (ZlogZ) and $p_0<1$ hold true. ☐

Proof of Theorem 2.14. Put $h_n(s)=\mathbb{E}(s^{Z_n}|Z_n>0)$ for $n\in\mathbb{N}$, so

$$\begin{aligned}
h_n(s) \;&=\; \frac{1}{\mathbb{P}(Z_n>0)}\int_{\{Z_n>0\}} s^{Z_n}\,d\mathbb{P}\\
&=\; \frac{f_n(s)-f_n(0)}{1-f_n(0)}\\
&=\; 1-\frac{1-f_n(s)}{1-f_n(0)}\\
&=\; 1-(1-s)\prod_{k=0}^{n-1}\left(\frac{1-r\circ f_k(s)/\mathsf{m}}{1-r\circ f_k(0)/\mathsf{m}}\right)
\end{aligned} \tag{2.29}$$

for all $s\in[0,1]$ and $n\in\mathbb{N}$, where (2.26) has been utilized for the last equality. Since r is nonincreasing and $f_k(s)\ge f_k(0)$ for all $s\in[0,1]$ and $k\in\mathbb{N}_0$, all factors in the above product are ≤ 1. Consequently, h_n decreases to a limit h as $n\to\infty$, viz.

$$h(s) \;=\; 1-(1-s)\prod_{k\ge 0}\left(\frac{1-r\circ f_k(s)/\mathsf{m}}{1-r\circ f_k(0)/\mathsf{m}}\right) \;\in\; [0,1]$$

for all $s\in[0,1]$. Obviously,

$$h(0)=0,\quad h(1)=1 \quad\text{and}\quad h(s)=\sum_{k\ge 1}\lambda_k s^k$$

for suitable $\lambda_k \ge 0$ and all $s \in [0,1)$, but it must still be verified that $h(s) \uparrow h(1) = 1$ as $s \uparrow 1$ which then ensures via $\sum_{k\ge 1} \lambda_k = 1$ that $(\lambda_k)_{k\ge 1}$ defines a proper distribution on $\mathbb{N}$. But $f_n(0) \to q = 1$ implies

$$h \circ f_k(0) \;=\; \lim_{n\to\infty} h_n \circ f_k(0) \;=\; \lim_{n\to\infty} 1 - \frac{1 - f_k \circ f_n(0)}{1 - f_n(0)} \;=\; 1 - \mathsf{m}^k \tag{2.30}$$

for any $k \in \mathbb{N}$ and thus, by letting $k \to \infty$,

$$\sum_{k\ge 1} \lambda_k \;=\; \lim_{k\to\infty} h \circ f_k(0) \;=\; 1.$$

In order to verify (2.22), recall from (2.27) that $c = \lim_{k\to\infty} \mathsf{m}^{-k}(1 - f_k(0))$ and use (2.30) to infer

$$\sum_{k\ge 1} k\lambda_k \;=\; h'(1) \;=\; \lim_{k\to\infty} \frac{1 - h \circ f_k(0)}{1 - f_k(0)} \;=\; \lim_{k\to\infty} \frac{\mathsf{m}^k}{1 - f_k(0)} \;=\; \frac{1}{c}.$$

By Theorem 2.13 the last constant is finite iff (ZlogZ) holds true.

Finally bound for a proof of (2.24) (the equivalence with (2.23) is left an exercise [☞ Problem 2.16]), we note that for all $s \in [0,1]$

$$\begin{aligned}
h \circ f(s) \;&=\; \lim_{n\to\infty} h_n \circ f(s) \\
&=\; 1 - \lim_{n\to\infty} \left(\frac{1 - f_{n+1}(s)}{1 - f_{n+1}(0)}\right)\left(\frac{1 - f \circ f_n(0)}{1 - f_n(0)}\right) && \text{(by (2.29))} \\
&1 - \lim_{n\to\infty} (1 - h_{n+1}(s)) \cdot \lim_{n\to\infty} \frac{1 - f \circ f_n(0)}{1 - f_n(0)} \\
&=\; 1 - (1 - h(s))\mathsf{m}
\end{aligned}$$

which is the desired result. □

Proof of Theorem 2.15. By (2.27) and (2.28), $f_n(0) = 1 - c_n \mathsf{m}^n$ for suitable $c_n \in [0,1]$ with positive limit c, for (ZlogZ) is assumed. Using $\mathbb{P}_k(T > n) = \mathbb{P}_k(Z_n > 0) = 1 - f_n(0)^k$, we obtain

$$\begin{aligned}
\frac{\mathbb{E}_k T}{\log_{1/\mathsf{m}} k} \;&=\; \frac{1}{\log_{1/\mathsf{m}} k} \sum_{n\ge 0} \mathbb{P}_k(T > n) \\
&=\; \frac{1}{\log_{1/\mathsf{m}} k} \sum_{n\ge 0} \left(1 - f_n(0)^k\right) \;=\; \frac{1}{\log_{1/\mathsf{m}} k} \sum_{n\ge 0} (1 - (1 - c_n \mathsf{m}^n))^k .
\end{aligned}$$

Fix any $\varepsilon \in (0,1)$, put

$$n_* = n_*(k,\varepsilon) = (1-\varepsilon)\log_{1/\mathsf{m}} k, \quad n^* = n^*(k,\varepsilon) = (1+\varepsilon)\log_{1/\mathsf{m}} k,$$

and divide the last sum in the previous display into three parts $S_1(k), S_2(k), S_3(k)$ with summation ranges $0,...,n_*-1$, $n_*,...,n^*-1$ and $n^*,n^*+1,...$, respectively. Each of these sums will now be estimated separately.

Choose $N \in \mathbb{N}$ so large that $\inf_{n\geq N} c_n \geq c/2$. Since $\log(1-x) \leq -x$ and $\mathsf{m}^{n_*} = k^{\varepsilon-1}$, it follows that

$$1-(1-c\mu^{n_*}/2)^k \;\geq\; 1-\exp(-ck^{\varepsilon}/2).$$

For $S_1(k)$ we obtain

$$\begin{aligned}
&\left(1-\varepsilon-\frac{\mathsf{m}}{\log_{1/\mathsf{m}} k}\right)(1-\exp(-ck^{\varepsilon}/2)) \\
&\qquad \leq\; \left(1-\varepsilon-\frac{\mathsf{m}}{\log_{1/\mathsf{m}} k}\right)\Big(1-(1-c\mu^{n_*}/2)^k\Big) \\
&\qquad \leq\; \frac{1}{\log_{1/\mathsf{m}} k}\sum_{n=N}^{n_*-1}\Big(1-(1-c\mu^{n}/2)^k\Big) \\
&\qquad \leq\; S_1(k) \\
&\qquad \leq\; \frac{1}{\log_{1/\mathsf{m}} k}\sum_{n=0}^{n_*-1}\Big(1-(1-c\mu^{n}/2)^k\Big) \;\leq\; 1-\varepsilon
\end{aligned}$$

and thus $\lim_{k\to\infty} S_1(k) = 0$. As for $S_2(k)$, it suffices to note that $0 \leq S_2(k) \leq 2\varepsilon$. Finally turning to $S_3(k)$, we obtain for sufficiently large k that

$$\begin{aligned}
0 \;\leq\; S_3(k) \;\leq\; &\frac{1}{\log_{1/\mathsf{m}} k}\sum_{n\geq 0}\Big(1-(1-\mathsf{m}^{n^*+n})^k\Big) \\
=\; &\frac{1}{\log_{1/\mathsf{m}} k}\sum_{n\geq 0}\Big(1-(1-k^{-(1+\varepsilon)}\mathsf{m}^{n})^k\Big) \\
\leq\; &\frac{1}{\log_{1/\mathsf{m}} k}\sum_{n\geq 0}\big(1-\exp(-2k^{-\varepsilon}\mathsf{m}^{n})\big) \\
\leq\; &\frac{2}{k^{\varepsilon}\log_{1/\mathsf{m}} k}\sum_{n\geq 0}\mathsf{m}^{n},
\end{aligned}$$

where $1-e^{-x} \leq x$ has been utilized for the last inequality. It follows that $\lim_{k\to\infty} S_3(k) = 0$. A combination of the previous estimation proves the theorem, for ε was chosen arbitrarily. $\square$

Problems

All problems in this section keep the usual notation and assume $(Z_n)_{n\geq 0}$ to be subcritical, given in a standard model, and with extinction epoch T.

Problem 2.16. Prove the equivalence of (2.23) and (2.24) in Yaglom's theorem.

Problem 2.17. Given the assumptions of Yaglom's theorem, prove that

$$\lambda_k \;=\; \lim_{n\to\infty} \mathbb{P}_i(Z_n = k | Z_n > 0)$$

for all $i,k \in \mathbb{N}$, which means that the Yaglom limit yields regardless of the number of ancestors. [Hint: Consider $h_{i,n}(s) = \mathbb{E}_i(s^{Z_n}|Z_n > 0)$ and prove that $h_{i,n}(s)/h_{1,n}(s) \to 1$ for all $s \in [0,1]$ and $i \in \mathbb{N}$.]

Problem 2.18. Given the assumptions of Yaglom's theorem, prove that:

(a) $\lambda = (\lambda_k)_{k\geq 0}$ is degenerate, i.e., a Dirac distribution, iff $p_0 + p_1 = 1$. [Hint: Use (2.23) or (2.24).]
(b) If λ is nondegenerate, then $\lambda_k > 0$ for infinitely many k.
(c) If $(p_n)_{n\geq 0}$ is aperiodic, the same holds true for λ.
(d) If (ZlogZ) holds, then λ has variance $\frac{1}{c}\left(\frac{f''(1)}{\mathsf{m}(\mathsf{m}-1)} - 1\right)$ which is finite iff the reproduction variance is finite.

Problem 2.19. In the situation of Yaglom's theorem, suppose that

$$h(s) \;=\; \frac{(1-a)s}{1-as} \quad \text{for some } a \in (0,1).$$

Show that $(p_n)_{n\geq 0}$ must be linear fractional and determine the parameters in terms of a and m.

Problem 2.20. Prove that, under $\mathbb{P} = \mathbb{P}_1$, the extinction epoch T satisfies the distributional equation

$$T \;\stackrel{d}{=}\; 1 + \min\{T_k : 1 \leq k \leq Z_1\}$$

where $Z_1, T_1, T_2, \ldots$ are independent and $T_1, T_2, \ldots$ are copies of T.

Problem 2.21. Prove that $\mathbb{E}\exp(\theta T) < \infty$ for any $\theta < \log(1/\mathsf{m})$ under the assumptions of Kolmogorov's theorem.

Problem 2.22. In the situtation of Theorem 2.15, let

$$T(m) \;=\; \inf\{n \geq 0 : Z_n \leq m\}$$

for $m \in \mathbb{N}$. Prove that the theorem remains valid with $\mathbb{E}_k T(m)$ instead of $\mathbb{E}_k T$.

Problem 2.23. Suppose that λ is any quasi-stationary distribution of $(Z_n)_{n\geq 0}$ in the sense of (2.25). Show that T then has a geometric distribution on $\mathbb{N}$ under $\mathbb{P}_\lambda$. [Hint: Prove that $\mathbb{P}_\lambda(T > n) = \mathbb{P}_\lambda(T > k)\mathbb{P}_\lambda(T > n-k)$ for all $0 \leq k \leq n$.]

2.3 Critical case: The Kolmogorov-Yaglom exponential limit theorem and the expected maximum

It should not be surprising that the critical case that we are going to study next is of some interest because it marks the ”boundary” between two cases of drastically different behavior. For most of the result presented here we assume that $(Z_n)_{n\ge0}$ has finite and positive offspring variance σ^2 [$\Rightarrow\ p_1<1$], here given by [☞ (1.6)]

$$\sigma^2 \;=\; f''(1)+f'(1)(1-f'(1)) \;=\; f''(1),$$

for $\mathsf{m}=f'(1)=1$. We already know from previous results that

$$\begin{aligned}&q=\mathbb{P}(Z_n=0\text{ eventually})=1;\\&\mathbb{E}Z_n=1\quad\text{for all } n\ge0;\\&\mathbb{V}\mathrm{ar}\,Z_n=n\sigma^2\to\infty,\quad\text{as } n\to\infty.\end{aligned}$$

They provide a first impression about the instabilities of critical GWP’s. The following exponential limit theorem, again due to KOLMOGOROV [17] and YAGLOM [33], sheds further light on their behavior. Its proof as opposed to the ones in the previous sections is relatively simple. Its main assertion is about the asymptotic distribution of $n^{-1}Z_n$ conditioned upon survival. A second, more difficult and recent result due to ATHREYA [3] determines the asymptotic behavior of the maximum $M_n=\max_{0\le k\le n}Z_k$, more precisely of $\mathbb{E}_kM_n$ as $n\to\infty$ for any $k\in\mathbb{N}$.

Theorem 2.24. [Kolmogorov-Yaglom] *For a critical GWP $(Z_n)_{n\ge0}$ with offspring variance $\sigma^2\in\mathbb{R}_>$ the following assertions hold true for all $k\in\mathbb{N}$ and $t\ge0$:*

$$\lim_{n\to\infty} n\,\mathbb{P}_k(Z_n>0) \;=\; \frac{2k}{\sigma^2}, \tag{2.31}$$

$$\lim_{n\to\infty} \mathbb{E}_k\left(\frac{Z_n}{n}\middle|Z_n>0\right) \;=\; \frac{\sigma^2}{2}, \tag{2.32}$$

$$\lim_{n\to\infty} \mathbb{P}_k\left(\frac{Z_n}{n}\le t\middle|Z_n>0\right) \;=\; 1-e^{-2t/\sigma^2}, \tag{2.33}$$

that is, the asymptotic distribution of $n^{-1}Z_n$ given $Z_n>0$ is exponential with parameter $2/\sigma^2$.

As already noted, our second theorem on the asymptotic growth of the expected maximum of a critical GWP was obtained by ATHREYA [3] following earlier work by PAKES [26] and WEINER [31].

Theorem 2.25. [Athreya] *Let* $(Z_n)_{n\ge 0}$ *be a critical GWP with offspring variance* $\sigma^2 \in \mathbb{R}_>$ *and maximum sequence* $M_n = \max_{0\le k\le n} Z_k$ *for* $n \in \mathbb{N}_0$*. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}_k M_n}{k \log n} \;=\; 1 \quad \textit{for all } k \in \mathbb{N}.$$

All three assertions of the first result will be deduced from the following basic lemma.

Lemma 2.26. *Under the assumptions of Theorem 2.24,*

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right) \;=\; \frac{\sigma^2}{2} \tag{2.34}$$

holds true for all $s \in [0,1)$ *and the convergence is uniform.*

Proof. Let us start by noting the following second order Taylor expansion of f in 1:

$$f(s) \;=\; 1 + (s-1) + \frac{\sigma^2}{2}(s-1)^2 + r_2(s)(s-1)^2 \;=\; s + \frac{\sigma^2}{2}(s-1)^2 + r_2(s)(s-1)^2$$

for a function r_2 satisfying $\lim_{s\uparrow 1} r_2(s) = 0$. It follows that

$$\begin{aligned}
\frac{1}{1-f(s)} - \frac{1}{1-s} \;&=\; \frac{f(s)-s}{(1-f(s))(1-s)}\\
&=\; \frac{(\sigma^2/2)(1-s)^2 + r_2(s)(1-s)^2}{(1-f(s))(1-s)}\\
&=\; \frac{1-s}{1-f(s)}\left(\frac{\sigma^2}{2} + r_2(s)\right)\\
&=\; \frac{\sigma^2}{2} + \rho(s)
\end{aligned}$$

for all $s \in [0,1)$ and some function ρ that also satisfies $\lim_{s\uparrow 1} \rho(s) = 0$. An iteration of this identity yields upon multiplication with $1/n$

$$\begin{aligned}
\frac{1}{n}\left(\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right) \;&=\; \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{1}{1-f\circ f_k(s)} - \frac{1}{1-f_k(s)}\right)\\
&=\; \frac{\sigma^2}{2} + \frac{1}{n}\sum_{k=0}^{n-1} \rho\circ f_k(s).
\end{aligned}$$

The assertion now follows because, by Corollary 1.8, $f_n(s) \uparrow 1$ uniformly in $s \in [0,1]$ and $\rho(s) \to 0$ as $s \uparrow 1$. $\square$

There is an alternative way to prove this lemma by means of a comparison argument. We refer to Problem 2.32.

Proof of Theorem 2.24. We confine ourselve to the case $k = 1$ (one ancestor) and leave the simple extension to general k as an exercise [☞ Problem 2.33]. For the proof of (2.31), write

$$n\mathbb{P}(Z_n > 0) \;=\; n(1-f_n(0)) \;=\; \left(\frac{1}{n}\left(\frac{1}{1-f_n(0)}-1\right)+\frac{1}{n}\right)^{-1}$$

and notice that, by Lemma 2.26, the last expression converges to $2/\sigma^2$ as $n \to \infty$. This may be further utilized to infer (2.32) via

$$\mathbb{E}\left(\frac{Z_n}{n}\middle| Z_n > 0\right) \;=\; \frac{1}{\mathbb{P}(Z_n > 0)}\,\mathbb{E}\left(\frac{Z_n}{n}\right) \;=\; \frac{1}{n\mathbb{P}(Z_n > 0)} \;\to\; \frac{\sigma^2}{2} \quad \text{as } n \to \infty.$$

Left with the proof of (2.33), pick an arbitrary $t \in \mathbb{R}_>$. Then

$$\begin{aligned}
\mathbb{E}(e^{-tZ_n/n}|Z_n > 0) \;&=\; \frac{1}{\mathbb{P}(Z_n > 0)}\Big(\mathbb{E}(e^{-tZ_n/n} - \mathbb{P}(Z_n = 0)\Big)\\
&=\; \frac{1}{1-f_n(0)}\Big(f_n(e^{-t/n}) - f_n(0)\Big)\\
&=\; 1 - \frac{1-f_n(e^{-t/n})}{1-f_n(0)}\\
&=\; 1 - \frac{1}{n(1-f_n(0))}\left(\frac{1}{n}\left(\frac{1}{1-f_n(e^{-t/n})} - \frac{1}{1-e^{-t/n}}\right) + \frac{1}{n(1-e^{-t/n})}\right)^{-1}\\
&\to\; 1 - \frac{\sigma^2}{2}\left(\frac{\sigma^2}{2}+\frac{1}{t}\right)^{-1} \qquad (n \to \infty)\\
&=\; \frac{1}{1+t\sigma^2/2}
\end{aligned}$$

where Lemma 2.26 has been used to infer

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1-f_n(e^{-t/n})} - \frac{1}{1-e^{-t/n}}\right) \;=\; \frac{\sigma^2}{2}.$$

Notice that this conclusion requires the uniform convergence in (2.34). We finally conclude (2.33) because $(1+t\sigma^2/2)^{-1}$ is the LT of the exponential distribution with parameter $2/\sigma^2$ and by the continuity theorem for LT's [newhand e.g. [9, Thm. 2 on p. 431]]. □

The proof of Theorem 2.25 is more difficult and has therefore been divided into several lemmata. As M_n increases to $M_\infty := \max_{n\geq 0} Z_n$, we first show that $\mathbb{E}_k M_\infty = \infty$ for all $k \in \mathbb{N}$ regardless of the finiteness of σ^2.

Lemma 2.27. *For any critical GWP $(Z_n)_{n\ge 0}$ with $p_1 \ne 1$ its supremum M_∞ has infinite mean, viz. $\mathbb{E}_k M_\infty = \infty$ for all $k \in \mathbb{N}$.*

Proof. Obviously, $Z_n \le M_n \le M_\infty$ for all $n \in \mathbb{N}_0$. Hence, $\mathbb{E}_k M_\infty = \infty$ would imply $\mathbb{E}_k Z_n \to 0$, for $\mathbb{P}_k(Z_n > 0) \to 0$. On the other hand, $(Z_n)_{n\ge 0}$ constitutes a martingale under each $\mathbb{P}_k$ and so $\mathbb{E}_k Z_n = \mathbb{E}_k Z_0 = k$ for all $n \in \mathbb{N}_0$. □

Lemma 2.28. *Let $(Z_n)_{n\ge 0}$ be a critical GWP with $p_1 \ne 1$ and put $\psi(x) = x\log x$. Then $\lim_{n\to\infty} \mathbb{E}_k \psi(Z_n) = \infty$ for all $k \in \mathbb{N}$ and, more precisely,*

$$\lim_{n\to\infty} (\mathbb{E}_k \psi(Z_n) - k\log n) \;=\; k\alpha, \quad \alpha \;:=\; \frac{4}{\sigma^4}\int_0^\infty \psi(x)\, e^{-2x/\sigma^2}\, dx \tag{2.35}$$

holds true if σ^2 is finite.

Proof. By Lemma 1.2, we obviously have $\mathbb{E}_k \psi(Z_n) \ge \mathbb{E}\psi(Z_n)$ for all $k,n \in \mathbb{N}$ so that we must prove the first assertion only for $k = 1$. Put $a_n = \mathbb{P}(Z_n > 0)^{-1}$. Then

$$\mathbb{E}\psi(Z_n) \;=\; \mathbb{E}(\psi(Z_n)|Z_n > 0)\,\mathbb{P}(Z_n > 0) \;=\; =\; \mathbb{E}(\psi(a_n^{-1} Z_n)|Z_n > 0) + \log a_n,$$

which upon using $c := \sup_{0<x<1} |\psi(x)| < \infty$ implies

$$\mathbb{E}\psi(Z_n) - \log a_n \;\ge\; \mathbb{E}(\psi(a_n^{-1} Z_n)\mathbf{1}_{\{0<Z_n<a_n\}}|Z_n > 0) \;\ge\; -c$$

and thereby

$$\liminf_{n\to\infty} (\mathbb{E}\psi(Z_n) - \log a_n) \;\ge\; -c.$$

Since $a_n \to \infty$, we arive at the desired conclusion $\mathbb{E}\psi(Z_n) \to \infty$ as $n \to \infty$.

Turning to the proof of (2.35), let now σ^2 be finite. Since, by Prop. 1.4, $\mathbb{E}_k Z_n^2 = \mathbb{V}\mathrm{ar}_k Z_n + (\mathbb{E}_k Z_n)^2 = k(n\sigma^2 + k)$ for any $n \in \mathbb{N}_0$, it follows with the help of (2.31) that

$$\sup_{n\ge 0} \mathbb{E}_k \left(\frac{Z_n^2}{n^2}\middle| Z_n > 0\right) \;=\; \sup_{n\ge 0} \frac{\mathbb{E}_k Z_n^2 \mathbf{1}_{\{Z_n>0\}}}{n^2 \mathbb{P}_k(Z_n > 0)} \;=\; \sup_{n\ge 0} \frac{k\sigma^2 + k^2/n}{n\,\mathbb{P}_k(Z_n > 0)} \;<\; \infty,$$

and so the L^2-boundedness of $n^{-1}Z_n|Z_n > 0$, $n \in \mathbb{N}_0$[1] under each $\mathbb{P}_k$. As a consequence, we have the uniform integrability of $\psi(n^{-1}Z_n)|Z_n > 0$, $n \in \mathbb{N}_0$, under each $\mathbb{P}_k$, which in combination with $\mathbb{E}_k Z_n = k$ and (2.33) finally implies

$$\begin{aligned} \mathbb{E}_k \psi(Z_n) - k\log n \;&=\; \mathbb{E}_k(\psi(Z_n) - Z_n \log n) \\ &=\; \mathbb{E}_k(\psi(n^{-1}Z_n)|Z_n > 0)\, n\,\mathbb{P}(Z_n > 0) \;\to\; k\alpha \end{aligned}$$

[1] Of course, this means that any sequence $(Y_n)_{n\ge 0}$ with $\mathbb{P}(Y_n \in \cdot) = \mathbb{P}(Z_n \in \cdot|Z_n > 0)$ for each n has this property.

for all $k \in \mathbb{N}_0$. □

Lemma 2.29. *Under the assumptions of Theorem 2.25,*

$$\limsup_{n\to\infty} \frac{\mathbb{E}_k M_n}{\mathbb{E}_k \psi(Z_n)} \;\le\; 1$$

holds true for each $k \in \mathbb{N}$.

Proof. Doob's maximal inequality [☞ [12, p. 14]], applied to the martingale $(Z_n)_{n\ge 0}$, provides us with the tail inequality

$$\mathbb{P}_k(M_n > t) \;\le\; \frac{1}{t}\int_{\{M_n>t\}} Z_n \, d\mathbb{P}_k$$

for any $t > 0$, whence we infer upon partial integration

$$\begin{aligned}
\mathbb{E}_k M_n \;&=\; \int_0^\infty \mathbb{P}_k(M_n > t)\, dt \\
&\le\; k \;+\; \int_k^\infty \frac{1}{t}\int \mathbf{1}_{\{M_n>t\}} Z_n \, d\mathbb{P}_k \, dt \\
&=\; k \;+\; \int Z_n \int_t^{M_n} \frac{dt}{t}\, d\mathbb{P}_k \\
&=\; k(1-\log k) \;+\; \mathbb{E}_k Z_n \log M_n.
\end{aligned}$$

A simple estimation, given as Problem 2.34 below, gives

$$a\log b \;\le\; \psi(a) + \frac{b}{c}\mathbf{1}_{[0,b/c]}(a) + \frac{b}{c}\mathbf{1}_{[b/c,b]}(a)$$

for all $0 \le a \le b, b > 0$ and $e < c < \infty$ [naturally, with $e = \exp(1)$]. Applying this inequality in the previous one to $\mathbb{E}_k Z_n \log M_n$ with $a = Z_n$, $b = M_n$ and arbitrary $c > e$, we obtain

$$\begin{aligned}
\mathbb{E}_k M_n \;&\le\; k(1-\log k) \;+\; \mathbb{E}_k \psi(Z_n) \;+\; \frac{\log c}{c}\int_{\{Z_n \le M_n/c\}} M_n \, d\mathbb{P}_k \\
&\quad + \frac{1}{e}\int_{\{M_n/c \le Z_n \le M_n\}} M_n \, d\mathbb{P}_k \\
&\le\; k\left(1 - \log k + \frac{c}{e}\right) \;+\; \mathbb{E}_k \psi(Z_n) \;+\; \frac{\log c}{c}\, \mathbb{E}_k M_n.
\end{aligned}$$

and thus after rearrangement and division by $\mathbb{E}_k \psi(Z_n)$ (which by Lemma 2.28 tends to ∞)

$$\left(1 - \frac{\log c}{c}\right) \limsup_{n\to\infty} \frac{\mathbb{E}_k M_n}{\mathbb{E}_k \psi(Z_n)} \;\le\; 1$$

for any $k \in \mathbb{N}$. By finally letting c tend to infinity, we arrive at the assertion of the lemma. □

Proof of Theorem 2.25 [Part 1]. A combination of the previous lemma with (2.35) of Lemma 2.28 obviously leads to

$$\limsup_{n\to\infty} \frac{\mathbb{E}_k M_n}{k \log n} \;\le\; 1$$

for all $k \in \mathbb{N}$ and thus leaves us with a proof of the reverse inequality

$$\liminf_{n\to\infty} \frac{\mathbb{E}_k M_n}{k \log n} \;\ge\; 1 \tag{2.36}$$

for any $k \in \mathbb{N}$ for which again we first provide two lemmata.

As in [1], a sequence $(S_n)_{n\ge 0}$ of sums of iid real-valued random variables with $S_0 = 0$ is called *standard random walk (SRW)* hereafter.

Lemma 2.30. *Let $(S_n)_{n\ge 0}$ be a SRW with $\mathbb{E}S_1 = 1$ and $\mathbb{E}S_1^2 < \infty$. Then*

$$\sum_{n\ge 1} \frac{1}{n}\, \mathbb{E}\, S_n \mathbf{1}_{\{S_n > \rho n\}} \;<\; \infty$$

for any $\rho > 1$.

Proof. Let $X_1, X_2, \ldots$ denote the iid increments of $(S_n)_{n\ge 0}$. Fixing any $\rho > 1$, consider the SRW $(\rho n - S_n)_{n\ge 0}$ whose increments $\rho - X_1, \rho - X_2, \ldots$ clearly have positive mean and finite variance. Denote by $\mathbb{U} = \sum_{n\ge 0} \mathbb{P}(\rho n - S_n \in \cdot)$ the associated *renewal measure*, which under the given assumptions satisfies [☞ e.g. [1]]

$$C \;:=\; \sup_{t\in\mathbb{R}} \frac{\mathbb{U}(t)}{t \vee 1} \;<\; \infty,$$

where $\mathbb{U}(t) \;:=\; \mathbb{U}((-\infty, t])$ is the so-called *renewal function*. Since $n^{-1} S_n = \mathbb{E}(X_1 | S_n)$ a.s., we now infer

$$\begin{aligned}\sum_{n\ge 1}\frac{1}{n}\mathbb{E}S_n\mathbf{1}_{\{S_n>\rho n\}} &= \sum_{n\ge 1}\mathbb{E}X_1\mathbf{1}_{\{S_n>\rho n\}}\\ &\le \sum_{n\ge 1}\int |x|\,\mathbb{P}(S_{n-1}\ge \rho n-x)\,\mathbb{P}(X_1\in dx)\\ &= \int |x|\,\mathbb{U}(x-\rho)\,\mathbb{P}(X_1\in dx)\\ &\le \int (|x|\vee 1)\,\mathbb{U}(x)\,\mathbb{P}(X_1\in dx)\\ &\le C\,\mathbb{E}(X_1^2\vee 1) \;<\; \infty\end{aligned}$$

which is the desired conclusion. □

As a final auxiliary result for the proof of (2.36) we need:

Lemma 2.31. *Let* $(Z_n)_{n\ge 0}$ *be a nonnegative martingale and*

$$\tau \;=\; \tau_{n,j} \;=\; \inf\{i\ge 1 : Z_i=0 \text{ or } Z_i\ge j\}\wedge n$$

for $j,n\ge 1$. *Then* $\mathbb{E}Z_\tau\mathbf{1}_{\{Z_\tau\ge j\}}\ge \mathbb{E}Z_n\mathbf{1}_{\{Z_n\ge j\}}$.

Proof. Since τ is a bounded stopping time for $(Z_n)_{n\ge 0}$, the optional sampling theorem [☞ [6, Thm. 5.10]] ensures $\mathbb{E}Z_\tau=\mathbb{E}Z_n$. Now use $\{0<Z_\tau<j\}=\{\tau=n,0<Z_n<j\}$ to obtain

$$\begin{aligned}\mathbb{E}Z_\tau\mathbf{1}_{\{Z_\tau<j\}} &= \mathbb{E}Z_\tau\mathbf{1}_{\{0<Z_\tau<j\}} \;=\; \mathbb{E}Z_n\mathbf{1}_{\{\tau=n,0<Z_n<j\}}\\ &\le \mathbb{E}Z_n\mathbf{1}_{\{0<Z_n<j\}} \;=\; \mathbb{E}Z_n\mathbf{1}_{\{Z_n<j\}}\end{aligned}$$

which in combination with $\mathbb{E}Z_\tau=\mathbb{E}Z_n$ clearly gives the asserted inequality. □

Proof of Theorem 2.25 [Part 2]. Fix $j,k,n\in\mathbb{N}$ and an arbitrary $\rho>1$. Since $(Z_n)_{n\ge 0}$ is a nonnegative martingale, an application of the previous lemma with $\tau=\tau_{n,j}$ as defined there and $a_{n,j}:=\mathbb{E}_kZ_\tau\mathbf{1}_{\{Z_\tau>\rho j\}}$ provides us with

$$\begin{aligned}\mathbb{E}_kZ_n\mathbf{1}_{\{Z_n\ge j\}} &\le \mathbb{E}_kZ_\tau\mathbf{1}_{\{Z_\tau\ge j\}}\\ &= \mathbb{E}_kZ_n\mathbf{1}_{\{j\le Z_n\le\rho j\}}+\mathbb{E}_kZ_n\mathbf{1}_{\{Z_n>\rho j\}}\\ &\le \rho j\,\mathbb{P}_k(M_n\ge j)+a_{n,j}. \qquad (2.37)\end{aligned}$$

Now, if $(S_n)_{n\ge 0}$ denotes a SRW with increment distribution $(p_n)_{n\ge 0}$ under any $\mathbb{P}_k$ and $\phi(i,j):=\mathbb{E}_kS_j\mathbf{1}_{\{S_i>\rho j\}}$, then we infer upon using $0=\phi(0,j)\le\ldots\le\phi(j,j)$ and the Markov property for $(Z_n)_{n\ge 0}$ that

$$
\begin{aligned}
a_{n,j} \;&=\; \sum_{i=0}^{n-1} \mathbb{E}_k Z_\tau \mathbf{1}_{\{Z_\tau > \rho j, \tau = i+1\}} \;=\; \sum_{i=0}^{n-1} \mathbb{E}_k Z_{i+1} \mathbf{1}_{\{Z_{i+1} > \rho j, M_i < k\}} \\
&=\; \sum_{i=0}^{n-1} \int_{\{M_i < k\}} \mathbb{E}_k \left(\mathbf{1}_{\{\sum_{l=1}^{Z_i} X_{i,l} > \rho j\}} \sum_{l=1}^{Z_i} X_{i,l} \middle| Z_i \right) \, d\mathbb{P}_k \\
&=\; \sum_{i=0}^{n-1} \mathbb{E}_k \phi(Z_i, j) \mathbf{1}_{\{M_i < k\}} \\
&\le\; \phi(j,j) \sum_{i=1}^{n} b_{i,j}, \quad \text{where } b_{i,j} \;:=\; \mathbb{P}_k(0 < Z_i < j).
\end{aligned}
\tag{2.38}
$$

The next thing to show is that

$$
\lim_{n\to\infty} \frac{1}{\log n} \sum_{l\ge 1} \frac{a_{n,l}}{l} \;=\; 0. \tag{2.39}
$$

To this end we make use of the results of Theorem 2.24 and particularly the following uniform version of (2.33)

$$
\lim_{n\to\infty} \sup_{t>0} \left| \mathbb{P}_k \left(\frac{Z_n}{n} \le t \middle| Z_n > 0 \right) - G(t) \right| \;=\; 0 \tag{2.40}
$$

with $G(t) := 1 - e^{-2t/\sigma^2}$, which holds because G is a continuous distribution function.

It easily follows from $b_{n,j} \le b_n := \mathbb{P}_k(Z_n > 0)$, (2.31) and $\sum_{i=1}^{n} i^{-1} \simeq \log n$ that

$$
\sup_{j,n\ge 1} \frac{1}{\log(n+1)} \sum_{i=0}^{n} b_{i,j} \;<\; \infty. \tag{2.41}
$$

Furthermore, for each $N \ge 1$ and $n > N$,

$$
\sum_{i=0}^{n-1} b_{i,j} \;\le\; jN \;+\; \sum_{i=jN}^{n-1} b_i \left| \mathbb{P}_k(Z_i < k | Z_i > 0) - G\left(\frac{j}{i}\right) \right| \;+\; G\left(\frac{1}{N}\right) \sum_{i=kN}^{n-1} b_i.
$$

Therefore, we infer

$$
\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=0}^{n-1} b_{i,j} \;=\; 0. \tag{2.42}
$$

from (2.40), (2.41) and $\lim_{t\downarrow 0} G(t) = 0$ by first choosing N large enough and then letting $n \to \infty$.

Since $\sum_{l\ge 1} l^{-1} \phi(l,l) < \infty$ by Lemma 2.30, a combination of (2.38), (2.41), (2.42) and the dominated convergence theorem allows us to infer

$$
\lim_{n\to\infty} \frac{1}{\log n} \sum_{l\ge 1} \frac{a_{n,l}}{l} \;\le\; \sum_{l\ge 1} \frac{\phi(l,l)}{l} \left(\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=0}^{n-1} b_{i,l} \right) \;=\; 0,
$$

that is (2.39).

Finally, we return to (2.37), which upon division by j and subsequent summation over j gives

$$\sum_{j\geq 1}\frac{1}{j}\,\mathbb{E}_k Z_n \mathbf{1}_{\{Z_n\geq j\}} \;\leq\; \rho\sum_{j\geq 1}\mathbb{P}_k(M_n\geq j)\;+\;\sum_{j\geq 1}\frac{a_{n,j}}{j}$$

for all $n\in\mathbb{N}$ and thus further

$$\frac{1}{\log n}\,\mathbb{E}_k\left(Z_n\sum_{j=1}^{Z_n}\frac{1}{j}\right)\;\leq\;\frac{\rho}{\log n}\,\mathbb{E}_k M_n\;+\;\frac{1}{\log n}\sum_{j\geq 1}\frac{a_{n,j}}{j}. \tag{2.43}$$

But $\sum_{j=1}^{n} j^{-1}\simeq\log n$ in combination with Lemma 2.28 implies

$$\lim_{n\to\infty}\frac{1}{k\log n}\mathbb{E}_k\left(Z_n\sum_{j=1}^{Z_n}\frac{1}{j}\right)\;=\;\lim_{n\to\infty}\frac{1}{k\log n}\mathbb{E}_k\,\psi(Z_n)\;=\;1,$$

which in turn may be combined with (2.39) in (2.43) to conclude

$$\liminf_{n\to\infty}\frac{\rho}{k\log n}\,\mathbb{E}_k M_n\;\geq 1.$$

The proof is herewith complete because $\rho>1$ was chosen arbitrarily. □

Problems

Problem 2.32. Let $f^{(1)}, f^{(2)}$ denote the gf's of two critical offspring distributions with variances $\sigma_1^2<\sigma_2^2<\infty$.

(a) **[Comparison lemma]** Show that, for suitable $n_1,n_2\in\mathbb{N}_0$,

$$f^{(1)}_{n+n_1}(s)\leq f^{(2)}_{n+n_2}(s)$$

for all $s\in[0,1]$ and $n\in\mathbb{N}_0$.

(b) Show that (2.34) in Lemma 2.26 is exactly fulfilled in the case of a linear fractional offspring distribution as defined in Section 1.5 (with $b=(1-p)^2$ by criticality), that is

$$\frac{1}{n}\left(\frac{1}{1-f_n(s)}-\frac{1}{1-s}\right)\;=\;\frac{\sigma^2}{2}$$

for all $s\in[0,1]$ and $n\in\mathbb{N}_0$.

(c) Use (b) and the comparison lemma to give an alternative proof of Lemma 2.26.

Problem 2.33. Prove that Theorem 2.24 holds true for any $k \in \mathbb{N}$.

Problem 2.34. Show that

$$a \log b \ \leq \ \psi(a) + \frac{b}{c}\mathbf{1}_{[0,b/c]}(a) + \frac{b}{c}\mathbf{1}_{[b/c,b]}(a)$$

holds true for all $0 \leq a \leq b$, $b > 0$ and $e < c < \infty$.

Problem 2.35. In the situation of Theorem 2.24 and with T denoting the extinction epoch of $(Z_n)_{n\geq 0}$, prove that

(a) for all $s \in [0,1)$

$$\lim_{n\to\infty} n^2 (f_{n+1}(s) - f_n(s)) \ = \ \frac{2}{\sigma^2}.$$

[Hint: Use the Taylor expansion of f given at the beginning of the proof of Lemma 2.26 together with (2.31).]

(b) for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} n^2\, \mathbb{P}_k(T = n+1) \ = \ \lim_{n\to\infty} n^2\, \mathbb{P}_k(Z_n > 0, Z_{n+1} = 0) \ = \ \frac{2k}{\sigma^2}.$$

[Hint: Use (a).]

What does (b) imply for $\mathbb{E}_k T$?

Problem 2.36. Under the assumptions of Theorem 2.24, show that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}_>$

$$\mathbb{P}_k\left(\frac{Z_n}{n} \leq t \middle| Z_{n+j} > 0\right) \ \to \ \mathrm{Exp}\left(\frac{2}{\sigma^2}\right) * \mathrm{Exp}\left(\frac{2(1+\alpha)}{\alpha\sigma^2}\right)([0,t])$$

as $j, n \to \infty$ in such a way that $j/n \to \alpha$.

Problem 2.37. In the situation of Theorem 2.24 and assuming additionally $p_1 > 0$, prove the following assertions:

(a) If $f_n^{(j)}$ denotes the j^{th} derivative of f_n, then

$$f_n^{(j)}(s) \ = \ a_{n,j}(s) + f'(f_{n-1}(s))\, f_n^{(j-1)}(s)$$

for all $j, n \in \mathbb{N}$, where $a_{n,j}(s)$ is a power series in s with nonnegative coefficients.

(b) For all $j \in \mathbb{N}$

$$\frac{\mathbb{P}(Z_1 = j)}{\mathbb{P}(Z_1 = 1)} \ \leq \ \frac{\mathbb{P}(Z_2 = j)}{\mathbb{P}(Z_2 = 1)} \ \leq \ \frac{\mathbb{P}(Z_3 = j)}{\mathbb{P}(Z_3 = 1)} \ \leq \ \ldots$$

[Hint: Use (a).]

(c) Defining

$$\pi_j \;=\; \lim_{n\to\infty} \frac{\mathbb{P}(Z_n = j)}{\mathbb{P}(Z_n = 1)} \quad \text{and} \quad \Pi(s) \;=\; \sum_{j\ge 1} \pi_j s^j$$

it follows that

$$\Pi(s) \;=\; \lim_{n\to\infty} \frac{f_n(s) - f_n(0)}{f_n'(0)} \;\le\; \lim_{n\to\infty} \frac{f_n'(s)}{f_n'(0)} \;<\; \infty \tag{2.44}$$

for all $s \in [0,1)$. [Hint: In order to prove that the last limit in (2.44) exists and is finite, verify first that $f_n'(s) = \prod_{j=0}^{n-1} f'(f_j(s)) \le \prod_{j=0}^{n-1} f'(f_{j+k}(0))$ for $s \in (0,1)$, some $k = k(s) \ge 0$ and all $n \ge 1$. Show then that

$$\prod_{j=0}^{n-1} \frac{f'(f_{j+k}(0))}{f'(f_j(0))} \;\le\; \prod_{j=0}^{n-1} \left(1 + \frac{\sigma^2}{p_1}(f_{j+k}(0) - f_j(0))\right)$$

for all $n \ge 1$ and use finally Problem 2.35 to infer the convergence of the right-hand product to a finite limit.]

(d) [SENETA] For all $j \ge 1$,

$$\theta_j \;:=\; \lim_{n\to\infty} \mathbb{P}(Z_n = j | Z_n > 0, Z_{n+1} > 0) \;=\; \frac{\pi_j p_0^j}{\Pi(p_0)}, \tag{2.45}$$

where $\sum_{j\ge 1} \theta_j = 1$ and $\Theta(s) := \sum_{j\ge 1} \theta_j s^j$ satisfies

$$\Theta(s) \;=\; \frac{\Pi(p_0 s)}{\Pi(p_0)}.$$

Problem 2.38. *(Continuation)* Show that the π_j in the previous problem fulfill the invariance equations

$$\pi_j \;=\; \sum_{i\ge 1} \pi_i \mathbb{P}_i(Z_1 = j) \;=\; \sum_{i\ge 1} \pi_i p_{ij}, \quad j \ge 1$$

and that the associated gf $\Pi(s)$ satisfies the functional equation

$$\Pi(f(s)) \;=\; \Pi(s) + \Pi(p_0)$$

for all $s \in [0,1]$ and furthermore $\Pi(1) = \sum_{j\ge 1} \pi_j = \infty$.

Problem 2.39. *(Continuation)* Use (2.44) and (2.45) for the explicit computation of the π_j and θ_j for the case of a critical linear fractional gf.

2.4 The total progeny

Given a GWP $(Z_n)_{n\ge 0}$ in a standard model with finite offspring mean m and offspring variance $\sigma^2 \in (0,\infty]$, this section will focus on the sequence

$$Y_n \;:=\; \sum_{k=0}^{n} Z_k, \quad n \in \mathbb{N}_0$$

with associated gf's h_n. Plainly, Y_n increases to the *total progeny* (or *total population size*)

$$Y_\infty \;:=\; \sum_{k\ge 0} Z_k$$

which is finite iff $Z_n = 0$ eventually, thus

$$\mathbb{P}(Y_\infty < \infty) \;=\; q.$$

Let h_∞ denote the gf of Y_∞. In the following, we will determine the distribution of Y_∞ in terms of the transition probabilities $p_{ij} = \mathbb{P}_i(Z_1 = j)$ for $i, j \in \mathbb{N}_0$ [☞ (1.2)] and the asymptotic behavior of Y_n on the event $\{Y_\infty = \infty\} = \{Z_n \to \infty\}$ (supercritical case) or conditioned upon $Z_n > 0$ (critical and subcritical case). The first result, giving the distribution of Y_∞, was obtained by DWASS [8].

Theorem 2.40. [Dwass] *Besides the stated assumptions let further $p_0 > 0$. Then*

$$\mathbb{P}_i(Y_\infty = j) \;=\; \frac{i}{j}\, p_{j,j-i} \tag{2.46}$$

for all $i \in \mathbb{N}$ and $j \ge i$, in particular

$$\mathbb{P}(Y_\infty = j) \;=\; \frac{1}{j}\, p_{j,j-1} \tag{2.47}$$

for all $j \in \mathbb{N}$.

The second result owing to PAKES [24] provides a good picture of the behavior of Y_n on $\{Z_n > 0\}$ as $n \to \infty$, however under the additional condition $\sigma^2 < \infty$ if $\mathsf{m} \le 1$.

Theorem 2.41. [Pakes] *Under the stated assumptions the following assertions hold true:*

(a) ***Supercritical case**: If k_n, W^* and W are defined as in Section 2.1, then*

$$\lim_{n\to\infty} \frac{Y_n}{k_n} \;=\; \frac{\mathsf{m}W^*}{\mathsf{m}-1} \quad \textit{and} \quad \lim_{n\to\infty} \frac{Y_n}{\mathsf{m}^n} \;=\; \frac{\mathsf{m}W}{\mathsf{m}-1} \quad \mathbb{P}\text{-a.s.} \tag{2.48}$$

(b) ***Subcritical case***: *If $\sigma^2 < \infty$ and $\theta := \frac{\sigma^2}{\mathsf{m}(1-\mathsf{m})}$, then*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n}{n} - \theta\right| > \varepsilon \,\middle|\, Z_n > 0\right) = 0 \tag{2.49}$$

for any $\varepsilon > 0$.

(c) ***Critical case***: *If $\sigma^2 < \infty$, then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{n^2} \le t \,\middle|\, Z_n > 0\right) = F(t) \tag{2.50}$$

for any $t \in \mathbb{R}_\ge$, where F is the distribution on $\mathbb{R}_>$ having LT

$$\varphi(t) = \int_{\mathbb{R}_>} e^{-ty}\, F(dy) = \frac{2u(t)e^{-u(t)}}{1-e^{-2u(t)}}, \quad t \in \mathbb{R}_>,$$

with $u(t) := \sigma(2t)^{1/2}$.

Towards a proof of these results we set out with a basic lemma about the relation between the gf's h_n and an entailed functional equation for h_∞ which once again reflects a distributional equation valid for the associated random variable Y_∞. As a direct consequence of Lemma 1.2, we note before-hand that

$$\mathbb{P}_k((Z_n,Y_n)_{n\ge 0} \in \cdot) = \mathbb{P}(Z_n,Y_n)_{n\ge 0} \in \cdot)^{*k} \tag{2.51}$$

for all $k \ge 0$ so that particularly $h_n(s)^k$ is the gf of Y_n with respect to $\mathbb{P}_k$.

Lemma 2.42. *Under the stated assumptions the following assertions hold true for the gf's h_n and h_∞: For all $n \ge 0$ and $s \in [0,1]$,*

$$h_{n+1}(s) = s\, f \circ h_n(s), \tag{2.52}$$
$$h_\infty(s) = s\, f \circ h_\infty(s). \tag{2.53}$$

If $p_0 > 0$ and thus $q > 0$, then h_∞ is given by

$$h_\infty(s) = q\widehat{h}^{-1}(s), \quad \widehat{h}(s) := \frac{sq}{f(sq)}, \tag{2.54}$$

for $s \in [0,1]$ and forms the unique solution in the class of gf's of possibly defective probability distributions on $\mathbb{N}_0$.

Proof. A simple conditioning argument gives

$$h_{n+1}(s) \;=\; s\,\mathbb{E}\left(\mathbb{E}(s^{Z_1+\ldots+Z_{n+1}}|Z_1)\right) \;=\; s\,\mathbb{E}h_n(s)^{Z_n} \;=\; s\,f\circ h_n(s)$$

for all $n\in\mathbb{N}_0$ and $s\in[0,1]$ and thus (2.52). The functional equation (2.53) then follows by letting n tend to infinity and using the continuity of f. Now, if g denotes another gf solving (2.53), we infer upon using the convexity of f that

$$|h_\infty(s)-g(s)| \;\le\; s\mathsf{m}\,|h_\infty - g(s)|$$

for all $n\ge 1$ and $s\in[0,1]$ and therefore $h_\infty = g$ on the interval $[0,\mathsf{m}^{-1})$. But then $h_\infty = g$ on $[0,1]$, for both functions are analytic on $(-1,1)$.

In order to prove (2.54) suppose first $\mathsf{m}\le 1$ and thus $q=1$. Replacing s with $h_\infty^{-1}(s)$, we find that $s=h_\infty^{-1}(s)$ and so $h_\infty^{-1}(s)=s/f(s)$ as claimed. If $\mathsf{m}>1$ and $q>0$, then consider the gf $\widehat{f}(s):=q^{-1}f(sq)$ and let $\widehat{h}_\infty$ denote the unique solution of (2.53) with $\widehat{f}$ instead of f, viz.

$$\widehat{h}_\infty(s) \;=\; s\,\widehat{f}\circ\widehat{h}_\infty(s) \tag{2.55}$$

for $s\in[0,1]$. Since $\widehat{f}'(1)=f'(q)<1$ [☞ Cor. 1.8], the gf $\widehat{f}$ belongs to a subcritical offspring distribution whence we obtain from what has been proved before that

$$\widehat{h}_\infty^{-1}(s) \;=\; \frac{s}{\widehat{f}(s)} \;=\; \frac{sq}{f(sq)} \;=\; \widehat{h}(s).$$

Finally rewriting (2.55) as $q\widehat{h}_\infty(s)=s\,f(q\widehat{h}_\infty(s))$, we see that $q\widehat{h}_\infty$ forms another solution to (2.53) and thus, by uniqueness, $h_\infty(s)=q\widehat{h}_\infty(s)=q\widehat{h}^{-1}(s)$. □

Remark 2.43. The distributional equation behind (2.53) is easily disclosed when observing that

$$Y_{n+1} \;=\; 1 \,+\, \sum_{k=1}^{Z_1} Y_n(k) \tag{2.56}$$

for all $n\ge 0$, where $Y_n(k)$ is the total size over the first n generations of the subpopulation stemming from the k^{th} individual in generation 1 of the whole population. The $Y_n(k)$, $k\ge 1$, are clearly independent copies of Y_n and further independent of Z_1. Therefore, upon letting n tend to infinity, we find that

$$Y_\infty \;=\; 1 \,+\, \sum_{k=1}^{Z_1} Y_\infty(k) \tag{2.57}$$

which in terms of gf's corresponds to (2.53). Similarly, eq. (2.56) corresponds to (2.52). The uniqueness assertion of the above lemma further shows that the law of Y_∞ is the only solution to (2.57) among all distributions on $\mathbb{N}_0\cup\{\infty\}$. For an extension of the last statement see Problem 2.48

As a further prerequisite for the proof of Thm. 2.40 by Dwass, we need the following result that is known in the literature as the *ballot theorem*.

Lemma 2.44. [Ballot theorem] *Let $(S_n)_{n\ge 0}$ be a SRW with nonnegative integer-valued increments $X_1,X_2,\ldots$ with common mean μ. Then*

$$\mathbb{P}(S_k < k \textit{ for } 1 \le k \le n \,|\, S_n) \;=\; \left(1-\frac{S_n}{n}\right)^+ \quad \textit{a.s.} \tag{2.58}$$

for all $n \ge 1$ and furthermore

$$\mathbb{P}(S_n < n \textit{ for all } n \ge 1) \;=\; (1-\mu)^+. \tag{2.59}$$

Remark 2.45. To give an explanation for the name of the result consider the special case that $\mathbb{P}(X_k=0)=1-\mathbb{P}(X_k=2)$. Interpret $X_k=0$ as a vote for candidate A and $X_k=2$ as a vote for candidate B in a ballot with n received votes. Obviously, the event "$S_k<k$ for $1\le k\le n$" then means that A is always ahead of B during the counting of the votes.

Proof. Since (2.59) follows from (2.58) by letting $n\to\infty$ and the SLLN, we must only verify the latter assertion which is trivially satisfied on the event $\{S_n \ge n\}$. Thus confining to the event $\{S_n<n\}$ and possibly after switching to the SRW with truncated increments $X_1\wedge n, X_2\wedge n,\ldots$, we may assume w.l.o.g. that μ is finite. Then it is well-known that

$$M_k \;:=\; \frac{S_{n+1-k}}{n+1-k}, \quad 1\le k\le n$$

forms a martingale with respect to its natural filtration $(\mathscr{F}_k)_{1\le k\le n}$, that is $\mathscr{F}_k = \sigma(S_{n+1-k},\ldots,S_n)$. Defining the stopping time $\tau=\inf\{k\ge 1 : M_k\ge 1 \text{ or } k=n\}$, it follows that

$$\begin{aligned} A \;:=\; \{S_k<k \text{ for } 1\le k\le n\} \;&=\; \{M_k<1 \text{ for } 1\le k\le n\} \\ =\; \{\tau=n, S_1<1\} \;&=\; \{\tau=n, S_1=M_n=0\} \end{aligned}$$

and therefore $M_\tau=0$ on A, while $M_\tau=1$ on $A^c\cap\{S_n<n\}=A^c\cap\{\tau>1\}$. For the last conclusion put $\nu:=n+1-\tau$ and observe that on $A^c\cap\{S_n<n\}=\{1<\tau<n\}\cup\{\tau=n, S_1\ge 1\}$ we have $\nu>1$ as well as

$$\nu\le S_\nu\le S_{\nu+1}<\nu+1 \quad \text{and therefore} \quad S_\nu=\nu.$$

Having thus shown $M_\tau=\mathbf{1}_{A^c}$ on $\{S_n<n\}\in\mathscr{F}_1$, we finally conclude by an appeal to the optional sampling theorem [☞ [6, Thm. 5.10]] that on this event

$$\mathbb{P}(A^c|S_n) \;=\; \mathbb{P}(A^c|\mathscr{F}_1) \;=\; \mathbb{E}(M_\tau|\mathscr{F}_1) \;=\; M_1 \;=\; \frac{S_n}{n} \quad \text{a.s.}$$

which is the desired result. □

Remark 2.46. For those readers with some basic knowledge of fluctuation theory of random walks we like to point out that there is another quite elegant argument showing (2.59). Consider the SRW $(S_n - n)_{n\ge 0}$ whose increments take values in $\{-1,0,1,...\}$. Let $\sigma_+ := \inf\{n \ge 1 : S_n - n \ge 0\}$ be the first weakly ascending ladder epoch and $\sigma_- := \inf\{n \ge 1 : S_n - n < 0\}$ its associated dual, the first strictly descending ladder epoch. Then duality relation to be exploited here is [☞ e.g. [7, Thm. 5.4.2]]

$$\mathbb{P}(\sigma_+ = \infty) \;=\; \frac{1}{\mathbb{E}\sigma_-}$$

with the usual convention that $1/\infty := 0$. Now the left-hand side is just the probability we want to compute, while the right-hand side does indeed equal $(1-\mu)^+$ as the following argument shows. It equals 0 if $\mathbb{E}\sigma_- = \infty$ which in turn holds iff $\mu - 1 \ge 0$, and it equals $1-\mu$ otherwise, for then $(\mu-1)\mathbb{E}\sigma_- = \mathbb{E}S_{\sigma_-} = -1$ by Wald's identity [☞ [7, Thm. 5.3.1]] and the fact that $(S_n - n)_{n\ge 0}$ can only make downward steps of size one, also called left skip-freeness.

Proof (of Theorem 2.40). We embark on the definition of the SRW's $(S_n(s))_{n\ge 0}$, $s \in [0,1]$, with generic increment $X(s)$ having distribution

$$\mathbb{P}(X(s) = k) \;=\; \frac{p_k s^k}{f(s)}, \quad k \in \mathbb{N}_0$$

and mean

$$\mathbb{E}X(s) \;=\; \frac{sf'(s)}{f(s)}$$

under $\mathbb{P} = \mathbb{P}_1$. Observe that $p_0 > 0$ ensures $f(s) > 0$ for any $s \in [0,1]$. Obviously, $\mathbb{P}(S_j(1) = j-i) = \mathbb{P}_j(Z_1 = j-i) = p_{j,j-i}$ for all $j \ge i \ge 1$. By (2.51), it therefore suffices to prove that

$$h_\infty(s)^i \;=\; \sum_{j\ge i} \frac{i}{j}\,\mathbb{P}(S_j(1) = j-i)\,s^j \tag{2.60}$$

for all $i \in \mathbb{N}$ and $s \in [0,1]$.

Let us first consider the case $\mathsf{m} \le 1$ and fix any $i \in \mathbb{N}$. Then $f'(s) < 1 < s^{-1}f(s)$ for $s \in [0,1)$ and so

$$\mathbb{E}X(s) < 1 \text{ for } s \in [0,1) \quad\text{and}\quad \mathbb{E}X(1) = \mathsf{m}.$$

The ballot theorem provides us with

$$\mathbb{P}(S_n(s) < n \text{ for all } n \ge 1) \;=\; 1 - \mathbb{E}X(s) \;=\; 1 - \frac{sf'(s)}{f(s)}$$

for all $s \in [0,1]$. Furthermore, $\lim_{n\to\infty} n^{-1}S_n(s) = \mathbb{E}X(s) < 1$ a.s. by the SLLN implies $\mathbb{P}(S_n(s) \ge n-i \text{ infinitely often}) = 0$ for any $s \in [0,1)$. Now use $S_{\rho(s)}(s) = \rho(s) - i$ for $\rho(s) := \sup\{n \ge 0 : S_n(s) \ge n-i\}$ in combination with $\mathbb{P}(S_n(s) = j) = \mathbb{P}(S_n(1) = j)\,s^j/f(s)^n$ [☞ Problem 2.49] to infer

$$\begin{aligned}
1 &= \sum_{n\ge i}\mathbb{P}(S_n(s)=n-i, S_{n+j}(s)<n+j-i \text{ for all } j\ge 1)\\
&= \mathbb{P}(S_j(s)<j \text{ for all } j\ge 1)\sum_{n\ge i}\mathbb{P}(S_n(s)=n-i) \qquad (2.61)\\
&= \left(1-\frac{sf'(s)}{f(s)}\right)\sum_{n\ge i}\mathbb{P}(S_n(1)=n-i)\,\frac{s^{n-i}}{f(s)^n}.
\end{aligned}$$

Next, setting $u:=h_\infty^{-1}(s)=s/f(s)$, it follows that

$$\frac{sf'(s)}{f(s)} \;=\; \frac{h_\infty(u)\,f'\circ h_\infty(u)}{f\circ h_\infty(u)} \;=\; u\,f'\circ h_\infty(u).$$

Differentiation of the functional equation (2.53) then further yields

$$h_\infty'(u) \;=\; f\circ h_\infty(u)+u\,f'\circ h_\infty(u)\,h_\infty'(u) \;=\; f(s)+\frac{sf'(s)}{f(s)}\,h_\infty'(u)$$

whence by another use of (2.53)

$$1-\frac{sf'(s)}{f(s)} \;=\; \frac{f(s)}{h_\infty'(u)} \;=\; \frac{f\circ h_\infty(u)}{h_\infty'(u)} \;=\; \frac{h_\infty(u)}{u\,h_\infty'(u)}$$

is obtained. By plugging this in (2.61) together with $u=s/f(s)$ and $s=h_\infty(u)$, we finally arrive at

$$1 \;=\; \frac{h_\infty(u)}{u\,h_\infty'(u)}\sum_{n\ge i}\mathbb{P}(S_n(1)=n-i)\,\frac{u^n}{h_\infty(u)^i}$$

or, equivalently,

$$i\,h_\infty(u)^{i-1}h_\infty'(u) \;=\; \sum_{n\ge i}\mathbb{P}(S_n(1)=n-i)\,u^{n-1}.$$

But this gives (2.60) upon integration with respect to u from 0 to s and noting that $h_\infty(0)=0$.

Now suppose $\mathsf{m}>1$ and $q>0$. As before, let $i\in\mathbb{N}$ be fixed. Recall from the proof of Lemma 2.42 that, if $\widehat{h}_\infty$ is the solution to (2.53) with the subcritical $\widehat{f}(s)=q^{-1}f(sq)$ instead of f, then $h_\infty(s)=q\,\widehat{h}_\infty(s)$. Denoting by $\{\widehat{p}_{jk}:k\ge 0\}$ the coefficients of the power series expansion of $\widehat{f}(s)^j$, the relation

$$\widehat{f}(s)^j \;=\; \frac{f(sq)^j}{q^j} \;=\; \sum_{k\ge 0}\mathbb{P}_j(Z_1=k)q^{k-j}s^k \;=\; \sum_{k\ge 0}p_{jk}q^{k-j}s^k$$

obviously implies $\widehat{p}_{jk}=p_{jk}q^{k-j}$ for all $j,k\in\mathbb{N}_0$. Consequently, by utilizing (2.60) for $\widehat{h}_\infty$ (naturally with $\widehat{p}_{j,j-i}$ instead of $p_{j,j-i}=\mathbb{P}(S_j(1)=j-i)$), we finally arrive at

$$h_\infty(s)^i \;=\; q^i\,\widehat{h}_\infty(s)^i \;=\; q^i\sum_{n\ge i}\frac{i}{n}\,\widehat{p}_{n,n-i}\,s^n \;=\; \sum_{n\ge i}\frac{i}{n}\,p_{n,n-i}\,s^n$$

and thus (2.60) for h_∞. □

Proof (of Thm. 2.41). (a) *Supercritical case*: As in Sect. 2.1, let g_n be the inverse of f_n on $[q,1]$. Then $k_n := (1-g_n(s))^{-1}$ for $n\ge 0$ and some $s\in(q,1)$ provides us with a Heyde-Seneta norming, that is $W_n^* = k_n^{-1}Z_n$ converges a.s. to a limit W^* which is positive on the event of survival $\{Z_n\to\infty\}$ [☞ the proof of Thm. 2.1]. Here we must verify that W^* satisfies the first statement of (2.48). To this end, we recall from (2.17) that, for any $\mathsf{m}_0\in(0,\mathsf{m})$, the inequality

$$\mathsf{m}_0^{n-j} \;\le\; \frac{1-g_j(s)}{1-g_n(s)} \;=\; \frac{k_n}{k_j} \;\le\; \mathsf{m}^{n-j}$$

holds true for all $n\ge j\ge J(s,\mathsf{m}_0)$ and a suitable choice $J(s,\mathsf{m}_0)\in\mathbb{N}_0$. Pick any $\varepsilon\in(0,\mathsf{m}-1)$ and put $J=J(s,\mathsf{m}-\varepsilon)$ and $\nu=\sup\{n\ge 0: W_n^*\ge(1+\varepsilon)W^*\}\vee J$. Then it follows that

$$\begin{aligned}
\frac{Y_n}{k_n} \;&=\; \frac{Y_\nu}{k_n} \;+\; \sum_{j=\nu+1}^{n}\frac{Z_j}{k_j}\cdot\frac{k_j}{k_n}\\
&\le\; \frac{Y_\nu}{k_n} \;+\; (1+\varepsilon)W^*\sum_{j=\nu+1}^{n}(\mathsf{m}-\varepsilon)^{j-n}\\
&\le\; \frac{Y_\nu}{k_n} \;+\; \frac{(1+\varepsilon)(\mathsf{m}-\varepsilon)W^*}{\mathsf{m}-\varepsilon-1}
\end{aligned}$$

for all $n\ge\nu$ and therefore

$$\limsup_{n\to\infty}\frac{Y_n}{k_n} \;\le\; \frac{\mathsf{m}W^*}{\mathsf{m}-1}\quad\text{a.s.}$$

For a reverse inequality, note that $k_{n-j}^{-1}k_n\to\mathsf{m}^j$ for $n\to\infty$ implies

$$\liminf_{n\to\infty}\frac{Y_n}{k_n} \;\ge\; \liminf_{n\to\infty}\sum_{j=0}^{N}\frac{Z_{n-j}}{k_{n-j}}\cdot\frac{k_{n-j}}{k_n} \;=\; W^*\sum_{j=0}^{N}\mathsf{m}^{-j}\quad\text{a.s.}$$

for any $N\ge 0$. Since the last sum converges to $\frac{\mathsf{m}}{\mathsf{m}-1}$ as $N\to\infty$ we have proved the first statement of (2.48). But the second one follows exactly the same way with $k_n=\mathsf{m}^n$ for $n\ge 0$.

(a) *Subcritical case*: It obviously suffices to verify that, for some $s\in(0,1)$,

$$\mathbb{E}\left(s^{Y_n/n}|Z_n>0\right) \;=\; \frac{h_n(s^{1/n})-h_n^0(s^{1/n})}{1-f_n(0)} \;\to\; s^\theta\quad(n\to\infty), \tag{2.62}$$

where $h_n^0(s) := \mathbb{E}(s^{Y_n}\mathbf{1}_{\{Z_n=0\}})$ for $n\in\mathbb{N}_0$. It is a simple exercise [☞ Problem 2.52] to show that, for $s\in[0,1]$,

$$h_0^0(s)=0,\quad h_{n+1}^0(s)=s\,f\circ h_n^0(s),\quad h_{n+1}^0(s)\ge h_n^0(s)\quad\text{and}\quad h_n^0(s)\uparrow h_\infty(s).$$

In particular, the h_n^0 fulfill the same recursion as the h_n [☞ (2.52)] and converge to the same limit, however from below, so that $h_n^0\le h_\infty\le h_n$ for all $n\ge 0$. Defining

$$\gamma(s)=s\,f'\circ h_\infty(s),\quad \varphi_n(s)=\frac{h_n(s)-h_\infty(s)}{\gamma(s)^n}\quad\text{and}\quad \psi_n(s)=\frac{h_\infty(s)-h_n^0(s)}{\gamma(s)^n},$$

we now infer

$$\frac{h_n(s)-h_n^0(s)}{1-f_n(0)}\;=\;\frac{\gamma(s)^n(\varphi(s)+\psi(s))}{1-f_n(0)}\tag{2.63}$$

for $s\in[0,1]$ and $n\in\mathbb{N}_0$. Using (2.52), (2.53) and $h_0(s)=s$, we obtain

$$\begin{aligned}\varphi_n(s)\;&=\;\frac{s-h_\infty(s)}{\gamma(s)^n}\prod_{j=1}^n\left(\frac{h_j(s)-h_\infty(s)}{h_{j-1}(s)-h_\infty(s)}\right)\\&=\;\frac{s-h_\infty(s)}{s\,f'\circ h_\infty(s)^n}\prod_{j=1}^n\left(\frac{f\circ h_{j-1}(s)-f\circ h_\infty(s)}{h_{j-1}(s)-h_\infty(s)}\right)\end{aligned}$$

for $s\in[0,1]$ and $n\in\mathbb{N}_0$ and thereupon by n-fold application of the mean value theorem

$$\varphi_n(s)\;=\;(s-h_\infty(s))\prod_{j=1}^n\left(\frac{f'\circ\eta_j(s)}{f'\circ h_\infty(s)}\right)\tag{2.64}$$

for appropriate $\eta_j(s)$ satisfying $h_\infty(s)\le\eta_j(s)\le h_j(s)$. The two simple estimates

$$\begin{gathered}0\;\le\;h_j-h_\infty\;\le\;f\circ h_{j-1}-f\circ h_\infty\;\le\;\mathsf{m}(h_{j-1}-h_\infty)\;\le\ldots\le\;\mathsf{m}^j,\\ f'\circ\eta_j-f'\circ h_\infty\;\le\;f'\circ h_j-f'\circ h_\infty\;\le\;f''(1)\mathsf{m}^j,\end{gathered}$$

valid on $[0,1]$, show the uniform convergence of the series $\sum_{j\ge1}(f'\circ h_j-f'\circ h_\infty)$ on $[0,1]$. As a consequence, the product in (2.64) converges uniformly on any compact subset of $(0,1]$, i.e., φ_n increases to a continuous function φ_∞ on this interval. By Dini's theorem [☞ [30, p. 143]], the convergence is also compactly uniform which in turn entails that

$$\lim_{n\to\infty}\varphi_n(s^{1/n})\;=\;\varphi_\infty(1)\;=\;\lim_{n\to\infty}\varphi_n(1)\;=\;0.\tag{2.65}$$

One can show in the same manner that ψ_n decreases to a continuous function ψ_∞ uniformly on compact subsets of $(0,1]$ [☞ Problem 2.52], whence

$$\lim_{n\to\infty}\psi_n(s^{1/n})\;=\;\psi_\infty(1)\;=\;\lim_{n\to\infty}\mathsf{m}^{-n}(1-f_n(0))\;=\;c,\tag{2.66}$$

where c denotes the limit of $\mathsf{m}^{-n}\mathbb{P}(Z_n>0)$, which by Kolmogorov's theorem 2.13 exists and is positive, the latter because of $\sigma^2\in\mathbb{R}_>$.

In view of (2.63) and (2.64) it remains to examine the behavior of $\gamma(s^{1/n})^n/(1-f_n(0))$ as $n\to\infty$. Since $h'_\infty(1)=(1-\mathsf{m})^{-1}$, which may easily be deduced from

(2.53), it follows that

$$(f'\circ h_\infty)'(1) \;=\; f''\circ h_\infty(1)h_\infty'(1) \;=\; \frac{f''(1)}{1-\mathsf{m}} \;=\; \frac{\sigma^2-\mathsf{m}(1-\mathsf{m})}{1-\mathsf{m}} \;=\; \mathsf{m}(\theta-1)$$

and thereby, for some $\rho(s)$ with $\lim_{s\uparrow 1}\rho(s)=0$,

$$f'\circ h_\infty(s) \;=\; \mathsf{m}\Big(1-(\theta-1-\rho(s))(1-s)\Big).$$

Finally noting that $\lim_{n\to\infty} n(1-s^{1/n})=\log s$, we obtain

$$\gamma(s^{1/n})^n \;=\; s\mathsf{m}^n\Big(1-(\theta-1-\rho(s^{1/n}))(1-s^{1/n})\Big)^n \;\simeq\; s\mathsf{m}^n e^{(\theta-1)\log s} \;=\; \mathsf{m}^n s^\theta,$$

and thus $\lim_{n\to\infty}\gamma(s^{1/n})^n/(1-f_n(0))=s^\theta/c$. Assertion (2.62) is now concluded by combining this result with (2.63), (2.65) and (2.66).

(c) *Critical case*: Considering the conditional LT of $n^{-2}Y_n$ given $Z_n>0$, it must me shown that, as $n\to\infty$,

$$\mathbb{E}(e^{-tY_n/n^2}|Z_n>0) \;=\; \frac{h_n(e^{-t/n^2})-h_n^0(e^{-t/n^2})}{1-f_n(0)} \;\to\; \frac{2\sigma(2t)^{1/2}e^{-\sigma(2t)^{1/2}}}{1-e^{-2\sigma(2t)^{1/2}}}$$

for all $t>0$. Despite many similarities to the arguments given for the subcritical case, the proof is too technical to be presented here. We refer instead to the original work by PAKES [24]. $\square$

Problems

Problem 2.47. Under the assumptions stated at the beginning of this section, prove the following assertions: For each $n\in\mathbb{N}_0$,

$$\mathbb{E}Y_n \;=\; \begin{cases} \dfrac{1-\mathsf{m}^{n+1}}{1-\mathsf{m}}, & \text{if } \mathsf{m}\neq 1,\\ n+1, & \text{if } \mathsf{m}=1,\end{cases}$$

and if the reproduction variance σ^2 is finite, then furthermore

$$\mathbb{V}\mathrm{ar}\,Y_n \;=\; \begin{cases} \dfrac{\sigma^2}{(1-\mathsf{m})^2}\left(\dfrac{1-\mathsf{m}^{2n+1}}{1-\mathsf{m}}-(2n+1)\mathsf{m}^n\right), & \text{if } \mathsf{m}\neq 1,\\ \dfrac{1}{6}n(n+1)(2n+1)\sigma^2, & \text{if } \mathsf{m}=1.\end{cases}$$

Problem 2.48. Consider the distributional equation (2.57) stated in Rem. 2.43, but now for nonnegative random variables Y_∞ that are not necessarily integer-valued. Let $\mathscr{L}$ denote the associated class LT's of distributions on $\mathbb{R}_\geq \cup \{\infty\}$. Recall that $\widehat{f}(s) = q^{-1} f(sq)$ and $\widehat{h}(s) = sq/f(sq) = s/\widehat{f}(s)$ for $s \in [0,1]$. Prove the following assertions:

(a) If φ denotes the LT of a solution Y_∞, then

$$\varphi(t) \;=\; e^{-t} f \circ \varphi(t) \tag{2.67}$$

for all $t \in \mathbb{R}_\geq$ and $\varphi(0+) = \lim_{t\downarrow 0} \varphi(t) = q$.

(b) If $\mathsf{m} \leq 1$, the solution to (2.67) in $\mathscr{L}$ is unique and given by $\varphi(t) = \widehat{h}^{-1}(e^{-t})$ for $t \in \mathbb{R}_>$.

(c) If $\mathsf{m} > 1$ and $q > 0$, then the unique solution to (2.67) in $\mathscr{L}$ is given by $\varphi(t) = q\widehat{\varphi}(t)$, where $\widehat{\varphi}$ denotes the unique (by part (b)) solution to (2.67) with $\widehat{f}$ instead of f, thus $\varphi(t) = q\widehat{h}^{-1}(e^{-t})$.

Problem 2.49. In the situation of the proof of Thm. 2.40, show that

$$\mathbb{P}(S_n(s) = k) \;=\; \mathbb{P}(S_n(1) = k)\,\frac{s^k}{f(s)^n}$$

for all $s \in [0,1]$ and $k, n \in \mathbb{N}_0$. [Hint: Find the gf of $S_n(s)$.]

Problem 2.50. Find the distribution of the total progeny Y_∞ in the cell-splitting case where $p_0 = 1 - p_2$.

Problem 2.51. Formulate the results of Thm. 2.41 in the case of $k \geq 2$ ancestors and explain.

Problem 2.52. In the situation of Thm. 2.41(b), show that

(a) [☞ after (2.62)] the functions h_n^0, $n \geq 0$, defined there satisfy the recursion (2.52) and are increasing to h_∞ on $[0,1]$.

(b) [☞ after (2.65)] the functions ψ_n, $n \geq 0$, defined there are decreasing to a continuous function ψ_∞ on $(0,1]$, the convergence being uniform on compact subsets of $(0,1]$.

Problem 2.53. For $s,t \in [0,1]$ and $n \in \mathbb{N}_0$, let $H_n(s,t) := \mathbb{E}(s^{Y_n} t^{Z_n})$ denote the two-dimensional gf of (Y_n, Z_n). Show that:

(a) $H_0(s,t) = st$ and

$$H_{n+1}(s,t) \;=\; s\, f \circ H_n(s,t)$$

for all $s,t \in [0,1]$ and $n \geq 0$.

(b) If σ^2 is finite and if κ_n, ρ_n denote the covariance of Y_n, Z_n and its correlation coefficient, respectively, then

$$\kappa_n = \begin{cases} \dfrac{\sigma^2}{1-\mathsf{m}}\left(n\mathsf{m}^{n-1} - \mathsf{m}^n\left(\dfrac{1-\mathsf{m}^n}{1-\mathsf{m}}\right)\right), & \text{if } \mathsf{m} \neq 1, \\ \dfrac{1}{2}n(n+1)\sigma^2, & \text{if } \mathsf{m} = 1. \end{cases}$$

for all $n \geq 1$ and

$$\lim_{n\to\infty} \rho_n = \begin{cases} 0, & \text{if } \mathsf{m} < 1, \\ \dfrac{3^{1/2}}{2}, & \text{if } \mathsf{m} = 1. \\ 1, & \text{if } \mathsf{m} > 1. \end{cases}$$